Hyper-Kähler Manifolds

Andrew S. Dancer

Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario L8S 4K1
Canada
dancer@icarus.math.mcmaster.ca

A Riemannian manifold (M, g) is hyper-Kähler if it possesses complex structures I and J such that

(i) the metric g is Kähler with respect to both I and J; and
(ii) the given complex structures anti-commute: $IJ = -JI$

Defining $K = IJ$, it then follows that g is also Kähler with respect to K. Moreover, I, J, and K automatically satisfy the usual quaternionic multiplication table. Each tangent space of M therefore becomes a quaternionic vector space, and the dimension of M is therefore $4n$ for some integer n. Condition (i) implies that parallel transport commutes with I and J, and hence with K. Thus the holonomy group is contained in $O(4n) \cap GL(n, \mathbb{H}) = Sp(n)$. Hyper-Kähler manifolds therefore correspond to one of the possibilities on Berger's list of holonomy groups.

In fact $aI + bJ + cK$ is also a Kähler structure on M whenever $a^2 + b^2 + c^2 = 1$, so a hyper-Kähler manifold actually carries a two-sphere of Kähler structures. It is this fact which lies at the heart of the twistor approach to hyper-Kähler geometry, outlined later in the article. Note in particular that a hyper-Kähler manifold is a *symplectic* manifold in many different ways.

The existence of this large family of parallel two-forms gives us more rigidity than in the Kähler case. For example, any Kähler metric admits an infinite-dimensional space of Kähler perturbations, for we can just add $\partial\bar{\partial}f$ to ω, where f is a function with $\partial\bar{\partial}f$ sufficiently small. There is no analogue of this in hyper-Kähler geometry. Another contrast with the Kähler situation is that for a hyper-Kähler manifold the complex structures, and hence the metric, are determined by knowledge of the triple of two-forms $\omega_I, \omega_J, \omega_K$. For, viewing these forms as living in $\mathrm{Hom}(TM, T^*M)$, we find that $K = -\omega_I^{-1}\omega_J$ et cetera.

It is often useful to know that if, for some metric, I, J, K are almost-Hermitian structures satisfying (ii), and if the associated two-forms are all closed, then I, J, K are in fact integrable and we have a hyper-Kähler structure. Again, the analogous result for Kähler manifolds is false.

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Let I be an arbitrary element of the two-sphere of complex structures, and let J, K be complex structures such that $IJ = K$ etc. Then the closed 2-form $\omega = \omega_J + i\omega_K$ is holomorphic with respect to I, and nondegenerate. That is, ω defines a *complex-symplectic structure* on (M, I).

Taking the $2n$th power of ω gives a trivialisation of the canonical bundle, so a hyper-Kähler manifold has $c_1 = 0$ (for any one of the two-sphere of complex structures). Moreover ω is parallel, so in fact we obtain a parallel trivialisation of the canonical bundle, and the Ricci tensor of M is zero. Alternatively, one may view this in terms of holonomy groups and observe that $Sp(n)$ is contained in $SU(2n)$. Note that in real dimension four, the hyper-Kähler condition is equivalent to Ricci-flat Kähler, as $Sp(1) = SU(2)$. In higher dimensions, however, the hyper-Kähler property is much stronger.

If a *compact* complex manifold M is complex-symplectic and Kähler, then it is hyper-Kähler [**4**]. For the complex-symplectic structure implies, as above, that c_1 vanishes, and so Yau's theorem [**91, 92**] tells us that M admits a Ricci-flat Kähler metric. A Bochner argument now shows that the holomorphic symplectic form is actually parallel with respect to the Yau metric. The Kähler form, together with the real and imaginary parts of the holomorphic symplectic form, spans the space of parallel two-forms which reduces the holonomy to $Sp(n)$.

It is still unclear in general when a complex-symplectic manifold admits a compatible hyper-Kähler structure. There are examples [**33, 34**] in all dimensions of compact complex-symplectic manifolds which do not support any hyper-Kähler metric (see §6). However, in this article we shall encounter many cases which *are* known to be hyper-Kähler; examples include coadjoint orbits of complex semsimple groups, spaces of representations of fundamental groups of surfaces into complex semisimple groups, and open subsets of cotangent bundles of Kähler manifolds. Looking for complex-symplectic manifolds and attempting to find a compatible hyper-Kähler structure has proved a very fruitful way of producing examples of hyper-Kähler manifolds.

1. Early Examples

In 1978 Eguchi and Hanson [**25**] found an example of a four-dimensional complete Ricci-flat metric which was of Bianchi IX type; that is, it admitted an isometric action of $SU(2)$ with generically three-dimensional orbits. In fact, their example is hyper-Kähler. Explicitly, the metric is

$$W^{-1} dr^2 + \frac{1}{4} r^2 \left(\sigma_1^2 + \sigma_2^2 + W \sigma_3^2 \right), \tag{1.1}$$

where

$$W = 1 - \frac{a^4}{r^4},$$

$\sigma_1, \sigma_2, \sigma_3$ are left-invariant one-forms, and a is a parameter.

If we denote by X_i the vector fields dual to σ_i, then one of the complex structures, I say, maps X_1 to X_2, and sends the normal $\partial/\partial r$ to $\frac{2}{rW} X_3$. We can pick an anticommuting complex structure J which maps the normal to $\frac{2}{r\sqrt{W}} X_1$ and X_2 to $\frac{1}{\sqrt{W}} X_3$.

If a is zero then (1.1) is just the Euclidean metric on $\mathbb{R}^4$. For nonzero a we obtain a smooth complete metric by letting r range over $[a, \infty)$. The apparent degeneracy at $r = a$ is just a coordinate singularity, where the three-dimensional orbits of $SU(2)$

collapse down to a special orbit which is a two-sphere. The underlying manifold of the Eguchi-Hanson metric is the cotangent bundle of this sphere, so the principal orbits are copies of $\mathbb{RP}^3$ rather than S^3. As the coefficients of σ_1^2 and σ_2^2 are equal, the full isometry group of the metric is actually $U(2)/\{\pm 1\}$, acting on T^*S^2 in the natural way. With respect to the complex structure I, the manifold is biholomorphic to the cotangent bundle of $\mathbb{CP}^1$, but for the other complex structures it is an affine algebraic variety, the nonsingular quadric in $\mathbb{C}^3$. We shall see a similar phenomenon for many other hyper-Kähler manifolds. Note that I is the only complex structure preserved by the full isometry group–the other complex structures are preserved by $SU(2)$ but rotated by the $U(2)$ action.

The Eguchi-Hanson metric was soon generalised in several ways. Calabi [**15**] constructed hyper-Kähler metrics on the cotangent bundle of $\mathbb{CP}^n$ for any n. One of the complex structures is just that induced from complex projective space. Indeed, Calabi proceeds by explicitly calculating a Kähler potential on this complex manifold such that the associated Kähler form and the natural complex-symplectic form on the cotangent bundle together define a hyper-Kähler structure. With respect to the other complex structures, however, M is biholomorphic to an affine variety, the coadjoint orbit $SL(n+1,\mathbb{C})/GL(n,\mathbb{C})$. Of course, this complex manifold has completely different properties from $T^*\mathbb{CP}^n$. For example, it has no compact complex submanifolds.

On another front, Belinskii, Gibbons, Page and Pope [**5**] found the general Bianchi IX hyper-Kähler metric in the case when the $SU(2)$ action preserves each Kähler structure. The metric may be written as

$$(abc)^2\, dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2, \tag{1.2}$$

where a, b, c must satisfy

$$2\frac{a'}{a} = b^2 + c^2 - a^2 \tag{1.3}$$

and the two equations obtained from (1.3) by cyclically permuting a, b, c. These equations are equivalent, after making the change of variables $f_1 = bc, f_2 = ac, f_3 = ab$, to the spinning top equations $\dot{f}_i = f_j f_k$, where (ijk) ranges over cyclic permutations of (123). These may be solved in terms of Jacobi elliptic functions, and the metric written down explicitly. The resulting metrics are generically *triaxial*, that is, the coefficients of σ_i^2 are all distinct so the isometry group is $SU(2)$ rather then $U(2)$. However the triaxial metrics are never complete. In fact, the only complete examples in the family considered by Belinskii et al. are Eguchi-Hanson and flat space. The former corresponds to the solution $f_1 = -\coth t, f_2 = f_3 = -\operatorname{csch} t$, while the latter comes from the solution $f_1 = f_2 = f_3 = -t^{-1}$. The other non-triaxial solution, corresponding to $f_1 = -\csc t, f_2 = f_3 = -\cot t$, gives an incomplete metric.

Also in four dimensions, Gibbons and Hawking [**29**] found the family of *multi-instanton metrics.* These admit a circle action preserving the hyper-Kähler structure, and can be written as

$$V d\underline{x}.d\underline{x} + V^{-1}(d\theta + \underline{\omega}.d\underline{x})^2. \tag{1.4}$$

Here θ is a coordinate on the circle fiber, $\underline{x}$ a coordinate on the three-dimensional quotient, $\underline{\omega}$ is the connexion form, and V is a real-valued function on the quotient. In order to obtain a hyper-Kähler metric we need $dV = *d\underline{\omega}$, so V is harmonic.

