Rigidity and Compactness of Einstein Metrics

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ABSTRACT. We survey some rigidity and compactness results for Einstein metrics, with special emphasis on the degeneration results of Michael Anderson.

1. Introduction

In this essay, we shall explain some rigidity and compactness phenomena for Einstein manifolds and, more generally, for manifolds with bounded Ricci curvature. Many of the compactness results are due to Anderson and Anderson-Cheeger. Our proofs are not always rigorous. However, we have tried not to cheat too much, and to at least present the most important ideas.

The introduction is followed by a short section on notation, just to nail down our conventions on the various curvature tensors and L^p norms. After that, we proceed to establish a very general Bochner-Lichnerowicz-Weitzenböck formula which contains almost all the formulae of this type (spinor fomulæ excepted). In section 3 we begin to examine the Einstein condition and prove some results. The main focus here is on how the Einstein condition helps us go from weak to strong pinching conditions. This leads us to several rigidity/gap results. In section 4 we discuss harmonic coordinates and how they can be used to introduce pseudo-norms of Riemannian manifolds. In section 5 we use all of the previous material to study compactness and degeneration phenomena for Einstein manifolds. Also we have two pinching theorems that use these compactness results. First we show how to prove the compactness theorem from [**1**], when one has a lower bound for the injectivity radius. Then we slowly work our way up to proving the orbifold degeneration result from [**1**] for manifolds with an $L^{n/2}$ bound for the curvature. This is used to get some nice corollaries for 4-manifolds. In the last section we given some examples of orbifold degeneration of Einstein metrics.

We shall not always be careful with giving the original references, instead will rely heavily on the two books [**5**] and [**16**] whenever it is convenient.

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2. Notation

On a Riemannian n-manifold (M, g), usually denoted M, we have the metric and torsion free connection ∇. This enables us to define covariant derivatives of all tensors, in particular we get the curvature tensor as

$$\begin{aligned} R(X,Y)Z &= \left(\nabla^2_{X,Y} - \nabla^2_{Y,X}\right)(Z) \\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y} Z - \nabla_Y \nabla_X Z + \nabla_{\nabla_Y X} Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned}$$

This gives us the Ricci tensor Ric which we think of as a $(1,1)$ tensor

$$\mathrm{Ric}(X) = \sum R(X, E_i) E_i,$$

where E_i is an orthonormal frame. The $(0,2)$ version of the Ricci tensor is defined by

$$\mathrm{ric}(X,Y) = g(\mathrm{Ric}(X), Y).$$

Finally we have the curvature operator $\mathfrak{R} : \Lambda^2 TM \to \Lambda^2 TM$ defined implicitly by

$$\langle \mathfrak{R}(X \wedge Y), V \wedge W \rangle = g(R(X,Y)W, V)$$

For a function $u : \Omega \to \mathbb{R}$ on a bounded domain in a Riemannian manifold M we define the L^p norm as

$$\|u\|_{p,\Omega} = \left(\frac{1}{\mathrm{vol}\Omega}\int_\Omega |u|^p \, d\mathrm{vol}\right)^{1/p},$$

where $d$vol denotes the Riemannian volume element. In case M is compact we usually let $\Omega = M$ and omit it as a subscript.

3. Bochner-Lichnerowicz-Weitzenböck Formulae

First we explain a very general Weitzenböck formula due to Lichnerowicz. In subsequent sections we shall see how this general formula behaves in some specific situations that lead to some interesting rigidity and regularity results for Einstein metrics.

Consider a Riemannian n-manifold M and a vector bundle E over M endowed with a metric and compatible connection. We shall use the notation ∇ and $R(X,Y)$ for the connection and curvature on both E and TM.

For an (E,p) tensor T on M, i.e., $T(X_1, \ldots, X_p)$ is a section of E for vector fields $X_1, \ldots, X_p$ on M, we have the covariant derivative and its dual defined by

$$\begin{aligned} (\nabla T)(X_0, \ldots, X_p) &= (\nabla_{X_0} T)(X_1, \ldots, X_p) \\ &= \nabla_{X_0} T(X_1, \ldots, X_p) - \sum_{i=1}^{p} T(X_1, \ldots, \nabla_{X_0} X_i, \ldots, X_p), \\ (\nabla^* T)(X_2, \ldots, X_p) &= -\sum_{i=1}^{n} (\nabla_{E_i} T)(E_i, X_2, \ldots, X_p), \end{aligned}$$

where E_i is an orthonormal frame. Note that if S has $p+1$ variables and T has p variables then we can construct a 1-form

$$\begin{aligned} \omega(X) &= \langle S(X, \cdots), T(\cdots) \rangle \\ &= \sum \left\langle S\left(X, E_{i_1}, \cdots, E_{i_p}\right), T\left(E_{i_1}, \cdots, E_{i_p}\right) \right\rangle. \end{aligned}$$

The divergence or codifferential of this form is $\nabla^*\omega$ and by the divergence theorem we know that the integral of $\nabla^*\omega$ is zero. This implies that ∇ and ∇^* are adjoints in L^2 since we have the identity

$$\nabla^*\omega = -\langle S, \nabla T\rangle + \langle \nabla^* S, T\rangle .$$

Note that this identity also gives us the basic identity

$$\begin{aligned}\Delta\langle T_1, T_2\rangle &= \nabla^*\nabla\langle T_1, T_2\rangle \\ &= -\langle \nabla T_1, \nabla T_2\rangle + \langle \nabla^*\nabla T_1, T_2\rangle + \langle T_1, \nabla^*\nabla T_2\rangle .\end{aligned}$$

When we can express the connection Laplacian $\nabla^*\nabla$ in terms of curvature and a different Laplacian this gives us a Bochner formula.

