Einstein-Weyl Geometry

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A Weyl manifold is a conformal manifold equipped with a torsion free connection preserving the conformal structure, called a *Weyl connection*. It is said to be *Einstein-Weyl* if the symmetric trace-free part of the Ricci tensor of this connection vanishes. In particular, if the connection is the Levi-Civita connection of a compatible Riemannian metric, then this metric is Einstein. Such an approach has two immediate advantages: firstly, the homothety invariance of the Einstein condition is made explicit by focusing on the connection rather than the metric; and secondly, not every Weyl connection is a Levi-Civita connection, and so Einstein-Weyl manifolds provide a natural generalization of Einstein geometry.

The simplest examples of this generalization are the locally conformally Einstein manifolds. A Weyl connection on a conformal manifold is said to be *closed* if it is *locally* the Levi-Civita connection of a compatible metric; but it need not be a global metric connection unless the manifold is simply connected. Closed Einstein-Weyl structures are then locally (but not necessarily globally) Einstein, and provide an interpretation of the Einstein condition which is perhaps more appropriate for multiply connected manifolds. For example, $S^1 \times S^{n-1}$ admits flat Weyl structures, which are therefore closed Einstein-Weyl. These closed structures arise naturally in complex and quaternionic geometry.

Einstein-Weyl geometry not only provides a different way of viewing Einstein manifolds, but also a broader setting in which to look for and study them. For instance, few compact Einstein manifolds with positive scalar curvature and continuous isometries are known to have Einstein deformations, yet we shall see that it is precisely under these two conditions that nontrivial Einstein-Weyl deformations can be shown to exist, at least infinitesimally.

The Einstein-Weyl condition is particularly interesting in three dimensions, where the only Einstein manifolds are the spaces of constant curvature. In contrast, three dimensional Einstein-Weyl geometry is extremely rich [**16, 68, 72**], and has an equivalent formulation in twistor theory [**34**] which provides a tool for constructing self-dual four dimensional geometries. In section 10, we shall discuss a construction relating Einstein-Weyl 3-manifolds and hyper-Kähler 4-manifolds [**40, 29, 50, 79**]. Twistor methods also yield complete self-dual Einstein metrics of negative scalar curvature with prescribed conformal infinity [**48, 35**]. An important special case of this construction is the case of an Einstein-Weyl conformal infinity [**34, 61**].

Although Einstein-Weyl manifolds can be studied, along with Einstein manifolds, in a Riemannian framework, the natural context is Weyl geometry [**23**]. We take this point of view seriously, because of the insight it provides into the formulation of ideas and results about Einstein-Weyl manifolds. For this reason, and because the approach is less familiar, the first few pages of this essay are devoted to a brief presentation of some concepts invaluable in Weyl geometry, such as densities, Weyl derivatives and conformal metrics; these concepts, despite being as basic as tangent vectors, linear connections and Riemannian metrics, are not common currency.

In section 3, Einstein-Weyl manifolds are introduced. After giving a few examples, we present the initial results of the theory. A key role is played by the contracted Bianchi identity, which implies the constancy of the scalar curvature in the Einstein case. The implications in Einstein-Weyl geometry are more subtle and are described in Theorem 3.6: closed Einstein-Weyl structures have parallel scalar curvature, and the converse holds in the compact case [**27**] or when the dimension is not four [**12**]. We also give the definition of Einstein-Weyl manifolds in two dimensions [**11**] and we indicate throughout how general results apply to this case.

Many of the general theorems about compact Einstein-Weyl manifolds follow from the existence of a distinguished compatible metric, the Gauduchon metric [**24**]. These results are given in section 4 and imply that, apart from in the Einstein case, the isometry group of the Gauduchon metric on a compact Einstein-Weyl manifold is at least one dimensional [**72**]. We also observe that the sign of the scalar curvature is constant in four or more dimensions [**67, 12**], contrary to some previous claims. In section 10 we show that this need not hold in dimensions two and three.

In section 6 we give an extensive supply of examples of Einstein-Weyl manifolds. These examples are often obtained from Riemannian submersions, which we discuss, in section 5, within the more general framework of conformal submersions. In particular we give an ansatz aimed at a study of submersions between Einstein-Weyl manifolds, which includes as special cases both circle bundles over Kähler-Einstein manifolds [**66, 56**] and hyper-complex 4-manifolds (which are Einstein-Weyl) over Einstein-Weyl 3-manifolds [**17, 29**]. We discuss this latter case in section 10, where we use submersions to give a direct proof of the result, of Jones and Tod [**40**], concerning the construction of three dimensional Einstein-Weyl spaces from self-dual 4-manifolds with a conformal vector field. The Jones-Tod construction was used in [**68**] to obtain the full moduli of Einstein-Weyl structures near the round metric on the 3-sphere. More generally, in section 7 we study Einstein-Weyl moduli spaces near Einstein metrics [**65**].

The material in section 8 illustrates how additional conditions on Einstein-Weyl manifolds often lead to closed structures (see [**39**] for another instance of this), and also highlights the role of Weyl structures in complex and quaternionic geometry [**58, 59, 63**]. This is further amplified in following section on four dimensions, where Weyl geometry and complex geometry are intimately linked. However, as shown by Gauduchon and Ivanov [**28**], in the compact case the Einstein-Weyl condition again gives only closed structures. Also in section 9 we discuss the interactions between four dimensional Weyl geometry and twistor theory [**25, 66**], and give a local formula for the Bach tensor on an Einstein-Weyl manifold [**12**]. In the compact case, a similar formula was given in [**67**], where it was used to show that compact Einstein-Weyl manifolds with self-dual Weyl curvature are closed. This fact was obtained in a different way in [**27**] and may be combined with the results of [**66**] to give a classification of the compact self-dual examples. It is now known that, even locally, half conformally flat Einstein-Weyl structures are "half-closed" [**12**] and therefore Einstein or locally hyper-complex [**66**]. We briefly discuss some local examples taken from [**4**].

It is far from being true, however, that the only compact Einstein-Weyl manifolds in dimension four are the closed ones. After discussing topological constraints given by an analogue of the Hitchin-Thorpe inequality [**64**], we end section 9 by presenting the classification [**54**] of four dimensional Einstein-Weyl manifolds with symmetry group of dimension at least four.

In section 10, after discussing the twistor theory of Einstein-Weyl 3-manifolds and the Jones-Tod construction, we present some special classes of three dimensional Einstein-Weyl geometries [**17, 29, 50, 79**] and place them in a unified framework [**14**]. We also explain why the possible geometries on compact 3-manifolds are all obtained as quotients of $\mathbb{R}^4$ and review their classification [**72**]. Finally we give the analogous classification result in dimension two [**11**].

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1. Density line bundles and Weyl derivatives

DEFINITION 1.1. Let V be a real n-dimensional vector space and w any real number. Then a *density of weight* w or w-*density* on V is defined to be a map $\rho\colon (\Lambda^n V)\smallsetminus 0 \to \mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{-w/n}\rho(\omega)$ for all $\lambda \in \mathbb{R}^\times$ and $\omega \in (\Lambda^n V)\smallsetminus 0$. The space of densities of weight w is denoted $L^w = L^w(V)$.

REMARKS. L^w naturally carries the representation $\lambda.\rho = |\lambda|^w\rho$ of the center of $\mathrm{GL}(V)$ or equivalently the representation $A.\rho = |\det A|^{w/n}\rho$ of $\mathrm{GL}(V)$. Note also:

- L^w is an oriented one dimensional linear space with dual space L^{-w}, and L^0 is canonically isomorphic to $\mathbb{R}$.
- The absolute value defines a map from $\Lambda^n V^*$ to L^{-n}. If V is oriented then the $(-n)$-densities can be identified with the volume forms.
- The densities of $L^{-1}\otimes V$ are canonically isomorphic to $\mathbb{R}$.

Now let M be any manifold. Then the *density line bundle* $L^w = L^w_{TM}$ of M is defined to be the bundle whose fiber at $x \in M$ is $L^w(T_xM)$. Equivalently it is the associated bundle $\mathrm{GL}(M) \times_{\mathrm{GL}(n)} L^w(n)$ where $\mathrm{GL}(M)$ is the frame bundle of M and $L^w(n)$ is the space of w-densities of $\mathbb{R}^n$.

One advantage of using densities is that they permit a simple geometric dimensional analysis to be carried out on tensors. Sections of $L = L^1$ are scalar fields with dimensions of length. More generally:

DEFINITION 1.2. The tensor bundle $L^w \otimes (TM)^j \otimes (T^*M)^k$ (and any subbundle, quotient bundle, element or section) will be said to have weight $w+j-k$, or dimensions of $[\text{length}]^{w+j-k}$.

It is quite common in the literature to call a section of such a bundle a tensor field of weight w, or perhaps $-w$, $w/2$, w/n ... various normalization are possible. In view (for instance) of the isomorphism $\Lambda^n T^*M \cong L^{-n}$ on an oriented manifold, such notions of weight would not permit a reasonable dimensional analysis.

On the other hand, the weight defined above can be interpreted invariantly as the representation of the center of $\mathrm{GL}(TM)$. It is additive under tensor product, compatible with contractions, and gives tangent vectors dimensions of length. Here "length" has been identified with weight $+1$, which is not the only reasonable choice. For instance in Fegan [**22**], the weight $+1$ is assigned to cotangent vectors.

NOTATION 1.3. When tensoring a vector bundle with some L^w, we shall often omit the tensor product sign.

The real line bundles L^w are oriented and hence trivializable. However, there is generally no preferred trivialization, and so we prefer to make such a choice explicit.

DEFINITION 1.4. A non-vanishing (usually positive) section of L^1 (or L^w for $w \neq 0$) will be called a *length scale* or *gauge* (of weight w).

It can be convenient in computation and examples to choose a length scale. Nevertheless, the following will be viewed as being more geometrically fundamental.

DEFINITION 1.5. A *Weyl derivative* is a covariant derivative D on L^1. It induces covariant derivatives on L^w for all w. The curvature of D is a real 2-form F^D which will be called the *Faraday curvature.* If $F^D = 0$ then D is said to be *closed*, and there exist local length scales μ with $D\mu = 0$. If such a length scale exists globally, then D is said to be *exact.*

Note that the Weyl derivatives form an affine space modeled on the space of 1-forms, while the spaces of closed and exact Weyl derivatives are modeled on the closed and exact 1-forms respectively.

A length scale μ induces an exact Weyl derivative D^μ such that $D^\mu\mu = 0$. Consequently we shall sometimes call an exact Weyl derivative a gauge, but note that $c\mu$ induces the same derivative for any $c \in \mathbb{R}^+$. If D is any other Weyl derivative then $D = D^\mu + \omega^\mu$ for the 1-form $\omega^\mu = \mu^{-1}D\mu$.

A *gauge transformation* on M is a positive function e^f which rescales a gauge $\mu \in \mathrm{C}^\infty(M, L^w)$ to give $e^{wf}\mu$. Gauge transformations also act on Weyl derivatives via $e^f \cdot D = e^f \circ D \circ e^{-f} = D - df$. However, we shall normally only consider the action on length scales, so that if, for a fixed Weyl derivative D and any length scale μ, we write $D = D^\mu + \omega^\mu = D^{e^f\mu} + \omega^{e^f\mu}$, then $\omega^{e^f\mu} = \omega^\mu + df$.

REMARKS. The theory of Weyl derivatives is a gauge theory with gauge group $\mathbb{R}^+$, and is a geometrization of classical electro-magnetism: the Faraday curvature represents the electro-magnetic field. Indeed this is the original gauge theory of "metrical relationships" introduced by Weyl [**80**]. As a model for electromagnetism, however, it was subsequently rejected in favor of a $\mathrm{U}(1)$ gauge theory. An unfortunate consequence of this is that Weyl derivatives have suffered a period of neglect in differential geometry, although there are several contexts in which they are useful.

EXAMPLE 1.6. Let Ω be a non-degenerate 2-form on M^n. Then Ω^m $(n = 2m)$ equips M with an orientation and a length scale (hence an exact Weyl derivative). Suppose instead that $\Omega \in \mathrm{C}^\infty(M, L^2\Lambda^2T^*M)$ and that Ω^m is a constant nonzero section of the orientation line bundle $L^n\Lambda^n T^*M$. Now $d\Omega$ is no longer well defined: for each Weyl derivative D on L^1 one can define $d^D\Omega$, but if γ is a 1-form then $d^{D+\gamma}\Omega = d^D\Omega + 2\gamma\wedge\Omega$. However, for $2m > 2$, the non-degeneracy of Ω implies that there is a unique Weyl derivative such that $\mathrm{tr}_\Omega\, d^D\Omega = 0$. In four dimensions this forces $d^D\Omega = 0$, so that every weightless almost symplectic form is "symplectic" with respect to a unique Weyl derivative: it is symplectic in the usual sense iff the Weyl derivative is exact. This Weyl derivative is a manifestly scale invariant version of the Lee form [**75**], which appears naturally in Hermitian geometry, since the Kähler form of an orthogonal almost complex structure on a conformal manifold is a weightless non-degenerate 2-form.

Weyl derivatives also arise in (oriented) contact and CR geometry, where they are induced by complementary subspaces to the contact distribution. The exact Weyl derivatives correspond to global contact forms.

Finally, whenever a geometry has a preferred family of linear connections affinely modeled on the space of 1-forms, these linear connections are usually parameterized by Weyl derivatives. This occurs in quaternionic geometry, projective geometry and the example of interest here: conformal geometry.

2. Conformal geometry

The modern approach to gauge theory has provided much geometrical clarification by identifying it as a theory of connections rather than potentials and gauge transformations. Yet this approach has not filtered back to conformal geometry, where the gauge is constantly being fixed by a metric, and then transformations under rescaling are considered. Part of the problem is that the standard definition of a conformal manifold is a manifold equipped with an equivalence class of Riemannian metrics. The very notation, $[g]$, for the conformal structure leads one to fix the gauge.

A conformal structure may alternatively be defined as a reduction of the frame bundle to a principal $\mathrm{CO}(n)$-bundle, just as a Riemannian metric is equivalently an $\mathrm{O}(n)$-structure. However, this definition has the disadvantage that although the group of invariance of the geometry is clear, it is not made clear exactly *what* remains invariant, and so a Riemannian metric is usually introduced. As counterpoint to the tendency to do conformal geometry in a Riemannian framework, we would like to suggest that a conformal structure is more fundamental than a Riemannian structure by defining the latter in terms of the former.

One motivation for this is that the notion of a Riemannian metric is dimensionally incorrect, since the length of a tangent vector should be a length, not a number. One can only turn it into a number by choosing a length scale.

DEFINITION 2.1. (See e.g., Hitchin [**33**]) A *conformal structure* on a manifold M is an L^2 valued inner product on TM. More precisely it is a section $\mathrm{c} \in C^\infty(M, L^2S^2T^*M)$ which is everywhere positive definite. Furthermore, we shall always take it to be normalized in the sense that $|\det \mathrm{c}| = 1$. Equivalently c is a normalized metric on the *weightless tangent bundle* $L^{-1}TM$. The normalization condition makes sense because the densities of $L^{-1}TM$ are canonically trivial.

In physics, where dimensional analysis is part of the culture, the determinant of a metric is often set to unity: physical metrics assign a length, not a number, to a vector. On the other hand a Riemannian metric is not dimensionless, and so it is meaningless to normalize it. Instead it defines a preferred length scale.

DEFINITION 2.2. A *Riemannian structure* on M is a conformal structure c together with a length scale μ. The metric is $g = \mu^{-2}\mathrm{c} \in C^\infty(M, S^2T^*M)$ and we write (c_g, μ_g) for the corresponding conformal structure and length scale.

This decomposition of a Riemannian structure into two pieces is reflected in the linearized theory: the bundle S^2T^*M is not irreducible under the orthogonal group, but decomposes into a trace and a trace-free part.

An alternative definition of a Riemannian structure is a conformal structure together with an exact Weyl derivative. Such a definition does not distinguish between homothetic metrics, which is often appropriate in practice. The existence and uniqueness of the Levi-Civita connection inducing this exact Weyl derivative is then a special case of the following foundational result.

