General Relativity

K.P.Tod

1. Introduction

1.1. Aims. In this essay, my brief is to describe some current research in general relativity which would be of interest to mathematicians working elsewhere in geometry. To achieve this, I shall need first to review a range of background material in modern general relativity, corresponding roughly to a second or graduate-level course. For reasons of space, I shall need to assume that the reader has had a first course in the subject. After the review, the choice of topics is my own.

1.2. A way in. One way into relativity for a mathematical audience is to compare and contrast Riemannian and Lorentzian geometry — what changes when the signature of the metric changes? What familiar things cease to be of interest and what new things become of interest? One may classify topics of interest in relativity by their relation to Riemannian geometry into one of three classes:

- **direct uses of Riemannian geometry.** e.g. space-like surfaces are intrinsically Riemannian, therefore so are questions to do with the Initial Value Problem; the classification of black holes is concerned with time-independent solutions, where the field equations become elliptic; the first proof of the Positive Mass theorem uses the methods of Riemannian geometry;
- **Lorentzian problems motivated by analogy with Riemannian ones.** e.g. there are Lorentzian Splitting theorems, motivated by analogy with the Riemannian ones, but with their own physical interpretation; with an indefinite metric, positive sectional curvature is not a helpful notion but certain conditions of positivity of the Ricci tensor are of crucial importance, and play an analogous role in forcing the existence of conjugate points on geodesics;
- **complete novelties.** e.g. anything explicitly hyperbolic, so existence theory for the Einstein equations; singularity theorems and cosmic censorship.

1.3. One difference. It is instructive to pursue one answer to the question 'what changes when the signature changes?', namely the answer 'the Hopf-Rinow theorem;' cf. e.g. [**9**]. In Riemannian geometry, the manifold becomes a metric space with the distance defined by the metric tensor, and the open sets in the manifold are determined by the metric. The Hopf-Rinow theorem asserts that the

©2000 International Press

manifold is complete as a metric space if and only if it is geodesically complete. Further, in this case, there will be a geodesic connecting any two points which achieves the (minimum) distance between them. When the metric tensor is indefinite, none of this works.

For the topology, one can seek instead to define the open sets by causal relations and one is led into a study of causal spaces, which represent an important layer of structure between the topological and the metric in relativity. For the completeness, one can distinguish a whole range of (independent) geodesic completenesses, and completeness for other types of curve. There is also the important condition of global hyperbolicity in relativity which implies the existence of maximal curves in appropriate circumstances. (Note 'maximal' rather than 'minimal': time-like geodesics locally maximise distance; it is always possible to join points by 'short' curves by making them nearly null.)

1.4. Physical arguments. Relativity is a theory of gravity, and an extremely accurate one; cf. e.g. the theory of binary pulsars [**95**, p.230]. This means on the one hand that physical concerns and heuristic arguments have a proper place in the subject, and on the other that physical insight can lead one to results which can then be proved to the most rigorous standards — physical insight can coincide with what is true in the theory. However as a mathematician, one may not want to delve too deeply into the physical aspects of the theory. There is a standard way to achieve this aim:

1.5. The Einstein inequalities. Recall the Einstein equations in the form

$$G_{ab} = 8\pi G T_{ab} \tag{1.1}$$

where G_{ab} is the Einstein tensor of some Lorentzian metric and T_{ab} is the energy-momentum tensor of some matter source. (Relativists commonly, though by no means invariably, use indices. In this article, where necessary, I shall use the *abstract index convention* of Penrose [**96**]. This allows one to use all the notations of local tensor calculus, so that one does not need to devise notational synonyms for tensor operations, while remaining perfectly invariant.)

The left-hand side of (1.1) is the mathematical side (the 'marble palace' of Einstein) and the right-hand side is the physical side (the 'wooden shed'). In many situations, one may regard (1.1) as producing a set of inequalitites by requiring of the right-hand side only that it have some positivity properties, and ignoring its details. The physical input of general relativity into geometry is then confined to demanding these positivity properties of the left-hand-side. These positivity properties are the various energy conditions: they express different conditions of positive energy locally, and most of what we shall see below is premised on one or another energy condition.

1.6. Conjugate point arguments. It is the energy conditions which make gravity attractive. One consequence of this attractiveness is that, given a large amount of mass in a small region, gravity may overwhelm the forces holding the matter up and bring about a gravitational collapse to a singularity. The mathematical counterpart of this physical argument is that energy conditions eventually lead to the existence of conjugate points on geodesics provided the geodesics can be extended to arbitrary values of affine parameter. Then these conjugate points can be shown to be inconsistent with other physical hypotheses which encode the fact of collapse, from which one is led to geodesic incompleteness. This is a paradigm *conjugate point argument.* A consequence of it is that relativists are obliged to consider manifolds which are geodesically incomplete or singular in other ways. Many of the arguments in sections 3 to 7 are conjugate point arguments in this sense.

1.7. Causality. One of the other ways to be singular is to have a closed time-like curve (or CTC). If such a thing existed in a space-time, then one could travel along it into one's past, when various, usually murderous, paradoxes could be generated. There is a whole range of causality pathologies which one might seek to forbid for physical reasons. Now a *compact* Lorentzian manifold necessarily has a CTC (in fact many, joining any point to any point), which is why these have traditionally held less interest for relativists.

1.8. Positive energy. An interesting problem historically has been how to derive global positive energy, as measured 'at infinity' and containing non-local contributions from the gravitational field, from an assumption of positive energy locally, expressed by an energy condition. (This problem is difficult because one expects gravitational energy to exist and so to contribute to total energy, but not to be the integral of any local quantity.) There are now three different ways to derive this result, two which work on space-like surfaces and are therefore 'elliptic' and a newer four-dimensional way. The result, the Positive Energy theorem, has subsequently been used to prove new results and strengthen old ones.

1.9. Cosmic censorship. Arguably the biggest unsolved problem in relativity is to prove or disprove the cosmic censorship hypothesis. In a weak form, this is the hypothesis that, while the formation of singularities in certain circumstances is inevitable, these singularities are hidden inside black holes and cannot be seen from large distances. In a strong form, the hypothesis is that only particular kinds of singularities can ever arise in an evolution of regular data. Either form is a hard problem, made harder by a physical consideration: these are supposed to be statements about the world so that one is interested in *generic* or *stable* sets of circumstances arising with *reasonable* matter, and all the italicised words are problematic.

1.10. Contents. This essay is organized as follows:

In §2, we describe the landscape of general relativity as it is now. The development here is inevitably condensed almost to the telegraphic but it sketches what is needed to locate the later sections.

In §3, we review various topological issues in relativity. These include those mentioned above, ideas related to dynamic topology, changing with time, and the recent notion of topological censorship, which is analogous to cosmic censorship.

In §4, we describe the Lorentzian Splitting Theorems and related material and in §5 we review what is known about existence for solutions of the Einstein equations.

In §6, we review work on the Black Hole Uniqueness theorems, where there has been a resurgence of interest recently, and finally in §7 we review work on the evidence for and against the Cosmic Censorship Hypothesis.

ACKNOWLEDGMENTS 1.10.1. In composing this review, I have benefitted from discussions with many people among whom I would like to mention Lars Andersson,

Piotr Chruściel, Helmut Friedrich, Lionel Mason, Vince Moncrief, Ted Newman, Roger Penrose, Alan Rendall and Bernd Schmidt.

1.11. Further reading. A review such as this, to be successful, needs to lead the reader onward and beyond itself. Thus a good text for Section 2 is [**137**]; more details in particular directions will be found in [**60**], which after 24 years is still the place to start, and in [**9**]. A useful resource in the near future will be the 'Living reviews' on various topics in relativity maintained by the Albert Einstein Institute in Potsdam at http://www.aei-potsdam.mpg.de, and much of the topical material discussed in this review first appeared in the gr-qc archive at http://xxx.lanl.gov/ or one of its mirrors.

2. Background Material

We will use conventions as in [**96**], so that the signature of the metric is $(+,-,-,-)$ and indices are abstract.

2.1. Infinity for flat space. We need a definition of isolated source in general relativity, which must convey the idea of asymptotic flatness at large distances. The idea is to define an infinity for flat space as a boundary, so that one may later define a space to be asymptotically flat if it has the same kind of infinity as flat space.

To this end, call flat space M and consider the metric of M in spherical polar coordinates:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}$$

Radially in- and out-going null geodesics have respectively

$$v = t + r = \text{ constant; } u = t - r = \text{ constant; } -\infty < u \leq v < \infty \tag{2.2}$$

Introduce u and v as coordinates in (2.1) to find:

$$ds^2 = dudv - \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3}$$

We shall add a boundary to M by first adding a point to the end of each radial null geodesic. This is achieved by introducing coordinates p and q via

$$u = \tan p;\ v = \tan q;\ \text{so that } -\frac{\pi}{2} < p \leq q < \frac{\pi}{2}$$

when (2.3) becomes

$$ds^2 = \frac{1}{4}\sec^2 p \sec^2 q[4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.4}$$

In this form, the metric can be conformally-rescaled to give a new metric on a larger manifold than M:

$$d\hat{s}^2 = \Omega^2 ds^2 = 4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

where

$$\Omega = 2\cos p \cos q$$

and now we can extend the range of p and q to include the end-points $\pm\frac{\pi}{2}$. The rescaled metric (2.5) is the product metric on $\mathbb{R} \times S^3$ as one may see by introducing one last set of coordinates via

$$T = p + q;\ R = p - q$$

when

$$ds^2 = dT^2 - dR^2 - \sin^2 R(d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.6}$$

In this context, this product metric is known as 'the Einstein static cylinder', having been at one time proposed as a cosmological model by Einstein.

We have found that flat space M is conformally related to the part of the Einstein static cylinder lying in the range

$$T + R > -\pi; \; T - R < \pi \tag{2.7}$$

The conformal structure of M extends to the boundary of this region in $\mathbb{R} \times S^3$, which is the locus where Ω from (2.5) vanishes. The boundary consists of the past null cone $\mathcal{I}^+$ (pronounced 'scri-plus') of the point i^+, which is also the future null cone of the point i^0, together with the past null cone $\mathcal{I}^-$ of i^0, which is also the future null cone of the point i^- (see figure 1). These symbols are conventional and

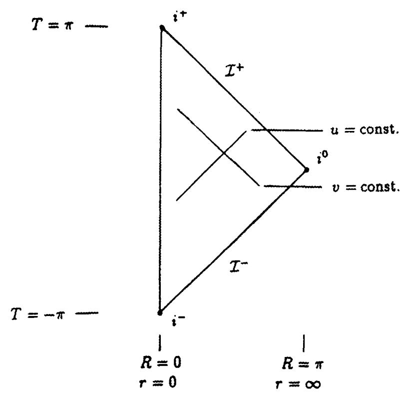

FIGURE 1. The (t, r)-half-plane of Minkowski space in the (T, R)-space of $\mathbb{R} \times S^3$; i^0 is antipodal to the origin on S^3.

are associated with the following names:

TERMINOLOGY 2.1.1.

- $\mathcal{I}^+$ is future null infinity;
- $\mathcal{I}^-$ is past null infinity;
- i^+ is future time-like infinity;
- i^- is past time-like infinity;
- i^0 is space-like infinity.

All null geodesics have a past end-point on $\mathcal{I}^-$ and a future one on $\mathcal{I}^+$; all time-like geodesics run from i^- to i^+; all space-like geodesics run from i^0 back to i^0. We may sometimes use $\mathcal{I}$ to mean the union $\mathcal{I}^+ \cup \mathcal{I}^-$.

We have defined this boundary using coordinates but invariant descriptions are possible.

2.2. Asymptotic simplicity. We use the work of the previous section to give a definition intended to capture the notion of asymptotic flatness.

DEFINITION 2.2.1. A space-time M with metric g is *asymptotically simple* if there is a smooth manifold $\hat{M}$ with boundary $\mathcal{I} = \partial\hat{M}$ and metric $\hat{g}$ and a scalar field Ω such that

- $M = \text{int } \hat{M}$;
- $\hat{g} = \Omega^2 g$ in M;
- Ω and $\hat{g}$ are smooth everywhere in $\hat{M}$;
- $\Omega > 0$ in M; $\Omega = 0$ and $d\Omega \neq 0$ on $\mathcal{I}$; and
- every null geodesic in M acquires a future and a past end-point on $\mathcal{I}$.

The last condition is needed to avoid trivial satisfaction of the conditions with $\mathcal{I}$ empty, but is too strong in practice since even the extended Schwarzschild solution will fail to be asymptotically simple. Thus one defines:

DEFINITION 2.2.2. A space-time M with metric g is *weakly asymptotically simple* (or WAS) if there is an asymptotically simple M' and a neighbourhood U of $\mathcal{I}$ in the corresponding $\hat{M}$ such that $U \cap M'$ is isometric to a subset of M.

2.3. Causal relations. Causal relations define a layer of structure prior to the smooth in a space-time. This section consists largely of definitions, made to introduce a convenient language.

DEFINITION 2.3.1. A Lorentzian manifold M is *time-orientable* if it is possible to make a consistent choice of future-light-cone at every point; M is *space-orientable* if it is possible to make a consistent choice of a right-handed triad of space-like vectors at every point.

If M is time and space orientable, then M is orientable but not conversely. If M admits spinors then M is orientable in all three senses.