The generalized Lichnerowicz Laplacian Δ_L on E-valued tensors is defined as

$$\begin{aligned}\Delta_L T &= \nabla^*\nabla T + \mathrm{Ric}T, \\ (\mathrm{Ric}T)(X_1, \ldots, X_p) &= \sum_{j=1}^{p}\sum_{i=1}^{n} (R(E_i, X_j) T)(X_1, \ldots, E_i, \ldots, X_k),\end{aligned}$$

where E_i replaces X_j in the j^{th} spot. Note that if the curvature of both M and E is bounded, then $\mathrm{Ric}T$ is also bounded in terms of T. Using the Lichnerowicz Laplacian we get the Bochner formula

$$\begin{aligned}\Delta\langle T_1, T_2\rangle &= -\langle \nabla T_1, \nabla T_2\rangle + \langle \nabla^*\nabla T_1, T_2\rangle + \langle T_1, \nabla^*\nabla T_2\rangle \\ &= \langle \Delta_L T_1, T_2\rangle + \langle T_1, \Delta_L T_2\rangle \\ &\quad -\langle \nabla T_1, \nabla T_2\rangle - \langle \mathrm{Ric}T_1, T_2\rangle - \langle T_1, \mathrm{Ric}T_2\rangle\end{aligned}$$

In case T is skew symmetric we also have the exterior covariant derivative defined by

$$(dT)(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i (\nabla_{X_i} T)\left(X_0, \ldots, \hat{X}_i, \ldots, X_p\right).$$

Note that ∇^* is dual to both d and ∇. In the case of the exterior covariant derivative d this dual is often also denoted as d^* or δ. Moreover, one easily checks that we have the Weitzenböck formula

$$\Delta_L = dd^* + d^*d.$$

This demonstrates that the Lichnerowicz Laplacian is nonnegative in these cases and also that it is the Hodge Laplacian on forms.

4. Rigidity

In this section we shall see how an Einstein metric gives us extra control over the curvature tensor and how this leads to some rigidity results.

Consider the special case where $E = \Lambda^2 TM$ and the tensor T is the Riemannian curvature operator $\mathfrak{R} : \Lambda^2 TM \to \Lambda^2 TM$. In fact we shall modify this tensor a little by subtracting a constant tensor so that we can study some pinching phenomena. Thus we let $T = \mathfrak{R} - \lambda I$. It follows from Bianchi's second identity that $d\mathfrak{R} = 0$, thus also $d(\mathfrak{R} - \lambda I) = 0$. Furthermore the fact that the metric is Einstein also tells us, again via Bianchi's second identity, that $d^*\mathfrak{R} = 0$. Therefore we get

$$\nabla^*\nabla T + \mathrm{Ric}T = 0.$$

This leads to a Bochner formula of the form

$$\begin{aligned}\Delta\frac{1}{2}|T|^2 &= \langle\nabla^*\nabla T, T\rangle - |\nabla T|^2\\ &= -\langle \mathrm{Ric}T, T\rangle - |\nabla T|^2\\ &\leq C_1(n)|\mathfrak{R}||T|^2 - |\nabla T|^2.\end{aligned}$$

Since we also have

$$\Delta\frac{1}{2}|T|^2 = |T|\Delta|T| - |\nabla|T||^2$$

Kato's inequality then tells us that

$$\begin{aligned}\Delta|T| &\leq C_1(n)|\mathfrak{R}||T|\\ &\leq C_1(n)|T|^2 + C_2(n)\alpha.\end{aligned}$$

We can now multiply both sides by $|T|^{2p-1}$, use integration by parts, and the Sobolev inequality

$$\left\|u - \frac{1}{\mathrm{vol}M}\int u\right\|_{\frac{2n}{n-2}} \leq C_S\|\nabla u\|_2,$$

to obtain an inequality of the form

$$\|T\|_{\frac{2pn}{n-2}} \leq C_3(n,p,\lambda,C_S)\left(\|T\|_{2p+1} + \left(\|T\|_{2p-1}\right)^{\frac{2p-1}{2p}}\right).$$

This does not look very promising for iteration purposes as we might have $2p+1 > \frac{2pn}{n-2}$. However given some sort of L^q bound for T we can make things look a little better. Namely we can use Hölder's inequality to obtain

$$\|T\|_{\frac{2pn}{n-2}} \leq C_3(n,p,\lambda,C_S)\left(\|T\|_q\|T\|_{\frac{2pq}{q-1}} + \left(\|T\|_{2p-1}\right)^{\frac{2p-1}{2p}}\right).$$