THEOREM 2.3 (The Fundamental Theorem of Conformal Geometry). **[80]** *On a conformal manifold M there is an affine bijection between Weyl derivatives and torsion free connections on TM preserving the conformal structure. More explicitly, the torsion free connection on TM is determined by the Koszul formula*

$$\begin{aligned}2\langle D_XY, Z\rangle &= D_X\langle Y, Z\rangle + D_Y\langle X, Z\rangle - D_Z\langle X, Y\rangle\\ &\quad + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle,\end{aligned}$$

where $\langle X, Y\rangle \in C^\infty(M, L^2)$ denotes the conformal inner product of vector fields. (Note also that we shall write $|X|^2$ for $\langle X, X\rangle$.) The corresponding linear map sends a 1-form γ to the $\mathfrak{co}(TM)$-valued 1-form Γ defined by $\Gamma_X = \gamma(X)\mathrm{id} + \gamma \vartriangle X$, where $(\gamma \vartriangle X)(Y) = \gamma(Y)X - \langle X, Y\rangle\gamma$. Here γ is viewed as a vector field of weight -1 using the natural isomorphism $\sharp\colon T^*M \to L^{-2}TM$ given by the conformal structure.

Henceforth, we identify a Weyl derivative on a conformal manifold with the induced "Weyl connection" on the tangent bundle and all associated bundles. We also use the sharp isomorphism freely, only writing it explicitly to avoid ambiguity.

DEFINITION 2.4. A conformal structure c and a Weyl derivative D define a *Weyl structure* on M, making it into a *Weyl manifold*. For each $w \in \mathbb{R}$, $R^{D,w}$ will denote the *curvature* alt D^2 of D on $L^{w-1}TM$: it is a section of $\Lambda^2T^*M \otimes \mathfrak{co}(TM)$. We write $R^D = R^{D,1}$ for the curvature of the torsion free connection D on TM.

A basic fact in Weyl geometry is the existence of a weight -2 tensor r^D, called the *normalized Ricci endomorphism* of the Weyl structure, such that the curvature of D decomposes as follows:

$$R^{D,w}_{X,Y} = W_{X,Y} + wF^D(X,Y)\mathrm{id} - r^D(X) \vartriangle Y + r^D(Y) \vartriangle X.$$

Here W is the *Weyl curvature* of c, which is independent of D and is trace-free.

One way to establish this (and hence find r^D) is to study the way in which the curvature $R^{D,w}$ depends upon the choice of D. Since this is useful for other reasons, we state the result explicitly.

PROPOSITION 2.5. *Suppose D and $\tilde{D} = D+\gamma$ are Weyl derivatives on a conformal manifold (M^n, c). Then the curvatures of D and $\tilde{D}$ are related by the formula:*

$$\begin{aligned} R^{\tilde{D},w}_{X,Y} = R^{D,w}_{X,Y} &+ w\, d\gamma(X,Y)\mathrm{id} \\ &+ \big(D_X\gamma - \gamma(X)\gamma + \tfrac{1}{2}\langle\gamma,\gamma\rangle X\big) \vartriangle Y \\ &- \big(D_Y\gamma - \gamma(Y)\gamma + \tfrac{1}{2}\langle\gamma,\gamma\rangle Y\big) \vartriangle X. \end{aligned}$$

The proof is a matter of computing $d^D\Gamma + \Gamma\wedge\Gamma$ where Γ is related to γ by 2.3. The first term, $w\,d\gamma$, is simply the change in the Faraday curvature F^D on L^w, while the remainder is given in terms of the expression $D\gamma - \gamma\otimes\gamma + \frac{1}{2}\langle\gamma,\gamma\rangle\mathrm{id}$. In order to find a tensor r^D transforming in this way, define, for each $w \in \mathbb{R}$, a section of $L^{-2}\,\mathrm{End}\,TM$ by

$$\mathrm{Ric}^{D,w}(X) = \textstyle\sum_i R^{D,w}_{X,e_i} e_i,$$

where e_i is a weightless orthonormal basis. This Ricci endomorphism is not necessarily symmetric: its skew part turns out to be $(w - \frac{n-2}{2})F^D$, where F^D is viewed as the endomorphism $X \mapsto \sharp\iota_X F^D = \sharp F^D(X,.)$. The symmetric part of $\mathrm{Ric}^{D,w}$ is independent of w and hence so is the trace scal^D, which is a section of L^{-2} called the *scalar curvature* of D. Let $r^D_0 = \frac{1}{n-2}\,\mathrm{sym}_0\,\mathrm{Ric}^{D,w}$ be the (normalized) symmetric trace-free part, and define

$$r^D = r^D_0 + \tfrac{1}{2n(n-1)}\mathrm{scal}^D\mathrm{id} - \tfrac{1}{2}F^D.$$

PROPOSITION 2.6. *If D and $\tilde{D} = D+\gamma$ are Weyl derivatives on (M^n, c) then:*

$$\begin{aligned} r^{\tilde{D}}_0 &= r^D_0 - \mathrm{sym}_0\, D\gamma + (\gamma\otimes\gamma - \tfrac{1}{n}\langle\gamma,\gamma\rangle\mathrm{id}) \\ \mathrm{scal}^{\tilde{D}} &= \mathrm{scal}^D - 2(n-1)\,\mathrm{tr}\, D\gamma - (n-1)(n-2)\langle\gamma,\gamma\rangle \\ r^{\tilde{D}} &= r^D - (D\gamma - \gamma\otimes\gamma + \tfrac{1}{2}\langle\gamma,\gamma\rangle\mathrm{id}). \end{aligned}$$

This follows from 2.5 by taking traces, and also shows that W is independent of D.

PROPOSITION 2.7. *On any Weyl manifold of dimension $n > 2$,*

$$\mathrm{div}^D\left(r^D_0 - \tfrac{1}{2n}\mathrm{scal}^D\mathrm{id} + \tfrac{1}{2}F^D\right) = 0,$$

where $\mathrm{div}^D = \mathrm{tr}_{\mathrm{c}} \circ D$ and in particular, $\mathrm{div}^D F^D = \sum_i (D_{e_i}F^D)(e_i,.)$.

This is a consequence of the differential Bianchi identity $d^D R^{D,0} = 0$.

The exterior divergence δ on sections of $L^{-n}\Lambda^k TM$ (multi-vector densities) is an invariant operator, just like the exterior derivative on forms. In fact, up to sign, these divergences form a complex formally adjoint to the deRham complex. Our convention is to define $\delta = \mathrm{tr}\, D$, the trace being taken with the first entry. On forms, div^D can therefore be identified with a twisted exterior divergence δ^D. Such twisted divergences no longer form a complex in general. One consequence of this is the following.

PROPOSITION 2.8. **[12]** *Let D be a Weyl derivative on a conformal n-manifold M. Then $(\delta^D)^2 F^D = -(n-4)|F^D|^2$. If $n \neq 4$ it follows that $\mathrm{div}^D F^D = 0$ iff $F^D = 0$.*

PROOF. F^D is a section of $\Lambda^2 T^*M \cong L^{n-4}L^{-n}\Lambda^2 TM$ and so the divergence has been twisted by D on L^{n-4}. The formula follows by direct computation using a trivialization of L^{n-4}. □

3. The Einstein-Weyl equation

We now come to the main definition of this essay.

DEFINITION 3.1. [**16, 34**] Let (M, c, D) be a Weyl manifold of dimension at least three. Then M is said to be *Einstein-Weyl* iff $r_0^D = 0$; equivalently, the symmetric trace-free part of the Ricci tensor vanishes.

EXAMPLES 3.2. We illustrate the three types of Einstein-Weyl manifold.

(i) M is Einstein-Weyl with D exact iff it is Einstein, in the sense that each length scale μ with $D\mu = 0$ defines an Einstein metric.

(ii) Suppose $M = S^1 \times S^{n-1} \cong (\mathbb{R}^n \setminus \{0\})/\mathbb{Z}$, where the $\mathbb{Z}$ action is generated by $x \mapsto 2x$. This action preserves the flat conformal structure and the flat Levi-Civita derivative on $\mathbb{R}^n$, but not the flat metric. Hence M has a natural flat Weyl structure, which is therefore Einstein-Weyl, but the Weyl derivative, although closed, is not exact [**66, 68**]. Note that $S^1 \times S^2$ and $S^1 \times S^3$ admit no Einstein metric [**3**], yet both are Einstein-Weyl in a simple way.

(iii) The simplest example of an Einstein-Weyl manifold with nonzero Faraday curvature is the following Weyl structure on the Berger sphere [**40**]:

$$g = d\theta^2 + \sin^2\theta d\phi^2 + a^2(d\psi + \cos\theta d\phi)^2$$
$$\omega = b(d\psi + \cos\theta d\phi).$$

Here $D = D^g + \omega$ and a, b are constants with $b^2 = a^2(1-a^2)$. This example is related to the Hopf fibration over S^2, and will be discussed again in section 6.

REMARK. In two dimensions, there is no symmetric trace-free Ricci tensor, and so the Einstein-Weyl condition is vacuous. A 2-manifold is usually said to be Einstein iff it has constant scalar curvature, since this follows from the contracted Bianchi identity in higher dimensions. There is a natural generalization in Einstein-Weyl geometry.

PROPOSITION 3.3. [**68, 27**] *Suppose M is Einstein-Weyl of dimension $n > 2$. Then $D\mathrm{scal}^D - n \operatorname{div}^D F^D = 0$. (As before, the trace is with the first entry of F^D.)*

This is immediate from 2.7, and suggests the following definition.

DEFINITION 3.4. A Weyl manifold (M, c, D) of dimension two is said to be *Einstein-Weyl* iff $D\mathrm{scal}^D - 2 \operatorname{div}^D F^D = 0$.

Another justification for this definition is that a Weyl derivative D on a conformal 2-manifold defines an (almost) Möbius structure [**11**], and this Möbius structure is integrable (in other words, a complex projective structure) iff D is Einstein-Weyl.

The contracted Bianchi identity has several useful consequences.

PROPOSITION 3.5. [**12, 67**] *Let M be an n-dimensional Einstein-Weyl manifold. Then $\Delta^D \mathrm{scal}^D = -n(n-4)|F^D|^2$, where $\Delta^D = \mathrm{tr}\, D^2$.*

We also obtain the following result, essentially given in [**27, 31**], although by using 2.8 compactness assumptions can be avoided except in dimension four [**12**].

THEOREM 3.6. *If (M^n, D) is Einstein-Weyl, the following are equivalent:*

(i) *Either D is closed or $n = 4$, M is non-compact and F^D is harmonic.*

(ii) $\operatorname{div}^D F^D = 0$.

(iii) $D\mathrm{scal}^D = 0$.

(iv) *Either D is exact or* scal^D *is identically zero.*

PROOF. (ii) and (iii) are equivalent by 3.3, and clearly (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) or (iii). The equivalence of (i) and (ii) follows from 2.8, together with the conformal invariance of the divergence on 2-forms in four dimensions, and the fact that an exact co-closed 2-form on a compact 4-manifold necessarily vanishes (write $F^D = d\gamma$ and integrate the section $|F^D|^2$ of L^{-4} by parts). □

4. The Gauduchon gauge

For electro-magnetism, it is common to fix the gauge by requiring the potential to be divergence free. In Weyl geometry, there are several possible ways to interpret this. However, it is the following gauge that has become the most important, thanks to its global existence and the wealth of results that follow from it [**24, 27, 72**].

DEFINITION 4.1. Let (M, c, D) be a Weyl manifold. Then a length scale μ is called a *Gauduchon gauge* iff $D = D^\mu + \omega^\mu$ with $\mathrm{tr}_{\mathrm{c}} D^\mu \omega^\mu = 0$. The exact Weyl derivative D^μ will be called the *Gauduchon derivative* and ω^μ the *Gauduchon 1-form.*

Note that it is the *gauge* derivative being used to define the divergence and so *a priori* this condition is nonlinear, except in two dimensions where the divergence on 1-forms is conformally invariant. However, in higher dimensions the condition is easily linearized by using a length scale of weight $2 - n$.

PROPOSITION 4.2. *Suppose* (M, c, D) *is a Weyl manifold of dimension* $n \geqslant 3$. *Then a length scale* λ *of weight* $2-n$ *is a Gauduchon gauge iff* $\mathrm{div}\, D\lambda := \mathrm{tr}\, D^2\lambda = 0$.

This follows from the invariance of the divergence on $L^{-n}TM \cong L^{2-n}T^*M$.

On an oriented 3-manifold, a Gauduchon gauge is an "Abelian monopole": the Gauduchon gauge condition means that $*D\lambda$ is a closed 2-form, which is locally equivalent to $*D\lambda = d\theta$ for some 1-form θ. On an Einstein-Weyl 4-manifold, Proposition 3.5 shows that the scalar curvature scal^D defines a Gauduchon gauge wherever it is nonzero. More generally, there is the following theorem.

THEOREM 4.3. [**24**] *A compact Weyl manifold admits a Gauduchon gauge, unique up to homothety* (*i.e., the Gauduchon derivative is uniquely determined*).

PROOF. If $n = 2$ a Gauduchon gauge is a co-closed representative for the space of 1-forms γ such that $D - \gamma$ is exact (in particular $d\gamma = F^D$). The result in this case is therefore a consequence of the Hodge decomposition. Now suppose $n > 2$.
• The formal adjoint of $\mathrm{tr}\, D^2\colon J^2L^w \to L^{w-2}$ is $\mathrm{tr}\, D^2\colon J^2L^{-w+2-n} \to L^{-w-n}$. Now let Δ_D denote this Weyl Laplacian on functions, and Δ_D^* its formal adjoint on sections of L^{2-n}. By Proposition 4.2 a positive section λ of L^{2-n} defines a Gauduchon gauge iff $\Delta_D^*\lambda = 0$.
• Since Δ_D and Δ_D^* have the same principal symbol (after trivializing L^1), they have the same index, which is therefore zero, since they are adjoints. Consequently $\dim\ker\Delta_D^* = \dim\ker\Delta_D = 1$ by the maximum principle.
• No $\phi \in \ker\Delta_D^*$ may change sign: if it did, its integral in a gauge could take any real value and so in particular there would exist positive sections of L^{-2} orthogonal to ϕ. However, the image of Δ^D cannot contain such a positive section, since the Hopf maximum principle implies that super-solutions of Δ^D must be constant. Therefore any $\phi \in \ker\Delta_D^*$ is everywhere nonnegative or non-positive, and so (by the

Hopf maximum principle again) any nonzero ϕ is nowhere vanishing, whence $\ker \Delta_D^*$ consists precisely of the constant multiples of some positive section of L^{2-n}. □

The Gauduchon gauge is particularly powerful on compact Einstein-Weyl manifolds, because it is a *Killing gauge* in the sense that the Gauduchon 1-form is dual to a Killing field. This result of Tod [**72**] is closely related to the existence of a *Gauduchon constant* [**27**] generalizing the constant scalar curvature on an Einstein manifold.

THEOREM 4.4. *Let M be a* compact *Einstein-Weyl n-manifold and suppose that $D = D^g + \omega^g$ in the Gauduchon gauge. Then the section*

$$\kappa = \mathrm{scal}^g - (n+2)|\omega^g|^2 = \mathrm{scal}^D + n(n-4)|\omega^g|^2$$

of L^{-2} is constant and $\sharp\omega^g$ is a Killing field with respect to D^g. The Ricci endomorphism of D^g is given by

$$\mathrm{Ric}^g = \tfrac{1}{n}\mathrm{scal}^D\mathrm{id} + (n-2)\big(\langle\omega^g,\omega^g\rangle\mathrm{id} - \omega^g \otimes \omega^g\big). \tag{4.1}$$

Conversely suppose that M is Riemannian with Levi-Civita derivative D^g and that ω^g is a 1-form such that $\sharp\omega^g$ is a Killing field and the Ricci tensor of D^g is of the above form, where $\mathrm{scal}^D = \mathrm{scal}^g - (n-1)(n-2)|\omega^g|^2$. *Then $D = D^g \pm \omega^g$ is Einstein-Weyl with Gauduchon derivative D^g. (In two dimensions it is also necessary to suppose that* $\mathrm{scal}^g - 4|\omega^g|^2$ *is constant with respect to D^g.)*

The proof of this theorem involves the contracted Bianchi identity for D^g. In general, let μ be a gauge on a conformal n-manifold. Then for $n > 2$, $r_0^\mu - \frac{1}{2n}\mathrm{scal}^\mu\mathrm{id}$ is divergence free. If M is Einstein-Weyl, r_0^μ may also be defined by the difference $\mathrm{sym}_0(D^\mu)^2 - \mathrm{sym}_0 D^2$, and this definition works in dimension two: the contracted Bianchi identity for D^μ is then a consequence of the two dimensional Einstein-Weyl equation. From these observations, the following identities are obtained.