DEFINITION 2.3.2. For points p and q in a time-orientable M define the relations:

$p \ll q$ (read 'p chronologically precedes q') iff there is a future-directed (non-empty) time-like path from p to q;

$p \prec q$ (read 'p causally precedes q') iff there is a future-directed (possibly empty) path from p to q which is everywhere non-space-like (i.e. is time-like or null at each point; call this a *causal path*).

DEFINITION 2.3.3. We define the sets:

- $I^+(p) = \{q | p \ll q\}$ the chronological future of p;
- $I^-(p) = \{q | p \gg q\}$ the chronological past of p;
- $J^+(p) = \{q | p \prec q\}$ the causal future of p;
- $J^-(p) = \{q | p \succ q\}$ the causal past of p.

'Time-like' is an open condition, whence it follows that $I^+(p)$ and $I^-(p)$ are open, but $J^+(p)$ and $J^-(p)$ are not necessarily closed (though they will be in Minkowski space).

In terms of these notions one can frame various causality conditions:

DEFINITION 2.3.4. M satisfies the *chronology condition* if it contains no closed time-like curves, equivalently if for no $p \in M$ is it true that $p \in I^+(p)$ or $p \in I^-(p)$.

DEFINITION 2.3.5. M satisfies the *causality condition* if it is never true that $p \prec q \prec p$ for distinct p and q.

A range of stronger conditions restricting causal pathologies is available. A useful one, needed in 3.1.1, which excludes almost closed causal paths is:

DEFINITION 2.3.6. M is *strongly causal* at p if there is a neighbourhood of p which no non-space-like path intersects more than once.

The strongest condition normally encountered is the following:

DEFINITION 2.3.7. M is *globally hyperbolic* if the strong causality condition holds everywhere and, for any $p, q \in M$, the set $J^+(p) \cap J^-(q)$ is compact.

Global hyperbolicity is related to Cauchy developments, so we need to define these:

DEFINITION 2.3.8. An *achronal set* S is one for which $I^+(S) \cap S = \emptyset$.

DEFINITION 2.3.9. The *future Cauchy development* or *future domain of dependence* $D^+(S)$ of an achronal set S in a space-time M is the set of $p \in M$ such that every past-inextendible non-space-like path through p intersects S.

DEFINITION 2.3.10. The *future Cauchy horizon* of S is the future boundary of $D^+(S)$, that is the set $H^+(S) = \overline{D^+(S)} - I^-(D^+(S))$ (writing $\bar{U}$ for the closure of U).

One defines $D^-(S)$ and $H^-(S)$ analogously, and then $D(S) = D^+(S) \cup D^-(S)$. The relation with global hyperbolicity is provided by the result:

PROPOSITION 2.3.11. [**60**, Prop 6.6.3] *If S is a closed achronal set then* int $D(S)$, *if non-empty, is globally hyperbolic.*

PROPOSITION 2.3.12. *An achronal set S is a Cauchy surface for M if $M = D(S)$.*

Thus if a space-time M has a Cauchy surface, then it is globally hyperbolic. We shall encounter a converse in 3.3.1.

An aspect of the role of global hyperbolicity as a completeness condition is provided by the result:

PROPOSITION 2.3.13. [**60**, Prop 6.7.1] *If p, q lie in a globally hyperbolic set U with $q \in J^+(p)$ then there is a non-space-like geodesic from p to q whose length is greater than or equal to the length of any other non-space-like curve from p to q.*

Finally in this section, we note that there is an invariant characterisation of $\mathcal{I}^-$ and $\mathcal{I}^+$ in terms of causal structure, so that these can be added as future and past *causal boundaries.*

2.4. The Schwarzschild solution.

SPACE-LORE 2.4.1. The Schwarzschild solution is characterised by Birkhoff's theorem [**60**] as *the* spherically-symmetric vacuum solution. It is weakly asymptotically simple, and *static*, which means that it admits a hypersurface-orthogonal Killing vector which is time-like at large distances (one reserves the term *stationary* for a solution with a time-like Killing vector which is not hypersurface-orthogonal). The solution depends on a single parameter which can be identified as the mass (see §2.11).

EXAMPLE 2.4.2 (Extending the Schwarzschild solution). In a first course on general relativity, the Schwarzschild solution is usually exhibited in coordinates as

$$ds^2 = (1 - 2\frac{m}{r})dt^2 - (1 - 2\frac{m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.8}$$

where the coordinate ranges are $-\infty < t < \infty$, $2m < r < \infty$.

This form of the metric is singular at $r = 2m$ but this is only a coordinate singularity. A first exercise is to solve the geodesic equations for radial null geodesics, when one readily finds that these geodesics run off the coordinate patch by arriving at $r = 2m$ at finite values of affine parameter, but infinite values of t. The strategy is now to mimic the process leading to equation (2.2), introducing coordinates u and v constant on out- and in-going radial null geodesics respectively, to arrive at an extended form of the metric:

$$ds^2 = \frac{32m^3}{r}\exp(-\frac{r}{2m})dudv - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.9}$$

where $uv = -(\frac{r}{2m} - 1)\exp(\frac{r}{2m})$.

The metric is no longer 'time-independent', the Killing vector K^a which was $\partial/\partial t$ has become

$$\frac{\partial}{\partial t} = \frac{1}{4m}(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}). \tag{2.10}$$

We may represent the manifold on which the metric is defined by its *Carter-Penrose diagram*, figure 2.

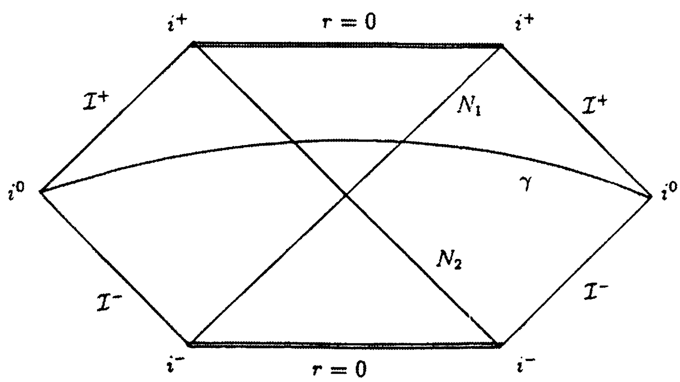

FIGURE 2. Carter-Penrose diagram of maximally analytically extended Schwarzschild solution; each point represents a 2-sphere; null-lines are at 45°; note initial and final $r = 0$ singularities, two asymptotic regions and two Killing horizons at $r = 2m$.

In figure 2 each point represents a 2-sphere of symmetry, and null directions are at 45°. The surprise about the diagram is the presence of two singularities, one in the past (at the bottom) and one in the future, and two asymptotic regions. The picture includes the two distinct $\mathcal{I}^+$'s and two distinct $\mathcal{I}^-$'s, where the conformal structure is regular. The conformal structure is singular at the points $i^\pm$, but also, perhaps surprisingly, at i^0.

The Killing vector (2.10) is time-like and future-pointing near the righthand asymptotic region and time-like, past-pointing near the left-hand one (choosing again the time-orientation which has the future towards the top of the page). The Killing vector becomes null on the pair of null hypersurfaces $N_1 : u = 0$ and $N_2 : v = 0$; each of these is a *Killing horizon:*

DEFINITION 2.4.3. A *Killing horizon* is a null hypersurface with a null Killing vector K^a tangent to the (null, geodesic) generators.

DEFINITION 2.4.4. The Killing horizon has a *surface gravity* κ defined by

$$\nabla_a(K^b K_b) = -2\kappa K_a.$$

Under quite general conditions the surface gravity is constant on the Killing horizon. For the Schwarzschild solution, $\kappa = 1/4m$.

DEFINITION 2.4.5. A Killing horizon is *degenerate* if it has zero surface gravity.

In the extended Schwarzschild case, there are two Killing horizons, which intersect in the *bifurcation surface* at $u = v = 0$.

A Killing horizon often defines an *event-horizon:*

DEFINITION 2.4.6. In a weakly asymptotically simple space-time, the event horizon (strictly, the future event horizon) is $\partial J^-(\mathcal{I}^+)$ if this is non-empty.

Thus if there is an event horizon, then it separates points from which there is a causal path to $\mathcal{I}^+$ from those where there is no such path i.e. it bounds the region from which one can 'escape' to infinity. In the extended Schwarzschild manifold, N_1 defines the event horizon for the $\mathcal{I}^+$ to the right.

We noted above that any point in figure 2 defines a 2-sphere. Furthermore, the area of the 2-sphere is $4\pi r^2$. Now consider a point in the top triangle, that is one with $u < 0$, $v > 0$, $r < 2m$; if the corresponding 2-sphere is moved in any direction normal to itself and into its own future then it will move to a smaller value of r and so its area will decrease (strictly speaking, one needs to calculate something to prove this). We define:

DEFINITION 2.4.7. A space-like 2-surface is said to be *trapped* if its area locally decreases in every future-pointing normal direction.

Now consider a line like γ running across the Carter-Penrose diagram from one i^0 to the other (and not necessarily through the bifurcation surface). This defines a spherically-symmetric space-like surface which is a Cauchy surface for the space-time. At the minimum value of r there will be a (stable) minimal surface, in the usual sense, but every sphere of constant r less than $2m$ will be trapped, while the spheres $r = 2m$ are *marginally-trapped* in that the area is non-increasing in every future-normal direction, and is strictly decreasing in all directions except one of the two null normal directions.

TERMINOLOGY 2.4.8. This Cauchy surface has the character of a *worm-hole* in that it connects two asymptotically flat regions through a minimal surface.

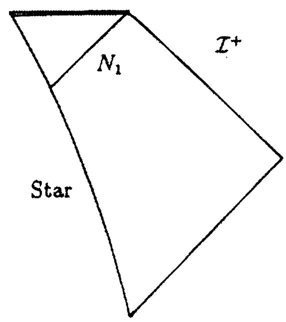

FIGURE 3. Carter-Penrose diagram of collapse of a star to a black hole; the solution outside the star is Schwarzschild and N_1 is the event horizon; the final singularity is formed in the collapse.

However it is not possible in the Schwarzschild manifold to follow a causal path through the worm-hole from one asymptotic region to the other (this can be seen from figure 2, which correctly shows causal relations).

On a Cauchy surface through the bifurcation surface, the bifurcation surface itself is both minimal and marginally-trapped with respect to both its null normals. This is a rather degenerate situation.

The collapse of a spherically symmetric body, say a star, surrounded by vacuum, to a singularity may be represented by a Carter-Penrose diagram, figure 3, consisting of the outer region of figure 2 joined across the surface of the star to another solution with matter. The matter solution cuts off the 'unphysical' past singularity. Now the null hypersurface N_1 defines the event horizon as the boundary of a black hole. A singularity forms in this collapse but it cannot be seen from infinity, that is to say no future causal path connects it to $\mathcal{I}^+$ - it is 'censored'.

An important property of the Schwarzschild solution is the following:

CONDITION 2.4.9. For any $p \in \mathcal{I}^-$, $I^+(p)$ contains all of $\mathcal{I}^+$.

This surprising result is a consequence of the phenomenon of time-delay in the passage of light past a massive body (equivalently 'of time-delay in the solutions of the null-geodesic equation in the Schwarzschild metric'). It is characteristic of positive mass - it is not true in flat space or in the negative-mass Schwarzschild solution.

2.5. The Reissner-Nordstrom solution.

SPACE-LORE 2.5.1. The *Reissner-Nordstrom solution* is characterised as the spherically-symmetric electrovac solution, which is to say a solution of the Einstein equations for which the energy-momentum tensor is that for electromagnetism, in which the spheres of symmetry vary in size. Again it is static and weakly asymptotically simple.

The metric is usually encountered first in the form

$$ds^2 = V(r)dt^2 - (V(r))^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.11}$$

where now $V = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$; e, m real constants, and we shall suppose that

$$e^2 < m^2 \tag{2.12}$$

The parameter m may be identified with the mass at infinity, while e is the charge.

Because V has two zeroes, the extension is more complicated. The Carter-Penrose diagram is as in figure 4. The Killing vector $K^a = \partial/\partial t$ is time-like in the (infinitely many) asymptotic regions, and again near the (infinitely many) singularities, being space-like in an intermediate region. There are inner and outer Killing horizons (with different surface gravities) where the Killing vector is null, intersecting at bifurcation surfaces where it vanishes.

A curve like γ defines a spherically-symmetric achronal surface S which is now not a Cauchy surface: note that S has a Cauchy horizon, because of the time-like character of the singularities.

Also because of the time-like character of the singularities, it is possible to follow a causal path from one asymptotic region down to small values of r then into the future and into a second asymptotic region. By identifying the diagram with a periodicity vertically one may therefore introduce closed time-like curves. (We shall see this again in §3.5.)

In the case $e^2 = m^2$, the zeroes of V coincide. The Carter-Penrose diagram simplifies to figure 5. The Killing vector is time-like everywhere except on the Killing horizon where it becomes null. The bifurcation surface has disappeared from the picture and the Killing horizon has become degenerate (these phenomena are related). There are internal points 'at infinity' in that the Riemannian distance on a (space-like) hypersurface of constant t from a value of r greater than m down to $r = m$ is infinite. Thus a constant t hypersurface is asymptotically flat at large distances, but asymptotic to an infinite cylinder as r tends to m. (Degenerate horizons will lead to problems in §6.)

