Kähler-Einstein Manifolds of Positive Scalar Curvature

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1. Introduction

A Riemannian metric on M is Kähler-Einstein if it is Kähler and its Ricci curvature form is a constant multiple of its Kähler form. Such a metric provides a special solution of the Einstein equation on Riemannian manifolds. In 50's, E. Calabi asked when a compact Kähler manifold M admits any Kähler-Einstein metrics.

Since the Ricci form represents the first Chern class $c_1(M)$, a necessary condition for the existence of Kähler-Einstein metrics is that $c_1(M)$ is definite. In fact, Calabi conjectured that any (1,1)-form representing $c_1(M)$ is the Ricci form of some Kähler metric on M (the Calabi conjecture). In particular, the conjecture implies the existence of Ricci-flat Kähler metrics in case $c_1(M) = 0$. The Calabi conjecture was solved by Yau in 1977 [**28**]. Around the same time, Aubin and Yau proved independently the existence of Kähler-Einstein metrics on Kähler manifolds with negative first Chern class [**1, 28**]. Therefore, it had been known by the middle of 70's that $c_1(M)$ being zero or negative is also sufficient for the existence of Kähler-Einstein metrics on the underlying manifold.

Back to early 50's, using the maximum principle, Calabi had proved the uniqueness of Kähler-Einstein metrics within a fixed Kähler class for Kähler manifolds with non-positive first Chern class. In 1986, Bando and Mabuchi proved the uniqueness of Kähler-Einstein metrics on compact Kähler manifolds with positive first Chern class.

We remain to study the existence problem of Kähler-Einstein metrics on a compact Kähler manifold M with $c_1(M) > 0$. Now Kähler-Einstein metric, if they exist, must have positive scalar curvature.

In algebraic geometry, a Kähler manifold M with $c_1(M) > 0$ is called a Fano manifold.

In these notes, we summarize and discuss basic results on Kähler-Einstein manifolds with positive scalar curvature. We will emphasize on recent progress, particularly, the existence problem. The materials chosen here may depend on the author's interest.

¿From now on, unless specified, M always denotes a compact Kähler manifold with positive first Chern class. We will identify a Kähler metric g with its Kähler

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form ω_g, or simply ω if there is no confusion. In local coordinates, if g is given by hermitian matrices $\{g_{i\bar{j}}\}$, then

$$\omega = \omega_g = \sum_{i,=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

where n is the complex dimension of M.

2. Holomorphic Obstructions

There are holomorphic obstructions to existence of Kähler-Einstein metrics with positive scalar curvature.

We denote by $\eta(M)$ the Lie algebra of all holomorphic vector fields on M. In 1957, Matsushima proved

THEOREM 2.1. [**16**] *If M admits a Kähler-Einstein metric with positive scalar curvature, then $\eta(M)$ is reductive.*

Its proof can be found in [**16**] or [**11**]. It follows from this theorem that there are compact Kähler manifolds with the first Chern class positive and which do not admit Kähler-Einstein metrics. For instance, if M is the blow-up of $\mathbb{CP}^2$ at one or two points, then $\eta(M)$ is not reductive, consequently, such an M does not have any Kähler-Einstein metrics.

In 1983, Futaki introduced another holomorphic obstruction: let M be as above. Since $c_1(M) > 0$, one can find a Kähler metric ω such that $\frac{\sqrt{-1}}{2\pi}\omega = \omega_g$ represents $c_1(M)$. If $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, then $\frac{\sqrt{-1}}{2\pi}\mathrm{Ric}(g)$ also represents $c_1(M)$, therefore, there is a real-valued function h_ω such that

$$\mathrm{Ric}(\omega) - \omega = \partial\bar{\partial} h_\omega, \quad \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

Futaki proved in [**10**] that for any X in $\eta(M)$, the integral

$$\int_M X(h_\omega)\omega^n$$

is independent of choices of ω. Hence, one can define a holomorphic invariant $f_M : \eta(M) \mapsto \mathbb{C}$ by

$$f_M(X) = \int_M X(h_\omega)\omega^n.$$

In fact, f_M is a Lie algebra character, so it descends to $\eta(M)/[\eta(M), \eta(M)]$. Moreover, we have

THEOREM 2.2. [**10**]*If M admits a Kähler-Einstein metric with positive scalar curvature, then $f_M \equiv 0$.*

In [**10**], Futaki constructed an example of 3-dimensional M with $c_1(M) > 0$ such that $\eta(M)$ is reductive and $f_M \neq 0$. Hence, such an M does not admit any Kähler-Einstein metrics.

The Futaki invariant can be computed by formulas similar to the Bott residue formula for Chern numbers.

DEFINITION 2.3. X is non-degenerate if the zero set of X is the disjoint union of smooth connected complex submanifolds $\{Z_\lambda\}_{\lambda\in\Lambda}$ and if at each $z \in Z_\lambda$, $DX : T_zM/T_zZ_\lambda \mapsto T_zM/T_zZ_\lambda$ is non-degenerate, i.e., $\det(DX|_{T_zM/T_zZ_\lambda}) \neq 0$.

We denote by $L_\lambda(X)$ the induced homomorphism $DX : T_zM/T_zZ_\lambda \mapsto T_zM/T_zZ_\lambda$. The following was proved in [**11**].

THEOREM 2.4. *Let M be a compact Kähler manifold with $c_1(M) > 0$. Then for any $X \in \eta(M)$,*

$$f_M(X) = \frac{1}{n+1} \sum_{\lambda \in \Lambda} \int_{Z_\lambda} \frac{\operatorname{tr}(L_\lambda(X) + c_1(M))^{n+1}}{\det(L_\lambda(X) + \frac{\sqrt{-1}}{2\pi} K_\lambda)},$$

where K_λ is the curvature form of the induced metric on TM/TZ_λ by ω.

REMARK 2.5. The Futaki invariant is only a special case of the Calabi-Futaki invariant defined for any Kähler class on any compact Kähler manifolds. The Calabi-Futaki invariant can be defined in a way similar to what we did above (cf. [**11**]). It is an obstruction to existence of Kähler metrics with constant scalar curvature. There is also a residue formula computing the Calabi-Futaki invariant analogous to the above one in general cases [**25**]. Also we refer readers to [**5, 14**] for some references on Kähler metrics with constant scalar curvature.

