

## THE MODULI SPACE OF ABELIAN VARIETIES AND THE SINGULARITIES OF THE THETA DIVISOR

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### Introduction

The object of study here is the singular locus of the theta divisor  $\Theta$  of a principally polarized abelian variety  $(X, \Theta)$ . The special case when  $(X, \Theta)$  is the Jacobian of a curve  $C$  shows that this is meaningful: singularities of  $\Theta$  are closely related to the existence of special linear systems on the curve  $C$  and for curves of genus  $g \geq 4$  the divisor  $\Theta$  is always singular. But for the general principally polarized abelian variety the theta divisor  $\Theta$  is smooth. In their pioneering work [1] Andreotti and Mayer introduced in the moduli space  $\mathcal{A}_g$  of principally polarized abelian varieties the loci  $N_k$  of those principally polarized abelian varieties for which  $\Theta$  has a  $k$ -dimensional singular locus:

$$N_k = \{[(X, \Theta)] \in \mathcal{A}_g : \dim \text{Sing}(\Theta) \geq k\} \quad k \geq 0.$$

(Warning: we shall use a slightly different definition of the  $N_k$  in this paper.) They were motivated by the Schottky problem of characterizing Jacobian varieties among all principally polarized abelian varieties. They showed that  $N_0$  is a divisor and that the image of the moduli space of curves under the Torelli map is an irreducible component of  $N_{g-4}$  for  $g \geq 4$ . In a beautiful paper [18] Mumford calculated the cohomology class of the divisor  $N_0$  and used it to show that the moduli space  $\mathcal{A}_g$  is of general type for  $g \geq 7$ . Another reason for interest in the loci  $N_k$  might be the study of cycles on the moduli space  $\mathcal{A}_g$ . The theory of automorphic forms seems to suggest that there exist many algebraic cycles, but it seems very difficult to find them. The  $N_k$  yield many interesting examples of cycles. Unfortunately, for  $k \geq 1$  our knowledge

of the codimension of the  $N_k$ , let alone of the irreducible components of  $N_k$ , is very limited.

In this paper we give a new result on the codimension of the  $N_k$  and formulate a conjectural lower bound for the codimension of the  $N_k$  in  $\mathcal{A}_g$ .

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It is a great pleasure for the authors to dedicate this paper to the illustrious four mathematicians Atiyah, Bott, Hirzebruch and Singer, who with their insight and personality changed mathematics in the second half of the 20th century in such a decisive way.

### 1. Review of known results

Let  $(X, \Theta_X)$  be a principally polarized abelian variety of dimension  $g$  over  $\mathbf{C}$ . We shall assume that  $g \geq 2$ . Then  $\Theta = \Theta_X$  is an ample effective divisor with  $h^0(\Theta) = 1$  and  $\Theta$  defines an isomorphism

$$\lambda : X \xrightarrow{\approx} \hat{X}, \quad x \mapsto [\Theta - \Theta_x],$$

of  $X$  with its dual abelian variety  $\hat{X}$ ; here  $\Theta_x$  stands for the translate of  $\Theta$  over  $x$ .

We denote by  $\mathcal{A}_g$  the moduli space of principally polarized abelian varieties over  $\mathbf{C}$ . This is an orbifold of dimension  $g(g+1)/2$ .

The basic objects that we are interested in here are the loci

$$N_k := N_{g,k} = \overline{\{[(X, \Theta_X)] \in \mathcal{A}_g(\mathbf{C}) : \dim \text{Sing}(\Theta_X) = k\}} \\ (0 \leq k \leq g-2)$$

where the overline means Zariski closure. Note that this definition is slightly different from the definition of Andreotti-Mayer and the one used by Mumford. Andreotti and Mayer introduced these loci in 1967 in relation with the Schottky problem. To explain the relation, we consider the moduli space  $\mathcal{M}_g$  of irreducible, smooth complete curves of genus  $g$  over  $\mathbf{C}$ . Then there is the Torelli map

$$t : \mathcal{M}_g(\mathbf{C}) \rightarrow \mathcal{A}_g(\mathbf{C})$$

given by

$$[C] \mapsto [(X = \text{Jac}(C), \Theta \subset \text{Pic}^{g-1}(C))],$$

where  $\Theta$  is the divisor of effective line bundles of degree  $g-1$  in  $\text{Pic}^{g-1}(C)$  and this defines a divisor up to translations, again denoted by  $\Theta$ , on  $\text{Jac}(C)$ . Riemann showed (see [1]) that for a Jacobian of genus  $g \geq 4$  the singular locus  $\text{Sing}(\Theta)$  has dimension  $g - 4$  unless the curve  $C$  is hyperelliptic; in that case  $\text{Sing}(\Theta)$  has dimension  $g - 3$  and there is even an explicit description of  $\text{Sing}(\Theta)$ :

$$\text{Sing}(\Theta) = g_2^1 + W_{g-3}^0 \subset \text{Pic}^{g-1}(C)$$

with  $W_{g-3}^0$  the locus of effective divisor classes (line bundles) of degree  $g - 3$ . We define

$$J_g := \overline{t(M_g)}, \quad \text{the Jacobian locus.}$$

It is an irreducible closed algebraic subset of  $\mathcal{A}_g$  of dimension  $3g - 3$ . We also need  $H_g$ , the hyperelliptic locus in  $\mathcal{M}_g$ , and we put

$$\mathcal{H}_g = \overline{t(H_g)}, \quad \text{the hyperelliptic locus in } \mathcal{A}_g.$$

It is irreducible and of dimension  $2g - 1$ . Now Riemann's results on the dimension of  $\text{Sing}(\Theta)$  for Jacobians imply:

$$J_g \subseteq N_{g,g-4} \quad \text{and} \quad \mathcal{H}_g \subseteq N_{g,g-3}.$$

**(1.1) Theorem. (Andreotti-Mayer).** *i) The Jacobian locus  $J_g$  is an irreducible component of  $N_{g,g-4}$ . ii) The hyperelliptic locus  $\mathcal{H}_g$  is an irreducible component of  $N_{g,g-3}$ .*

The fact that  $N_0$  is a divisor was first noticed in an unpublished version of the Andreotti-Mayer paper. It was also proved by Beauville ([3]) for  $g = 4$  and Mumford observed that Beauville's proof works in the general case.