The multi-instanton metrics are obtained by setting

$$V(\underline{x}) = \sum_{i=1}^{m} | \underline{x} - p_i |^{-1}, \tag{1.5}$$

for some points $p_1, \dots, p_m$ in $\mathbb{R}^3$. With this choice of V we get a *complete* metric, provided the p_i are all distinct.

The underlying manifold fibres over $\mathbb{R}^3$ with generic fiber S^1, but at the points p_i the circle collapses to a point (metrically these are the points where V^{-1}, the length of the circle, is zero). The topology of the manifold is generated by the $m-1$ two-spheres which fiber over the closed line segment from p_i to p_{i+1}. If $m = 1$ we just get Euclidean space, and if $m = 2$ we recover the Eguchi-Hanson metric. The multi-instanton metrics were also obtained, using twistor methods, by Hitchin [**41**].

At about the same time, progress was made in a different, less explicit direction. Yau's proof of the Calabi conjecture [**91, 92**] showed that the $K3$ surface admitted a hyper-Kähler metric. This was the first *compact* hyper-Kähler manifold known, except for flat tori. We shall discuss the compact case further in the last section of this article.

2. Hyper-Kähler Quotients

The hyper-Kähler quotient construction, due to Hitchin, Karlhede, Lindström and Roček [**45**], is at present the most useful method of constructing complete hyper-Kähler manifolds. The beauty of the technique is that, starting from flat quaternionic space, it produces highly nontrivial manifolds. We shall see, for example, that the Calabi and multi-instanton spaces can be produced in this way.

The construction is inspired by the symplectic quotient of Marsden and Weinstein, a good account of which is given in the book of Guillemin and Sternberg [**36**]. If a group G acts on a symplectic manifold M preserving the symplectic form, then in good cases (for example, if G is semisimple), one obtains a *moment map* $\mu : M \to \mathfrak{g}^*$. The moment map is a G-equivariant map such that

$$d\mu(v)(X) = \omega(X^*, v) \tag{2.1}$$

for all tangent vectors v to M and all elements X of $\mathfrak{g}$. Here X^* denotes the vector field on M associated to X by the group action.

The equivariance of μ shows that if λ lies in the center of $\mathfrak{g}^*$ then G acts on the level set $\mu^{-1}(\lambda)$. The key result is that if λ is in addition a regular value of μ and G acts freely and properly on the level set, then the quotient $\mu^{-1}(\lambda)/G$ is a symplectic manifold.

The starting point for the hyper-Kähler quotient is a hyper-Kähler manifold M admitting an action of a Lie group G preserving the hyper-Kähler structure. This means that the action of G is isometric and holomorphic with respect to all the complex structures. In particular, the action of G preserves the Kähler forms $\omega_I, \omega_J, \omega_K$. If the associated three symplectic moment maps exist, then we can combine them into a single hyper-Kähler moment map $\mu : M \to \mathbb{R}^3 \otimes \mathfrak{g}^*$. It is useful to observe that if $IJ = K$ etc. then $\mu_J + i\mu_K$ is holomorphic with respect to the complex structure I. The level sets of this map are therefore Kähler with respect to I.

The main theorem of [**45**] says that if we pick λ_i in the center of $\mathfrak{g}^*$, and if $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is a regular value of μ and G acts freely and properly on $\mu^{-1}(\underline{\lambda})$, then the quotient $\mu^{-1}(\underline{\lambda})/G$ is hyper-Kähler. To see this, observe that the quotient can be viewed as the symplectic quotient by G of $(\mu_J + i\mu_K)^{-1}(\lambda_2 + i\lambda_3)$. Now, the symplectic quotient of a Kähler manifold by an action preserving the Kähler structure is in fact Kähler, so $\mu^{-1}(\underline{\lambda})/G$ has a Kähler structure induced by I. The other Kähler structures are produced in the same way, by considering different combinations of μ_I, μ_J, μ_K.

Before we mention some examples, let us make a few remarks about the construction.

(i) It is sufficient that G should be semisimple for the moment maps to exist. This condition is definitely not necessary, as some of our later examples will show.
(ii) If the action of G on $\mu^{-1}(\underline{\lambda})$ is free, then $\underline{\lambda}$ is automatically a regular value of the moment map. The dimension of the quotient in this case is $\dim M - 4\dim G$.
(iii) If M is complete and G is compact, then the quotient, when smooth, is also complete.
(iv) If G has a nontrivial center, then, typically, the action of G on the level set is free for generic values of $\underline{\lambda}$. At special values, we get singularities.
(v) Although the quotient construction is a powerful method for showing existence of hyper-Kähler metrics, finding the metric *explicitly* may be difficult.

The hyper-Kähler quotient procedure is well-adapted to finding complex structures on hyper-Kähler manifolds. The tool used is the equivalence between Kähler and algebro-geometric quotients [**52**]. More precisely, if N is a Kähler manifold with an action of G preserving the Kähler structure, then the symplectic quotient of N by G can be identified with the quotient by $G_{\mathbb{C}}$ of the set of *stable* points–those points whose $G_{\mathbb{C}}$ orbits meet the zero set of the moment map for the G action.

In many concrete situations this notion of stability coincides with some more natural notion, for example Mumford stability of vector bundles. In any event, the crucial point is always to ascertain which orbits of the complex group meet the zero set of the moment map.

In the hyper-Kähler case, we just observe, as above, that the hyper-Kähler quotient $\mu^{-1}(0, \lambda_2, \lambda_3)/G$ is the symplectic quotient by G of the variety $(\mu_J + i\mu_K)^{-1}(\lambda_2 + i\lambda_3)$ and apply the previous discussion.

As an example of the construction, let $M = \mathbb{H}^n = \mathbb{C}^n \times \mathbb{C}^n$, and let $G = U(1)$ act by $e^{it}.(z, w) = (e^{it}z, e^{-it}w)$. The moment map is $\mu : (z, w) \mapsto (\frac{1}{2}(|z|^2 - |w|^2), \mathrm{Re}\langle z, w\rangle, \mathrm{Im}\langle z, w\rangle)$. Taking $\underline{\lambda} = (0, \mathrm{Re}\,\tau, \mathrm{Im}\,\tau)$, we find that the hyper-Kähler quotient is biholomorphic to the algebro-geometric quotient $\{(z, w) : \langle z, w\rangle = \tau\}//\mathbb{C}^*$, where $\mathbb{C}^*$ acts by $s.(z, w) = (sz, s^{-1}w)$. If we take τ to be nonzero then all points (z, w) with $\langle z, w\rangle = \tau$ are stable for this action. The quotient can be identified as a complex manifold with the cotangent bundle of $\mathbb{CP}^{n-1}$, and the hyper-Kähler metric is that found by Calabi.

There are some particularly interesting four-dimensional examples of hyper-Kähler quotients. The best understood case is that of asymptotically locally Euclidean (ALE) hyper-Kähler four-manifolds, studied by Kronheimer [**57, 58**]. The ALE condition means that the metric is asymptotically a quotient by a finite group of the Euclidean metric

$$dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \tag{2.2}$$

on $\mathbb{R}^4$. More precisely, we require that the metric should agree with the quotient of the Euclidean metric modulo terms whose p^{th} derivatives decay at least as fast as r^{-4-p}.

The construction is begun by choosing a finite subgroup Γ of $SU(2)$. There are two infinite families, the cyclic and binary dihedral groups, together with the binary tetrahedral, octahedral and icosahedral groups. We can rephrase this by saying that the finite subgroups of $SU(2)$ have an A-D-E classification, that is, they correspond to the simply-laced Dynkin diagrams. Moreover the nodes of the extended Dynkin diagram correspond to the irreducible representations of Γ.

For each such Γ, let V denote the regular representation of Γ and let M be the Γ-invariant part of $\text{End}(V) \otimes \mathbb{C}^2$, where Γ acts on $\mathbb{C}^2$ via inclusion into $SU(2)$. Let G be the subgroup of $PU(V)$ commuting with the action of Γ on $\text{End}(V)$. Then the hyper-Kähler quotient of M by G is, for generic choices of level set, a smooth ALE hyper-Kähler four-manifold.

The zero level set yields the Kleinian singular space $\mathbb{C}^2/\Gamma$, with the flat metric, and the smooth quotients are desingularisations of this space. Their topology is generated by embedded two-spheres of self-intersection -2, intersecting according to the non-extended Dynkin diagram corresponding to Γ.

Asymptotically, the metric approaches the flat metric on $\mathbb{R}^4/\Gamma$. The analysis of the asymptotics relies on the homogeneity of the moment map with respect to dilations of M. Roughly speaking, this means that going out to infinity in the hyper-Kähler manifold corresponds to scaling the level set $\underline{\lambda}$ towards zero. The flatness of the space corresponding to the zero level set, together with the flatness of the connexion controlling the first variation of the metric about $\underline{\lambda} = \underline{0}$, yields the ALE property.

