

Mirror Principle I

Bong H. Lian,¹ Kefeng Liu,² and Shing-Tung Yau³

Abstract. We propose and study the following *Mirror Principle*: certain sequences of multiplicative equivariant characteristic classes on stable map moduli spaces can be computed in terms of certain hypergeometric type classes. As applications, we compute the equivariant Euler classes of obstruction bundles induced by any concave bundles – including any direct sum of line bundles – on \mathbf{P}^n . This includes proving the formula of Candelas-de la Ossa-Green-Parkes for the instanton prepotential function for quintic in \mathbf{P}^4 . We derive, among many other examples, the so-called multiple cover formula for GW invariants of \mathbf{P}^1 . We also prove a formula for enumerating Euler classes which arise in the so-called local mirror symmetry for some noncompact Calabi-Yau manifolds. At the end we interpret an infinite dimensional transformation group, called the mirror group, acting on Euler data, as a certain *duality group* of the linear sigma model.

Appeared in Asian Journ. Math. Vol 1, No. 4 (1997).

¹Department of Mathematics, Brandeis University, Waltham, MA 02154.

²Department of Mathematics, Stanford University, Stanford, CA 94305.

³Department of Mathematics, Harvard University, Cambridge, MA 02138.

Contents

1. Introduction	406
1.1. The Mirror Principle	406
1.2. Enumerative problems and the Mirror Conjecture	409
1.3. Acknowledgements	412
2. Euler Data	413
2.1. Preliminaries and notations	413
2.2. Eulerity	415
2.3. Concave bundles	421
2.4. Linked Euler data	428
2.5. The Lagrange map and mirror transformations	431
3. Applications	437
3.1. The first convex example: The Mirror Conjecture	440
3.2. First concave example: multiple-cover formula	443
3.3. Second concave example: $K_{\mathbf{P}^2}$	444
3.4. A concave bundle on \mathbf{P}^3	445
3.5. A concave bundle on \mathbf{P}^4	446
3.6. General concave bundles	447
3.7. Equivariant total Chern class	448
3.8. Concluding remarks	449

1. Introduction

1.1. The Mirror Principle

In section 2, we develop a general theory of Euler data, and give many examples. In particular we introduce the notions of a convex and a concave bundles on \mathbf{P}^n , and show that they naturally give rise to Euler data. In section 3 we apply our method to compute the equivariant Euler classes, and their nonequivariant limits, of obstruction bundles induced by a convex or a concave bundle.

We briefly outline our approach for computing multiplicative equivariant characteristic classes on stable map moduli. This outline also fixes some notations used later. Our approach is partly inspired by the idea of mirror transformations (Candelas et al), the idea of the linear sigma model (Witten, Morrison-Plesser, Jinzenji-Nagura,..), and the idea of using the torus action and equivariant techniques (Ellinsrud-Stromme, Kontsevich, Givental,..). All these ideas are synthesized with what we call the *Mirror Principle*, which we now explain.

Let M be a projective manifold and $\beta \in H^2(M, \mathbf{Z})$. Let $\bar{\mathcal{M}}_{g,k}(\beta, M)$ be the stable map moduli space of degree β , arithmetic genus g , with k marked points. For a good introduction to stable maps, see [16]. Throughout this paper, we shall only deal with the case with $g = 0$.

We begin by analyzing two distinguished types of fixed points under an induced torus T action on $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$. Both types of fixed points reflect the structure of the stable map moduli space. A smooth fixed point we consider is a degree d cover of a T -invariant \mathbf{P}^1 joining two fixed points p_i, p_j in \mathbf{P}^n . A singular fixed point we consider is in the compactification divisor. It is given by gluing together two 1-pointed maps $(f_1, C_1, x_1) \in \bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$ and $(f_2, C_2, x_2) \in \bar{\mathcal{M}}_{0,1}(d-r, \mathbf{P}^n)$ at the marked points with $f_1(x_1) = f_2(x_2) = p_i \in \mathbf{P}^n$ (p_i being a T -fixed in \mathbf{P}^n), resulting in a degree d stable map (f, C) . We consider two types of T -equivariant bundles V on \mathbf{P}^n , which we called convex and concave respectively (Definition 2.7). To be brief, we consider the convex case in this outline. A convex bundle $V \rightarrow \mathbf{P}^n$ induces on $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ an obstruction bundle U_d whose fiber at (f, C) is the section space $H^0(C, f^*V)$. First we have the exact sequence over C

$$0 \rightarrow f^*V \rightarrow f_1^*V \oplus f_2^*V \rightarrow V|_{p_i} \rightarrow 0.$$

Passing to cohomology, we have

$$0 \rightarrow H^0(C, f^*V) \rightarrow H^0(C_1, f_1^*V) \oplus H^0(C_2, f_2^*V) \rightarrow V|_{p_i} \rightarrow 0.$$

Hence we obtain a similar exact sequence for the U_d restricted to a suitable fixed point set. Let b_T be any multiplicative T -equivariant characteristic class [23] for vector bundles. The exact sequence on the fixed point set above gives rise to the identity, which we call the *gluing identity*:

$$b_T(V) \cdot b_T(U_d) = b_T(U_r) \cdot b_T(U_{d-r}).$$

Let $M_d := \bar{\mathcal{M}}_{0,0}((1, d), \mathbf{P}^1 \times \mathbf{P}^n)$. This space has a $G = S^1 \times T$ action. There is a natural equivariant contracting map $\pi : M_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ given by

$$\pi : (f, C) \in M_d \mapsto (\pi_2 \circ f, C') \in \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$$

where π_2 denotes the projection onto the second factor of $\mathbf{P}^1 \times \mathbf{P}^n$. Here $C' = C$ if $(\pi_2 \circ f, C)$ is still stable; and if $(\pi_2 \circ f, C)$ is unstable, this is the case when C is of the form $C = C_1 \cup C_0 \cup C_2$ with $\pi_2 \circ f(C_0)$

a point in \mathbf{P}^n , then C' is obtained from C by contracting the unstable component C_0 .

Now via π , we pull back to M_d all the information obtained above on $\bar{M}_{0,0}(d, \mathbf{P}^n)$. The reason is that there is an collapsing map $\varphi : M_d \rightarrow N_d = \mathbf{P}^{(n+1)d+n}$ which then allows us to perform computations on the linear object N_d . We call N_d *the linear sigma model* and M_d *the nonlinear sigma model*. There is a natural G action on N_d such that φ is G -equivariant. For example, to determine an equivariant cohomology class ω on N_d , we only need to know its restrictions $\iota_{p_{ir}}^*(\omega)$ to the $(n+1)(d+1)$ G -fixed points $\{p_{ir}\}$ in N_d . The corresponding weights of the G action on N_d are $\lambda_i + r\alpha$. Let Q_d be the push-forward of $b_T(V_d) = \pi^*b_T(U_d)$ into N_d , ie. $Q_d = \varphi_* b_T(V_d)$. Then the classes Q_d inherit the gluing identity (Theorem 2.8)

$$\iota_{p_{i,0}}^*(Q_0) \cdot \iota_{p_{i,r}}^*(Q_d) = \iota_{p_{i,r}}^*(Q_r) \cdot \iota_{p_{i,r}}^*(Q_{d-r})$$

which is an identity in the ring $H_G^*(pt) = H^*(BG)$. The sequence Q_d is an example of what we call an *Euler data* (Definition 2.3). We summarize the various ingredients used, now and later, in our constructions:

$$\begin{array}{ccccccc}
 & & V_d = \pi^*U_d & & U_d & & \rho^*U_d \\
 & & \downarrow & & \downarrow & & \downarrow \\
 N_d & \xleftarrow{\varphi} & M_d & \xrightarrow{\pi} & \bar{M}_{0,0}(d, \mathbf{P}^n) & \xleftarrow{\rho} & \bar{M}_{0,1}(d, \mathbf{P}^n) & \xrightarrow{ev} & \mathbf{P}^n
 \end{array}$$

where ρ forgets, and ev evaluates at, the marked point of a 1-pointed stable map. Also note that $U_d = \rho_!(ev^*V)$, ie. the direct image of ev^*V .

The gluing identity is not enough to determine all Q_d . In order to get further information we localize the Q_d to a fixed point in N_d whose inverse image in M_d is a *smooth* fixed point, and compute Q_d at the special values of $\alpha = (\lambda_i - \lambda_j)/d$. The property of $b_T(U_d)$ at a smooth fixed point comes into play here. For simplicity in this outline, let's consider for example the case when b_T is the equivariant Euler class e_T and $V = \mathcal{O}(l)$. In this case we find that, at $\alpha = (\lambda_i - \lambda_j)/d$,

$$\iota_{p_{i,0}}^*(Q_d) = \prod_{m=0}^{ld} (l\lambda_i - m(\lambda_i - \lambda_j)/d).$$

This immediately tells us that the Q_d should be compared with the sequence of classes

$$P : P_d = \prod_{m=0}^{ld} (l\kappa - m\alpha)$$

(Theorem 2.10), which has $\iota_{p_i,0}^*(P_d) = \iota_{p_i,0}^*(Q_d)$ at $\alpha = (\lambda_i - \lambda_j)/d$. We find that P is another example of an Euler data. This Euler data will then naturally give rise to a generating series of hypergeometric type, hence explains the very origin of these functions in enumerative problems on stable moduli! The same holds true for a large class of vector bundles V on \mathbf{P}^n .

Finally, under a suitable bound on $c_1(V)$, we can completely determine the Q_d , in terms of P_d by means of a mirror transformation argument (Theorem 3.9). We can in turn use Q_d to compute $e_T(U_d)$ and their nonequivariant limits (Theorem 3.2). Our approach, thus, makes the roles of three objects and their relationships quite transparent: certain fixed point sets, equivariant multiplicative characteristic classes, and series of hypergeometric type. Our method works well for many characteristic classes such as the Euler class and the total Chern class.

We summarize this *Mirror Principle*. Let b_T be a multiplicative equivariant characteristic class, V an equivariant vector bundle on \mathbf{P}^n which induces a sequences of vector bundles $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$.

1. (Euler Data) The behaviour of the U_d at a singular fixed point gives rise to the gluing identity. In turn this defines an Euler data Q_d on the linear sigma model N_d .
2. (Linking) The behaviour of the U_d at a smooth fixed point allows us to read off the restrictions of Q_d at certain fixed points of N_d for special values of weights. The restriction values determine a distinguished Euler data P_d to be compared with the Q_d .
3. (Mirror Transformations) Compute the Q_d and the $b_T(U_d)$ in terms of the P_d explicitly by means of a mirror transformation argument.

Our approach outlined here can be applied to a rather broad range of cases by replacing \mathbf{P}^n by other manifolds. They include manifolds with torus action and their submanifolds. We will study the cases of toric varieties and homogeneous manifolds in a forth-coming paper [37]. On the other hand, we can even go beyond equivariant multiplicative characteristic classes. In a future paper, we will study a sequence of equivariant classes T_d of geometric origin on stable moduli satisfying our gluing identity.