By matching to a collapsing spherically-symmetric charged body, the Reissner-Nordstrom solution can be interpreted as a black hole solution.

2.6. The Majumdar-Papapetrou solutions. The Majumdar-Papapetrou solutions are electrovac solutions generalising the Reissner-Nordstrom solution with $e^2 = m^2$. They may be written

$$ds^2 = V^2dt^2 - V^{-2}(dx^2 + dy^2 + dz^2) \tag{2.13}$$

where V is harmonic

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In the special case

$$V = 1 + \sum_{i=1}^{n} \frac{m_i}{|r - r_i|} \tag{2.14}$$

the solution represents a superposition of charged black holes, the i-th having mass m_i and charge e_i satisfying $e_i^2 = m_i^2$, the same sign taken for all. The locations are

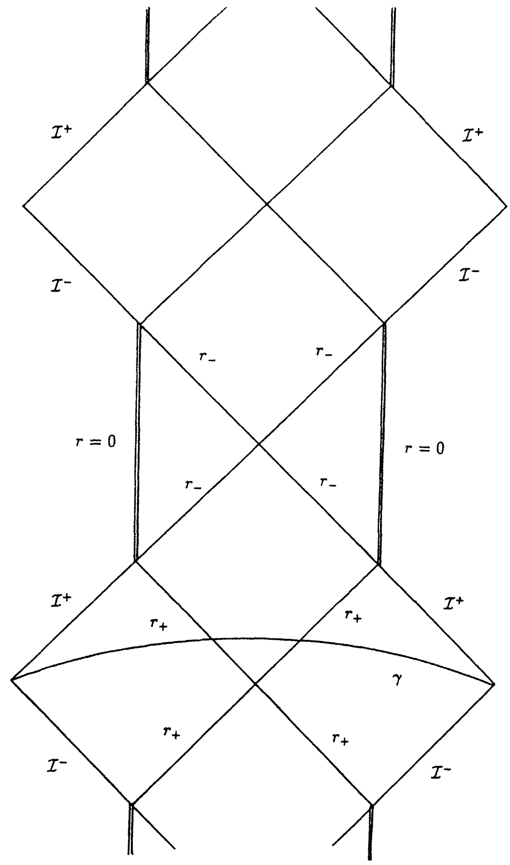

FIGURE 4. Carter-Penrose diagram of the maximally analytically extended Reissner-Nordstrom solution with $e^2 < m^2$; note the $r = 0$ singularity has become time-like and there are infinitely many asymptotic regions, also infinitely many Killing horizons at $r = r_+$ and $r = r_-$; note also the occurrence of a Cauchy horizon at $r = r_-$ for the surface γ.

freely specifiable since, physically speaking, the mutual gravitational attractions are balanced by the electrostatic repulsions. All the black holes have degenerate horizons.

2.7. Homogeneous and isotropic cosmologies. We shall need these in §5.

TERMINOLOGY 2.7.1. The *Robertson-Walker* (or FRW) metric is the metric

$$ds^2 = dt^2 - (R(t))^2 d\sigma_k^2 \tag{2.15}$$

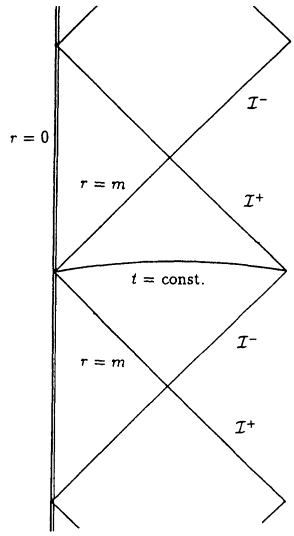

FIGURE 5. Carter-Penrose diagram of the maximally analytically extended Reissner-Nordstrom solution with $e^2 = m^2$; the Killing horizons of figure 4 merge in pairs and become degenerate at $r = m$; the surfaces of contant t have points at infinite distance at $r = m$.

where $R(t)$ is the *scale factor* and the spatial part $d\sigma_k^2$ is the metric of a 3-space of constant curvature k.

With any scale factor, this metric has a 6-dimensional isometry group transitive on the surfaces of constant time t, hence the title of this section. The metric is always conformally-flat and so is conformal to part of the Einstein static cylinder.

Particular examples are:

EXAMPLE 2.7.2. The *de Sitter space*, for which $k = 1$ and $R(t) = a\cosh(t/a)$, or $k = 0$ and $R(t) = \exp(t/a)$.

EXAMPLE 2.7.3. The (universal cover of the) *anti-de Sitter space*, for which $k = -1$ and $R(t) = a\cos(t/a)$.

These are in fact Lorentzian symmetric spaces. De Sitter space is conformal to the region $0 < T < \pi$ on the Einstein static cylinder, so it has a space-like $\mathcal{I}^-$ at $T = 0$ and a space-like $\mathcal{I}^+$ at $T = \pi$. Anti-de Sitter space is conformal to the region $R < \pi/2$ on the Einstein cylinder and has a time-like $\mathcal{I}$ at $R = \pi/2$. It is a consequence of the definition of weak asymptotic simplicity 2.2.2 that, if the Einstein equations in the form

$$R_{ab} - \lambda g_{ab} = 0$$

hold near $\mathcal{I}$, then $\mathcal{I}$ is time-like, space-like or null according as the cosmological constant λ is greater than, less than or equal to zero (there is a choice of convention in the sign of the Ricci tensor R_{ab} which can confuse this issue; see §5.4).

2.8. Energy conditions. With the Einstein equations as in (1.1):

CONDITION 2.8.1. The stress-energy tensor T_{ab} is said to satisfy the *weak energy condition* if $T_{ab}t^a t^b \geq 0$ for every time-like vector t^a.

CONDITION 2.8.2. The stress-energy tensor T_{ab} is said to satisfy the *strong energy condition* if $T_{ab}t^a t^b \geq \frac{1}{2}T^c_c g_{ab}t^a t^b$ for every time-like vector t^a.

CONDITION 2.8.3. The stress-energy tensor T_{ab} is said to satisfy the *dominant energy condition* if $T_{ab}t^a$ is a non-spacelike, future-pointing vector for every future-pointing time-like vector t^a.

These conditions can all be regarded as reasonable conditions on (classical) matter. From them and the Einstein equations one deduces:

CONDITION 2.8.4. If T_{ab} satisfies the weak energy condition, then the Ricci tensor R_{ab} satisfies the *null convergence condition*: $R_{ab}n^a n^b \geq 0$ for every null vector n^a.

CONDITION 2.8.5. If T_{ab} satisfies the strong energy condition then R_{ab} satisfies the *time-like convergence condition*: $R_{ab}t^a t^b \geq 0$ for every time-like vector t^a.

The dominant energy condition is the one needed in the first two proofs of the positive energy theorem; the others are relevant to the existence of conjugate points, which we turn to next.

2.9. Geodesic deviation. We need some formalism here. Suppose γ is a time-like geodesic with unit future-pointing tangent vector T^a. Write $D = T^a\nabla_a$ for the directional derivative along γ and s for proper time along γ, and let $e^a_i = \{e^a_1, e^a_2, e^a_3\}$ be an orthonormal basis of vectors orthogonal to T^a and parallely-propagated along γ. A Jacobi field X^a is a vector field defined at points of γ and satisfying the geodesic deviation equation. If we assume that X^a is orthogonal to T^a and expand it in the triad e^a_i then geodesic deviation is the equation

$$D^2X^i = \Phi^i_j X^j \tag{2.16}$$

where

$$X^a = X^i e^a_i;\text{ and } \Phi^j_i e^a_j = -R_{bcd}{}^a T^b T^d e^c_i. \tag{2.17}$$

We wish to consider simultaneously all Jacobi fields vanishing at a point p taken as $s = 0$. These can be represented by the columns of a matrix $A = (A^j_i)$ satisfying

$$D^2A = \Phi A \tag{2.18}$$

where we adopt a matrix notation and write $\Phi = (\Phi^j_i)$.

Introduce the matrices M and Σ and the scalar θ by

$$\begin{aligned} DA &= MA \\ M &= \Sigma + \frac{1}{3}\theta I \end{aligned} \tag{2.19}$$

where Σ is trace-free and I is the identity. Then (2.18) implies

TERMINOLOGY 2.9.1. the *Raychaudhuri equation*:

$$D\theta + \theta^2 + tr(\Sigma^2) = tr\Phi \tag{2.20}$$

and a (nameless) propagation equation for Σ:

$$D\Sigma + \frac{2}{3}\theta\Sigma + \Sigma^2 - \frac{1}{3}Itr(\Sigma^2) = \Phi - \frac{1}{3}Itr(\Phi) \tag{2.21}$$

Note that $tr(\Phi)$ in (2.20) is, by (2.17), equal to $-R_{ab}T^aT^b$ which is nonpositive if we have the time-like convergence condition, so $D\theta + \theta^2$ in (2.20) is non-positive. Now a point q is conjugate to a point p iff there is a (non-trivial) Jacobi field vanishing at p and at q. This will happen iff $\det A$ satisfying (2.18) vanishes at q, but from (2.19)

$$\theta = trM = D(\log \det A) \tag{2.22}$$

Thus q is conjugate to p iff θ is infinite at q. The idea is to prove from (2.20) and (2.21) that this is inevitable: by (2.20) θ will become infinite along γ if it once becomes negative, and by (2.21) if Φ is non-zero somewhere on γ, then that will produce Σ which will enter (2.20) to reduce θ. This can be made precise:

PROPOSITION 2.9.2. [**60**, Prop 4.4.2] *Given*

(i) *the time-like convergence condition;*
(ii) *the generic condition:* $R_{abcd}T^aT^c \neq 0$ *at some point of each time-like geodesic;*
(iii) *time-like geodesic completeness;*

then every time-like geodesic contains a pair of conjugate points.

A similar formalism can be developed for geodesic deviation along null geodesics with one slight difference: one concentrates on Jacobi fields representing infinitesimally-neighbouring geodesics 'abreast' of the fiducial one, which is to say lying in a null hypersurface with it. This entails that the matrix A in this case is 2×2 rather than 3×3. The proposition analogous to 2.9.2 can be proved:

PROPOSITION 2.9.3. [**60**, Prop 4.4.5] *Given*

(i) *the null convergence condition;*
(ii) *the generic condition:* $T_{[a}R_{b]ef[c}T_{d]}T^eT^f \neq 0$ *at some point of each null geodesic;*
(iii) *null geodesic completeness;*

every null geodesic contains a pair of conjugate points.

The role of the generic condition is to constrain the relevant term for the modification of (2.21).

The significance of conjugate points is their relation to maximising properties of geodesics. One has:

LEMMA 2.9.4. (i) *a time-like geodesic curve* γ *from* p *to* q *is maximal iff there is no point conjugate to* p *along* γ *in* (p, q)*;*
(ii) *if* p *and* q *lie on a null geodesic* γ *and there is a point* r *conjugate to* p *between them, then there is a time-like curve from* p *to* q.

As an application of (ii) used below, consider the boundary of the future of p, $\partial I^+(p)$; near p this is ruled by the null geodesics generating the null cone at p; if one of these generators meets a point r conjugate to p, then, by (ii), beyond r it lies inside $I^+(p)$ and no longer on $\partial I^+(p)$. This observation is the key ingredient in the proof of 2.12.1.

2.10. The Cauchy problem for general relativity. Here the problem is to express the Einstein equations as the evolution of something, and then to prove existence and uniqueness of solutions. The idea is to decompose tensorial quantities with respect to a foliation by hypersurfaces of constant 'time', t say, in the knowledge that the choice of this foliation usually has a great deal of arbitrariness in it. The variables are the first and second fundamental forms of the 3-surfaces of constant t, say h_{ij} and K_{ij}, where the indices are abstract but 3-dimensional, together with whatever matter variables are needed. One needs the Gauss and Codazzi equations to relate 3-dimensional and 4-dimensional tensors.

Suppose that the normal to the 3-surfaces is N^a and take the Einstein equations to be

$$G_{ab} = 8\pi T_{ab} \tag{2.23}$$

These decompose into 'constraints plus evolution'. The (time,time) component, using the Gauss equation twice-contracted, is

$$^3R + K^2 - K_{ij}K^{ij} = 2G_{ab}N^aN^b = 16\pi T_{ab}N^aN^b \tag{2.24}$$

where ^3R is the 3-dimensional scalar curvature and $K = h_{ij}K^{ij}$. This is known as *the Hamiltonian constraint.* The (time, space) component, using the Codazzi equation once contracted is

$$D_jK_i^j - D_iK = G_{ai}N^a = 8\pi T_{ai}N^a \tag{2.25}$$

where D_i is the intrinsic 3-dimensional Levi-Civita derivative. This is the *momentum constraint.* These are four constraints: they are conditions on the data which must hold at each time and so in particular must hold initially.

The (space,space) components are the evolution equations, determining the time-derivative of K_{ij}, equivalently the second derivative of h_{ij}. There will also be matter evolution equations, and possibly matter constraints too.

The equations will not be strictly hyperbolic until the diffeomorphism invariance (or coordinate freedom) has been constrained. One then needs to verify that all the constraints are preserved by the evolution, which usually follows from the contracted Bianchi identities.