Mumford calculated in [18] the class of  $N_0$  in the Chow group  $CH^1(\tilde{\mathcal{A}}_g^{(1)})$  for the canonical partial compactification  $\tilde{\mathcal{A}}_g^{(1)}$ . If  $\lambda = \lambda_1$  denotes the first Chern class of the determinant of the Hodge bundle (a line bundle the sections of which are modular forms) and  $\delta$  is the class of the boundary then the result is this:

**(1.2) Theorem. (Mumford).** *The class of  $N_0$  in  $CH^1(\tilde{\mathcal{A}}_g^{(1)})$  is given by*

$$[N_0] = \left( \frac{(g+1)!}{2} + g! \right) \lambda - \frac{(g+1)!}{12} \delta.$$

A variation of the notion of Jacobian varieties is given by *Prym varieties*: take a double étale cover

$$\pi : \tilde{C} \rightarrow C$$

of smooth irreducible curves, where  $\tilde{C}$  has genus  $2g + 1$  and  $C$  has genus  $g + 1$ . The morphism  $\pi$  induces a norm map  $\text{Nm} = \pi_* : \text{Pic}(\tilde{C}) \rightarrow \text{Pic}(C)$ . We now look at the restriction to the degree  $2g$  part

$$\tilde{J} \cong \text{Pic}^{2g}(\tilde{C}) \xrightarrow{\text{Nm}} \text{Pic}^{2g}(C) \cong J$$

and define the Prym variety  $P$  as the connected component of zero in the kernel of  $\text{Nm}$ :

$$P = P(\tilde{C}/C) = \ker \text{Nm}(\tilde{J} \rightarrow J)^0.$$

As it turns out we can also write it as

$$P \cong \{L \in \text{Pic}^{2g}(\tilde{C}) : \text{Nm}(L) = K_C, h^0(L) \equiv 0 \pmod{2}\}.$$

and this is an abelian variety of dimension  $g$ . A principal polarization on  $P$  is provided by the divisor of effective line bundles

$$\Xi = \{L \in P : \text{Nm}(L) = K_C, h^0(L) \equiv 0 \pmod{2}, h^0(L) > 0\}.$$

Let  $R\mathcal{M}_{g+1}$  be the moduli space of such double covers  $\tilde{C} \rightarrow C$ . It is an orbifold of dimension  $3g$  with a natural map  $R\mathcal{M}_{g+1} \rightarrow \mathcal{M}_{g+1}$  defined by forgetting the cover  $\tilde{C}$  of  $C$ . The Torelli map has an analogue for this situation, the Prym-Torelli map:

$$p : R\mathcal{M}_{g+1} \rightarrow \mathcal{A}_g \quad (\tilde{C} \rightarrow C) \mapsto P(\tilde{C}/C).$$

**(1.3) Theorem.** (Friedman-Smith [15], Donagi [11], [12]). *The morphism  $p$  is dominant for  $g \leq 5$ ; it is birational to its image for  $g \geq 6$ , but not injective.*

Note that the non-injectivity follows immediately from Mumford's description of the Prym varieties of hyperelliptic curves, see [17].

We define a locus in  $\mathcal{A}_g$ :

$$P_g := \overline{p(R\mathcal{M}_{g+1})}, \quad \text{the Prym locus in } \mathcal{A}_g$$

This new locus is of dimension  $3g$  for  $g \geq 5$  and it contains the Jacobian locus:

**(1.4) Theorem. (Wirtinger, Beauville [3]).** *The Prym locus contains the Jacobian locus:  $J_g \subseteq P_g$ .*

The classical result of Riemann on the singular locus of  $\Theta$  for Jacobians has an analogue for Prym varieties. The singular points of  $\Xi$  are of two types. If  $L \in P$  then we have  $L \in \text{Sing}(\Xi)$  if and only if

- i)  $h^0(L) \geq 4$ , or
- ii)  $L$  is of the form  $\pi^*(E) + M$ , where  $M \geq 0$  and  $h^0(E) \geq 2$ .

The singularities of type i) are called *stable* and those of type ii) are called *exceptional*. Welters and Debarre proved that the singular locus of the divisor  $\Xi$  has dimension  $\geq g - 6$ , see [25], [7]. It follows from their work and that of Debarre ([8]) that for a generic Prym variety every singular point of  $\Xi$  is stable and  $\text{Sing}(\Xi)$  is irreducible of dimension  $g - 6$  for  $g \geq 7$ , reduced of dimension 0 for  $g = 6$  and empty if  $g \leq 5$ . Mumford showed that if  $\dim \text{Sing}(\Xi) \geq g - 4$  then  $\text{Sing}(\Theta)$  has an exceptional component and the curve  $C$  is either hyperelliptic, trigonal, bi-elliptic, a plane quintic or a genus 5 curve with an even theta characteristic. By work of Debarre we know that if  $C$  is not a 4-gonal curve then  $\dim \text{Sing}_{\text{exc}}(\Xi) \leq g - 7$  for  $g \geq 10$ . He also gives a beautiful description of exceptional singular locus of 4-gonal curves.

**(1.5) Theorem. (Debarre [8]).** *The Prym locus  $P_g$  is an irreducible component of  $N_{g,g-6}$  for  $g \geq 7$ .*

This shows once more that the components of  $N_{g,k}$  give geometrically meaningful cycles on the moduli space.

What do we know about the structure of the loci  $N_{g,k}$  ? Let us start with  $k = 0$ .

**(1.6) Theorem. (Debarre [9]).** *The divisor  $N_{g,0} = N_0$  consists of two irreducible components for  $g \geq 4$ :*

$$N_0 = \theta_{\text{null}} + 2N'_0.$$

Some explanation is in order here. The generic point of the irreducible divisor  $\theta_{\text{null}}$  corresponds to a polarized abelian variety  $(X, \Theta)$  for which  $\Theta$  has one singularity, a double point at a point of order 2 of  $X$ , while the generic point of the irreducible divisor  $N'_0$  corresponds to an abelian variety  $(X, \Theta)$  where  $\Theta$  has two singularities. Mumford has shown in [17] how the divisor  $N_0$  can be defined scheme-theoretically so

that it comes with multiplicities. The divisor  $\theta_{\text{null}}$  is given as the zero divisor of the modular form given by the product of the  $2^{g-1}(2^g + 1)$  even thetanulls  $\theta_{\left[\begin{smallmatrix} \epsilon_1 \\ \epsilon_2 \end{smallmatrix}\right]}(\tau, z)$ .

**(1.7) Example.** If  $g = 4$  the component  $N'_0$  is the Jacobian locus  $J_4$  as Beauville showed. For  $g = 5$  the component  $N'_0$  can be identified with the locus of intermediate Jacobians of double covers of  $\mathbb{P}^3$  ramified along a quartic surface with 5 nodes, cf. [20], [9].

Mumford showed that for  $k \geq 1$  none of the  $N_k$  have codimension 1:

$$\text{codim}_{\mathcal{A}_g} N_k > 1 \quad \text{if } k \geq 1.$$

At the other extreme we have  $N_{g,g-2}$ . We call a principally polarized abelian variety *decomposable* if it is a product of (positive-dimensional) principally polarized abelian varieties. The singular locus of  $\Theta$  for a decomposable abelian variety has codimension 2. There are natural maps

$$\mathcal{A}_{i_1} \times \dots \times \mathcal{A}_{i_r} \rightarrow \mathcal{A}_g, \quad ([X_{i_1}], \dots, [X_{i_r}]) \mapsto [X_{i_1} \times \dots \times X_{i_r}].$$

We denote the image by  $\mathcal{A}_{i_1, \dots, i_r}$ . Let

$$\Pi_g = \bigcup_{1 \leq i \leq g/2} \mathcal{A}_{i, g-i}$$

be the locus of decomposable abelian varieties in  $\mathcal{A}_g$ . This is a closed algebraic subset whose components have codimension  $\geq g - 1$  in  $\mathcal{A}_g$ .

**(1.8) Proposition.**  $N_{g,g-2} = \Pi_g$  with  $\Pi_g$  the locus of decomposable abelian varieties.

This proposition is a corollary to a fundamental result by Kollár and a result of Ein-Lazarsfeld.