1.2. Enumerative problems and the Mirror Conjecture

For the remarkable history of the Mirror Conjecture, see [14]. In 1990, Candelas, de la Ossa, Green, and Parkes [8] conjectured a formula for counting the number n_d of rational curves in every degree d on a general quintic in \mathbf{P}^4 . The conjectured formula gives a

generating function for the n_d explicitly in terms of certain hypergeometric functions. Their computation is based on the existence of “mirror manifolds”, conjectured by a number of physicists including Dixon and Lerche, Vafa and Warner on the basis of physical intuition. Mirror symmetry took a dramatic turn upon the appearance of the papers of Greene-Plesser [21] and of Candelas et al [8]. In [21], they established the existence of mirror manifolds for a particular class of Calabi-Yau manifolds including the quintic.

On the mathematical side, it has been conjectured earlier by Clemens that the number of rational curves in every degree in a general quintic is finite. The conjectured formula agrees with a classical result in degree 1, an earlier computation by S. Katz in degree 2, and has been verified in degree 3 by Ellingsrud-Stromme. In 1994, using results of Uhlenbeck-Sacks, Gromov, Parker-Wolfson, and Witten, Ruan-Tian introduced the notions of J -holomorphic maps and symplectic Gromov-Witten (GW) invariants. Motivated by the same works, Kontsevich proposed an algebraic geometric analogue of GW invariants and stable maps. The latter is the algebraic geometric counterpart of J -holomorphic maps. Generalizations have been given by Li-Tian and Behrend-Fantuzzi. A recent paper of Li-Tian shows that the symplectic version and the algebraic geometric version of the GW theory are essentially the same in the projective category. Beautiful applications of ideas from quantum cohomology and GW theory have recently been done by Caporaso-Harris [10], Crauder-Miranda [11], DiFrancesco-Itzykson [12], Bryan-Leung [2], and others solving many important enumerative problems.

Closer to mirror symmetry, it is known that the degree k GW invariants for \mathbf{P}^1 (the so-called multiple-cover contribution) is given by k^{-3} . This was conjectured in [8], justified in [1][44] using a different compactification, and in [38] using the stable map compactification. According to [34], the number

$$K_d = \sum_{k|d} n_{d/k} k^{-3}$$

is the degree of the Euler class $c_{top}(U_d)$ for $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^4)$ induced by $\mathcal{O}(5) \rightarrow \mathbf{P}^4$.

By means of the torus action on \mathbf{P}^4 , Ellingsrud-Stromme [15] and Kontsevich [34] have verified that n_3 and n_4 respectively agree with the conjectured formula. In some recent papers [18][19], Givental introduces some ideas which emphasize the use of equivariant version of quantum cohomology (see also Kim [27]). Unfortunately, the proof in [18] for the formula of Candelas et al, which has been read by many

prominent experts, is incomplete. In light of our present paper, it is now clear that essentially all ingredients (equivariant cohomology theory, the Atiyah-Bott formula, equivariant Euler classes, obstruction bundles, fixed point structures, the linear sigma model, the graph construction, hypergeometric series, mirror transformations, etc) required to prove the formula of Candelas et al have all been well-known to experts since at least 1995. Thus the remaining nontrivial task, is to set up and put together in details all the pieces in a correct way to give a complete proof. In our present paper, we have introduced the substantial new ideas (beyond the known ingredients) needed to give the first complete proof. It is conceivable that our new ideas here can furnish the necessary parts missing in the paper [18]. More recently, fixed point method has also been applied by Graber-Pandharipande [20] to study GW invariants of \mathbf{P}^n .

We emphasize that the formula of Candelas et al, albeit conjectural, not only computes the K_d in terms of hypergeometric functions, but also provides a crucial guide for putting together various technical inputs in our proof. While applying the Atiyah-Bott localization formula on the stable moduli and on the linear sigma model yield important pieces of information about the cohomology classes in question, it is the formula of Candelas et al which provides the clues for how to put the pieces together – in terms of hypergeometric functions (which are periods of a holomorphic 3-form on a mirror manifold). Earlier attempts on the conjectured formula using quantum cohomology seems to have confused the issue considerably.

We believe that the machinery introduced in this paper will be useful for many other enumerative problems, aside from proving the formula of Candelas et al. In fact we have applied our machinery to problems in local mirror symmetry proposed by C. Vafa, S. Katz, and others. ⁴

We now formulate one of our main theorems in this paper. Let

$$K_d = \int_{\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^4)} c_{top}(U_d), \quad F(T) = \frac{5T^3}{6} + \sum_{d>0} K_d e^{dT}.$$

Consider the fourth order hypergeometric differential operator [8]:

$$L := \left(\frac{d}{dt}\right)^4 - 5e^t \left(5\frac{d}{dt} + 1\right) \cdots \left(5\frac{d}{dt} + 4\right).$$

S. Katz has informed us that A. Elezi has also studied a similar problem.

By the Frobenius method, it is easy to show that

$$f_i := \frac{1}{i!} \left(\frac{d}{dH} \right)^i |_{H=0} \sum_{d \geq 0} e^{d(t+H)} \frac{\prod_{m=1}^{5d} (5H+m)}{\prod_{m=1}^d (H+m)^5}, \quad i = 0, 1, 2, 3,$$

form a basis of solutions to the differential equation $L \cdot f = 0$. Let

$$T = \frac{f_1}{f_0}, \quad \mathcal{F}(T) = \frac{5}{2} \left(\frac{f_1}{f_0} \frac{f_2}{f_0} - \frac{f_3}{f_0} \right).$$

Theorem 1.1. (*The Mirror Conjecture*) $F(T) = \mathcal{F}(T)$.

The transformation on the functions f_i given by the normalization

$$f_i \mapsto \frac{f_i}{f_0}$$

and the change of variables

$$t \mapsto T(t) = \frac{f_1}{f_0}$$

are known as the *mirror transformation*. By the construction of Candelas et al, the functions f_0, \dots, f_3 are periods of a family of Calabi-Yau threefolds. By the theorem of Bogomolov-Tian-Todorov, these periods in fact determine the complex structure of the threefold.

A similar Mirror Conjecture formula holds true for a three dimensional Calabi-Yau complete intersection in a toric Fano manifold [37]. This will turn out to agree with the many beautiful mirror symmetry computations [40][36][5][9][4][30][31].

In this paper, to make the ideas clear we restrict ourselves to the simplest case, genus 0 stable moduli space for \mathbf{P}^n . In a forth-coming paper [37], we extend our discussions to toric varieties, homogeneous manifolds, and general projective manifolds. We also hope to eventually understand from our point of view the far reaching results for higher genus of [6], and the beautiful computations [17][13] for elliptic GW invariants.

1.3. Acknowledgements

We thank A. Todorov, A. Strominger, C. Vafa for helpful discussions. Our special thanks are due to J. Li who has been very helpful throughout our project.

2. Euler Data

One of the key ingredients in our approach is the linear sigma model, first introduced by Witten [45], and later used to study mirror symmetry by Morrison-Plesser [29], Jinzenji-Nagura [25], and others, resulting in new insights into the origin of hypergeometric series. In this paper, we consider the $S^1 \times T$ -equivariant cohomology of the linear sigma model.

2.1. Preliminaries and notations

Let T be an r -dimensional real torus with a complex linear representation on \mathbf{C}^{N+1} . Let β_0, \dots, β_N be the weights of this action. We consider the induced action of T on \mathbf{P}^N , and the T -equivariant cohomology with coefficients in \mathbf{Q} , which we shall denote by $H_T^*(-)$. Now $H_T^*(pt)$ is a polynomial algebra in r -variables, and β_i may be regarded as elements of $H_T^2(pt)$. *Throughout this paper, we shall follow the convention that such generators have degree 1.* It is known that the equivariant cohomology of \mathbf{P}^N is given by [32]

$$H_T(\mathbf{P}^N) = H_T(pt)[\zeta] / \left(\prod_{i=0}^N (\zeta - \beta_i) \right).$$

Here ζ , which we shall call the equivariant hyperplane class, is a fixed lifting of the hyperplane class of \mathbf{P}^N . Each one-dimensional weight space in \mathbf{C}^{N+1} becomes a fixed point p_i in \mathbf{P}^N . *We shall identify the rings $H_T^*(p_i)$ and $H_T^*(pt) = H^*(BT)$.* There are $N + 1$ canonical restriction maps $\iota_{p_i}^* : H_T(\mathbf{P}^N) \rightarrow H_T(pt)$, given by $\zeta \mapsto \beta_i$, $i = 0, \dots, N$. There is also a push-forward map $H_T(\mathbf{P}^N) \rightarrow H_T(pt)$ given by integration over \mathbf{P}^N . By the localization formula, it is given by

$$\omega \mapsto \int_{\mathbf{P}^N} \omega = \text{Res}_\zeta \frac{\omega}{\prod_{i=0}^N (\zeta - \beta_i)}.$$

Two situations arise frequently in this paper. First consider the standard action of $T = (S^1)^{n+1}$ on \mathbf{C}^{n+1} , and let $\lambda = (\lambda_0, \dots, \lambda_n)$ denote the weights. On \mathbf{P}^n , there are $n + 1$ isolated fixed points p_0, \dots, p_n . We shall denote the equivariant hyperplane class by p , the canonical restrictions by $\iota_{p_i}^* : \omega \mapsto \iota_{p_i}^*(\omega) = \omega(\lambda_i)$, and the push-forward by

$$pf : H_T(\mathbf{P}^n) \rightarrow H_T(pt) = \mathbf{Q}[\lambda].$$

We shall use the evaluation map $\lambda \mapsto 0$ on the ring $H_T^*(\mathbf{P}^n)$, and shall call this the *nonequivariant limit*. In this limit, p becomes the ordinary hyperplane class $H \in H^*(\mathbf{P}^n)$.

We now consider the second situation. For each $d = 0, 1, 2, \dots$, consider the following complex linear action of the group $G := S^1 \times T$ on $\mathbf{C}^{(n+1)(d+1)}$. We let the group act on the (ir) -th coordinate line in $\mathbf{C}^{(n+1)(d+1)}$ by the weights $\lambda_i + r\alpha$ (λ_i being the weights of T as before, and α being a weight of S^1), $i = 0, \dots, n, r = 0, \dots, d$. Thus there are $(n+1)(d+1)$ isolated fixed points p_{ir} on the projective space $\mathbf{P}^{(n+1)d+n}$, given by those coordinate lines. In this case, we shall denote the equivariant hyperplane class by κ , the canonical restrictions by $\iota_{p_{ir}}^*$: $\omega \mapsto \iota_{p_{ir}}^*(\omega) = \omega(\lambda_i + r\alpha)$, and the push-forward by

$$pfd : H_G(\mathbf{P}^{(n+1)d+n}) \rightarrow H_G(pt) = \mathbf{Q}[\alpha, \lambda].$$

Here we have abused the notation κ , using it to represent a class in $H_G(\mathbf{P}^{(n+1)d+n})$ for every d . But it should present no confusion in the context it arises.