PROPOSITION 4.5. *Let M be Einstein-Weyl and let μ be any gauge. Then*

$$r_0^\mu - \tfrac{1}{2n}\mathrm{scal}^\mu\mathrm{id} = \mathrm{sym}_0 D^\mu\omega^\mu - \omega^\mu \otimes \omega^\mu + \tfrac{1}{n}\big(\langle\omega^\mu,\omega^\mu\rangle - \tfrac{1}{2}\mathrm{scal}^\mu\big)\mathrm{id}$$

is divergence free with respect to μ and consequently:

$$\begin{aligned}
&\mathrm{div}^\mu(\mathrm{sym}_0 D^\mu\omega^\mu)\\
&\qquad = 2\langle\mathrm{sym}_0 D^\mu\omega^\mu,\omega^\mu\rangle + \tfrac{n+2}{n}(\mathrm{div}^\mu\omega^\mu)\omega^\mu + \tfrac{1}{2n}D^\mu\big(\mathrm{scal}^\mu - (n+2)|\omega^\mu|^2\big)\\
&\mathrm{div}^\mu\Big(\langle\mathrm{sym}_0 D^\mu\omega^\mu,\omega^\mu\rangle - \tfrac{1}{2n}\big(\mathrm{scal}^\mu + (n-2)|\omega^\mu|^2\big)\omega^\mu\Big)\\
&\qquad = 2|\mathrm{sym}_0 D^\mu\omega^\mu|^2 - \tfrac{1}{2n}\big(\mathrm{scal}^\mu - (n+2)|\omega^\mu|^2\big)\,\mathrm{div}^\mu\omega^\mu.
\end{aligned}$$

On a compact manifold, taking μ to be a Gauduchon gauge and integrating the second of these identities immediately gives the main part of Theorem 4.4. The rest of the theorem is now straightforward.

If (M, D) is an Einstein-Weyl manifold with Killing gauge $D = D^g + \omega^g$ then $2D^g\omega^g = F^D$ and $D^g\langle\omega^g,\omega^g\rangle = -F^D(\omega^g,.)$. Consequently the contracted Bianchi identity 3.3 and the constancy of κ imply:

$$2\,\mathrm{tr}(D^g)^2\omega^g = \mathrm{div}^g F^D = -\tfrac{2}{n}\mathrm{scal}^D\omega^g \tag{4.2}$$

$$\Delta^g|\omega^g|^2 + \tfrac{2}{n}\mathrm{scal}^D|\omega^g|^2 = |F^D|^2 \tag{4.3}$$

$$\Delta^g\mathrm{scal}^D - 2(n-4)|\omega^g|^2\mathrm{scal}^D = -n(n-4)|F^D|^2 \tag{4.4}$$

We now collect some geometrical consequences.

THEOREM 4.6. **[12, 27, 37, 67, 68, 72]** *Let M^n be a* compact *Einstein-Weyl manifold with $D = D^g + \omega^g$ in the Gauduchon gauge. Then*

(i) *If D is not exact, then the isometry group of the Gauduchon metric is at least one dimensional.*

(ii) *Contracting* (4.2) *with ω^g and integrating gives:*

$$\int_M |F^D|^2 = \frac{2}{n} \int_M \mathrm{scal}^D |\omega^g|^2.$$

Consequently, if $\mathrm{scal}^D \leqslant 0$ *then D is closed.*

(iii) *D closed $\implies$ $D^g \omega^g = 0$ and, if D is not exact, $|\omega^g|^{-1}$ is a Gauduchon gauge.*

(iv) *If* $\mathrm{scal}^D > 0$ *then* $\mathrm{Ric}^g > 0$, *while if* $\mathrm{scal}^D \geqslant 0$ *then* $\mathrm{Ric}^g \geqslant 0$, *and* scal^g *is strictly positive if $n \geqslant 4$ or $n = 3$ and $\kappa \neq 0$.*

(v) *The Hopf maximum principle applied to* (4.4) *implies that if $n \geqslant 4$ and* scal^D *is not everywhere positive, then it is constant in the Gauduchon gauge.*

Theorems 3.6 and 4.6 together give the following rough classification result.

THEOREM 4.7. *If M is compact Einstein-Weyl, one of the following holds:*

(i) scal^D *is negative and D is exact.*

(ii) scal^D *is identically zero, D is closed and if D is not exact, M admits a metric of positive scalar curvature (zero scalar curvature in two dimensions).*

(iii) scal^D *is positive and M admits a metric of positive Ricci curvature.*

(iv) scal^D *is of non-constant sign,* $\dim M \leqslant 3$, *$\kappa \leqslant 0$ and F^D is nonzero.*

We also obtain topological consequences of the Einstein-Weyl condition.

THEOREM 4.8. **[27, 66]** *Let M be a compact Einstein-Weyl manifold. Then if* scal^D *is positive, M has finite fundamental group. Also if D is not exact and M is a spin manifold, then the $\hat{A}$-genus of M vanishes.*

THEOREM 4.9. **[27, 67]** *Let M^n be a compact Einstein-Weyl manifold* ($n > 2$) *with D closed but not exact. Then the parallel 1-forms on M are precisely the multiples of the Gauduchon 1-form and so the first Betti number of M is one. Also, the universal cover of M is $\mathbb{R} \times \Sigma$ where Σ is a simply connected Einstein manifold of positive scalar curvature. If $n \leqslant 4$ then $\Sigma = S^{n-1}$ and D is flat.*

PROOF. These results all follow easily from the formula (4.1) for the Ricci endomorphism of the Gauduchon gauge, together with the fact that ω^g is D^g-parallel. The first part can be proven either by a Bochner argument **[67]** or as a consequence of the second part **[27]**. The flatness of D for $n = 3$ is immediate from its Ricci-flatness, while for $n = 4$ it follows because $\mathbb{R} \times S^3$ is conformally flat. □

The flat non-exact compact Weyl manifolds, or manifolds of type $S^1 \times S^{n-1}$, therefore exhaust the closed Einstein-Weyl manifolds in dimension less than or equal to four. A detailed study of the four dimensional case can be found in **[27]**.

5. Conformal submersions

As in Einstein geometry, many examples of Einstein-Weyl manifolds arise from submersions. Although we shall mainly focus on Riemannian submersions with totally geodesic fibers **[3]**, we would like to place these in a conformal context.

DEFINITION 5.1. Let $\pi\colon M \to B$ be a smooth surjective map between conformal manifolds and let the *horizontal bundle* $\mathcal{H}$ be the orthogonal complement to the vertical bundle $\mathcal{V}$ of π in TM. Then π will be called a *conformal submersion* iff for all $x \in M$, $d\pi_x|_{\mathcal{H}_x}$ is a nonzero conformal linear map.

It is not at all necessary to restrict attention to submersions in the following. The base could, for instance, be an orbifold, or be replaced altogether by the horizontal geometry of a foliation (see [**60**]). However, since we are primarily interested in the local geometry, we shall, for convenience of exposition, take the base to be a manifold. A bundle $\mathcal{H}$ complementary to $\mathcal{V}$ is often called a connection on π.

PROPOSITION 5.2. *If $\pi\colon M \to B$ is a submersion onto a conformal manifold B, then conformal structures on M making π into a conformal submersion correspond bijectively to triples $(\mathcal{H}, c^{\mathcal{V}}, \rho)$, where $\mathcal{H}$ is a connection on π, $c^{\mathcal{V}}$ is a conformal structure on the fibers, and $\rho\colon \pi^* L^1_B \cong L^1_{\mathcal{H}} \to L^1_{\mathcal{V}}$ is a (positive) isomorphism.*

The final ingredient ρ in this construction will be called a *relative length scale*, since it allows vertical and horizontal lengths to be compared. The freedom to vary ρ generalizes the so called "canonical variation" of a Riemannian submersion, in which the fiber metric is rescaled, while the base metric remains constant.

DEFINITION 5.3. Let $\pi\colon M \to B$ be a conformal submersion and D a Weyl derivative on M. Then, following O'Neill [**57**], we define fundamental forms A^D, II^D by $A^D(X,Y) = (D_X Y)^{\mathcal{V}}$ for $X, Y \in \mathcal{H}$ and $\mathrm{II}^D(U,V) = (D_U V)^{\mathcal{H}}$ for $U, V \in \mathcal{V}$, where $(\ldots)^{\mathcal{V}}$ and $(\ldots)^{\mathcal{H}}$ denote the vertical and horizontal components.

A remarkable feature of conformal submersions is the existence of a preferred Weyl derivative, much like the Bott connection of a foliation.

PROPOSITION 5.4. *Suppose M is conformal and $TM = \mathcal{V} \oplus_\perp \mathcal{H}$ with $\mathcal{V}, \mathcal{H}$ nontrivial. Then if D is any Weyl derivative, $U \mapsto \mathrm{tr}_{\mathcal{H}} DU$ and $X \mapsto \mathrm{tr}_{\mathcal{V}} DX$ are tensorial for $U \in \mathcal{V}$ and $X \in \mathcal{H}$, and there is a unique $D = D^0$ such that $\mathcal{V}$ and $\mathcal{H}$ are minimal, in the sense that these mean curvature tensors are zero.*

(The last part follows by comparing the mean curvature tensors of D and $D + \gamma$.)

For a conformal submersion, D^0 will be called the *minimal Weyl derivative*, and the corresponding fundamental forms will be denoted II^0 and A^0. The integrability of $\mathcal{V}$ implies that for any D, II^D is symmetric in U, V (it is just the second fundamental form of the fibers), and so II^0 is symmetric and trace free. On the other hand, the conformal property of π implies that the symmetric part of $\langle A^D(X,Y), U\rangle = -\langle D_X U, Y\rangle$ is a pure trace, and so A^0 is skew in X, Y. If D^0 is exact, then in this gauge, the submersion is Riemannian and the fibers are minimal submanifolds.

The O'Neill formulae [**57, 30**] carry over to the conformal setting without substantial change, but here we restrict attention to the case of one dimensional fibers. A foliation of a conformal manifold with oriented one dimensional leaves is equivalently given by the weightless unit vector field tangent to the leaves. In this case the properties of D^0 can be reinterpreted as follows.

PROPOSITION 5.5. *Let ξ be a weightless unit vector field on a conformal manifold. Then the minimal Weyl derivative of the corresponding foliation is characterized by $D^0_\xi \xi = 0$ and $\mathrm{tr}\, D^0 \xi = 0$ and the foliation is (locally) a conformal submersion iff $D^0\xi$ is skew. D^0 is exact iff there is a conformal vector field K with*

$K = |K|\xi$, in which case $D^0|K| = 0$ and so D^0 is the Levi-Civita derivative of $g = |K|^{-2}c$ and K is a unit Killing field.

In other words, if a conformal submersion is given by the flow of a non-vanishing conformal vector field K then D^0 is the *constant length gauge* of K.

Let $\pi\colon M^{n+1} \to B^n$ be a conformal submersion with one dimensional fibers and D^0 exact (this is equivalently a Riemannian submersion with totally geodesic fibers and D^0 is the Levi-Civita derivative). Then D^0 is well defined on the base, and any other Weyl derivative on B is of the form $D^0 + \omega$ for some 1-form ω. On the total space M, we now consider the Weyl derivative $D = D^0 + \frac{n-2}{n-1}\pi^*\omega + \lambda\xi$ where ξ is the weightless (co)tangent vector to the fibers and λ is a section of L^{-1}.

Using the well known submersion formulae for the Ricci tensor of D^0 [**3**], together with the formulae in 2.6 we obtain the following.

PROPOSITION 5.6. *Let $D = D^0 + \frac{n-2}{n-1}\pi^*\omega + \lambda\xi$. Then:*

$$\begin{aligned}
\operatorname{sym}\operatorname{Ric}^D_M(X,Y) &= \operatorname{sym}\operatorname{Ric}^{D^0+\omega}_B(X,Y) - 2\langle A^0_X, A^0_Y\rangle - \big(D^0_\xi\lambda + (n-1)\lambda^2\big)\langle X,Y\rangle\\
&\quad - \tfrac{n-2}{n-1}\,\omega(X)\omega(Y) + \tfrac{1}{n-1}\big(\operatorname{div}^0\omega + (n-2)|\omega|^2\big)\langle X,Y\rangle\\
\operatorname{sym}\operatorname{Ric}^D_M(\xi,X) &= \textstyle\sum_i\langle (D^0_{e_i}A^0)(e_i,X),\xi\rangle - \tfrac12(n-1)D^0_X\lambda\\
&\quad + (n-2)\big(\omega(X)\lambda - \langle A^0(X,\omega),\xi\rangle\big)\\
\operatorname{sym}\operatorname{Ric}^D_M(\xi,\xi) &= |A^0|^2 - nD^0_\xi\lambda\\
&\quad - \tfrac{n-2}{n-1}\big(\operatorname{div}^0\omega + (n-2)|\omega|^2\big)
\end{aligned}$$

where X, Y are horizontal, $e_1, ..e_n, \xi$ is a weightless orthonormal basis with ξ vertical, $\langle A^0_X, A^0_Y\rangle = \sum_i\langle A^0(X,e_i), A^0(Y,e_i)\rangle$ and $|A^0|^2 = \sum_i\langle A^0_{e_i}, A^0_{e_i}\rangle$.

The factor $\frac{n-2}{n-1}$ eliminates the difficult terms involving $D^0\omega$. It occurs naturally in the case of a hyper-complex 4-manifold over an Einstein-Weyl 3-manifold [**29**], which we shall discuss in section 10. In this section, though, we shall only treat the case $\omega = 0$, as considered by Pedersen and Swann [**66**].

THEOREM 5.7. [**66, 56**] *Let $\pi\colon M^{n+1} \to B^n$ be a Riemannian submersion, over an Einstein manifold B, with complete totally geodesic one dimensional fibers. Suppose that M admits an Einstein-Weyl structure of the form $D = D^0 + \lambda\xi$. Then*

(i) *$\operatorname{scal}^0_B \geqslant (n+2)|A^0|^2 + n(n-1)\lambda^2$ and $\operatorname{scal}^D_M \geqslant n|A^0|^2$, with equality (in both) iff λ is constant on the fibers (which necessarily holds if the fibers are compact).*

(ii) *In the D^0 gauge, A^0 defines a symplectic form on the open subset of B where it is nonzero, and so n is even unless A^0 is identically zero. If $|A^0|^2$ is a nonzero constant then B is almost Kähler and M is almost Sasakian.*

PROOF. By the submersion formulae, the Einstein-Weyl condition gives rise to the following three equations:

$$\begin{aligned}
\langle A^0_X, A^0_Y\rangle &= \tfrac1n|A^0|^2\langle X,Y\rangle\\
\textstyle\sum_i\langle (D^0_{e_i}A^0)(e_i,X),\xi\rangle &= \tfrac12(n-1)D^0_X\lambda\\
n(n-1)(\lambda^2 - D^0_\xi\lambda) &= \operatorname{scal}^0_B - (n+2)|A^0|^2.
\end{aligned}\tag{5.1}$$

The last equation and the completeness of the fibers together imply that along each fiber, λ is either constant or a negative hyperbolic tangent with respect to

D^0. Hence $D^0_\xi\lambda$ is non-positive and the first part readily follows. The first equation implies that $(X,Y) \mapsto \langle A^0(X,Y), \xi\rangle$ is either zero or non-degenerate at each point. Also, if μ is a D^0-parallel length scale, then $\langle A^0(X,Y), \mu^{-1}\xi\rangle = -\frac{1}{2}d(\mu^{-1}\xi)$, which is a closed basic 2-form $\pi^*\Omega$. If A^0 is nonzero, the metric $\mu|A^0|\mathrm{c}$ on B is almost Hermitian with Kähler form Ω and so if $D^0|A^0| = 0$ then B is almost Kähler and M is almost Sasakian. □

Examining this theorem more closely, we see that the Einstein-Weyl equations on M have in fact been encoded on B, suggesting that there should be an inverse construction. In fact one parameter families of Einstein-Weyl structures can be found on S^1-bundles in this way.