**(1.9) Theorem. (Kollár [16]).** *The pair  $(X, \Theta)$  is log canonical.*

Kollár's result implies that

$$\Theta^{(r)} := \{x \in \Theta : \text{mult}_x(\Theta) \geq r\}$$

has codimension  $\geq r$  in  $X$ . Moreover, Ein and Lazarsfeld prove in [14]

$$\text{codim}_X \Theta^{(r)} = r \iff X \text{ is decomposable as product of } r \text{ p.p.a.v.}$$

and this then implies the proposition, cf. also [22]. Ein and Lazarsfeld also proved that if  $\Theta$  is irreducible then  $\Theta$  is normal and the singularities are rational.

In the following table we collect what is known about components of the  $N_k$ . We do not give multiplicities.

Table for Low Genera

$g \setminus N_k$	$A_g$	$N_0$	$N_1$	$N_2$	$N_3$	$N_4$	$N_5$
2	$A_2 = J_2$	$\Pi_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$A_3 = J_3$	$\mathcal{H}_3$	$\Pi_3$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	$A_4 = P_4$	$\theta_{\text{null}} + J_4$	$\mathcal{H}_4$	$\Pi_4$	$\emptyset$	$\emptyset$	$\emptyset$
5	$A_5 = P_5$	$\theta_{\text{null}} + N'_0$	$J_5 + A + B + C$	$\mathcal{H}_5$	$\Pi_5$	$\emptyset$	$\emptyset$
6	$A_6 = ?$	$\theta_{\text{null}} + N'_0$	$?$	$J_6 + ?$	$\mathcal{H}_6 + ?$	$\Pi_6$	$\emptyset$
7	$A_7 = ?$	$\theta_{\text{null}} + N'_0$	$P_7 + ?$	$?$	$J_7 + ?$	$\mathcal{H}_7 + ?$	$\Pi_7$

It is known by work of Debarre that the three irreducible components of  $N_{5,1}$  different from  $J_5$  have dimensions 10, 9 and 9, cf. [5], [13]. Moreover, in [5] Debarre constructed components of  $N_{g,g-4}$  (resp.  $N_{g,g-6}$ ) (resp.  $N_{g,g-8}$ ) of codimension  $g'(g - g')$  for  $2 \leq g' \leq g/2$  and  $g \geq 5$  (resp. for  $3 \leq g' \leq g/2$  and  $g \geq 7$ ) (resp. for  $4 \leq g' \leq g/2$  and  $g \geq 9$ ) and these thus are part of the question marks in the table at positions  $N_{5,1}$ ,  $N_{6,2}$ ,  $N_{7,3}$  and  $N_{7,1}$ .

2. Bounds and a conjecture on the codimension

As the review of the preceding section may show, very little is known about the components of the loci  $N_{g,k}$ . Apart from Mumford's estimate that  $\text{codim } N_k \geq 2$  for  $k \geq 1$  we know almost nothing about the codimension of the  $N_{g,k}$ . Our new results give some lower bounds for the codimension. Debarre proved in an unpublished manuscript independently that  $\text{codim } N_{g,k} \geq k + 1$  for  $k \geq 1$ .

**(2.1) Theorem.** *Let  $g \geq 4$ . Then for  $k$  with  $1 \leq k \leq g - 3$  we have  $\text{codim } N_{g,k} \geq k + 2$ .*

**(2.2) Theorem.** *Let  $g \geq 5$ . If  $k$  satisfies  $g/3 < k \leq g - 3$  then  $\text{codim } N_{g,k} \geq k + 3$ .*

The first theorem is sharp for  $k = 1$  and  $g = 4, 5$ . However, we do not expect that this is an accurate description of reality and believe that Theorems (2.1) and (2.2) are never sharp for  $k = 1$  and  $g \geq 6$ , or for  $k \geq 2$ . We conjecture the following much stronger bound.

**(2.3) Conjecture.** *If  $1 \leq k \leq g - 3$  and if  $M$  is an irreducible component of  $N_{g,k}$  whose generic point corresponds to a simple abelian variety then  $\text{codim } M \geq \binom{k+2}{2}$ . Moreover, equality holds if and only if  $g = k + 3$  (resp.  $g = k + 4$ ) and then  $M = \mathcal{H}_g$  (resp.  $M = J_g$ ).*

Note that by work of Beauville and Debarre ([3], [5]) the conjecture is true for  $g = 4$  and  $g = 5$ .

We now describe some corollaries of this. Let  $\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$  be the universal family of principally polarized abelian varieties. The reader should view this as a stack, or replace  $\mathcal{A}_g$  by a fine moduli space, e.g. the moduli space of principally polarized abelian varieties with a level 3 structure. We can view  $\mathcal{X}_g$  as the universal family of pairs  $(X, \Theta)$ . In it we can consider the algebraic subset  $S_g$  where the morphism  $\pi|_{\Theta}$  is not smooth. If we write  $\mathcal{A}_g$  as the orbifold  $\mathbf{H}_g/\mathrm{Sp}(2g, \mathbf{Z})$  with  $\mathbf{H}_g$  the upper half plane and  $\mathcal{X}_g$  as the orbifold

$$\mathbf{H}_g \times \mathbf{C}^g / \mathrm{Sp}(2g, \mathbf{Z}) \ltimes \mathbf{Z}^{2g}$$

then  $\Theta$  is given in  $\mathbf{H}_g \times \mathbf{C}^g$  by the vanishing of Riemann's theta function  $\theta(\tau, z) = 0$ , with

$$\theta(\tau, z) = \sum_{n \in \mathbf{Z}^g} e^{\pi i n^t \tau n + 2\pi i n^t z}$$

and  $S_g$  is defined in  $\mathbf{H}_g \times \mathbf{C}^g$  by the  $g + 1$  equations

$$\theta = 0, \quad \frac{\partial \theta}{\partial z_i} = 0 \quad i = 1, \dots, g.$$

Therefore,  $S_g$  has codimension  $\leq g + 1$ . Theorem (2.1) implies that the codimension is equal to  $g + 1$ :

**(2.4) Theorem.** *Every irreducible component of  $S_g = \mathrm{Sing}(\Theta) \subset \mathcal{X}_g$  has codimension  $g + 1$  in  $\mathcal{X}_g$ , hence  $S_g$  is locally a complete intersection.*

*Proof.* Take an irreducible component  $S$  of  $S_g$  and let  $N$  be its image under the natural map  $\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$ . We first assume that  $N$  is not contained in  $\Pi_g$ . Suppose now that  $N$  is contained in  $N_{g,k}$  for some  $k \geq 1$ , and we may assume by (1.8) that  $k \leq g - 3$ . Then by Theorem (2.1) the codimension of  $N$  in  $\mathcal{A}_g$  is at least  $k + 2$ . This implies that the codimension of  $S$  in  $\mathcal{X}_g$  is at least  $g + 2$ , which is impossible. Hence generically, the fibres of  $\pi|_S : S \rightarrow N$  are 0-dimensional and  $N$  must have codimension  $\leq 1$  in  $\mathcal{A}_g$ . So  $S$  maps dominantly to a component of  $N_{g,0}$ , a divisor and we get  $\mathrm{codim} \geq g + 1$ . Finally, if  $N$  is contained in  $\Pi_g$  we observe that  $\Pi_g$  has codimension  $g - 1$  and the fibres have dimension  $g - 2$  leading also to  $\mathrm{codim} \geq g + 1$  and this concludes the proof.