Let N_d be the space of nonzero $(n+1)$ -tuple of degree d homogeneous polynomials in two variables w_0, w_1 , modulo scalar. There is a canonical way to identify N_d with $\mathbf{P}^{(n+1)d+n}$. Namely, a point $z \in \mathbf{P}^{(n+1)d+n}$ corresponds to the polynomial tuple $[\sum_r z_{0r} w_0^r w_1^{d-r}, \dots, \sum_r z_{nr} w_0^r w_1^{d-r}] \in N_d$. This identification will be used throughout this paper. The natural T -action on $(n+1)$ -tuples together with the S^1 -action on $[w_0, w_1] \in \mathbf{P}^1$ by weights $(\alpha, 0)$, induces a $S^1 \times T$ -action on N_d given by $[f_0(w_0, w_1), \dots, f_n(w_0, w_1)] \mapsto [e^{\lambda_0} f_0(e^\alpha w_0, w_1), \dots, e^{\lambda_n} f_n(e^\alpha w_0, w_1)]$. This coincides with the $S^1 \times T$ -action on $\mathbf{P}^{(n+1)d+n}$ described earlier.

Definition 2.1. (Notations) We call the sequence of projective spaces $\{N_d\}$ the linear sigma model for \mathbf{P}^n . Here are some frequently used notations: $G := S^1 \times T$, $\mathcal{R} := \mathbf{Q}(\lambda)[\alpha]$, $\mathcal{R}^{-1} := \mathbf{Q}(\lambda, \alpha)$, $\mathcal{R}H_G^*(N_d) := H_G^*(N_d) \otimes_{\mathbf{Q}[\lambda, \alpha]} \mathcal{R}$, $\mathcal{R}^{-1}H_G^*(N_d) := H_G^*(N_d) \otimes_{\mathbf{Q}[\lambda, \alpha]} \mathcal{R}^{-1}$, and $\text{deg}_\alpha \omega$ means the degree in α of $\omega \in \mathcal{R}$.

Obviously the maps $\iota_{p_{ir}}^*$, pfd defined linearly over $\mathbf{Q}[\lambda, \alpha]$, can be extended \mathcal{R} - or \mathcal{R}^{-1} -linearly. There are two natural equivariant maps between the N_d , given by

$$I : N_{d-1} \rightarrow N_d, \quad [f_0, \dots, f_n] \mapsto [w_1 f_0, \dots, w_1 f_n]$$

$$\bar{\cdot} : N_d \rightarrow N_d, \quad [f_0(w_0, w_1), \dots, f_n(w_0, w_1)] \mapsto [f_0(w_1, w_0), \dots, f_n(w_1, w_0)].$$

The second map induces on equivariant cohomology $\mathcal{R}^{-1}H_G^*(N_d)$, $\bar{\kappa} = \kappa - d\alpha$, $\bar{\alpha} = -\alpha$, $\bar{\lambda}_i = \lambda_i$. In particular any $x \in \mathcal{R}$ has the form

$x = x_- + x_+$ with $\bar{x}_\pm = \pm x_\pm$. We also extend $\bar{\cdot}$ to the power series ring $\mathcal{R}[[e^t]]$ by leaving t invariant.

Composing a chain of d I 's, we get a canonical inclusion $N_0 = \mathbf{P}^n \xrightarrow{I_d} N_d$. Note that the image of the fixed point p_i is $p_{i,0}$. For $\omega \in \mathcal{R}^{-1}H_G^*(N_d)$, we shall denote by $I_d^*(\omega) \in \mathcal{R}^{-1}H_G^*(N_0)$ the restriction of ω to N_0 . Since S^1 acts trivially on \mathbf{P}^n , we can write

$$H_G^*(N_0) = H_T^*(\mathbf{P}^n)[\alpha].$$

In particular note that $H_T^*(\mathbf{P}^n)$ is invariant under $\bar{\cdot}$, and that $I_d^*(\kappa) = p$.

Obviously the set of classes $\omega \in H_T^*(\mathbf{P}^n)$ with $\iota_{p_i}^*(\omega) \neq 0$ for all i , is closed under multiplication. We localize the ring $H_T^*(\mathbf{P}^n)$ by allowing to invert such elements ω . We denote the resulting ring by $H_T^*(\mathbf{P}^n)^{-1}$.

Definition 2.2. (Notations) The degree in α of a class $\omega \in H_T^*(\mathbf{P}^n)^{-1}[\alpha]$ will be denoted by $\text{deg}_\alpha \omega$. A class $\Omega \in H_T^*(\mathbf{P}^n)^{-1}$ with $\iota_{p_i}^*(\Omega) \neq 0$ for all i will be called invertible. Throughout this paper, Ω will denote a fixed but arbitrary invertible class. \mathcal{S} denotes the set of sequences of cohomology classes $Q : Q_d \in \mathcal{R}^{-1}H_G^*(N_d)$, $d = 1, 2, \dots$

2.2. Eulerity

Definition 2.3. We call a sequence $Q : Q_d \in \mathcal{R}H_G^*(N_d)$, $d = 1, 2, \dots$, an Ω -Euler data if for all d , and $r = 0, \dots, d$, $i = 0, \dots, n$,

$$(*) \quad \iota_{p_i}^*(\Omega) \iota_{p_{i,r}}^*(Q_d) = \overline{\iota_{p_{i,0}}^*(Q_r)} \iota_{p_{i,0}}^*(Q_{d-r}),$$

where $Q_0 := \Omega$. We denote by \mathcal{A}^Ω the set of Ω -Euler data.

When dealing only with one fixed class Ω at a time, we shall say Euler rather than Ω -Euler, and shall write \mathcal{A} for the set of Euler data.

More explicitly, condition (*) can be written as $\Omega(\lambda_i)Q_d(\lambda_i + r\alpha) = \overline{Q_r(\lambda_i)}Q_{d-r}(\lambda_i)$. Applying this at $r = d$, we find that

$$Q_d(\lambda_i + d\alpha) = \overline{Q_d(\lambda_i)}. \tag{2.1}$$

Putting $\alpha = (\lambda_j - \lambda_i)/d$, we see that $Q_d(\lambda_j)$ at $\alpha = (\lambda_j - \lambda_i)/d$ coincides with $\overline{Q_d(\lambda_i)}$ at $\alpha = (\lambda_i - \lambda_j)/d$. Applying both (*) and (2.1) at $\alpha = (\lambda_j - \lambda_i)/r$ (hence $\lambda_j = \lambda_i + r\alpha$), we get

$$\Omega(\lambda_i)Q_d(\lambda_j) = Q_r(\lambda_j)Q_{d-r}(\lambda_i) \text{ at } \alpha = (\lambda_j - \lambda_i)/r.$$

Lemma 2.4. (*Reciprocity Lemma*) *If Q is an Euler data, then for $i, j = 0, \dots, n, r = 0, 1, \dots, d, d = 0, 1, 2, \dots$, we have*

- (i) $Q_d(\lambda_i + d\alpha) = Q_d(\lambda_i)$.
- (ii) $Q_d(\lambda_j)$ at $\alpha = (\lambda_j - \lambda_i)/d$ coincides with $Q_d(\lambda_i)$ at $\alpha = (\lambda_i - \lambda_j)/d$ for $d \neq 0$.
- (iii) $\Omega(\lambda_i)Q_d(\lambda_j) = Q_r(\lambda_j)Q_{d-r}(\lambda_i)$ at $\alpha = (\lambda_j - \lambda_i)/r$ for $r \neq 0$.

Example 1. Let l be a positive integer. Put

$$P : P_d = \prod_{m=0}^{ld} (l\kappa - m\alpha) \in H_G^*(N_d).$$

That this is an lp -Euler data follows from the identity:

$$l\lambda_i \prod_{m=0}^{ld} (l\lambda_i + (r - m)\alpha) = \prod_{m=0}^{lr} (l\lambda_i + m\alpha) \times \prod_{m=0}^{l(d-r)} (l\lambda_i - m\alpha).$$

This will arise naturally in the problem of computing the equivariant Euler classes of the obstruction bundles induced by $\mathcal{O}(l) \rightarrow \mathbf{P}^n$. (See below.)

Example 2. Put $\Omega = p^{-2}$, and

$$P : P_d = \prod_{m=1}^{d-1} (\kappa - m\alpha)^2 \in H_G^*(N_d).$$

This Euler data will arise in the problem of computing the so-called multiple cover contributions, ie. the GW invariants for \mathbf{P}^1 .

Example 3. Put $\Omega = (-3p)^{-1}$, and

$$P : P_d = \prod_{m=1}^{3d-1} (-3\kappa + m\alpha) \in H_G^*(N_d).$$

This Euler data will arise in the problem of computing the equivariant Euler classes of the obstruction bundles induced by the canonical bundle of \mathbf{P}^2 .

Example 4. Put $\Omega = -1$, and

$$P : P_d = \prod_{m=0}^{2d} (2\kappa - m\alpha) \times \prod_{m=1}^{2d-1} (-2\kappa + m\alpha) \in H_G^*(N_d).$$

This Euler data will arise in the problem of computing the equivariant Euler classes of the obstruction bundles induced by $\mathcal{O}(2) \oplus \mathcal{O}(-2)$ on \mathbf{P}^3 .

Example 5. It is easy to show that if Q is Ω -Euler and $a \in \mathbf{Q}(\lambda)$ is any nonzero element, then the data $Q' : Q'_d = a \cdot Q_d$ is $a\Omega$ -Euler. Similarly, the data $Q' : Q'_d = (-1)^{d+1}Q_d$ is also $-\Omega$ -Euler.

Example 6. We observe that the set of Euler data is a monoid, ie. it is closed under the product $Q_d Q'_d$, and has the unit given by $Q_d = 1$ for all d . Hence the product of an Ω -Euler with an Ω' -Euler data is an $\Omega\Omega'$ -Euler data. In the geometrically setting, this multiplicative property sometimes corresponds to taking intersection of two suitable projective manifolds. In this case, the class $\Omega \in H_T^*(\mathbf{P}^n)$ plays the role of the equivariant Thom class of the normal bundle of such a projective manifold.

Example 7. Let $Q_d = \kappa(\kappa - d\alpha) \in H_G^*(N_d)$. Then it is again trivial to check that Q is κ^2 -Euler.

Example 8. Let Q be an Ω -Euler data, and Q' be an Ω' -Euler data. Suppose $Q_d/Q'_d \in \mathcal{RH}_G^*(N_d)$ for all $d > 0$. Then it is immediate that they form a sequence, denoted by Q/Q' , which is Ω/Ω' -Euler. As a special case, let $Q_d = l^2\kappa(\kappa - d\alpha)$ as in Example 7, and let P as in Example 1. Then P/Q is a $(lp)^{-1}$ -Euler data. Example 2 is obtained by squaring this. Examples 3 and 4 can also be obtained in a similar way.

Example 9. Introduce a formal variable x . We can extend everything above by adjoining x , ie. by replacing the ground field \mathbf{Q} by $\mathbf{Q}(x)$. For example, it is easy to show that

$$P : P_d = \prod_{m=0}^{ld} (x + l\kappa - m\alpha) \in H_G^*(N_d)(x)$$

satisfies the gluing identity as in Example 1, thus is an Euler data in a more general sense. Such Euler data will appear in the computations of equivariant total Chern classes.