As explained in [**57**], one can use the work of McKay [**70**] to give a more concrete description of the quotient construction. Let Δ be the extended Dynkin diagram of Γ, with vertices $v_1, \ldots, v_m$, and when $i \leq j$ and v_i is adjacent to v_j, let p_{ij} denote the edge of Δ from v_i to v_j. To each vertex v_i we attach an integer n_i, the dimension of the irreducible representation of Γ corresponding to v_i. Then M is the direct sum over the edges of $\mathbb{H}^{n_i n_j}$, while G is the product over the vertices of $U(n_i)$, factored out by the circle subgroup which acts trivially on M. To define the action, for each edge p_{ij} we view $\mathbb{H}^{n_i n_j}$ as $\text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \oplus \text{Hom}(\mathbb{C}^{n_j}, \mathbb{C}^{n_i})$ and let $U(n_i)$ and $U(n_j)$ act in the natural way. In this picture the relation $A + C = 2I$ between the adjacency and Cartan matrices of the Dynkin diagram shows that the real dimension of M is $4|\Gamma|$. On the other hand, the dimension of G is $(\sum n_i^2) - 1$, which is $|\Gamma| - 1$ from elementary representation theory, so the hyper-Kähler quotients are four-dimensional, as required.

In the A_n case, corresponding to $\Gamma = \mathbb{Z}_{n+1}$, all the n_i are equal to 1, since Γ is abelian. Hence M is just $\mathbb{H}^{n+1}$ and $G = U(1)^n$. The resulting hyper-Kähler metrics are the Gibbons-Hawking multi-instanton series. Note that the description of the topology of the ALE spaces above agrees with our discussion of the multi-instanton spaces in the preceding section.

The multi-instanton metrics may be written down explicitly as in (1.4) because they admit a circle action preserving each Kähler structure. In terms of the algorithm for constructing the ALE spaces as a quotient, this is due to the fact that in the A_n case there are as many edges as vertices in the extended Dynkin diagram. This means that there is still a residual circle action after we perform the quotient. In the other cases the number of vertices is one greater than the number of edges, and we do not obtain a hyper-Kähler action on the quotient. No explicit form of the metric is known in these cases.

For each choice of Γ we get a family of hyper-Kähler metrics, depending on the choice of level set. The action of the isometry group of M means that certain choices of level set will give equivalent quotients, so not all these parameters are effective. For A_n, we can parametrise the metrics by the locations of the points $p_1, \dots, p_{n+1}$ of (1.5), modulo translations and rotations of $\mathbb{R}^3$. The points must be distinct if we are to obtain a smooth metric. For $n = 1$ (the Eguchi-Hanson metric) we have a single parameter. Another way of viewing the parametrisation is via the *periods*, the numbers obtained by pairing the cohomology classes of the three Kähler forms with the homology classes represented by the embedded two-spheres. The periods determine the hyper-Kähler metric [**58**].

In fact, Kronheimer shows that these examples are the only ALE hyper-Kähler four-manifolds. His proof exploits the fact that ALE metrics admit an orbifold compactification. Hyper-Kähler metrics in dimension four are always anti-self-dual, so this compactification carries an anti-self-dual conformal structure and the methods of Penrose's twistor theory [**78**] may be applied. If the ALE condition is dropped, one may get other examples, the simplest of which are the *multi Taub-NUT* metrics. This family, discovered by Hawking [**39**], is given as in (1.4), but now we have $V = 1 + \sum_{i=1}^{m} \mid \underline{x} - p_i \mid^{-1}$.

If $m = 1$, we obtain a $U(2)$-invariant metric, the celebrated Taub-NUT metric. This may be written in $SU(2)$-invariant form as

$$\frac{r+1}{4r}\, dr^2 + r(r+1)(\sigma_1^2 + \sigma_2^2) + \frac{r}{r+1}\sigma_3^2, \tag{2.3}$$

where $r \in [0, \infty)$. At $r = 0$ the $SU(2)$ orbit collapses to a point (a *nut* in the terminology of general relativity). The change of variables $r = s^2$ shows that the metric extends smoothly over the nut, and we obtain a metric defined on $\mathbb{R}^4$. Although complete, the Taub-NUT metric is not ALE, because the coefficient of σ_3^2 approaches a constant as r tends to infinity rather than growing like r^2. This kind of asymptotic behaviour, which is shared by the other multi-Taub-NUT metrics, is called Asymptotically Locally Flat (ALF). There is a detailed discussion of the Taub-NUT metric by LeBrun [**62**], who shows that all the complex structures are equivalent to that of $\mathbb{C}^2$. There is a circle subgroup of $U(2)$ preserving each complex structure, but the action of $SU(2)$ permutes them, making it hard to write them down explicitly in the coordinates of (2.3). The Taub-NUT example shows that a complex manifold ($\mathbb{C}^2$ in this case) may admit non-homothetic complete Ricci-flat Kähler metrics, in contrast to the situation for negative Einstein constant [**17, 93**].

One can also produce the multi-Taub-NUT metrics by replacing an $\mathbb{H}$ factor in the Kronheimer construction by an $\mathbb{R}^3 \times S^1$. It is interesting to note that these metrics may also be obtained as hyper-Kähler quotients by a noncompact group. For example, if $\mathbb{R}$ acts on $\mathbb{C}^2 \times \mathbb{C}^2$ by $t.(z_1, z_2, w_1, w_2) = (e^{it}z_1, e^{-it}z_2, w_1 + t, w_2)$, then the hyper-Kähler quotient is the Taub-NUT metric [**6**].

There are many higher-dimensional examples produced as hyper-Kähler quotients of $\mathbb{H}^n$ by subgroups of $Sp(n)$. For example, Lindström and Roček found hyper-Kähler metrics defined on the cotangent bundles of Grassmannians $SU(m+n)/S(U(m) \times U(n))$ by taking a hyper-Kähler quotient of $\mathbb{H}^{m(n+m)}$ by $U(m)$. The case $m = 1$ corresponds to the Calabi metric. We shall see later that cotangent bundles of arbitrary generalised flag manifolds also carry a hyper-Kähler structure.

One class of quotients whose properties can be calculated in detail is motivated by Delzant's construction of compact Kähler toric varieties as Kähler quotients [**22, 35**]. The starting point for Delzant's work is a polytope Δ in $\mathbb{R}^n$ defined by inequalities $\langle x, u_k \rangle \geq \lambda_k$ for vectors $u_1, \dots, u_d$ and scalars $\lambda_1, \dots, \lambda_d$. Here d is the number of faces of Δ. This data defines a subtorus N of T^d, for we can take the Lie algebra of N to be the kernel of the map from $\mathbb{R}^d$ to $\mathbb{R}^n$ given by sending the kth element of the standard basis to u_k. Delzant now considers the Kähler quotient of $\mathbb{C}^d$ by N, using the scalars λ_k to fix the level set. If the polytope satisfies certain constraints, then the resulting Kähler quotient is a smooth compact toric variety X of real dimension $2n$. Moreover X admits a Kähler action of $T^n \cong T^d/N$, and the image of the associated moment map is just Δ.

By performing a hyper-Kähler quotient of $\mathbb{H}^d$ by N in an analogous fashion, one obtains [**30, 8**] families of hyper-Kähler manifolds associated to hyperplane arrangements in $\mathbb{R}^n$. (We do not necessarily assume these define a polytope). The hyper-Kähler manifolds have real dimension $4n$ and have an action of $T^n \cong T^d/N$ preserving the hyper-Kähler structure. The methods of [**45, 77**] may therefore be applied to give rather explicit formulae for the metric and Kähler form. The manifolds are noncompact, but their topology is generated by a union of compact toric varieties. The Calabi spaces correspond to the hyperplanes defining the standard n-simplex, while the multi-instanton metrics correspond to a set of points in $\mathbb{R}$. There are many other complete examples given by more complicated hyperplane arrangments. One also obtains, by restriction, hyper-Kähler metrics on the cotangent bundles of toric varieties, but these metrics are usually incomplete. Taking the zero level set yields a singular quotient, but, in good cases, this is a cone over one of the *smooth* 3-Sasakian manifolds of [**12**]. Bielawski has investigated the interesting topological properties of these examples [**7**].

This may be a good point to mention a difference between symplectic and hyper-Kähler quotients. When performing symplectic quotients by a group with nontrivial center Z, the usual picture is that the set of critical values of the moment map is of real codimension one in $\mathfrak{z}^*$. We therefore have a wall-crossing phenomenon; as the level set crosses a critical value the symplectic quotient can change topology by a sequence of symplectic blowups and blowdowns. This situation has been extensively analysed in the beautiful paper of Guillemin and Sternberg [**38**].