Suppose $\pi\colon M \to B$ is a fibration over an almost Kähler-Einstein manifold of positive scalar curvature and that it has a connection $\mathcal{H}$ with curvature $k\pi^*\Omega \otimes U$, where Ω is the Kähler form on B, U is a non-vanishing vertical vector field and k is constant. If for some choice of relative length scale, M becomes a Riemannian submersion with totally geodesic fibers and U constant, then the same holds for any constant multiple of this relative length scale, giving a one parameter family of metrics $g_t = \pi^* g_B + t^2 g(U, .)^2$ called the *canonical variation.* The equations (5.1) with constant λ may be satisfied provided $\mathrm{scal}^0_B \geqslant (n+2)|A^0|^2_t$. If $A^0 = 0$ this holds for all t, while for $A^0 \neq 0$, it is only possible for $0 < t \leqslant t_0$ where g_{t_0} is an Einstein metric.

THEOREM 5.8. [**66**] *Let B be a Kähler-Einstein manifold of positive scalar curvature and let M be a principal S^1-bundle with connection whose curvature is a multiple of the Kähler form. Then M admits a one parameter family of Einstein-Weyl structures.*

These results fit in with the idea that Einstein-Weyl geometry is a natural deformation of Einstein geometry, which we shall discuss again in section 7.

6. Examples

Examples of Einstein-Weyl structures on S^1-bundles include the following.

1. The basic example of a nontrivial S^1-bundle over a Kähler-Einstein base is the Hopf fibration $S^3 \to S^2$. If σ is a left invariant 1-form on S^3 then the bi-invariant (round) metric is $g = \pi^* g_{S^2} + \sigma^2$ where π is the Riemannian submersion generated by the Killing field dual to σ. If we now consider the U(2) invariant Berger metric $g_a = \pi^* g_{S^2} + a^2\sigma^2$, we find that for $0 < a < 1$ there is a unique b up to sign such that $D = D^{g_a} + b\sigma$ is Einstein-Weyl. This is the example given in 3.2 and it easily generalizes to the higher dimensional Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$. The Einstein-Weyl structures are parameterized by a point (a, b) on an ellipse, where the two points on the axis of symmetry $b = 0$ are respectively degenerate and Einstein [**65**].

2. The unit tangent bundle T_1S^n of S^n is an S^1-bundle over the Grassmannian $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{n+1})$ of oriented 2-planes in $\mathbb{R}^{n+1}$. Since $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{n+1})$ is Kähler-Einstein, T_1S^n admits a one parameter family of Einstein-Weyl structures.

3. The twistor space Z of a quaternionic manifold M possesses a natural S^1-bundle S. If M is quaternionic Kähler with positive scalar curvature, then Z is Kähler-Einstein and S is a 3-Sasakian manifold admitting a one parameter family of Einstein-Weyl structures which fiber over M with Berger 3-spheres as fibers.

Riemannian submersions have also been used [67] to construct Einstein-Weyl structures on S^2- or $\mathbb{R}P^2$-bundles over compact Kähler-Einstein manifolds of positive scalar curvature. For instance, we find Einstein-Weyl structures with $\mathrm{scal}^D > 0$ on $P(\mathcal{O}(k) \oplus \mathcal{O})$ over $\mathbb{C}P^n$ for $|k| \leqslant n$ (i.e., $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ when $n = 1$).

In all these bundle constructions the base manifold may be taken to be a *product* $M_1 \times \cdots \times M_m$ of Kähler-Einstein manifolds (M_i, g_i) with $c_1(M_i)$ positive and proportional to an indivisible class α_i:

1. Let $\pi\colon P \to M_1 \times \cdots \times M_m$ be a principal T^r-bundle with characteristic classes $\beta_i = \sum_{j=1}^m b_{ij} \pi_j^* \alpha_j$, for $i = 1, \ldots, r \leqslant m + 1$, where$(b_{ij})$ is a matrix of integers of rank at least $r - 1$. Then there is a family of Einstein-Weyl structures (g, ω) on P such that π is a Riemannian submersion with flat totally geodesic fibers, the metric on B is of the form $x_1 g_1 + \cdots + x_m g_m$, and the 1-form ω is vertical.

More explicitly, for $r = 1$, let θ be the principal connection and set $\omega = f\theta$ for some function f. Let g be the metric $x_1 \pi^* g_1 + \cdots + x_n \pi^* g_n + \theta^2$. Then, using the Riemannian submersion formula 5.6, the Einstein-Weyl equation forces the function f to be constant. A fixed point argument modeled on that of Wang and Ziller [78] shows the existence of a solution. For general r the technical condition on the rank of (b_{ij}) turns out to be equivalent to the necessary condition $b_1(P) \leqslant 1$ for the existence of an Einstein-Weyl solution [66].

2. Similarly, there are solutions on S^2- or $\mathbb{R}P^2$-bundles over $M_1 \times \cdots \times M_m$. For integers q_i, the cohomology class $q_1\alpha_1 + \cdots + q_m\alpha_m$ is the Euler class of a principal circle bundle P with curvature $\Omega = \sum_i q_i \Omega_i$ where Ω_i is the Kähler form on M_i. Then there are Einstein-Weyl structures on $M = P \times_{S^1} S^2$, of the form

$$h = dt^2 + f(t)^2 \sigma^2 + \textstyle\sum_i h_i(t)^2 g_i$$
$$\omega = A\, dt + B f \sigma,$$

where $d\sigma$ is the pull-back of Ω [77].

Another important reservoir of examples is provided by looking for Einstein-Weyl manifolds with a high degree of symmetry [52, 54]. The natural group of symmetries on a Weyl manifold is the group of automorphisms preserving both the conformal structure c (the conformal transformations) and the Weyl connection D (the affine transformations). One may argue that it is equally natural to consider projective transformations but on a Weyl manifold conformal projective transformations are automatically affine: if D^1 and D^2 are projectively equivalent then $D^1_X - D^2_X = \alpha(X)\mathrm{id} + \alpha \otimes X$, whereas if D^1 and D^2 are both compatible with the conformal structure, then $D^1_X - D^2_X$ has to be a section of $\mathfrak{co}(TM)$ for all X, which forces $\alpha = 0$. Furthermore, we can generally assume that the symmetry groups preserve the Gauduchon metrics:

PROPOSITION 6.1. *Let G act by symmetries on a compact Weyl manifold M. Then G preserves the Gauduchon derivative D^g and so acts by homotheties of each Gauduchon metric. If G is compact then it acts by isometries. (If M is not the n-sphere then G is compact by the theorems of Obata and Lelong-Ferrand* [46].)

PROOF. The Gauduchon gauge D^g satisfies $\mathrm{tr}\, D^g \omega^g = 0$. For each $a \in G$ we have $D = a^*D = a^*D^g + a^*\omega^g$. The pull-back of an exact Weyl derivative is exact and $\mathrm{tr}(a^*D^g)a^*\omega^g = a^* \mathrm{tr}\, D^g \omega^g = 0$, so uniqueness implies that $a^*D^g = D^g$ and $a^*\omega^g = \omega^g$. The action of G on the homothety class of Gauduchon metrics is thus described by a homomorphism $\rho\colon G \to \mathbb{R}$. If G is compact, ρ is constant. □

COROLLARY 6.2. *Let G be the symmetry group of a compact Weyl manifold M with Weyl derivative $D = D^g + \omega^g$ in the Gauduchon gauge. Then G preserves ω^g and hence for each $x \in M$ either $\omega^g_x = 0$ or the isotropy representation at x has a trivial summand tangent to the orbit.*

This greatly restricts the possible non-exact homogeneous Weyl structures. If one then imposes the Einstein-Weyl equation, only a few examples are known. For instance, there are the S^1-bundles S^{2n+1} and T_1S^n with $\mathrm{scal}^D > 0$ given above, and examples of type $S^1 \times S^{n-1}$ with $\mathrm{scal}^D = 0$. Together with an additional family on S^{4n+3}, these are the only homogeneous examples on symmetric spaces [**43**].

We turn now to some explicit examples of compact Einstein-Weyl n-manifolds M of co-homogeneity one under a group of symmetries G. The principal orbits are therefore homogeneous submanifolds G/H of dimension $n-1$ and M/G is either a closed interval or a circle. In the latter case M will not have finite fundamental group and therefore (for $n \geqslant 4$) the Weyl derivative is closed. Consequently we restrict attention to the case $M/G = [0,\ell]$.

Motivated by the possible actions in low dimensions (in particular see Theorem 9.13 for a classification result in dimension four) we look for Einstein-Weyl manifolds of co-homogeneity one under $\mathrm{SO}(n)$, $S^1 \times \mathrm{SO}(n-1)$ or $\mathrm{U}(2m)$ $(n = 2m)$ with principal orbits covered by S^{n-1}, $S^1 \times S^{n-2}$ or S^{2m-1} respectively.

The case of $\mathrm{SO}(n)$ gives only Einstein manifolds by 6.2, since the principal orbits are isotropy irreducible. In the other cases, non-closed Einstein-Weyl structures may be constructed explicitly via the solution of ODEs with boundary conditions.

Consider, for example, $S^1 \times \mathrm{SO}(n-1)$ symmetry. First of all we study the possible $S^1 \times \mathrm{SO}(n-1)$-invariant Einstein-Weyl structures on $(0,\ell) \times S^1 \times S^{n-2}$. In the Gauduchon gauge they are easily seen to be of the form:

$$\begin{aligned} g &= dt^2 + f(t)^2 d\theta^2 + h(t)^2 g_{\mathrm{can}} \\ \omega &= A f(t)^2 d\theta \end{aligned}$$

where g_{can} is the round metric on S^{n-2} of sectional curvature one. Here f and h are smooth functions on $[0,\ell]$ with $f, h > 0$ on $(0,\ell)$ and A is constant. With this Ansatz the Einstein-Weyl equation becomes:

$$\begin{aligned} -\frac{f''}{f} - (n-2)\frac{h''}{h} &= \Lambda \\ -\frac{f''}{f} - (n-2)\frac{f'h'}{fh} + (n-2)A^2 f^2 &= \Lambda \\ -\frac{h''}{h} - (n-3)\frac{h'^2}{h^2} - \frac{f'h'}{fh} + \frac{n-3}{h^2} &= \Lambda. \end{aligned}$$

At the boundary points $0,\ell$ we seek subgroups K such that $\mathrm{SO}(n-2) < K \leqslant S^1 \times \mathrm{SO}(n-1)$ with $K/\mathrm{SO}(n-2)$ a sphere. For instance, $M = S^n$ is obtained if we take $K_1 = \mathrm{SO}(n-1)$ at $t = 0$ and $K_2 = S^1 \times \mathrm{SO}(n-2)$ at $t = \ell$. The boundary conditions at $t = 0$ are then seen to be $f > 0$, $f', h, h'' = 0$, $h' = 1$ while at $t = \ell$, we have $h > 0$, $f, f'', h' = 0$, $f' = -1$. Solutions matching these boundary conditions can be found explicitly. In particular, when $n = 4$ we find the following solutions on S^4.

$$
(6.1)\quad
\begin{aligned}
g &= \frac{1 - a\cot a}{y\cot y - a\cot a}dy^2 + \frac{4(1-a\cot a)(y\cot y - a\cot a)}{(a + a\cot^2 a - \cot a)^2}d\theta^2 + \sin^2 y\, g_{\mathrm{can}}\\
\omega &= \frac{2(y\cot y - a\cot a)}{a + a\cot^2 a - \cot a}d\theta.
\end{aligned}
$$

Here $\sin y = h(t)$ and $(y, \theta) \in (0, a) \times (0, 2\pi)$, where $0 \leqslant a < \pi$ and $a = 0$ corresponds to the standard Einstein metric on S^4.

In this way we find one parameter families of $S^1 \times \mathrm{SO}(n-1)$ symmetric Einstein-Weyl structures on S^n and $S^2 \times S^{n-2}$. Relaxing the boundary conditions gives solutions on line bundles over compact manifolds [**52**].

A similar calculation leads to families of solutions with $\mathrm{U}(m)$ symmetry on S^{2m}, $\mathbb{C}P^m$ and $P(\mathcal{O}(k) \oplus \mathcal{O})$ $(0 < |k| < m)$ over $\mathbb{C}P^{m-1}$. This last case fits into the framework of Einstein-Weyl structures on S^2-bundles discussed earlier (the Fubini-Study metric on $\mathbb{C}P^{m-1}$ being the Kähler-Einstein base).

One motivation for studying these highly symmetric examples is that the principal orbits provide an interesting family of submanifolds [**65**]. For instance, in the case of S^n with principal orbits $S^1 \times S^{n-2}$ there exists $t_0 \in [0, \ell]$ such that the corresponding $S^1 \times S^{n-2}$ is minimal in the Gauduchon metric of S^n. This generalizes the Clifford torus in the round 3-sphere. Likewise S^{2m} with the Gauduchon metric has a totally geodesic equator S^{2m-1} $(t = \frac{\ell}{2})$.

It should be pointed out, however, that the induced structures on the submanifolds $S^1 \times S^{n-2}$ in S^{n-1} and and S^{2m-1} in S^{2m} are *not* Einstein-Weyl. Of course, both $S^1 \times S^{n-2}$ and S^{2m-1} are Einstein-Weyl with respect to other Weyl structures and in fact these structures *do* sit as minimal hypersurfaces in *some* Einstein-Weyl space due to the following theorem which is inspired by the work of Koiso [**44**].

THEOREM 6.3. [**65**] *Let (M, c, D) be a real analytic Weyl manifold with an analytic symmetric bilinear form β taking values in a real line bundle over M. Then, there is a germ unique Einstein-Weyl space $(\bar{M}, \bar{\mathrm{c}}, \bar{D})$ in which (M, c, D) is embedded as a hypersurface with second fundamental form β. In particular the embedding could be minimal or totally geodesic.*

7. Moduli spaces of Einstein-Weyl structures

A possible motivation for studying Einstein-Weyl geometry in arbitrary dimensions is that Einstein manifolds with Killing fields often admit continuous families of Einstein-Weyl structures, as discussed in section 5. Since such Einstein manifolds are often rigid [**3**], the Einstein-Weyl condition may provide nontrivial deformations which would otherwise be lacking. One might then hope to get new Einstein metrics by going to the boundary of the Einstein-Weyl moduli space. So far, though, only known Einstein metrics have been obtained in this way.

Let M be compact and let the diffeomorphism group $\mathrm{Diff}(M)$ act on Weyl structures (c, D) by pull-back. Since the quotient space is not a manifold, we need to fix a slice to this action. One way of doing this is to describe Weyl structures in the Gauduchon gauge and use the Ebin slice [**21**] near a suitable Gauduchon metric g_0. The homothety factor of this metric may be fixed by specifying the Gauduchon constant κ. To do this, note that for $n \geqslant 4$, an Einstein-Weyl structure with $\kappa \leqslant 0$ is either Einstein or belongs to the known family of four dimensional manifolds of type $S^1 \times S^3$. In dimensions two and three there is a classification of Einstein-Weyl geometries on compact manifolds (see section 10), and so we may focus here on the case of positive Gauduchon constant and fix $\kappa = 1$. The Ebin slice $\mathcal{S}(g_0)$ is now given infinitesimally by $\operatorname{div}^{g_0} \dot{g} = 0$, where $\dot{g}$ is a tangent vector at g_0 to the space of metrics. This fixes the action of $\operatorname{Diff}(M)$ up to isometries of g_0.

DEFINITION 7.1. Suppose (g_0, ω_0) is an Einstein-Weyl solution in the Gauduchon gauge on M and let $\mathcal{S}(g_0)$ be the Ebin slice. The subset $\widetilde{\mathcal{M}}$ of $\mathcal{S}(g_0) \times \Omega^1 M$ given by $\operatorname{div}^g \omega = 0$, $\kappa = 1$ and $\operatorname{ric}^g = \frac{1}{n}\operatorname{scal}^D g + (n-2)(|\omega|^2 g - \omega \otimes \omega)$ is called the *premoduli space* of Einstein-Weyl structures around (g_0, ω_0). The local *moduli space* $\mathcal{M}$ near (g_0, ω_0) is the quotient of $\widetilde{\mathcal{M}}$ by the isometry group of g_0.