When $q < n/2$ this doesn't give anything useful. However, as long as $q > n/2$ this can easily be iterated to yield a bound of the type

$$|T| \leq C_4(n,p,\lambda,C_S)\|T\|_q.$$

When $q = n/2$ we can bring the first term on the right-hand side to the left-hand side to obtain

$$\left(1 - C_3(n,p,\lambda,C_S)\|T\|_{\frac{n}{2}}\right)\|T\|_{\frac{2pn}{n-2}} \leq C_3(n,p,\lambda,C_S)\left(\|T\|_{2p-1}\right)^{\frac{2p-1}{2p}}.$$

Provided $1 - C_3(n,p,\lambda,C_S)\|T\|_{n/2} > 0$, or in other words that $\|T\|_{n/2}$ is sufficiently small, this can be iterated to yield

$$|T| \leq C_5(n,p,\lambda,C_S)\|T\|_{n/2}.$$

Thus an L^q, $q > n/2$ bound for the curvature tensor of an Einstein metric immediately leads to a C^0 bound on the curvature. This is one of the crucial ingredients in all of our compactness results for Einstein metrics. As for pinching we see that if the $L^{n/2}$ norm of T is small then T is itself small and hence the eigenvalues for the curvature operator are pinched to be near λ. In order for these estimates to be truly interesting we must also have bounds for the Sobolev constant. Thanks to Gromov and Gallot (see **[9]**, **[10]**) we now know that upper diameter bounds and lower Ricci curvature bounds suffice to give bounds for C_S. In fact, this was recently generalized to the case where one allows for the Ricci curvature lying below a certain constant to be small in L^p, $p > n/2$ (see [**18**]).

Our first rigidity/gap theorem is the following $L^{n/2}$ pinching result.

THEOREM 4.1. *Let $\lambda > 0$ be given, there is a constant $\varepsilon(n, \lambda) > 0$ so that any Einstein metric with $\|\mathfrak{R} - \lambda I\|_{n/2} \leq \varepsilon$ has constant curvature.*

PROOF. Given a bound for the Sobolev constant the above result tells us that the curvature operator has eigenvalues close to one, in particular they are positive. One knows from a result of Tachibana that any Einstein metric with positive curvature operator has constant curvature (see [**16**, Chapter 7]). To get a bound for the Sobolev constant we first need a bound for the Ricci curvature. However, the $L^{n/2}$ pinching for the curvature tells us that the Einstein constant must be bigger than $(n-1)\lambda - C(n)\varepsilon$. Thus for small ε we get a positive lower bound for the Ricci curvature. Then Myers' Theorem gives us a diameter bound as well. □

N.B. Our conventions on L^p norms, established in §2, involve a volume normalization which is by no means standard. Without this normalization, the above result, and many other results in this article, would be false in the stated form. Readers accustomed to other conventions should thus exercise appropriate care in interpreting the results herein.

In case $\lambda = 0$ we cannot expect such a nice result since any Einstein metric can be scaled so as to have small $L^{n/2}$ norm on curvature. However, if we bound the diameter as well we get

THEOREM 4.2. *There is a constant $\varepsilon(n, D) > 0$ so that any Ricci flat metric with $\|\mathfrak{R}\|_{n/2} \leq \varepsilon$ and* diam $\leq D$ *is flat.*

PROOF. Simply observe that we get pinched curvature as in the almost flat manifold theorem of Gromov. In particular, the manifold must be $K(\pi, 1)$ (see [**12**]). Then it follows from the Cheeger-Gromoll splitting theorem that the manifold is flat. □

We shall later obtain a similar gap theorem for complete Ricci flat metrics which give us some very interesting compactness results. There is also a gap theorem when $\lambda < 0$ which is proved below (see 5.4).

To prepare for the non-compact result let us see what the above iterations can do for us. Let M be a complete Ricci flat manifold with $\mathrm{vol}B(p,r) \geq v \cdot r^n$ and $\int |R|^{n/2}\, d\mathrm{vol} \leq Q$. Note that these three condition are scaling invariant, i.e., if we multiply the metric by a constant these conditions will still hold with the same v and Q. We need to get some sort of smallness for the $L^{n/2}$ norm of the curvature. This is achieved as follows. Absolute volume comparison tells us that annuli of the form $A(r) = B(p, 2r) - B(p, r)$ satisfy $\mathrm{vol}A(r) \geq v' \cdot r^n$ for some $v'(n, v)$. Moreover since $\int |R|^{n/2}\, d\mathrm{vol} \leq Q$ we must have that $\int_{A(r)} |R|^{n/2}\, d\mathrm{vol} \to 0$ as $r \to \infty$. The volume estimate for the annuli then tells us that

$$\|R\|_{n/2, A(r)} \leq \varepsilon(r) \cdot r^{-2},$$

where $\varepsilon(r) \to 0$ as $r \to \infty$. We now claim that this, in analogy with the compact case, gives us an inequality of the form

$$\|R\|_{\infty, A(r)} \leq C(n, v) \cdot \varepsilon(r) \cdot r^{-2},$$

in other words the curvature decays faster that quadratically at infinity. In order to prove this we have to use a Sobolev inequality of the form

$$\|u\|_{\frac{2n}{n-2},\Omega} \leq C_S \|\nabla u\|_{2,\Omega},$$

where u has compact support in the bounded domain $\Omega \subset M$. Results of Croke (see [**7**]) tell us that the volume growth condition and nonnegative Ricci curvature yield a bound for this Sobolev constant. The next problem is to bump $|R|$ down so that it has compact support in $A(r)$. To this end one selects an appropriate bump function ϕ with compact support in $A(r)$ and then multiply the inequality $\Delta|R| \leq C_1(n)|R|^2$ by $\phi^{2p}|R|^{2p-1}$. After some calculations and an iteration as above one then gets an estimate of the form

$$\|\phi R\|_{\infty,A(r)} \leq C(n,v) \cdot \varepsilon(r) \cdot r^{-2}.$$

While this is not precisely the promised estimate is good enough to give us the desired curvature decay condition. In section 5 we shall see how this curvature decay condition is used to "classify" all of the manifolds with these conditions and also how the space is Euclidean space provided Q is sufficiently small.