In the hyper-Kähler case, the set of critical values is typically of codimension three or more in $\mathbb{R}^3 \otimes \mathfrak{z}^*$, so one expects the smooth quotients to all be diffeomorphic (although not isometric). For example, the ALE spaces of [**57**] are, when smooth, all diffeomorphic to the minimal resolution of the appropriate Kleinian singularity.

3. Moduli Spaces

One of the most important features about hyper-Kähler structures is that there are many naturally occurring examples. In particular, the moduli spaces arising in gauge theory often carry a hyper-Kähler metric. The key fact is that such moduli spaces may often be viewed as finite-dimensional hyper-Kähler quotients of an infinite-dimensional space of connexions by an infinite-dimensional gauge group.

As an example, let us consider connexions on a principal G-bundle over $\mathbb{R}^4$. The space $\mathcal{A}$ of connexions is an infinite-dimensional affine space modelled on $\Omega^1(\mathbb{R}^4, \mathfrak{g})$. We put a quaternionic structure on $\mathcal{A}$ by writing connexions as $(\partial_0 + A_0) + i(\partial_1 + A_1) + j(\partial_2 + A_2) + k(\partial_3 + A_3)$, where ∂_a denotes the a^{th} partial derivative and

$\sum_{a=0}^{3} A_a dx_a$ is a $\mathfrak{g}$-valued one-form. The gauge group $\mathcal{G}$ may be identified with $C^\infty(\mathbb{R}^4, G)$.

If we restrict to a space $\mathcal{A}_0$ of connexions which are standardised at infinity, in some suitable sense, then we can put an L^2 metric on $\mathcal{A}_0$ which makes it into an infinite-dimensional hyper-Kähler manifold. Furthermore, if we consider the subgroup $\mathcal{G}_0$ of gauge transformations tending to the identity at infinity, then the action preserves the hyper-Kähler structure and admits a moment map. An integration by parts argument, together with the conditions at infinity, shows that the equations for the vanishing of the moment map are

$$[\partial_0 + A_0, \partial_i + A_i] = [\partial_j + A_j, \partial_k + A_k], \tag{3.1}$$

where (ijk) ranges over cyclic permutations of (123). The commutator terms are of course components of the curvature form of the connexion, and (3.1) is the self-dual Yang-Mills equation. Hence the moduli space of based instantons on $\mathbb{R}^4$ is a hyper-Kähler quotient of $\mathcal{A}_0$ by $\mathcal{G}_0$. Although $\mathcal{A}_0$ and $\mathcal{G}_0$ are infinite-dimensional, M is finite-dimensional, as alluded to above.

There are many variations on this theme. For example, one may replace Euclidean $\mathbb{R}^4$ by some other hyper-Kähler base manifold. Itoh [**48**] has considered moduli spaces of instantons on the two compact hyper-Kähler four-manifolds, the four-torus and the K3 surface. Instantons on ALE hyper-Kähler four-manifolds have been studied by Kronheimer and Nakajima [**61**]. In subsequent work, Nakajima has shown that the middle cohomology of these moduli spaces provides weight spaces for representations of affine Kac-Moody algebras [**74**].

Another modification, which has proved extremely fruitful, is to consider self-dual connexions invariant under some group of transformations of $\mathbb{R}^4$ preserving the hyper-Kähler structure. Some examples follow.

3.1. Nahm's Equations. These are the self-duality equations with $\mathbb{R}^3$ translation-invariance imposed. The equations then become the system of ordinary differential equations

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k], \tag{3.2}$$

for quadruples (T_0, T_1, T_2, T_3) of $\mathfrak{g}$-valued functions. As in (3.1), the triple (ijk) ranges over cyclic permutations of (123).

The simplest case is when T_i are smooth maps defined on a finite interval, say $[0,1]$. The group of maps from $[0,1]$ to G acts on solutions to (3.2) by gauge transformations

$$T_0 \mapsto \mathrm{Ad}(g)T_0 - \frac{dg}{dt}g^{-1}, \quad T_i \mapsto \mathrm{Ad}(g)T_i \quad (i = 1, 2, 3). \tag{3.3}$$

The quotient of the space of solutions by the group of gauge transformations vanishing at the endpoints is hyper-Kähler. We may see this by viewing the equations as the vanishing of a moment map for the action of this group on the space of maps $[0,1] \to \mathfrak{g}^4$.

There are many modifications of this construction. One can have the Nahm matrices defined on a union of intervals, and impose boundary conditions at the endpoints. In particular, one often requires the Nahm matrices to have poles of prescribed residues at the endpoints of the intervals. In this case it is necessary to fix the residues for there to be a finite L^2 hyper-Kähler metric on the moduli space.

3.2. Self-Duality Equations on a Riemann Surface. If we look at self-dual G-connexions with $\mathbb{R}^2$ translation invariance, we get a system of equations involving a connexion A and a complex Higgs field Φ on $\mathbb{R}^2$. The Higgs field is regarded as a $(1,0)$-form taking values in the complex adjoint bundle associated to the principal G-bundle. These equations are conformally invariant, so also make sense on an arbitrary Riemann surface Σ. Hitchin [**43**] has carried out an intensive study of the resulting moduli space for $G = SU(2)$. If g is the genus of the Riemann surface, the moduli space M is a hyper-Kähler manifold of real dimension $12(g-1)$. For a special choice of complex structure, I say, M can be identified with a moduli space of stable Higgs pairs (V, Φ). Here V is a rank two holomorphic vector bundle on Σ, and Φ is a holomorphic section of $\mathrm{End}(V) \otimes K$, where K denotes the canonical bundle. The stability condition for such pairs is that the degree of any Φ-invariant rank one subbundle L of V should be less than half the degree of V. The cotangent bundle of the moduli space N of stable rank two bundles on Σ sits inside M as an open dense subset. We can regard N itself as the subset of M consisting of configurations with zero Higgs field. The stability condition for pairs reduces at such configurations to the ordinary stability condition for bundles.

There is an isometric circle action preserving I, defined by scalar multiplication of the Higgs field. The moment map for this action is just the L^2 norm squared of the Higgs field, and is a perfect Morse function on M. Hitchin uses this to get formulae for the Betti numbers of the moduli space. Note that the critical points for the Morse function, or equivalently, fixed points for the circle action, need not occur only when the Higgs field is zero. This is because we get a fixed point on the moduli space whenever (A, Φ) is *gauge equivalent* to $(A, e^{i\theta}\Phi)$ for all θ. The upshot is that as well as the critical submanifold N we also get critical submanifolds of higher index, and the Poincaré polynomial of M becomes quite complicated.

As with the Calabi examples, the other complex structures, which are all equivalent, make M into a Stein manifold. With respect to such a complex structure, the moduli space may be regarded as a space of representations of a central extension of $\pi_1(\Sigma)$ into $G_{\mathbb{C}}$ modulo conjugation. This is an example of a space with a natural complex-symplectic structure turning out to admit a hyper-Kähler metric.

Several authors have generalised these results to moduli spaces of parabolic Higgs bundles on Riemann surfaces (or equivalently orbifold Higgs bundles on orbifold Riemann surfaces) [**55, 75, 11**]. The dimension of the hyper-Kähler moduli spaces depends not just on the genus but also on the marked points and the parabolic data there. One obtains, by judicious choice of the various pieces of data, hyper-Kähler manifolds of dimension $4n$ for every n. In particular, taking Σ to be the Riemann sphere with four marked points gives a hyper-Kähler four-manifold fibering over $\mathbb{C}$. The generic fiber is an elliptic curve, and the fiber over the origin is the union of five rational curves intersecting like the extended Dynkin diagram of D_4. Hitchin has also constructed this manifold by twistor methods [**42**].

3.3. Monopoles. If instead we require $\mathbb{R}$-translation invariance, we obtain from the self-dual Yang-Mills equations the Bogomolny equations

$$*F_A = D_A \Phi, \tag{3.4}$$

for a connexion A and Higgs field Φ on $\mathbb{R}^3$. The Higgs field is a section of the real adjoint bundle associated to the principal G-bundle on which the connexion lives. Solutions to (3.4) with appropriate behaviour at infinity are called *monopoles*, and as with instantons we can study moduli spaces of these objects. Monopole moduli spaces are hyper-Kähler and admit an action of $SO(3)$, induced from spatial rotation in $\mathbb{R}^3$, which permutes the complex structures. Each complex structure identifies the moduli space with a space of based rational maps from the Riemann sphere into some flag manifold [**23, 47**].

In all these cases the moduli space has a nice description as a complex manifold. Formally, this is shown by the same kind of argument as in the finite-dimensional case, discussed in §2. One splits up the equations defining the moduli space into a real and a complex equation; in terms of the moment maps these are just $\mu_I = 0$ and $\mu_J + i\mu_K = 0$. The real equation is invariant under the action of the gauge group, but the complex equation is preserved by the action of the complexified gauge group. Then one shows that, subject to a stability condition, every orbit of the complexified gauge group in the set of solutions to the complex equations contains a solution to the real equations, unique up to the action of the real gauge group. The moduli space is thus identified with the quotient of $(\mu_J + i\mu_K)^{-1}(0)$ by the action of the complexified gauge group. In this infinite-dimensional setting, establishing the existence of a solution to the real equation in each orbit of stable points involves some analytical work.