We now study the Einstein-Weyl moduli space near an Einstein metric.

THEOREM 7.2. **[65]** *Suppose (g_t, ω_t) is a smooth curve in $\tilde{\mathcal{M}}$ with $(g_0, \omega_0) = (g, 0)$, so that g is Einstein with* $\operatorname{scal}^g = 1$, *and let $(\dot{g}, \dot{\omega})$ be the tangent at $t = 0$.*

Then $\dot{\omega}$ is a Killing field of g and $\dot{g}$ satisfies the linearized Einstein equation. This equation is elliptic and the space of Einstein-Weyl deformations is finite dimensional. In particular, if the Einstein metric has no infinitesimal Einstein deformations then (M, g) has at most an m-dimensional family of infinitesimal Einstein-Weyl deformations, where m is the rank of the isometry group of g.

PROOF. The Killing condition $\operatorname{sym} D^{g_t} \omega_t = 0$ implies that $\operatorname{sym} D^g \dot{\omega} = 0$, so $\dot{\omega}$ is dual to a Killing field. Next, differentiating the equation defining the Gauduchon constant and using $\dot{\kappa} = 0$ gives $\dot{\operatorname{scal}}^D = 0$ as the derivative of the quadratic term involving ω_t vanish. Differentiating the Einstein-Weyl equation

$$\operatorname{ric}^{g_t} = \tfrac{1}{n}\operatorname{scal}^{D^t} g_t + (n-2)(|\omega_t|^2 g_t - \omega_t \otimes \omega_t)$$

now gives the linearized Einstein equation since $\omega_t = 0$ and $\operatorname{scal}^{D^t} = 1$ at $t = 0$. Together with the infinitesimal Ebin slice condition, this equation is known to be elliptic **[3]**, which gives the finite dimensionality.

If the Einstein metric has no infinitesimal deformations then there remain only the deformations of ω, namely the Killing fields of g modulo isometries. The dimension of a generic orbit is the co-rank of the isometry group (since the stabilizer is a maximal torus), which gives the bound on the dimension of the moduli space. □

REMARK. More generally, following **[18]**, it has been shown **[68, 67]** that the Einstein-Weyl equation is elliptic in harmonic coordinates, at least once supplemented by the Bianchi identities 2.7, 3.5. Consequently, Einstein-Weyl manifolds are real analytic, and on compact manifolds the moduli space is finite dimensional.

We will now give some examples which show that at least some of these deformations may be integrated to give a nontrivial moduli space.

1. Theorem 7.2 implies in particular that the number of infinitesimal Einstein-Weyl deformations of the standard n-sphere is equal to the rank $\lfloor (n+1)/2 \rfloor$ of $\mathrm{SO}(n+1)$.

On S^3 the number of infinitesimal deformations is two and all these have been integrated **[68]**, using a relationship between three dimensional Einstein-Weyl manifolds and four dimensional self-dual manifolds, which we shall describe in section 10.

On S^4, as we discussed in section 6, there is a one parameter family with $S^1 \times \mathrm{SO}(3)$ symmetry and a one parameter family with $\mathrm{U}(2)$ symmetry, but so far a two parameter family of solutions integrating all the infinitesimal deformations has not been found.

2. Further examples where the rank of the isometry group agrees with the dimension of the space of known deformations can be found amongst the Einstein-Weyl structures on r-torus bundles over products of $m \geqslant r$ Kähler-Einstein manifolds, as described in section 6. The family of solutions is r-dimensional and the Einstein-Weyl structures are close to the Einstein metrics found by Wang and Ziller [**78**]. In particular, if the base manifold is a product of Kähler-Einstein manifolds without a continuous family of isometries (such as those found by Tian and Yau [**71**] on k-fold blow-ups of $\mathbb{C}P^2$ for $4 \leqslant k \leqslant 8$) then the isometry group of the Einstein metric on the torus bundle is the torus T^r itself, which has rank r. Note however, that the Einstein metrics on these T^r-bundles are not known to be rigid, so the full moduli space of Einstein-Weyl deformations could be larger.

3. Let (M, g_0) be a locally symmetric Einstein manifold of compact type and let $\prod_{a=1}^N M_a$ be the irreducible decomposition of the universal Riemannian covering manifold $\widetilde{M}$. Consider the following lists of compact symmetric manifolds.

$A.$ $\dfrac{\mathrm{SU}(p+q)}{S(\mathrm{U}(p)\times\mathrm{U}(q))}$ $(p \geqslant q \geqslant 2)$, $\dfrac{E_6}{F_4}$, $\dfrac{\mathrm{SU}(\ell)}{\mathrm{SO}(\ell)}$, $\dfrac{\mathrm{SU}(2\ell)}{\mathrm{Sp}(\ell)}$, $\mathrm{SU}(\ell)$ $(\ell \geqslant 3)$;

$B.$ $\dfrac{G_2}{SO(4)}$, G_2, or a Hermitian symmetric space of dimension $\geqslant 4$;

$C.$ S^2.

If $N = 1$ and M_a is not on list A, or $N = 2$ and M_a is not on lists A–B, or $N = 3$ and M_a is not on the lists A–C, then (M, g_0) has no infinitesimal Einstein deformations [**45**]. Thus, for example, $S^2 \times S^2$ can have at most a two parameter family of Einstein-Weyl solutions near the Einstein metric. An explicit one parameter family can be found using the constructions of section 6 [**54**].

As an example of a moduli of Einstein-Weyl structures away from an Einstein metric, we should mention the moduli of flat Weyl structures on the manifold $S^1 \times S^{n-1}$. In four dimensions, all Einstein-Weyl structures on $S^1 \times S^3$ are flat, as we shall see in section 9.

8. Complex and quaternionic structures

DEFINITION 8.1. A conformal manifold (M, c) will be called *Kähler Weyl* iff it is equipped with an almost complex structure $J \in \mathrm{C}^\infty(M, \mathfrak{so}(TM))$ and a Weyl connection D such that $DJ = 0$.

REMARK. These manifolds were called "Hermitian Weyl" in [**63**], since representative metrics for c are generally only Hermitian. From the perspective adopted here, however, properties of representative metrics are less relevant, and so, since $DJ = 0$ is a Kähler condition on D, we would like to advocate this change of terminology, which is also consonant with the term "locally (conformally) Kähler".

Since D is torsion-free, $DJ = 0$ implies that J is integrable. It also implies that $d^D\Omega = 0$ where Ω is the weightless Kähler form associated to J using c. Therefore $0 = (d^D)^2\Omega = 2F^D \wedge \Omega$ and so F^D vanishes in dimension $2m > 4$ [**63, 75**]. It then follows that if μ is a parallel local length scale, the metric $g = \mu^{-2}\mathrm{c}$ is a local Kähler metric. If D is exact, such a length scale exists globally and M is Kähler.

Conversely, a locally Kähler manifold is Kähler Weyl: the complex structure is the one given and the Weyl connection is locally the Levi-Civita connection of the compatible local Kähler metrics.

Since we shall discuss the four dimensional case in section 9, we confine ourselves here to the case that D is closed. Therefore if $D = D^g + \omega^g$ in the Gauduchon gauge, then ω^g is harmonic with respect to D^g. If also M is compact and Einstein-Weyl and D is not exact, then ω^g is a nontrivial D^g-parallel 1-form (see 4.6) and we may take $g = |\omega^g|^2 c$ as the Gauduchon metric.

One easily sees that $\sharp\omega^g, J\sharp\omega^g$ are commuting holomorphic Killing fields. Let $\mathcal{B}$ be the foliation generated by $\sharp\omega^g$ and let $\mathcal{E}$ be generated by $\sharp\omega^g$ and $J\sharp\omega^g$.

PROPOSITION 8.2. [**63**] *Let M be a compact Kähler Einstein-Weyl manifold of dimension $n = 2m > 4$ which is not exact. If the leaves of $\mathcal{B}$ and $\mathcal{E}$ are compact then there is a commutative diagram*

$$\begin{array}{ccc} M & \longrightarrow & P \\ & \searrow & \downarrow \\ & & N \end{array}$$

where $P = M/\mathcal{B}$ and $N = M/\mathcal{E}$ are Einstein orbifolds with positive scalar curvature and N is Kähler.

PROOF. Let ν be the 1-form $-\omega^g \circ J$. Then $h = g - (\omega^g)^2 - \nu^2$ descends to a Hermitian metric on N with Kähler form $\Omega_g + \omega^g \wedge \nu = d\nu$, where Ω_g is the Kähler form of J with respect to g. Since $d\omega^g = 0$, the Einstein-Weyl equation on M implies that $g_P = h + \nu^2$ is an Einstein metric on P with scalar curvature $(n-1)(n-2)$. Similarly, the submersion formulae for $P \to N$ show that N is also Einstein. □

The example to keep in mind is $M = S^1 \times S^{2n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ where the $\mathbb{Z}$ action is generated by $x \mapsto 2x$. Then $P = S^{2n-1}$ and $N = \mathbb{C}P^{n-1}$.

Conversely, if N is a Kähler-Einstein manifold, then the Calabi metric on $\mathcal{L}\backslash 0$, where $\mathcal{L}$ is a maximal root of the canonical bundle of N, gives a Kähler Einstein-Weyl structure on the universal cover of M [**63**].

Next we turn to the quaternions.

DEFINITION 8.3. A conformal manifold (M, c) of dimension $n > 4$ will be called *quaternion Kähler Weyl* iff it is equipped with a rank 3 sub-bundle $\mathcal{Q} \leqslant \mathfrak{so}(TM)$ pointwise isomorphic to $\operatorname{Im}\mathbb{H} = \mathfrak{sp}(1)$, and a Weyl connection D preserving $\mathcal{Q}$. It is (locally) *hyper-Kähler Weyl* iff the induced covariant derivative on $\mathcal{Q}$ is (locally) trivial. (We discuss the four dimensional case in section 9.)

Since D is torsion free, a quaternion Kähler Weyl manifold is quaternionic and a (locally) hyper-Kähler Weyl manifold is (locally) hyper-complex.

PROPOSITION 8.4. [**63**] *Let M be a conformal manifold with $\dim M > 4$. Then M is quaternion Kähler Weyl iff it is locally quaternion Kähler, in which case it is closed Einstein-Weyl. A non-exact quaternion Kähler Weyl manifold is locally hyper-Kähler Weyl, and any locally hyper-Kähler Weyl manifold is locally hyper-Kähler.*

PROOF. If M is quaternion Kähler Weyl then the weightless 4-form Ω of the quaternionic structure satisfies $d^D\Omega = 0$, so $F^D \wedge \Omega = 0$ and therefore $F^D = 0$ if $\dim M > 4$. Parallel local length scales are therefore Einstein, and so D is Einstein-Weyl. If D is not exact then scal^D must vanish by Theorem 3.6. □

Assuming D is not exact, we can again use the Gauduchon metric with $|\omega^g| = 1$ and consider the foliation $\mathcal{B}$, as in the complex case. Also, let $\mathcal{D}$ be the foliation given by the quaternionic span of $\sharp\omega^g$. Then Proposition 8.2 has a quaternionic analogue. We concentrate on the following results of Ornea and Piccinni [**59**].

PROPOSITION 8.5. *Let M be a compact quaternion Kähler Weyl manifold such that the foliations $\mathcal{B}$ and $\mathcal{D}$ have compact leaves. Then there is a finite hyper-Kähler Weyl covering $\tilde{M}$ of M and a commutative diagram*

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{S^1} & \tilde{P} & \xrightarrow{S^3/H} & \tilde{N} \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{S^1} & P & \xrightarrow{S^3/G} & N \end{array}$$

with finite coverings as vertical arrows and Riemannian submersions over orbifolds as horizontal arrows. The orbifolds P and $\tilde{P}$ carry respectively local and global 3-Sasakian structures, while N and $\tilde{N}$ are quaternion Kähler orbifolds with positive scalar curvature. The fibers of $P \to N$ and $\tilde{P} \to \tilde{N}$ are spherical space forms, respectively locally and globally homogeneous. On M there is a global integrable compatible complex structure and $M/\mathcal{E}$ is the twistor space of N (see Proposition 8.2).

PROOF. Let $M \xrightarrow{\pi} P$ be a flat S^1-bundle with connection ω^g. If $(\phi_\alpha, \xi_\alpha)$ is a locally defined 3-Sasakian structure on P, a quaternionic structure on M may be defined by

$$I_\alpha Y = -\phi_\alpha Y - g(\xi_\alpha, Y)\sharp\omega^g, \quad I_\alpha \sharp\omega^g = \xi_\alpha$$

and this is compatible with the metrics $g_M = \pi^* g_P + (\omega^g)^2$.

Now since all the leaves of $P \to N$ are spherical space forms S^3/G, each leaf has a global Sasakian structure induced by a conjugate complex structure on $S^3 \subseteq \mathbb{C}^2$. The unit vector field also belongs to the locally 3-Sasakian distribution in TP. Thus P has a global Sasakian structure which may be lifted to a complex structure J on M using the formulae above. Then $\sharp\omega^g$ and $J\sharp\omega^g$ generate the foliation $\mathcal{E}$ and $M/\mathcal{E}$ is the twistor space of N. □

REMARKS. 1. The relation to 3-Sasakian geometry leads to a classification of compact homogeneous hyper-Kähler Weyl manifolds using the classification of homogeneous 3-Sasakian manifolds [**10, 58**].

2. Similarly, results on Betti numbers of 3-Sasakian geometry imply topological constraints and relations on M and N above [**58, 59**]. One such constraint is $b_1(M) = 1$, which is also known to hold for closed non-exact Einstein-Weyl manifolds for other reasons.

EXAMPLE 8.6. Let G be a discrete subgroup of $\mathrm{GL}(1,\mathbb{H})\mathrm{Sp}(1) = \mathrm{CO}^+(4)$ and let $M = (\mathbb{H}^n \setminus \{0\})/G$, where G is acting diagonally. Equip M with the metric $g = \left(\sum_\alpha q_\alpha \bar{q}_\alpha\right)^{-1} \sum_\alpha dq_\alpha \otimes d\bar{q}_\alpha$. Then (M, g) is a quaternion Kähler Weyl manifold (the Weyl connection coincides locally with the Levi-Civita connection of a quaternion Kähler metric). In fact $G \leqslant \mathrm{GL}(n,\mathbb{H})\mathrm{Sp}(1)$ and by choosing G so it is contained in $\mathrm{GL}(n,\mathbb{H})$, M becomes hyper-Kähler Weyl. If G is chosen inside $\mathrm{GL}(n,\mathbb{H})\mathrm{U}(1) \subseteq \mathrm{GL}(2n,\mathbb{C})$ there is a global integrable complex structure on M [**58**].

9. Einstein-Weyl geometry in four dimensions

On an oriented conformal 4-manifold, the bundle $\Lambda^2 T^*M$ decomposes as the direct sum of the bundles of self-dual and anti-self-dual 2-forms, $\Lambda^2_+ T^*M$ and $\Lambda^2_- T^*M$. For any Weyl derivative D there is a corresponding decomposition $F^D = F^D_+ + F^D_-$ of the Faraday curvature. The same is true of the bundle of Weyl tensors, and so the Weyl curvature W of the conformal structure splits into two components W^+ and W^-. When one of these components vanishes, it is well known [**2, 3**] that the conformal geometry of M may be studied in terms of the holomorphic geometry of an associated complex manifold or "twistor space".

DEFINITION 9.1. The *twistor space* Z of M is the bundle of negatively oriented orthogonal almost complex structures on TM, which is a sphere bundle in $L^2\Lambda^2_- T^*M$. The fibers are the real *twistor lines* of Z.

There is also a spinorial representation of the twistor space. Equipping M (at least locally) with a spin structure, there are weightless spinor bundles V_+, V_- (complex symplectic with rank 2) such that the complexified tangent bundle $\mathbb{C}TM$ is isomorphic to $L^1 \otimes (V_+ \otimes_{\mathbb{C}} V_-)$.

The twistor space Z is then isomorphic to $P(L^w V_-)$ for any w, although the tautological line bundle over this projectivized bundle will depend on the weight. Because the twistor operator is conformally invariant on $L^{1/2}V_- \cong (L^{-1/2}V_-)^*$, the following choice of projective structure is the "right" one.