5. Harmonic coordinates

Harmonic coordinates give us even better control over the metric than we had in the previous section. A more in-depth account can be found in [**16**, Chapter 10].

Suppose we have harmonic coordinates $x = (x^1, \dots, x^n)$ on some open set $U \subset M$, i.e., $\Delta x^i = 0$. The Weitzenböck formula for the gradient fields ∇x^i then tells us that $\nabla^*\nabla(\nabla x^i) = \mathrm{Ric}(\nabla x^i)$. From this we can derive a Bochner formula

$$\frac{1}{2}\Delta\langle\nabla x^k, \nabla x^l\rangle = -\langle\nabla^2 x^k, \nabla x^l\rangle - \langle\nabla x^k, \nabla^2 x^l\rangle - \mathrm{ric}(\nabla x^k, \nabla x^l).$$

If we write out the gradient in terms of the coordinate vector fields ∂_i and define $g_{ij} = \langle\partial_i, \partial_j\rangle$, then one obtains

$$\frac{1}{2}\Delta g_{ij} = Q(g, \partial g) - \mathrm{ric}(\partial_i, \partial_j)$$

for some universal function Q that depends on the metric coefficients and its derivatives. What is interesting about this equation is that if one had C^1 bounds for the metric coefficients and C^0 bounds for the Ricci curvature then standard elliptic estimates tell us that one in fact has $C^{1,\alpha}$ bounds for the metric for any $\alpha \in (0,1)$. Moreover, if the metric is Einstein then one gets $C^{k,\alpha}$ bounds for any k and $\alpha \in (0,1)$. This is similar in spirit to what we saw above for the curvature tensor.

In order to make these estimates a little more precise and useful we introduce some more notation. Let $O \subset M$ be a subset. We say that the $C^{k,\alpha}$ norm of $O \subset M$ on the *scale of* r is bounded by K, denoted $\|O \subset M\|_{C^{k,\alpha},r} \leq K$, if we can find a covering U_s of O by *harmonic* coordinate charts such that

1. $x_s : U_s \to B(0,r) \subset \mathbb{R}^n$ is a diffeomorphism,
2. for each $x \in O$ the ball $B(x, re^{-K})$ lies in some chart U_s,
3. $|Dx| \leq e^K$, $\left|(Dx)^{-1}\right| \leq e^K$, and
4. g_{ij} as functions on $B(0,r)$ satisfy $r^{k+\alpha}\left\|\sum_{|I|=k}\partial^I g_{ij}\right\|_{C^\alpha} \leq K$.

This norm has many important and interesting properties. First we mention what happens with the above (interior) elliptic estimates. If $|\text{Ric}| \leq \Lambda$, then for any scale $\bar{r} < r$ we have

$$||O \subset M||_{C^{1,\alpha},\bar{r}} \leq C(n, \alpha, \Lambda, r, \bar{r}) \, ||O \subset M||_{C^{1},r}.$$

Moreover, if the metric is Einstein $\text{Ric} = \lambda I$, then

$$||O \subset M||_{C^{k,\alpha},\bar{r}} \leq C(n, k, \alpha, \lambda, r, \bar{r}) \, ||O \subset M||_{C^{1},r}.$$

We mention some further important facts about the norm:

1. If we multiply the metric on M by λ^2 we get a new Riemannian manifold N whose norm satisfies

$$||O \subset N||_{C^{k,\alpha},\lambda r} = ||O \subset M||_{C^{k,\alpha},r}.$$

2. If (M_i, p_i) converges to (M, p) in the pointed $C^{k,\alpha}$ topology, then for any set $O \subset M$ we can find sets $O_i \subset M_i$ such that

$$||O_i \subset M_i||_{C^{k,\alpha},r} \to ||O \subset M||_{C^{k,\alpha},r}.$$

3. Given r the norm $||O \subset M||_{C^{k,\alpha},r}$ is realized at some $p \in O$ if O is compact, in other words

$$||O \subset M||_{C^{k,\alpha},r} = ||\{p\} \subset M||_{C^{k,\alpha},r}.$$

For the latter norm it suffices to use one chart. In case O is open or unbounded we can at least find $p \in O$ such that

$$\frac{1}{2} ||O \subset M||_{C^{k,\alpha},r} \leq ||\{p\} \subset M||_{C^{k,\alpha},r} \leq ||O \subset M||_{C^{k,\alpha},r}.$$

4. For a compact set O the norm satisfies

$$||O \subset M||_{C^{k,\alpha},r} \to 0 \text{ as } r \to 0.$$

5. Euclidean space is the only Riemannian manifold such that all of its norms are zero on all scales r. In fact

$$||M||_{C^{k,\alpha},r} \to \infty \text{ as } r \to \infty$$

unless M is Euclidean space.