The residue formula in Theorem 2.4 can be expressed more explicitly in the following two cases:

1. If X only has isolated zeroes so $\dim Z_\lambda = 0$ and hence the only terms that will contribute to the integral are the degree 0 terms and therefore
$$f_M(X) = \frac{1}{n+1} \sum_\lambda \frac{\operatorname{tr}(L_\lambda(X))^{n+1}}{\det(L_\lambda(X))}.$$
In particular if M has a Kähler-Einstein metric then we know that $f_M = 0$ and therefore this puts a constraint on the zero set of the vector field.
2. If M is a complex surface, then $\Lambda = \Lambda_0 \cup \Lambda_1$ where $\Lambda_i = \{\lambda \in \Lambda; \dim Z_\lambda = i\}$, and
$$f_M(X) = \frac{1}{3} \sum_{\lambda \in \Lambda_0} \frac{\operatorname{tr}(L_\lambda(X))^3}{\det(L_\lambda(X))} + \frac{2}{3} \sum_{\lambda \in \Lambda_1} L_\lambda(X)(c_1(M)(Z_{\mathrm{I}}) + 1).$$

Let us apply this to the blow-up M of $\mathbb{CP}^2$ at one point, say $[1,0,0]$. Choose the vector field X obtained by differentiating automorphisms:

$$\phi(\lambda)([z_0, z_1, z_2]) = [z_0, \lambda z_1, \lambda z_2]$$

at $\lambda = 1$. The zero set of X consists of two connected components E_0 and E_∞, where E_0 is the exceptional divisor of M and E_∞ is the pull-back of the divisor $\{z_0 = 0\}$ in $\mathbb{CP}^2$. It is easy to see that $L_\lambda(X) = 1$ along E_0 and -1 along E_∞. Hence, we have

$$f_M(X) = \frac{2}{3}(2-8) = -4.$$

The Futaki invariant of M is not zero. In a similar way, one can show that the blow-up of $\mathbb{CP}^2$ at two points has non-vanishing Futaki invariant.

In [**8**], Ding and Tian defined the generalized Futaki invariant for any almost Fano variety (possibly singular). They used this new invariant to construct new obstructions to existence of Kähler-Einstein metrics. Such obstructions were refined and used to define K-stability in [**24**]. The simplest case of these obstructions can be described as follows:

Let $\pi : N \mapsto D$ is a holomorphic submersion, such that M is biholomorphic to a fiber $N_z = \pi^{-1}(z)$ for some $z \in D$, where D is the unit disk in $\mathbb{C}^1$. We further assume that there is a family of automorphisms $\sigma(\lambda)$ of N which can descend to the dilations $z \mapsto \lambda z$ on D ($\lambda \leq 1$). Usually, we denote by X_N the holomorphic vector field on N_0 induced by those automorphisms. More precisely, $X_N = -\sigma'(1)$. If N is holomorphically different from $M \times D$, we say that $\pi : N \mapsto D$ is non-trivial. Otherwise, we say that N is trivial.

THEOREM 2.6. *If M admits a Kähler-Einstein metric with positive scalar curvature, then for any fibration $\pi : N \mapsto D$ as above, we have* $\mathrm{Re}(f_{N_0}(X_N)) \geq 0$*, the equality holds if and only if N is trivial, i.e., X_N does not preserve M.*

Theorem 2.6 can be used to disprove a long-standing conjecture in the case of complex dimensions higher than two. The conjecture claims that *any compact Kähler manifold M with $c_1(M) > 0$ and $\eta(M) = \{0\}$ admits a Kähler-Einstein metric.* A counterexample can be briefly described as follows (see section 6 for details): let $G(4,7)$ be the complex Grassmannian manifold consisting of all 4-dimensional subspaces in $\mathbb{C}^7$, for any 3-dimensional subspace $P \subset \wedge^2\mathbb{C}^7$, one can define a sub-variety X_P in $G(4,7)$ by

$$X_P = \{U \in G(4,7) \mid P \text{ projects to zero in } \wedge^2 (C^7/U) \}$$

For a generic P, X_P is a smooth 3-fold with $c_1(X_P) > 0$. These manifolds were first constructed by Iskovskih (cf. [**12**], [**17**]).

Take P_a to be the subspace spanned by bi-vectors

$$3e_1 \wedge e_6 - 5e_2 \wedge e_5 + 6e_3 \wedge e_4 + \textstyle\sum_{j+k\geq 8} a_{1jk}e_j,$$
$$3e_1 \wedge e_7 - 2e_2 \wedge e_6 + e_3 \wedge e_5 + \textstyle\sum_{j+k\geq 9} a_{2jk}e_j,$$
$$e_2 \wedge e_7 - e_3 \wedge e_6 + e_4 \wedge e_5 + \textstyle\sum_{j+k\geq 10} a_{3jk}e_j,$$

where e_i are euclidean basis of $\mathbb{C}^7$ and $a = \{a_{ijk}\}$.

Then we can deduce the following from Theorem 2.6

COROLLARY 2.7. *For generic a, X_{P_a} has neither nontrivial holomorphic vector fields nor Kähler-Einstein metrics.*

3. Kähler-Einstein Metrics and Complex Monge-Ampere Equations

Before 1987, only known Kähler-Einstein metrics were either homogeneous or of cohomogeneity one. They can be reduced to solving an either algebraic or ODE equation (cf. [**19**]). Kähler-Einstein metrics were first constructed on manifolds without any holomorphic vector fields by Tian [**21**], Tian-Yau [**26**] and Siu [**20**] by solving certain complex Monge-Ampere equations. Those manifolds have only finite automorphism group. They include Fermat hypersurfaces in $\mathbb{CP}^{n+1}$ of degree n and $n+1$, complex surfaces with symmetries. The method in [**21**] and [**26**] was based on the following holomorphic invariant introduced in [**21**]. More examples of Kähler-Einstein manifolds were later constructed by A. Nadel in [**18**]. In particular, Nadel proved that there are Kähler-Einstein metrics on Fermat hypersurfaces in $\mathbb{CP}^{n+1}$ of degree $d \geq 1 + n/2$.

In this section, we give an analytic criterion for the existence of Kähler-Einstein metrics with positive scalar curvature. We will also show that every Fermat hypersurface admits a Kähler-Einstein metric.

Let M be a compact Kähler manifold with $c_1(M) > 0$, and ω be a Kähler metric with the Kähler class $c_1(M)$. We define $P(M,\omega)$ to be the set of smooth functions φ satisfying: $\omega + \partial\overline{\partial}\varphi > 0$. It is essentially the set of Kähler metrics with the Kähler class $c_1(M)$.