NOTATION 9.2. If M is a spin manifold, then the twistor space Z of M will be identified with $P(L^{-1/2}V_-)$ and $\mathcal{O}(-1)$ will denote the corresponding tautological line bundle. Note that for m even, $\mathcal{O}(m)$ makes sense globally even if M is not spin. In particular the canonical bundle of Z is $K_Z \cong \mathcal{O}(-4)$. Let $L_{\mathbb{C}}$ denote the complexified pull-back of $L^1 \to M$ to Z. The Euler sequence on each fiber then implies that the vertical tangent bundle $\mathcal{V}$ of $Z \to M$ is $L_{\mathbb{C}}^{-1}\mathcal{O}(2) = L_{\mathbb{C}}^{-1}K_Z^{-1/2}$.

Let us now compare the standard twistor theory on a conformal manifold with twistor theory on a Weyl manifold.

THEOREM 9.3. *Let M be an oriented conformal manifold with twistor space Z.*

(i) [**2, 33**] *Z carries a natural almost complex structure, which is integrable if and only if $W^- = 0$. The complex line bundles $\mathcal{O}(m)$ are then holomorphic, and their (real) holomorphic sections correspond to solutions of conformally invariant differential equations on M* [**22**].

(ii) [**25, 26, 27, 66**] *Suppose now that $W^- = 0$ and let D be a Weyl derivative on M. Then for $w \neq 0$, the complex line bundle $L^w_{\mathbb{C}}$ over Z carries a pre-holomorphic structure depending on D which is integrable if and only if $F^D_- = 0$. Holomorphic sections of $L^w_{\mathbb{C}} \otimes \mathcal{O}(m)$ for $w \neq 0$ then correspond to solutions of differential equations on M depending on D.*

In particular if $W^- = F^D_- = 0$, then D defines a holomorphic structure on $\mathcal{V} = L_{\mathbb{C}}^{-1}\mathcal{O}(2)$ and the holomorphic sections correspond to solutions of a twistor-type equation on the bundle of anti-self-dual endomorphisms of TM. Gauduchon [**25, 26**] has used this equation to prove that a compact self-dual conformal manifold which admits a gauge of negative scalar curvature does not admit a global anti-self-dual complex structure. In [**27**] he also uses twistor theory to analyze conformal vector fields on Weyl manifolds, and to study manifolds of type $S^1 \times S^3$.

The above theorem suggests that we will only be able to obtain a twistor interpretation of the Einstein-Weyl condition when $W^- = F^D_- = 0$. Following the constructions in the metric case [**3**] *mutatis mutandis*, we obtain:

THEOREM 9.4. [**27, 66**] *Let M be an Einstein-Weyl 4-manifold with $W^- = F^D_- = 0$. Then there is a twisted 1-form $\theta \in H^0(Z, \Omega^1 \otimes \mathcal{V})$ which is holomorphic iff M is Einstein-Weyl. Furthermore, $\theta \wedge d\theta \in H^0(Z, L_{\mathbb{C}}^{-2})$ is a nonzero multiple of the pull-back of* scal^D. *In particular θ defines a holomorphic map $Z \to \mathbb{C}P^1$ iff the symmetrized Ricci endomorphism of D is identically zero.*

On a compact 4-manifold M, any Weyl derivative D with self-dual Faraday curvature is closed (since self-dual exact 2-forms must vanish). If M is also Einstein-Weyl then D is flat by 4.9. The self-duality of F^D therefore seems very restrictive, but is inevitable in view of the following result.

THEOREM 9.5. [**12**] *Let M, D be an Einstein-Weyl 4-manifold with self-dual Weyl tensor. Then F^D is also self-dual.*

COROLLARY 9.6. [**27, 67**] *Let M, D be a compact Einstein-Weyl 4-manifold with self-dual Weyl tensor. Then D is closed.*

This corollary may be established directly by using the classification of 4-manifolds admitting self-dual metrics with positive Ricci curvature [**27**], or by using the vanishing of the Bach tensor [**67**].

Even without assuming compactness, Theorem 3.6 shows that if D is Einstein-Weyl with $F^D_- = 0$, then $D\mathrm{scal}^D = 0$ and so either scal^D is identically zero or M is Einstein. In the self-dual case this can be seen on the twistor space as follows [**66**].

Let S be the divisor of $\theta \wedge d\theta$. Then S is either empty or it meets every twistor line [**70**]. However, $L_{\mathbb{C}}^{-2}$ is a pull-back from M and so it is trivial on twistor lines. Therefore, if S is non-empty, it must contain all twistor lines and therefore be all of Z, meaning that $\theta \wedge d\theta$ is identically zero. Otherwise, $\theta \wedge d\theta$ is nowhere vanishing, which implies the existence of a section of $\Omega^1 \otimes K_Z^{-1/2}$ and therefore, by standard twistor theory, an Einstein gauge of nonzero scalar curvature.

Consequently, all non-exact self-dual Einstein-Weyl 4-manifolds are scalar flat with self-dual Faraday curvature. The twistor space fibers over $\mathbb{C}P^1$ and D defines a flat connection on the bundle of anti-self-dual complex structures. Conversely if D is flat on this bundle, W is self-dual and $F^D \wedge \Omega_J = 0$ for each anti-self-dual complex structure J, so F^D is also self-dual. In fact we have the following equivalence.

PROPOSITION 9.7. [**66**] *A Weyl 4-manifold is Einstein-Weyl with $W^- = F^D_- = \mathrm{scal}^D = 0$ iff it is locally hyper-complex with Obata connection D.*

The anti-self-dual complex structures on a 4-manifold M form a bundle isomorphic to the imaginary quaternions and any Weyl derivative D preserves this bundle. By analogy with Theorem 8.4, we say that M is *quaternion Kähler Weyl* iff it Einstein-Weyl with $W^- = F^D_- = 0$ and locally *hyper-Kähler Weyl* if also $\mathrm{scal}^D = 0$.

The compact examples may be classified as follows [**66, 67**].

THEOREM 9.8. *A compact self-dual Einstein-Weyl 4-manifold is isometric to S^4, $\mathbb{C}P^2$ or an Einstein manifold of negative scalar curvature, or is covered by a flat torus, a $K3$ surface or a coordinate quaternionic Hopf surface* [**3, 8**].

One might hope to obtain more examples by replacing self-duality by the vanishing of the *Bach tensor*. This is a symmetric trace-less bilinear form B obtained from the Weyl tensor by applying a conformally invariant second order differential operator. It arises on compact manifolds as the gradient of the functional $c \mapsto \int_M |W^c|^2$. In terms of an arbitrary Weyl derivative

$$B(X,Y) = \sum_i (D_{e_i} C^D_{e_i,X} Y - \langle W_{e_i,X}\, r^D(e_i), Y\rangle)$$

where C^D is the *Cotton-York* tensor of D, which is a vector valued 2-form defined by $C^D_{X,Y} := D_X r^D(Y) - D_Y r^D(X) = -\sum_i D_{e_i} W_{X,Y} e_i$ (by the second Bianchi identity). The Bach tensor may also be computed by applying the same formula to W^+ or W^- and doubling it (see [**12**]) and it therefore vanishes if c is (anti)self-dual. If M is Einstein-Weyl then we can compute B with the help of Proposition 3.3 to obtain the following result.

THEOREM 9.9. [**12, 67**] *Let M be an Einstein-Weyl 4-manifold. Then*

$$B(X,Y) = \tfrac{1}{24}(D^2_{X,Y}\mathrm{scal}^D + D^2_{Y,X}\mathrm{scal}^D) - \langle F^D_+(X), F^D_-(Y)\rangle - \langle F^D_-(X), F^D_+(Y)\rangle$$

and so if $B = 0$ and $\mathrm{scal}^D = 0$ then F^D is (anti)self-dual [**12**]. *When M is compact the formula for B in the Gauduchon gauge $D = D^g + \omega^g$ becomes*

$$B(X,Y) = \tfrac{2}{3}\mathrm{scal}^D(\omega^g \otimes_0 \omega^g)(X,Y) - \langle F^D_+(X), F^D_-(Y)\rangle - \langle F^D_-(X), F^D_+(Y)\rangle$$

and it follows that if $B = 0$ then D is closed [**67**].

We now turn to four dimensional Kähler Weyl geometry, which is richer than the higher dimensional case of the previous section. Indeed if (c, J) is *any* conformal Hermitian structure on a 4-manifold M, it follows from Example 1.6 that there is a unique Weyl derivative D with $DJ = 0$ [**75**]. We choose the orientation so that J is anti-self-dual. Therefore $\langle F^D_-, J\rangle = 0$, and since $[R^D, J] = 0$, r^D_0 is J-invariant and also (see [**1**])

$$W^- = \tfrac{1}{4}\mathrm{scal}^D\big(\tfrac{1}{3}\mathrm{id}_{\Lambda^2_-} - \tfrac{1}{2}\Omega_J \otimes \Omega_J\big) - \tfrac{1}{2}(JF^D_- \otimes \Omega_J + \Omega_J \otimes JF^D_-).$$

In particular, W^- vanishes iff F^D_- and scal^D both vanish, and so a compact self-dual Hermitian 4-manifold is locally scalar flat Kähler. (See [**7**], and also [**69**] for a twistor proof.)

Despite the wide generality of Kähler Weyl geometry, the Einstein-Weyl condition is much more restrictive. The following result is due to Gauduchon and Ivanov [**28**], although we sketch a different proof.

THEOREM 9.10. *Let M be a compact Kähler Einstein-Weyl 4-manifold. Then the Weyl derivative D is closed.*

PROOF. By 4.4, $D = D^g + \omega^g$ with $K = \sharp\omega^g$ a Killing field, which implies

$$\langle R^g_{JK,K} e_i, Je_i\rangle = \langle D^g_{JK}(D^g K)_{e_i}, Je_i\rangle = -\langle D^g_{e_i} K, (D^g_{JK} J)e_i\rangle.$$

This vanishes, since $D^g_X J + [\omega^g \vartriangle X, J] = D_X J = 0$ and when $X = JK$ we have $[\omega^g \vartriangle JK, J] = 0$. Now $\langle R^D_{JK,K} e_i, Je_i\rangle$ is also zero, since $R^D_{X,Y}$ commutes with J. Comparing these using 2.5, we find that $\mathrm{scal}^D|\omega^g|^2 = 0$ and so either D is exact or $\mathrm{scal}^D = 0$. In the latter case D is closed by 3.6. □

This conclusion continues to hold even if the complex structure is not integrable: Kamada [**42**] shows that a compact almost Kähler Einstein-Weyl manifold with non-negative scalar curvature is in fact Kähler Einstein-Weyl.

In contrast to these negative results we now present some non-compact examples and examples which are not (anti)self-dual. One interesting class of examples are those of Bianchi type IX, i.e., admitting a (possibly local) SU(2) action with three dimensional orbits. Apart from the Einstein case, the solutions are all diagonal, biaxial and conformally Kähler [**5, 53**] and so there is actually a (local) U(2) action. Madsen [**53**] obtains the (anti)self-dual examples, and the general solutions can be found in [**4, 54**], although the latter reference is concerned with the compact case, to which we shall return at the end of this section.

We write the solutions in the form $D = D^g + \omega$, where g is a Kähler metric, and we have reorganized Bonneau's parameters to simplify and unify the various cases. We use the coordinates of Madsen; in particular, σ_i are invariant 1-forms with $d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc.

$$\begin{aligned} (9.1)\quad g &= V(\rho)^{-1}d\rho^2 + \tfrac{1}{4}\rho^2(\sigma_1^2 + \sigma_2^2 + V(\rho)\sigma_3^2) \\ \omega &= -\frac{\rho(b+2c\rho^2)}{a+b\rho^2+c\rho^4}d\rho \pm \tfrac{1}{2}\rho^2 V(\rho)\frac{\sqrt{4ac-b^2}}{a+b\rho^2+c\rho^4}\sigma_3 \\ V(\rho) &= \frac{(a+b\rho^2+c\rho^4)\big(a+c\rho^4+\lambda(a-c\rho^4)\big)}{2ac\rho^4} \\ &\quad + \mu\frac{a+b\rho^2+c\rho^4}{\rho^4(b+2c\rho^2)}\left[1-\frac{4c(a-c\rho^4)}{4ac-b^2}\left(1-\frac{b+2c\rho^2}{\sqrt{4ac-b^2}}\operatorname{arccot}\frac{b+2c\rho^2}{\sqrt{4ac-b^2}}\right)\right] \end{aligned}$$

The parameters are constrained by $4ac \geqslant b^2$ and the requirement that $V(\rho)$ should be somewhere positive. The solution is homogeneous in (a, b, c) and is also invariant under the transformation $\rho \mapsto k\rho$, $a \mapsto k^2 a$, $c \mapsto k^{-2}c$, $\mu \mapsto k^2\mu$, $g \mapsto k^2 g$, so there are only three independent parameters. The Einstein case occurs when $b^2 = 4ac$: note that $1 - x\operatorname{arccot} x \approx 1/(3x^2)$ for x large, and so the μ term has a well defined limit. In this situation, and also when $\mu = 0$, $V(\rho)$ is of the form

$$V(\rho) = 1 + A_+\rho^2 + B_+\rho^4 + \frac{A_-}{\rho^2} + \frac{B_-}{\rho^4}$$

where $A_+A_- = 4B_+B_-$ in the Einstein case and $A_+^2B_- + A_-^2B_+ = A_+A_-$ in the case $\mu = 0$. The latter constraint is simply the condition for V to factorize as $(a + b\rho^2 + c\rho^4)\big((1+\lambda)a + (1-\lambda)c\rho^4\big)/2ac\rho^4$. The scalar curvature of the Kähler metric is $-24(A_+ + 2B_+\rho^2)$, but note that replacing ρ by $1/\rho$ and rescaling by ρ^4 gives a metric of the same form with A_+ and B_+ interchanged with A_- and B_-. Consequently these conformal metrics admit Kähler complex structures of both orientations. The $\mu = 0$ Einstein-Weyl structures satisfy $\mathrm{scal}^D = 0$, giving Ricci flat metrics when $b^2 = 4ac$. Also they are conformally Einstein for $\lambda = \pm 1$, when the Weyl curvature is (anti)self-dual. If we fix the orientation by $W^- = 0$, these self-dual conformal structures each admit:

- a compatible Einstein metric [**62**]
- a compatible scalar flat Kähler metric [**49**]
- a compatible self-dual Kähler metric [**4**]
- a compatible hyper-complex structure [**53**].

On the other hand, for $\mu = 0$, $\lambda \neq \pm 1$, $4ac > b^2$, we have examples of scalar flat Einstein-Weyl structures where F^D is *not* (anti)self-dual [**4**].

We turn now to the search for more compact examples and begin by noting that there are topological constraints on compact 4-manifolds admitting Einstein-Weyl structures, given by an analogue of the Hitchin-Thorpe inequality [**32**]. Related to this is a a generalization of the Lafontaine inequality [**47**], and also the fact that four dimensional Einstein-Weyl manifolds minimize a quadratic total curvature functional. These constraints were previously established using the Gauduchon gauge [**64, 65**], but we sketch here how they can be obtained in Weyl geometry. One advantage of this approach is that we find a quadratic total curvature functional minimized by all Einstein-Weyl structures, not just the closed ones.

The key idea is that a Weyl connection is a metric connection on $L^{-1}TM$. Since L^{-1} is a trivializable bundle, the Euler characteristic of M is given by the integral of a multiple of the Pfaffian of $R^{D,0}$. This may be computed by viewing $R^{D,0}$ as a weight -2 endomorphism of Λ^2T^*M and splitting into self-dual and anti-self-dual parts. The Pfaffian of $R^{D,0}$ reduces to $\langle R^{D,0}, *R^{D,0}*\rangle$. In block diagonal form $R^{D,0}$ may be written $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$, where $A_\pm$ is given by the action of $W^\pm$, scal^D and $F^D_\pm$, whereas B is given by the action of r^D_0. Hence the Pfaffian integrand is $|R^{D,0}|^2$ with the r^D_0 term negated. A straightforward computation of $\sum_{i<j,k<l} \langle R^{D,0}_{e_i,e_j} e_k, e_l\rangle^2$ gives:

$$|R^{D,0}|^2 = |W^+|^2 + |W^-|^2 + |F^D_+|^2 + |F^D_-|^2 + \frac{1}{24}\left(\mathrm{scal}^D\right)^2 + 2|r^D_0|^2,$$

where $S^2_0T^*M$ is given the tensor product norm, and Λ^2T^*M its usual norm.