This Corollary of Theorem (2.1) was obtained independently by Debarre in an unpublished note.

**(2.5) Corollary.**  *$N_0$  is a divisor properly containing  $\cup_{k \geq 1} N_k$ .*



This raises the problem about the respective positions for higher  $N_k$ .

**(2.6) Problem.** Is it true that  $N_k$  properly contains  $\cup_{i \geq k+1} N_i$  ?

For the generic point of a component  $N_0$  we know the singularities of  $\Theta$ . In general we know almost nothing about the nature of the singular locus  $\text{Sing}(\Theta)$  of a generic point of a component of  $N_{g,k}$ . For a discussion of the case  $N_1$  we refer to Section 8.

### 3. Deformation theory and the heat equation

In this section we explain Welters' interpretation of the Heat Equation for the theta function, cf. [24]. The Heat Equation is one of the tools for obtaining our estimates on the codimension.

Let  $(X, \Theta)$  be a principally polarized abelian variety of dimension  $g$ . We denote the invertible  $O_X$ -module  $O_X(\Theta)$  associated to  $\Theta$  by  $L$ . The space  $\text{Def}(X)$  of linear infinitesimal deformations of the algebraic variety  $X$  has a well-known cohomological interpretation:

$$\text{Def}(X) \cong H^1(X, T_X),$$

where  $T_X$  denotes the tangent sheaf of  $X$ . The space of linear infinitesimal deformations of the pair  $(X, \Theta)$  or equivalently of the pair  $(X, L)$ , where we consider  $\Theta$  or  $L$  up to translations on  $X$ , is given by

$$\text{Def}(X, L) \cong H^1(X, \Sigma_L),$$

where  $\Sigma_L$  is the sheaf of germs of differential operators of order  $\leq 1$  on  $L$  (sums of functions and derivations). Given now a section  $s \in \Gamma(L)$  we obtain a complex

$$0 \rightarrow \Sigma_L \xrightarrow{d^1 s} L \rightarrow 0, \quad \text{with } d^1 s : D \mapsto D(s)$$

on  $X$  given by associating to a differential operator  $D$  the section  $D(s)$  of  $L$ . The cohomological interpretation of the space of linear infinitesimal deformations of the triple  $(X, L, s)$  is

$$\text{Def}(X, L, s) \cong \mathbb{H}(d^1 s),$$

the hypercohomology of the complex. Explicitly, it can be given as follows: if  $(X_\epsilon, L_\epsilon, s_\epsilon)$  is an infinitesimal deformation, then on a suitable open cover  $U_j[\epsilon]$  of  $X_\epsilon$  the section  $s_\epsilon$  is given as  $s_j + \epsilon \sigma_j$  with

$\sigma_j - \sigma_i = \eta_{ij}(s)$ , where  $\eta_{ij}(s)$  is a cocycle whose class in  $H^1(X, \Sigma_L)$  determines the deformation  $(X_\epsilon, L_\epsilon)$ . So we obtain a 1-cocycle  $(\{\sigma_i\}, \{\eta_{ij}\}) \in C^0(U, L) \oplus C^1(U, \Sigma_L)$  of the total complex associated with

$$\begin{array}{ccccc} C^0(U, \Sigma_L) & \longrightarrow & C^1(U, \Sigma_L) & \longrightarrow & \dots \\ & & \downarrow d^1 s & & \downarrow -d^1 s \\ C^0(U, L) & \longrightarrow & C^1(U, L) & \longrightarrow & \dots \end{array}$$

and we thus have an element of  $\mathbb{H}^1(d^1 s)$ .

The central point is now the following:

**(3.1) Claim.** An element of  $H^0(X, \text{Sym}_2 T_X)$  determines canonically a linear infinitesimal deformation of  $(X, L)$  and  $(X, L, s)$ .

This follows from the first connecting homomorphism of the exact sequence of hypercohomology of the short exact sequence of complexes

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Sigma_L & \longrightarrow & \Sigma_L^{(2)} & \longrightarrow & \text{Sym}_2 T_X \longrightarrow 0 \\ & & \downarrow d^1 s & & \downarrow d^2 s & & \downarrow \\ 0 & \longrightarrow & L & \xrightarrow{\text{id}} & L & \longrightarrow & 0 \longrightarrow 0, \end{array}$$

where  $\Sigma_L^{(2)}$  stands for differential operators of order  $\leq 2$  on  $L$  and  $\text{Sym}_2 T_X$  is the *subspace* of elements fixed by the involution  $(x_1, x_2) \mapsto (x_2, x_1)$  on  $T_X \otimes T_X$ . We thus have the connecting homomorphism of the upper exact sequence of (1)

$$b : H^0(X, \text{Sym}_2 T_X) \longrightarrow H^1(X, \Sigma_L)$$

and the connecting homomorphism of the short exact sequence of complexes (1)

$$\beta : H^0(X, \text{Sym}_2 T_X) \longrightarrow \mathbb{H}^1(d^1 s)$$

such that  $b = f \cdot \beta$  with  $f : \mathbb{H}^1(d^1 s) \longrightarrow H^1(X, \Sigma_L)$  the forgetful map, and we find the morphisms

$$(2) \quad H^0(X, \text{Sym}_2 T_X) \longrightarrow \mathbb{H}^1(d^1 s) \longrightarrow H^1(X, \Sigma_L) \longrightarrow H^1(X, T_X).$$

But it is well-known that we can identify  $H^0(X, \text{Sym}_2 T_X)$  with  $H^1(X, \Sigma_L)$ : we have

$$H^1(X, T_X) \cong H^1(X, \mathcal{O}_X) \otimes T_{X,0} \cong T_{\hat{X},0} \otimes T_{X,0}$$

and using the polarization  $\lambda : X \xrightarrow{\sim} \hat{X}$  we see that the subspace corresponding to deformations preserving the polarizations is

$$\text{Sym}_2 T_{X,0} \subset T_{X,0} \otimes T_{X,0} \cong T_{\hat{X},0} \otimes T_{X,0}.$$