If G is a maximal compact subgroup of the automorphism group of M, then we may take ω to be G-invariant and define

$$P_G(M,\omega) = \{\varphi \in P(M,\omega) \mid \sigma^*\varphi = \varphi, \forall \sigma \in G\}.$$

For any $\varphi \in P(M,\omega)$, we put $\omega_\varphi = \omega + \partial\overline{\partial}\varphi$. If ω_φ is a Kähler-Einstein metric, then

$$-\partial\overline{\partial}\log\left(\frac{\omega_\varphi^n}{\omega^n}\right) = \omega + \partial\overline{\partial}\varphi - \mathrm{Ric}(\omega),$$

since $\mathrm{Ric}(\omega) - \omega = \partial\overline{\partial}h_\omega$, we can derive from the above

$$(\omega + \partial\overline{\partial}\varphi)^n = e^{h_\omega - \varphi}\omega^n,\ \omega + \partial\overline{\partial}\varphi > 0. \tag{3.1}$$

This is a complex Monge-Ampere equation. The existence of Kähler-Einstein metrics on M is equivalent to solvability of (3.1).

In order to solve (3.1), we used the continuity method and consider

$$(\omega + \partial\overline{\partial}\varphi)^n = e^{h_\omega - t\varphi}\omega^n,\ \omega + \partial\overline{\partial}\varphi > 0, \tag{3.2}$$

where $t \in [0,1]$; for a given t, we will refer to this as equation $(3.2)_t$. Let S be the set of t in $[0,1]$ such that $(3.2)_t$ is solvable. Then by Yau's solution for the Calabi conjecture [**28**], $0 \in S$.

LEMMA 3.1. *Let φ be a solution of $(3.2)_t$, and $\omega_t = \omega_\varphi$. Then $\mathrm{Ric}(\omega_t) \geq t\omega_t$, and the equality holds if and only if $t = 1$.*

PROOF. . Taking $\partial\overline{\partial}$ on both sides of $(3.2)_t$, we obtain

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

Then the lemma follows. □

Combining this with a Bochner identity, Aubin [**2**] proved in that the first nonzero eigenvalue of ω_t is bigger than t for $t < 1$. On the other hand, the linearization of $(3.2)_t$ at φ is given by $\Delta_t + t$, where Δ_t denotes the Laplacian operator of ω_t. Therefore, the linearization is invertible if $t < 1$. So by using the Implicit Function Theorem, Aubin concluded in [**2**] that S is open in $[0,1]$.

It remains to show that S is closed. It amounts to establishing an a priori C^3-estimate for solutions of $(3.2)_t$. Repeating Yau's computations for C^2 in [**28**], one can show

LEMMA 3.2. *There is a uniform constant $c > 0$, such that for any solution φ of 3.2, we have*

$$||\varphi||_{C^2} \leq ce^{c\sup_M|\varphi|}.$$

Then, using Calabi's computations for C^3 in [**28**], one can further show that $||\varphi||_{C^3}$ can be bounded uniformly by an a prior C^0-estimate for any solution φ of $(3.2)_t$.

However, such a C^0-estimate does not exist due to those obstructions in Section 2. Therefore, it is of fundamental importance to find a correct geometric condition for validity of such a C^0-estimate.

A sufficient condition can be described as follows: we define

$$\alpha(M) = \sup\{\alpha \mid \text{there is a } C(\alpha) > 0, s.t. \frac{1}{V}\int_M e^{\alpha(\varphi - \sup_M\varphi)}\omega^n \leq C(\alpha), \tag{3.3}$$

One can show that it is always positive and independent of G and ω, so it is a holomorphic invariant [**21**]. It was shown in [**21**] (cf. [**7**]) that *M has a Kähler-Einstein metric whenever $\alpha(M) > \frac{n}{n+1}$*. The existence of Kähler-Einstein metrics was shown in [**21**], [**26**] and [**18**] by establishing $\alpha(M) > \frac{n}{n+1}$. In particular, Nadel proved elegant vanishing theorems involving multiplier sheaves if $\alpha(M)$ is less than 1.

Unfortunately, $\alpha(M)$ may not exceed $\frac{n}{n+1}$ even if M has a Kähler-Einstein metric.

It turns out that (3.1) is the Euler-Lagrange equation of a functional F_ω, which will be defined as follows: for any $\varphi \in P(M, \omega)$,

$$\begin{aligned} F_\omega(\varphi) &= J_\omega(\varphi) - \tfrac{1}{V}\int_M \varphi\omega^n - \log\left(\tfrac{1}{V}\int_M e^{h_\omega - \varphi}\omega^n\right) \\ J_\omega(\varphi) &= \sum_{i=0}^{n-1} \tfrac{i+1}{n+1}\tfrac{1}{V}\int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i \wedge (\omega + \partial\overline{\partial}\varphi)^{n-i-1}, \end{aligned}$$

where $V = \int_M \omega^n = c_1(M)^n$. An easy computation shows that (3.1) is just the first variation of F_ω. Clearly, $J_\omega(\varphi) \geq 0$ whenever $\varphi \in P(M, \omega)$. So we may consider J_ω as a generalized energy.

DEFINITION 3.3. We say that F_ω is proper on $P_G(M, \omega)$, if there is an increasing function $\mu : \mathbb{R} \mapsto [c(\omega), \infty)$, such that $\lim_{t\to\infty} \mu(t) = \infty$ and $F_\omega \geq \mu(J_\omega)$ on $P_G(M, \omega)$, where $c(\omega)$ is some constant.

An easy computation shows that F_ω satisfies the following cocycle condition:

$$F_\omega(\varphi) - F_{\omega_\psi}(\varphi - \psi) = F_\omega(\psi).$$

It follows that for any $\psi \in P_G(M, \omega)$, if F_ω is proper, so is F_{ω_ψ}. Clearly, it is also independent of choices of the maximal compact subgroup G. Therefore, the properness of F_ω is an intrinsic property of the underlying manifold M.

THEOREM 3.4. [**24**] *Let M be a compact Kähler manifold with $c_1(M) > 0$. Then M has a Kähler-Einstein metric if and only if F_ω is proper on $P_G(M, \omega)$.*

This gives an analytic criterion for the existence of Kähler-Einstein metrics. The properness can often be checked. In general, we believe that the properness is equivalent to certain stability of underlying manifolds in Geometry Invariant Theory.

REMARK 3.5. The arguments in [**7**] shows that F_ω is essentially proper if $\alpha(M) > \frac{n}{n+1}$.

Now let us apply this theorem to proving that any Fermat hypersurfaces admit Kähler-Einstein metrics.

A Fermat hypersurface of degree d is of the form

$$M = \{z_0^d + \ldots + z_{n+1}^d = 0\} \subset \mathbb{CP}^{n+1}.$$

Since $c_1(M) > 0$, we have $d < n + 2$. Note that for other d, M has non-positive $c_1(M)$, so it has a Kähler-Einstein metric with non-positive scalar curvature.