6. Compactness

In this section we shall work our way towards understanding certain classes of manifolds with bounded Ricci curvature. The first part of the material is covered in [**16**, Chapter 10] for the rest be have supplied references to the appropriate research articles.

We begin by mentioning the following finiteness and compactness theorem essentially due to Cheeger.

THEOREM 6.1. *Given n, k, α and $r, K, D > 0$ we have that the class of Riemannian n-manifolds with $||M||_{C^{k,\alpha},r} \leq K$ and* diam $\leq D$ *contains only finitely many diffeomorphism types and is compact in the $C^{k,\beta}$ topology for any $\beta < \alpha$.*

In case we allow for complete manifolds and don't have a diameter bound we can no longer get finiteness for diffeomorphism types, but we can still get compactness in the pointed $C^{k,\beta}$ topology. More precisely this means that for each sequence M_i with $\|M_i\|_{C^{k,\alpha},r} \leq K$ and $p_i \in M_i$ we can find a subsequence (again indexed by i) and a limit manifold M with a point $p \in M$ such that for each $R > 0$ there are embeddings $\phi_i : B(p,R) \to M_i$ which contain $B(p_i, R)$ and such that the pull-back metrics $\phi_i^* g_i \to g$ in the $C^{k,\beta}$ topology on $B(p,R)$.

In order for all this to be useful it is of course necessary that we have some sort of method that allows us to get bounds for these norms. This is a achieved by a very interesting rescaling argument which was first explored in detail by Anderson. The simplest result along these lines is (see [**1**] and [**16**, Chapter 10])

THEOREM 6.2. *Given n, i_0, D, and Λ the class of Riemannian n-manifolds with*

$$\begin{array}{rcl} \text{inj} & \geq & i_0 \\ \text{diam} & \leq & D \\ |\text{Ric}| & \leq & \Lambda \end{array}$$

has the property that for every $K > 0$ we can find $r(n, i_0, D, \Lambda, K, \alpha) > 0$ such that any manifold in this class satisfies $\|M\|_{C^{1,\alpha},r} \leq K$. In particular, this class is compact in any $C^{1,\beta}$ topology.

PROOF. The proof goes by contradiction. Thus suppose that we have a sequence of manifolds M_i in this class such that $\|\{p_i\} \subset M_i\|_{C^{1,\alpha},r_i} = \|M_i\|_{C^{1,\alpha},r_i} = K$ for a sequence $r_i \to 0$. If we rescale these Riemannian manifolds by r_i^{-2}, the norms stay the same on the new scale of 1. Thus we have a new sequence of Riemannian manifolds N_i which satisfy

$$\begin{array}{rcl} \|\{p_i\} \subset N_i\|_{C^{1,\alpha},1} & = & K \\ \text{inj} N_i & \to & \infty \\ |\text{Ric} N_i| & \to & 0 \end{array}$$

From the above compactness theorem we can conclude that a subsequence (not renumbered) will converge in the pointed $C^{1,\beta}$ topology to some complete Riemannian manifold N. First we note that since all of the manifolds have bounded Ricci curvature we can in fact assume that their $C^{1,\gamma}$ norms are bounded for any $\gamma \in (0,1)$, but on some slightly smaller scale. Thus we can also assume that the manifolds converge in the $C^{1,\alpha}$ topology. Since the $C^{1,\alpha}$ norm is continuous with respect to the $C^{1,\alpha}$ topology we therefore get that $\|\{p\} \subset N\|_{C^{1,\alpha},1} = K$, in particular, the manifold cannot be Euclidean space. On the other hand if we look at the formula for the Ricci tensor in harmonic coordinates on N_i we see that the limit metric must be a weak solution to

$$\frac{1}{2}\Delta g = Q(g, \partial g),$$

since $|\text{Ric} N_i| \to 0$. Elliptic regularity theory then tells us that the metric on N is smooth of any order and Ricci flat. Now we come to the crucial point. Since $\text{inj} N_i \to \infty$ the limit manifold also has $\text{inj} N = \infty$, thus the Cheeger-Gromoll splitting theorem tells us that the manifold is the standard Euclidean space. We have therefore arrived at a contradiction. □

If we insist on only considering Einstein metrics the class becomes compact in the $C^{k,\alpha}$ topology for any k, α. In the Einstein case we can also get a similar result which lies closer to some of the stuff we are aiming for.

THEOREM 6.3. *Given $n, q > n/2, v_0, D, Q$ and λ the class of Einstein n-manifolds with*

$$\begin{array}{rcl} \mathrm{vol} & \geq & v_0 \\ \mathrm{diam} & \leq & D \\ \mathrm{Ric} & = & \lambda I \\ \|R\|_q & \leq & Q \end{array}$$

has the property that for every $K > 0$ we can find $r(n, q, v_0, D, Q, \Lambda, K, k, \alpha) > 0$ such that any manifold in this class satisfies $\|M\|_{C^{k,\alpha},r} \leq K$. In particular, this class is compact in the $C^{k,\beta}$ topology.