The composition  $H^0(X, \text{Sym}_2 T_X) \rightarrow \mathbb{H}^1(d^1 s) \rightarrow H^1(X, \Sigma_L)$  is therefore an isomorphism. The first spectral sequence for the hypercohomology gives us an exact sequence

$$H^0(\Sigma_L) \rightarrow H^0(L) \xrightarrow{\alpha} \mathbb{H}^1(d^1 s) \rightarrow H^1(\Sigma_L) \rightarrow H^1(L),$$

where  $\alpha(t) = (X = X_\epsilon, L = L_\epsilon, s + t\epsilon)$ . So we get an exact sequence

$$0 \rightarrow H^0(L)/\mathbf{C} \cdot s \rightarrow \mathbb{H}^1(d^1 s) \rightarrow H^1(\Sigma_L) \rightarrow 0$$

which shows that for principally polarized abelian varieties the forgetful map

$$(3) \quad f : \mathbb{H}^1(d^1 s) \rightarrow H^1(X, \Sigma_L)$$

is also an isomorphism: *every deformation  $(X_\epsilon, L_\epsilon)$  of  $(X, L)$  canonically determines a deformation  $s_\epsilon$  of  $s$ .* This is Welters' interpretation of the classical Heat Equations. If we represent  $X$  as a complex torus  $X = \mathbf{C}^g/\Lambda$  with  $\Lambda = \mathbf{Z}^g \tau + \mathbf{Z}^g$ ,  $\tau \in \mathbf{H}_g$ ,  $z \in \mathbf{C}^g$  and  $\Theta$  as before by

$$\theta(\tau, z) = \sum_{n \in \mathbf{Z}^g} e^{\pi i n^t \tau n + 2\pi i n^t z}$$

then it satisfies the relation

$$2\pi i(1 + \delta_{ij}) \frac{\partial \theta}{\partial \tau_{ij}} = \frac{\partial^2 \theta}{\partial z_i \partial z_j}$$

where  $\delta_{ij}$  denotes the Kronecker delta. These are the classical "Heat Equations" for Riemann's theta function.

#### 4. Singularities of theta and quadrics

The tangent cone of a singular point  $x$  of  $\Theta$  with multiplicity 2 defines after projectivization and translation to the origin a quadric  $Q_x$  in  $\mathbf{P}^{g-1} = \mathbf{P}(T_{X,0})$ . Another description is obtained as follows. The singular points of  $\Theta \subset X$  are the points  $x$  where the map  $d^1 s : \Sigma_L \rightarrow L$  of Section 3 vanishes. Replace now in diagram (1) all sheaves by their fibres at  $x$  and denote the resulting maps by the suffix  $(x)$ . Then  $(d^1 s)(x) = 0$  and diagram (1) implies then that at such points  $x$  the map  $(d^2 s)(x)$  factors through

$$(4) \quad (\text{Sym}_2 T_X)_x \longrightarrow L_x.$$

This gives an element of  $L_x \otimes \text{Sym}_2(T_X)_x^\vee \cong H^0(\text{Sym}^2(\Omega_X^1)) \otimes L_x$ . We can view this as an equation  $q_x$  for the projectivized tangent cone  $Q_x$  of  $\Theta$  at  $x$  (if the multiplicity of the point is 2; otherwise it is zero). Note that if  $x \in \text{Sing}(\Theta)$  then  $\mathbb{H}^1((d^1s)(x))$  can be identified with  $L_x$  and we can identify (4) with

$$H^0(X, \text{Sym}_2 T_X) \longrightarrow (\text{Sym}_2 T_X)_x \longrightarrow \mathbb{H}^1((d^1s)(x)) = L_x.$$

The map  $(d^2s)(x) : \text{Sym}_2 T_X \rightarrow L_x$  sends an element  $w$  to 0 if and only if  $q_x(w) = 0$ , i.e., if and only if the quadric  $q_x$  and the dual quadric  $w$  are orthogonal.

Suppose that we have an element  $w \in H^0(X, \text{Sym}_2 T_X)$  determining by (3) an element of  $\mathbb{H}^1(d^1s)$  with corresponding deformation  $(X_\epsilon, L_\epsilon, s_\epsilon)$  of  $(X, L, s)$ . This is given by a cocycle  $(\sigma_i, \eta_{ij})$  representing an element of  $\mathbb{H}^1(d^1s)$ . With respect to a suitable covering  $\{U_i\}$  of  $X_\epsilon$  we can write the section  $s_\epsilon$  as

$$s_i + \sigma_i \epsilon.$$

Identifying  $\mathbb{H}^1((d^1s)(x))$  with  $L_x$  we see that the corresponding element of  $L_x$  is given by  $\sigma_i(x)$ . Suppose that  $x \in X$  deforms to  $x_\epsilon$ . The condition that  $s_\epsilon(x_\epsilon) = 0$  can be translated as follows:

$$s_i(x) + (v_x s_i + \sigma_i(x))\epsilon = 0,$$

where  $v_x$  is the tangent vector to  $X$  at  $x$  corresponding to  $x_\epsilon$ . Since  $s_i(x) = 0$  and  $v_x s_i = 0$  because  $x$  is a singular point, the condition is  $\sigma_i(x) = 0$ , i.e.,  $q_x(w) = 0$ . We thus see:

**(4.1) Lemma.** *Let  $x$  be a quadratic singularity of  $\Theta$ . The infinitesimal deformations of  $(X, \Theta)$  which keep  $x$  on  $\Theta$  are the deformations contained in  $Q_x^\perp \subset \text{Sym}_2(T_X)$ . In particular, the deformations that keep  $x$  a singular point of  $\Theta$  are contained in  $Q_x^\perp$ .*

Let  $R$  be an irreducible component of the locus  $\text{Sing}^{(2)}(\Theta)$  of quadratic singularities of  $\Theta$ , where we are assuming that  $\text{Sing}^{(2)}(\Theta)$  is not empty. We now consider the map

$$\phi : R \rightarrow \mathbf{P}(\text{Sym}_2(T_X)_x^\vee) = \mathbf{P}(T_{\mathcal{A}_g, [X]}^\vee) \quad x \mapsto Q_x$$

given by associating to  $x \in R$  the quadric  $Q_x \subset \mathbf{P}^{g-1}$ . We identified the space  $\text{Sym}_2(T_{X,0})$  with the tangent space  $T_{\mathcal{A}_g, [X]}$  to the moduli space  $\mathcal{A}_g$  at  $[(X, \Theta)]$ .

Since  $\theta = 0$  and all derivatives  $\partial_j \theta$  vanish on  $R$  the partial derivatives  $\partial_i \partial_j \theta$  are sections of  $O_R(\Theta)$ :

**(4.2) Proposition.** *The map  $\phi$  is given by sections of  $\mathcal{O}_R(\Theta)$ .*

Another way to interpret this is using the exact sequence

$$0 \rightarrow \mathcal{T}_\Theta \rightarrow \mathcal{T}_{X|\Theta} \rightarrow \mathcal{N}_{\Theta, X} \rightarrow \mathcal{T}_\Theta^1 \rightarrow 0,$$

where  $\mathcal{T}_\Theta \cong \mathcal{O}_\Theta^g$  is the tangent sheaf,  $\mathcal{N}_{\Theta, X} \cong \mathcal{O}_\Theta(\Theta)$  is the normal sheaf, and  $\mathcal{T}_\Theta^1 \cong \mathcal{O}_{\text{Sing}(\Theta)}(\Theta)$  is the first higher tangent sheaf of deformation theory and the middle arrow sends  $\partial/\partial z_i$  to  $\partial\theta/\partial z_i$ . The induced Kodaira-Spencer map is

$$\delta : T_{\mathcal{A}_g, X} \rightarrow H^0(\mathcal{T}_\Theta^1) = H^0(\mathcal{O}_{\text{Sing}(\Theta)}(\Theta))$$

which maps  $\partial/\partial\tau_{ij}$  to  $\partial\theta/\partial\tau_{ij}$ , cf. [23]. In this interpretation, for a singular point  $x$  the deformation  $v \in T_{\mathcal{A}_g, X}$  keeps the point  $x$  on  $\Theta$  if and only if  $\delta(v)(x) = 0$ . If we assume for simplicity that  $\text{Sing}(\Theta) = \text{Sing}^{(2)}(\Theta)$  then the image of  $\delta$  is a linear system on  $\text{Sing}(\Theta)$  and we thus find a map

$$\sigma : \text{Sing}(\Theta) \rightarrow \mathbb{P}(H^0(\mathcal{T}_\Theta^1)^\vee) \xrightarrow{\delta^\vee} \mathbb{P}(T_{\mathcal{A}_g, X}^\vee).$$

The Heat Equations tell us that this can be identified with the map  $\phi$  that associates to  $x \in \text{Sing}(\Theta)$  the quadric defined by  $\sum_{i,j} (\partial^2\theta/\partial z_i\partial z_j) x_i x_j$ .