Notice from the Hamiltonian constraint that, if a 3-surface is *maximal,* which is to say that the trace $K = h^{ij}K_{ij}$ is zero, then either the weak energy condition or the dominant energy condition implies that the 3-dimensional Ricci scalar is non-negative.

2.11. Definitions of mass and positive energy theorems. In §2.4, we mentioned the 'mass' of the Schwarzschild solution. How is this defined? Without going into details, let us note that there is a definition of mass 'at infinity' on asymptotically flat hypersurfaces in asymptotically flat space-times. This is the ADM mass and is, roughly speaking, read off from the $O(1/r)$ terms in the metric. In an analogous way, one can define a mass at any (topologically spherical) section (or *cut*) of $\mathcal{I}^+$ or $\mathcal{I}^-$ in a weakly asymptotically simple space-time. This is the Bondi mass, and it decreases as the cut is moved into the future on $\mathcal{I}^+$, or into the past on $\mathcal{I}^-$. In both these cases, the mass is more properly called the energy as it is the time-like component of a 4-vector at infinity, the total energy-momentum.

In a stationary space-time with a Killing vector K_a one may associate a mass with any 2-surface by the Komar integral

$$m = \frac{1}{4\pi}\int *dK \tag{2.26}$$

The integrand is closed given the Einstein vacuum equations. On a sphere at large distances the Komar integral gives the Bondi or ADM mass (which are equal in a stationary space-time). In a vacuum space-time containing one or more black holes, the Komar integral gives the formula:

$$M = \frac{1}{4\pi}\sum_i \kappa_i A_i \tag{2.27}$$

in terms of the individual surface gravities κ_i and areas A_i of the black holes (of course, one does not expect there to be multiple static vacuum black hole solutions, but the extension of this formula to charged, rotating holes is a significant resource).

There has been a great deal of work with the aim of defining a mass or energy-momentum vector to be associated with an arbitrary 2-surface in an arbitrary space-time [**59, 94, 5, 129**]. Usually such a mass is called 'quasi-local' since one does not expect it to be the integral of a local density over a spanning 3-surface - gravitational mass-energy is not a local quantity - but one does require that it be determined by geometrical quantities at the 2-surface.

Given one of the definitions of total energy-momentum, one can seek to prove that the vector is time-like given some local energy condition, and vanishes only in flat space. We call such a result a Positive Energy Theorem.

The first proof that the ADM momentum is time-like and vanishes only for flat space given the dominant energy condition and an asymptotically flat maximal space-like hypersurface diffeomorphic to $\mathbb{R}^3$ was given by [**112**]. In a sequence of extensions, they subsequently dropped the condition of maximality, allowed the hypersurface to have an inner boundary which was minimal, and extended the result to the Bondi mass [**113, 114**]. They use methods of Riemannian geometry applied to the data for the space-time on the maximal hypersurface: they show that nonpositive mass together with non-negative Ricci scalar (which follows from the Hamiltonian constraint) permit the existence of a particular kind of minimal surface, which in turn forces the data to be data for flat space.

Under the assumptions of the dominant energy condition and the existence of an asymptotically flat space-like hypersurface diffeomorphic to $\mathbb{R}^3$, the same result was proved by Witten in a very different way [**145**]. He uses a 2-component spinor field and an identity, quadratic in the spinor field, which relates a component of the ADM energy-momentum to an integral over the space-like hypersurface. This integral is manifestly non-negative if the spinor field satisfies a linear equation, a modification of the 3-dimensional Dirac equation generally known now as the Witten or Sen-Witten equation. The problem is therefore reduced to the existence theory for the Witten equation. This has been established, and the Witten-style proof has been extended to permit inner boundaries and to prove positivity of the Bondi energy [**86, 75, 54, 109, 61**].

There is a third approach to the positive energy theorem [**97**]. First we need a definition:

DEFINITION 2.11.1. For a weakly asymptotically simple space-time M, define the *domain of outer communications* $D = I^+(\mathcal{I}^-) \cap I^-(\mathcal{I}^+)$.

These authors prove:

PROPOSITION 2.11.2. *In a WAS space-time M, if D is globally hyperbolic and every null geodesic in D possesses a pair of conjugate points then the ADM energy-momentum is future-pointing.*

The idea is to exploit the result noted in 2.4.9: causal properties of the point i^0 are quite different if the ADM mass is positive or negative; for positive mass and any point $p \in \mathcal{I}^-$, all of $\mathcal{I}^+$ is contained in $I^+(p)$; for negative mass, this is not true and there is a $q \in \partial I^+(p) \cap \mathcal{I}^+$; in this case one then shows that there is a null geodesic γ from p to q lying in the boundary $\partial I^+(p)$; but γ contains a pair of conjugate points and so cannot remain on the boundary $\partial I^+(p)$ by 2.9.4 (ii) yielding a contradiction. The existence of conjugate points follows from 2.9.3 given the Einstein equations, an energy condition and the generic condition.

2.12. Singularity Theorems. We saw in §2.4 how the Schwarzschild singularity may be seen to 'form' in gravitational collapse to a black hole. It was at one time argued that the formation of singularities was a very special circumstance, attributable possibly to the high degree of symmetry in the Schwarzschild solution. This position changed after the first singularity theorem appeared.

PROPOSITION 2.12.1. [**87**] *The following conditions on a space-time M cannot hold simultaneously:*

(i) *M has a non-compact Cauchy surface S;*
(ii) *M contains a closed trapped surface T;*
(iii) *M is null geodesically complete;*
(iv) *the null convergence condition holds in M.*

This is a 'singularity theorem' to the extent that geodesic incompleteness is taken as the criterion of singularity. We sketch the proof: by (ii) the outgoing null geodesics orthogonal to T are converging at T; (iii) and (iv) then enforce the appearance of a point conjugate to T along each such geodesic by a version of the argument leading to 2.9.3; beyond this conjugate point, the geodesic is in the interior of $I^+(T)$ by a modification of 2.9.4; thus the boundary $\partial I^+(T)$ is compact; this is incompatible with (i) - to see this, choose a smooth time-like vector field on M and use the integral curves of it to map $\partial I^+(T)$ continuously into S, which is not compact.

This was the first of the *conjugate point arguments* which have been crucial in mathematical relativity. There have been many more singularity theorems proved under different assumptions, for example different energy conditions, dropping global hyperbolicity, allowing causality violations, allowing compact spatial sections. The proofs typically derive contradictions from the simultaneous existence of conjugate points and some geometric condition implying collapse.

3. Topological Issues; Topological Censorship

3.1. The Alexandrov topology. As observed in §1, in a space-time M the spacetime metric does not define a topological metric. One may seek instead to define the open sets of the manifold by causal properties. The Alexandrov topology is the one generated by open sets of the form $I^+(p) \cap I^-(q)$; when does it coincide with the manifold topology (which will always be assumed to be Hausdorff)?

PROPOSITION 3.1.1. [**89**] *The following are equivalent:*

(i) *M is strongly causal*
(ii) *the Alexandrov topology agrees with the manifold topology;*
(iii) *the Alexandrov topology is Hausdorff.*

Clearly some causal condition is needed, and strong causality turns out to be the right one.

3.2. Compact Lorentzian manifolds. Historically, relativists have not been much interested in compact space-times. There are several reasons for this:

PROPOSITION 3.2.1. [**8**] *Any compact M contains closed time-like curves.*

PROOF. Take an open cover of M by sets $I^+(p)$ and contemplate a finite subcover. □

Next:

PROPOSITION 3.2.2. [**8**] *The 4-manifold M admits a Lorentzian metric iff M admits an everywhere time-like direction field. If M is compact this happens iff the Euler characteristic is zero, so in particular would imply that M is not simply-connected.*

Hawking and Ellis [**60**] interpret 3.2.2 as meaning that a compact spacetime is 'really' a non-compact space-time with identifications. Against this view is Tipler's 'No-return' theorem [**125**]: call a space-time M with a Cauchy surface S *time-periodic* if M admits an infinite cyclic group of isometries $G = \{\theta_i | i \in Z\}$ with $\theta_i(S) \cap \theta_j(S) = \emptyset$ for all i, j. Then:

PROPOSITION 3.2.3. *If M admits a compact Cauchy surface S and the generic and time-like convergence conditions hold in M then M cannot be time-periodic.*

PROOF. Note first that the generic and convergence conditions imply the existence of conjugate points on time-like geodesics; now one connects copies S_i and S_j of the Cauchy surface under the isometry by maximising time-like geodesics; take a limit, then the limit geodesic has conjugate points which contradicts the maximality. □

From Tipler's no-return theorem, [**82**] deduces another pathology of compact space-times:

PROPOSITION 3.2.4. *If M is compact and satisfies the null and time-like convergence and generic conditions, then M cannot admit a closed, embedded, edgeless, space-like hypersurface.*

The proof shows that, if it did, 3.2.3 would be violated in a suitable covering space.

3.3. Topology change. The idea that space-like hypersurfaces might have nontrivial topology which, furthermore, might change with time has long interested relativists. Typically, though, there are problems with topology change:

PROPOSITION 3.3.1. [**48**] *If M is globally hyperbolic then M admits a Cauchy surface S and M is homeomorphic (in fact diffeomorphic) to $\mathbb{R} \times S$.*

Thus the topology cannot change with time if M is globally hyperbolic. In the absence of conditions, however, topology can change:

PROPOSITION 3.3.2. [**103, 46**] *Any two compact (not necessarily connected) 3-manifolds S and S' are* Lorentz cobordant*: there is a compact M, whose boundary is the disjoint union $S \coprod S'$, and which admits a Lorentzian metric in which S and S' are space-like.*

But causality is necessarily violated if topology does change:

PROPOSITION 3.3.3. [**46**] *With M, S and S' as in 3.3.2, if M is time-oriented and contains no closed time-like curves then S and S' are diffeomorphic.*

The idea for 3.3.3 is to use the time-like direction field which M admits (by 3.2.2) to map S to S'.

Even giving up causality is not enough:

PROPOSITION 3.3.4. [**123, 124**] *With M, S and S' as in 3.3.2, if the null convergence and null generic conditions hold in M then S and S' are diffeomorphic and M is $\mathbb{R} \times S$.*

A different kind of difficulty with topology change was found by Gibbons and Hawking [**55**]. This is the problem of defining spinors on a topology changing space-time.

PROPOSITION 3.3.5. *There is a mod 2 invariant $u(S)$ of 3-manifolds such that, with M, S and S' as in 3.3.2, M will admit $SL(2,\mathbb{C})$ spinors iff $u(S) = u(S')$. Here $u(S)$ is the Kervaire invariant:*

$$u(S) = \dim_{\mathbb{Z}_2}(H_0(S;\mathbb{Z}_2) \oplus H_1(S;\mathbb{Z}_2)) \mod 2$$

Thus, for example, a Lorentzian metric can be defined on the topology-changing space-time M with $S = S^3$ and $S' = S^3 \amalg S^3$, but M will not admit spinors. Gibbons and Hawking argue that failing to admit spinors is a more serious defect in a space-time than having closed time-like curves.

3.4. Obstructions to spatial topology. Given that it is difficult to change spatial topology, are there are obstructions to having it at all? The answer is, "No, but ... "

PROPOSITION 3.4.1. [**144**] *Every closed 3-manifold occurs as a space-like hypersurface in a vacuum space-time; every closed 3-manifold minus a point occurs as an asymtotically flat initial data set for a vacuum space-time.*

The proof is by an explicit construction of a solution of the constraints for the vacuum field equations exhibited in §2.10.

However, if one seeks to impose the extra condition that the hypersurface is maximal then there is a problem: the Hamiltonian constraint implies that the (3-dimensional) Ricci scalar is positive. Thus:

PROPOSITION 3.4.2. [**144**] *Any closed oriented 3-manifold with a $K(\pi, 1)$ as a prime factor admits no metric with $R > 0$ and only flat metrics with $R \geq 0$; thus there are many space-times (vacuum or with matter satisfying an energy condition) with no maximal slice.*

A simple explicit example of an asymptotically flat space-time containing no maximal surface due to Brill [**12**] contains an asymptotically flat space-like hypersurface which is topologically T^3 minus a point. It is constructed by joining part of the Schwarzschild solution to a piece of the $k = 0$ dust-filled FRW universe across a collapsing sphere, and then identifying the FRW part to a torus. This example in turn has been generalised by Bartnik [**4**] to give a space-time with spatial topology $T^3 \# T^3$ which admits no space-like hypersurface of constant mean curvature for any value of the constant.

There is current interest in the existence of foliations by constant-mean-curvature or CMC hypersurfaces: see §5. There are at present no examples in the literature of vacuum space-times which admit no CMC hypersurfaces.

3.5. Topological censorship. We met cosmic censorship in §2.4 and will meet it again in §7. Topological censorship [**35**] is a related idea, that an asymptotically flat space-time may well have complicated topology close in, but this fact cannot be communicated to large distances. The starting point is the singularity theorem of Gannon [**45**], which needs a definition:

DEFINITION 3.5.1. A space-like hypersurface S in an asymptotically flat space is *regular near infinity* if it satisfies the following three conditions:

(i) $S = \bigcup_{i=1}^{\infty} W_i$, $W_i \subset W_{i+1}$ and each W_i is is a compact 3-manifold with boundary homeomorphic to a 2-sphere;
(ii) $S - \mathrm{int} W_i$ is homeomorphic to $\partial W_i \times \mathbb{R}^+$;
(iii) the ingoing null geodesics normal to ∂W_i are converging everywhere on ∂W_i.