PROOF. We know from above that the curvature is bounded not just in L^q but in C^0. Cheeger's lemma then tells us that this class must have a lower bound for the injectivity radius (see [**16**, Chapter 10]). The above theorem then takes care of the rest. □

From the proof of this theorem we can also get the promised gap theorem for negative Einstein metrics.

THEOREM 6.4. *Given n, D and $\lambda < 0$ there is an $\varepsilon(n, D, \lambda) > 0$ such that any Einstein metric with* diam $\leq D$ *and $\|\mathfrak{R} - \lambda I\|_{n/2} \leq \varepsilon$ has constant curvature.*

PROOF. The iterations from the above section tell us that the metric satisfies $|\mathfrak{R} - \lambda I| \leq C \cdot \varepsilon$. In particular, the manifolds have pinched negative curvature. This together with the diameter bound tells us that the manifold has a lower volume bound (see [**11**]). Therefore, if the theorem were false we would have a sequence of Einstein metrics converging to a hyperbolic metric in any $C^{k,\alpha}$ topology. This however contradicts a rigidity result of Koiso (see [**5**, 12.F and 12.H]). □

It is interesting to see what happens if relax the Einstein condition so that we only have $|\mathrm{Ric}| \leq \Lambda$. In this case we still get compactness in the $C^{1,\alpha}$ topology. The argument goes by contradiction and uses rescaling as above. The way in which we get that the limit manifold is flat is to note that $\int |R_i|^q \to 0$ after the rescaling (a slightly stronger convergence coming from elliptic L^p estimates is needed here (see [**13**, p167-202])). Also, as we have a global diameter bound and a lower volume bound, relative volume comparison gives us that the limit manifold has a volume growth condition $\mathrm{vol}B(p, r) \geq v \cdot r^n$ for some $v(n, v_0, \Lambda)$. Now the only flat manifold with such volume growth is $\mathbb{R}^n$.

Finally we could try to examine the borderline case $q = n/2$. We already studied what happened when $\|R - \lambda I\|_{n/2}$ was small in the Einstein case. Furthermore we showed that Ricci flat manifolds with volume growth $\mathrm{vol}B(p, r) \geq v \cdot r^n$ and $\int |R|^{n/2} \leq Q$ have faster than quadratic curvature decay.

Suppose we have a sequence of manifolds with

$$\begin{array}{rcl} \mathrm{vol} & \geq & v_0, \\ \mathrm{diam} & \leq & D, \\ |\mathrm{Ric}| & \leq & \Lambda, \\ \int |R|^{n/2} & \leq & Q. \end{array}$$

If we blow up these metrics as above, then we have that the volume condition gives us a volume growth condition in the limit, the Ricci curvature makes the limit Ricci flat, and finally since $\int |R|^{n/2}$ is a scale invariant quantity the limit satisfies $\int |R|^{n/2} \leq Q$. The next result tells us what happens when Q is small.

THEOREM 6.5. *Given n, v there exists $\varepsilon(n,v) > 0$ such that any Ricci flat manifold with* $\mathrm{vol}B(p,r) \geq v \cdot r^n$ *and* $\int |R|^{n/2} \leq \varepsilon$ *is Euclidean space.*

PROOF. First we exploit the fact that the metric has faster than quadratic curvature decay. Note that relative volume comparison implies that the volume growth condition is independent of the base point. Thus we have good lower volume bounds for all balls in M. If we take x so that $d(x,p) = r$, then the ball $B(x, r/4)$ will have volume $\geq v \cdot (r/4)^n$ and the curvature on this ball will be smaller than $\varepsilon(r) r^{-2}$. These two facts tell us that the injectivity radius at x must be larger than $c(n,v) \cdot r$ (this requires a slight improvement on Cheeger's original argument which can be found in [**6**]). In particular, any of the norms $\|M\|_{C^{k,\alpha},r}$ will be finite and bounded uniformly in terms of v and $\int |R|^{n/2}$.

To prove the theorem we now proceed by contradiction. Thus suppose we have a sequence of non-flat Ricci flat manifolds M_i satisfying $\mathrm{vol}B(p_i, r) \geq v \cdot r^n$ and $\int |R|^{n/2} = \varepsilon_i \to 0$ as $i \to \infty$. Since any manifold which is not Euclidean space has the property that its norm goes to infinity as the scale goes to infinity, we can find r_i such that $\|M_i\|_{C^{k,\alpha},r_i} = 1$. Now rescale these metrics by r_i^{-2} so that we have new manifolds N_i with $\|N_i\|_{C^{k,\alpha},1} = 1$. Thus we can find $q_i \in N_i$ such that $\|\{q_i\} \subset N_i\|_{C^{k,\alpha},1} \geq 1/2$. Since all of the conditions for M_i are scale invariant we also have that $\|N_i\|_{C^{k,\beta},1} \leq K\left(n, v, \int |R|^{n/2}\right)$, where $\beta > \alpha$. This means that we can assume that N_i converges in the pointed $C^{k,\alpha}$ topology to a complete Ricci flat manifold with $\mathrm{vol}B(q,r) \geq v \cdot r^n$. Moreover, since the norm is continuous in this topology we must also have $\|\{q\} \subset N\|_{C^{k,\alpha},1} \geq 1/2$. On the other hand $\int |R|^{n/2} = \varepsilon_i \to 0$, showing that the limit space is flat. This together with the volume growth condition tells us that the manifold is Euclidean space. Thus we have arrived at a contradiction. □

We can use this result to obtain a very general pinching theorem.

