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SINGULARITIES IN THE WORK OF FRIEDRICH HIRZEBRUCH

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My first love in mathematics was the theorem of Hirzebruch-Riemann-Roch. In my second year as an undergraduate in Munich I took a course on sheaf theory by Helmut Röhrl. Afterwards, Röhrl told us that if he went on for another year, he could tell us about the theorem of Riemann-Roch. I was deeply impressed, and when I asked Karl Stein where I should go to learn about this theorem, I was advised to go to Bonn.

The summer term 1959 was my first semester in Bonn. I enrolled for Hirzebruch's seminar on "Geometry and Topology" and for his course on "Algebraic Topology". I remember the first day of that course. Our teacher standing in front of the class was a very friendly, very young man, less formal than the German professors I had known so far. My first thought was, that it must be his assistant. However, it was Friedrich Hirzebruch himself. In my first letter to my mother from Bonn I wrote "Professor Hirzebruch ist mir sehr sympathisch. Er ist noch sehr jung". I became Hirzebruch's student, and since then my sympathy, my admiration and gratitude has grown continuously.

Hirzebruch had come to Bonn in 1956, after an offer of a chair in Göttingen had been withdrawn as the result of intervention by Carl Ludwig Siegel. Siegel had failed to recognize the significance of the new methods employed so successfully in Hirzebruch's Habilitationsschrift "Neue topologische Methoden in der algebraischen Geometrie", which appeared in 1956. This book, which culminates in the proof of the theorem of Riemann-Roch for complex projective algebraic manifolds, is dedicated to the teachers of Friedrich Hirzebruch, Heinrich Behnke and Heinz Hopf.

After the end of the war, Behnke had quickly restored his contacts with mathematicians in other countries, in particular with Henri Cartan, who together with Jean Pierre Serre applied the modern methods of sheaf theory introduced by Jean Leray in their investigation of Stein manifolds and of algebraic manifolds. Behnke's contacts had also made it possible for the young student Hirzebruch to visit Heinz Hopf in Zürich, who became his second teacher.

From 1952 to 1954 Friedrich Hirzebruch was at the Institute for Advanced Study in Princeton. This was certainly the most important period in his mathematical development, a period of learning, of intensive exchange and cooperation with Armand Borel, Kunihiko Kodaira and D. C. Spencer and, by letter, with René Thom and J. P. Serre. Many important results were obtained during this time, in particular the theorem of Riemann-Roch and large parts of the joint papers with A. Borel on characteristic classes and homogeneous spaces.

I believe that besides his great mathematical ability Friedrich Hirzebruch's personality, his friendly, open-minded, sincere character must have helped in establishing mathematical cooperation and in making friends in the mathematical world only a few years after the horrible crimes committed by Germans in the time of the Third Reich. As an example let me mention that Nicolaas Kuiper once told me that Friedrich Hirzebruch was the first German mathematician who he was able to speak to after the end of the German occupation of his country.

When Hirzebruch came to Bonn he began, of course, to build up a group of students. Both his lectures and his seminars played an important role in this. I have always admired his wonderful and unique style of lecturing. Every new idea appears at the same time spontaneously and naturally exactly at the right place, so much that one feels that one could almost have had the ideas oneself. I remember a talk by Hirzebruch in his seminar on the theorem of Riemann-Roch after which we almost had the impression that we could have discovered it ourselves. The clarity of these lectures becomes even more surprising when one looks at the notes made in preparation for them – just a few formulas scattered on one page or maybe nothing at all. Many lectures were prepared during the five-minute-walk from Hirzebruch's home to the mathematical institute.

For me, Hirzebruch's seminars were even more important than his lectures. One seminar was always called *"Seminar über Geometrie und Topologie"*. It dealt however with a wide variety of modern subjects. There we learnt about such notions as manifolds, fibre bundles, characteristic classes and theories such as homotopy theory, obstruction theory, Morse theory, the index theorem and much, much more. We learned at the same time modern conceptual forms of mathematical thought and the interplay between such general theories and the analysis of wellchosen interesting concrete problems and examples. It was exciting for us to have famous mathematicians like John Milnor in Bonn, lecturing to Hirzebruch's students on the latest theories. The most exciting week in the year was always the *Mathematische Arbeitstagung.*

The Arbeitstagung

The first Arbeitstagung took place in 1957. The participants were Michael Atiyah, Hans Grauert, Alexander Grothendieck, Friedrich Hirzebruch, Nicolaas Kuiper and Jacques Tits. In subsequent years, more names of first rank were added to the list of participants. Instead of trying to make a complete list, let me mention some of those who became particularly faithful friends of the Arbeitstagung. Raoul Bott, Michel Kervaire, John Milnor, Jean-Pierre Serre and René Thom were added to this list in 1958, Frank Adams, Armand Borel and Serge Lang in 1959. In the sixties, James Eells, Günter Harder, Wilfried Schmid and C.T.C. Wall were added to those who frequently contributed to the program of the Arbeitstagung. Of course, more names come to my mind: Palais, Quillen, Remmert, Smale, Van de Ven, Zagier ... Let me stop at this point. The program was decided on in a public program discussion chaired with subtle guidance by Friedrich Hirzebruch. The first lecture was usually given by Michael Atiyah, who contributed more to the Arbeitstagung than anybody else.

Altogether there were thirty meetings of the Arbeitstagung organized by Hirzebruch. The last one took place in 1991. There is now a second series, organized by G. Faltings, G. Harder, Y. Manin, and D. Zagier, but this is another story. Hirzebruch's Arbeitstagung was a unique phenomenon in the mathematics of the second half of the twentieth century. A large part of the history of mathematics of that period is reflected in the annals of the Arbeitstagung, and some of it was written during its meetings. For example, in his Arbeitstagung lecture given 16 July 1962 on "Harmonic Spinors and Elliptic Operators" Atiyah formulated the problem of expressing the index of elliptic operators in terms of topological invariants associated to their symbol and stated the fundamental conjecture for the Dirac operator *"that* $\mathrm{spin}(X, E) = \hat{A}(X, E)$*, where* $\hat{A}$ *is the so-called* $\hat{A}$*-genus (cf. Hirzebruch Ergebnisse book)."* He explained that this included as special cases the Hirzebruch index theorem and the theorem of Riemann-Roch for Kähler manifolds with zero first Chern class.

A few months later, in February 1963, Atiyah and Singer announced the general index formula for elliptic operators on closed manifolds and indicated the main steps of a proof in a note in the Bulletin of the American Mathematical Society. This first proof was modelled closely on Hirzebruch's proof of the Riemann-Roch theorem. K-theory, which gave the essential framework for the statement of the index theorem, had been introduced by Atiyah and Hirzebruch following Grothendieck's lead in their 1959 paper *Riemann-Roch theorems for differentiable manifolds.* In their paper *Vector bundles and homogeneous spaces* they had given the first systematic exposition of this new cohomology theory. The "central and deep point" of this new cohomology theory was the Bott isomorphism.

Bott's famous periodicity theorem $\pi_k(\mathbf{U}) \cong \pi_{k+2}(\mathbf{U})$ published October 1957 in the Proceedings of the National Academy of Sciences had been suggested to Bott by results of Borel and Hirzebruch published later in the paper *Homogeneous spaces and characteristic classes* and by computations of homotopy groups of Lie groups done by Toda. In the paper of Atiyah and Hirzebruch on Riemann-Roch for differentiable manifolds, Bott's theorem for the unitary group was reformulated as an isomorphism

$$K(X \times S^2) \cong K(X) \otimes K(S^2).$$

In this or similar forms, it was applied also in the subsequent paper of Atiyah and Hirzebruch on the Riemann-Roch for analytic embeddings and in the original proof of the Atiyah-Singer index theorem as well as in the later proof by embedding. Conversely, further generalization of the index theorem led Atiyah and Bott to a beautiful elementary proof of the periodicity theorem, which was presented by Hirzebruch during the Arbeitstagung 1963.

The fusion of analysis and topology in the development leading from the theorem of Riemann-Roch to the index theorem and the Lefschetz fixed point formula for elliptic differential operators was one of the most exciting achievements during the three decades of the Arbeitstagung organized by Friedrich Hirzebruch. It was characterized by a vivid interaction between a small group of leading mathematicians, and some part of that interaction happened during the Arbeitstagung in Bonn.

The work of Michael Atiyah presented in Bonn was not the only work discussed at these meetings that won a Fields Medal. Half of the medalists who won the award between 1950 and 1990 gave lectures at the meetings of the first series of the Arbeitstagung.

Of course, Hirzebruch's students tried to make themselves acquainted with the exciting new mathematics presented at the Arbeitstagung. Thus, in 1963, we had a seminar on the Atiyah-Singer index theorem, working hard on trying to understand the details of the proof. Two of us, Karl Heinz Mayer and Klaus Jänich, wrote their PhD-thesis on related subjects. Mayer constructed certain elliptic differential operators and applied the index theorem in order to get an integrality theorem containing as special cases all the integrality theorems previously proved by Borel and Hirzebruch. The possibility of such a unified proof had been indicated by Atiyah in his talk in the Séminaire Bourbaki in May 1963.

Klaus Jänich constructed an isomorphism

$$[X, \mathcal{F}\,] \longrightarrow K(X)$$

where $[X, \mathcal{F}\,]$ is the ring of homotopy classes of maps of a compact space X into the space $\mathcal{F}$ of Fredholm operators of a separable Hilbert space. Jänich presented his result during the Mathematische Arbeitstagung 1964. In the proof he used a new theorem on which Nicolaas Kuiper had lectured during the same Arbeitstagung: "The unitary group of Hilbert space is k-connected". Klaus Jänich and Detlef Gromoll, who spoke on exotic spheres and metrics of positive curvature, were the first students of Hirzebruch to talk at the Arbeitstagung. The index map $[X, \mathcal{F}\,] \to K(X)$ was also constructed in a slightly more general form by Atiyah and was used in the definition of a map $K(S^2 \times X) \to K(X)$ leading to a new simple proof of Bott periodicity.

My own thesis written in 1962 dealt with subjects more in line with the previous work of my teacher. Its first part was a theorem on complex quadrics which was an analogue of a theorem on projective spaces proved by Hirzebruch and Kodaira in 1957. The proof was an application of the theorem of Riemann-Roch, and I had been given that problem because I was in love with this theorem. The second part of my thesis generalized work of one part of Hirzebruch's own thesis, in which he had investigated a particularly nice class of simply connected complex surfaces, namely $\mathbb{P}_1$-bundles over $\mathbb{P}_1$. My generalization dealt with $\mathbb{P}_n$-bundles over $\mathbb{P}_1$, which were also investigated from the new viewpoint of the deformation theory of Kodaira and Spencer.

The Thesis

The thesis of Friedrich Hirzebruch was written in 1950. In the year 2000 we celebrated the 50^{th} anniversary of that event in Münster, the "Goldenes Doktorjubiläum", as it is called in Germany, and there Hirzebruch gave a talk on the other part of his thesis, which has been published under the title *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen.* At this point returning to the beginning of Hirzebruch's work, I am finally approaching the subject given in the title: Singularities in the Work of Friedrich Hirzebruch.

The fame of great mathematicians is justly founded on their great achievements, the creation of new theories and the depth, originality and strength of their mind shown in formulating and solving problems of outstanding importance for the development of our science. In this way the achievements of Friedrich Hirzebruch have been described in the laudations given on the many occasions when he received awards of the highest rank. Instead of repeating such praise I shall try to understand some features in the work of my teacher by asking the questions: What were the objects that he liked? How did he look at them? What did he see?

These questions are not quite as harmless as they might appear, since any attempt to explain the meaning of the words "mathematical objects" must lead to deep philosophical problems. I remember discussions on such matters revealing the belief underlying a whole life devoted to mathematics.

Matthias Kreck has claimed that obviously manifolds are the central objects in Hirzebruch's work. Indeed, manifolds do occur in every work in the two volumes of his collected papers, and in one of these papers he himself writes

"Seit mehr als 30 Jahren beschäftige ich mich mit Mannigfaltigkeiten, besonders mit algebraischen Mannigfaltigkeiten."

But in the same place Hirzebruch mentions *"die Theorie der Singularitäten, die mich seit langem interessiert"*. This interest in singularities began with Hirzebruch's thesis. In the first volume of the collected papers the thesis is the only paper in which singularities play an essential role. However, for the second volume the situation is different; singularities appear in three out of four papers, and in some cases they even appear in the title. So singularities are obviously objects Hirzebruch is interested in. They were among the first objects which he studied, et l'on revient toujours à ses premiers amours.

I think that a case could also be made for yet another and more fundamental entity: number. Integrality problems, divisibility properties, the calculation of integral invariants, and relations between number theory and other fields such as topology, algebraic geometry and analysis on manifolds play a role in many ways in Hirzebruch's work. Finally, we must add to the list of things which Hirzebruch likes, things which are symmetric.

The most venerable symbols of symmetry are the platonic solids, and in particular the icosahedron. In Plato's Timaios the world was conceived as cosmos ordered and shaped by numbers and figures in the best possible way. Anything good had to be beautiful, and beauty was not possible without symmetry. The platonic solids were the elements of Plato's cosmology which has played a very important role in the evolution of European science. Today the icosahedron is to be seen at the entrance of the Max Planck Institute for Mathematics in Bonn. It was founded in 1980 with Friedrich Hirzebruch as its first director for the first fifteen years.