For example, in the case of Nahm moduli spaces we may introduce the complex Nahm matrices

$$\alpha = T_0 - iT_1, \quad \beta = T_2 + iT_3.$$

Then the Nahm equations are equivalent to the real equation

$$\frac{d}{dt}(\alpha + \alpha^*) + [\alpha, \alpha*] + [\beta, \beta^*] = 0, \tag{3.5}$$

together with the complex equation

$$\frac{d\beta}{dt} + [\alpha, \beta] = 0. \tag{3.6}$$

In the case $G = SU(2)$ Donaldson showed by a variational argument that every solution to (3.6) can be transformed by a complex gauge transformation, unique up to the action of the real gauge group, into a configuration which also solves the real equation.

The next step, therefore, is to describe the solutions to the complex equation modulo (complex) gauge equivalence. Now, on the set where the Nahm matrices are smooth the complex equation may be trivialised in the sense that α may be gauged to zero, and hence β is gauged to a constant element of $\mathfrak{g}_{\mathbb{C}}$ The moduli space is in fact parametrised by an element of $\mathfrak{g}_{\mathbb{C}}$ and some further data which describes how the transformation gaugeing α to zero behaves at an endpoint of the interval. Donaldson uses this parametrisation to describe the moduli space as a set of rational maps. This result has been generalised to other groups by Hurtubise [**47**].

Moduli spaces of instantons on $\mathbb{R}^4$ are isometric, via the ADHM transform [**2, 66**], to moduli spaces of solutions to a system of matrix equations satisfying certain nondegeneracy conditions. These equations are in fact obtained by imposing $\mathbb{R}^4$-translation invariance on the self-dual Yang-Mills equations (3.1). Alternatively, they can be interpreted as the vanishing of a hyper-Kähler moment map for the action of a finite-dimensional group on a finite-dimensional vector space.

Similarly, the Nahm transform relates moduli spaces of monopoles and solutions to Nahm's equations [**72, 73**]. Given a monopole (A, Φ), one defines a family of Dirac operators coupled to (A, Φ), parametrised by a real number. This gives rise to an index bundle over a subset of $\mathbb{R}$, and projection from the trivial bundle defines a connexion which, written out in components, gives a solution to the Nahm equations. A similar procedure allows one to go in the reverse direction and build a monopole from Nahm matrices. The intervals on which the Nahm matrices are defined, and the boundary conditions, determine the gauge group and charge of the monopole. For example, in the case of $SU(2)$ monopoles the Nahm matrices are defined on a single interval and take values in $\mathfrak{u}(k)$, where k is the charge of the monopole. Moreover T_1, T_2, T_3 should have poles at the endpoints of the interval and their residues at each endpoint define the irreducible k-dimensional representation of $SU(2)$. Unlike the situation with instantons, as yet no finite-dimensional procedure for obtaining monopole spaces is known.

Monopole moduli spaces have been the focus of particular attention, because of the Manton programme for deducing information about monopole dynamics from geodesic flow on the moduli spaces [**67**]. In the simplest case, when $G = SU(2)$, the moduli space of charge k monopoles is $4k$-dimensional and is biholomorphic to the space of based rational maps of degree k from the Riemann sphere to itself. If $k = 1$ we just get the flat hyper-Kähler manifold $\mathbb{R}^3 \times S^1$, but for higher k the metric is nontrivial. After taking a cover, the moduli space splits as a product of $\mathbb{R}^3 \times S^1$ and a $(4k-4)$-dimensional hyper-Kähler manifold $\tilde{M}_k^0$ which is the universal cover of the moduli space M_k^0 of monopoles with fixed centre. Of course, this factor contains the interesting geometry.

Atiyah and Hitchin [**3**] studied the $k = 2$ case in detail. This example is particularly interesting because the metric can be found completely explicitly. This is possible because M_2^0 is now four-dimensional and, as mentioned earlier, admits an isometric action of $SO(3)$ permuting the Kähler structures. The orbits are generically three-dimensional, so the metric is of Bianchi IX type. Writing the metric as

$$(abc)^2 dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{3.7}$$

it turns out that the coefficients a, b, c must satisfy the equation

$$2\frac{a'}{a} = (b-c)^2 - a^2 \tag{3.8}$$

and its cyclic permutations. (We can compare this with the equations (1.3) considered by Belinskii, Gibbons, Page and Pope [**5**] in their study of Bianchi IX hyper-Kähler metrics where the action preserves each complex structure individually).

The equations (3.8) are more difficult than those of [**5**], but they can be solved in terms of elliptic functions. There are three solutions corresponding to complete metrics; one gives flat space, one the Taub-NUT metric (2.3), and the last and most complicated gives the Atiyah-Hitchin moduli space metric. This metric is triaxial, but in the asymptotic region of M_2^0 approaches a $U(2)$-invariant metric. This is the so-called negative-mass version of the Taub-NUT metric. Although well-behaved in the asymptotic region, negative-mass Taub-NUT becomes singular in the core region, in contrast both to the Atiyah-Hitchin metric and the version of Taub-NUT given in (2.3). Of course, in the core region, negative mass Taub-NUT is no longer a good approximation to the Atiyah-Hitchin metric, which is smooth and complete.

In fact, Manton has shown that the asymptotic behaviour of the Atiyah-Hitchin metric can be deduced from the dynamics of well-separated monopoles [**68**]. This is a reversal of the usual procedure, in which geometric information is used to study the dynamics.

The underlying manifold of the Atiyah-Hitchin metric is diffeomorphic to $S^4 - \mathbb{RP}^2$. In fact, this identification can be made $SO(3)$-equivariant. We regard S^4 as the sphere in the real irreducible five-dimensional representation of $SO(3)$, which we can take concretely to be the set of traceless symmetric real three-by-three matrices with $SO(3)$ acting by conjugation. The generic stabiliser of the induced $SO(3)$ action on S^4 is just conjugate to the viergruppe of diagonal matrices in $SO(3)$, but there are two special orbits, obtained by considering matrices with two equal positive or two equal negative eigenvalues. Each special orbit is a copy of $\mathbb{RP}^2$. One of these is the $\mathbb{RP}^2$ we remove to obtain the Atiyah-Hitchin manifold, while the other is the unique special orbit in the Atiyah-Hitchin space. Physically, this orbit represents the set of axisymmetric monopoles; there is a unique such monopole for each choice of axis in $\mathbb{R}^3$.

Many other examples of hyper-Kähler manifolds may be obtained by considering Nahm or monopole moduli spaces. A particularly useful example is found by considering $\mathfrak{g}$-valued Nahm matrices smooth on a closed interval. Then, as observed by Kronheimer, the hyper-Kähler moduli space is biholomorphic to the cotangent bundle of $G_{\mathbb{C}}$. The complex description is especially simple here, because the smoothness of the Nahm matrices on the whole interval means that we can globally trivialise the complex Nahm equation–after a gauge transformation, its solutions are constant. The moduli space admits two commuting actions of G, obtained by considering gauge transformations which are the identity at one end of the interval but not the other. The moment maps are just evaluation of the Nahm matrices at the respective ends of the interval. Taking hyper-Kähler quotients of $T^*G_{\mathbb{C}}$ by a subgroup H of one copy of G leads [**20**] to a complete hyper-Kähler metric on a neighbourhood of the zero section in the vector bundle $G \times_H (\mathfrak{m}^* \otimes \mathbb{R}^3)$ over G/H, where $\mathfrak{m}$ is an $\mathrm{Ad}\, H$-invariant complement for $\mathfrak{h}$ in $\mathfrak{g}$. These metrics are hyper-Kähler analogues of the symplectic examples of Guillemin and Sternberg [**37**].

One can obtain a one-parameter family of deformations of the double cover $\tilde{M}_2^0$ of the Atiyah-Hitchin space by taking a hyper-Kähler quotient of a moduli space of $SU(3)$ monopoles by a circle action [**19**]. Making the deformation breaks the $SO(3)$ symmetry of $\tilde{M}_2^0$ down to a circle action. This action, although isometric, does not preserve all the complex structures, so the metric is not one of the Gibbons-Hawking family. The infinitesimal variation of the Kähler forms of the deformation is controlled by an $SO(3)$-invariant anti-self-dual two-form on $\tilde{M}_2^0$. This two-form can also be regarded as the curvature form on the pullback to $\tilde{M}_2^0$ of the index bundle on M_2^0, whose fiber over a monopole is the kernel of the Dirac operator associated to that monopole as in the Nahm transform [**72**]. The index bundle plays a key role in the study by Manton and Schroers of monopoles coupled to fermions [**69**]. In their work on the quantum dynamics of such particles, the wavefunction is interpreted as a section of the index bundle.