THEOREM 6.6. *Given $n, q > n/2, v_0, D$, and Λ, λ we can find $\varepsilon(n, q, v_0, D, \Lambda, \lambda) > 0$ such that any n-manifold with*

$$\begin{array}{rcl} \mathrm{vol} & \geq & v_0, \\ \mathrm{diam} & \leq & D, \\ \|\mathrm{Ric}\|_q & \leq & \Lambda, \\ \|R - \lambda I\|_{n/2} & \leq & \varepsilon \end{array}$$

is C^α, $\alpha < 2 - n/q$ close to a metric with constant curvature λ.

PROOF. For this to work we actually need to work with $L^{k,p}$ norms of manifolds as in [**13**, p167-202]. Also in order to get relative volume comparison as in [**17**] we need to know that the amount of Ricci curvature below a certain constant is small in $L^{q'}$ for some $q' > n/2$. However the fact that $\|R - \lambda I\|_{n/2} \leq \varepsilon, \|\mathrm{Ric}\|_q \leq \Lambda$ implies that for any $q' \in [n/2, q)$ we have $\|\mathrm{Ric} - (n-1)\lambda I\|_{q'} \leq C(n, q, \Lambda, \lambda, \varepsilon)$. As $\varepsilon \to 0$ also $C(n, q, \Lambda, \lambda, \varepsilon) \to 0$, so we get the desired pinching. This means we have relative volume comparison and hence that the above arguments kick in to finish the proof. □

We are now ready to study the more general class of manifolds which satisfy

$$\begin{array}{rcl} \text{vol} & \geq & v_0, \\ \text{diam} & \leq & D, \\ |\text{Ric}| & \leq & \Lambda, \\ \|R\|_{n/2} & \leq & Q. \end{array}$$

When rescaling (blowing up) manifolds in this class we end up, as already pointed out, with complete Ricci flat metrics satisfying $\text{vol}B\,(p,r) \geq v \cdot r^n$ and $\int |R|^{n/2} \leq Q$. Such metrics are called *almost locally Euclidean gravitational instantons.* If Q is not small these manifolds are not necessarily Euclidean space. The Eguchi-Hanson metric is an example of a non-flat limit space of this type. Still, one can say something intelligent about these spaces. We already know that they have faster than quadratic curvature decay. This implies that if we multiply the metric by a constant $\varepsilon \to 0$, then the curvature will converge to zero outside any compact set. Moreover, we also showed that the injectivity radius was large outside compact sets. Thus we get that the metric outside any compact set converges (in any topology) to a flat metric with Euclidean volume growth. This means that the limit as $\varepsilon \to 0$ must look like a cone over a space form of the type S^{n-1}/Γ, where Γ is a finite group of isometries acting freely on the sphere (see also [**4**]). The ALE gravitational instanton is then Euclidean space precisely when Γ is trivial. Note that when the dimension is odd, the space from must be even dimensional, hence it is either a sphere (total space is Euclidean) or a real projective space. The latter case, however, cannot occur as an even dimensional projective space cannot be the boundary of a compact manifold. As for the Eguchi-Hanson metric we know that it is a Ricci flat metric on TS^2, so when we scale the metric it will converge to the flat cone over $\mathbb{R}P^3$. The type of cone singularities that develop in this way are also known as orbifold point singularities.

From the above observations we obtain the following result of Anderson (see [**1**])

THEOREM 6.7. *A sequence of n-manifolds satisfying*

$$\begin{array}{rcl} \text{vol} & \geq & v_0, \\ \text{diam} & \leq & D, \\ |\text{Ric}| & \leq & \Lambda, \\ \|R\|_{n/2} & \leq & Q. \end{array}$$

will subconverge to an orbifold with finitely many point singularities. Away from these points the convergence is in $C^{1,\alpha}$. In case n is odd the limit space is a manifold and the class is compact in the $C^{1,\alpha}$ topology.

It is important that one only gets finitely many singularities. The reason for this is that each singularity takes up a certain amount of $\int |R|^{n/2}$, otherwise the above gap theorem tells us that there isn't a singularity. Given that $\int |R|^{n/2}$ is bounded we can therefore only develop singularities at a given number of points.

Studying in detail this special type of possible degeneration Anderson and Cheeger (see [**3**]) were able to obtain a very attractive finiteness theorem for this class of manifolds.

THEOREM 6.8. *The class of n-manifolds satisfying*

$$\begin{array}{rcl} \mathrm{vol} & \geq & v_0, \\ \mathrm{diam} & \leq & D, \\ |\mathrm{Ric}| & \leq & \Lambda, \\ \|R\|_{n/2} & \leq & Q. \end{array}$$

contains only finitely many diffeomorphism types.