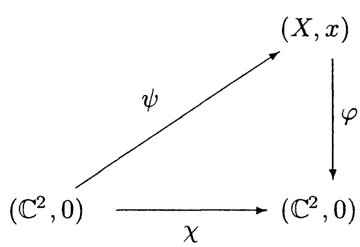
In Hirzebruch's papers singularities are mostly not studied as isolated objects for their own sake. Almost always they occur together with interesting manifolds, frequently in relation to certain symmetric configurations or group actions or in a number theoretic context. There is such a rich variety of beautiful constructions of modern context and relations to classical mathematics that it will be completely impossible to do justice to this work in a few pages. All I can do, is to present some of the themes. For the one which I know best, I shall also try to describe its evolution.

Let me begin with Hirzebruch's thesis. Adding our present knowledge about complex spaces we might summarize its contents as being a constructive resolution of the singularities of 2-dimensional complex spaces. However, at the time when this thesis was written, Heinrich Behnke, Karl Stein, and Henri Cartan had just begun to lay the foundations for the theory of complex spaces. In 1951, Behnke and Stein published a paper in the Mathematische Annalen entitled *"Modifikation komplexer Mannigfaltigkeiten und Riemannscher Gebiete"*, in which they introduced two new notions: the notion of complex space, defined by means of analytic coverings of domains in $\mathbb{C}^k$, and the notion of modification. Also in 1951, Cartan introduced his notion of complex space, modelled on normal analytic subsets of $\mathbb{C}^\ell$. In 1955, Serre allowed arbitrary analytic subsets, so that Cartan's spaces became what is now called normal complex spaces. The relation between the two notions of complex spaces was clarified by Hans Grauert and Reinhold Remmert. In their paper *"Komplexe Räume"*, published 1958 in Mathematische Annalen they proved that the notions of complex space in the sense of Behnke and Stein and in the sense of Cartan were coextensive.

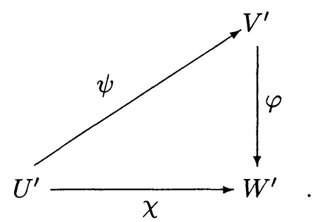
Grauert and Remmert also clarified a question that Hirzebruch had to leave unanswered in his thesis. They proved that every k-dimensional normal complex space can be presented locally as an algebroid covering of a domain in $\mathbb{C}^k$. This means that locally it is the normalization of a Weierstraßcovering defined by an irreducible Weierstraßpolynomial in $\mathbb{C}\{z_1, \ldots, z_k\}[z_{k+1}]$. Hirzebruch's method of resolution uses such local presentations of 2-dimensional complex spaces as algebroid coverings of domains in $\mathbb{C}^2$. This is possible because normal singularities of 2-dimensional complex spaces are isolated so that in dimension two resolution is a local problem.

The discriminant of the Weierstraßpolynomial describing a 2-dimensional algebroid covering defines a curve in a domain of $\mathbb{C}^2$. The first step in Hirzebruch's resolution process consists in resolving the singularities of this curve by a sequence of σ-processes so that the total transform has only normal crossings. The notion of σ-process had been introduced by Hirzebruchs teacher Heinz Hopf in 1951 as a local process of modification of complex manifolds. It modifies a k-dimensional complex manifold X in a point p by replacing p by the $(k-1)$-dimensional complex projective space of tangent directions at p. Hopf knew that in algebraic geometry quadratic transformations were an old and successful method of modifying varieties. Zariski had introduced the notion of a quadratic transformation at a point p of a surface in 1939 in his paper *"The reduction of the singularities of an algebraic surface"*, and in 1943 he defined general monoidal transformations in his paper *"Foundations of a general theory of birational correspondences"*. In dimension two, the σ-process replaces a point by a Riemann sphere, and that is the reason for its name, σ being the first letter of the greek word $\sigma\varphi\alpha\tilde{\iota}\rho\alpha$.

Now let $\varphi : (X, x) \to (\mathbb{C}^2, 0)$ be an algebroid m-fold covering defined by an irreducible Weierstraßpolynomial of degree m in $\mathbb{C}\{z_1, z_2\}[z_3]$ such that the reduced discriminant curve has the equation $z_1 z_2 = 0$. The irreducibility implies that all points of $X' = X - \{x\}$ over $z_1 z_2 = 0$ are branch points of the same order $a-1$ over $z_1 = 0$ and of order $b-1$ over $z_2 = 0$, with integers $a, b > 1$. Define a ramified covering $\chi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\chi(\zeta_1, \zeta_2) = (\zeta_1^a, \zeta_2^b)$. Let (Y, y) be the fibered product of $(\mathbb{C}^2, 0)$ and (X, x) with respect to χ and φ. The normalization of (Y, y) is a multigerm unbranched over $\mathbb{C}^2 - \{0\}$. So it consists of m germs isomorphic to $(\mathbb{C}^2, 0)$. Therefore we have a holomorphic lifting



such that ψ is an unramified covering over a suitable punctured neighbourhood $V_0 - \{x\}$ of x. If n is the degree of ψ, we have $m \cdot n = a \cdot b$. We want a precise description of ψ. Choose a polycylinder $W \subset \mathbb{C}^2$ sufficiently small so that $V = \varphi^{-1}(W) \subset V_0$. Let $U \subset \mathbb{C}^2$ be the polycylinder $\chi^{-1}(W)$. Let W', V', U' be the spaces obtained by removing the axes $z_1 z_2 = 0$ and their inverse images. These spaces have the homotopy type of $\mathbb{C}^* \times \mathbb{C}^*$. In particular, the fundamental groups are abelian. So we have a diagram of regular unramified coverings



Let G and H be the groups of covering transformations of χ and ψ. The group $G \subset GL(2, \mathbb{C})$ is the group of diagonal matrices with diagonal entries the a-th and b-th roots of unity. The subgroup H of order n is also the group of covering tranformations of the regular unramified covering $\psi : U - \{0\} \to V - \{x\}$. Therefore it is a "small" subgroup of $GL(2, \mathbb{C})$ which means that its nontrivial elements have no eigenvalue 1. Therefore H must be one of the cyclic groups of order n

$$C_{n,q} = \left\langle \begin{pmatrix} e^{2\pi i q/n} & 0 \\ 0 & e^{2\pi i 1/n} \end{pmatrix} \right\rangle$$

where q is an integer $0 < q < n$ relatively prime to n. So we obtain the result that the germ (X, x) is isomorphic to the cyclic quotient singularity $(X_{n,q}, 0)$ where $X_{n,q} = \mathbb{C}^2 / C_{n,q}$ may be described as the algebroid covering given by the Weierstraßpolynomial

$$z_3^n - z_1 z_2^{n-q}.$$

Hirzebruch borrows this result from an article by Heinrich W. E. Jung which appeared in 1908 in Crelles Journal. Of course, the modern terminology used above does not occur in Jung's paper. In particular the notion of the quotient of a complex space with respect to a properly discontinuous group of automorphisms was introduced not until 1953/54 when it appeared in the Séminaire Cartan.

In view of the result obtained above, all we have to do is to resolve the singularity of $X_{n,q}$. Hirzebruch constructs a resolution by means of an algorithm taken from the paper of Jung. I shall try to motivate this construction and present it so as to show its relation to the theory of toroidal embeddings developed by Kempf, Knudsen, Mumford and Saint-Donat in 1973. As a matter of fact, Mumford was partly motivated by later work of Hirzebruch on cusp singularities which may be seen as a natural continuation of his thesis.

Let T be the standard complex algebraic torus $\mathbb{C}^{*2} \subset \mathbb{C}^2$. The basic fact is that $X_{n,q}$ contains the algebraic torus $T_{n,q} = T/C_{n,q}$. We shall construct the resolution $\tilde{X}_{n,q} \to X_{n,q}$ by gluing several copies of $\mathbb{C}^2$ which map to $X_{n,q}$ so that T is mapped isomorphically onto $T_{n,q}$.

Let $\overline{X}_{n,q} \subset \mathbb{C}^3$ be the Weierstraßspace given by the equation $z_3^n - z_1 z_2^{n-q} = 0$. Let $X_{n,q} \to \overline{X}_{n,q}$ be the normalization map induced by the map $\mathbb{C}^2 \to \overline{X}_{n,q}$ given by $(z_1, z_2, z_3) = (t_1^n, t_2^n, t_1 t_2^{n-q})$. This maps $T_{n,q}$ isomorphically onto its image $\overline{T}_{n,q} \subset \overline{X}_{n,q}$. Therefore isomorphisms $T \to T_{n,q}$ can be given by

$$\begin{aligned} z_1 &= u^{\lambda} v^{\lambda'} \\ z_2 &= u^{\mu} v^{\mu'} \\ z_3 &= u^{\nu} v^{\nu'} \end{aligned}$$

where the exponents have to satisfy the conditions

$$|\mu\nu' - \mu'\nu| = 1$$

and λ, λ' are determined by the other exponents. Let N be the 2-dimensional lattice of algebraic homomorphisms $\mathbb{C}^* \to T$. The canonical homomorphisms $t \mapsto (t, 1)$ and $t \mapsto (1, t)$ form a canonical basis $(1, 0)$ and $(0, 1)$ of $N = \mathbb{Z}^2$. If we compose $T \to \overline{T}_{n,q}$ with the projection $\overline{T}_{n,q} \to T$ given by $(z_1, z_2, z_3) \to (z_2, z_3)$, we get an isomorphism $T \to T$, and $T \to T_{n,q}$ is determined by the induced map $N \to N$, i.e., by the images (μ, ν) and (μ', ν') of the canonical basis. These two vectors span a sector σ in $\mathbb{R}^3$, consisting of their linear combinations with nonnegative coefficients.

Consider the sector $S \subset \mathbb{R}^2$ spanned by $(0, 1)$ and $(n, n - q)$. If we identify N with $\mathrm{Hom}(\mathbb{C}^*, T_{n,q})$ via the projection $\overline{T}_{n,q} \to T$, the points in $S \cap N$ correspond to those algebraic isomorphisms $\mathbb{C}^* \to T_{n,q}$ which have a limit in $X_{n,q}$ for $t \to 0$. Therefore, an isomorphism $T \to T_{n,q}$ will extend to a holomorphic map $\mathbb{C}^2 \to X_{n,q}$ if and only if its sector σ is contained in S. If we try to construct the resolution $\tilde{X}_{n,q} \to X_{n,q}$ by gluing a finite number $s + 1$ of copies of $\mathbb{C}^2$ with maps $\mathbb{C}^2 \to X_{n,q}$, the condition that $\tilde{X}_{n,q} \to X_{n,q}$ has to be proper, means that the corresponding sectors $\sigma_0, \ldots, \sigma_s$ have to cover S. Such a covering is minimal if the sectors $\sigma_0, \ldots, \sigma_s$ form a subdivision of S, and if their number is minimal. As a matter of fact, there is a unique subdivision

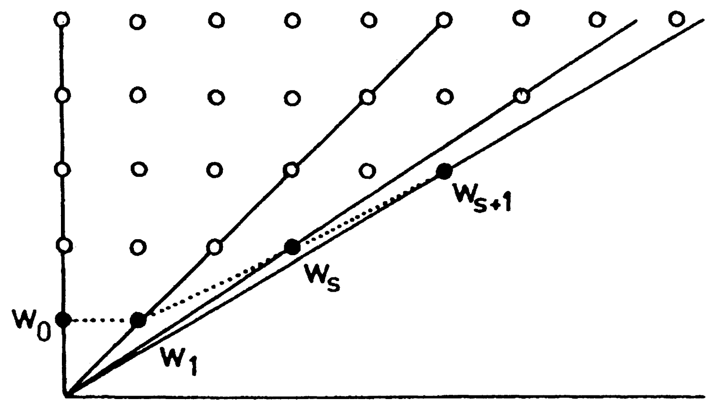


FIGURE 1

of S with that property. Let Σ be the convex hull of $S \cap N - \{0\}$, and let $w_0, w_1, \dots, w_{s+1}$ be the points of N on $\partial\Sigma$ between $w_0 = (0,1)$ and $w_{s+1} = (n, n-q)$. Then S is subdivided by the rays $\mathbb{R}^+ \cdot w_k$ and the sectors between them. Figure 1 illustrates this for Jung's example $(n,q) = (5,2)$.

The vectors $w_k = (\mu_k, \nu_k)$ are computed recursively as follows:

$$\begin{aligned} w_{k+1} &= b_k w_k - w_{k-1} \quad , \quad k = 1, \dots, s \\ w_0 &= (0,1) \\ w_1 &= (1,1). \end{aligned}$$

Here b_k are natural numbers larger than 1 computed from the continued fraction

$$\frac{n}{n-q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_s}}} \quad .$$

This is the Hirzebruch-Jung algorithm. $\tilde{X}_{n,q}$ is obtained by gluing $s+1$ copies of $\mathbb{C}^2$. The gluing transformation from the $(k-1)$-th copy to the k-th copy is

$$\begin{aligned} u_k &= u_{k-1}^{b_k} v_{k-1} \\ v_k &= u_{k-1}^{-1}. \end{aligned}$$

Gluing of these two copies gives the total space of the b_k-th tensor power of the Hopf line bundle over the Riemann sphere. Thus the inverse image

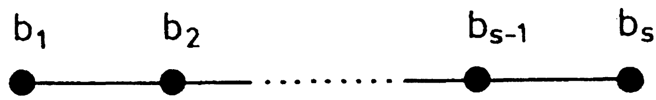


FIGURE 2

of the singular point of $X_{n,q}$ in $\tilde{X}_{n,q}$ is a chain of s nonsingular rational curves with selfintersection numbers $-b_k$, such that only subsequent curves intersect, and they intersect transversely.