One can also construct a series of metrics on deformations of the Kleinian singularity corresponding to the dihedral group D_n for $n \geq 2$. R. Kobayashi showed the existence of such metrics by analytical methods [**54**]. Alternatively [**18**], one may replace an $\mathbb{H}^4$ factor in Kronheimer's construction for dihedral ALE spaces by a copy of $T^*GL(2, \mathbb{C})$ with the hyper-Kähler structure mentioned above. This is analogous to obtaining the multi-Taub-NUT family by replacing an $\mathbb{H}$ factor by an $\mathbb{R}^3 \times S^1$ in the construction of the multi-instanton series. This dihedral series has been further studied by Houghton [**46**], from the Nahm viewpoint, and by Chalmers [**16**], using twistor methods.

In dimension four, one may also mention examples of infinite topological type constructed in [**1, 31**] and [**32**]. These may be viewed as A_∞ and D_∞ limits of the ALE spaces, but, although complete, are not themselves ALE. They can be obtained using an infinite-dimensional hyper-Kähler quotient, or, in the A_∞ case, by a modification of the Gibbons-Hawking method to allow infinitely many points p_i in (1.5). In order to obtain a smooth, complete metric, the sequence of points must go off to infinity fast enough to ensure that the series $\sum \mid \underline{x} - p_i \mid^{-1}$ converges. The D_∞ examples can be realised as desingularisations of the quotient of the A_∞ spaces by an involution. The examples of [**1**] were the first examples of complete Ricci-flat Kähler manifolds of infinite topological type. The last property means that they cannot live on the complement of a divisor in a projective variety, answering in the negative a question posed by Yau.

As a final illustration of gauge theory methods, let us mention an example involving loop spaces. The space ΩG of based loops in a compact Lie group can be made into an infinite-dimensional symplectic (in fact Kähler) manifold [**79**]. Donaldson has shown that the corresponding complex-symplectic structure on $\Omega G_{\mathbb{C}}$ in fact comes from a hyper-Kähler structure on this infinite-dimensional space. The proof relies on interpreting $\Omega G_{\mathbb{C}}$ as a hyper-Kähler quotient of the space of $G_{\mathbb{C}}$ connexions on the trivial $G_{\mathbb{C}}$-bundle over the Poincaré disc by G-valued gauge transformations equal to the identity on the boundary. There is a detailed discussion in Hitchin's survey article [**44**].

4. Coadjoint Orbits

One particularly interesting family of higher-dimensional hyper-Kähler spaces is that of coadjoint orbits for complex semisimple Lie groups. It is well known that a coadjoint orbit of a compact Lie group G carries a symplectic structure. Explicitly, this is defined by the Kirillov-Kostant-Souriau form, given at $\beta \in \mathfrak{g}^*$ by

$$\omega(X^*(\beta), Y^*(\beta)) = \beta([X, Y]) \tag{4.1}$$

where X^*, Y^* are the vector fields on the orbit associated by the action to $X, Y \in \mathfrak{g}$. Similarly, a coadjoint orbit of the complexification $G_{\mathbb{C}}$ carries a complex-symplectic form. But more is true, it in fact admits a compatible hyper-Kähler metric.

The first work on this subject was done by Kronheimer [**59, 60**]. The central idea is to realise the coadjoint orbit as a moduli space of solutions to Nahm's equations. This space carries a hyper-Kähler structure via an infinite-dimensional quotient construction, as discussed in the preceding section.

As an example, let us consider the case of regular semsimple orbits. Let (τ_1, τ_2, τ_3) be a triple of elements in a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and suppose that the intersection of their stabilisers is precisely the corresponding maximal torus. Kronheimer considers the Nahm equations for quadruples (T_0, T_1, T_2, T_3) of functions from $(-\infty, 0]$ to $\mathfrak{g}$. The gauge group of functions from $(-\infty, 0]$ to G acts on solutions as in (3.3). Clearly $(0, \tau_1, \tau_2, \tau_3)$ is a constant solution to the equations. We can obtain other constant solutions by applying the action of a constant element of G to this configuration. Kronheimer now looks at a moduli space M of solutions which, after conjugation by a constant element of G, approach $(0, \tau_1, \tau_2, \tau_3)$ exponentially fast as $t \to -\infty$. The group by which we quotient consists of gauge transformations which are the identity at 0 and have suitable decay at $-\infty$. The equations can then be interpreted, as usual, as the vanishing condition for the hyper-Kähler moment map for this gauge group. The analysis by Donaldson of the Nahm equations, as discussed in §3, carries over to give a description of a complex structure on the moduli space. It turns out that for a suitable choice of τ_2, τ_3, the moduli space M is biholomorphic (for some complex structure) to the orbit of $G_{\mathbb{C}}$ through $\tau_2 + i\tau_3$. This orbit may be viewed as a homogeneous space $G_{\mathbb{C}}/T_{\mathbb{C}}$, and the subset of constant Nahm matrices forms a copy of G/T inside the orbit. The Eguchi-Hanson space is the special case $G = SU(2)$. Burns [**14**], using twistor methods, had earlier shown the existence of a hyper-Kähler metric on a neighbourhood of G/T inside $G_{\mathbb{C}}/T_{\mathbb{C}}$, but Kronheimer's method also tells us the metric is complete.

Kronheimer also showed that nilpotent orbits admit a hyper-Kähler structure. More recently, Biquard and Kovalev [**9, 56**] have independently shown that *any* coadjoint orbit of $G_{\mathbb{C}}$ is hyper-Kähler. The basic strategy is the same as before–to identify the orbit with a suitable Nahm moduli space. One now considers triples (τ_1, τ_2, τ_3) in the Cartan algebra $\mathfrak{t}$ such that the Lie algebra $\mathfrak{h}$ of their common centraliser is allowed to strictly include $\mathfrak{t}$. The correct moduli space to work with consists of solutions to the Nahm equations with T_i asymptotic (possibly after applying the action of G) to $\tau_i + \rho(e_i)t^{-1}$ for $i = 1, 2, 3$. Here e_1, e_2, e_3 form a standard basis for $\mathfrak{su}(2)$, and ρ is a homomorphism from $\mathfrak{su}(2)$ to $\mathfrak{h}$. If ρ is zero we get a semisimple orbit. The nilpotent orbits, on the other hand, are obtained by setting each τ_i to zero. In this case the orbit we get is the one containing $\rho(e_2 + ie_3)$, where we have extended ρ to a homomorphism defined on $\mathfrak{sl}(2, \mathbb{C})$. This illustrates the principle that nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ correspond to conjugacy classes of homomorphisms $\rho : \mathfrak{su}(2) \to \mathfrak{g}$.

The hyper-Kähler metrics obtained are complete if and only if the orbit is semisimple. These orbits are diffeomorphic to the cotangent bundles of flag manifolds for G. In fact for a suitable choice of τ_i they are even biholomorphic, though of course the complex structure here is not the same as that which comes from viewing the manifold as a $G_{\mathbb{C}}$ orbit. Donaldson and Kronheimer have an alternative construction of these hyper-Kähler structures, using gauge theory in two dimensions. In the special case when the flag manifold is a hermitian symmetric space, it is possible to construct the metric more explicitly without the use of gauge-theoretic or twistor methods [**10, 21**]. One also obtains in this way an *incomplete* hyper-Kähler metric on a neighbourhood of the zero section in the cotangent bundle of *noncompact* hermitian symmetric spaces.

If G is a classical group, the nilpotent orbits of $G_{\mathbb{C}}$ may also be obtained by a finite-dimensional hyper-Kähler quotient construction [**53**]. The relevant quotients are singular, but they have a smooth open dense set, which may be identified with a nilpotent orbit.

Nilpotent orbits have a particularly nice interpretation in terms of quaternionic Kähler geometry. Swann [**82**] has shown that given a quaternionic Kähler manifold N with positive scalar curvature, one can define a hyper-Kähler manifold $\mathcal{U}(N)$ whose underlying space is the total space of a $\mathbb{H}^*$- or $(\mathbb{H}^*/\mathbb{Z}_2)$-bundle over N. Now $\mathcal{U}(N)$ admits a free isometric action of $Sp(1)$ or $SO(3)$ permuting the complex structures. Moreover, if A is a complex structure and X_A the Killing field of the circle subgroup of $Sp(1)$ or $SO(3)$ fixing A, then AX_A is independent of A. Conversely, any hyper-Kähler manifold with such an action is $\mathcal{U}(N)$ for some quaternionic Kähler space N. As well as the $Sp(1)$ action, $\mathcal{U}(N)$ also admits a homothetic action of $\mathbb{R}^*$ whose conformal Killing field is AX_A. The manifolds $\mathcal{U}(N)$ may also be characterised as the hyper-Kähler manifolds which admit a hyper-Kähler potential, that is, a function $f : M \to \mathbb{R}$ which is a Kähler potential for each Kähler structure simultaneously.