It might happen that all singularities of  $\Theta$  are of higher order. In order to deal with this case we extend the approach to the partial derivatives of  $s = \theta$  which are sections of  $L$  when restricted to singular points of  $\Theta$ . We define

$$R^{(j)} := \{x \in X : m_x(s) \geq j\},$$

with  $m_x$  the multiplicity at  $x$ , the set of points of multiplicity  $\geq j$  of  $\Theta$ ; so  $R^{(0)} = X$ ,  $R^{(1)} = \Theta$ , etc. Suppose that  $s$  is a non-zero-section of  $L$ . Then any partial derivative  $\eta = \partial_v s$  ( $v \in \text{Sym}^{(j)}(T_X)$ ) of weight  $j$  defines a section of  $L|R^{(j)}$ .

If  $\eta$  is a partial derivative of  $\theta$  then it satisfies again the Heat Equation

$$2\pi i(1 + \delta_{ij}) \frac{\partial\eta}{\partial\tau_{ij}} = \frac{\partial^2\eta}{\partial z_i\partial z_j}.$$

The algebraic interpretation is as follows. Given a partial derivative  $\eta$  of weight  $j$  we apply the formalism of Section 3 to  $\eta$  and find a map

$$\text{Sym}_2 T_{X,0} \rightarrow H^0(R^{(j)}, \text{Sym}_2 T_{R^{(j)}}) \rightarrow \mathbb{H}^1(d^1\eta) \rightarrow H^1(R^{(j)}, \Sigma_{R^{(j)}})$$

We claim that  $\eta$  satisfies the heat equation: any linear infinitesimal deformation of  $(X, L)$  determines canonically a deformation  $\eta_\epsilon$  of  $\eta$ . This can be deduced in a way very similar to the earlier case by extending Welters' analysis.

### 5. The tangent space to $N_k$

Instead of looking at a component of  $N_0$  (with its reduced structure) we may look at the space  $\tilde{N}_0$  of triples defined by

$$S_g = \tilde{N}_0 = \{[(X, \Theta, x)] : x \in \text{Sing}(\Theta)\} \subset \mathcal{X}_g,$$

where  $\mathcal{X}_g$  is the universal abelian variety over  $\mathcal{A}_g$ . This has to be taken in the sense of stacks or one has to work with level structures. We have a natural map  $\pi : \tilde{N}_0 \rightarrow N_0$ . By Lemma (4.1) the image under  $d\pi$  of the Zariski tangent space of  $\tilde{N}_0$  at a point  $[(X, x)]$  is contained in the space  $q_x^\perp \subset \text{Sym}_2(T_{X,0})$ .

We shall call an abelian variety  $X$  simple if it does not contain abelian subvarieties  $\neq X$  of positive dimension. The reason to consider simple abelian varieties is that we then can use the non-degeneracy of the *Gauss map*:

**(5.1) Theorem.** *If  $Z \subset X$  is a positive-dimensional smooth subvariety of a simple abelian variety then the span of the tangent spaces to  $Z$  translated to the origin is not contained in a proper subspace of  $T_{X,0}$ .*

Suppose that for  $(X, \Theta)$  we have  $\text{Sing}^{(2)}(\Theta) \neq \emptyset$  and that  $N_k$  is smooth at  $[(X, \Theta)]$ . The Zariski tangent space to  $N_k$  (with its reduced structure) at  $[(X, \Theta)]$  is contained in the subspace of  $\text{Sym}_2(T_{X,0})$  orthogonal to the linear span in  $\text{Sym}^2(\Omega_X^1)$  of the quadrics  $q_x$  with  $x \in R$  for some  $k$ -dimensional irreducible subvariety  $R$  of  $\text{Sing}(\Theta)$ .

By sending  $x \in R$  to the quadric  $Q_x$  we get a natural map

$$R \longrightarrow \mathbf{P}(N_{N_k}), \quad x \mapsto Q_x$$

with  $\mathbf{P}(N_{N_k})$  the projectivized conormal space to  $N_k$ . The image quadrics have rank  $\leq g - k$  because of the following lemma.

**(5.2) Lemma.** *The Zariski tangent space to  $\text{Sing}^{(2)}(\Theta)$  at a point  $x$  equals  $\text{Sing}(Q_x)$ .*

*Proof.* In local coordinates  $z_1, \dots, z_g$  a local equation of  $\Theta$  at a point  $x$  is

$$f = q_x + \text{higher order terms.}$$

By putting  $q_x = \sum a_{ij} z_i z_j$  we get for  $v = (v_1, \dots, v_g)$  that  $f(z + v) = \sum_{i,j} a_{ij} v_j z_i + \dots$  and we see  $v \in \text{Sing}(Q_x)$ , i.e.,  $\sum_{i,j} a_{ij} v_j z_i = 0$ , is equivalent to  $f(x + v)$  having no linear term, i.e.,  $v \in T_{\text{Sing}^{(2)}(\Theta), x}$ .

**(5.3) Proposition.** *Let  $X$  be a simple principally polarized abelian variety and let  $S$  be an open part of a component of  $\text{Sing}^{(2)}(\Theta)$  where*

the rank of  $Q_x$  is constant, say  $g - d$ . Then the map  $S \rightarrow \text{Gras}(d, g)$ ,  $x \mapsto \text{vertex}(Q_x)$  has finite fibres.

*Proof.* We first note by (5.2) that the tangent space at a point  $x$  to the reduced variety  $S_{\text{red}}$  is contained in the vertex of  $Q_x$ . If  $F$  denotes a fibre of the map  $x \mapsto Q_x$  then the tangents spaces to  $F$  are contained in the subspace which is the vertex of the constant  $Q_x$ . The result then follows from (5.1).

**(5.4) Proposition.** *Let  $X$  be simple and let  $x$  be a quadratic singularity of  $\Theta$  and a smooth point of  $\text{Sing}^{(2)}(\Theta)$ . The general deformation  $w \in Q_x^\perp$  preserves only finitely many singularities of  $\text{Sing}^{(2)}(\Theta)$ .*

*Proof.* The deformations  $w \in Q_x^\perp$  preserving  $y \in \text{Sing}(\Theta)$  are  $Q_x^\perp \cap Q_y^\perp$ . Hence all deformations preserving  $x$  preserve  $y$  if and only if  $Q_x = Q_y$ . But the map  $x \mapsto \text{vertex}(Q_x)$  has finite fibres. q.e.d.