Following Hopf's example, Hirzebruch describes the configuration of exceptional curves by a weighted dual graph

There are two interesting extreme cases: $q = 1$ and $q = n - 1$. For $q = 1$, the resolution graph is just one point with value n, and the exceptional curve in $X_{n,1}$ identifies with the zero section in the n-th power of the Hopf bundle. Compactifying that bundle by adding a point at infinity for each complex line of the bundle gives the Σ_n-surface treated in the other half of Hirzebruch's thesis.

The case $q = n - 1$ is characterized by the fact that in this case $C_{n,q}$ is a subgroup of $SL(2, \mathbb{C})$. It is also characterized by the fact that in this case all b_k are equal to two. This can be interpreted as follows. Up to a sign the intersection matrix of the exceptional curve in the resolution of $X_{n,n-1}$ is the Cartan matrix of the root system of type A_{n-1}, and the resolution graph is the Coxeter-Dynkin diagram of type A_{n-1}. This correspondence between $C_{n,n-1}$ and A_{n-1} is part of a perfect correspondence between conjugacy classes of subgroups $G \subset SL(2, \mathbb{C})$ and isomorphism classes of their quotient singularities $(\mathbb{C}^2/G, 0)$ on one hand and simple Lie algebras of type A_n, D_n, E_6, E_7, E_8 on the other hand. In their book "Compact Complex Surfaces" Barth, Peters and Van de Ven say the following about this:

The relation between simple singularities and simple Lie groups is one of the most beautiful discoveries in mathematics. It is impossible to attribute it to a single author.

Friedrich Hirzebruch is one of those who have a share in this discovery, and it is due to him that I too got involved in this on-going story of more than a century.

The importance of Hirzebruch's thesis from a historic point of view is perhaps not primarily to be seen in the fact that he proves the existence of a resolution of singularities of complex surfaces. As a matter of fact, Robert J. Walker had given the first rigorous proof for algebraic surfaces as early as 1935 in a paper in the Annals of Mathematics apparently not known to Hirzebruch at the time when he wrote his thesis. Walker had used essentially the same approach as Hirzebruch, quadratic transformations and Jung's algorithm. Hirzebruch's solution has the merit of clarity and simplicity made possible by his strictly local complex analytic approach as opposed to the projective algebraic methods of the previous proofs. However, the primary importance of this thesis probably is to be seen in the fact that it contained certain germs unfolded in future work of Hirzebruch and his students.

One of these germs is Hirzebruch's remark that the singular point $X_{n,q}$ has a neighbourhood in $X_{n,q}$ bounded by the lens space $L(n,q) = S^3/C_{n,q}$. Lens spaces, constructed by Poul Heegaard in 1898 and by Heinrich Tietze in 1908 were the first examples of closed orientable 3-manifolds not determined by their fundamental groups. In 1918 Alexander proved that $L(5,1)$ and $L(5,2)$ are not homeomorphic although both have the same fundamental group. In 1935 Kurt Reidemeister proved that $L(n,q)$ and $L(n',q')$ are homeomorphic if and only if $n = n'$ and $q' \equiv \pm q \mod n$ or $qq' \equiv \pm 1 \mod n$. On the other hand Hirzebruch proves that the singularities of $X_{n,q}$ and $X_{n',q'}$ are isomorphic if and only if $n = n'$ and $q = q'$ or $qq' \equiv 1 \mod n$.

The interest in the topology of singularities can be traced back to the last decade of the 19^{th} and the first decade of the 20^{th} century, when Poul Heegaard wanted to develop topological tools for the investigation of algebraic surfaces, and when Wilhelm Wirtinger adopted Felix Klein's geometric view of the theory of analytic functions and tried to understand the topology of the ramification of functions of two variables. The fascinating story how this led to the first result of modern knot theory, Tietze's proof that the trefoil knot is not trivial, is told in Moritz Epple's book "Die Entstehung der Knotentheorie". The story is too long to be told here. Let me just indicate in moderately unhistoric terms what Wirtinger did. He studied the algebraic function z of two variables x, y defined by the equation

$$z^3 + 3xz + 2y = 0.$$

In modern terms: the projection of the surface X with this equation to the (x, y)-plane is the semiuniversal unfolding of the 0-dimensional A_2-type singularity $z^3 = 0$. The discriminant curve $D \subset \mathbb{C}^2$ has the equation

$$x^3 + y^2 = 0.$$

The fundamental group of $\mathbb{C}^2 - D$ operates on the fibre over the base point by the monodromy representation. Wirtinger calculates $\pi_1(\mathbb{C}^2 - D)$

and finds a presentation with two generators and one relation $sts = tst$. In modern terms: π_1 is the braid group on 3 strings. The monodromy representation is the canonical homomorphism of this group to the symmetric group S_3.

The group S_3 is the Weyl group of A_2 operating on the plane $z_1 + z_2 + z_3 = 0$ by permutations. If we map this plane to the (x, y)-plane by means of the elementary symmetric functions σ_2, σ_3 and lift the covering of the (x, y)-plane by means of this base extension, we get $(z - z_1)(z - z_2)(z - z_3) = 0$. So we get a trivial covering over the complement of the discriminant $\Pi(z_i - z_j) = 0$. The fundamental group of that complement is the coloured braid group, i.e., the kernel of $B_3 \to S_3$. It is part of the beautiful relation between simple singularities and simple Lie algebras that all this generalizes to all types A_k, D_k, E_6, E_7, E_8.

In his computation of $\pi_1(\mathbb{C}^2 - D)$, Wirtinger used an idea of Heegaard. Heegaard reduced the complex geometry of an algebroid covering $(X, x) \to (\mathbb{C}^2, 0)$ with a singularity $(D, 0)$ of the discriminant to a situation of 3-dimensional topology. He considered a small 4-ball $B \subset \mathbb{C}^2$ centered at 0 with boundary $\partial B = S^3$, a 3-sphere. The intersection $L = D \cap S^3$ is a knot or link in S^3. In Wirtinger's example it is the trefoil knot. $D \cap B \subset B$ is homeomorphic to the cone over L. Therefore $B - D \cap B$ has the same fundamental group as the complement $S^3 - L$ of the link. Let $U \subset X$ be the inverse image of B and $\partial U = M$ the inverse image of S^3. Then M is a 3-manifold, which is a ramified covering of S^3 ramified over L. Moreover, M is a boundary of the neighbourhood U of x in X, and U is homeomorphic to the cone over M. This established a link between the geometry of singularities of complex surfaces and 3-dimensional topology which turned out to be very fruitful both for complex analytic geometry and topology.

The title of Hirzebruch's paper "Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen" and the 1928 paper of Wirtinger's student Brauner in the list of references are indications that Hirzebruch's thesis is to be seen in this context.

$\mathbf{E}_8$

The gist of the story that I want to tell now is expressed in the titles of a talk by Hirzebruch in the Séminaire Bourbaki, given November 1966, and of a paper by myself published in the same year in volume 2 of Inventiones Mathematicae. Hirzebruch's title was *"Singularities and Exotic Spheres"* and mine was *"Beispiele zur Differentialtopologie von*

Singularitäten".

The story begins in the year 1956 with John Milnor's sensational discovery that there are 7-dimensional differentiable manifolds which are homeomorphic but not diffeomorphic to the 7-dimensional sphere S^7. This discovery was certainly one of the most germinal achievements of mathematicians in the twentieth century. In less than seven years a theory describing these exotic differentiable structures was developed by an extraordinary meshing of the results of mathematicians working in diverse parts of topology.

Several fundamental tools and constructions had already been prepared during the two decades preceeding Milnor's discovery. Fibre bundles and the characteristic classes of Stiefel-Whitney, Chern and Pontryagin were at the disposition of differential topology. Cobordism and framed cobordism had been introduced by Thom and Pontryagin, and the signature theorem had been proved by Hirzebruch. In homotopy theory Freudenthal had proved the stability of $\pi_{d+n}(S^d)$ for $d > n+1$ in 1937, and the resulting stable groups Π_n had been proved to be finite for $n > 0$ and had been computed for low values by Serre and Toda. The stable homotopy groups of the classical groups had been computed by Bott. A link between these groups had been established by George Whitehead in 1942. Generalizing a construction by Heinz Hopf in his 1935 paper *"Über die Abbildung von Sphären auf Sphären niedrigerer Dimension"* Whitehead defined a homomorphism

$$J_n : \pi_n(\mathbf{SO}) \longrightarrow \Pi_n$$

from the stable homotopy groups of the orthogonal group to the stable n-stem of the homotopy groups of spheres. Work of Pontryagin culminating in his 1955 paper *"Smooth manifolds and their applications in homotopy theory"* identifies Π_n with the framed cobordism group of framed embedded n-manifolds. The canonical homomorphism from the framed cobordism group to Π_n is defined by means of the Thom-Pontryagin construction.

Some of the most outstanding results of the period following Milnor's discovery were the proofs of the Poincaré conjecture in dimensions greater than four by Stallings, Zeeman and Smale, the development of handlebody theory and the proof of the h-cobordism theorem by Smale and the determination of the image of the J-homomorphism by Frank Adams.

The h-cobordism theorem allows the identification of oriented diffeomorphism classes of topological n-spheres with h-cobordism classes for $n \geq 5$. Let Θ_n be the set of h-cobordism classes of closed oriented

C^∞-manifolds homotopy equivalent to S^n. This is a group with respect to the connected sum operation. Using Bott's calculation of $\pi_n(\mathbf{SO})$, Hirzebruch's signature theorem and the results of Adams, Kervaire and Milnor showed that homotopy spheres are stably parallelizable. Therefore, they can apply the Thom-Pontryagin construction in order to define a homomorphism

$$p : \Theta_n \longrightarrow \text{coker } J_n.$$

The kernel of p is the group bP_{n+1} of classes of oriented homotopy-spheres bounding parallelizable manifolds. The cokernel of p is trivial for $n \not\equiv 2 \mod 4$ and trivial or of order 2 if $n \equiv 2 \mod 4$. Kervaire and Milnor apply the technique of surgery developed by Milnor in order to determine the group bP_{n+1}. For n even bP_{n+1} is trivial. For n odd bP_{n+1} is finite cyclic. The order is 1 or 2 if $n = 4k + 1$. For $n = 4k - 1$, the order ist

$$\sigma_k/8 = 2^{2k-2}(2^{2k-1} - 1) \text{ numerator } (4B_k/k),$$

where B_k denotes the k-th Bernoulli number. Thus Θ_n, $n \neq 3$, is always a finite abelian group, and for n odd the calculation of its order is reduced to the calculation of the order of Π_n (up to a factor 2 if $n = 4k+1$). The first non-zero group Θ_n, $n \neq 3$, is Θ_7. In this case coker J_7 is trivial, and $\Theta_7 = bP_8$ is cyclic of order 28. The first nontrivial group bP_{n+1} with $n = 4k + 1$ is bP_{10}.

An isomorphism $bP_{4k} \to \mathbb{Z}/(\sigma_k/8\mathbb{Z})$ is obtained as follows. Let Σ be a homotopy sphere bounding a $(2k - 1)$-connected parallelizable $4k$-manifold W with signature σ. The intersection form on $H_{2k}(W, \mathbb{Z})$ is symmetric, even and unimodular. Therefore, its signature σ is divisible by 8. The isomorphism maps the class of Σ in bP_{4k} to $\sigma/8 \mod \sigma_k/8$. In particular one obtains a generator for bP_{4k} if $\sigma = 8$. The minimal rank for an even unimodular quadratic form with signature 8 is 8, and up to isomorphism there is only one form with these properties, namely that of the root lattice of E_8.

In a mimeographed manuscript dated Princeton, January 23, 1959, Milnor constructed such a manifold W with this quadratic form. However, the choice of a basis of the lattice with which he begins his construction is not the simplest possible choice, since the graph describing the intersection matrix contains cycles. Up to isomorphism, there is only one choice where the graph is a tree and all intersection numbers are non-negative and 2 on the diagonal. The corresponding graph is the famous Coxeter-Dynkin diagram of E_8, shown in Figure 3.

Hirzebruch noticed the possibility of simplifying Milnor's construction and presented it in a colloquium lecture in Bonn in the winter

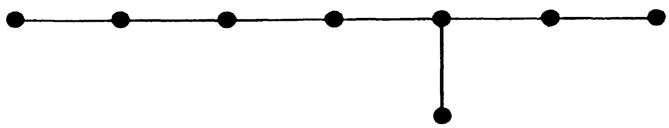


FIGURE 3

1960/61 and in a lecture in Vienna on November 18, 1960. The corresponding publication *"Zur Theorie der Mannigfaltigkeiten"* is Hirzebruch's shortest publication, just one page in the Internationale Mathematische Nachrichten. Milnor adopted Hirzebruch's "lovely" construction and used it in his essay *"Differential topology"* in the book *"Lectures on Modern Mathematics"* edited by Saaty.