These two results take a particularly nice form on dimension 4. Namely, given the bound on the Ricci curvature one can obtain a bound on the $L^{n/2} = L^2$ norm of the curvature from the Euler characteristic. This is done using the Allendoerfer-Weil formula for the Gauss-Bonnet integrand in dimension 4

$$\begin{aligned} \chi(M) &= \frac{1}{8\pi^2}\int \left(|R|^2 - \left|\mathrm{Ric} - \frac{\mathrm{scal}}{4}I\right|^2\right) \\ &= \frac{1}{8\pi^2}\int |R|^2 - \frac{1}{8\pi^2}\int \left|\mathrm{Ric} - \frac{\mathrm{scal}}{4}I\right|^2. \end{aligned}$$

Thus

$$\int |R|^2 \leq C\left(|\chi(M)| + \Lambda\right).$$

This means that in these two theorems we can replace the $L^{n/2}$ bound on curvature by a bound on the Euler characteristic. Note that for an Einstein metric $\mathrm{Ric} = \frac{\mathrm{scal}}{4}I$, so in this case we don't need to know the Einstein constant in order to bound the $L^{n/2}$ norm of the curvature. Moreover, the Einstein constant is actually bounded by the Euler characteristic provided we have a lower volume bound.

From the discussion on Euler characteristic and L^2 norms on curvature we now obtain

COROLLARY 6.9. *Given D, v, C there are only finitely many diffeomorphism classes of Einstein 4-manifolds satisfying*

$$\begin{array}{rcl} \mathrm{vol} & \geq & v, \\ \mathrm{diam} & \leq & D, \\ \chi & \leq & C. \end{array}$$

Moreover, a sequence of such manifolds always has a subsequence which converges to an orbifold.

Having such a finiteness theorem means that we only need to study metrics on a fixed manifold. Thus we are left with studying the moduli of Einstein metrics on a fixed 4-manifold M. It is standard practice to fix something like the volume, but here it seems more natural to fix the Einstein constant. This leaves us with 3 different cases according to whether the Einstein constant is positive, zero or negative. Furthermore in the positive/negative case we can fix the constant to be ± 3. In the positive case the degeneration results then take the form

COROLLARY 6.10. *Let $v > 0$ be given. A sequence of 4-dimensional Einstein manifolds with Einstein constant 3 and volume $\geq v$ has a subsequence which converges to an orbifold with only point singularities.*

One can also establish a Gauss-Bonnet type formula for the ALE gravitational instantons. First we note that the volume growth together with the nonnegative Ricci curvature imply that the fundamental group is finite. From Poincaré duality it then follows that the Euler characteristic is $\chi = 1 + b_2 = 1 + \dim H^2(M, \mathbb{R})$. The Gauss-Bonnet formula now gives a formula for the modified Euler characteristic

$$\tilde{\chi}(M) = \left(1 - \frac{1}{|\Gamma|}\right) + b_2 = \frac{1}{8\pi^2}\int |R|^2 ,$$

where Γ is the finite fundamental group at infinity. The formula is obtained by applying the Gauss-Bonnet formula to a suitable sequence of sets that exhaust M. The $1/|\Gamma|$ term comes from the boundary terms as we pass to the limit. This means that the L^2 norm of the curvature is quantized, i.e., can only take certain values.

Moreover, if

$$\frac{1}{8\pi^2}\int |R|^2 < \frac{1}{2},$$

then the space must be Euclidean space. The Eguchi-Hanson metric in fact has a free $\mathbb{Z}_2$ action by isometries. If we pass to the quotient $TS^2/\mathbb{Z}_2$ we obtain a complete manifold where the Betti numbers satisfy $b_1 = b_2 = b_3 = b_4 = 0$. The topology of the example dictates that $\frac{1}{8\pi^2}\int |R|^2 = \frac{3}{4}$. It is not hard to show that this is the smallest possible nontrivial value for $\frac{1}{8\pi^2}\int |R|^2$.

7. Examples

It is worthwhile to study the orbifold degeneration in a little more detail. First we briefly mention some examples which show that such degeneration does occur.

The first examples of orbifold degeneration on compact manifolds come from [**14**]. Consider the standard flat 4-torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$. Let $\sigma : T^4 \to T^4$ be a Cartan involution, i.e., an involution in a point. Such an isometry has 16 fixed points. If we divide out by σ then we get a flat orbifold T^4/σ with 16 singularities each of which looks like a cone over $\mathbb{R}P^3$. We can blow each of these singularities up to get a K3 surface. On this K3 surface it is now possible to construct a sequence of Ricci flat metrics which converge in the above sense to the flat orbifold T^4/σ.

Tian in [**19**] shows that surfaces of the type $\mathbb{C}P^2\sharp k\overline{\mathbb{C}P^2}$, where $3 \leq k \leq 8$, admit positive Kähler-Einstein metrics. Moreover, it is also established that when $5 \leq k \leq 8$ it is possible for these Kähler-Einstein metrics to degenerate to orbifolds. Tian even gives very explicit possibilities for the exact type of degenerations that might occur. We just mention a folklore example which looks similar to the above K3 surface example. Take X to be the orbifold obtained from $S^2 \times S^2$ by dividing out by the Cartan involution that rotates π on each sphere. One can blow up the 4 resulting singularities to obtain $M = \mathbb{C}P^2\sharp 5\overline{\mathbb{C}P^2}$, and on this manifold it is possible using Tian's work to construct Kähler-Einstein metrics which converge to X.

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