Consider the group G_0 generated by

$$\sigma_i : [z_0, \ldots, z_{n+1}] \mapsto [z_0, \ldots, e_d z_i, \ldots, z_{n+1}]$$

where $e_d = e^{\frac{2\pi\sqrt{-1}}{d}}$. Clearly, M is invariant under G_0. Let G be a maximal compact subgroup of $\mathrm{Aut}(M)$ containing G_0 and ω be a G-invariant metric. We will show that F_ω is proper on $P_G(M, \omega)$. Let

$$\pi_i : M \to \mathbb{CP}^n_i$$

be the projection onto $\mathbb{CP}_i^n = \{[z_0, ..., z_{i-1}, 0, z_{i+1}, ..., z_{n+1}]\} \simeq \mathbb{CP}^n$. Note that this map is well-defined because $[0, ..., 0, 1, 0, ..., 0] \notin M$. So any $\phi \in P_G(M, \omega)$ is of the form $(n+2-p)\pi_i^*\varphi$ for some $\varphi \in P(M, \omega_{FS})$, where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^n$. Note that $c_1(M) = (n+2-p)[\pi^*\omega_{FS}]$. We may normalize $\int_M e^{h_\omega - \phi}\omega^n = V$.

For any Kähler metric ω and $\phi \in P(M, \omega)$, we put

$$F_\omega^0(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n.$$

This functional also satisfies the cocycle condition, so

$$F_\omega(\phi) = F_\omega^0(\phi) = F^0_{(n+2-d)\pi_i^*\omega_{FS}}(\phi) - F_\omega^0(u),$$

where $(n+2-d)\pi_i^*\omega_{FS} = \omega + \partial\overline{\partial}u$ may not be a Kähler metric, but u and consequently, $F_\omega^0(u)$ are bounded. Furthermore, one can check that

$$F^0_{(n+2-d)\pi_i^*\omega_{FS}}(\phi) = \frac{n+2-d}{n+1}F^0_{(n+1)\omega_{FS}}(\frac{n+1}{n+2-d}\varphi).$$

Since $\mathbb{CP}^n$ has a canonical Kähler-Einstein metric $(n+1)\omega_{FS}$, by Theorem 2.1, we know that $F_{(n+1)\omega_{FS}}$ is bounded from below on $P(\mathbb{CP}^n, (n+1)\omega_{FS})$. This implies that

$$F_\omega(\phi) \geq \frac{n+2-d}{n+1}\log\left(\frac{1}{V}\int_M e^{-\frac{n+1}{n+2-d}\phi}(n+2-d)^n\pi_i^*\omega_{FS}^n\right) - C.$$

Note that C always denotes a uniform constant. Consequently,

$$F_\omega(\phi) \geq \frac{n+2-p}{n+1}\log\left(\frac{1}{V}\int_M e^{-\frac{n+1}{n+2-d}\phi}(n+2-d)^n\sum_{i=0}^{n+1}\pi_i^*\omega_{FS}^n\right) - C,$$

since $\sum_{i=0}^{n+1}\pi_i^*\omega_{FS}^n \geq c\omega$ for some $c > 0$, we have

$$F_\omega(\phi) \geq \frac{n+2-d}{n+1}\log\left(\frac{1}{V}\int_M e^{-\frac{d-1}{n+2-d}\phi}e^{h_\omega - \phi}\omega^n\right) - C.$$

We may assume that $d > 1$.

Claim 1. There is a uniform $C > 0$ such that

$$\sup_M \phi \leq C\left(1 + \log\left(\frac{1}{V}\int_M e^{-\frac{d-1}{n+2-d}\phi}e^{h_\omega-\phi}\omega^n\right)\right).$$

Proof. Fix any α in $(0, \alpha(M))$, then for any $\phi \in P(M, \omega)$,

$$\frac{1}{V}\int_M e^{-\alpha(\phi - \sup_M\phi)}\omega^n \leq C(\alpha).$$

Choose $\delta > 0$ such that

$$\alpha = \frac{\delta(n+1)}{\delta(n+2-d)+d-1}.$$

Then by the Hölder inequality, we have

$$\begin{aligned} 1 &= \tfrac{1}{V}\textstyle\int_M e^{h_\omega-\phi}\omega^n \\ &\leq e^{-\delta sup_M \phi}\left(\tfrac{1}{V}\textstyle\int_M e^{\alpha(h_\omega - \phi + \sup_M\phi)}\omega^n\right)^{\frac{\delta}{\alpha}}\left(\tfrac{1}{V}\textstyle\int_M e^{\frac{d-1}{n+2-d}(h_\omega-\phi)}e^{h_\omega-\phi}\omega^n\right)^{\frac{(1-\delta)(n+2-d)}{n+1}}. \end{aligned}$$

This shows the claim. □

It follows from this claim that F_ω is proper on $P_G(M,\omega)$, and consequently, M admits a Kähler-Einstein metric.

In fact, if M has a Kähler-Einstein metric, the functional F_ω is not only proper, but satisfies the following nonlinear inequality.

THEOREM 3.6. *Let (M,ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$. Then there are constants $\delta = \delta(n)$ and $c = c(n, \lambda_2(\omega) - 1)$, such that for any $\phi \in P(M,\omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \geq J_\omega(\phi)^\delta - c,$$

which is the same as

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \leq e^c e^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - J_\omega(\phi)^\delta},$$

where Λ_1 is the space of eigenfunctions with eigenvalue 1, and $\lambda_2(\omega)$ is the next eigenvalue of ω after 1.

This was proved in [**24**] under a further condition on ϕ. This condition was removed later by Zhu and myself.

Indeed, it was conjectured in [**24**] that *for any $\phi \in P(M,\omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \geq \delta J_\omega(\phi) - c,$$

where $\delta = \delta(n) > 0$ and $c = c(n, \lambda_2(\omega) - 1) > 0$ are as above.

Finally, we would like to cite the uniqueness theorem of Bando and Mabuchi.

THEOREM 3.7. [**4**] *The solution of (3.1) is unique modulo the connected component $Aut_0(M)$ of $Aut(M)$ containing the identity if it exists, where $Aut(M)$ denotes the group of all holomorphic automorphisms of M. In particular, it implies the uniqueness of the Kähler-Einstein metric on M if it exists.*

4. The Case of Complex Surfaces

In this section, we assume that $n = 2$, i.e., M is a complex surface with positive first Chern class. By the classification theory of complex surfaces, M is either $\mathbb{CP}^1 \times \mathbb{CP}^1$ or the blow-up of $\mathbb{CP}^2$ at k points $(0 \leq k \leq 8)$.