Let me describe the construction in somewhat greater generality with a historical perspective. The first examples of exotic 7-spheres constructed by Milnor in 1956 were S^3-bundles over S^4. These may be viewed as boundaries of D^4-disk-bundles over S^4. Thus it may appear to be natural to consider more generally orientable D^m-bundles over S^m. In his paper *"Differentiable Structures on Spheres"* in the Annals of Mathematics 1959 Milnor took such bundles as basic building blocks for a certain construction of manifolds with boundary. For a suitable choice of the building blocks the boundary is an exotic sphere.

The construction is as follows. Take two D^m-bundles $p_i : W_i \to S^m$, $i = 1, 2$ with structure group $SO(m)$. Choose m-disks $U_i \subset S^m$ and trivializations $\varphi_i : D^m \times D^m \to p_i^{-1}(U)$. Let W be obtained from the disjoint union of W_1 and W_2 by identifying $\varphi_1(x, y)$ with $\varphi_2(y, x)$. The result of this is a bounded manifold with corners. Unbending of the corners finally gives a smooth compact orientable manifold W with boundary.

The boundary ∂W may be obtained by gluing two copies of $D^m \times S^{m-1}$ along their boundaries by means of a suitable diffeomorphism. For example, gluing $D^m \times S^{m-1}$ and $S^{m-1} \times D^m$ by means of the identity $S^{m-1} \times S^{m-1} \to S^{m-1} \times S^{m-1}$ gives S^{2m-1}. This was already observed by Hopf in 1935 in his paper mentioned above. The archetypical case of this construction is the Heegaard decomposition of S^3 into two solid tori. This is the case $m = 2$.

However, the construction is older yet. There are reasons to believe that the case $m = 1$ was known to Gauss. Some evidence for this is to be seen in the following two figures. The one on the left hand side

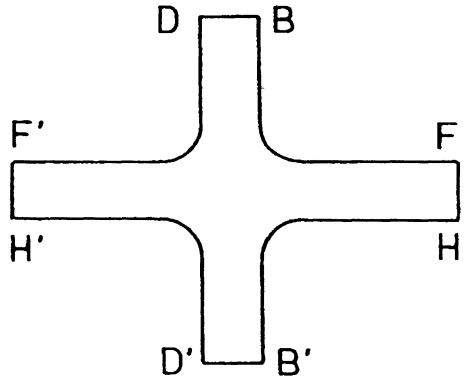


FIGURE 4

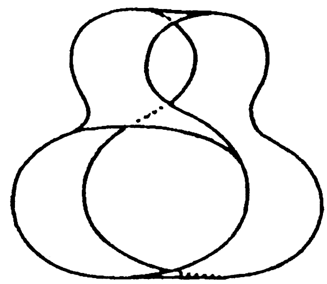


FIGURE 5

dates from the years 1858-60 and is to be found in the collected works of August Ferdinand Moebius, volume 2, page 541. The figure on the right hand side is to be found in the essay *"Der Census räumlicher Komplexe"* published in 1861 by Gauss' student Johann Benedikt Listing. So the simplest case of the construction introduced by Milnor in 1959 was at least one hundred years old and probably known to Gauss.

For a suitable choice of the disk bundles used in Milnor's construction the resulting manifold W will be parallelizable. In particular this will be so, if both copies are the unit-disc bundles in the tangent bundle of S^m. In 1960 Kervaire used the 10-dimensional manifold W obtained in this way for $m = 5$ in order to construct a manifold W_0 which does not admit any differentiable structure. W_0 is the union of W with the cone over ∂W. The boundary ∂W is homomorphic to S^9, but it follows from Kervaire's result that it is not diffeomorphic to it and it is thus the exotic Kervaire sphere generating the cyclic group of order two bP_{10}. An analogous construction can be done for all odd numbers m, and the resulting $(2m-1)$-dimensional Kervaire sphere is the nontrivial element of bP_{2m} whenever bP_{2m} is not trivial.

Milnor's construction can be generalized in various ways. In the first place, one may use more than two disc bundles in the construction, with identifications along disjoint copies of $D^m \times D^m$. The scheme for the construction may be given by a weighted graph. The vertices with weights specify the bundles, the edges are the prescription for the gluing. The graph has to be a tree if we want the resulting manifold M to be highly connected. A further generalization consists in admitting disk-bundles over more general bases such as for example Riemann surfaces of arbitrary genus instead of the Riemann sphere.

A construction of this kind was introduced in a germinal paper by David Mumford submitted to the Publications Mathématiques of the Institut des Hautes Etudes Scientifiques in May 1960 and published in 1961. The title of the paper was *"The topology of normal singularities of an algebraic surface and a criterion for simplicity"*. In this paper Mumford describes certain "good" neighbourhoods of normal singular points x of a complex algebraic surface X in two different ways leading to the same result. One way is to embed (X, x) in some affine space $\mathbb{C}^n$ and to intersect with a sufficiently small ball B^{2n} with center x. The resulting neighbourhood $V = X \cap B^{2n}$ has a boundary $K = \partial V$ which is a 3-dimensional closed orientable manifold. Later work of Hassler Whitney published in 1965 shows that this construction can be generalized to isolated singularities (X, x) of arbitrary dimensions, that the resulting neighbourhood boundary $K = \partial V$ is essentially uniquely determined by (X, x) and that V is homeomorphic to the cone over K.

The second description uses a good resolution $(Y, E) \to (X, x)$ of the 2-dimensional singularity. The exceptional curve E in the complex surface Y is a divisor with normal crossings. Its components E_i are Riemann surfaces intersecting transversely in at most one point. Mumford constructs a smooth boundary M of a tubular neighbourhood of E in Y from building blocks obtained from the normal S^1-bundles of the curves E_i by removing the inverse image of small disks around points where E_i intersects some E_j, $j \neq i$. These building blocks are "patched" by a *"standard plumbing fixture"*

$$\{(x, y, u, v) \,|(x^2 + y^2) \leq \ 1/4, (u^2 + v^2) \leq \ 1/4,$$
$$(x^2 + y^2)^n (u^2 + v^2)^m = \ \varepsilon < 1/8^{n+m}\}.$$

The plumbing fixture is obviously homeomorphic to $S^1 \times S^1 \times [0, 1]$.

Mumford uses this description of the neighbourhood boundary M to derive a presentation of its fundamental group. He then proves the theorem that $\pi_1(M)$ is nontrivial if $x \in X$ is not a regular point. This implies that a normal complex surface which is a topological manifold must be nonsingular.

In the last paragraph of his paper Mumford studies an interesting example. He looks at surfaces in $\mathbb{C}^3$ defined by an equation

$$0 = x^p + y^q + z^r,$$

where p, q and r are pairwise relatively prime, $p < q < r$. Mumford does not resolve these singularities. Instead, he notices that the neighbourhood boundary K is an r-fold branched covering of S^3 branched over a torusknot of type (p,q). Mumford then refers to Herbert Seifert's paper *"Topologie dreidimensionaler gefaserter Räume"* in Acta mathematica 60, 1932, where it is proved that K is a homology 3-sphere. Among these homology spheres, there is only one with finite fundamental group, namely the one for $(p,q,r) = (2,3,5)$. Its fundamental group is the binary icosahedral group, and it is the spherical dodecahedral space, as proved by Seifert and Threlfall in part II of *"Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes"*. Mumford studies the singular point x of the surface X defined by the equation

$$0 = x^2 + y^3 + z^5.$$

He proves that for a resolution $\pi : Y \to X$ we have $(R^1\pi\mathcal{O}_Y)_x = 0$. This is done without an explicit description of the exceptional divisor. In terms of Michael Artin's 1960 Harvard thesis this means that the singularity is rational. This, together with the fact that the neighbourhood boundary is a homology sphere, implies that the local ring $\mathcal{O}_{X,x} = \mathbb{C}\{x,y,z\}/(x^2+y^3+z^5)$ is a unique factorization domain. Actually it is the only nonregular two-dimensional analytic local ring with that property, as I was able to show some years later.

I have now described what was probably known to Hirzebruch when he found the beautiful construction of Milnor's exotic sphere generating bP_{4k}. The construction consists in gluing 8 copies of the tangent-disc-bundle of S^{2k} according to the E_8-scheme. The resulting parallelizable $4k$-manifold is $(2k-1)$-connected and has signature 8. Therefore, its boundary is the Milnor generator of bP_{4k}.

It would be interesting to know whether Hirzebruch's construction had its origin in the remarkable temporal coincidence of the constructions of Milnor and Mumford, one coming from differential topology and the other one from algebraic geometry. There is some evidence for such a fusion of ideas. In February 1963 Hirzebruch gave a talk in the Séminaire Bourbaki reporting on Mumford's paper with a final section "Further remarks", in which he mentions his E_8-construction of the Milnor sphere and points out that certain singularities given by equations have resolution graphs of type A_n, D_n, E_6, E_7, E_8. In this Bourbaki talk Hirzebruch adopts Mumford's term "plumbing" for the construction of manifolds by gluing disk bundles. However, the construction is presented in the way of Milnor, with bending of corners, instead of fitting in Mumford's "plumbing fixture".

In the one page paper *"Zur Theorie der Mannigfaltigkeiten"* Hirzebruch first refers to Milnor's mimeographed notes *"Differentiable manifolds which are homotopy spheres"*, but finishes with the sentences:

"Die Konstruktion wurde motiviert durch die Singularität der affinen algebraischen Fläche $z_1^2 + z_2^3 + z_3^5 = 0$ *in* $(0,0,0)$. *Löst man auf, dann wird der singuläre Punkt aufgeblasen in einen* E_8*-Baum von* 8 *nichtsingulären rationalen Kurven der Selbstschnittzahl* -2.*"*

It is also interesting to take notice of Hirzebruch's commentary on this paper in his collected works. There he writes:

"In dem 2. Teil meiner Dissertation [...] hatte ich zwar die Flächensingularitäten aufgelöst, aber leider, abgesehen von den Quotientensingularitäten $A_{n,q}$ *[...], keine konkreten Beispiele behandelt. Um 1960 lernte ich die heute so berühmten "einfachen" Singularitäten kennen, deren Auflösungsbäume die aus der Theorie der Lieschen Gruppen bekannten Diagramme* $A_{n-1}, D_{n+2}, E_6, E_7, E_8$ *sind* $(n \geq 2;\ A_{n-1} = A_{n,n-1})$. *Ich benutzte die älteren Arbeiten von Patrick Du Val [...]. Später kamen dann sein Buch (Homographies, quaternions and rotations, Oxford University Press 1964) und eine interessante Korrespondenz mit Du Val hinzu, wodurch ich auch die Beziehungen zur Invariantentheorie nach F. Klein kennenlernte. Die Singularitäten wollte ich dann mittels "plumbing" in höheren Dimensionen "imitieren". So kam ich auf die* E_8*-Konstruktion der Milnorschen exotischen Sphäre."*

It is impossible to present in a few pages the historical development to which Hirzebruch alludes in these sentences. I shall restrict myself to a few comments on the names mentioned by Hirzebruch and to a narration of some part of the story in which Hirzebruch and I myself were involved.

Klein's invariant theory came into being in 1874. In his paper *"Über binäre Formen mit linearen Transformationen in sich selbst"* in Mathematische Annalen 9 we find among other things a relation between three invariants T, f and H of the binary icosahedral group acting on the ring of polynomials in two variables. In 1877 Klein writes this relation as

$$T^2 = 12f^5 - 12^4H^3.$$

Essentially the same relation had been found a few years earlier by Hermann Amandus Schwarz. In his paper *"Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt"* published in Crelles Journal, vol. 75 (1872/73) Schwarz considers three polynomials $\varphi_{12}, \varphi_{20}$ and φ_{30} whose roots correspond to the vertices, the midpoints of the faces and the midpoints of the edges of an icosahedron inscribed in the Riemann sphere.

He obtains the identity

$$\varphi_{20}^3 - 4^3 \cdot 3^3 \varphi_{12}^5 = \varphi_{30}^2.$$

Today, we see this as the defining relation between three generators of the ring of invariants $\mathbb{C}[u,v]^G$ of the binary icosehedral groups G acting on $\mathbb{C}^2$, and we identify this ring with the ring of functions on the affine variety $\mathbb{C}^2/G$ imbedded in $\mathbb{C}^3$ and given by such an equation. However, it took a long time until it was possible to see things this way.

An important step relating singularities to root systems and groups generated by reflections was a paper by Patrick Du Val published in 1934 in the Proceedings of the Cambridge Philosophical Society. Du Val considered surface singularities which have a resolution such that all components of the exceptional curve are nonsingular with self-intersection number -2. He classified them by their resolution-graphs, which are, in modern terminology, the Coxeter-Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8. He also describes these singularities as singularities of double coverings of the plane with a description of the singularity of the branch curve. This amounts to writing down equations of the form $z^2 = f(x,y)$. For E_8, or U_{10}^* in Du Val's notation $f(x,y) = y^3 - x^5$. Du Val notices the analogy between his classification and Coxeter's classification of finite groups generated by reflections obtained in the years 1931/34. He shows that the reflection groups, i.e., the Weylgroups of type A_n, D_n, E_6, E_7, E_8 can be used in a systematic discussion of exceptional curves of the first kind and of exceptional configurations of A-D-E-type on rational surfaces. Modern accounts of these matters were given by Manin in his book *"Cubic forms"* and by Demazure in his four talks on Del Pezzo surfaces in the *"Séminaire sur les Singularités des Surfaces"*, 1976/77, dedicated to P. Du Val.