It turns out that all nilpotent orbits are associated to a quaternionic Kähler manifold via the Swann construction. In terms of the Nahm data description of the nilpotent orbit, the $Sp(1)$ action is just the natural action on triples (T_1, T_2, T_3), while the homothetic action is the rescaling which replaces $T_i(t)$ by $cT_i(ct)$. Note that the boundary conditions at $-\infty$ prevent such actions from occurring in the non-nilpotent case. We can partially order the nontrivial nilpotent orbits of $G_{\mathbb{C}}$ by saying $\mathcal{O}_1 \leq \mathcal{O}_2$ if and only if $\mathcal{O}_1$ is contained in the closure of $\mathcal{O}_2$. There is a unique nontrivial nilpotent orbit minimal with respect to this ordering, and it may be identified with $\mathcal{U}(N)$ where N is the Wolf space [**90**] associated to G. In general, the quaternionic Kähler manifold associated to a nilpotent orbit has a description due to Kobak and Swann [**53**], [**83**]. They define a functional on the Grassmannian $\mathrm{Gr}_3(\mathfrak{g})$ of oriented 3-planes in $\mathfrak{g}$ by $\psi(V) = \langle e_1, [e_2, e_3] \rangle$, where e_1, e_2, e_3 is an oriented orthonormal basis for V. As mentioned above, a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ corresponds to a conjugacy class in G of homomorphisms $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. The associated quaternionic Kähler manifold can then be identified with the stable manifold for ψ of the critical set containing the image of ρ.

Monopole moduli spaces, even though they have a Nahm description and hence admit an $Sp(1)$ action permuting the complex structures, do not fit into the above picture. In fact they do not admit the homothetic $\mathbb{R}^*$-action, as the intervals on which the Nahm matrices are defined are now finite. However, moduli spaces of instantons on $\mathbb{R}^4$ do arise from the Swann construction [**66**].

5. Twistors

The Penrose twistor construction [**78**] has been very successful in translating problems concerning self-dual four-manifolds into questions about the complex geometry of the twistor space. More recently it was realised [**80**] that the twistor construction could be generalised to higher dimensions for manifolds with a quaternionic structure.

A manifold M is *quaternionic* if there is a rank three subbundle $\mathcal{G}$ of $\mathrm{End}(TM)$, preserved by a torsion-free connexion, and locally spanned by sections I, J, K satisfying the relations $I^2 = J^2 = K^2 = -1$, $IJ = K$ etc. The twistor space is now the sphere bundle of $\mathcal{G}$. This space is a complex manifold in a natural way, and the fibers of the twistor projection $\pi : Z \to M$ (the *twistor lines*) are rational curves with normal bundle $2n\mathcal{O}(1)$, where $4n$ is the real dimension of M. There is also a *real structure* τ, an antiholomorphic involution defined by taking the antipodal map on the twistor lines.

If M is hyper-Kähler, then the almost complex structures exist globally, and are integrable. This means that the twistor fibration π is trivial, and that I, J, K define divisors in Z. We may regard Z as the smooth manifold $M \times \mathbb{CP}^1$, with complex structure given at (m, a, b, c) by $(aI + bJ + cK, I_0)$ where I_0 is the standard complex structure on S^2. As well as the twistor fibration, we have a holomorphic fibration $p : Z \to \mathbb{CP}^1$, whose fiber over $(a, b, c) \in \mathbb{CP}^1$ is just M endowed with the complex structure $aI + bJ + cK$. Each point of M defines a section of this fibration.

Of course this picture will arise even if we just have a hyperhermitian structure, that is, a triple of integrable hermitian structures multiplying like the quaternions. The extra information of a hyper-Kähler metric is encoded in a holomorphic section of the second exterior power of the cotangent bundle to the fibers of p, twisted by $p^*\mathcal{O}(2)$. Explicitly, this is given by

$$\omega = \omega_J + i\omega_K - 2\zeta\omega_I - \zeta^2(\omega_J - i\omega_K),$$

where ζ is an affine coordinate on $\mathbb{CP}^1$.

THEOREM 5.1. [**45**] *Let Z be the twistor space of a hyper-Kähler $4n$-manifold M. Then*

(i) *there is a holomorphic fibration $p : Z \to \mathbb{CP}^1$;*
(ii) *the fibration has a family of rational sections with normal bundle $2n\mathcal{O}(1)$;*
(iii) *there is a holomorphic section ω of $\Lambda^2 T_F^* \otimes p^*\mathcal{O}(2)$, defining a symplectic form on each fiber (T_F denotes the tangent bundle to the fibers); and*
(iv) *there is an antiholomorphic involution τ of Z, compatible with* (i–iii) *and inducing the antipodal map on $\mathbb{CP}^1$.*

Conversely any such Z is the twistor space of a hyper-Kähler $4n$-manifold M. This manifold may be identified with the set of real sections of the fibration p.

The procedure for recovering the hyper-Kähler metric from the twistor data goes as follows. Elements of the complexified tangent space to M at m may be regarded as sections of the normal bundle to the twistor line P_m over m. But $H^0(P_m, N)$ can be identified with $H^0(P_m, T_F)$, which in turn may be identified with $H^0(P_m, T_F(-1)) \otimes H^0(P_m, \mathcal{O}(1))$, since $T_F \cong 2n\mathcal{O}(1)$. The $\mathcal{O}(2)$-valued form ω defines a symplectic structure on the first factor, while we define a symplectic form on the second by $\langle a_1 + b_1\zeta, a_2 + b_2\zeta\rangle = a_1 b_2 - a_2 b_1$. The upshot is that we have a symmetric complex bilinear form on $T_m M \otimes \mathbb{C}$, and using the real structure we obtain a positive definite inner product on the real tangent space.

The twistor construction is compatible with the hyper-Kähler quotient in the following way. If G acts on M preserving the hyper-Kähler structure, then we can lift to a holomorphic action of G on Z preserving the twisted complex-symplectic form ω. In good cases, this action extends to a global action of the complexified group $G_{\mathbb{C}}$ and we obtain a moment map μ, which is a section of $\mathcal{O}(2) \otimes \mathfrak{g}_{\mathbb{C}}^*$. The quotient $\mu^{-1}(0)/G_{\mathbb{C}}$ is the twistor space of the hyper-Kähler quotient of M by G.

The simplest example of a twistor space is when $M = \mathbb{H}^n$. Then Z is the total space of $2n\mathcal{O}(1) \to \mathbb{CP}^1$. Many other examples, including the twistor space of the Taub-NUT metric, are worked out in [**42, 6, 45**]. As twistor spaces of hyper-Kähler manifolds fiber over $\mathbb{CP}^1$, they may locally be constructed by patching together copies of $\mathbb{C}^{2n} \times U$ and $\mathbb{C}^{2n} \times \tilde{U}$, where $U, \tilde{U}$ are the standard patches on the Riemann sphere. Often it is convenient to work with a Hamiltonian function for a vector field which exponentiates to give the patching diffeomorphism. A particularly nice case is that of hyper-Kähler $4n$-manifolds with an isometric action of T^n preserving each Kähler structure. The metric is then determined by a solution to a system of linear equations on $\mathbb{R}^{3n}$, and these functions are given in terms of a contour integral involving the Hamiltonian function. Alternatively, one can express the Kähler potential of the metric in terms of this integral. The Gibbons-Hawking metrics are the $n = 1$ case of this theory. As mentioned in §2, the toric hyper-Kähler manifolds considered in [**8**] provide a large class of complete examples where these techniques may be applied.

The twistor method is more general than that of hyper-Kähler quotients, although it has the disadvantage that global properties of the metric are difficult to check in the twistor picture. A recent use of twistor techniques is the result of B. Feix [**26**] that, given a real-analytic Kähler manifold N, a hyper-Kähler metric will exist on some neighbourhood of the zero section in T^*N. This has also been proved independently by Kaledin, using different methods [**51**]. The metric is compatible with the standard complex-symplectic structure on T^*N, and is preserved by the natural circle action obtained by scalar multiplication in each cotangent space. Again, we have an example of an obviously complex-symplectic manifold in fact admitting a hyper-Kähler structure. This metric will not, however, always be complete. Indeed, the Milnor-Wolf theorem shows that a necessary condition for completeness is that finitely generated subgroups of $\pi_1(N)$ must have polynomial growth.

6. Compact Examples

The preceding techniques have usually lead to non-compact manifolds. In this section, we shall consider the compact case, where fewer examples are known.

The two basic examples are the four-torus with a flat metric and a $K3$ surface with the Yau metric. These are, in fact, the only compact four-dimensional hyper-Kähler manifolds. Note that any such manifold is Ricci-flat and anti-self-dual, so gives equality in the Hitchin-Thorpe inequality $3|\tau| \leq 2\chi$ for compact Einstein four-manifolds [**40**]. Hitchin' s list of examples where equality is attained also includes quotients of the above examples, but these are only *locally* hyper-Kähler.