It follows that we have the following integral formulae for the Euler characteristic, the signature and the trivial characteristic of L^1:

$$2\chi(M) = \frac{1}{4\pi^2}\int_M |W^+|^2 + |W^-|^2 + |F^D_+|^2 + |F^D_-|^2 + \frac{1}{24}\left(\mathrm{scal}^D\right)^2 - 2|r^D_0|^2$$

$$3\tau(M) = \frac{1}{4\pi^2}\int_M |W^+|^2 - |W^-|^2, \qquad 0 = \frac{1}{4\pi^2}\int_M |F^D_+|^2 - |F^D_-|^2.$$

THEOREM 9.11. *Let M be a compact 4-manifold. Then the quadratic total curvature functional $\int_M |R^{D,0}|^2$ is minimized by Einstein-Weyl structures and also by half conformally flat, scalar flat, closed Weyl structures. If D is Einstein-Weyl then*

$$2\chi(M) \geqslant 3|\tau(M)| + \frac{1}{2\pi^2}\int_M |F^D_\pm|^2$$

with equality iff $\mathrm{scal}^D = 0$ and W is (anti)self-dual. Similarly, if M, D is a Weyl manifold with $\mathrm{scal}^D = 0$ and W (anti)self-dual, then the reverse inequality holds, with equality iff M is Einstein-Weyl.

It follows from this [**64, 67**], that if M is a torus or $K3$ surface then M admits no non-exact Einstein-Weyl structures, and $M\#M$ admits no Einstein-Weyl structures at all. Also $k\mathbb{C}P^2$ can only be Einstein-Weyl for $k \leqslant 3$. Finally, any Einstein-Weyl structure on $S^1 \times S^3$ is closed and therefore flat by 4.9.

We end this section with the classification of compact Einstein-Weyl 4-manifolds with large symmetry group. First let us consider the homogeneous case.

THEOREM 9.12. [**54**] *A compact homogeneous Einstein-Weyl 4-manifold is either finitely covered by $S^1 \times S^3$ with its standard Einstein-Weyl structure or is a homogeneous Einstein manifold.*

PROOF. Assume D is not exact and let $M = G/H$ where G is the symmetry group. Theorem 9.8 implies that the only conformally flat Einstein-Weyl structures on S^4 are the Einstein metrics, and so (as noted in 6.1) we may assume G is compact. Let $\mathfrak{m}$ be an Ad_H invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$. Then $\mathfrak{m} = \ker\omega^g \oplus (\ker\omega^g)^\perp$ where ω^g is the Gauduchon 1-form. Therefore $\mathfrak{h} < \mathfrak{o}(3) \oplus \mathfrak{o}(1)$ so the rank of $\mathfrak{h}$ is at most 1 and $\dim\mathfrak{g}$ at most 7. The classification of compact Lie groups now implies that we only need to consider a few cases which either gives M finitely covered by $S^1 \times S^3$ or $b_1(M) \geqslant 2$. But we have seen that $b_1(M) \leqslant 1$ for non-exact Einstein-Weyl manifolds with equality iff M is flat (see 4.9 and 9.11). Indeed, the manifolds of type $S^1 \times S^3$ exhaust the compact closed Einstein-Weyl manifolds. □

Inspired by the work of Bérard-Bergery [3] on Einstein manifolds with large symmetry group, we now consider the following situation.

THEOREM 9.13. [54] *Let G be the symmetry group of a compact four dimensional inhomogeneous Einstein-Weyl manifold with non-closed structure and assume that* $\dim G \geqslant 4$. *Then the Einstein-Weyl structure is of co-homogeneity one and it is defined on S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ or some of their finite quotients. The solutions in each case come in one dimensional families.*

PROOF. If M is not homogeneous then as G preserves the metric on the principal orbit P^n, and so we must have $4 \leqslant \dim G \leqslant \frac{1}{2}n(n+1)$ and hence $n = \dim P = 3$. There are now only the following cases to consider:

- $\mathrm{SO}(4)$ with principal orbit $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$
- $S^1 \times \mathrm{SO}(3)$ with orbit $S^1 \times S^2 = S^1 \times \mathrm{SO}(3)/\mathrm{SO}(2)$
- $\mathrm{U}(2)$ with orbit $S^3 = \mathrm{U}(2)/\mathrm{U}(1)$

or finite quotients of these. We have studied Einstein-Weyl manifolds with this kind of symmetry in section 6: the Einstein-Weyl equation reduces to a collection of ODEs over a closed interval or circle, the latter case yielding only closed structures. When $M/G = [0, \ell]$, it is convenient to write $M = [\,G/K_1 \,|\, G/H \,|\, G/K_2\,]$ for the manifold with principal orbit G/H and special orbits G/K_i, $i = 1, 2$ at the endpoints.

For each symmetry group we classify the possible diffeomorphism types using Lie theory and the known topological constraints on Einstein-Weyl geometry. Then we impose the appropriate boundary conditions on the ODEs and solve explicitly. The case of $\mathrm{SO}(4)$ symmetry yields only closed Einstein-Weyl structures so let us consider $S^1 \times \mathrm{SO}(3)$ symmetry.

Some of the topologies here do not carry any Einstein-Weyl solutions. Firstly, if M/G is a circle then M is finitely covered by $T^2 \times S^2$ which cannot be Einstein-Weyl. When M/G is an interval with special orbits $\mathbb{R}P^1 \times S^2$ we have:

$$\begin{aligned} M &= [\,\mathbb{R}P^1 \times S^2 \,|\, S^1 \times S^2 \,|\, \mathbb{R}P^1 \times S^2\,] \\ &= [\,\mathbb{R}P^1 \,|\, S^1 \,|\, \mathbb{R}P^1\,] \times S^2 \\ &= \big([\,\mathbb{R}P^1 \,|\, S^1 \,|\, \mathrm{pt}\,] \# [\,\mathrm{pt} \,|\, S^1 \,|\, \mathbb{R}P^1\,]\big) \times S^2 \\ &= K^2 \times S^2, \end{aligned}$$

where K^2 is the Klein bottle. However, K^2 is double-covered by T^2 and $T^2 \times S^2$ is not Einstein-Weyl. Also, not all finite quotients of $S^1 \times S^2$ are possible principal orbits. For instance:

$$M = [\,\mathbb{R}P^1 \times \mathbb{R}P^2 \,|\, S^1 \times \mathbb{R}P^2 \,|\, \mathbb{R}P^1 \times \mathbb{R}P^2\,] \\ = K^2 \times \mathbb{R}P^2$$

which again cannot be Einstein-Weyl. The remaining cases give one parameter families on S^4, $S^2 \times S^2$ and some finite quotients, such as $\mathbb{R}P^2 \times S^2$. The family on S^4 was given in (6.1).

The U(2) symmetric examples are obtained from the family given in (9.1). For S^4, $\mathbb{C}P^2$, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, the boundary value problem leads to one parameter families of solutions, and these and these descend to the finite quotients $\mathbb{R}P^4$ and $\mathbb{C}P^2 \# \mathbb{R}P^4$.

We refer to [**54**] for the full details of all the cases, but note that this reference contains some errors in the U(2) case, corrected by Bonneau [**6**]. □

10. Einstein-Weyl geometry in three dimensions

In three dimensions, there is also a twistor theory of Einstein-Weyl manifolds, but unlike the four dimensional case, where twistor methods are limited to the self-dual structures, in three dimensions "mini-twistor theory" applies to all Einstein-Weyl spaces. Indeed this was the case first studied, by Cartan [**16**], who showed that the Einstein-Weyl equation is the integrability condition for the existence, in a complex three dimensional Weyl manifold, of a two parameter family of totally geodesic null hypersurfaces. Consequently, the space of oriented geodesics in a real three dimensional Einstein-Weyl manifold is a complex surface. Hitchin showed that this surface contains projective lines with normal bundle $\mathcal{O}(2)$ and conversely, that given such a complex surface (with a real structure), the real points in the Kodaira moduli space of these lines form a three dimensional Einstein-Weyl manifold [**34**]. In other words there is a twistor construction, the *Hitchin correspondence*, for three dimensional Einstein-Weyl manifolds, in terms of a class of complex surfaces called *mini-twistor spaces*. The conformal structure of the Einstein-Weyl space is given by the condition for nearby "mini-twistor lines" to intersect to second order, and the Weyl derivative can be obtained via a construction of projective structures on moduli spaces [**55**].

For example, the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$, together with the plane sections, generates the Einstein space S^3 or H^3 depending on the real structure, and the mini-twistor space of $\mathbb{R}^3$ is the punctured cone $T\mathbb{P}^1$, together with its sections over $\mathbb{P}^1$. The following result shows that other mini-twistor spaces are more complicated.

PROPOSITION 10.1. *A mini-twistor space which is an open set of a compact surface generates the Einstein-Weyl geometry of a space of constant curvature. The compact surface can be taken to be the cone or the quadric surface.*

Despite this, we can construct mini-twistor spaces locally by taking blow-ups and branched covers. For instance, a $(1,n)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is rational with normal bundle $\mathcal{O}(2n)$. If we take a branched n-fold covering, then in the covering the normal bundle is $\mathcal{O}(2)$ and we have a mini-twistor space [**61**], although the covering cannot extend to all of the quadric.

There are close connections between mini-twistor theory and twistor theory in four dimensions. In [**40**], Jones and Tod observed that, given a self-dual conformal 4-manifold M with a conformal vector field K, the quotient of the twistor space Z

of M by the induced holomorphic vector field is a mini-twistor space. They then wrote down a Weyl structure on the orbit space $B = M/K$ and showed that this agreed with the Weyl structure coming from the Hitchin correspondence. In other words, the quotient of a self-dual conformal manifold by a conformal vector field is Einstein-Weyl.

Although such a result would have been difficult to find without twistor theory, the twistor theoretic proof that the Jones-Tod Weyl structure is Einstein-Weyl is rather indirect. More direct arguments, sometimes only in special cases, have been given in [**14, 29, 41, 50**] and we would like to sketch the approach of [**14**], which has the advantage that it extends to a more general class of conformal submersions [**13**], although we shall treat only conformal vector fields here.

Let M be a self-dual conformal manifold with a conformal vector field K, and by restricting to an open set if necessary, assume K is nowhere vanishing. Then $|K|$ is a length scale on M and induces an exact Weyl derivative D^0, the constant length gauge of K. One can compute D^0 in terms of an arbitrary Weyl derivative D by the formula

$$D^0 = D - \frac{\langle DK, K\rangle}{\langle K, K\rangle} = D - \frac{1}{4}\frac{(\mathrm{tr}\, DK)K}{\langle K, K\rangle} + \frac{1}{2}\frac{(d^D K)(K, .)}{\langle K, K\rangle}.$$

Note that $\langle D^0 K, .\rangle$ is a weightless 2-form. The crucial observation is that there is a unique Weyl derivative D^{sd} on M such that $\langle D^{sd} K, .\rangle$ is a weightless *self-dual* 2-form. One way to see this is to observe that $\omega = (*d^D K)(K, .)/\langle K, K\rangle$ is a 1-form independent of the choice of D and define:

$$D^{sd} = D^0 - \frac{1}{2}\omega = D - \frac{1}{4}\frac{(\mathrm{tr}\, DK)K}{\langle K, K\rangle} + \frac{1}{2}\frac{(d^D K)(K, .) - (*d^D K)(K, .)}{\langle K, K\rangle}.$$

Since D is arbitrary, we may take $D = D^{sd}$ to see that $(D^{sd}K - *D^{sd}K)(K, .) = 0$ from which it is immediate that $D^{sd}K = *D^{sd}K$ since an anti-self-dual 2-form is uniquely determined by its contraction with a nonzero vector field.

Next recall that for any vector field K and torsion free connection D on TM, $(\mathcal{L}_K D)_X = D_X DK - R^D_{X,K}$. There is an analogous formula for Weyl derivatives.

PROPOSITION 10.2. *Let X be a vector field, μ a section of L^w and D a Weyl derivative on M^n. Then $\mathcal{L}_X \mu = D_X \mu - \frac{w}{n}(\mathrm{div}^D X)\mu$ and so the Lie derivative of the Weyl derivative on L^1 is: $(\mathcal{L}_K D)_X = \frac{1}{n}\partial_X(\mathrm{div}^D K) - F^D(X, K)$.*

Now if K is conformal then the Lie derivative (along K) of a Weyl connection D on TM is given by the linearized Koszul formula applied to the Lie derivative of D on L^1. Hence

$$D_X DK = R^D_{X,K} + \gamma_K(X)\mathrm{id} + \gamma_K \vartriangle X,$$

where $\gamma_K = \frac{1}{n}d(\mathrm{tr}\, DK) + F^D(K, .)$. (This formula also appears in [**27**].)

Applying this with $D = D^{sd}$ and decomposing the curvature gives:

$$D^{sd}_X D^{sd} K = W_{X,K} + r^{sd}(K) \vartriangle X - r^{sd}(X) \vartriangle K + F^{sd}(K, .) \vartriangle X.$$

Now $D^{sd}_X D^{sd} K$ and $W_{X,K}$ are both self-dual 2-forms and hence so is the sum of the remaining terms. This implies that if $\langle X, K\rangle = \langle Y, K\rangle = 0$ then

$$r^{sd}(X, Y)\langle K, K\rangle + r^{sd}(K, K)\langle X, Y\rangle = *\big(K \wedge (r^{sd} + F^{sd})(K) \wedge X \wedge Y\big).$$

Symmetrizing in X, Y, we see that the horizontal part of the symmetric Ricci endomorphism of D^{sd} is a multiple of the identity. It now looks as if $D^{sd} = D^0 - \frac{1}{2}\omega$

might be the Einstein-Weyl structure we seek. In fact this is not the case: instead it is $D^0 - \omega$ which is Einstein-Weyl on B.

THEOREM 10.3. **[40]** *Suppose M is a self-dual 4-manifold and K a conformal vector field such that $B = M/K$ is a manifold. Let D^0 be the constant length gauge of K and $\omega = 2(*D^0K)(K,.)/\langle K, K\rangle$. Then $D = D^0 - \omega$ is Einstein-Weyl on B and D^0 is a Gauduchon gauge.*

Conversely, if (B, D) is an Einstein-Weyl 3-manifold and $\mathrm{w} \in C^\infty(B, L^{-1})$ is a non-vanishing solution of the monopole equation $d{*}D\mathrm{w} = 0$ then there is a self-dual 4-manifold M with symmetry over B such that $*D\mathrm{w}$ is the curvature of the connection defined by the horizontal distribution.

PROOF. The conformal structure and Weyl derivative descend to B because K is Killing in the constant length gauge and ω is a basic 1-form. The first submersion formula in 5.6 relates the Ricci curvature of D on B to that of D^{sd} on M:

$$\operatorname{sym}\operatorname{Ric}^D_B(X,Y) = \operatorname{sym}\operatorname{Ric}^{sd}_M(X,Y) + 2\langle D^0_X K, D^0_Y K\rangle + \tfrac{1}{2}\omega(X)\omega(Y) + \mu^{-2}\langle X, Y\rangle$$

for some section μ of L^1. We have shown that $\operatorname{sym}\operatorname{Ric}^{sd}(X,Y)$ is a multiple of $\langle X, Y\rangle$. Since $D^0_K K = 0$, ω vanishes on the plane spanned by D^0K, and so by comparing the lengths of ω and D^0K one verifies that $2\langle D^0_X K, D^0_Y K\rangle + \frac{1}{2}\omega(X)\omega(Y)q$ is also a multiple of $\langle X, Y\rangle$, and hence B is Einstein-Weyl. Now D^0K is a closed 2-form with respect to D^0 on M, so ω is co-closed with respect to D^0 on B and D^0 is a Gauduchon gauge. Finally one sees that no information is lost in this construction. Indeed if $*D\mathrm{w} = d\theta$ (locally) then the metric $g_M = \pi^*\mathrm{w}^2 c_B + (dt + \theta)^2$ is self-dual and $\partial/\partial t$ is a unit Killing field. (More invariantly, let G be the group of D^0-parallel sections of L^1 under addition so that M is a principal G-bundle. Then the monopole equation $*D\mathrm{w} = \Omega$, with Ω closed, couples a relative length scale $\mathrm{w}: L^1 \to M \times_G \mathfrak{g}$ to the curvature Ω of a principal connection on M.) □

Two special cases of this construction have received particular attention. The first is the case of a scalar flat Kähler 4-manifold with a Killing field. In this case, the Einstein-Weyl structure on B, which we call a *LeBrun-Ward* geometry **[50, 79]**, is given locally by

$$g = e^u(dx^2 + dy^2) + dz^2 \qquad (10.1)$$
$$\omega = -u_z dz$$

where $D = D^g + \omega$ and where u satisfies the Toda equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0.$$

Consequently these Einstein-Weyl geometries are also said to be *Toda*. Examples can be found in **[15, 73, 79]**.