Du Val's 1934 paper had established a link between singularities of type A-D-E and Weyl groups of type A-D-E. On the other hand, around 1960 Hirzebruch realized that these singularities have a relation to the finite subgroups G of $SU(2)$, since their neighbourhood boundaries have the same topological properties as the spherical space forms S^3/G. The exchange with Du Val finally clarified the situation. Du Val identified these singularities with the quotient singularities $\mathbb{C}^2/G$. Those of type A_n correspond to the cyclic groups, the ones of type D_n to the binary octahedral groups, and E_6, E_7, E_8 correspond to the binary tetrahedral, octahedral and icosahedral groups.

When Hirzebruch speaks of "simple" singularities, he is referring to a beautiful discovery of Vladimir Igorevich Arnold made in 1972. In a paper entitled *"Normal forms of functions near degenerate critical points,* *the Weyl groups A_k, D_k, E_k, and Lagrange singularities"* Arnold proved the following theorem:

Every 0-modal germ of an analytic function with an isolated singularity is stably equivalent to one of the germs of type A, D or E at zero; these germs are themselves 0-modal.

Two germs are stably equivalent, if they become equivalent when one adds a number of squares of new variables. Thus the germs equivalent to surface singularities of type A_2 or of type E_8 look like this:

$$z_1^3 + z_2^2 + z_3^2 + \cdots + z_n^2,$$
$$z_1^5 + z_2^3 + z_3^2 + \cdots + z_n^2.$$

These stabilized germs, which were characterized by Arnold by a property of their semiuniversal unfolding or deformation had already appeared in an entirely different context. In 1955 J. Herszberg had characterized them in his thesis by a property of their resolution: They are the only absolutely isolated double points on hypersurfaces. An isolated singular point is called absolutely isolated if it can be resolved by a sequence of monoidal transformations with 0-dimensional centre. For absolutely isolated double points of surfaces this theorem had already been obtained by D. Kirby and had been published in three parts in the Proceedings of the London Mathematical Society 1955-1957. The title was *"The Structure of an Isolated Multiple Point of a Surface"*. Herszberg and Kirby were aware of the earlier work of Du Val.

Later work of Hirzebruch, Milnor and myself was to show that these higher-dimensional singularities of type E_8 and A_2 have a very close relation to the Milnor and Kervaire spheres and to their plumbing construction, a relation going beyond the intentions of Hirzebruch when he wanted to "mimic" the 2-dimensional singularities by plumbing in higher dimensions. This development came as a surprise while I was struggling for the solution of another problem related to E_8.

It began when I asked Hirzebruch for a problem for my first postdoctoral work. This was at some time in 1963. Hirzebruch gave me a 7-page paper by Michael Atiyah published in 1958 in the Proceedings of the Royal Society. The title was *"On analytic surfaces with double points"*. Hirzebruch suggested that I might try to generalize this from ordinary double points, i.e., surface singularities of type A_1, to the other surface singularities of type A_n, D_n, E_6, E_7, E_8. At that time, there was correspondence between Hirzebruch and Du Val about these singularities, there were two Ph.D.-theses on plumbing written by two of Hirzebruch's students, Arlt and von Randow, and there were mimeographed notes of lectures by Hirzebruch at the University of California, Berkeley in 1962 entitled *"Differentiable manifolds and quadratic forms"*. In these notes the A-D-E singularities were treated as twofold algebroid coverings of the plane and resolved by Hirzebruch's method.

Atiyah's paper dealt with Kummer surfaces. The history of these surfaces is too long to be told here. I shall say only a few words about it. The first example of a Kummer surface appeared long before Kummer in the work of Fresnel between 1820 and 1830. He introduced a surface now called Fresnel surface describing the expansion of light in a crystal. Around 1860 this surface appeared in another context in the work of Kummer who investigated focal surfaces of algebraic ray systems. In 1865 Kummer proved that the focal surface of a ray system of order 2 in complex projective 3-space is a surface of degree 4 in $P_3(\mathbb{C})$ with 16 ordinary double points or a degeneration of such a surface. Conversely any surface of degree 4 in $P_3(\mathbb{C})$ with 16 ordinary double points is the focal surface of a ray system of order 2. Subsequent work of Weber, Borchardt, Rohn and Klein showed that these Kummer surfaces of degree 4 in $P_3(\mathbb{C})$ with 16 double points are exactly the surfaces $\mathrm{Jac}\,(C)/\{\pm 1\}$, where $\mathrm{Jac}\,(C)$ is the Jacobian of a Riemann surface of genus 2.

One of the results in Atiyah's 1958 paper is that the nonsingular complex surfaces obtained by the minimal resolution of the 16 double points of a Kummer surface are diffeomorphic to the nonsingular quartic surfaces in $P_3(\mathbb{C})$. This theorem was one of the starting points of the fabulous development of the theory of $K3$-surfaces which began at that time. André Weil refers to Atiyah's theorem in his final report on Contract No. AF 18(603)-57 and to his exchange with Atiyah. He says that he had observed independently that the minimal resolution of a surface in $P_3(\mathbb{C})$ with one ordinary double point and a nonsingular surface in $P_3(\mathbb{C})$ of the same degree are diffeomorphic. The reason for the name $K3$-surface introduced by Weil is given in his comment on that final report: "ainsi nommés en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne $K2$ au Cachemire". Today, a $K3$-surface may be defined as a compact complex surface with trivial canonical bundle and first Betti number 0. The minimal resolutions of Kummer surfaces are very special $K3$-surfaces, which have remarkable symmetry properties, and which are used in the analysis of moduli problems of $K3$-surfaces. I refer to the chapter on $K3$-surfaces in the beautiful book *"Compact complex surfaces"* of Barth, Peters and Van de Ven.

Atiyah used the following basic facts. Consider the quadric cone V in $\mathbb{C}^4$ given by the equation

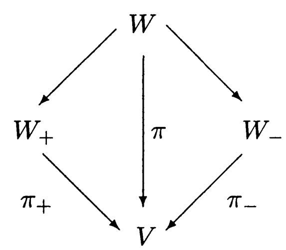
$$x_1x_2 - x_3x_4 = 0.$$

Let $Q \subset P_3(\mathbb{C})$ be the quadric of complex lines $\ell \subset V$. In the Grassmannian of complex planes in $\mathbb{C}^4$ there are two projective lines $P_\pm$ of planes $p \subset V$. Each line $\ell \in Q$ is contained in two uniquely determined planes $p_\pm(\ell) \in P_\pm$. This defines two projections $p_\pm : Q \to P_\pm$ and an isomorphism $Q \to P_+ \times P_-$.

Now define three modifications W_+, W_- and W of V as follows:

$$W_\pm = \{(x,p) \in X \times P_\pm \mid x \in p\},$$
$$W = \{(x,\ell) \in X \times Q \mid x \in \ell\}.$$

There is an obvious diagram of holomorphic maps



The modifications π, π_+, π_- are three different resolutions of the singularity $(V, 0)$. Whereas π replaces the singular point by the divisor Q in W, the resolutions π_+ and π_- are "small" resolutions. They replace the singular point only by the 2-codimensional curves $P_\pm$ in $W_\pm$. The resolution π is obtained by blowing up the maximal ideal of the local ring $\mathcal{O}_{V,0}$. This local ring is not a unique factorization domain. Its divisor class group is infinite cyclic. The divisors $p \in P_+$ represent one generator, those in P_- the other one. W_+ and W_- may be obtained by blowing up the nonprincipal ideals corresponding to these divisors, e.g. (x_1, x_3) and (x_1, x_4). The transition from W_+ to W_- is the simplest example of what people working on complex 3-manifolds nowadays call a flop.

Atiyah uses these modifications as follows. Let $f : (X, x) \to (S, s)$ be the germ of a map from a 3-dimensional complex manifold to a 1-dimensional complex manifold. Assume that the fibre has an ordinary double point of type A_1 at x. Let $\varphi : (T, t) \to (S, s)$ be a double covering by a smooth germ (T, t) ramified in t. Then the fibred product $(T \times_S X, t \times x)$ is isomorphic to the quadric cone $(V, 0)$. Choosing an isomorphism and choosing one of the two modifications W_+, W_-, we get a modification X' of $T \times_S X$. Choosing suitable representatives, we get a diagram of holomorphic maps of complex manifolds

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[\varphi]{} & S \end{array}$$

with the following properties: (i) f' is a regular map, i.e., without singularities, (ii) φ is a ramified covering, (iii) ψ is proper and surjective, (iv) for each fibre X'_t of f' the map $X'_t \to X_{\varphi(t)}$ is a resolution of the singularities of the fibre $X_{\varphi(t)}$ of f. Let us call such a diagram a simultaneous resolution of the singularities of the fibres of f.

The construction of Atiyah indicated above gives the existence of simultaneous resolutions for maps $f : X \to S$ of 3-manifolds X to 1-manifolds S with only A_1-type singularities. His theorem on Kummer surfaces is an easy consequence of this.

I took it that my task was to generalize this to all surface singularities of type A_n, D_n, E_6, E_7, E_8. One difficulty in the beginning was that it was not quite clear what was meant by "the" A-D-E-singularities, since a priori the definition by the resolution graph was wider than the other definitions (i.e., by equations, or as quotient singularities or as absolutely isolated double points). Correspondence on this with Du Val and Kirby was not conclusive, but in 1964/65 the situation was clarified by means of Michael Artin's new work on rational singularities published in 1966 in the American Journal. The A-D-E-singularities were identified with Artin's rational double points and were determined up to isomorphism by the corresponding diagram.

When $f(x, y, z) = 0$ is the equation of such a singularity, the fibered product for a base extension by a covering of degree d will have the equation

$$f(x, y, z) - t^d = 0.$$

Thus, in the cases A_n, E_6 and E_8 this leads to equations of the form

$$x^a + y^b + z^c + t^d = 0.$$

I tried to find small modifications of these 3-dimensional singularities by mapping them to others such as the quadric cone and inducing the small modification from another one already constructed. For example mapping to the quadric cone V meant writing $F = f(x, y, z) - t^d$ in the form $\phi_1\phi_2 - \phi_3\phi_4$. With such methods and encouraged by my teacher I constructed in 1964 the simultaneous resolutions for A_n, D_n, E_6 and E_7. It also became clear that for maps $f : X \to S$ of 3-manifolds to 1-manifolds simultaneous resolutions could only exist for the A-D-E-singularities. So the only case in question was E_8.

My calculations in the other cases had shown that somehow the right number d for the base extension was the Coxeter number for the corresponding root system. Thus the equation to consider for E_8 was

$$x^2 + y^3 + z^5 + t^{30} = 0.$$

I was unable to treat this case with the methods which I had used for the other cases. During the Arbeitstagung in 1965 I talked about this with Heisuke Hironaka. He suggested that I should study the divisor class groups of the local rings of the 3-dimensional singularities which I wanted to modify by blowing up the ideals of nonprincipal divisors. In particular, I should study the cohomology of the neighbourhood boundaries of these singularities, since the divisor class group of the local ring of a 3-dimensional isolated Cohen Macaulay singularity injects into the second cohomology group of the neighbourhood boundary with integer coefficients.

Shortly afterwards I set sail for New England because following Hirzebruch's advice I had successfully applied for a C.L.E. Moore instructorship at MIT. This was a very good place for me, since Michael Artin was at MIT and David Mumford at Harvard. Michael was very friendly and always ready to help, and I learned a lot from the many discussions which we had. David was also very friendly and in a number of discussions gave me very valuable ideas. The two years in Boston and Cambridge are among the best in my mathematical life.

Shortly after my arrival in Boston I intended to calculate the divisor class group for

$$x^2 + y^3 + z^5 + t^{30} = 0.$$

However, Mumford suggested that I should first look at the simpler example

$$x^2 + y^3 + z^5 + t^2 = 0.$$

This is the 3-dimensional E_8-singularity. I decided to do first a much simpler case, namely, the 3-dimensional A_2-singularity

$$z_1^2 + z_2^2 + z_3^2 + z_4^3 = 0.$$

I discovered quickly that it was factorial because the second cohomology group of the neighbourhood boundary was zero. Then I did the E_8-case suggested by Mumford, which was much more tedious, since the resolution by a sequence of monoidal transforms with the singular points as centres leads to an exceptional divisor with a dozen components. Again I found that the second cohomology group was zero and the divisor class group trivial. I was not happy about this, since I wanted nontrivial divisor class groups. Anyhow, I had a closer look at the topology of the neighbourhood boundary of the 3-dimensional A_2-singularity, and in September 1965 I made the irritating discovery that this singularity was topologically trivial. Its neighbourhood boundary is homeomorphic to S^5. Thus, there was no analogue of Mumford's theorem for singularities of dimension higher than two. Of course, I told this immediately to Mumford, and I also wrote a letter to Hirzebruch. In that letter dated September 28, I speculated about the E_8-singularity and possible connections with Hirzebruch's E_8 plumbing contruction and exotic spheres.