Example 10. Let $M_d^0 := \mathcal{M}_{0,0}((1, d), \mathbf{P}^1 \times \mathbf{P}^n)$ be the moduli space of holomorphic maps $\mathbf{P}^1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^n$ of bidegree $(1, d)$. Recall that M_d is the stable map compactification of M_d^0 . Each map $f \in M_d^0$ can be represented by $f : [w_0, w_1] \mapsto [w_1, w_0] \times [f_0(w_0, w_1), \dots, f_n(w_0, w_1)]$, where f_i are degree d homogeneous polynomials. So there is an obvious map

$$\varphi : M_d^0 \rightarrow N_d, f \mapsto [f_0, \dots, f_n]$$

which is $G = S^1 \times T$ equivariant. For convenience we define $M_0 := N_0 = \mathbf{P}^n$. With a bit of work (see below), it can be shown that

the map φ has an equivariant regular extension to $\varphi : M_d \rightarrow N_d$. Let $(f, C) \in M_d$. Then C is an arithmetic genus 0 curve of the form $C = C_0 \cup C_1 \cup \dots \cup C_N$ such that $\pi_1 \circ f : C_0 \xrightarrow{\sim} \mathbf{P}^1$, where π_1, π_2 are projections from $\mathbf{P}^1 \times \mathbf{P}^n$ to the first and second factors. Each $C_j, j > 0$, is glued to C_0 at some point $x_j \in C_0$. The map $\pi_2 \circ f : C_j \rightarrow \mathbf{P}^n$ is of degree d_j with $\sum_j d_j = d$, and $\pi_1 \circ f : C_j \rightarrow \mathbf{P}^1$ is constant map with image $\pi_1 \circ f(x_j) \in \mathbf{P}^1$. If we denote by $[\sigma_0, \dots, \sigma_n]$ the degree d_0 polynomials representing $\pi_2 \circ f : C_0 \rightarrow \mathbf{P}^n$, then $\varphi : (f, C) \mapsto [\sigma_0 g, \dots, \sigma_n g]$, where $g = \prod_j (a_j w_0 - b_j w_1)^{d_j}$ with $\pi_1 \circ f(x_j) = [a_j, b_j]$. Thus φ collapses all but one component of C .

The idea of using a collapsing map relating two moduli problems is not new. The map φ was known to Tian [43], and a similar map also appeared in [35] in which a collapsing map was used to relate two moduli spaces. The map φ was also used in [18]. Similar maps have also been studied in [24][26].

Let $V = \mathcal{O}(l)$ for $l > 0$, and consider the induced bundle $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$. Pulling this back via the projection π , we get a bundle $V_d \rightarrow M_d$ (see Introduction). Let $e_T(V_d)$ be the equivariant Euler class of V_d , and let $\varphi_! e_T(V_d)$ be its push-forward via φ . The following theorem will be a special case of a general theorem proved in the next subsection.

Theorem 2.5. *The sequence $\varphi_! e_T(V_d) \in H_G^*(N_d)$ above is an lp -Euler data.*

We now return to the map φ . The reader who wishes to skip technical details can safely omit the proof.

Lemma 2.6. *There exists a morphism $\varphi : M_d \rightarrow N_d$. Moreover φ is equivariant with respect to the induced action of $S^1 \times T$.*

Proof: The following proof is given by J. Li. Let M_d be the moduli space of stable morphisms $f : C \rightarrow \mathbf{P}^1 \times \mathbf{P}^n$ from arithmetic genus 0 curves to $\mathbf{P}^1 \times \mathbf{P}^n$ of bi-degree $(1, d)$, and let N_d be the space of equivalence classes of $(n+1)$ -tuples (f_0, \dots, f_n) , where f_i are degree d homogeneous polynomials in two variables, and $(f_0, \dots, f_n) \sim (f'_0, \dots, f'_n)$ if there is a constant $c \neq 0$ such that $f_i = c \cdot f'_i$ for all i . We first define the morphism $\varphi : M_d \rightarrow N_d$. For convenience, we let \mathcal{S} be the category of all schemes of finite type (over \mathbf{C}) and let

$$\mathcal{F} : \mathcal{S} \longrightarrow (\text{Set})$$

be the the contra-variant functor that send any $S \in \mathcal{S}$ to the set of families of stable morphisms

$$F : \mathcal{X} \longrightarrow \mathbf{P}^1 \times \mathbf{P}^n \times S$$

over S , where \mathcal{X} are families of connected arithmetic genus 0 curves, modulo the obvious equivalence relation. Note that \mathcal{F} is represented by the moduli stack M_d . Hence to define φ it suffices to define a transformation

$$\Psi : \mathcal{F} \longrightarrow \text{Mor}(-, N_d).$$

We now define such a transformation. Let $S \in \mathcal{S}$ and let $\xi \in \mathcal{F}(S)$ be represented by (\mathcal{X}, F) . We let p_i be the composite of F with the i -th projection of $\mathbf{P}^1 \times \mathbf{P}^n \times S$ and let p_{ij} be the composite of F with the projection from $\mathbf{P}^1 \times \mathbf{P}^n \times S$ to the product of its i -th and j -th components. We consider the sheaf $p_2^* \mathcal{O}_{\mathbf{P}^n}(1)$ on \mathcal{X} and its direct image sheaf

$$\mathcal{L}_\xi = p_{13*} p_2^* \mathcal{O}_{\mathbf{P}^n}(1).$$

We claim that \mathcal{L}_ξ is flat over S . Indeed, by argument in the proof of Theorem 9.9 in [22], it suffices to show that $\pi_{S*}(\mathcal{L}_\xi \otimes \pi_{\mathbf{P}^1}^* \mathcal{O}_{\mathbf{P}^1}(m))$ are locally free sheaves of \mathcal{O}_S -modules for $m \gg 0$, where $\pi_{\mathbf{P}^1}$ and π_S are the first and the second projections of $\mathbf{P}^1 \times S$. Clearly, this sheaf is isomorphic to $p_{3*}(p_2^* \mathcal{O}_{\mathbf{P}^n}(1) \otimes p_1^* \mathcal{O}_{\mathbf{P}^1}(m))$, which is locally free because

$$R^i p_{3*}(p_2^* \mathcal{O}_{\mathbf{P}^n}(1) \otimes p_1^* \mathcal{O}_{\mathbf{P}^1}(m)) = 0$$

for $i > 0$ and $m \gg 0$. For the same reasoning, the sheaves \mathcal{L}_ξ satisfy the following base change property: let $\rho : T \rightarrow S$ be any base change and let $\rho^*(\xi) \in \mathcal{F}(T)$ be the pull back of ξ . Then there is a canonical isomorphism of sheaves of \mathcal{O}_T -modules

$$\mathcal{L}_{\rho^*(\xi)} \cong (\mathbf{1}_{\mathbf{P}^1} \times \rho)^* \mathcal{L}_\xi. \tag{2.2}$$

Since \mathcal{L}_ξ is flat over S , we can define the determinant line bundle of \mathcal{L}_ξ , denoted by $\det(\mathcal{L}_\xi)$ [28]. The sheaf $\det(\mathcal{L}_\xi)$ is an invertible sheaf over $\mathbf{P}^1 \times S$. Using the Riemann-Roch theorem, one computes that its degree along fibers over S are d . Further, because \mathcal{L}_ξ has rank one, there is a canonical homomorphism

$$\mathcal{L}_\xi \longrightarrow \det(\mathcal{L}_\xi), \tag{2.3}$$

so that its kernel is the torsion subsheaf of \mathcal{L}_ξ , denoted by $\text{Tor}(\mathcal{L}_\xi)$. Let w_0, \dots, w_n be the homogeneous coordinate of \mathbf{P}^n chosen before. w_0, \dots, w_n form a basis of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. Then their pull backs provide a collection of canonical sections of \mathcal{L}_ξ , and hence a collection of canonical sections

$$\sigma_{\xi,0}, \dots, \sigma_{\xi,n} \in H^0(S, \pi_{S*} \det(\mathcal{L}_\xi)).$$

based on (2.3). Then after fixing an isomorphism

$$\det(\mathcal{L}_\xi) \cong \pi_S^* \mathcal{M} \otimes \pi_{\mathbf{P}^1}^* \mathcal{O}_{\mathbf{P}^1}(d) \quad (2.4)$$

for some invertible sheaf \mathcal{M} of \mathcal{O}_S -modules, we obtain a section of

$$\pi_{S*}(\pi_{\mathbf{P}^1}^* \mathcal{O}_{\mathbf{P}^1}(d)) \otimes_{\mathcal{O}_S} \mathcal{M} \equiv H_{\mathbf{P}^1}^0(\mathcal{O}_{\mathbf{P}^1}(d)) \otimes_{\mathbb{C}} \mathcal{M}.$$

Finally, we let w_0, w_1 be the homogeneous coordinate of \mathbf{P}^1 . Then the space $H_{\mathbf{P}^1}^0(\mathcal{O}_{\mathbf{P}^1}(d))$ is the space of degree d homogeneous polynomials in variables w_0 and w_1 . This way, we obtain a morphism

$$\Psi(S) : S \longrightarrow N_d$$

that is independent of the isomorphisms (2.4). It follows from the base change property (2.2) that the collection $\Psi(S)$ defines a transformation

$$\Psi : \mathcal{F} \longrightarrow \text{Mor}(-, N_d),$$

thus defines a morphism φ as desired.

It remains to check that for any $w \in S$, the sections

$$\sigma_{\xi,0}(w), \dots, \sigma_{\xi,n}(w) \in H^0(\mathbf{P}^1, \det(\mathcal{L}_\xi) \otimes_{\mathcal{O}_S} k_w)$$

has the described vanishing property. Because of the base change property of \mathcal{L}_ξ , it suffices to check the statement when S is a point and $\xi \in \mathcal{F}_d(S)$ is the stable map $f: C \rightarrow \mathbf{P}^1 \times \mathbf{P}^n$. Let x_1, \dots, x_N be the set of points in \mathbf{P}^1 so that $p_1: C \rightarrow \mathbf{P}^1$, where $p_1 = \pi_{\mathbf{P}^1} \circ f$, is not flat over these points. Let C_i be $p_1^{-1}(x_i)$ and let m_i be the degree of $f([C_i]) \in H_2(\mathbf{P}^n)$. Then $\mathcal{L}_\xi = p_{1*} p_2^* \mathcal{O}_{\mathbf{P}^n}(1)$ is locally free away from x_1, \dots, x_N and has torsion of length m_i at x_i . Then $\mathcal{L}_\xi / \text{Tor}(\mathcal{L}_\xi)$ is locally free of degree $k - \sum m_i$. It is direct to check that the canonical inclusion

$$\mathcal{L}_\xi / \text{Tor}(\mathcal{L}_\xi) \longrightarrow \det(\mathcal{L}_\xi) \cong \mathcal{O}_{\mathbf{P}^1}(d)$$

has cokernel supported on the union of x_1, \dots, x_N whose length at x_i is exactly m_i . The statement about the vanishing of $\sigma_{\xi,0}(w), \dots, \sigma_{\xi,n}(w)$ follows immediately.