THEOREM 4.1. [**22**] *Let M be a complex surface with $c_1(M) > 0$. Then M admits a Kähler-Einstein metric if and only if its Lie algebra $\eta(M)$ of holomorphic fields is reductive.*

Theorem 4.1 gives the complete solution to the existence of Kähler-Einstein metrics on complex surfaces.

Now we outline the proof of Theorem 4.1 in [**22**]. First, by Theorem 1.1, if M has a Kähler-Einstein metric, then $\eta(M)$ is reductive. So we need to only prove the sufficient part. By the classification theory of complex surfaces, we may assume that M is a blow-up of $\mathbb{CP}^2$ at k points $(1 \leq k \leq 8)$, since $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ have homogeneous Kähler-Einstein metrics. A direct computation shows that $\eta(M)$ is not reductive if $k = 1$ or 2. It was shown in [**26**] that $\alpha(M) \geq 1$ when $k = 3$ and $\alpha(M) \geq 3/4$ when $k = 4$, so M has a Kähler-Einstein metric when $k = 3, 4$. The case that $k = 3$ was also proved independently by Siu [**20**].

Now, we may assume that $k \geq 5$. Then the moduli space of complex structures on the differentiable manifold underlying M is connected and $(k-3)$-dimensional. We denote by $\mathcal{M}_k$ such a moduli space.

Let M be a complex surface with positive first Chern class. By the main theorem in [**26**], each $\mathcal{M}_k$ contains at least one Kähler-Einstein surface M_k'. Choose a smooth family of Kähler surfaces $\{M_t\}_{0\leq t\leq 1}$ with $c_1(M_t) > 0$, such that $M_1 = M$ and $M_0 = M_k'$ admits a Kähler-Einstein metric g_0.

Now we define

$$E = \{t \in [0,1] \mid M_{t'} \text{ has a K} - \text{E metric for any } t' \leq t\}.$$

Then E is nonempty since $0 \in E$.

Since each M_t has no non-trivial holomorphic vector fields, by the Implicit Function Theorem, one can easily show that E is open in $[0,1]$.

Hence, in order for M to have a Kähler-Einstein metric, we suffice to prove that E is closed. The closure of E is the most difficult part in the proof of Theorem 4.1. We accomplish this in the following two steps.

Without loss of generality, we may assume that $E = [0,1)$. Let ω_t $(0 \leq t < 1)$ be the Kähler-Einstein metric on M_t. We need to show that ω_t converges to a metric ω_1 on M in C^2-topology. Since $\{M_t\}_{0\leq t\leq 1}$ is a smooth family of Kähler surfaces with $c_1(M) > 0$, there are Kähler metrics $\tilde{\omega}_t$ on M_t $(0 \leq t \leq 1)$ such that their Kähler class is $C_1(M)$. We may further assume that for each $t < 1$, there is a unique function φ_t such that $\omega_t = \tilde{\omega}_t + \partial\bar{\partial}\varphi_t$ and

$$\omega_t^2 = e^{\tilde{h}_{\tilde{\omega}_t} - \varphi_t}\, \tilde{\omega}_t^2 \text{ on } M_t \cdot t \tag{4.1}$$

As we argued in last section, if there is an a prior C^0-estimate of φ_t $(0 \leq t < 1)$, then by taking a subsequence if necessary, we may have that ω_t converges to a metric on M.

Instead of deriving such a C^0-estimate directly, we will first give a partial C^0-estimate for φ_t.

First we introduce a notion of height functions for line bundles. Let L be a positive Hermitian line bundle over M. Let ω be a Kähler metric with $c_1(L)$ as its Kähler class. We define a height function $h_L(\cdot,\omega)$ by

$$h_L(x,\omega) = \sup_{0\neq s\in H^0(M,L)} \sqrt{\frac{a(s,\bar{s})(x)}{\frac{1}{V}\int_M a(s,\bar{s})\omega^n}},$$

where $x \in M$, $V = \int_M \omega^n$ and $a(\cdot,\cdot)$ is a Hermitian metric of L with the curvature form ω. Though such a hermitian metric $a(\cdot,\cdot)$ is not unique, the above h_L will remains the same. If L is very ample, then h_L is strictly positive. In fact, it was proved in [**23**] that $m^{-1}h_{L^m}(\cdot, m\omega)^{\frac{1}{m}}$ converges to 1 uniformly on M as m tends to infinity.

In our situation, L is the anti-canonical bundle K_M^{-1}. We simply write h_m for $h_{K_M^{-m}}$. The following was first proposed in [**25**].

Conjecture 1. For any $\epsilon > 0$, there are $m = m(n,\epsilon)$ and $c = c(m,n,\epsilon) > 0$ such that $h_m(\cdot, m\omega) \geq c$ for any Kähler metric ω with the Kähler class $c_1(M)$ on any compact Kähler manifold M of dimension n, provided that $\mathrm{Ric}(\omega) \geq \epsilon\omega$.

It is possible that the conclusion in the above conjecture is true for any sufficiently large m. This conjecture should be also true for a general ample line bundle L over a much larger class of Kähler manifolds. For example, it is possible that *for any $d > 0$, there are $m = m(n,d)$ and $c = c(m,n,d) > 0$ such that $h_{L^m}(\cdot, m\omega) \geq c$ for any Kähler metric ω with the Kähler class $c_1(L)$ on any compact Kähler manifold M of dimension n, provided that ω has constant scalar curvature* *and its diameter* diam(ω) *is bounded from above by* d. Recently, Zhiqin Lu was able to verify this for hyperbolic Riemann surfaces.

The following was proved in **[22]** (though it was stated differently)

THEOREM 4.2. *The above conjecture holds for any Kähler-Einstein surfaces of complex dimension two. In fact, for* $m > 0$ *and* $m \equiv 0 \bmod 6$, *there is a constant* $C(m)$ *such that* $h_m(\cdot, m\omega) \geq C(m)$ *for any Kähler-Einstein surfaces.*

The proof of this theorem used the Cheeger-Gromov's compactness Theorem, Uhlenbeck's estimates for Yang-Mills connections and L^2-estimates for $\bar{\partial}$-operators.

Now we explain what this means to φ_t. Let m be given in the above theorem. Choose any orthonormal basis $\{S_i(t)\}_{0\leq i\leq N_m}$ of $H^0(M, K_M^{-m})$ with respect to the inner product induced by ω_t. Then

$$\log(\sum_{i=0}^{N_m} a_t(S_i, \bar{S}_i)) \geq \log C(m), \tag{4.2}$$

where $a_t(\cdot,\cdot)$ denotes the Hermitian metric on K_M^{-1} defined by ω_t^n.