At that time, Hirzebruch was at a conference in Rome. He gave a very nice talk entitled *"Über Singularitäten komplexer Flächen"*, where he explained the A-D-E surface singularities and many related subjects. Among other things, he reported on my work on simultaneous resolution, and on the recent discovery announced in my letter.

Meanwhile I continued my efforts to construct the missing simultaneous resolution for E_8. I tried to gather strength by looking at the beautiful crystal icosahedron on the mantelpiece of my apartment on Beacon-Hill, but I did not get anywhere, and I got more and more depressed. In December 1965 I wrote to my mother that I was abandoning E_8. I also wrote about an experience intensifying my melancholy mood, namely, meeting John Nash. I knew that he had done extraordinary things before he got ill. In 1965/66 he was in Brandais and back in MIT, and he was able to do mathematics. Sometimes late in the evening we met in the long high corridors of MIT and started to talk about mathematics. Nash was interested in the resolution of singularities of complex algebraic varieties. Some traces of our conversation may be seen in a draft of a paper entitled *"Arc structure of singularities"*, where absolutely isolated double points of dimensions 2 and 3 serve as examples illustrating his distinction between essential and inessential components of the exceptional set in a resolution. What made me sad to the extent of being terrified was the feeling that he had lost his strength. I felt that this once powerful mind had broken wings.

In February 1966 I gave a talk in Cornell on the topology of singularities showing my example

$$z_1^2 + \ldots + z_k^2 - z_0^3 = 0 \quad , \quad k > 1 \text{ odd},$$

for a singular normal complex space which is a topological manifold. I had been invited by my friend, Hirzebruch's student, Klaus Jänich. We had a good time together, thus feeling at home in a foreign country. I do not remember whether we talked about Jänich's work. If we did, we certainly didn't anticipate what happened next.

In a letter dated March 24, 1966 Hirzebruch told me that he had found out that there were very close connections between my work and that of Jänich, which were explained in a provisional manuscript of eight pages.

Jänich was studying actions of compact Lie groups G on connected differentiable manifolds X without boundary. Such a G-manifold X was called "special", if for each $x \in X$ the action of the isotropy group G_x on the normal space in x to the orbit Gx is the direct sum of a trivial and a transitive representation. "Transitive" means transitive on the set of rays. The orbit space X/G of a special G-manifold is canonically a differentiable manifold M with boundary. Let M^0 be the interior of M and A be the set of boundary components. The orbit structure of X associates to M^0 the conjugacy class of isotropy groups $H(x) = G_x$ of $x \in X$ over M_0 and to $\alpha \in A$ the conjugacy class of isotropy groups $U_\alpha(x) = G_x$ of points x over α. Jänich defined a notion of admissible fine orbit structures in terms of data $H \subset G$, $H_\alpha \subset G$, $\alpha \in A$. His main result was a classification of special G-manifolds X with quotient M in terms of these fine orbit structures. The result was published in Topology 5, 1966. Hirzebruch used the manuscript of this paper. At about the same time, there was independent closely related work of Wu-Chung Hsiang and Wu-Yi Hsiang announced in the Bulletin of A.M.S.

Hirzebruch applied Jänich's result to a very special class of examples. Motivated by my work he looked at the neighbourhood boundaries of absolutely isolated double points of type A_{d-1}. Thus, he considered the differentiable manifolds $W^{2n-1}(d)$ in $\mathbb{C}^{n+1}$ given by the equations

$$\begin{aligned} z_0^d + z_1^2 + \cdots + z_n^2 &= 0 \\ |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 &= 2. \end{aligned}$$

He noticed that there is an obvious operation of the orthogonal group $O(n)$ on $W^{2n-1}(d)$. The operation is obvious indeed, but only if you have the idea of looking for it. Hirzebruch proved that $W^{2n-1}(d)$ is a special $O(n)$-manifold with orbit space D^2, the 2-disk. The isotropy groups are conjugate to $O(n-1)$ for the special orbits, i.e., those with $|z_0| = 1$ and to $O(n-2)$ for the general orbits.

Hirzebruch applied Jänich's classification result to the special case of special $O(n)$-manifolds, $n \geq 2$, with this orbit structure $H = O(n-2)$, $U = O(n-1)$. According to Jänich, they are classified up to equivariant diffeomorphism by an integer $d \geq 0$. For $d = 0$ one has the diagonal action on $S^n \times S^{n-1}$. For $d > 0$ Hirzebruch proved that one gets exactly the $O(n)$-manifolds $W^{2n-1}(d)$, $d \geq 1$.

Certain $O(n)$-manifolds M_k^{2n-1} with orbit type $(O(n-2), O(n-1))$ had been studied by Bredon in a paper in Topology 3, 1965. Hirzebruch noticed that they are special, and thus he could identify M_k^{2n-1} with $W^{2n-1}(2k+1)$. Using Bredon's results, he could prove that $W^{2n-1}(d)$ is a homology sphere if and only if d is odd, and that for fixed n and different $d's$ one gets different knots.

But the most exciting result was derived from a result of Bredon on his M_1^9 derived from a result of Kosinski: M_1^9 is an exotic sphere. Hence Hirzebruch got

Theorem 3. *The manifold*

$$W^9(3) = \{(z_0, \ldots, z_5) \in \mathbb{C}^6 \mid z_0^3 + z_1^2 + \ldots + z_5^2 = 0 \ , \ \|z\| = 1\}$$

is an exotic 9-sphere.

$W^{2n-1}(d)$ is obviously embedded in S^{2n+1}. Kervaire had proved in a paper which appeared in the volume *"Differential and Combinatorial Topology"*:

A homotopy m-sphere can be imbedded in S^{m+2} if and only if it bounds a parallelizable manifold.

Thus it was clear that $W^9(3)$ is the 9-dimensional Kervaire sphere. But where was the highly connected parallelizable manifold obtained by plumbing two disc-bundles with boundary $W^9(3)$? And how about the absolutely isolated singularities of type E_8? The parallelizable manifolds were not expected to be found by resolution. Hirzebruch speculated on this question in a postscript referring to my own speculation on his E_8-construction. Hirzebruch expected to deal with the E_8-case by means of a certain generalization of Jänich's result to $O(n)$-manifolds with 3 types of orbits and to show in this way that the neighbourhood boundaries of the n-dimensional E_8-singularities were the Milnor generators of bP_{2n} for n even.

The joy about these wonderful results was so great that I was almost beside myself, and I wrote in a letter to Hirzebruch that I could not imagine a more beautiful interplay between teacher and students.

Two weeks later a new player appeared. On April 16, John Nash showed me a letter to him from John Milnor, dated Ann Arbor 4-13-66. As far as I know, this letter has not been published. I hope that it is not improper when I publish it here. It is also a sign of gratitude to John

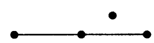
Nash. The text is as follows.

Dear John,

I enjoyed talking to you last week. The Brieskorn example is fascinating. After staring at it a while I think I know which manifolds of this type are spheres, but the statement is complicated and the proof doesn't exist yet. Let $\Sigma(p_1, \ldots, p_n)$ be the locus

$$z_1^{p_1} + \ldots + z_n^{p_n} = 0 \ , \ |z_1|^2 + \ldots + |z_n|^2 = 1$$

where $p_j \geq 2$. It is convenient to introduce the graph G which has one vertex for each p_i and one edge for each p_j, p_k which have $G.C.D.$ greater than one.

E.g. (4 , 6, 7, 15) $\longmapsto$ 

Assertion. $\Sigma(p_1, \ldots, p_n)$ is a topological $(2n-3)$-sphere if and only if $n \neq 3$ and either

a) G has at least two isolated points

or b) G has one isolated point and one component consisting of an odd number of p_j's, any two of which have $G.C.D. = 2$.

For example $\Sigma(2,2,2,25)$ and $\Sigma(2,2,2,3,5)$ are topological spheres, but $\Sigma(4,6,7,15)$ or $\Sigma(2,2,2,2,3)$ is not.

In the case $2n-3 \equiv 1 \mod 8$ one can describe which are exotic spheres and which not; but I can't handle the other dimensions.

Are results of this type known to Brieskorn or Hirzebruch?

Note: The conjecture I mentioned about $(\text{disk}, \text{disk} \cap \Gamma) \cong (\text{slab}, \text{slab} \cap \Gamma)$ is true and not so difficult.

Regards

Jack

There was a little figure about 1 cm in diameter on the margin next to the last sentence. Figure 6 is a magnified facsimile.

Initially I must have overlooked this figure or failed to realize that it was a key for understanding Milnor's approach. For, when I sent a copy of the letter to Hirzebruch I wrote that I had no idea how he was going to prove his assertion.

But even without knowing Milnor's ideas I was able to prove his assertion by good luck in less than two weeks. On the shelf for the latest journals in the library of MIT I found an article by Frédéric Pham, submitted to the Bulletin de la Société Mathématique de France in March 1965. The title was *"Formules de Picard-Lefschetz généralisées et ramification des integrales"*. Pham, who at that time was working at the Service de Physique théorique in Saclay was motivated by problems which

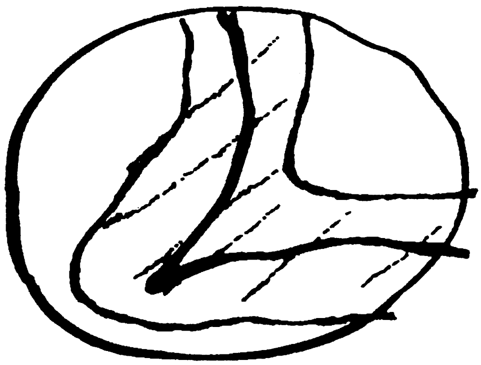


FIGURE 6

at first sight seemed to be unrelated to what we were doing. The paper was a contribution to efforts of theoretical physicists aiming at a better understanding of the singularities and discontinuities of Feynman integrals by applying methods of algebraic topology developed for the topological analysis of algebraic manifolds. Thus these efforts had their mathematical roots in the two volume treatise *"Théorie des fonctions algébriques de deux variables indépendantes"* published by Picard-Simart in 1897 and 1906 and in Lefschetz' 1924 monograph *"L'analysis situs et la géométrie algébrique"*. Pham was also influenced by work of Leray and Thom. The first part of Pham's paper was a generalization of the classical Picard-Lefschetz formula. This formula describes the monodromy transformation on the homology of a general member of a pencil of algebraic varieties such that the singularities of the special members are at most ordinary double points. Pham generalized the Picard-Lefschetz formula exactly to the class of singularities considered in Milnor's letter to Nash.

Let $a = (a_0, \ldots, a_n)$ be a tuple of positive integers. Pham considers the pencil of affine hypersurfaces

$$\Xi^a(t) = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \ldots + z_n^{a_n} = t\}.$$

Let $\omega_k = e^{2\pi i/a_k}$ and C_{a_k} the cyclic group of unit roots generated by ω_k. Pham constructs a simplicial complex in $\Xi^a(1)$ which is a deformation retract of $\Xi^a(1)$. As an abstract complex it is the iterated join

$$C_{a_0} * \ldots * C_{a_n}.$$

$\Xi^a(0)$ is the only singular member of the pencil. Intersecting with the sphere $\|z\| = 1$ one gets Milnor's $\Sigma(a_0, \dots, a_n)$. Removing the singular fibre on gets a fibre bundle $\mathbb{C}^{n+1} - \Xi^a(0) \to \mathbb{C} - \{0\}$. There is a geometric monodromy $(z_0, \dots, z_n) \mapsto (\omega_0 z_0, \dots, \omega_n z_n)$ acting in the obvious way on the complex $C_{a_0} * \dots * C_{a_n}$. Using these facts Pham calculates the homology of $\Xi^a(1)$ and the monodromy transformation on the only nontrivial reduced homology group $H_n(\Xi^a(1))$ with integral coefficients. He also calculates the intersection form on the homology.

From these results I could painlessly deduce the homological part of Milnor's assertion. All I had to do was to calculate the characteristic polynomial of the monodromy

$$\Delta_a = \prod_{0<i_k<a_k} \left(x - \omega_0^{i_0} \dots \omega_n^{i_n}\right)$$

and to show that $\Delta_a(1) = \pm 1$ is equivalent to Milnor's condition on the graph G_a associated to a. This was done on four pages by April 25 and sent to Milnor and Hirzebruch. Until the end of the month I had also proved on two more pages that $\pi_1(\Sigma(a_0, \dots, a_n))$ is trivial for $n > 2$. Thus the proof of Milnor's assertion was complete and I went on to show that Pham's result also allowed to calculate the signature σ_a of $\Xi^a(1)$ for n even and thus to determine the differentiable structure when $\Sigma(a_0, \dots, a_n)$ was an exotic sphere of dimension $4m - 1$. I did the case $\Sigma(2, 2, 2, 3, 5)$ explicitly and found that it was Milnor's generator of bP_8 constructed by Hirzebruch's E_8-plumbing construction.

Meanwhile by the end of April Milnor had completed a manuscript entitled *"On isolated singularities of hypersurfaces"*. It did not give a complete proof of the assertion about the class of singularities considered in his letter to Nash, but it contained foundational results on arbitrary isolated hypersurface singularities.