**(5.5) Example.** Let  $C$  be a curve of genus  $g$  and  $L \in \Theta \subset \text{Jac}(C)$  a quadratic singularity of  $\Theta$ . This means that the linear system  $|L|$  defined by  $L$  is a  $g_{g-1}^1$ , i.e., has degree  $g - 1$  and projective dimension 1.

i)  $C$  is hyperelliptic. Then  $L$  is of the form  $g_2^1 + D$ , where  $D$  is a divisor of degree  $g - 3$  on  $C$ . The quadric  $Q_L$  is then the cone projecting the canonical image of  $C$  from the span of  $D$ . The image  $\Sigma$  of the map  $\phi : \text{Sing}(\Theta) \rightarrow \mathbb{P}^{\binom{g+1}{2}-1}$  can be identified with the quadratic Veronese  $V : \mathbb{P}^{g-3} \rightarrow \mathbb{P}^{\binom{g-1}{2}-1} \subset \mathbb{P}^{\binom{g+1}{2}-1}$ . So the normal space to  $N_{g,g-3}$  at  $[\text{Jac}(C), \Theta]$  is the subspace spanned by  $\Sigma \cong \mathbb{P}^{\binom{g-1}{2}-1}$ . Since  $\text{codim}_{A_g} \mathcal{H}_g = \binom{g+1}{2} - \binom{g-1}{2}$  we see that  $\mathcal{H}_g$  is a component of  $N_{g,g-3}$ .

ii)  $C$  is not hyperelliptic. By a theorem of M. Green the space spanned by the quadrics  $Q_x$  with  $x \in \text{Sing}^{(2)}(\Theta)$  is the space of quadrics containing the canonical curve. Since the space of quadratic differentials on  $C$  has dimension  $3g - 3$  it follows that the normal space to  $N_{g,g-4}$  at  $[\text{Jac}(C), \Theta]$  has dimension  $g(g + 1)/2 - (3g - 3)$ . We thus see that  $J_g$  is a component of  $N_{g,g-4}$ .

## 6. A result on pencils of quadrics

One of the ingredients of our proofs is a classical result of Corrado Segre on pencils of quadrics. Judging from the reactions of experts this theorem seems to have been completely forgotten.

**(6.1) Theorem. (C. Segre, 1883).** *Let  $L$  be a linear pencil of singular quadrics of rank  $\leq n + 1 - r$  in  $\mathbb{P}^n$  with  $n \geq 2$  whose generic member has rank  $n + 1 - r$  (i.e., the vertex  $\cong \mathbb{P}^{r-1}$ ). We assume that*

the vertex is not constant in this pencil. Then the Zariski closure of the generic vertex in this pencil

$$V_L = \left( \bigcup_{\text{rk}(Q)=n+1-r} \text{Vertex}(Q) \right)^-$$

is a variety of dimension  $r$  and degree  $m - r + 1$  in a projective linear subspace  $\mathbb{P}^m \subset \mathbb{P}^n$  with  $m \leq (n + r - 1)/2$  and  $r \leq (n + 1)/3$ .

If  $L$  is a pencil of quadric cones whose generic member has rank  $n$  in  $\mathbb{P}^n$  and such that the vertex does not stay fixed then the Zariski closure of the union of the vertices of the rank  $n$  quadrics is contained in the base locus of the family and it is a rational normal curve of degree  $m$  contained in a linear subspace  $\mathbb{P}^m \subset \mathbb{P}^n$  with  $m \leq n/2$ .

For the proof we refer to Segre [19, p. 488-490]. It would be desirable to have an extension of this theorem to higher dimensional linear families of quadrics.

## 7. Sketch of the proof

We now sketch a proof of Theorem (2.1). Let  $M$  be an irreducible component of  $N_k$ . We choose a smooth point  $\xi \in M$  corresponding to a pair  $(X, \Theta)$ . Note that we may assume that the abelian variety  $X$  is simple since the loci of non-simple abelian varieties have codimension  $\geq g - 1$  in  $\mathcal{A}_g$  and  $g - 1 \geq k + 2$  by our assumption on  $k$ . We choose a  $k$ -dimensional subvariety  $R$  of  $\text{Sing}(\Theta_X)$  which deforms. For simplicity we start with the case when the generic point  $x$  of  $R$  is a quadratic singularity of  $\Theta$ .

The construction of the preceding section yields a rational map

$$\begin{array}{c} \phi : R \longrightarrow \mathbb{P}(N_{M/\mathcal{A}_g}^\vee), \quad x \mapsto Q_x \\ \parallel \\ \mathbb{P}^\nu \end{array}$$

Here  $N_{M/\mathcal{A}_g}^\vee$  is the normal space of the component  $M$  in  $\mathcal{A}_g$  and we are assuming that the codimension of  $M$  in  $\mathcal{A}_g$  is  $\nu + 1$ . We also have the Gauss map

$$(5) \quad \gamma : R_{\text{smooth}} \longrightarrow \text{Gras}(k, g), \quad x \mapsto \text{vertex}(Q_x) = \mathbb{P}(T_{R,x})$$

which associates to a smooth point of  $R$  its projectivized tangent space, or equivalently the vertex of the quadric  $Q_x$ . The non-degeneracy of the



Gauss map  $\gamma : R \rightarrow \text{Gras}(k, g)$  implies as in (5.3) that the fibres of  $\phi$  must be zero-dimensional, and this gives immediately  $\nu \geq k$ .

To go further we assume that  $\nu = k$ . Again, using the Gauss map we see that  $\phi$  maps dominantly to  $\mathbb{P}^\nu$ . We consider a pencil  $L$  of quadrics, i.e., a  $\mathbb{P}^1 \subset \mathbb{P}^\nu$ . By (5.2) these quadrics have rank  $\leq g - k$ . The theorem of Segre implies that the Gauss map  $\gamma$  restricted to  $\phi^{-1}(L)$  is degenerate since the vertices lie in a linear subspace of  $\mathbb{P}^{g-1}$ , contradicting (5.1).

If the generic point of  $R$  has higher multiplicity, say  $r$ , then we apply the preceding to the partial derivatives  $\partial_v \theta$  with  $|v| = r - 2$  instead of to the section  $\theta$ . These satisfy the Heat Equations and we proceed with these as with  $\theta$  before. This completes the sketch of proof of Theorem (2.1).

For the proof of Theorem (2.2) we let  $M$  as before be a component of  $N_k$ , we pick a point  $\xi \in M$  corresponding to  $(X, \Theta)$  with  $X$  simple and let  $R$  be a  $k$ -dimensional subvariety of  $\text{Sing}(\Theta)$  which deforms. Assume then that  $\text{codim}(M) = k + 2$  in  $\mathcal{A}_g$ . We get again a rational map of  $R$  to a projective space

$$\phi : R \longrightarrow \mathbf{P}^{k+1}$$

whose image is a hypersurface  $\Sigma$ . We have to distinguish two cases:

- i) Not every quadric corresponding to a point of  $\mathbb{P}^{k+1}$  is singular.
- ii) The general quadric corresponding to a point of  $\mathbb{P}^{k+1}$  is singular, say of rank  $g - r$ .