Note that (iii) is what you would expect on a large 2-sphere — this is not a 'trapped' condition. Then:

PROPOSITION 3.5.2. [**45**] *If a space-time M admits a Cauchy surface which is regular near infinity and not simply-connected, and if the time-like convergence condition is satisfied in M, then M is not null geodesically complete.*

PROOF. The idea is to consider, in the universal covering space $\bar{M}$ of M, a copy A of one of the large spheres ∂W_i lying on a copy $\bar{S}$ of S; the ingoing null geodesics normal to A define a submanifold N which is part of the boundary $\partial J^+(A)$; by the argument in 2.12.1 they leave the boundary after passing conjugate points if they are complete, so that N is compact and $A = \partial N$; now a time-like direction-field maps N down to S, but A cannot bound a compact 3-manifold in S. □

Topological censorship deals with a weakly asymptotically simple space-time M and causal curves from $\mathcal{I}^-$ to $\mathcal{I}^+$. Let γ be such a curve which lies in a simply-connected neighbourhood of $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$.

PROPOSITION 3.5.3. [**35**] *If M is WAS and globally hyperbolic and the null convergence condition holds in M then every causal curve from $\mathcal{I}^-$ to $\mathcal{I}^+$ is homotopic to γ.*

The idea is that, if Γ is a causal curve from $\mathcal{I}^-$ to $\mathcal{I}^+$ not homotopic to γ then, in the universal cover of M, Γ connects different asymptotic regions; to do this Γ must pass through a trapped surface T say on its way to $\mathcal{I}^+$; one derives a contradiction from a conjugate point argument applied to a null geodesic generator of the boundary of the future of T, $\partial\mathcal{I}^+(T)$, which meets $\mathcal{I}^+$.

The interpretation of 3.5.3 is that topological complexity close in in an asymptotically flat space-time satisfying an energy condition collapses 'too fast' for an observer outside to probe the topology, and in particular therefore, too fast for the observer to pass through any wormholes and escape safely. A result equivalent to 3.5.3 due to Galloway is:

PROPOSITION 3.5.4. [**41**] *If M is WAS, the null convergence condition holds in M, and the domain of outer communication $D = I^+(\mathcal{I}^-) \cap I^-(\mathcal{I}^+)$ is globally hyperbolic, then D is simply connected.*

In this form, the result will be seen to be relevant to the study of black holes in §6.

Finally, there is a version of topological censorship due to Galloway and Woolgar [**44**] which drops the condition of global hyperbolicity, replacing it with a form of cosmic censorship and a causal condition at i^0.

A testing example of a *traversable wormhole* was provided by Schein and Aichelburg [**111**]. Their electrovac solution can be interpreted as an exterior consisting of a 2-body Majumdar-Papapetrou solution containing two topologically-spherical charged shells, joined across the shells to an interior consisting of part of the extended Reissner-Nordstrom solution; the trick is that the two shells are in two different asymptotic regions in the Reissner-Nordstrom solution, one later than the other. The matching is done without violating energy conditions. Now it is possible to follow a causal curve through one shell at a certain time t_0, move forward in time in the Reissner-Nordstrom part but re-emerge into the Majumdar-Papapetrou exterior from the second shell at a time earlier than t_0: there are closed time-like curves through every point of the space-time; the wormhole is traversable but the energy conditions are not violated.

3.6. Signature change. Signature change, while not a topological issue, is related to the idea of topology change. The question is can the Einstein equations have solutions in which the signature of the metric changes from Riemannian to Lorentzian or vice-versa? The motivation for considering the possibility has come from the Hartle-Hawking 'No-boundary' proposal in quantum gravity [**58**]. There is a need for care because the metric must degenerate to change signature.

Gibbons and Hartle [**53**] consider the general theory, showing that the signature can only change across an umbilic (equivalently, a totally geodesic) space-like hypersurface S. Then the Hamiltonian constraint again constrains the topology of S as in 3.4.2. Ellis *et al* [**31**] present some explicit solutions of the Einstein equations with matter which do change signature.

4. Lorentzian Splitting Theorems; Related Matters

4.1. Yau's question. For this we first need a definition:

DEFINITION 4.1.1. A time-like *line* is an inextendible time-like geodesic which maximises the distance between any two of its points.

Yau [**147**] posed the problem, slightly rephrased here, of proving that a geodesically complete space-time M in which the time-like convergence condition holds and which contains a time-like line is isometrically the product of the line and a space-like hypersurface. This was proposed as an analogue of the Cheeger-Gromoll splitting theorem in Riemannian geometry.

The Lorentzian Splitting Theorem in this form was proved by Eschenburg [**32**], with the extra assumption that M is globally hyperbolic, by a modification of the Riemannian proof. Galloway [**40**] proved the theorem with the assumption of global hyperbolicity but dropping the assumption of time-like geodesic completeness. Then Newman [**83**] proved the theorem precisely in Yau's form, with the assumption of time-like geodesic completeness and without the assumption of global hyperbolicity. (See [**9**] for a more detailed account of this history.)

4.2. Geroch's suggestion. A related set of ideas is associated with the suggestion of Geroch [**46, 49**] that most closed universes should be flat or become singular. This was interpreted by Galloway and Horta [**43**] as 'spatially closed space-times should fail to be flat only under exceptional circumstances'. Geroch supported his contention with a singularity theorem, which we give in a modified form due to Bartnik [**4**]:

PROPOSITION 4.2.1. *Suppose the time-orientable space-time M has a compact Cauchy surface S and that the time-like convergence condition holds in M; suppose that there is at least one point $p \in S$ with no horizon in the sense that $M - (I^+(p) \cup I^-(p))$ is compact; then M is time-like geodesic incomplete or splits as a metric product.*

The 'no-horizon' condition means roughly that every observer can exchange communications with p. The idea of the proof is to move the surface S until it has everywhere negative or everywhere zero expansion; then use a conjugate point argument on the geodesic normals to S to prove incompleteness, or find a time-like line. Geroch assumed a stronger form of time-like convergence, namely that $R_{ab}t^a t^b \geq 0$ for all time-like t^a, and $R_{ab}t^a t^b = 0$ for some time-like t^a only if $R_{ab} = 0$. With this, the split case is actually flat.

For this section only, and following [**4**], call a space-time 'cosmological' if it is globally hyperbolic with a compact Cauchy surface and satisfies the time-like convergence condition. Then Bartnik [**4**] further conjectures that:

CONJECTURE 4.2.2. Any cosmological space-time is time-like geodesically incomplete or splits as a metric product.

One approach to this would be to find a maximal surface and use a conjugate point argument to prove incompleteness. Another would be to prove that a time-like line exists. The difficulty with the second strategy is that one can seek to construct the line as a limit, only to have the limit become null, a Lorentzian difficulty not existing in Riemannian geometry. With extra assumptions, the second route has been successfully followed by Eschenburg and Galloway [**33, 42**].

A related splitting theorem, due to Andersson *et al* [**2**], is concerned with warped products. They show that a globally hyperbolic space-time satisfying an energy condition with a negative cosmological constant (positive with their conventions) and having a finite but long enough time-like line is a warped product. With some more assumptions, they characterise anti-de Sitter space by this route.

5. Existence and Uniqueness

Questions of existence and uniqueness for the Einstein equations split into problems with the constraints, which are usually elliptic, and problems with evolution, which are hyperbolic. Solution of the constraints on constant-mean-curvature hypersurfaces, either compact, asymptotically flat or asymptotically hyperbolic, is well understood. Local-in-time solution of the evolution equations is also well understood. References to this material can be found in the 'Living review' *Existence theorems for the Einstein equations* by A.D.Rendall at http://www.aei-potsdam.mpg.de. For an earlier review, see [**34**]; for the situation with matter, see [**108**].

The big question now is global or long-time existence. In cosmological solutions, one expects initial and sometimes final singularities to form; in asymptotically flat solutions one expects gravitational collapse to be possible, resulting in singularities. Thus there is often no expectation that a solution obtained from Cauchy data will exist forever. Rather one hopes to investigate and perhaps constrain the kinds of singularities that are formed, and to prove existence up to the singularity. In particular, one would like to know whether Cauchy horizons ever arise in an evolution, or equivalently whether the maximal development of a set of data is globally hyperbolic; cf. e.g. [**20**]. Here we are getting close to cosmic censorship, discussed below in §7.

Results on long-time existence can be classified by the amount of symmetry a solution possesses:

5.1. Spatially-homogeneous cosmologies. These have isometry group transitive on space-like hypersurfaces (they are 'cohomogeneity-one') so the Einstein equations reduce to a system of ordinary differential equations. This system can often be solved; cf. e.g. [**135**]. Rendall [**105**] gives existence theorems for some symmetry types and matter models, which expand forever from an initial singularity, or expand and recollapse, with no Cauchy horizons. This is a 'large-data, long-time' theorem.

5.2. 1+1 reductions. Reductions with two commuting space-like symmetries lead to partial differential equations with one time and one space variable. These include Einstein-Rosen cylindrically-symmetric gravitational waves [**10, 146**] and Gowdy vacuum cosmologies [**80, 19, 24**], where the group orbits are compact . In both cases there are large-data, long-time existence theorems.

5.3. Spherical symmetry. Here collapse is possible. In a long series of papers, Christodoulou has investigated spherically-symmetric solutions with scalar-fields, which also lead to (1+1)-pdes; see [**16**] for references. He has produced a very complete picture, reviewed by Wald [**138**] and described at greater length in 7.2.3. From sufficiently small, asymptotically flat initial data, solutions last forever with a complete $\mathcal{I}^+$.

Rein et al. [**102**] established long-time existence for spherically-symmetric and other (1+1)-solutions of the Einstein-Vlasov equations, again for small data.

5.4. No symmetry. Christodoulou and Klainerman [**18**] have proved the global existence of solutions to the vacuum equations with data close to flat on an asymptotically flat initial hypersurface (see the chapter by Christodoulou in this volume). They find that, with generic (small) asymptotically flat data the conformal structure is not smooth at $\mathcal{I}^+$.

Friedrich, in a long series of papers, has studied the vacuum equations with cosmological constant. We noted in 2.7.3 that the nature of $\mathcal{I}^+$ depends on the sign of the cosmological constant (though note that Friedrich's conventions have the opposite sign for the cosmological constant). For positive cosmological constant, (negative with his conventions) Friedrich [**36**] shows that, with data on S^3 close to the data for de Sitter space, the solution exists globally, and is asymptotically simple. For negative cosmological constant (positive with his conventions), he poses an initial-boundary-value problem, with boundary data on a finite interval of the (time-like) $\mathcal{I}$, and initial data on a ball and proves existence of an asymptotically simple space-time generalising the anti-deSitter metric, with no assumption of smallness. In vacuum, he first considers 'hyperboloidal initial data' which is data on an asymptotically hyperbolic surface S spanning a cut of $\mathcal{I}^+$ [**36, 1**]. He proves

[**36**] that the solution exists to the future of S and that the smoothness of $\mathcal{I}^+$ is preserved by the evolution. Most recently [**38**], he considers the case of data for the vacuum equations on an asymptotically flat hypersurface S. What one wants to know here is what, if any, conditions on the data lead to hyperboloidal data, or equivalently to a smooth $\mathcal{I}^+$. This requires an intricate analysis of the geometry near i^0. Friedrich has a necessary condition on the data for the evolution to admit a smooth $\mathcal{I}^+$, but it is not yet known whether the condition is sufficient. For more details, see [**39**].

It is worth remarking that one knows already from the study of Maxwell's equations in flat space that, to obtain a solution which is smooth at $\mathcal{I}^+$, one needs to impose conditions on the data on an asymptotically flat hypersurface which are stronger than naive asymptotic flatness: crudely speaking with increasing k each 2^k-pole must fall off at a faster rate in r; equivalently the solution must be smooth at i^0. It seems reasonable that one should expect something similar in the vacuum equations.

There do exist radiating electrovac solutions which are smooth at $\mathcal{I}^+$. Cutler and Wald [**27**] give a spherically-symmetric solution of the constraints for an electrovac (actually 'magnetovac') solution. The data is asymptotically flat, and in fact is data for the Schwarzschild solution outside of a certain radius. These authors are able to show that the evolution therefore leads to hyperboloidal data which by Friedrich [**36**] evolves to have a complete $\mathcal{I}^+$.

5.5. Isotropic singularities. Another class of cosmological space-times where the solution is known 'up to' the singularity is the cosmologies with an isotropic singularity [**56, 130, 84**]. In fact, these are the other way round: data is given at the singularity, then local existence is proved; the data is unconstrained, but less data can be given than at a finite surface. Existence and uniqueness has been proved for some perfect fluid matter models [**26, 3**] and for the spatially homogeneous massless Einstein-Vlasov equations [**3**].