Consider holomorphic functions f defined in a neighbourhood U of 0 in $\mathbb{C}^{n+1}$ with an isolated singularity at 0 and $f(0) = 0$. Let $B_\varepsilon \subset U$ be the ball $\|z\| \leq \varepsilon$ and $S_\varepsilon = \partial B_\varepsilon$. For $\eta > 0$ let $D_\eta \subset \mathbb{C}$ be the disk $|t| < \eta$. Consider the "slab"

$$N_{\varepsilon,\eta} = B_\varepsilon \cap f^{-1}(D_\eta).$$

This is the slab shown by the figure in the letter to Nash. f defines a map

$$f : N_{\varepsilon,\eta} \longrightarrow D_\eta.$$

Let F_t denote the fibre of this map over $t \in D_\eta$. The fibre F_0 is singular for small ε. The intersection

$$K_\varepsilon = F_0 \cap S_\varepsilon$$

is a closed manifold and the diffeomorphism type of this neighbourhood boundary does not depend on ε. Fix ε and choose η sufficiently small. Then

$$f : N_{\varepsilon,\eta} - F_0 \longrightarrow D_\eta - \{0\}$$

is a differentiable locally trivial fibre bundle. The fibre F_t is parallelizable and $(n-1)$-connected with boundary diffeomorphic to K_ε. It has the homotopy type of a wedge of n-spheres. ∂F_t is $(n-2)$-connected. It is a homology sphere iff $\Delta(1) = \pm 1$ for the characteristic polynomial Δ of the fibre bundle. For n odd, the value $\Delta(-1) \mod 8$ determines the Arf invariant and hence the class of K in bP_{2n}, if K is a homotopy sphere. For homotopy spheres K with n even, the class of K in bP_{2n} is determined by the signature of F_t divided by 8.

There is a homeomorphism $N_{\varepsilon,\eta} \to B_\varepsilon$ keeping K pointwise fixed. It identifies $\partial N_{\varepsilon,\eta}$ with S_ε. The part of $\partial N_{\varepsilon,\eta}$ lying over ∂D_η is identified with the complement of a tubular neighbourhood of K in S_ε which therefore becomes a fibre bundle over a circle with typical fibre F_t. The fibration may be defined by $z \to f(z)/\|f(z)\|$. Thus $K \subset S_\varepsilon$ is a fibred knot. These fibrations are nowadays called *"Milnor fibrations"* and Milnor's results or analogues of them are fundamental for nearly all work on the topology of singularities.

Later on a local Picard-Lefschetz theory was developed which allows to represent certain bases of the homology of the Milnor fibre by vanishing cycles which are embedded n-spheres with tubular neighbourhoods isomorphic to their tangent disc bundles. In the case of the absolutely isolated double points of type A_k, D_k, E_6, E_7, E_8 a suitable choice of such a basis of vanishing cycles allows to identify the Milnor fibre directly with the corresponding parallelizable manifold constructed by plumbing.

Meanwhile Hirzebruch had been pursuing his idea of dealing with stabilized curve singularities

$$f(x,y) + z_1^2 + \ldots + z_n^2 = 0$$

via certain $O(n)$-manifolds with three types of orbits. Nearly simultaneously with Milnor's manuscript I got a manuscript from Hirzebruch dated May 1, 1966 with many beautiful results on these manifolds.

The general theory of such knot-G-manifolds was developed in the last paragraph of Jänich's article. Consider a closed connected $(2n+1)$-dimensional $O(n)$-manifold M with the following properties:

(i) The isotropy groups are conjugate to $O(n-2)$, $O(n-1)$ or $O(n)$.

(ii) The set F of fixed points is not empty and for any $x \in F$

the operation of $O(n)$ is the diagonal operation on $\mathbb{R}^{2n}$
plus a trivial operation on $\mathbb{R}$.

(iii) $M - F$ is a special $O(n)$-manifold.

These conditions imply that the orbit space $M/O(n)$ is a 4-dimensional manifold with boundary. M is called a knot $O(n)$-manifold, if there is a diffeomorphism of $M/O(n)$ with the 4-disk D^4 sending the image of F in the orbit space bijectively onto a knot $k \subset S^3 = \partial D^4$. Jänich proves that for each knot $k \subset S^3$ and each integer $n \geq 2$ there is a welldetermined $(2n+1)$-dimensional knot $O(n)$-manifold $\gamma_n(k)$. For $n = 1$ we define $\gamma_1(k)$ as the 2-fold covering of S^3 branched along k.

Hirzebruch announced the following result. $\gamma_n(k)$ is an $(n-1)$-connected manifold bounding a parallelizable manifold which can be constructed explicitly from a Seifert diagram of the knot. When $\gamma_n(k)$ is a homotopy sphere the invariants determining its class in bP_{2n+2} can be calculated from invariants of the knot.

Now let k be an algebraic knot, i.e., a knot associated to a plane curve singularity at the origin

$$k = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0 \ , \ |x|^2 + |y|^2 = 1\}.$$

Then $\gamma_n(k)$ is the neighbourhood boundary of the corresponding stabilized singularity and is imbedded in the standard sphere S^{2n+3} in $\mathbb{C}^{n+2}$:

$$\gamma_n(k) = \{(x, y, z_1, \ldots, z_n) \in S^{2n+3} \mid f(x,y) + z_1^2 + \ldots + z_n^2 = 0\}.$$

In particular for a torus knot $k = t(p,q)$ one obtains

$$\gamma_n(k) = \Sigma(p, q, 2, \ldots, 2).$$

Hirzebruch indicated a tentative way for calculating the signature and obtained an explicit formula for $t(p,q)$. In particular, he also concluded that for $\Sigma(3,5,2,\ldots,2)$ for n odd is the Milnor generator of bP_{2n+2}.

When Hirzebruch got my letter referring to Pham's paper, he saw very quickly how to calculate the signature of the evendimensional varieties of Pham by using Pham's description of the intersection form. He told me the result together with the proof in a letter dated May 9, 1966. The result is as follows. For $a = (a_0, \ldots, a_n)$, n even, the signature σ_a of $\Xi^a(1)$ is

$$\sigma_a = \sigma_a^+ - \sigma_a^-$$

where σ_a^+ and σ_a^- are the numbers of tuples $(j_0, \ldots, j_n)$ with $0 < j_k < a_k$ such that

$$\begin{array}{rcll} 0 & < & \Sigma j_k/a_k < 1 \mod 2 & \text{for } \sigma_a^+, \\ -1 & < & \Sigma j_k/a_k < 0 \mod 2 & \text{for } \sigma_a^-. \end{array}$$

The proof given in my paper in Inventiones is Hirzebruch's proof.

Later on the formula for σ_a went through a remarkable metamorphosis. In March 1970 I got a letter from Don Zagier who had been a student of Atiyah and had attended lectures on singularities and exotic spheres which I gave in Oxford in 1969. Don Zagier had discovered a formula for σ_a resembling closely the form of the Atiyah-Bott fixed point theorems and Atiyah-Singer G-signature theorems. Here is the formula:

$$\sigma_a = \frac{(-1)^{n/2}}{N} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2N} \cot\frac{\pi j}{2N} \cot\frac{\pi j}{2a_0} \cot\frac{\pi j}{2a_1} \dots \cot\frac{\pi j}{2a_n}$$

where n is even and N is any common multiple of $a_0, \dots, a_n$. I sent Zagier's letter to Hirzebruch, who found Zagier's result very interesting, since he had studied similar questions and had tried to get σ_a through the G-signature theorem. In March 1970 Hirzebruch had been taking part in the inauguration of the new Fine Hall in Princeton and had given a talk entitled *"The signature theorem: Reminiscences and recreations"*. The underlying theme was *"More and more number theory in topology"*. In that talk Hirzebruch dealt with Dedekind sums and reciprocity theorems and Markoff triples and tried to establish relations with the Atiyah-Bott-Singer index theorem and fixed point theorem. When he got Zagier's formula, he pointed out to Zagier that it could be deduced from a formula of Eisenstein, which Hirzebruch had found in Rademacher's lectures on Analytic Number Theory: Let $((x)) = x - [x] - \frac{1}{2}$ for $x \in \mathbb{R} - \mathbb{Z}$ and $((x)) = 0$ for $x \in \mathbb{Z}$. Then Eisenstein's formula expresses $((x))$ for rational $x = p/q$ with positive integers p, q by a trigonometric sum:

$$((p/q)) = \frac{i}{2q} \sum_{k=1}^{q-1} \cot\frac{\pi j}{q}\, e^{2\pi i k p/q}.$$

Zagier's formula can be deduced from this formula since σ_a can easily be expressed as a sum of values of (( )) for rational numbers.

Hirzebruch invited Zagier to discuss these matters with him in Bonn, and this was the beginning of a cooperation that led to Zagier's Lecture Notes *"Equivariant Pontrjagin Classes and Applications to Orbit Spaces"*, to the joint monograph *"The Atiyah-Singer theorem and elementary number theory"* and to joint papers on Hilbert modular surfaces. Zagier proves in his lecture notes a signature theorem expressing $\mathrm{Sign}(g, \Xi^a(1))$ as a trigonometric sum, where g is any element of the group $C_{a_0} \times \dots \times C_{a_n}$ operating on Pham's $\Xi^a(1)$. For $g = 1$ this specializes to his formula for σ_a. Other specializations include a result of Hirzebruch and Jänich published in 1969 in their joint paper *"Involutions and*

Singularities". The description of exotic spheres bounding parallelizable manifolds as neighbourhood boundaries $\Sigma(a_0, \ldots, a_n)$ gives lots of exotic actions of finite groups on these manifolds. It is natural to try to distinguish them by invariants. One such invariant is the Browder-Livesay-invariant. Hirzebruch and Jänich identify it with an invariant introduced by Hirzebruch in his paper *"Involutions on manifolds"*. They calculate this invariant in certain cases for the involution T on $\Sigma(a_0, \ldots, a_n)$ given by $T(z) = -z$, where all a_j have the same parity. For even parity they have a general formula, and this formula turned out to be a special case of Zagier's theorem.

When the problem relating to the manifolds $\Sigma(a_0, \ldots, a_n)$ had been clarified by Hirzebruch, Milnor and myself, I returned to my old problem of constructing simultaneous resolutions for the rational double point E_8, the icosahedral singularity. Now I was finally able to solve it. I found that the number of solutions of the problem is

$$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7.$$

This is the order of the Weyl group of type E_8. The divisor class group of the local ring of the singularity

$$x^2 + y^3 + z^5 + t^{30} = 0$$

has the structure of the lattice of weights of the root system E_8, and the different solutions correspond to the Weyl chambers. For each chamber one obtains a solution by blowing up any ideal class in the chamber. The construction of the solution used very classical algebraic geometry, an old paper of Max Noether on rational double planes from 1889 and properties of the exceptional curves on rational surfaces obtained by blowing up 8 points on a plane cubic. Some of these facts had been explained to Hirzebruch and to me by Du Val, and Hirzebruch had mentioned them in his talk in Rome.

In May 1969 Grothendieck read my papers on simultaneous resolution. He told me some interesting conjectures on related problems. Whereas I had been considering simultaneous resolutions of a very special kind of deformation of the A-D-E-singularities, he suggested to look at the semiuniversal deformation. He conjectured that this deformation was to be found in the adjoint quotient map of the simple Lie algebras of type A-D-E, and that a universal simultaneous resolution was to be obtained by means of a generalization of the Springer resolution of the nilpotent variety. I proved these conjectures with the help of Tits and explained it at the ICM in Nice 1970. Later developments were beautiful extensions to all simple Lie algebras by Slodowy and characterizations of the universal simultaneous resolution as universal deformation of the resolution of the rational double points by Michael Artin and Huikeshoven. Recently there has been very interesting work of Slodowy and Helmke on the relation between loop groups and elliptic singularities. But this is a different story.

Cusps

Let me return to the singularities in the work of Friedrich Hirzebruch. In the hierarchy of singularities as described by Arnold there is an interesting class of singularities lying between the simple singularities of A-D-E type and the simply elliptic singularities of type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. These are the singularities T_{pqr} with equation

$$x^p + y^q + z^r + xyz = 0,$$

where $1/p + 1/q + 1/r < 1$. These belong to a class of singularities which Hirzebruch discovered in 1970.

Hirzebruch has given four talks in the Séminaire Bourbaki. It is a remarkable fact that in three of them singularities played an important role. The first of these Bourbaki lectures was the report on the work of Mumford and the higher dimensional E_8-plumbing construction. The second was on singularities and exotic spheres. Finally the third lecture, delivered in June 1971, had the title: *"The Hilbert modular group, resolution of the singularities at the cusps and related problems"*.

This contribution of Hirzebruch is on one hand a direct continuation of work in his thesis and on the other hand has its origin in work of David Hilbert in 1893/94 and in the Habilitationsschrift of Hilbert's first student Otto Blumenthal. In his thesis Hirzebruch had considered surface singularities which are resolved by a chain of rational curves. Now the objects to be studied are surface singularities which are resolved by a cycle of rational curves. In the printed version of his thesis Hirzebruch had claimed without proof that there could be no cycles in the resolution graph of a surface singularity. He had soon noticed that this was wrong, and now singularities resolved by a cycle of rational curves became objects with which he occupied himself during a whole decade, from 1970 until 1980.