The fact that $\varphi : M_d \rightarrow N_d$ is $S^1 \times T$ -equivariant as stated follows immediately from the fact that φ is induced by the transformation Ψ of functors. This completes the proof. \square

2.3. Concave bundles

Definition 2.7. We call a T -equivariant vector bundle $V \rightarrow \mathbf{P}^n$ convex (resp. concave) if the T -equivariant Euler class $e_T(V)$ is invertible and if $H^1(C, f^*V) = 0$ (resp. $H^0(C, f^*V) = 0$) for every 0-pointed genus 0 stable map $f : C \rightarrow \mathbf{P}^n$. We call V a concave bundle if it is a direct sum of a convex and a concave bundles. We denote by V^\pm the convex and concave summands of V . By convention, we consider the zero bundle to be both convex and concave so that concavity includes both convexity and concavity.

The convexity of a bundle is analogous to the notion of convexity of a projective manifold introduced by Behrend-Manin.

For example $\mathcal{O}(l) \rightarrow \mathbf{P}^n$ is convex if $l > 0$, and concave if $l < 0$. Given any concave bundle $V \rightarrow \mathbf{P}^n$, we have a sequence of induced bundles

$$U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$$

whose fiber at (f, C) is the space $H^0(C, f^*V^+) \oplus H^1(C, f^*V^-)$. Pulling back U_d via the contracting map $\pi : M_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$, we get a sequence of bundles

$$V_d := \pi^*U_d \rightarrow M_d.$$

We denote by $e_T(V_d)$ the equivariant Euler class of V_d . We also introduce the notations:

$$\Omega^V := \frac{e_T(V^+)}{e_T(V^-)}$$

$$Q_d := \varphi_! e_T(V_d), \quad Q_0 := \Omega^V.$$

By convention, if V is the zero bundle, we set $e_T(V_d) = 1$, $e_T(V) = 1$, $\Omega^V = 1$.

Theorem 2.8. The sequence $\varphi_! e_T(V_d) \in H_G^*(N_d)$ is an Ω^V -Euler data.

Proof: We first discuss some preliminaries. Let M and N be two compact smooth manifolds with the action of a torus T , and $\varphi : M \rightarrow N$ be an equivariant map. Let F be one component of the fixed submanifold in N and i_F be the inclusion map F in N . Let $\phi_F = i_{F!}(1) \in H_T^*(N)$ denote the equivariant Thom class of the normal bundle of F in N . We then have, for any $\omega \in H_T^*(M)$

$$\int_M \omega \varphi^*(\phi_F) = \int_N \varphi_!(\omega) \phi_F = \int_F i_F^*(\varphi_!(\omega)).$$

On the other hand, let $\{P\}$ be the components of the fixed submanifold contained in $\varphi^{-1}(F)$. By the Atiyah-Bott [7][3] localization formula on M , we get

$$\sum_P \int_P i_P^*(\omega \varphi^*(\phi_F)) / e_T(P/M) = \int_F i_F^*(\varphi_!(\omega)).$$

Here $e_T(P/M)$ denotes the equivariant Euler class of the normal bundle of P in M . The reason is that the contribution of the fixed point sets not contained in $\varphi^{-1}(F)$ is clearly zero. Actually if Q is a component not contained inside $\varphi^{-1}(F)$, its contribution to the localization is given by

$$\int_Q i_Q^*(\omega \varphi^*(\phi_F)) / e_T(Q/M).$$

But by the naturality of the pull-backs, we have

$$i_Q^* \varphi^*(\phi_F) = \varphi_0^* i_E^*(\phi_F) = 0$$

where $E = \varphi(Q)$ is a fixed submanifold in N and φ_0 denotes the restriction of φ to Q . Note that if F is an isolated point, then $i_P^* \varphi^*(\phi_F)$ can be pulled out of the integral.

The above formula will be applied to the collapsing map $\varphi : M_d \rightarrow N_d$. All manifolds involved here are at worst orbifolds with finite quotient singularities, so the localization formula remains valid without any change as long as we consider the corresponding integrals in the orbifold sense.

We consider the S^1 -action on the \mathbf{P}^1 factor in $\mathbf{P}^1 \times \mathbf{P}^n$ with weights $\alpha, 0$. Combining with the natural T -action on \mathbf{P}^n , we get the naturally induced $G = S^1 \times T$ -actions on M_d and N_d , with respect to which the collapsing map φ is equivariant. As described in section 2.1, the G -fixed points in N_d are all of the form

$$p_{ir} = [0, \dots, 0, w_0^r w_1^{d-r}, 0, \dots, 0]$$

in which the only nonzero term is in the i -th position.

For each $r > 0$, let $\{F_r\} \subset \bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$ denote the T -fixed point components in $\bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$ with the marked point mapped onto the fixed point p_i in \mathbf{P}^n . Let $N(F_r) = N_{F_r/\bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)}$ denote the normal bundle of F_r in $\bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$.

Let π_1, π_2 be the projections from $\mathbf{P}^1 \times \mathbf{P}^n$ onto the first and second factors. From the construction of φ , we see that the G -invariant submanifold that is mapped to p_{ir} consists of the following degree $(1, d)$ stable maps $f : C \rightarrow \mathbf{P}^1 \times \mathbf{P}^n$ with $C = C_1 \cup C_0 \cup C_2$. Here $C_0 \simeq \mathbf{P}^1$ and

$$\pi_2 \circ f(C_0) = [0, \dots, 0, 1, 0 \dots, 0] = p_i \in \mathbf{P}^n$$

where 1 is at the i -th position. The map $\pi_1 \circ f : C_0 \rightarrow \mathbf{P}^1$ is an isomorphism and maps $x_1 = C_0 \cap C_1$ and $x_2 = C_0 \cap C_2$ to 0 and ∞ respectively. Actually

$$\pi_1 \circ f(C_1) = 0, \quad \pi_1 \circ f(C_2) = \infty \text{ in } \mathbf{P}^1,$$

ie. the curves C_1 and C_2 are respectively mapped to the points 0 and ∞ of \mathbf{P}^1 .

The maps $\pi_2 \circ f$ restricted to C_j for $j = 1, 2$ are stable maps in $\bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$ and $\bar{\mathcal{M}}_{0,1}(d-r, \mathbf{P}^n)$ respectively. We consider $F_r \times F_{d-r}$ as a G -fixed submanifold of M_d by gluing each pair to C_0 at x_1 and x_2 respectively as above. It is easy to see that $\{F_r \times F_{d-r}\}$ are the G -fixed point sets in M_d whose image under φ is the fixed point p_{ir} .

We first consider a convex bundle V on \mathbf{P}^n , and the case $r \neq 0, d$. Then we have

$$Q_d(\lambda_i + r\alpha) = \int_{N_d} \phi_{p_{ir}} Q_d = \int_{M_d} \varphi^*(\phi_{p_{ir}}) e_T(V_d). \tag{2.5}$$

Here $\phi_{p_{ir}}$ denote the equivariant Thom class of the G -fixed point p_{ir} in N_d . We will apply the localization formula to compute the right hand side of (2.5). First we need to know the normal bundle of the fixed points, which is, in the equivariant K -group of $F_r \times F_{d-r}$ [34][20],

$$\begin{aligned} N_{F_r \times F_{d-r}/M_d} &= N(F_r) + N(F_{d-r}) + [H^0(C_0, (\pi_1 \circ f)^*TP^1)] \\ &\quad + [L_r \otimes T_{x_1}C_0] + [L_{d-r} \otimes T_{x_2}C_0] - [T_{p_i}P^n] - [A_{C_0}]. \end{aligned}$$

Here L_r denotes the line bundle on $\bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$ whose fiber at (f_1, C_1, x_1) is the tangent line at x_1 . Likewise for L_{d-r} . Their contributions correspond to the deformation of the nodal points x_1 and x_2 . The term $H^0(C_0, (\pi_1 \circ f)^*TP^1)$ corresponds to the deformation of f restricted to C_0 . The term $[A_{C_0}]$ is the bundle representing the infinitesimal automorphism of C_0 fixing the two points x_1, x_2 . The term $-[T_{p_i}P^n]$ comes from gluing F_r and F_{d-r} onto C_0 and the property that $\pi_2 \circ f(C_0) = p_i$.

This gives the following formula for the corresponding equivariant Euler classes:

$$\begin{aligned} e_T(F_r \times F_{d-r}/M_d) &= e_T(N(F_r))e_T(N(F_{d-r}))e_T(L_r \otimes T_{x_1}C_0) \\ &\quad \times e_T(L_{d-r} \otimes T_{x_2}C_0)e_T(T_{p_i}\mathbf{P}^n)^{-1} \\ &\quad \times e_T(H^0(C_0, (\pi_1 \circ f)^*TP^1))e_T^{-1}(A_{C_0}). \end{aligned}$$

Each term in this formula can be explicitly calculated. We clearly have $e_T(T_{p_i}\mathbf{P}^n) = \prod_{j \neq i} (\lambda_i - \lambda_j)$; the weights of $T_{x_1}C_0$ and $T_{x_2}C_0$ are α and $-\alpha$ respectively, therefore

$$e_T(L_r \otimes T_{x_1}C_0) = \alpha + c_1(L_r), \quad e_T(L_{d-r} \otimes T_{x_2}C_0) = -\alpha + c_1(L_{d-r})$$

where $c_1(L_r)$, $c_1(L_{d-r})$ are the restriction to F_r and F_{d-r} of the equivariant Chern classes of the line bundles L_r and L_{d-r} with respect to the induced T actions on $\bar{M}_{0,1}(r, \mathbf{P}^n)$ and $\bar{M}_{0,1}(d-r, \mathbf{P}^n)$. To compute $e_T(H^0(C_0, (\pi_1 \circ f)^*TP^1))$ and $e_T(A_{C_0})$, first note that we have the standard exact sequence

$$0 \rightarrow O \rightarrow \mathcal{O}(1) \otimes \mathbf{C}^2 \rightarrow TP^1 \rightarrow 0,$$

with O being the trivial bundle. From this we get

$$0 \rightarrow O \rightarrow H^0(C_0, \mathcal{O}(1)) \otimes \mathbf{C}^2 \rightarrow H^0(C_0, (\pi_1 \circ f)^*(TP^1)) \rightarrow 0.$$

The weights of $H^0(C_0, \mathcal{O}(1))$ are $\alpha, 0$, the weights of \mathbf{C}^2 are $-\alpha, 0$ and the weight of O is 0. Therefore one finds that the weights of $H^0(C_0, (\pi_1 \circ f)^*(TP^1))$ are $\alpha, -\alpha, 0$.