Corresponding to a solution of the monopole equation $d{*}D\mathrm{w} = 0$ on B, is the scalar-flat Kähler manifold M given by the metric

$$g = e^u\mathrm{w}(dx^2 + dy^2) + \mathrm{w}dz^2 + \mathrm{w}^{-1}(dt + \theta)^2$$

and $\partial/\partial t$ is a Killing field. In this gauge, the monopole equation turns out to be equivalent to the linearized Toda equation

$$\mathrm{w}_{xx} + \mathrm{w}_{yy} + (e^u\mathrm{w})_{zz} = 0.$$

It follows that $\mathrm{w} = u_z$ is a distinguished monopole on B; if this monopole is used to construct M, then M is found to be hyper-Kähler **[9, 50]**.

The LeBrun-Ward spaces may be characterized invariantly as the Einstein-Weyl spaces locally fibering as a conformal submersion with geodesic one dimensional fibers and integrable horizontal distribution (i.e., they admit a shear-free, twist-free congruence of geodesics). In the above description, these geodesics are the curves of constant (x, y) [**73**].

The extra data on the mini-twistor space S given by this Toda structure is a real holomorphic section of $K_S^{-1/2}$ and the particular form in (10.1) is obtained by choosing a holomorphic coordinate $x + iy$ on the corresponding divisor.

Mini-twistor theory can be used to prove some of these claims [**50, 51**]. The monopole solution w is given, via the mini-twistor Ward correspondence, by a holomorphic line bundle $\mathcal{L} \to S$ with $c_1(\mathcal{L}) = 0$. If N denotes the normal bundle to the lifted mini-twistor lines, then the obstruction to the splitting of

$$0 \to \mathcal{O} \to N \to \mathcal{O}(2) \to 0$$

over a twistor line $\mathbb{C}P^1_x$ is an element of $H^1(\mathbb{C}P^1_x, \mathcal{O}(-2))$ and may be identified with $\mathrm{w}(x)$. Therefore, for $\mathrm{w}(x) > 0$, $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ and so $\mathcal{L}\backslash 0$ is a twistor space. The two orientations of the distinguished family of geodesics correspond to two curves $\mathcal{C}, \overline{\mathcal{C}}$ in S and $x + iy$ is a complex coordinate on $\mathcal{C}$. We shall now show that the line bundle represented by the divisor $\mathcal{C} + \overline{\mathcal{C}}$ is $K_S^{-1/2}$.

Choose a monopole (w, θ) and consider the twistor space Z of the corresponding scalar-flat Kähler metric. The vector field $\partial/\partial t$ lifts to Z so we may assume Z is a line bundle over S. Let $\mathcal{D} \subseteq Z$ be the section of $Z \xrightarrow{\pi} M$ corresponding to the complex structure on M. The projection $Z \to S$ maps a complex structure J at a point of M to the geodesic in B in the direction $J\frac{\partial}{\partial t}$. The image of $\mathcal{D}$ is therefore $\mathcal{C}$ and $\overline{\mathcal{D}}$ maps to $\overline{\mathcal{C}}$. From [**69**] we know that $[\mathcal{D} + \overline{\mathcal{D}}] = K_Z^{-1/2}$ and so, since the vertical tangent bundle of $Z \to S$ is trivial, it follows that $[\mathcal{C} + \overline{\mathcal{C}}] = K_S^{-1/2}$.

The second special case is the case of hyper-complex 4-manifolds with tri-holomorphic conformal vector fields. These were studied in connection with local heterotic geometries by Chave, Tod and Valent in [**17**]—see also [**74**]. In [**29**], Gauduchon and Tod showed that the Einstein-Weyl quotients arising in this situation are characterized by the presence of what might be called a "scalar curvature monopole": the scalar curvature is nonnegative and if $\kappa^2 = \frac{1}{6}\mathrm{scal}^D$ then κ satisfies the special monopole equation $*D\kappa = \frac{1}{2}F^D$. Together with the Einstein-Weyl equation, this is equivalent to the flatness of the connection $D - \kappa * 1$ on $L^{-1}TM$ and the parallel weightless unit vector fields are shear-free divergence-free geodesic congruences. We call these Einstein-Weyl spaces *Gauduchon-Tod* geometries or say that they are *hyper-CR*. Their mini-twistor spaces fiber over $\mathbb{C}P^1$ and the only compact examples, apart from the manifolds of constant nonnegative curvature, are $S^1 \times S^2$, the Berger spheres [**29**], and some finite quotients of these.

The total space M of an arbitrary monopole over a Gauduchon-Tod geometry carries a hyper-complex structure, and this provides an example of the Ansatz we have given in 5.6. If the scalar curvature monopole itself is used, then M turns out to be hyper-Kähler with a tri-holomorphic homothetic vector field.

There are clearly close parallels between these two cases.

- The LeBrun-Ward structures arise as quotients of scalar flat Kähler manifolds by a holomorphic Killing field. They have a special monopole (given in terms of a solution to the Toda field equation) which may be used to construct a hyper-Kähler manifold with a holomorphic Killing field.

- Hyper-complex manifolds with tri-holomorphic conformal Killing fields give rise to Gauduchon-Tod structures on the space of orbits. Again there is a special monopole (namely the scalar curvature monopole) leading to a hyper-Kähler metric, this time with a tri-holomorphic homothetic vector field.

The two situations may be unified and generalized by considering a self-dual 4-manifold with an anti-self-dual complex structure and a holomorphic conformal vector field. It can be shown [**14**] that the complex structure induces a shear-free geodesic congruence on the quotient Einstein-Weyl geometry B. The twist κ and divergence τ of this congruence turn out to be "special" monopoles on B. The κ monopole, if nonzero, gives a scalar flat Kähler 4-manifold over B with a holomorphic conformal vector field, while the τ monopole, if nonzero, gives a hyper-complex 4-manifold over B with a holomorphic conformal vector field. Hyper-Kähler manifolds are obtained when κ and τ are linearly dependent.

We have seen that hyper-Kähler 4-manifolds with special conformal vector fields give rise to interesting Einstein-Weyl geometries. It is natural to ask which geometries arise as (local) quotients of $\mathbb{R}^4$. Now, $\mathbb{R}^4$ is conformal to S^4 (minus a point) and so this question has been answered by Pedersen and Tod in [**68**]. Viewing S^4 as the light-cone in $\mathbb{R}^{5,1}$, conformal vector fields correspond to elements of the Lie algebra $\mathfrak{so}(5,1)$. There are no globally non-vanishing conformal vector fields and so, since conjugate elements of $\mathfrak{so}(5,1)$ will produce equivalent quotients, we may conjugate into a normal form in which they vanish at ∞ and then stereographically project. There are essentially three distinct cases: the hyperbolic elements (with a nontrivial infinitesimal dilation); the elliptic elements (generating rotations); and the parabolic elements (generating transrotations). The corresponding Einstein-Weyl geometries are given explicitly in [**68**], as Cases $(1, a \neq 0)$, $(1, a = 0)$ and (2) respectively. The generic case is the hyperbolic case, which gives a two dimensional moduli space of Einstein-Weyl structures near the Einstein metric on S^3.

In fact, these quotients of $\mathbb{R}^4$ exhaust the possible geometries on compact Einstein-Weyl manifolds.

THEOREM 10.4. *Let B, D be an Einstein-Weyl 3-manifold with Killing gauge $D = D^g + \omega^g$. Then B is locally isomorphic, as a Weyl manifold, to the quotient of an open subset of $\mathbb{R}^4$ by a conformal vector field with its induced Einstein-Weyl structure.*

PROOF. Let M be the total space of the monopole given by D^g. Then, by the inverse Jones-Tod construction, M is a self-dual conformal 4-manifold. However, since D^g is a Killing gauge, $D^g - \omega^g$ is also Einstein-Weyl. Now if $*\omega^g = dA$ then $*(-\omega^g) = d(-A)$ and $dt + (-A) = -(d(-t) + A)$. Hence changing the sign of ω^g does not alter the conformal structure on M, only the orientation. Therefore M is both self-dual and anti-self-dual, and thus conformally flat. The local isomorphisms are now given by conformal charts on M. □

This theorem was originally established by Tod [**72**] as a consequence of his classification of the possible local geometries on compact Einstein-Weyl 3-manifolds. He did this by solving the Einstein-Weyl equation in the Gauduchon gauge, using the fact that B fibers locally over a surface since ω^g is a Killing field. The freedom in the choice of isothermal coordinates on this surface may be used to reduce the Einstein-Weyl equation to an ODE, which is readily integrated. It is perhaps worth remarking that the additional symmetry which arises comes from the Faraday 2-form: generically $*F^D$ and ω^g are dual to linearly independent Killing fields. These generic solutions are of the form [**72**]:

$$g = P(v)^{-1}dv^2 + P(v)dy^2 + v^2(dt + Cv^{-2}dy)^2$$
$$\omega = 2\lambda v^2(dt + Cv^{-2}dy),$$

where
$$P(v) = -\lambda^2 v^4 + Av^2 + B - C^2 v^{-2}$$

and λ, A, B, C are arbitrary constants. The isothermal coordinates (x, y) can be found by solving the equation $v'(x) = P(v)$. Another change of coordinates relates these geometries to the quotients of S^4 in [**68**]. The parameters λ, A, B, C above are related to the parameters a, b, c in [**68**] by:

$$A = -a^2 + b^2 + c^2, \quad \lambda^2 B = a^2b^2 + a^2c^2 - b^2c^2, \quad \lambda^4 C^2 = a^2b^2c^2.$$

Examining Tod's argument, we find that the Gauduchon constant is $-6A$ and so $\mathrm{scal}^D = -6A + 3|\omega|^2 = 6(a^2 - b^2 - c^2 + 2\lambda^2 v^2)$. Also, the range of $\lambda^2 v^2$ when $P(v) \geqslant 0$ is the interval $[b^2, c^2]$. Therefore, for $|b^2 - c^2| > a^2$, the scalar curvature has non-constant sign [**12**].

In particular, there are Einstein-Weyl geometries globally defined on S^3 with scalar curvature of non-constant sign, contrary to remarks made in [**67, 68**]. Such examples are "far" from the Berger spheres, which are given by $b^2 = c^2$ (and $a^2 \neq 0$), but include some examples in the one parameter family given in [**52**].

We would also like to emphasize that, although most of the solutions above are globally defined on S^3, Tod's result [**72**] claims only to classify the local forms of solutions which can exist on compact manifolds. It should be possible to work out which compact 3-manifolds carry which local forms using the co-homogeneity one torus action given by the Killing fields $\partial/\partial y$ and $\partial/\partial t$, but care needs to be taken when considering the possible flows of $\partial/\partial t$ in this torus.

We now briefly treat the two dimensional case, where matters are simplified by the fact that the only compact 2-manifolds admitting metrics with Killing fields are S^2 and $S^1 \times S^1$. Hence only these manifolds can admit non-exact Einstein-Weyl structures.

In the Gauduchon gauge (g, ω), $\sharp_g\omega$ is holomorphic, so we may locally choose a complex coordinate $x + it$ such that $\sharp_g\omega = \partial/\partial t$. The Einstein-Weyl equation 3.4 immediately reduces to an ODE for a function of x, and we find [**11**]

$$g = P(v)^{-1}dv^2 + v^2dt^2$$
$$\omega = Av^2\,dt,$$

where
$$P(v) = -A^2v^4 + Bv^2 + C$$

and A, B, C are arbitrary constants. This time $v'(x)^2 = P(v)v^2$, but it is perhaps simpler to introduce a new coordinate r by $v'(r)^2 = P(v)$. The metric is now

$$g = dr^2 + v(r)^2dt^2$$

and $v(r)$ is an elliptic function since P is a quartic polynomial. In terms of Jacobian elliptic functions (assuming $P(v)$ is somewhere positive),

$$v(r) = \begin{cases} \lambda\,\mathrm{cn}(\mu r + \alpha, k) \quad \text{or} \quad \lambda\,\mathrm{sd}(\mu r + \alpha, k) & \text{if } C > 0 \qquad (1) \\ \lambda\,\mathrm{dn}(\mu r + \alpha, k) \quad \text{or} \quad \lambda\,\mathrm{nd}(\mu r + \alpha, k) & \text{if } C < 0 \qquad (2), \end{cases}$$

where α is a constant of integration and λ, μ, k are constants depending on A, B, C. The two forms given in each case are equivalent by period translation, but behave differently in the limit $k \to 1$ when the (real) period becomes infinite. The Gauduchon constant is $-2B$ which is always negative in (2), but is proportional to $1 - 2k^2$ in (1). If w ranges over a half-period of cn or sd, (1) gives a family of global solutions on S^2, whereas dn and nd are periodic and non-vanishing, and so the solutions in (2) are defined on $S^1 \times S^1$. In particular, there are non-closed Einstein-Weyl structures on $S^1 \times S^1$ in stark contrast to the situation on $S^1 \times S^{n-1}$ for $n \geqslant 4$.

11. Further horizons

Notwithstanding the pioneering work of Cartan and Hitchin, a detailed understanding of the nature of Einstein-Weyl spaces from a differential geometric point of view has only matured during the last fifteen years. We believe that there is now a good supply of concepts, examples and results about Einstein-Weyl geometry. Nevertheless many basic questions remain unanswered and there are interesting avenues still to be explored.

1. Is there a Lagrangian for the Einstein-Weyl equations? The calculations made to date suggest that the Einstein-Weyl equations may not be the Euler-Lagrange equations of a natural functional, but there are at least interesting functionals which have Einstein-Weyl spaces as a special class of minima (see [**36**] for the Euler-Lagrange equations of total curvature functionals).

2. Three dimensional Einstein-Weyl manifolds with a Gauduchon gauge correspond to four dimensional self-dual manifolds with symmetry, and these minimize the L^2-norm of the Weyl curvature. The critical points of this functional are the Bach flat manifolds. What is the symmetry reduction of this functional and of the Bach flatness condition?

3. The classification of compact three dimensional Einstein-Weyl manifolds needs to be completed.

4. So far, interesting interactions of *non-closed* Einstein-Weyl geometry with special conditions on *compact* 4-manifolds have not been found. Nevertheless there is a good supply of highly symmetric examples and so we would like to know what special properties they have.

5. We have seen that there are global obstructions to the existence of Einstein-Weyl structures, but the question of local existence of Einstein-Weyl structures compatible with a given conformal structure is a nontrivial problem. In [**19, 20**], Eastwood and Tod have shown that there are conformal structures which do not admit compatible Einstein-Weyl structures even locally. For instance, in three dimensions, the general left invariant metric on S^3 and Thurston's Sol geometry do not admit Einstein-Weyl structures. In four dimensions the product of two spheres of different sizes is not locally Einstein-Weyl. Nevertheless, local questions still remain. We would like to know, for instance, whether the scalar curvature of a Bach flat Einstein-Weyl structure is necessarily zero. A similar question can be asked about Kähler Einstein-Weyl structures.

6. The theory of submersions between Einstein-Weyl spaces with one dimensional fibers has not been studied when the total space is even dimensional, except in the case of self-dual 4-manifolds, when we have hyper-complex structures and self-dual

Einstein metrics with symmetry (the latter are conformally scalar flat Kähler). The Ansatz we have presented in 5.6 might be useful for generalizing these situations.

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