Let us consider a finite sequence of nonnegative integers $b_1, \ldots, b_q$. We want to construct a surface singularity with a cyclic resolution by nonsingular rational curves with self-intersection numbers $-b_k$. We want to do a construction in the style of Hirzebruch's thesis by using toroidal embeddings.

A cyclic configuration of exceptional curves with mutual intersection numbers 0 or 1 has at least 3 elements. So we assume $q \geq 3$. For $q > 3$ an exceptional curve of the first kind in such a configuration can be blown down, and the resulting configuration is still cyclic. So for $q > 3$ we can assume $b_k \geq 2$ for all k. Thus for $q \geq 3$ we admit all sequences such that $b_k \geq 2$ for all k, but not all b_k equal 2. For $q = 3$ we admit also sequences of the form $(a{+}3, 2, 1)$ with $a \geq 3$ and $(a_1{+}1, 1, a_2{+}1)$ with $a_1, a_2 \geq 2$ and $a_1 \geq 3$ or $a_2 \geq 3$. In these two cases blowing down of exceptional curves of the first kind leads to "reduced" sequences (a) and (a_1, a_2) of length 1 and 2 respectively. Hirzebruch gives two constructions for singularities with cyclic resolution. The first one is analogous to the construction in his thesis and uses the nonreduced sequence $b = (b_1, \dots, b_q)$. The second one uses the reduced sequence, which I shall again denote by $(b_1, \dots, b_q)$. I admit that this is an abuse of notation.

Here is the first construction. We define a doubly infinite sequence of integers b_k, $k \in \mathbb{Z}$ by $b_j \equiv b_k$ if $j \equiv k \mod q$. Now we proceed in strict analogy with the construction in Hirzebruch's thesis. We construct a 2-dimensional complex manifold Y_b by gluing an infinite number of copies of $\mathbb{C}^2$, one for each integer k. The transformation from the $(k-1)$-th copy to the k-th copy is

$$\begin{aligned} u_k &= u_{k-1}^{b_k} v_{k-1} \\ v_k &= u_{k-1}^{-1}. \end{aligned}$$

In Y_b we have an infinite chain of nonsingular rational curves with self-intersection numbers $-b_k$, and the complement of this system of curves is an algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$. Because of the periodicity of the sequence b_k we have a transformation $T : Y_b \to Y_b$ identifying the k-th copy of $\mathbb{C}^2$ canonically with the $(k+q)$-th copy. T has a fixed point in $(1,1) \in \mathbb{C}^* \times \mathbb{C}^*$, but there is a T-invariant tubular neighbourhood Y_b^0 of the chain of exceptional curves on which T acts freely. $\tilde{X}_b = Y_b^0 / \langle T \rangle$ is a complex manifold with a cyclic configuration of q nonsingular rational curves with self-intersection numbers $-b_1, \dots, -b_q$. The conditions on these sequences imply that the intersection matrix is negative definite. So, according to Grauert, one can blow these curves down. Thus we get a normal complex space X_b with a singular point x, and we have constructed a singularity (X_b, x) with cyclic resolution $\tilde{X}_b \to X_b$.

The second construction is somehow analogous to the description of the $X_{n,q}$ in Hirzebruch's thesis as quotient singularities $\mathbb{C}^2/C_{n,q}$, where the group $C_{n,q}$ is constructed from the sequence of self-intersection numbers by means of a continued fraction. Now the singularity of X_b will be constructed as a partial compactification of a quotient $\mathbb{H}^2/G_b$, where $\mathbb{H}$ is the upper half plane and the group G_b acting on $\mathbb{H} \times \mathbb{H}$ is defined by means of infinite continued fractions.

We start with the doubly infinite sequence of integers $b_k \geq 2$ with reduced period q generated from the reduced sequence associated to b. (Recall that this is an abuse of notation.)

For any integer k we define a real number a_k by an infinite periodic continued fraction

$$a_k = b_k - \cfrac{1}{b_{k+1} - \cfrac{1}{b_{k+2} \ddots}} \quad .$$

These numbers are totally positive algebraic numbers in the real quadratic field $K = \mathbb{Q}(a_1)$. In K we consider the lattice

$$M = \mathbb{Z}a_1 \oplus \mathbb{Z}.$$

In M we consider the sector of totally positive elements

$$M^+ = \{w \in M \mid w > 0, w' > 0\},$$

where w' is obtained from w by the nontrivial automorphism of K.

$$M^+ = \{y - xa_1 \mid (x,y) \in \mathbb{Z}^2, y - xa_1 > 0, y - xa_1' > 0\}.$$

The boundary of the convex hull of M^+ is an infinite polygon with vertices $w_k \in M^+$ which may be computed recursively by $w_0 = 1$ and $w_{k+1} = a_{k+1}^{-1} w_k$. Any pair (w_k, w_{k+1}) is a basis of the lattice M generating a sector contained in M^+, and the system of these sectors is a subdivision of M^+. The manifold constructed from these data by toroidal embeddings is exactly the manifold Y_b in the first construction, if the original sequence b was already reduced. The next figure illustrates the situation for the simplest reduced sequence $b_k = 3$ for all k and $q = 1$, where $a_k = (3 + \sqrt{5})/2$ for all k. This case corresponds to the singularity $T_{2,3,7}$. Figures of this kind occur already in Felix Klein's lectures *"Elementarmathematik vom höheren Standpunkte aus"*.

Now we define the group G_b. Let ε be the product $\varepsilon = a_1 \dots a_q$. This is a totally positive unit in the ring of integers of the field K. Multiplication with ε is an automorphism of the abelian group M. Thus we may form the semidirect product

$$G_b = M \rtimes \langle \varepsilon \rangle$$

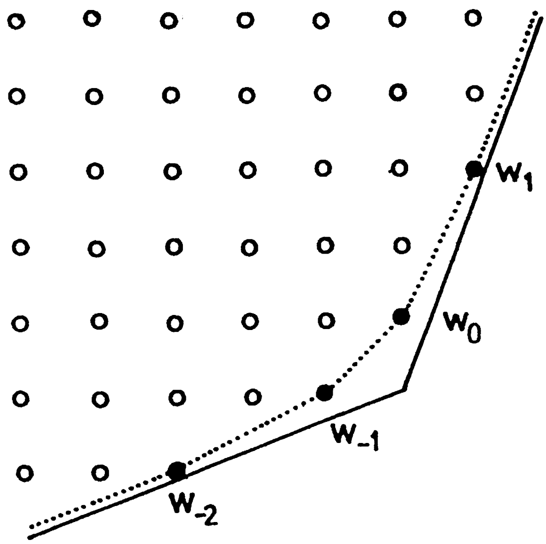


FIGURE 7

with the infinite cyclic group $\langle\varepsilon\rangle$ generated by ε. We identify G_b with a subgroup of $SL(2,K)$

$$G_b = \left\{ \begin{pmatrix} \varepsilon^n & \mu \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z}, \mu \in M \right\}.$$

$SL(2,K)$ operates on $\mathbb{H} \times \mathbb{H}$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'\Pi z_2 + b'}{c'\coprod z_2 + d'} \right).$$

The action of G_b on $\mathbb{H} \times \mathbb{H}$ is properly discontinuous, so that the orbit space $\mathbb{H}^2/G_b$ is a 2-dimensional normal complex space. We define a partial compactification

$$\overline{\mathbb{H}^2/G_b} = \mathbb{H}^2/G_b \cup \{\infty\}.$$

A basis of neighbourhoods of ∞ is given by the sets $\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 > d$, where d is any positive real number. A complex valued function on an open neighbourhood U of ∞ is holomorphic if it is continuous and holomorphic on $U - \{\infty\}$. With these definitions $\overline{\mathbb{H}^2/G_b}$ is a 2-dimensional normal complex space, and Hirzebruch proves

$$\overline{\mathbb{H}^2/G_b} \cong (X_b, x).$$

These singularities with cyclic resolution occur as cusp singularities of compactified orbit spaces $\overline{\mathbb{H}^2/G}$, where the groups G are certain discrete groups operating on $\mathbb{H}^2$ such as $SL(2, \underline{o})$, where $\underline{o}$ is the ring of integers in a real quadratic field K over $\mathbb{Q}$. By resolving all singularities of $\overline{\mathbb{H}^2/G}$, one gets the Hilbert modular surfaces. Hirzebruch has made a detailed investigation of such surfaces in a series of papers written between 1970 and 1980. Some of these papers were joint work with Don Zagier and with Van de Ven. As an example let me quote the main result of the joint paper with Van de Ven in Inventiones dedicated to Karl Stein on the occasion of his sixtieth birthday. Let $Y(p)$ be the Hilbert modular surface associated to $K = \mathbb{Q}(\sqrt{p})$, where p is a prime congruent 1 mod 4.

Theorem. *The surfaces $Y(p)$ are rational for $p = 5, 13, 17$; blown up elliptic K_3-surfaces for $p = 29, 37, 41$; honestly elliptic surfaces for $p = 53, 61, 73$ and surfaces of general type for $p \geq 89$.*

It is a pity that I am unable to render adequately the wealth of results in these papers on Hilbert modular surfaces. Let me mention at least one more beautiful result which I think is very typical of Hirzebruch's way of looking at mathematical objects. It is related to classical results of Clebsch and Klein. In 1873 Klein had proved that the famous diagonal surface of Clebsch, which is the surface in $P_4(\mathbb{C})$ with equations

$$\begin{aligned} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 &= 0 \\ x_0 + x_1 + x_2 + x_3 + x_4 &= 0, \end{aligned}$$

can be obtained from $P_2(\mathbb{C})$ by blowing up 6 points in $P_2(\mathbb{C})$ in a special position, namely the 6 points in $P_2(\mathbb{R}) = S^2/\{\pm 1\}$ corresponding to the 12 vertices of an icosahedron inscribed in S^2. Now Hirzebruch blows up 10 more points, namely those corresponding to the 20 vertices of the dual dodecahedron. The resulting surface Y can also be obtained from the Clebsch diagonal surface by blowing up 10 Eckhardt points, that is points, where 3 of the 27 lines on the surface meet. In a paper dedicated to P. S. Aleksandrov, this classical surface is identified with a Hilbert modular surface. Let $\underline{o} \subset \mathbb{Q}(\sqrt{5})$ be the ring of integers and $\Gamma \subset SL(2, \underline{o})$ the congruence subgroup mod 2. Hirzebruch proves: The icosahedral surface Y is the minimal resolution of $\overline{\mathbb{H}^2/\Gamma}$.

After his work on Hilbert modular surfaces Hirzebruch wrote a series of papers in which the problem of the existence of complex manifolds with invariants satisfying certain conditions was related to the problem of the existence of various types of geometric configurations and in particular to the problem of the existence of certain configurations of singularities

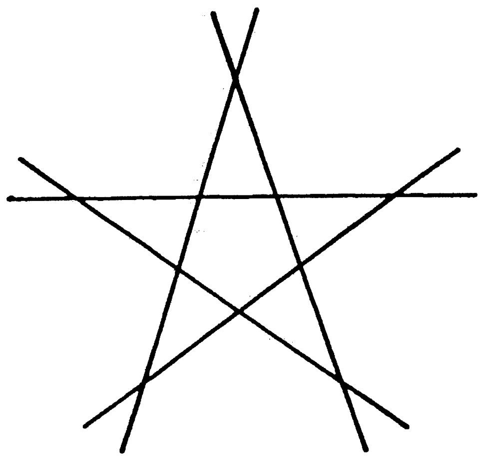


FIGURE 8

on certain algebraic manifolds. These papers contain such a wealth of beautiful geometry with relations to classical configurations of the 19-th century, but also to modern theoretical physics, that I am unable to produce an adequate summary. Instead, let me mention just one example which is taken from the last paper of the Collected Works published in 1987.

Consider hypersurfaces of degree d in complex projective n-space with singularities which are only ordinary double points of type A_1. Let $\mu_n(d)$ be the maximal number of double points that can occur on some hypersurface. For example $\mu_4(5) \leq 135$ by a theoretical estimate of Varchenko. It was not known whether this number is attained. In 1986 C. Schoen had constructed a quintic with 125 double points. In 1987 Hirzebruch constructed a quintic with one more double point. The construction is as follows. Consider a configuration of five lines in the real Euclidean plane forming a regular pentagram. Let $f(u,v)$ be a polynomial of degree 5 describing this configuration and invariant under its group of symmetries. f has 10 critical points of level 0 in the 10 intersection points of the five lines, 5 critical points of a certain level $a \neq 0$ in the triangles and one critical point of level b, $0 \neq b \neq a$, at the center of the pentagon. Now consider the quintic in $P_4(\mathbb{C})$ with the affine equation

$$f(u,v) - f(z,w) = 0.$$

Obviously this quintic has only ordinary double points and their number is

$$10 \cdot 10 + 5 \cdot 5 + 1 \cdot 1 = 126.$$

I have tried to show how singularities figure in the work of Friedrich Hirzebruch. I have also tried to show how much I owe to him. Many people, students and mathematicians from all parts of the world owe him thanks. He is always ready to listen, to give advice and to help. He has done an enormous amount of work organizing mathematical research and teaching and international cooperation. In the midst of all that he still has time and energy for wonderful mathematics When I asked him how he does it, he just said: I enjoy it.

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