First we treat the case i). Consider the discriminant locus  $\Delta \subset \mathbb{P}^{k+1}$ . This is a hypersurface of degree  $g$  containing the image  $\Sigma$  of  $R$  with multiplicity at least  $k$ :

$$\Delta = k\Sigma + \Phi.$$

In order to be able to apply Segre's result we use the following well-known Lemma:

**(7.1) Lemma.** *A hypersurface of degree  $\leq 2n - 3$  in  $\mathbb{P}^n$  contains a line.*

So if the degree of the hypersurface  $\Sigma$  satisfies  $\leq 2k - 1$  we have again a line and we can apply Segre's result. Note that since the degree of  $\Delta$  equals  $g$  we have  $\text{deg}(\Sigma) \leq g/k$ , so that  $g/k \leq 2k - 1$  suffices and this follows from  $g/3 < k$ . This rules out case i) if  $\dim(\text{Sing}^{(2)}(\Theta)) \geq k$ . If the generic singularity of  $\Theta$  has higher order we apply the procedure to the higher derivatives as before.

To treat the remaining case ii), where all quadrics parametrized by  $\mathbb{P}^{k+1}$  are singular note that  $r \leq k$  because our quadrics generically have rank  $g - k$  by (5.2) and this should be less than  $g - r$ , the generic rank of the whole family  $\mathbb{P}^{k+1}$ .

If  $r = k$  then by Segre's result we get  $k \leq g/3$ , contrary to our assumption. If  $r < k$  we shall use a refinement of Segre's theorem which says that the number of quadrics in a pencil of quadrics in  $\mathbb{P}^n$  where the rank drops equals  $n + r - 2m - 1$  in the notation of (6.1). Here one has to count a quadric with multiplicity  $d$  if the rank drops by  $d$ . In our case this yields

$$(k - r) \deg \Sigma \leq (g - 1) + r - 2m - 1 \leq g - 2 - r$$

since  $m \geq r$ . We get  $\deg \Sigma \leq (g - 2 - r)/(k - r) \leq 2k - 1$  and this assures us that  $\Sigma$  contains a line. With this line we can apply Segre's theorem to get a contradiction to the non-degeneracy of the Gauss map. This concludes our sketch of proof.

A closer analysis shows that we can draw stronger conclusions from the proof. If  $[X, \Theta]$  is a point of an irreducible component of  $N_1$  such that  $X$  is simple and  $\dim \text{Sing}(\Theta) = 1$  then  $[X, \Theta]$  admits a linear deformation in codimension 3 at most. If  $\text{codim}(N) = 3$  and  $[X, \Theta] \in \text{Sing}(N)$  then the singularities of  $\Theta$  must "get worse."

This approach to getting estimates on the codimension of components of  $N_k$  is by no means exhausted. For example, if  $\text{codim}(N) = k + 3$  then the image of a component  $R$  of  $\text{Sing}(\Theta)$  under the Gauss map is a codimension 2 variety  $\Sigma$  in  $\mathbb{P}^{k+2}$ . We now can use the variety spanned by the secant lines instead of  $\Sigma$  and apply Segre's result to that. We hope to return to this point in the future (joint work with A. Verra)

## 8. An approach to the conjecture for $N_1$

We now restrict to the case of  $N_1$ . Then the codimension is at least 3 and this is sharp: the case when the codimension is 3 occurs.

**(8.1) Example.** Let  $g = 4$ . We consider the hyperelliptic locus  $\mathcal{H}_4$ . If  $X = \text{Jac}(C)$ , the Jacobian of a hyperelliptic curve of genus 4, then the singular locus  $\text{Sing}(\Theta) = g_2^1 + W_1^0$  is a copy of  $C$  as explained above. The class of this curve in the cohomology is  $\Theta^3/3!$ . By associating to each point  $x \in \text{Sing}(\Theta)$  the vertex of the singular quadric  $Q_x \subset \mathbb{P}^3$  we obtain the Gauss map  $C \rightarrow \Gamma = \mathbb{P}^1$  and the image  $\Gamma$  is the rational normal curve of degree 3 in  $\mathbb{P}^3$ . The quadrics containing  $\Gamma$  form a net  $\mathbb{P}^2$  of quadrics. In general for a net of quadrics the curve of vertices is a

curve of degree 6 in  $\mathbb{P}^3$  and the discriminant curve of singular quadrics in  $\mathbb{P}^2$  is a curve  $\Delta$  of degree 4. But in our case the map  $\text{Sing}(\Theta) \rightarrow \Gamma$  is of degree 2 to a rational curve and  $\Delta$  is a conic with multiplicity 2.

**(8.2) Example.** Let  $g = 5$ . We consider the Jacobian locus  $J_5$ . If  $X = \text{Jac}(C)$ , the Jacobian of a curve of genus 5, then the singular locus  $\text{Sing}(\Theta) = W_4^1$  is a smooth curve  $\tilde{D}$  of genus 11 and class  $\Theta^4/4!$  if  $C$  is not trigonal and with no semi-canonical pencils. The quotient of  $\tilde{D}$  under the involution  $-1$  is a curve  $D$  and  $\tilde{D} \rightarrow D$  is a double unramified cover. The Gauss map  $\tilde{D}$  is the Prym canonical map of  $\tilde{D}$  to  $\mathbb{P}^4$ . The map  $\phi : \tilde{D} \rightarrow \mathbb{P}^2$  is a map of degree 2 to a plane quintic  $\Delta$ .

Our Conjecture says that a component of  $N_1$  is of codimension 3 if and only if  $g = 4$  (resp.  $g = 5$ ) and the component in question is  $\mathcal{H}_4$  (resp.  $J_5$ ). A tentative approach to proving this might be the following.

Take an irreducible component  $N$  of  $N_1$  and assume it has codimension 3 in  $\mathcal{A}_g$ . Let  $[X, \Theta]$  be a general point of  $N$  and  $R$  a 1-dimensional component of  $\text{Sing}(\Theta)$ .

**Step i)** Try to prove that the generic point of an irreducible component of  $R$  is a double points of  $\Theta$ . If it is not, we should be able to prove that  $\text{codim}(N)$  is higher than 3.

**Step ii)** Assume that the general quadric  $Q_x$  of the span of  $\Sigma$  is smooth; otherwise use Segre's result. Let  $\Delta$  be the discriminant locus of degree  $g$  in  $\mathbf{P}^2$ . The map  $\phi : R \rightarrow \Sigma \subset \Delta$  has degree  $\geq 2$  since it factors through  $-1$ . Prove that the degree is 2. This seems difficult. It would imply that  $\Theta \cdot R \leq 2g$ .

**Step iii)** We now assume that the class of  $R$  is a multiple  $m\alpha$  of the minimal class  $\alpha = \Theta^{g-1}/(g-1)! \in H^2(X, \mathbf{Z})$ . If it is not, then  $\text{End}(X) \neq \mathbf{Z}$  and this implies that  $\text{codim}(N) \geq g-1$ . By the preceding step we now find  $m \leq 2$ .

**Step iv)** Apply now the Matsusaka-Ran criterion or a result of Welters. This implies that  $X$  is a Jacobian or a Prym variety. The cases  $N \neq \mathcal{H}_4, J_5$  can then be ruled out by the results of Beauville.

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