5.6. CMC foliations. Under certain circumstances, a space-time will admit a foliation by constant-mean-curvature space-like hypersurfaces, one for each value of the mean curvature in some range (e.g. the range $(-\infty, +\infty)$ in the $k = 1$ FRW solutions; the range $(-\infty, 0)$ in the $k = 0$ FRW solution). The value of the mean curvature is then a useful time coordinate, and 'global in CMC-time' can be thought of as the canonical existence problem. For spatially compact space-times, and given the right energy condition, such a foliation is unique if it exists [**77**]; see also [**6**] for the asymptotically flat case, and [**30**] for application to numerical relativity. We saw in 3.4.2 that there are space-times without such a foliation. There are examples where only part of the space-time is covered, and cases where the existence of the foliation is assured. See [**107**] for a recent review.

One may also seek foliations of a space-like hypersurface by constant mean curvature 2-surfaces (in fact 2-spheres). This is a wholly Riemannian problem. Huisken and Yau [**66**] use a mean curvature flow and the positive energy theorem to prove that a unique stable foliation exists in a neighbourhood of infinity on an asymptotically flat space-like hypersurface. The spheres approach a family of Euclidean spheres at large distances, all with the same centre which these authors interpret as a 'centre of mass'. See [**148**] for a similar result.

6. Black-Hole Uniqueness

6.1. The problem. A time-independent, asymptotically flat but not flat, vacuum space-time cannot be everywhere non-singular [**74**]; cf. [**21**]. Roughly speaking, the (non-zero) gravitational field needs a source. However, as we saw in the example of the Schwarzschild metric in §2.4, it can be non-singular everywhere outside a horizon. The problem of black hole uniqueness is the problem of first finding all time-independent solutions of the Einstein field equations which are asymptotically flat outside a horizon, and then showing that these are indeed all. The field equations are allowed to have one of a small number of matter sources corresponding to various fields. (This is hard to make precise and indeed the rules of the game evolve: the idea is that there are no sources in the sense of non-zero T_{ab} outside the hole except for fields generated by 'charges' attributable to the hole.)

SPACE-LORE 6.1.1 (Classical Results). The first results were Israel's characterisations [**68, 69**] of the Schwarzschild (respectively, Reissner-Nordstrom) solutions as the only static vacuum (respectively, electrovac) black holes. Then, at the end of a long chain of results, with contributions from Carter, Hawking, Robinson, Bunting and Mazur, the Kerr and Kerr-Newman solutions were characterised as the corresponding stationary black holes. A sequence of arguments shows first that one need only consider stationary, axisymmetric metrics with a single, topologically spherical hole, and then that the system of non-linear PDEs to which the Einstein equations reduce has the corresponding unique solutions, depending on a small number of constants. References can be found in the excellent recent monograph of Heusler [**63**].

SPACE-LORE 6.1.2 (No Hair). The Kerr-Newman solution depends on three constants, interpretable as the mass, charge and specific angular momentum. The black hole uniqueness theorem is often aphoristically stated as 'a black hole has no hair' [**79**], being characterised uniquely by its values of these constants.

There has been a resurgence of interest in black hole uniqueness. This is partly with the aim of proving the uniqueness theorems under weaker conditions, or of proving stronger theorems, and partly because new solutions have been found with other fields.

Under the first heading, see the critical account due to Chruściel [**21**]. According to this author, weak links in the proof of the theorem as it stood at his time of writing included:

- the proof that stationary black holes are topologically spherical; the horizon is a null hypersurface, but one thinks of the black hole as being a space-like cross-section of the horizon and so 2- dimensional;
- the proof that stationary black holes are axisymmetric if not static: this requires the metric to be analytic on the horizon; it also uses a physical argument about ergoregions;
- the assumption that the black holes are connected — i.e. that there is only one hole.

We first consider progress in these areas. We recall the definition of domain of outer communication (DOC) from 2.11.1, then a recent result due to Galloway [**41**] met in 3.5.4 is the following:

PROPOSITION 6.1.3. *If in an asymptotically flat space-time M the null convergence condition is satisfied and the DOC is globally hyperbolic, then the DOC is simply connected.*

Galloway notes that this is equivalent to the Friedman-Schleich-Witt topological censorship theorem 3.5.3 (the proof is similar; note that there is no assumption of stationarity). In the context of this section, 6.1.3 shows that all black holes are topologically spherical.

EXAMPLE 6.1.4. Chrusciel and Galloway [**22**] show by an example that a Cauchy horizon can be nowhere differentiable. Analyticity for stationary black holes is proved where the Killing vector is time-like so that the Einstein equations become elliptic. However, at the horizon no Killing vector is time-like so there is a real question of whether elliptic regularity holds 'up to the boundary'.

6.2. Multiple static black holes. The example of the Majumdar-Papapetrou solutions shows that time-independent solutions corresponding to multiple black holes are possible, that is the horizon need not be connected. The physical explanation of the Majumdar-Papapetrou solutions is that electrostatic repulsion balances gravitational attraction, so that this should not happen with vacuum solutions. Bunting and Masood-ul-Alam [**14**] show that, indeed, in the vacuum case there cannot be multiple, static black holes with all components non-degenerate. The proof, which is extremely elegant, is an application of the positive mass theorem with black holes; cf. e.g. [**54, 61**].

Ruback [**110**] extended their work to show that, in the electrovac case, there could not be multiple black holes with every horizon non-degenerate. Here the proof uses a positive energy theorem for charged black holes [**54**].

However, the Majumdar-Papapetrou black holes have all components degenerate. Heusler [**64**] shows that, if all components are degenerate and the charges of all holes have the same sign, then a multiple-black-hole static electrovac solution is necessarily in the Majumdar-Papapetrou family. There are still open questions about mixtures of degenerate and non-degenerate holes.

6.3. Multiple stationary black holes. One may ask if there exist solutions corresponding to multiple rotating black holes. The physical idea would be that there is a spin-spin repulsion which could balance gravitational attraction. There are 2-body solutions in the literature [**28, 65**] but the solutions are extremely complicated and hard to analyse.

In a series of papers, Weinstein [**139, 140, 141, 142**] has analysed the problem of multiple, rotating black holes. The solutions are stationary and axisymmetric, with the holes strung out along the axis. He shows that, for each n, there is a $4n - 1$ parameter family of solutions containing n charged, rotating holes, where the parameters are related to mass, charge, and angular momentum of the n holes and their $n - 1$ separations. The solutions are asymptotically flat and regular everywhere except that there may be conical singularities on the axis segments. Physically, the idea is that 'rods' or 'struts' may be needed on the axis to keep the black holes apart. It is not yet known whether the axis can be regular for suitable parameter values. The black holes are all non-degenerate.

6.4. Yang-Mills fields and other sources. One may seek black hole solutions in theories with various other matter fields. Very often, one finds that the

Kerr or Schwarzschild solutions are still the only regular black holes. See [**63**] for this with various scalar field and harmonic map sources.

The situation is different when the source is the Yang-Mills field. Bartnik and McKinnon [**7**] described a numerical study which found spherically symmetric, asymptotically flat, static solutions of the Einstein-Yang-Mills equations with a regular centre. This paper generated a great deal of excitement. Spherical solutions with black holes were subsequently found numerically by Bizon [**11**] and Volkov and Galtsov [**133**], and a countably infinite family of solutions describing spherical black holes was found numerically by Künzle and Masood-ul-Alam [**73**].

Proofs that the solutions really do exist were given by Smoller *et al* [**120**], for the solutions with a regular centre, by Smoller and Wasserman [**118**] for a countably infinite family of solutions with a regular centre, and by Smoller, Wasserman and Yau [**121**] for black hole solutions. The problem is to show that solutions exist to the boundary-value problem for the coupled non-linear ODEs which the field equations reduce to. The infinite families are associated with a winding number.

Smoller and Wasserman [**119**] showed that the extreme Reissner-Nordstrom metric is the unique degenerate black hole among the $SU(2)$-Yang-Mills solutions. Brodbeck and Straumann [**13**] have shown that both the solutions with a regular centre and the black holes are unstable. This indicates that the solutions are probably not significant physically.

Other black hole solutions with matter sources related to the Yang-Mills field continue to appear in the literature, and may be found at the gr-qc archive.

7. Cosmic Censorship

7.1. Terminology. The term is due to Penrose [**88**]: 'Does there exist a "cosmic censor" who forbids the appearance of naked singularities, clothing each one in an absolute event horizon?'. This would now be called the weak cosmic censorship hypothesis, that singularities will form in gravitational collapse, but they will be hidden behind horizons, while the strong cosmic censorship hypothesis is the suggestion that space-time is globally hyperbolic. This distinction is also due to Penrose [**92**]; for example, weak cosmic censorship would allow time-like singularities to form inside horizons while strong cosmic censorship would not allow them anywhere.

7.2. Counterexamples. Attempts to disprove the cosmic censorship hypothesis (CCH) inevitably centre on finding counterexamples. A counterexample would be an evolution of regular data which results in an asymptotically flat space-time with a singularity in $J^-(\mathcal{I}^+)$. So far, proposed counterexamples have served mostly to hone the 'correct' statement of the CCH (a process which the unsympathetic may regard as moving the goal-posts).

CONDITION 7.2.1. The notion of a '*tame*' matter model has emerged [**108, 81**]. Suppose one has a putative counterexample to the CCH with some matter model; for this to be a real threat to the CCH the matter model should be 'tame' in the sense that it would not lead to the same kind of singularities without gravity - it is not reasonable to expect general relativity to remove a pathology from a matter model which produces singularities already in special relativity. Thus the shell-crossing singularities of the first counterexamples [**149, 150**] occur already with perfect fluids in flat-space: perfect fluids are not tame.

EXAMPLE 7.2.2. The Einstein-Vlasov equations are tame, because the Vlasov equation is linear, and also because the Newtonian limit has long-time existence **[98]**; cf. **[101]**. Shell-crossing singularities cannot occur in the spherically-symmetric Einstein-Vlasov equations **[102]**, although shell-focussing singularities may arise **[30]**. Shapiro and Teukolsky **[117]** have presented numerical evidence of a violation of the weak CCH with solutions of the Einstein-Vlasov equations. However, it is difficult to be certain that the CCH is violated in their examples, and their initial distribution function is non-smooth; cf. **[104]**.

EXAMPLE 7.2.3. Scalar fields are tame.

Christodoulou has investigated collapsing, spherically symmetric, massless scalar field configurations in a long and ongoing series of papers; for the references, see **[16]**. He gives data on a future light cone, centred at the origin, and shows **[16]** that there are choices of asymptotically flat initial data which evolve to solutions with a naked singularity. The singularity forms first at the origin and then propagates out to $\mathcal{I}^+$ along a singular null cone arriving at a finite (retarded) time. In a recent preprint **[17]**, he obtains a very complete picture according to which one of three things happens:

(i) long-time existence with a complete $\mathcal{I}^+$;
(ii) a singularity forms, surrounded by a horizon, and again $\mathcal{I}^+$ is complete;
(iii) neither of the above;

and (iii) includes naked singularities, but the third case is non-generic: Christodoulou exhibits an arbitrarily small perturbation of the data converting (iii) to (ii).

7.3. Evidence for the CCH.

EXHIBIT 7.3.1 (stability of black holes). If the time-independent black hole solutions were unstable, then they could not be the (stable) endpoint of collapse and it is hard to see how the weak CCH could be true. However first the Schwarzschild solution **[134, 99, 72]** and later the Kerr solution **[143]** have been shown to be (linearly) stable.

EXHIBIT 7.3.2 (Existence Theorems). In addition to the work of Christodoulou described in 7.2.3, there are other existence theorems supporting various aspects of the CCH. Much of what is described in §5 can be interpreted in this light: for example, Christodoulou and Klainerman **[18]** prove the CCH for small data and vacuum, Friedrich **[36, 37]** proves it for vacuum plus cosmological constant and small data. Strong cosmic censorship is the claim that the maximal evolution of Cauchy data is a globally hyperbolic space-time, possibly with singularities but with no Cauchy horizons, probably with a requirement that the data be 'generic'. Proofs of strong cosmic censorship have been given in some restricted cases **[25, 105]**. Note that Cauchy horizons can exist in solutions of the Einstein equations (e.g. in the Reissner-Nordstrom solution; see figure 4) but the expectation is that they are necessarily unstable, **[57]**, and therefore non-generic.

EXHIBIT 7.3.3 (Area Theorem for Black Holes). For the next pieces of evidence, we first need the following result **[60**, Prop 9.2.7]: Given the null convergence condition in a WAS, black-hole space-time M in which the weak CCH holds in the sense that $\mathcal{I}$ is in the domain of dependence $D(S)$ of a Cauchy surface S, then the area of any connected space-like cross-section of the event horizon $\partial J^-(\mathcal{I}^+)$ increases into the future.

The proof uses a conjugate point argument: if the area starts to decrease then it goes to zero in finite time and a naked singularity will appear. Chrusciel and Galloway [**23**] have emphasised that the present proof also assumes smoothness of the horizon.

Now one may attempt to devise gedanken experiments which reduce the area of a black hole and therefore violate the weak CCH, for example by firing in charged [**136**], dyonic [**116**] or spinning [**136, 126**] particles. The details of the particle trajectories turn out to foil these attempts.