LEMMA 4.3. *For* $S \in H^0(M_t, K_M^{-m})$ $(t < 1)$, *we have*

$$\sup_{M_t} a_t(S, \bar{S})(x) \leq \frac{C(m)}{V} \int_{M_t} a_t(S, \bar{S}) \omega_t^n.$$

Note that we always denote by $C(m)$ *a constant depending only on* m.

PROOF. An easy computation shows that

$$-\Delta_t \sqrt{a_t(S, \bar{S})} \leq m \sqrt{a_t(S, \bar{S})},$$

where Δ_t denotes the Laplacian operator of ω_t. Since $\mathrm{Ric}(\omega_t) = \omega_t$, the Sobolev constant can be bounded independent of t. So the lemma follows from the above differential inequality and a standard iteration. □

It follows from this lemma and (4.2) that

$$|\log(\sum_{i=0}^{N_m} a_t(S_i, \bar{S}_i))| \leq \log C(m). \tag{4.3}$$

Let $\tilde{a}_t$ be the Hermitian metric with the curvature $\tilde{\omega}_t$ with

$$\sup_{M_t} \log(\sum_{i=0}^{N_m} \tilde{a}_t(S_i, \bar{S}_i)) = 0.$$

Then we have

$$\varphi_t - \frac{1}{m} \log(\sum_{i=0}^{N_m} \tilde{a}_t(S_i, \bar{S}_i)) = \frac{1}{m} \log(\sum_{i=0}^{N_m} a_t(S_i, \bar{S}_i)).$$

It follows from this and (4.3) that

$$\|\varphi_t - \sup_M \varphi_t - \frac{1}{m} \log(\sum_{i=0}^{N_m} \tilde{a}_t(S_i, \bar{S}_i))\|_{C^0(M_t)} \leq C(m) \tag{4.4}$$

By using orthogonal transformations if necessary, we may assume that $S_i = \lambda_i \tilde{S}_i$ for an orthonormal basis $\{\tilde{S}_i\}$ of $H^0(M, K_M^{-m})$ with respect to the inner product induced by $\tilde{\omega}_t$. Moreover, we may arrange $0 < \lambda_0 < \cdots < \lambda_{N_m}$. Then (4.4) implies that each $\varphi_t - \sup_{M_t} \varphi_t$ is uniformly bounded in any compact subset outside the zero locus of $S_{N_m}(t)$. For this reason, we call (4.4) a partial C^0-estimate of φ_t.

To finish the proof of Theorem 4.1, we need a numerical criterion for the existence of a Kähler-Einstein metric on M.

Let $P_{G,m,k}(M, \tilde{\omega}_1)$ be the collection of all G-invariant functions of the form $\frac{1}{m}\log(\sum_{i=1}^k \tilde{a}_1(S_i, \bar{S}_i))$, where $\{S_i\}_{0\le i\le N_m}$ is an orthonormal basis of $H^0(M, K_M^{-m})$ with respect to the metric $\tilde{\omega}_1$.

Define

$$\alpha_{m,k}(M) = \sup\{\alpha \mid \exists C_\alpha > 0, \text{ s.t. } \frac{1}{V}\int_M e^{-\alpha(\varphi - \sup_M \varphi)}\tilde{\omega}_1^2 \le C_\alpha, \forall \varphi \in P_{G,m,k}(M, \tilde{\omega}_1)\}$$

THEOREM 4.4. *Let m be given in Theorem 4.2. If either $\alpha_{m,1}(M) > 2/3$ or $\alpha_{m,2}(M) > 2/3$ and $\alpha_{m,1}(M) > l_0/(l_0+1)$ where $l_0 = \max\{0, 2 - \frac{3\alpha_{m,2}(M)-2}{\alpha_{m,2}(M)}\}$, then M admits a Kähler-Einstein metric.*

Indeed, the conditions in Theorem 4.4 assures the properness of $F_{\tilde{\omega}_t}$ discussed in last section.

By direct computations, we proved in [**24**] that $\alpha_{6,1}(M) \ge 2/3$, $\alpha_{6,2}(M) > 2/3$ for any complex surface M with $c_1(M) > 0$. Thus Theorem 4.1 is proved.

The proof of Theorem 4.1 can be simplified by using Theorem 3.4 in the cases that M is a blow-up of $\mathbb{CP}^2$ at k points with $k = 5, 7, 8$. The case that $k = 6$ still needs to use Theorem 4.2. Let us show this by establishing directly existence of Kähler-Einstein metrics on any blow-up M of $\mathbb{CP}^2$ at 7 points in general position (i.e., $c_1(M) > 0$).

Let M be a blow-up of $\mathbb{CP}^2$ at 7 points. Then it is a double covering $\pi : M \mapsto \mathbb{CP}^2$ with branch locus along a smooth quartic curve. Assume that G contains the deck transformation of π. As before, ω denotes a fixed, G-invariant Kähler metric with the Kähler class $c_1(M)$. Then for any $\varphi \in P_G(M, \omega)$, $\pi^*\pi_*\varphi = 2\varphi$. We normalize φ such that

$$\frac{1}{V}\int_M e^{h_\omega - \varphi}\omega^2 = 1.$$

Put $f = \pi^*\omega_{FS}^2/\omega^2$. It is non-negative and vanishes along the branch locus of π. Moreover, we have

$$\frac{1}{V}\int_M |f|^{-\frac{4}{5}}\omega^2 \le c < \infty.$$

Using the Hölder inequality, we deduce from this

$$\frac{1}{V}\int_M e^{-\frac{4}{3}\varphi}\omega^2 \le c\left(\frac{1}{V}\int_M e^{-3\varphi}\pi^*\omega_{FS}^2\right)^{\frac{4}{9}}.$$

Note that c always denotes a uniform constant. By using Theorem 3.4 for $\mathbb{CP}^2$, we can show (cf. the example in last section) that

$$F_\omega(\varphi) \ge \frac{1}{3}\log\left(\frac{1}{V}\int_M e^{-3\varphi}\pi^*\omega_{FS}^2\right) - c.$$

It follows from the above inequalities that

$$F_\omega(\varphi) \ge \frac{1}{4}\sup_M \varphi - c.$$

Therefore, F_ω is proper on $P_G(M,\omega)$, and consequently, M admits a Kähler-Einstein metric.