For $[A_{C_0}]$, we have the exact sequence

$$0 \rightarrow A_{C_0} \rightarrow H^0(C_0, (\pi_1 \circ f)^*(TP^1)) \rightarrow T_{x_1}C_0 \oplus T_{x_2}C_0 \rightarrow 0.$$

The weights of $T_{x_1}C_0$ and $T_{x_2}C_0$ are α and $-\alpha$ respectively. So $[A_{C_0}]$ contributes a 0 weight space which cancels with the 0 weight space of $[H^0(C_0, (\pi_1 \circ f)^*(TP^1))]$. Therefore we will ignore the zero weights in our formulas and write as

$$e_T(H^0(C_0, (\pi_1 \circ f)^*(TP^1))e_T^{-1}(A_{C_0}) = -\alpha^2.$$

When V_d is restricted to $F_r \times F_{d-r}$ considered as a fixed point set of M_d as before, we have the exact sequence:

$$0 \rightarrow V_d \rightarrow V_r|_{F_r} \oplus V_{d-r}|_{F_{d-r}} \rightarrow V|_{p_i} \rightarrow 0.$$

Note that $V_r|_{F_r}$ and $V_{d-r}|_{F_{d-r}}$ is the same as $\rho^*U_r|_{F_r}$ and $\rho^*U_{d-r}|_{F_{d-r}}$ which respectively are the restrictions to F_r and F_{d-r} of the pull-backs to $\bar{\mathcal{M}}_{0,1}(r, \mathbf{P}^n)$ and $\bar{\mathcal{M}}_{0,1}(d-r, \mathbf{P}^n)$ of the corresponding bundles on $\bar{\mathcal{M}}_{0,0}(r, \mathbf{P}^n)$ and $\bar{\mathcal{M}}_{0,0}(d-r, \mathbf{P}^n)$. Here $V|_{p_i}$ denotes the fiber of V at $p_i \in \mathbf{P}^n$.

Here comes the important point. The multiplicativity of equivariant Euler classes gives us

$$\Omega^V(\lambda_i) \cdot e_T(V_d) = e_T(V_r) \cdot e_T(V_{d-r}) = \rho^*e_T(U_r) \cdot \rho^*e_T(U_{d-r}),$$

when restricted to $F_r \times F_{d-r} \subset M_d$. Here $\rho : \bar{\mathcal{M}}_{0,1}(d, \mathbf{P}^n) \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ is the forgetting map (same notation for all d). Note that the above equality is just the pull-back via π from $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ of the gluing identity discussed in the Introduction.

For the case of $r = 0$ or d , there is only one of the curves C_1 or C_2 , ie. C is of the form $C_0 \cup C_2$ or $C_1 \cup C_0$. In this case we identify F_d as the fixed point set in M_d by gluing its marked point to C_0 at x_1 or x_2 . The normal bundle in these two cases are respectively given by

$$N_{F_d/M_d} = N(F_d) + [H^0(C_0, (\pi_1 \circ f)^*T\mathbf{P}^1)] + [L_d \otimes T_{x_j}C_0] - [A_{C_0}]$$

in the K -group of F_d . Here L_d is the restriction to F_d of the line bundle on $\bar{\mathcal{M}}_{0,1}(d, \mathbf{P}^n)$ whose fiber is the tangent line at the marked point. For simplicity we write L as L_d in the following. For $j = 1, 2$, $T_{x_j}C_0$ is the tangent line of C_0 at the corresponding marked point x_j . In these two cases, one easily shows in the same way as above that the term $e_T(A_{C_0})$, except the 0 weight, contributes one nonzero weight $-\alpha$ or α respectively. Its 0 weight space still cancels with the 0 weight space of $[H^0(C_0, (\pi_1 \circ f)^*T\mathbf{P}^1)]$.

By putting all of the above computations together and combining

with (2.5), we get, for $r \neq 0, d$,

$$\begin{aligned} \Omega^V(\lambda_i)Q_d(\lambda_i + r\alpha) &= \Omega^V(\lambda_i) \int_{M_d} \varphi^*(\phi_{p_{ir}}) e_T(V_d) \\ &= -\alpha^{-2} \prod_{j \neq i} (\lambda_i - \lambda_j) e_T(p_{ir}/N_d) \\ &\times \sum_{F_r} \int_{F_r} \frac{\rho^* e_T(U_r)}{e_T(N(F_r))(\alpha + c_1(L_r))} \\ &\times \sum_{F_{d-r}} \int_{F_{d-r}} \frac{\rho^* e_T(U_{d-r})}{e_T(N(F_{d-r}))(-\alpha + c_1(L_{d-r}))}. \end{aligned} \tag{2.6}$$

Here $e_T(p_{ir}/N_d) = \iota_{p_{ir}}^*(\phi_{p_{ir}})$. Note that $\varphi^*(\phi_{p_{ir}})$ restricted to $F_r \times F_{d-r}$ is the same as $e_T(p_{ir}/N_d)$ which is a polynomial only in α and λ as given below. Similarly for $r = d$, we have

$$Q_d(\lambda_i + d\alpha) = \alpha^{-1} e_T(p_{id}/N_d) \sum_{F_d} \int_{F_d} \frac{\rho^* e_T(U_d)}{e_T(N(F_d))(\alpha + c_1(L))} \tag{2.7}$$

and for $r = 0$

$$Q_d(\lambda_i) = -\alpha^{-1} e_T(p_{i0}/N_d) \sum_{F_d} \int_{F_d} \frac{\rho^* e_T(U_d)}{e_T(N(F_d))(-\alpha + c_1(L))}. \tag{2.8}$$

We can easily compute $e_T(p_{ir}/N_d)$:

$$e_T(p_{ir}/N_d) = \prod_{j=0}^n \prod_{m=0}^d \prod_{(j,m) \neq (i,r)} (\lambda_i - \lambda_j + (r - m)\alpha).$$

For $r = 0$ and d we have

$$\begin{aligned} e_T(p_{i0}/N_d) &= \prod_{j=0}^n \prod_{m=0}^d \prod_{(j,m) \neq (i,0)} (\lambda_i - \lambda_j - m\alpha) \\ e_T(p_{id}/N_d) &= \prod_{j=0}^n \prod_{m=0}^d \prod_{(j,m) \neq (i,d)} (\lambda_i - \lambda_j + (d - m)\alpha) \\ &= \prod_{j=0}^n \prod_{m=0}^d \prod_{(j,m) \neq (i,0)} (\lambda_i - \lambda_j + m\alpha). \end{aligned}$$

The last two identities together with (2.7), (2.8) clearly gives us

$$Q_d(\lambda_i + d\alpha) = \overline{Q_d(\lambda_i)} \tag{2.9}$$

where $\bar{\alpha} = -\alpha$, $\bar{\lambda}_i = \lambda_i$. Finally our asserted quadratic relation:

$$\Omega^V(\lambda_i)Q_d(\lambda_i + r\alpha) = \overline{Q_r(\lambda_i)}Q_{d-r}(\lambda_i) \tag{2.10}$$

follows from (2.6), (2.7), (2.8), (2.9), and the following elementary identity:

$$\begin{aligned} & \prod_{j=0}^n \prod_{j \neq i} (\lambda_i - \lambda_j) \times \prod_{j=0}^n \prod_{m=0}^d \prod_{(j,m) \neq (i,r)} (\lambda_i - \lambda_j + (r - m)\alpha) \\ &= \prod_{j=0}^n \prod_{m=0}^r \prod_{(j,m) \neq (i,0)} (\lambda_i - \lambda_j + m\alpha) \\ & \quad \times \prod_{j=0}^n \prod_{m=0}^{d-r} \prod_{(j,m) \neq (i,0)} (\lambda_i - \lambda_j - m\alpha) \end{aligned}$$

Note that the last identity is just the interesting identity

$$e_T(T_{p_i}, \mathbf{P}^n) \cdot e_T(p_{i_r}/N_d) = e_T(p_{i_r}/N_r) \cdot e_T(p_{i_0}/N_{d-r}).$$

When V is concave, the fiber at $(f, C) \in \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ of the bundle U_d is $H^1(C, f^*V)$, and we need only one change in the above argument. The gluing exact sequence in the concave case is

$$0 \rightarrow V|_{p_i} \rightarrow V_d \rightarrow V_r|_{F_r} \oplus V_{d-r}|_{F_{d-r}} \rightarrow 0$$

instead. Therefore the gluing identity for equivariant Euler classes becomes

$$e_T(V_d) = e_T(V_r) \cdot e_T(V_{d-r}) \iota_{p_i}^* e_T(V).$$

Since $\Omega^V = 1/e_T(V)$ for concave V , the quadratic relation (2.10) remains valid in this case.

The case when V is a direct sum of a convex and a concave bundle is also similar. \square

2.4. *Linked Euler data*

Definition 2.9. *Two sequences $P, Q \in \mathcal{S}$ are said to be linked if $\iota_{p_i,0}^*(P_d - Q_d) \in \mathcal{R}^{-1}$ vanish at $\alpha = (\lambda_i - \lambda_j)/d$ for all $j \neq i, d > 0$.*

Let V be a T -equivariant concavex bundle on \mathbf{P}^n , and $C \cong \mathbf{P}^1$ be any T -invariant line in \mathbf{P}^n . By Grothendieck’s principle, we have the form

$$V|_C = \oplus_{a=1}^{N^+} \mathcal{O}(l_a) \oplus \oplus_{b=1}^{N^-} \mathcal{O}(-k_b)$$

for some positive integers l_a, k_b . (0 cannot occurs because $e_T(V)$ is invertible, by definition.) Assume that $\{l_a\}$ and $\{k_b\}$ are independent of C . This is the case, for example, if V is uniform [41]. We call the numbers $(l_1, \dots, l_{N^+}; k_1, \dots, k_{N^-})$ the *splitting type* of V . With this notations, we have

Theorem 2.10. *Let $Q_d = \varphi_! e_T(V_d)$ as before. At $\alpha = (\lambda_j - \lambda_i)/d, i \neq j$, we have*

$$\iota_{p_j,0}^*(Q_d) = \prod_a \prod_{m=0}^{l_a d} (l_a \lambda_j - m(\lambda_j - \lambda_i)/d) \times \prod_b \prod_{m=1}^{k_b d - 1} (-k_b \lambda_j + m(\lambda_j - \lambda_i)/d).$$

In particular the Euler data Q is linked to

$$P : P_d = \prod_a \prod_{m=0}^{l_a d} (l_a \kappa - m\alpha) \times \prod_b \prod_{m=1}^{k_b d - 1} (-k_b \kappa + m\alpha).$$

Proof: Since we shall evaluate the class $\iota_{p_j,0}^*(Q_d)$ at $\alpha = (\lambda_j - \lambda_i)/d$, we introduce the notation $Q_d(\kappa, \alpha) = Q_d$, and denote the value of the class above by $Q_d(\lambda_j, (\lambda_j - \lambda_i)/d)$. First consider the case $V = \mathcal{O}(l)$ on \mathbf{P}^n . We consider a smooth point in $(f, C) \in M_d$ with $C = \mathbf{P}^1$ and in coordinates

$$f : C \rightarrow \mathbf{P}^1 \times \mathbf{P}^n, \quad [w_0, w_1] \mapsto [w_1, w_0] \times [0, \dots, w_0^d, \dots, w_1^d, \dots, 0].$$

where in the last term, w_0^d is in the i -th position, w_1^d is in the j -th position, and all of the other components are 0. The image of (f, C) in N_d under φ is the smooth point

$$P_{ij} = [0, \dots, w_0^d, \dots, w_1^d, \dots, 0].$$

It is easy to see that, if the weight of S^1 in the group $G = S^1 \times T$ is $\alpha = (\lambda_j - \lambda_i)/d$, then P_{ij} is fixed by the action of the subgroup of G with $\alpha = (\lambda_j - \lambda_i)/d$. So (f, C) is a smooth point in M_d fixed by the subgroup in $G = S^1 \times T$ with $\alpha = (\lambda_j - \lambda_i)/d$. The class $Q_d(\kappa, \alpha)$ restricted to P_{ij} is just $Q_d(\lambda_j, (\lambda_j - \lambda_i)/d)$.