7.4. The Penrose inequality.

EXHIBIT 7.4.1 (The Inequality). There is a whole cycle of ideas around this prediction of the CCH. The weak CCH, together with the null convergence condition, implies via the area theorem 7.3.2 an inequality between mass and area of a black hole: suppose a black hole forms in a gravitational collapse, and then settles down to a stationary or static one; in the process its area A will increase, but its Bondi mass m will decrease; when it has settled down, black hole uniqueness tells us that it will be a Kerr or Kerr-Newman solution and these quantities will satisfy the following inequality, which can readily be seen to be true for the Kerr family:

$$A \leq 16\pi m^2 \tag{7.1}$$

Thus this inequality must be true at all earlier times too.

In fact there is a whole range of inequalities like (7.1) in the literature where A may be the area of a trapped or marginally-trapped surface and m may be the Bondi, ADM or even quasi-local mass. Not all of these proposed inequalities are strictly speaking predictions of the CCH.

EXHIBIT 7.4.2 (Special Cases). Ludvigsen and Vickers [**75**], Tod [**127, 128**], Jezierski [**71**] and Malec and O'Murchadha [**76**] have proved versions of the inequality under different assumptions. Herzlich [**62**] proves something very like (7.1) using a modification of the Witten positive energy theorem on an umbilic space-like hypersurface S (as we saw in §2.4, a marginally-trapped surface is then a minimal surface; A is the area of the minimal surface and m is the ADM mass. Herzlich's inequality has a different constant from (7.1), related to a Sobolev constant).

Gibbons [**52**] recently completed a programme started by himself [**51**] and Penrose [**90**] to prove (7.1) when A is the area of a marginally trapped surface and m is Bondi mass at $\mathcal{I}^-$ in an idealised model of gravitational collapse. Here (7.1) is implied by other geometric inequalities, like Minkowski's inequality for convex bodies as recently generalised by Trudinger [**132**].

Huisken and Ilmanen [**67**] prove (7.1) in the same setting as Herzlich; they show that an inverse mean curvature flow proposed with the aim of proving the positive energy theorem by Geroch [**50**] and modified to prove (7.1) by Jang and Wald [**70**] does indeed work, evolving in from infinity to the (outermost) minimal surface.

EXHIBIT 7.4.3 (Converse). There are physical reasons for hoping for a converse to (7.1): that a black hole will form if matter of mass m is squeezed into a small region, for example one enclosed by area A. This is is hard to make precise. There is a result of this form due to Schoen and Yau [**115**], modified by O'Murchadha [**85**]. Rather than area, they have a sophisticated measure of the size of a region in terms of the largest torus which it can contain. The proof uses minimal surface techniques.

CONJECTURE 7.4.4 (Hoop Conjecture). Related to the Penrose inequality and its converse is the *hoop conjecture* of Thorne [**122**]. The idea is that a black hole will form when and only when matter of mass m is squeezed into a region whose every circumference C satisfies

$$C \leq 4\pi m \tag{7.2}$$

Part of the idea here is that there could be a collapse of a long, thin object to a singularity without the formation of a horizon, therefore with a violation of the CCH. The study of Shapiro and Teukolsky [**117**] was presented as just such a violation of the CCH and a vindication of the hoop conjecture. Note [**131**] that there are real difficulties with making (7.2) precise and some formulations of it are false.

References

[1] L. Andersson, P.T. Chruściel and H. Friedrich, *On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations.* Comm. Math. Phys. **149** (1992) 587–612.

[2] L. Andersson, G.J. Galloway and R. Howard, *A strong maximum principle for weak solutions of quasi-linear elliptic equations with applications to Lorentzian and Riemannian geometry.* Comm. Pure. App. Math. **51** (1998) 581–624.

[3] K. Anguige, **Isotropic cosmological singularities**. D.Phil. thesis, Oxford University, 1997.

[4] R. Bartnik, *Remarks on cosmological space-times and constant mean curvature hypersurfaces.* Comm. Math. Phys. **117** (1988) 615–624.

[5] R. Bartnik, New definition of quasi-local mass. Phys. Rev. Lett. **62** (1989) 2346

[6] R. Bartnik, P.T. Chruściel and N. O'Murchadha, On maximal surfaces in asymptotically flat space-times. Comm. Math. Phys. **130** (1990) 95–109.

[7] R. Bartnik and J. McKinnon, *Particle-like solutions of the Einstein-Yang-Mills equations.* Phys. Rev. Lett. **61** (1988) 141–143.

[8] R.W. Bass and L. Witten, *Remark on cosmological models.* Rev. Mod. Phys. **29** (1957) 452–453.

[9] J.K. Beem, P.E. Ehrlich and K.L. Easley, **Global Lorentzian Geometry**, Second Edition, Marcel Dekker inc: New York, 1996.

[10] B.K. Berger, P.T. Chruściel and V. Moncrief, *On "asymptotically flat" space-times with G_2-invariant Cauchy surfaces*, Ann. Phys. **237** (1995) 322–54.

[11] P. Bizon, *Colored black holes.* Phys. Rev. Lett. **64** (1990) 2844–2847.

[12] D. Brill, *On space-times without maximal surfaces*, in **Proc. 3rd Marcel Grossman Meeting**, ed. H. Ning, North Holland, Amsterdam, 1982.

[13] O. Brodbeck and N. Straumann, *Instability proof for the Einstein-Yang-Mills solitons and black holes with arbitrary gauge groups*, J. Math. Phys. **37** (1996) 1414–1433.

[14] G.L. Bunting and A.K.M. Masood-ul-Alam, *Nonexistence of multiple black holes in asymptotically Euclidean vacuum space-times*, Gen. Rel. Grav. **19** (1987) 147–154.

[15] B. Carter, *The general theory of the mechanical, electromagnetic and thermodynamical properties of black holes*, in **General Relativity**, eds S.W. Hawking and W. Israel, Cambridge University Press, 1979.

[16] D. Christodoulou, *Examples of naked singularity formation in the gravitational collapse of a scalar field.* Ann. Math. **140** (1994) 607–653.

[17] D. Christodoulou, *The instability of naked singularities in the gravitational collapse of a scalar field*, Ann. Math. **149** (1999) 183–217

[18] D. Christodoulou and S. Klainerman, **The Global Nonlinear Stability of Minkowski Space**. Princeton University Press, 1993.

[19] P.T. Chruściel, *On space-times with $U(1)\Theta U(1)$ symmetric compact Cauchy surfaces*, Ann. Phys. **202** (1990) 100–150.

[20] P.T. Chruściel, *On uniqueness in the large of solutions of Einstein's equations ("Strong cosmic censorship")*, Proc. Centre for Math. and Appns. ANU **27** (1991)

[21] P.T. Chruściel, "No hair" theorems — folklore, conjectures, results. Contemporary Mathematics **170** (1994) 23–49.

[22] P.T. Chruściel, Uniqueness of stationary, electro-vacuum black holes revisited. Helv. Phys. Acta **69** (1996) 529–552.

[23] P.T. Chruściel and G.J. Galloway. "Nowhere" differentiable horizons, e-print gr-qc/9611032, 1996.

[24] P.T. Chruściel, J. Isenberg and V. Moncrief, *Strong cosmic censorship in polarised Gowdy space-times*, Class. Quant. Grav. **7** (1990) 1671–1680

[25] P.T. Chruściel and A.D. Rendall, *Strong cosmic censorship in vacuum space-times with compact, locally homogeneous Cauchy surfaces.* Ann. Phys. **242** (1995) 349–385.

[26] C.M. Claudel and K. Newman, *The Cauchy problem for quasi-linear hyperbolic evolution problems with a singularity in the time.* Proc. Roy. Soc. Lond. **A 454** 1073–1107

[27] C. Cutler and R.M. Wald, *Existence of radiating Einstein-Maxwell solutions which are regular on all of* I^+ *and* I^-. Class. Quant. Grav. **6** (1989) 452–466

[28] W. Dietz and C. Hoenselaers, *Two mass solutions of Einstein's vacuum equations: the Double Kerr solution.* Ann. Phys. **165** (1985) 319–383

[29] D.M. Eardley, *Naked singularities in spherical gravitational collapse in Gravitation*, in **Astrophysics** eds B.Carter and J.B.Hartle, Plenum, 1987.

[30] D.M. Eardley and L. Smarr, *Time functions in numerical relativity: marginally bound dust collapse.* Phys. Rev. **D19** (1979) 2239–59

[31] G.F.R. Ellis, A. Sumeruk, D. Coule and C. Hellaby, *Change of signature in classical relativity.* Class. Quant. Grav. **9** (1992) 1535–1554

[32] J.-H. Eschenburg,The splitting theorem for space-times with strong energy conditions. J. Diff. Geom. **27** (1988) 477–491

[33] J.-H. Eschenburg and G.J. Galloway, *Lines in space-time.* Comm. Math. Phys. **148** (1992) 209–216

[34] A.E. Fischer and J.E. Marsden, *The initial value problem and the dynamical formulation of general relativity*, in **General Relativity: an Einstein Centennary Survey** eds S.W. Hawking and W. Israel, Cambridge University Press, (1973)

[35] J. Friedman, K. Schleich and D.W. Witt. *Topological censorship.* Phys. Rev. Lett. **71** (1993) 1486–89

[36] H. Friedrich, *On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure.* Comm. Math. Phys. **107** (1986)587–609

[37] H. Friedrich, *Einstein equations and conformal structure: existence of anti-de-Sitter type space-times.* J. Geom. Phys. **17** (1995) 125–184

[38] H. Friedrich, *Gravitational fields near space-like and null infinity AEI.* Preprint 022, Albert Einstein Institute, Potsdam, 1996.

[39] H. Friedrich, *Einstein's equations and conformal structure*, in **The Geometric Universe** eds S. Huggett, L. Mason, K.P. Tod, S.T. Tsou and N.M.J. Woodhouse, Oxford Univ.Press, 1998.

[40] G.J. Galloway, *The Lorentzian splitting theorem without completeness assumptions.* J. Diff. Geom. **29** (1989) 373–387

[41] G.J. Galloway, *On the topology of the domain of outer communication.* Class. Quant. Grav. **12** (1995) L99–L101

[42] G.J. Galloway, *Some rigidity results for spatially closed space-times*, in **Mathematics of Gravitation 1** ed P. Chruściel, Banach Center Publications 41, 1997.

[43] G.J. Galloway and A. Horta, *Regularity of Lorentzian Busemann functions.* Trans. A.M.S. **348** (1996) 2063–2084.

[44] G.J. Galloway and E. Woolgar, *The cosmic censor forbids naked topology.* Class. Quant. Grav. **14** (1997) L1–L7

[45] D. Gannon, *Singularities in nonsimply connected space-times.* J. Math. Phys. **16** (1975) 2364–67.

[46] R.P. Geroch, *Singularities in closed universes.* Phys. Rev. Lett. **17** (1966) 445–447.

[47] R.P. Geroch, *Topology in general relativity.* J. Math. Phys. **8** (1969) 782–786.

[48] R.P. Geroch, *The domain of dependence.* J. Math. Phys. **11** (1970) 437–439

[49] R.P. Geroch, *Singularities*, in **Relativity** eds M.Carmeli, S.Fickler and L.Witten, Plenum Press, 1970.

[50] R.P. Geroch, *Energy extraction.* Ann. N.Y. Acad. Sci. **224** (1973) 108–117

[51] G.W. Gibbons, *The isoperimetric and Bogomolny inequalities for black holes*, in **Global Riemannian geometry** eds T.Willmore and N.Hitchin, Ellis Horwood, 1984.

[52] G.W. Gibbons, *Collapsing shells and the isoperimetric inequality for black holes.* Class.Quant.Grav. **14** (1997) 2905–2915.

[53] G.W. Gibbons and J.B. Hartle, *Real tunnelling geometries and the large scale topology of the universe.* Phys. Rev. **D42** (1990) 2458–68

[54] G.W. Gibbons, S.W. Hawking, G.T. Horowitz and M.J. Perry, *Positive mass theorems for black holes.* Comm. Math. Phys. **88** (1983) 259–308

[55] G.W. Gibbons and S.W. Hawking, *Selection rules for topology change.* Comm. Math. Phys. **148** (1992) 342–352

[56] S.W. Goode and J. Wainwright, *Isotropic singularities in cosmological models.* Class. Quant. Grav. **9** (1985) 445–455

[57] J.B. Hartle and S. Chandrasekhar, *On crossing the Cauchy horizon of a Reissner-Nordstrom black hole.* Proc. Roy. Soc. Lond. **A384** (1982) 301–315

[58] J.B. Hartle and S.W. Hawking, *The wave-function of the universe.* Phys. Rev. **D28** (1983) 2960–2975

[59] S.W. Hawking, *Gravitational radiation in an expanding universe.* J. Math. Phys. **9** (1968) 598–604

[60] S.W. Hawking and G.F.R. Ellis, **The Large-Scale Structure of Space-Time**, Cambridge University Press, 1973

[61] M. Herzlich, *The positive mass theorem for black holes revisited.* To appear in Jour. Geom. Phys.

[62] M. Herzlich, *A Penrose-like inequality for the mass of Riemannian asymptotically flat manifolds.* Comm. Math. Phys. **188** (1997) 121–133.

[63] M. Heusler, **Blackhole uniqueness theorems**. Cambridge University Press, 1996.

[64] M. Heusler, *On the uniqueness of the Majumdar-Papapetrou metric.* Class. Quant.Grav. **14** (1997) L129–L134.