5. Kähler-Ricci Solitons

Let M be a compact Kähler manifold. A Kähler-Ricci soliton is a Kähler metric ω such that $\mathrm{Ric}(\omega) - \omega = L_X\omega$, where X is a complex vector field and L_X is its Lie derivative. Since both the Ricci curvature and the metric are real and of type $(1,1)$, the field X has to be holomorphic and its imaginary part is a Killing field. Since ω is d-closed, $L_X\omega = \partial i_X\omega$ and $\overline{\partial} i_x\omega = 0$. It follows that there is a function θ such that $L_X\omega = \partial\overline{\partial}\theta$. This implies that $c_1(M) > 0$, i.e., Kähler-Ricci solitons can exist only on a compact Kähler manifold with positive first Chern class. We also notice that a Kähler-Ricci soliton is Kähler-Einstein if and only if the Futaki invariant vanishes.

The soliton-type solutions have been widely studied in the regularity theory of nonlinear parabolic equations. Ricci-solitons were first studied by R. Hamilton in his works on the Ricci flow. Its Kähler version was studied by H. Cao in his works on Kähler-Ricci flow. Interesting examples of Kähler-Ricci solitons were constructed by Koiso [**13**], H. Cao [**6**].

As in the case of Kähler-Einstein metrics, we can also reduce the existence of Kähler-Ricci solitons to solving a complex Monge-Ampere equation as follows.

We may assume that $c_1(M) > 0$, in particular, M is simply-connected. Choose a Kähler metric ω with the Kähler class $c_1(M)$. Let θ_X be the function defined by

$$i_X\omega = \overline{\partial}\theta_X, \ \frac{1}{V}\int_M e^{\theta_X}\omega^n = 1.$$

Then for any $\varphi \in P(M,\omega)$, we have

$$i_X\omega_\varphi = \overline{\partial}(\theta_X + X(\varphi)).$$

Now if ω_φ is a Kähler-Ricci soliton, then

$$\mathrm{Ric}(\omega_\varphi) - \omega_\varphi = \partial\overline{\partial}(\theta_X + X(\varphi)).$$

Hence,

$$(\omega + \partial\overline{\partial}\varphi)^n = e^{h_\omega - \varphi - \theta_X - X(\varphi)}\omega^n. \tag{5.1}$$

If M has no nontrivial holomorphic fields, then this equation reduces to (3.1).

Recently, it was proved by X.H.Zhu and the author that the solution of (5.1) is unique modulo automorphisms of M. This extends clearly the uniqueness theorem of Bando-Mabuchi.

The equation (5.1) bears considerable resemblance to (3.1), and is, moreover, the Euler-Lagrange equation of a functional similar to F_ω. However, few results are known pertaining to the solvability of (5.1). If M has sufficiently many symmetries, then one can reduce it to an ordinary differential equation and solve it (cf. [**13**]). On the other hand, the counterexample in Section 2 shows that (5.1) is not always solvable. A plausible conjecture is the following: *Given any compact Kähler manifold M with $c_1(M) > 0$, either M has a unique Kähler-Ricci soliton or there is a smooth holomorphic fibration $\pi : N \mapsto D$, where D is the unit disc in $\mathbb{C}$, such that M is biholomorphic to each $\pi^{-1}(z)$ with $z \neq 0$ and $\pi^{-1}(0)$ admits a unique Kähler-Ricci soliton.*

Note that M can be separated from the central fiber $\pi^{-1}(0)$ in the moduli space of complex structures. Also unless N is a trivial fibration, N and consequently, $\pi^{-1}(0)$, has a nontrivial holomorphic field.

6. CM-Stability

In this section, we show a connection between the existence of Kähler-Einstein metrics and the stability of underlying manifolds. It was proved by Donaldson-Uhlenbeck-Yau [**9**], [**27**] hat the existence of Hermitian-Yang-Mills connections is equivalent to the stability of the underlying holomorphic bundle. Inspired by this, in late 80's, S.T. Yau proposed the following

CONJECTURE 2 (Yau). *The existence of Kähler-Einstein metrics should correspond to certain stability of the underlying manifold in the geometric invariant theory.*

This stability of algebraic manifolds should be described in terms of an action of a linear group G on a Hilbert scheme together with a G-linearized line bundle L. There is often a clear choice of such a Hilbert scheme, which parameterizes all manifolds of type similar to that of M. However, there may be possibly many stability conditions which are specified by G-linearized line bundles. Therefore, to solve the above conjecture, we first need to identify the stability corresponding to Kähler-Einstein metrics, or equivalently, the G-linearized line bundle L defining the stability. Next we need to show how Kähler-Einstein metrics are related to the stability. Now let us briefly describe how these can be done.

In fact, our arguments also show a connection between Kähler metrics with constant scalar curvature and the stability on any compact Kähler manifolds.

Let $G = SL(N+1,\mathbb{C})$. Consider a G-equivariant holomorphic fibration $\pi : \mathcal{X} \mapsto Z$ with equi-dimensional fibers. We further assume that there is a G-equivariant line bundle $\mathcal{L}$ on $\mathcal{X}$ with a hermitian metric h, such that its curvature $R(h)$ restricts to a Kähler metric over each fiber. We let Z_0 be the sub-variety of Z consisting of those smooth fibers. Clearly, Z_0 is G-invariant.

Consider the virtual bundle

$$\mathcal{E} = (n+1)(\mathcal{K}^{-1} - \mathcal{K}) \otimes (\mathcal{L} - \mathcal{L}^{-1})^n - n\mu(\mathcal{L} - \mathcal{L}^{-1})^{n+1}, \tag{6.1}$$

where $\mathcal{K} = K_{\mathcal{X}} \otimes K_Z^{-1}$ is the relative canonical bundle, and μ is equal to

$$\frac{c_1(M) \wedge c_1(L)^{n-1}}{c_1(L)^n},$$

where $M = \pi^{-1}(z)$ for a fixed $z \in Z_0$ and $L = \mathcal{L}|_M$.

We define L_Z to be the inverse of the determinant line bundle $\det(\mathcal{E}, \pi)$.