At the points $(f, C) \in M_d$ and $P_{ij} \in N_d$, the map φ is a canonical identification. From definition, $Q_d(\kappa, \alpha)$ restricted to P_{ij} is the same as $e_T(V_d)$ restricted to $\varphi^{-1}(P_{ij}) = (f, C) \in M_d$, which by definition, is the same as $e_T(U_d)$ restricted to the T -fixed point $(\pi_2 \circ f, C)$ in $\bar{M}_{0,0}(d, \mathbf{P}^n)$. Note that $(\pi_2 \circ f, C)$ is the degree d cover of the T -invariant line joining p_i and p_j in \mathbf{P}^n . Explicitly

$$\pi_2 \circ f : [w_0, w_1] \rightarrow [0, \dots, w_0^d, \dots, w_1^d, \dots, 0].$$

So the induced action of T on $C = \mathbf{P}^1$ has weights $\{\lambda_i/d, \lambda_j/d\}$.

Now let us compute $e_T(U_d)$ restricted to $(\pi_2 \circ f, C)$. When restricted $(\pi_2 \circ f, C) \in \bar{M}_{0,0}(d, \mathbf{P}^n)$ the fiber of U_d is just the section space $H^0(C, (\pi_2 \circ f)^* \mathcal{O}(l)) = H^0(C, \mathcal{O}(ld))$. It has an explicit basis $\{w_0^k w_1^{ld-k}\}$ with $k = 0, \dots, ld$. Since the T -weight of $w_0^k w_1^{ld-k}$ is $k\lambda_i/d + (ld - k)\lambda_j/d$, by multiplying them together we get

$$Q_d(\lambda_j, (\lambda_j - \lambda_i)/d) = \prod_{m=0}^{ld} (l\lambda_j - m(\lambda_j - \lambda_i)/d).$$

For a general convex vector bundle V on \mathbf{P}^n , the fiber of the induced bundle V_d restricted to $\varphi^{-1}(P_{ij}) = (f, C) \in M_d$ is $H^0(\mathbf{P}^1, (\pi_2 \circ f)^* V)$. By Grothendieck's principle, V restricted to the line spanning p_i, p_j splits into direct sum of line bundles $\{\mathcal{O}(l_a)\}$. Pulling them back to C via the degree d map $\pi_2 \circ f$, we get the direct sum of $\{\mathcal{O}(l_a d)\}$. Since V is convex, each $l_a > 0$. By applying the same computation to each summand, we get

$$Q_d(\lambda_j, (\lambda_j - \lambda_i)/d) = \prod_a \prod_{m=0}^{l_a d} (l_a \lambda_j - m(\lambda_j - \lambda_i)/d).$$

For a concave bundle V , we need only one minor change in the above argument. We leave to the reader as an exercise to check that for $V = \mathcal{O}(-k)$, $k > 0$,

$$Q_d(\lambda_j, (\lambda_j - \lambda_i)/d) = \prod_{m=1}^{kd-1} (-k\lambda_j + m(\lambda_j - \lambda_i)/d),$$

by using either the Atiyah-Bott fixed point formula or by writing down an explicit basis for $H^1(\mathbf{P}^1, \mathcal{O}(-kd))$. So for an arbitrary concave bundle V , we have the form

$$Q_d(\lambda_j, (\lambda_j - \lambda_i)/d) = \prod_b \prod_{m=1}^{k_b d - 1} (-k_b \lambda_j + m(\lambda_j - \lambda_i)/d).$$

Similarly for an arbitrary concave bundle V , we have the form

$$\begin{aligned} Q_d(\lambda_j, (\lambda_j - \lambda_i)/d) &= \prod_a \prod_{m=0}^{l_a d} (l_a \lambda_j - m(\lambda_j - \lambda_i)/d) \\ &\quad \times \prod_b \prod_{m=1}^{k_b d - 1} (-k_b \lambda_j + m(\lambda_j - \lambda_i)/d). \quad \square \end{aligned}$$

Theorem 2.11. *Suppose P, Q are any linked Ω -Euler data. If*

$$\text{deg}_\alpha \iota_{p_i, 0}^*(P_d - Q_d) \leq (n + 1)d - 2$$

for all $i = 0, \dots, n$ and $d = 1, 2, \dots$, then $P = Q$.

Proof: By definition, $P_0 = Q_0 = \Omega$. We will show that $P_d = Q_d$, assuming that

$$P_r = Q_r, \quad r = 0, \dots, d - 1. \tag{2.11}$$

Since the \mathcal{R} -valued pairing $pf_d(u \cdot v)$ on $\mathcal{R}H_G^*(N_d)$ is nondegenerate, it suffices to show that

$$L_s := pf_d(\kappa^s \cdot (P_d - Q_d))$$

is zero for all $s = 0, 1, 2, \dots$. By the localization formula for pf_d , we get

$$L_s = \sum_{i=0}^n \sum_{r=0}^d (\lambda_i + r\alpha)^s \frac{\iota_{p_{i,r}}^*(P_d - Q_d)}{\prod_{k=0}^n \prod_{m=0, (k,m) \neq (i,r)}^d (\lambda_i - \lambda_k + (r - m)\alpha)}.$$

Since P is an Euler data, it follows that $\iota_{p_{i,r}}^*(P_d)$, for each $r = 1, \dots, d - 1$, is expressible in terms of P_1, \dots, P_{d-1} . Likewise for Q . Thus

by the inductive hypothesis (2.11), the sum over r above receives contributions only from the $r = 0, d$ terms. Applying the Reciprocity Lemma (i), we further simplify L_s to

$$L_s = \sum_{i=0}^n \left(\frac{\lambda_i^s A_i(\alpha)}{\alpha^d} + \frac{(\lambda_i + d\alpha)^s A_i(-\alpha)}{(-\alpha)^d} \right) \tag{2.12}$$

$$A_i(\alpha) := \frac{(-1)^d \iota_{p_i,0}^*(P_d - Q_d)}{d! \prod_{k \neq i} (\lambda_i - \lambda_k) \prod_{k \neq i} \prod_{m=1}^d (\lambda_i - \lambda_k - m\alpha)}.$$

Since P, Q are linked Euler data, we have

$$\iota_{p_i,0}^*(P_d - Q_d) = 0 \tag{2.13}$$

at $\alpha = (\lambda_i - \lambda_k)/d, k \neq i$. By the inductive hypothesis (2.11) and the Reciprocity Lemma (iii), (2.13) holds at $\alpha = (\lambda_i - \lambda_k)/m$ for $m = 1, \dots, d$ as well. This shows that $A_i \in \mathcal{R} = \mathbf{Q}(\lambda)[\alpha]$ for all i . By assumption $\deg_\alpha A_i < (n + 1)d - 1 - nd = d - 1$. But since $L_s \in \mathcal{R}$ ie. polynomial in α , for all s , it follows easily that the A_i must be identically zero. \square

2.5. The Lagrange map and mirror transformations

Throughout this subsection, we fix an invertible class Ω and shall denote by $\mathcal{A} = \mathcal{A}^\Omega$ the set of Ω -Euler data.

Definition 2.12. *An invertible map $\mu : \mathcal{A} \rightarrow \mathcal{A}$ is called a mirror transformation if for any $P \in \mathcal{A}$, $\mu(P)$ is linked to P . We call $\mu(P)$ a mirror transform of P .*

Definition 2.13. *(Notations) \mathcal{S}_0 denotes the set of sequences $B : B_d \in \mathcal{R}^{-1}H_G^*(N_0), d = 1, 2, \dots$. We define the map $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{S}_0, P \mapsto \mathcal{I}(P) = B$ where $B_d = I_d^*(P_d)$.*

Recall that any equivariant cohomology class $\omega \in \mathcal{R}^{-1}H_G^*(N_d)$ is determined by its restrictions $\iota_{p_{i,r}}^*(\omega) \in \mathcal{R}^{-1}, i = 0, \dots, n, r = 0, \dots, d$. Conversely given any collection $\omega_{i,r} \in \mathcal{R}^{-1}$, there exists a unique class $\omega \in \mathcal{R}^{-1}H_G^*(N_d)$ such that $\iota_{p_{i,r}}^*(\omega) = \omega_{i,r}$ for all i, r . In fact,

$$\omega = \sum_{i=0}^n \sum_{r=0}^d \omega_{i,r} \prod_{(j,m) \neq (i,r)} \frac{\kappa - \lambda_j - m\alpha}{\lambda_i - \lambda_j - (m - r)\alpha}.$$

In particular given a sequence $B \in \mathcal{S}_0$, then for each d there is a unique class $P_d \in \mathcal{R}^{-1}H_G^*(N_d)$ such that

$$\iota_{p_i}^*(P_d) = \iota_{p_i}^*(\Omega)^{-1} \overline{\iota_{p_i}^*(B_r)} \iota_{p_i}^*(B_{d-r}), \quad i = 0, \dots, n, \quad r = 0, \dots, d, \tag{2.14}$$

where we have set $B_0 := \Omega$. This defines a sequence $P \in \mathcal{S}$, hence a map

$$\mathcal{L}_\Omega : \mathcal{S}_0 \rightarrow \mathcal{S}, \quad B \mapsto \mathcal{L}_\Omega(B) = P.$$

We shall call $\mathcal{L} = \mathcal{L}_\Omega$ the Lagrange map. By (2.14) at $r = 0$, we get

$$\iota_{p_i}^*(B_d) = \iota_{p_i,0}^*(P_d) = \iota_{p_i}^* I_d^*(P_d), \quad i = 0, \dots, n. \tag{2.15}$$

First, this implies that the two classes $B_d, I_d^*(P_d)$ on $N_0 = \mathbf{P}^n$ coincide for each d . Thus

$$B = \mathcal{I}(P) = \mathcal{I} \circ \mathcal{L}(B). \tag{2.16}$$

Thus $\mathcal{L} : \mathcal{S}_0 \rightarrow \mathcal{S}$ is a section of the onto map $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{S}_0$. Second, substituting (2.15) into (2.14), we get

$$\iota_{p_i}^*(\Omega) \iota_{p_i}^*(P_d) = \overline{\iota_{p_i,0}^*(P_r)} \iota_{p_i,0}^*(P_{d-r}). \tag{2.17}$$

If, furthermore, we have $P_d \in \mathcal{R}H_G^*(N_d)$ rather than in $\mathcal{R}^{-1}H_G^*(N_d)$, then eqn. (2.17) says that P is an Euler data.