[65] C. Hoenselaers, *The double Kerr solution: a survey*, in **Proceedings of the 4th M. Grossman meeting on general relativity** ed R.Ruffini, North Holland, 1986.

[66] G. Huisken and S.T. Yau, *Definition of centre of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature.* Invent. Math. **124** (1996) 281–311.

[67] G. Huisken and T. Ilmanen, *The Riemannian Penrose inequality.* Int. Math. Res. Not. **20** (1997) 1045–1058.

[68] W. Israel, *Event horizons in static vacuum space-times.* Phys. Rev. **164** (1967) 1776–1779.

[69] W. Israel, *Event horizons in static electrovac space-times.* Comm. Math. Phys. **8** (1968) 245–260.

[70] P.S. Jang and R.M. Wald, *The positive energy conjecture and the cosmic censorship hypothesis.* J. Math. Phys. **18** (1977) 41–44.

[71] J. Jezierski, *Positivity of mass for certain space-times with horizons.* Class. Quant. Grav. **6** (1989) 1535–1539.

[72] B.S. Kay and R.M. Wald, *Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-surface.* Class. Quant. Grav. **4** (1987) 893–898.

[73] H.P. Künzle and A.K.M. Masood-ul-Alam, *Spherically symmetric static $SU(2)$ Einstein-Yang-Mills fields.* J. Math. Phys. **31** (1990) 928–935,

[74] A. Lichnerowicz, **Théories Relativistes de la Gravitation et de l'Électromagnétisme**, Masson et Cie, Paris, 1955.

[75] M. Ludvigsen and J. Vickers, *An inequality relating total mass and area of a trapped surface in general relativity.* J. Phys. **A16** (1983) 3349–3353.

[76] E. Malec and N. O'Murchadha, *Trapped surfaces and the Penrose inequality in spherically symmetric geometries.* Phys. Rev. **D49** (1994) 6931–6934.

[77] J.E. Marsden and F.J. Tipler, *Maximal hypersurfaces and foliations of constant mean curvature in general relativity.* Phys. Rep. **66** (1980) 109–139.

[78] P.O. Mazur, *Proof of uniqueness of the Kerr-Newman black hole solution.* J. Phys. **A15** (1982) 3173–3180.

[79] C. W. Misner, K. S. Thorne and J. A. Wheeler, **Gravitation**, W.H. Freeman, 1973.

[80] V. Moncrief, *Global properties of Gowdy space-times with T^3 topology.* Ann. Phys. **132** (1981) 87–107.

[81] V. Moncrief, *Space-time singularities and cosmic censorship*, in **Proc. 14th Conference on Gen. Rel. Grav.** ed M. Francaviglia et al, World Scientific, 1997.

[82] R.P.A.C. Newman, *Compact space-times and the no-return theorem.* Gen. Rel. Grav. **18** (1986) 1181–1186.

[83] R.P.A.C. Newman, *A proof of the splitting conjecture of S.T. Yau.* J. Diff. Geom. **31** (1990) 163–184.

[84] R.P.A.C. Newman, *On the structure of conformal singularities in classical general relativity I and II.* Proc. Roy. Soc. Lond. **A443** (1993) 473–492, 493–515.

[85] N. O'Murchadha, *How large can a star be?* Phys. Rev. Lett. **57** (1986) 2466–2469.

[86] T. Parker and C.H. Taubes, *On Witten's proof of the positive energy theorem.* Comm. Math. Phys. **84** (1982) 223–238.

[87] R. Penrose, *Gravitational collapse and space-time singularities.* Phys. Rev. Lett. **14** (1965) 57–59.

[88] R. Penrose, *Gravitational collapse: the role of General Relativity.* Riv. del N. Cim. **1** (1969) 252–276.

[89] R. Penrose, **Techniques of Differential Topology in Relativity**, SIAM, 1972.

[90] R. Penrose, *Naked singularities.* Ann. NY Acad. Sci. **224** (1973) 125–134.

[91] R.Penrose, *Singularities of space-times*, in **Theoretical Principles in Astrophysics and Relativity,** eds. N.R.Lebowitz *et al.* University of Chicago Press, 1978.

[92] R. Penrose, *Singularities and time-asymmetry* in **General Relativity: an Einstein Centennial Survey**, eds S.W.Hawking and W.Israel, Cambridge University Press, 1979.

[93] R. Penrose, *On Schwarzschild causality*, in **Essays in General Relativity**, ed F.J.Tipler, 1980.

[94] R. Penrose, *Quasi-local mass and angular momentum in general relativity.* Proc. Roy. Soc. Lond. **A381** (1982) 53–63.

[95] R. Penrose, **Shadows of the Mind**, Oxford Univ. Press, 1994.

[96] R. Penrose and W. Rindler, **Spinors and Spacetime**, volumes 1 and 2, Cambridge University Press, 1984.

[97] R. Penrose, R.D. Sorkin and E. Woolgar, *A positive mass theorem based on the focussing and retardation of null geodesics.* gr-qc/9301015, 1993.

[98] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data.* J. Diff. Eqns **95** (1992) 281–303.

[99] R. Price, *Nonspherical perturbations of relativistic gravitational collapse I and II.* Phys. Rev. **D5** (1972) 2419–2438, 2439–2454.

[100] G. Rein, *Cosmological solutions of the Vlasov-Einstein system with spherical, plane and hyperbolic symmetry.* Math. Proc. Cam. Phil. **119** (1996) 739–762.

[101] G. Rein , *Self-gravitating systems in Newtonian theory — the Vlasov-Poisson system*, in **Mathematics of Gravitation,** ed P.Chruściel, Banach Center Publications **41**, 1997.

[102] G. Rein, A.D. Rendall and J. Schaeffer, *A regularity theorem for solutions of the spherically symmetric Einstein-Vlasov system.* Comm. Math. Phys. **168** (1995) 467–478.

[103] B.L. Reinhart,*Cobordism and the Euler number.* Topology **2** (1963) 173–177.

[104] A.D. Rendall, *Cosmic censorship and the Vlasov equation.* Class. Quant. Grav. **9** (1992) L99–L101.

[105] A.D. Rendall, *Cosmic censorship for some spatially-homogeneous cosmological models.* Ann. Phys. **233** (1994) 82–96.

[106] A.D. Rendall, *Crushing singularities in space-times with spherical, plane and hyperbolic symmetry.* Class. Quant. Grav. **12** (1995) 1517–1535.

[107] A.D. Rendall, *Constant mean curvature foliations in cosmological spacetimes.* Helv. Phys. Acta **69** (1996) 490–500.

[108] A.D. Rendall, *Solutions of the Einstein equations with matter*, in **Proc.14th Conference on Gen. Rel. Grav.** ed M. Francaviglia et al. World Scientific, 1997.

[109] O. Reula and K.P. Tod, *Positivity of the Bondi energy.* J. Math. Phys. **25** (1984) 1004–1008.

[110] P. Ruback , *A new uniqueness theorem for charged black holes.* Class. Quant. Grav. **5** (1988) L155–L159.

[111] F. Schein and P.C. Aichelburg, *Traversable wormholes in geometries of charged shells.* Phys. Rev. Lett. **77** (1996) 4130–4133.

[112] R. Schoen and S.T. Yau, *On the proof of the positive mass conjecture in general relativity.* Comm. Math. Phys. **65** (1979) 45–76.

[113] R. Schoen and S.T. Yau, *Proof of the positive mass theorem II.* Comm. Math. Phys. **79** (1981) 231–260.

[114] R. Schoen and S.T. Yau, *Proof that the Bondi mass is positive.* Phys. Rev. Lett. **48** (1982) 369–371.

[115] R. Schoen and S.T. Yau, *The existence of a black hole due to condensation of matter.* Comm. Math. Phys. **90** (1983) 575–579.

[116] I. Semiz, *Dyon black holes do not violate cosmic censorship.* Class. Quant. Grav. **7** (1990) 353–359.

[117] S.L. Shapiro and S.A. Teukolsky, *Formation of naked singularities: the violation of cosmic censorship.* Phys. Rev. Lett. **66** (1991) 994–997.

[118] J. Smoller and A. Wasserman, *Existence of infinitely many smooth static global solutions of the Einstein-Yang-Mills equations. Comm. Math. Phys.* **151** (1993) 303–325.

[119] J. Smoller and A. Wasserman, *Uniqueness of zero surface gravity SU(2) Einstein-Yang-Mills black holes.* J. Math. Phys. **37** (1995) 1434–1484.

[120] J. Smoller, A. Wasserman, S.T. Yau and J.B. Mcleod, *Smooth static solutions of the Einstein-Yang-Mills equations.* Comm. Math. Phys. **143** (1991) 115–147.

[121] J. Smoller, A. Wasserman and S.T. Yau, *Existence of black hole solutions of the Einstein-Yang-Mills equations.* Comm. Math. Phys. **154** (1993) 137–401.

[122] K.S. Thorne, *Non-spherical gravitational collapse: a short review*, in Magic without magic: John Archibald Wheeler, ed J. Klauder, W.H. Freeman, 1972.

[123] F.J. Tipler, *Singularities and causality violation.* Ann. Physics **108**(1977) 1–36.

[124] F.J. Tipler, *Singularities and causality violation.* Gen. Rel. Grav. **10** (1979) 983–984.

[125] F.J. Tipler, *General relativity and the Eternal Return*, in Essays in general relativity ed F.J.Tipler, Academic Press, 1980.

[126] K.P. Tod, F. de Felice and M. Calvani, *Spinning test particles in the field of a black hole.* Nuovo Cim. **34B** (1976) 365–379.

[127] K.P. Tod, *Penrose's quasi-local mass and the isoperimetric inequality for static black holes.* Class. Quant. Grav. **2** (1985) L65–L68.

[128] K.P. Tod, *More on Penrose's quasi-local-mass.* Class.Quant.Grav. **3** (1986) 1169–1189.

[129] K.P. Tod, *Penrose's quasi-local mass*, in **Twistors in Mathematics and Physics**, eds T.N. Bailey and R.J. Baston, LMS Lecture Notes **156**, Cambridge Univ. Press, 1990.

[130] K.P. Tod, *Isotropic singularities.* Rend. Sem. Mat. Uni. Pol. Tor. **50** (1992) 69–93.

[131] K.P. Tod, *The hoop conjecture and the Gibbons-Penrose construction of trapped surfaces.* Class. Quant. Grav. **9** (1992) 1581–1591.

[132] N. Trudinger, *Isoperimetric inequalities for quermassintegrals.* Ann. Inst. H. Poincare **11** (1994) 411–425.

[133] D. V. Volkov and M. S. Galtsov, *Black holes in Einstein-Yang-Mills theory.* Sov. Jour. Nucl. Phys. **51** (1990) 747–753.

[134] C.V. Vishveshwara, *Stability of the Schwarzschild metric.* Phys. Rev. **D1** (1970) 2870–2879.

[135] J. Wainwright and G.F.R. Ellis, **Dynamical Systems in Cosmology**, Cambridge University Press, 1997.

[136] R.M. Wald, *Gedanken experiments to destroy a black hole.* Ann. Phys. **82** (1974) 548–556.

[137] R.M. Wald, **General Relativity**, Univ of Chicago Press, 1984.

[138] R.M. Wald, *Gravitational collapse and cosmic censorship.* gr-qc/9710068, 1997.

[139] G. Weinstein, *On rotating black holes in equilibrium in general relativity.* Comm. Pure Appl. Maths. **XLIII** (1990) 903–948.

[140] G. Weinstein, *The stationary axisymmetric two-body problem in general relativity.* Comm. Pure Appl. Maths **XLV** (1990) 1183–1203.

[141] G. Weinstein, *On the force between rotating coaxial black holes.* Trans. A.M.S. **343** (1994) 899–906.

[142] G. Weinstein, *Black hole stationary and axisymmetric solutions of the Einstein-Maxwell equations.* Comm. P.D.E. **21** (1996) 1389–1430.

[143] B. Whiting, *Mode stability of the Kerr black hole.* J. Math. Phys. **30** (1989) 1301–1305.

[144] D.M. Witt, *Vacuum space-times that admit no maximal slice.* Phys. Rev. Lett. **57** (1986) 1386–1389.

[145] E. Witten, *A new proof of the positive energy theorem.* Comm. Math. Phys. **80** (1981) 381–402.

[146] N.M.J. Woodhouse, *Integrability and Einstein's equations* in **Mathematics of Gravitation**, ed P.T.Chruściel, Banach Center Publications **41**, 1997.

[147] S.T. Yau, *Problem section* in **Seminar on differential geometry**, Ann. Math. Studs. **102**, Princeton University Press, 1982.

[148] R. Ye, *Foliation by constant mean curvature spheres on asymptotically flat manifolds* in **Geometric Analysis and Calculus of Variations**, ed J.Jost, Internat. Press, Cambridge, MA, 1996.

[149] P. Yodzis, H-J. Seifert and H. Muller zum Hagen, *On the occurrence of naked singularities in general relativity*. Comm. Math. Phys. **34** (1973) 135–148.

[150] P. Yodzis, H-J. Seifert and H. Muller zum Hagen, *On the occurrence of naked singularities in general relativity*. Comm. Math. Phys. **37** (1974) 29–40.

MATHEMATICAL INSTITUTE, 24–29 ST. GILES, OXFORD OX1 3LB, ENGLAND
E-mail address: tod@maths.ox.ac.uk