A straightforward computation shows that

$$\begin{aligned} & ch_{n+1}((n+1)(\mathcal{K}^{-1} - \mathcal{K}) \otimes (\mathcal{L} - \mathcal{L}^{-1})^n - n\mu(\mathcal{L} - \mathcal{L}^{-1})^{n+1} \\ = \; & 2^{n+1}\left((n+1)c_1(\mathcal{K}^{-1})c_1(\mathcal{L})^n - n\mu c_1(\mathcal{L})^{n+1}\right)\end{aligned} \tag{6.2}$$

Therefore, by the Grothendieck-Riemann-Roch Theorem,

$$c_1(L_Z) = 2^{n+1}\pi_*\left((n+1)c_1(\mathcal{K})c_1(\mathcal{L})^n + n\mu c_1(\mathcal{L})^{n+1}\right) \tag{6.3}$$

We also denote by $\mathcal{L}_Z^{-1}$ the total space of the line bundle L_Z^{-1} over Z. Then G acts naturally on $\mathcal{L}_Z^{-1}$. Recall that $M = \pi^{-1}(z)$ ($z \in Z_0$) is weakly CM-stable with respect to L, if the orbit $G \cdot \tilde{z}$ in $\mathcal{L}_Z^{-1}$ is closed, where $\tilde{z}$ is any nonzero vector in the fiber of L_Z^{-1} over z; If, in addition, the stabilizer G_z of z is finite, then M is CM-stable. We also recall that M is CM-semistable, if the 0-section is not in the closure of $G \cdot \tilde{z}$. Clearly, this G-stability (resp. G-semistability) is independent of choices of $\tilde{z}$.

THEOREM 6.1. **[24]** *Let $\pi : \mathcal{X} \mapsto Z$ be as above. Assume that M has a Kähler-Einstein metric and $c_1(M) = \mu c_1(L)$. Then M is weakly CM-stable. If X_z has no nontrivial holomorphic vector fields, it is actually CM-stable with respect to L.*

EXAMPLE 6.2. Let us apply Theorem 6.1 to proving Corollary 2.7.

Let Q be the universal quotient bundle over $G(4,7)$, which consists of all 4-subspaces in $\mathbb{C}^7$.

Let π_i $(i = 1,2)$ be the projection from $G(4,7) \times G(3, H^0(G(4,7), \wedge^2 Q))$ onto its i^{th}-factor, and let S be the universal bundle over $G(3, H^0(G(4,7), \wedge^2 Q))$. Then there is a natural endomorphism over $G(4,7) \times G(3, H^0(G(4,7), \wedge^2 Q))$

$$\Phi : \pi_2^* S \mapsto \pi_1^* \wedge^2 Q, \ \Phi(v)|_{(x,P)} = v_x \in \wedge^2 Q. \tag{6.4}$$

Naturally, one can regard Φ as a section in $\pi_2^* S^* \otimes \pi_1^*(\wedge^2 Q)$.

We define

$$\mathcal{X} = \{(x, P) \in G(4,7) \times G(3, H^0(G(4,7), \wedge^2 Q))$$

One can show that $\mathcal{X}$ is smooth.

If $L = \det(Q)$, then $c_1(L)$ is the positive generator of $H^2(W, \mathbb{Z})$.

Consider the G-equivariant fibration $\pi = \pi_2|_{\mathcal{X}} : \mathcal{X} \mapsto Z$, where $G = SL(7, \mathbb{C})$ and $Z = G(3, H^0(G(4,7), \wedge^2 Q))$. Its generic fibers are smooth and of dimension 3. Then Z_0 parameterizes all Fano 3-folds X_P.

Using the Adjunction Formula, one can show

$$c_1(\mathcal{K}) \;=\; -\pi_1^* c_1(L) - 3\pi_2^* c_1(S).$$

Therefore, it follows that

$$c_1(L_Z) = 16\pi_* \left(12\pi_2^* c_1(S^*) \pi_1^* c_1(L)^3 \;-\; \pi_1^* c_1(L)^4\right).$$

So L_Z is ample.

By the definition of P_a, one can show that none of $G \cdot P_a$ is closed. It implies that any generic X_{P_a} admits no Kähler-Einstein metrics, Furthermore, any generic X_{P_a} admits no nonzero holomorphic fields. In particular, we have proved Corollary 2.7.

The ideas in the proof of Theorem 6.1 can be outlined as follows. For simplicity, we assume that M has no nontrivial holomorphic vector fields.

We will start with a simple criterion for stability.

LEMMA 6.3. *Let $||\cdot||_Z$ be any fixed hermitian metric on L_Z^{-1}. Given any z in Z_0, define a function F_0 on G by*

$$F_0(\sigma) = \log\left(||\sigma(\tilde{z})||_Z\right), \ \sigma \in G.$$

Then M is CM-stable if and only if F_0 is proper on G.

Next we recall **[3, 15]** the definition of the K-energy. Let ω be any Kähler metric representing a positive multiple of $c_1(L)$. Then for any $\varphi \in P(M, \omega)$, the K-energy ν_ω is defined by

$$\nu_\omega(\varphi) = -\frac{1}{V} \int_0^1 \int_M \dot{\varphi}_t (\mathrm{Ric}(\omega_t) - \omega_t) \wedge \omega_t^{n-1} \wedge dt$$

where $\{\varphi_t\}_{0\leq t\leq 1} \subset P(M,\omega)$ is any path with $\varphi_0 = 0$ and $\varphi_1 = \varphi$, and $\omega_t = \omega + \partial\bar{\partial}\varphi_t$.

Let h be a hermitian metric on $\mathcal{L}$ over $\mathcal{X}$ such that its curvature $R(h)$ restricts to Kähler metrics on fibers. Fix a function θ such that $R(h)|_M = \omega + \partial\bar{\partial}\theta$. Define φ_σ by

$$\varphi_\sigma(x) = \theta(x) + \log\left(\frac{h(x)}{h(\sigma(x))}\right).$$

Then we have a function D on G given by

$$D_\omega(\sigma) = \nu_\omega(\varphi_\sigma).$$

Next, we compare D_ω with F_0 on G.

PROPOSITION 6.4. *There is a uniform constant c such that*

$$F_0(\sigma) \geq (n+1)2^{n+1}D_\omega(\sigma) - c.$$

This is the most technical part in the proof of Theorem 6.1 (cf. [**24**]). In fact, the conclusion holds for any ample line bundles over any compact Kähler manifolds.

Now, in order to complete the proof of Theorem 6.1, we suffice to prove that ν_ω is proper. If $\mu \leq 0$, it follows from the inequality

$$\nu_\omega(\varphi) \geq \frac{1}{V}\int_M \left(\log\left(\frac{\omega_1^n}{\omega^n}\right)\omega_1^n + h_\omega(\omega^n - \omega_1^n)\right).$$

If $\mu > 0$, then we have

$$\nu_\omega(\varphi) \geq F_\omega(\varphi) + \frac{1}{V}\int_M h_\omega\omega^n.$$

Then the properness of ν_ω follows from this and Theorem 3.6.

We conjecture that *the K-energy is proper on any compact Kähler manifold with constant scalar curvature.* This is true for any Kähler class which admits a Kähler-Einstein metric.

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