Of course, the existence of a hyper-Kähler metric on $K3$ follows from Yau's proof of the Calabi conjecture, together with the fact that in dimension four a Ricci-flat Kähler metric is hyper-Kähler. There are other interesting ways of viewing this hyper-Kähler metric, however.

The basic idea goes back to Page's 1978 article [**76**]. A classical construction due to Kummer shows that some $K3$ surfaces may be obtained by quotienting the four-torus by an involution and resolving the singularities. Each of the 16 singularities gives rise to a (-2) curve in the $K3$. Page suggested that this picture could be made to work on the Riemannian level. The idea was to glue copies of the Eguchi-Hanson metric on $T^*\mathbb{CP}^1$ to the flat metric on the complement of the singularities in $T^4/\mathbb{Z}_2$, and then smooth out so as to obtain a hyper-Kähler metric on $K3$.

More recently LeBrun and Singer [**64**] gave a rigorous treatment of this idea (see also [**84**]). They use a generalisation to the orbifold case of the Donaldson-Friedman technique for producing self-dual metrics on connected sums [**24**]. In the case of the $K3$ the orbifolds are $T^4/\mathbb{Z}_2$ and 16 copies of the one-point compactification of Eguchi-Hanson, and the resulting singular twistor space can be deformed to a smooth twistor space corresponding to a hyper-Kähler metric on the $K3$. Geometrically, one obtains sequences of $K3$ metrics which converge on an open dense set to the flat metric on $T^4/\mathbb{Z}_2$. Suitable rescalings of the metric at the fixed points converge to Eguchi-Hanson in a bubbling-off phenomenon.

A Kummer-type construction was used by Joyce to construct compact manifolds with exceptional holonomy in dimension 7 and 8 (see [**49, 50**] and the article in this volume). In this case twistor arguments are replaced by direct analytical methods. Joyce has also given a proof along these lines of the existence of a hyper-Kähler metric on the K3 surface.

Compact hyper-Kähler manifolds cannot arise from a finite-dimensional hyper-Kähler quotient [**44**]. It remains an open question whether the $K3$ surface can be produced as a quotient in an infinite-dimensional setting, for example as a gauge theory moduli space. In this connection, it is interesting to note that Braam, Maciocia and Todorov [**13**] have shown that some hyper-Kähler structures on $K3$ surfaces may be obtained by realising $K3$ as a desingularisation of a compactification of a moduli space of instantons on the four-torus. More precisely, they consider the moduli space of $SO(3)$-instantons with $p_1 = -4$ and $(w_2)^2 \equiv 0 \bmod 4$. This is an eight-dimensional space, but it admits an action of the four-torus induced by translations on the base manifold. The four-dimensional quotient $\mathcal{M}$ has some $\mathbb{Z}_2$ orbifold singularities arising from finite stabilisers of the action. There may also be orbifold singularities due to reducible instantons. Finally, compactifying $\mathcal{M}$ introduces a further orbifold singularity corresponding to the quaternion group of order eight. Resolving these singularities yields a K3 surface, which is not in general the Kummer surface of the torus.

Higher dimensional irreducible hyper-Kähler manifolds have been constructed from the four-dimensional examples by Fujiki [**27**] in dimension 8, and later in all dimensions by Mukai [**71**] and Beauville [**4**]. We describe Beauville's method here.

If S is a compact complex manifold, we let $S^{[r]}$ denote the Hilbert scheme whose points are finite analytic subschemes of S of length r. There is a birational map between $S^{[r]}$ and the r^{th} symmetric product of S. If S has dimension one, the symmetric powers are nonsingular, and the map is an isomorphism. If S has dimension two, the symmetric powers are no longer smooth, but $S^{[r]}$ is a desingularisation. We shall take S to be two-dimensional from now on.

Beauville shows that if S admits a complex-symplectic form, so does $S^{[r]}$. The corresponding statement is also true for Kähler structures, so, as S is assumed to be compact, it follows that whenever S is hyper-Kähler, so is $S^{[r]}$. Applying this construction to the $K3$ surface, Beauville obtains compact simply-connected irreducible hyper-Kähler manifolds of dimension $4r$ for each r. The case $r = 2$ was also found by Fujiki, who obtained it as a quotient by an involution of the manifold obtained by blowing up the diagonal in a product of two copies of $K3$. If instead we start with the four-torus, we get a family of spaces which are locally reducible. However, they admit a map onto the four-torus, defined by the group law, and the fiber is a compact, simply-connected irreducible hyper-Kähler manifold of dimension $4r - 4$. If $r = 2$ this is a $K3$ surface. There are also hyper-Kähler deformations of the members of these two families of irreducible hyper-Kähler manifolds.

It is interesting to note that the moduli space of charge r $SU(2)$ monopoles may be viewed as $S^{[r]}$ where S is the flat *noncompact* hyper-Kähler manifold $\mathbb{R}^3 \times S^1$, the charge one moduli space. The physical interpretation is that monopoles are not pure point particles, so the charge r moduli space is not the r^{th} symmetric power of the charge one moduli space, but rather a smoothed out version of this space.

Guan's construction of compact complex-symplectic manifolds admitting no hyper-Kähler metric starts by applying Beauville's method to a Kodaira surface [**33**]. The resulting manifolds inherit their complex-symplectic structure from the Kodaira surface, but, like this surface, have odd first Betti number so admit no Kähler metric. Guan constructs further examples by resolving singularities of complex-symplectic quotients of his initial examples, and by taking deformations [**34**].

A fair amount is now known about the topology of compact hyper-Kähler manifolds. One early observation [**4**] was that compact hyper-Kähler manifolds of dimension $4n$ with holonomy *equal* to $Sp(n)$ are simply connected. Ricci-flatness, and an argument involving the Cheeger-Gromoll splitting theorem, implies that the universal cover is compact. The Bochner argument shows that every holomorphic form on M is parallel, so the holonomy assumption implies that the only holomorphic forms are scalar multiples of powers of the complex-symplectic form ω. The same assertion holds on $\tilde{M}$, so M and $\tilde{M}$ have the same holomorphic Euler characteristic. As this quantity is multiplicative under covers, the assertion follows.

Wakakuwa and Fujiki studied the Hodge and Betti numbers of compact hyper-Kähler manifolds [**89, 28**]. In particular, Fujiki observed that the Hodge numbers satisfy the relation $h^{p,q} = h^{2m-p,q}$, $(0 \leq p, q \leq 2m)$, where $2m$ is the complex dimension of the manifold. In fact, wedging with the $(m-p)^{\text{th}}$ power of the complex-symplectic form provides an isomorphism between the relevant spaces.

Recently Salamon [**81**] has discovered some interesting relations between the Betti numbers of a compact hyper-Kähler manifold. Riemann-Roch techniques yield identities relating Hodge and Chern numbers on a compact Kähler manifold [**65**], and in the Ricci-flat case we have $c_1 = 0$ so a relation on the Hodge numbers follows. For hyper-Kähler manifolds, Fujiki's result allows this to be rewritten as the following equation satisfied by the *Betti* numbers of M:

$$2\sum_{j=1}^{2m}(-1)^j(3j^2 - m)b_{2m-j} = mb_{2m}.$$

Salamon deduces that the Euler characteristic, signature and middle Betti number are even unless the real dimension of M is divisible by 32. Another consequence is that m times the Euler characteristic is divisible by 24. A familiar illustration of these relations is the $K3$ surface, when $m = 1$, $b_1 = 0$, $b_2 = 22$, $\chi = 24$ and $\tau = 16$. As an example in higher dimensions, we may consider the eight-dimensional manifold with holonomy $Sp(2)$ obtained from the four-torus by the Beauville construction. The Poincaré polynomial of this space is $1 + 7t^2 + 8t^3 + 108t^4 + 8t^5 + 7t^6 + t^8$, and the signature is 84.

Verbitsky has also studied the cohomology of compact hyper-Kähler manifolds [**85, 86, 87, 88**]. As the complex structures are parallel they define an action of $\mathfrak{su}(2)$ on $H^*(M)$, which one may integrate to an action of $SU(2)$. Classes which are invariant under this action are of particular importance. If N is a complex submanifold of M with respect to one complex structure I, and if the class represented by the Poincaré dual of (N, I) is invariant, then in fact N is a hyper-Kähler submanifold. Verbitsky uses this to show that for generic elements I of the two-sphere of complex structures, any I-complex submanifold is hyper-Kähler. An interesting consequence is that for such a complex structure M has no divisors, so (M, I) is not projective algebraic,

It is interesting to contrast compact hyper-Kähler manifolds with compact quaternionic Kähler manifolds of positive scalar curvature. In the latter case considerable progress has been made in classification using twistor methods, exploiting the Fano property of the twistor space [**63**]. For hyper-Kähler manifolds, whose twistor spaces do not have such nice properties, much less is currently known about the classification problem.

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Current address: Jesus College, Oxford, OX1 3DW, **United Kingdom**