(A) *The image $P = \mathcal{L}(B)$ of a given $B \in \mathcal{S}_0$ under the Lagrange map is an Euler data if $P_d \in \mathcal{R}H_G^*(N_d)$, $d > 0$.*

On the other hand, it is trivial to show that if $Q \in \mathcal{A} \subset \mathcal{S}$ then

$$Q = \mathcal{L} \circ \mathcal{I}(Q). \tag{2.18}$$

Now using \mathcal{L} we can lift any map $\mu_0 : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ to a map

$$\mu = \mathcal{L} \circ \mu_0 \circ \mathcal{I} : \mathcal{S} \rightarrow \mathcal{S},$$

which we shall call the Lagrange lift of μ_0 . Thus from eqns. (2.16) and (2.18), we have

(B) *Let $\mu_0 : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ be invertible with inverse ν_0 , and let μ, ν be their respective Lagrange lifts. Then $\mu \circ \nu = \nu \circ \mu = id_{\mathcal{A}}$ when restricted to Euler data.*

We now discuss the relationship between Euler data and series of hypergeometric type.

Definition 2.14. Given any $B \in \mathcal{S}_0$, define

$$HG[B](t) := e^{-pt/\alpha} \left(\Omega + \sum_{d>0} \frac{B_d e^{dt}}{\prod_{k=0}^n \prod_{m=1}^d (p - \lambda_k - m\alpha)} \right)$$

where $p \in H_G^*(N_0)$ is the equivariant hyperplane class of $N_0 = \mathbf{P}^n$.

Note that $HG[B](t)$ is a cohomology valued formal series. If $P : P_d = \prod_{m=0}^{(n+1)d} (l\kappa - m\alpha)$ as in Example 1, it is obvious that in the limit $\lambda \rightarrow 0$, we have

$$HG[\mathcal{I}(P)](t) = e^{-Ht/\alpha} \sum_{d \geq 0} \frac{\prod_{m=0}^{(n+1)d} ((n+1)H - m\alpha)}{\prod_{m=1}^d (H - m\alpha)^{n+1}} e^{dt}$$

where $H \in H^*(\mathbf{P}^n)$ on the right hand side is the hyperplane class of \mathbf{P}^n . The coefficients of $(-\frac{H}{\alpha})^i$ for $i = 1, \dots, n$, are exactly solutions to a hypergeometric differential equation discussed in the Introduction.

We now consider a construction of mirror transformations. Let $B \in \mathcal{S}_0$, and set $B_0 := \Omega$. Given any power series $g \in e^t \mathcal{R}[[e^t]]$, there is a unique $\tilde{B} \in \mathcal{S}_0$ such that

$$HG[B](t + g) = HG[\tilde{B}](t).$$

In fact, since

$$HG[B](t + g) = e^{-pt/\alpha} e^{-pg/\alpha} \sum_{d \geq 0} \frac{B_d e^{dt} e^{dg}}{\prod_{k=0}^n \prod_{m=1}^d (p - \lambda_k - m\alpha)},$$

if we write $e^{dg} = \sum_{s \geq 0} g_{d,s} e^{st}$, $g_{d,s} \in \mathcal{R}$ and $e^{-pg/\alpha} = \sum_{s \geq 0} g'_s e^{st}$, $g'_s \in \mathcal{R}[p/\alpha]$, then it is straightforward to find that

$$\tilde{B}_d = B'_d + \sum_{r=0}^{d-1} g'_{d-r} B'_r \prod_{j=0}^n \prod_{m=r+1}^d (p - \lambda_j - m\alpha) \tag{2.19}$$

$$B'_d := B_d + \sum_{r=0}^{d-1} g_{r,d-r} B_r \prod_{j=0}^n \prod_{m=r+1}^d (p - \lambda_j - m\alpha).$$

Thus we have an invertible transformation $\mu_0 : \mathcal{S}_0 \rightarrow \mathcal{S}_0$, $B \mapsto \tilde{B}$. Similarly, given any power series $f \in e^t \mathcal{R}[[e^t]]$ we have an invertible transformation $\mu_0 : \mathcal{S}_0 \rightarrow \mathcal{S}_0$, $B \mapsto \tilde{B}$, such that

$$e^{f/\alpha} HG[B](t) = HG[\tilde{B}](t).$$

Again if we write $e^{f/\alpha} = \sum_{s \geq 0} f_s e^{st}$, $f_s \in \mathcal{R}[\alpha^{-1}]$, then

$$\tilde{B}_d = B_d + \sum_{r=0}^{d-1} f_{d-r} B_r \prod_{j=0}^n \prod_{m=r+1}^d (p - \lambda_j - m\alpha). \tag{2.20}$$

We now make an important observation about the transformation μ_0 in each case above. For $d > 0$ the class $\prod_{j=0}^n (p - \lambda_j - d\alpha)$ always vanishes when restricted to the fixed points $p_i \in \mathbf{P}^n$, at $\alpha = (\lambda_i - \lambda_j)/d$. It follows immediately from (2.19) and (2.20) that $\iota_{p_i}^*(B_d)$, $\iota_{p_i}^*(\tilde{B}_d)$ always agree (whenever defined for all d) at $\alpha = (\lambda_i - \lambda_j)/d$, for $j \neq i$. To summarize:

(C) Given $g, f \in e^t \mathcal{R}[[e^t]]$, let $\mu_0 : \mathcal{S}_0 \rightarrow \mathcal{S}_0$, $B \mapsto \tilde{B}$, be the invertible transformation defined by

$$e^{f/\alpha} HG[B](t + g) = HG[\tilde{B}](t).$$

Suppose B is such that all values of $\iota_{p_i}^*(B_d)$ are well-defined at $\alpha = (\lambda_i - \lambda_j)/d$, $j \neq i$. Then these values are preserved under μ_0 .

Obviously if P is an Euler data and $B = \mathcal{I}(P)$, then the restrictions $\iota_{p_i}^*(B_d) = \iota_{p_{i,0}}^*(P_d) \in \mathcal{R}$ are polynomial in α . Hence they are always well-defined at $\alpha = (\lambda_i - \lambda_j)/d$, $j \neq i$.

Lemma 2.15. Let μ be the Lagrange lift of the above transformation $\mu_0 : \mathcal{S}_0 \rightarrow \mathcal{S}_0$, $B \mapsto \tilde{B}$. Then μ is a mirror transformation. In particular, if P is an Euler data, then $\tilde{P} = \mu(P)$ is an Euler data with

$$e^{f/\alpha} HG[\mathcal{I}(P)](t + g) = HG[\mathcal{I}(\tilde{P})](t).$$

Proof: The second assertion follows from the first assertion and the fact that $\mathcal{I} \circ \mu = \mathcal{I} \circ \mathcal{L} \circ \mu_0 \circ \mathcal{I} = \mu_0 \circ \mathcal{I}$.

It suffices to consider the two cases $f = 0$, $g = 0$, separately. In each case we let P be an Euler data, and denote

$$\tilde{P} = \mu(P), \quad B = \mathcal{I}(P), \quad \tilde{B} = \mu_0(B).$$

Since $\mu := \mathcal{L} \circ \mu_0 \circ \mathcal{I}$, we have $\tilde{P} = \mathcal{L}(\tilde{B})$. In each case we will show that \tilde{P} is an Euler data. We claim that this suffices. First, by statement

(B) above, μ is invertible as a transformation on the set \mathcal{A} of Euler data. Second, by statement (C) above, the restrictions $\iota_{p_i}^*(B_d) = \iota_{p_{i,0}}^*(P_d)$ and $\iota_{p_i}^*(\tilde{B}_d) = \iota_{p_{i,0}}^*(\tilde{P}_d)$ agree at $\alpha = (\lambda_i - \lambda_j)/d, j \neq i$. Thus \tilde{P} is linked to P . So, by definition, μ is a mirror transformation.

We now proceed to checking Eulerity of \tilde{P} . Since $P = \mathcal{L}(B)$, (2.14) holds. Multiply both sides of eqn. (2.14) by the respective sides of the following identity:

$$\begin{aligned} & e^{d\tau} \frac{e^{(\lambda_i+r\alpha)(t-\tau)/\alpha}}{\prod_{j=0}^n \prod_{m=0}^d \prod_{(j,m) \neq (i,r)} (\lambda_i + r\alpha - \lambda_j - m\alpha)} \\ &= \frac{1}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \times e^{\lambda_i t/\alpha} \frac{e^{r\tau}}{\prod_{j=0}^n \prod_{m=1}^r (\lambda_i - \lambda_j + m\alpha)} \\ & \times e^{-\lambda_i \tau/\alpha} \frac{e^{(d-r)\tau}}{\prod_{j=0}^n \prod_{m=1}^{d-r} (\lambda_i - \lambda_j - m\alpha)}, \end{aligned}$$

and then sum over $i = 0, \dots, n$, and $r = 0, \dots, d$. The result is

$$\begin{aligned} & e^{d\tau} pf_d(P_d e^{\kappa(t-\tau)/\alpha}) \\ &= \sum_{r=0}^d pf \left(\Omega^{-1} \left[e^{-pt/\alpha} \frac{B_r e^{r\tau}}{\prod_{j=0}^n \prod_{m=1}^r (p - \lambda_j - m\alpha)} \right] \right) \\ & \times \left[e^{-p\tau/\alpha} \frac{B_{d-r} e^{(d-r)\tau}}{\prod_{j=0}^n \prod_{m=1}^{d-r} (p - \lambda_j - m\alpha)} \right]. \end{aligned}$$

Now summing this over $d = 0, 1, 2, \dots$, we get:

$$\sum_{d \geq 0} e^{d\tau} pf_d(P_d e^{\kappa(t-\tau)/\alpha}) = pf \left(\Omega^{-1} \overline{HG[B]}(t) HG[B](\tau) \right). \tag{2.21}$$

Likewise, of course, for \tilde{P} and \tilde{B} .

First case:

$$HG[\tilde{B}](t) = HG[B](t + g(e^t)). \tag{2.22}$$

By (2.21), we have

$$\begin{aligned} & pf \left(\Omega^{-1} \overline{HG[\tilde{B}]}(t) HG[\tilde{B}](\tau) \right) = \sum_{d \geq 0} e^{d\tau} pf_d(\tilde{P}_d e^{\kappa(t-\tau)/\alpha}) \\ & pf \left(\Omega^{-1} \overline{HG[B]}(t + g(e^t)) HG[B](\tau + g(e^\tau)) \right) \\ &= \sum_{d \geq 0} e^{d(\tau + g(e^\tau))} pf_d \left(P_d e^{\kappa(t + \bar{g}(e^t) - \tau - g(e^\tau))/\alpha} \right). \end{aligned}$$

By (2.22), we can equate the two right hand sides above. Setting $q = e^\tau$, $\zeta = (t - \tau)/\alpha$, we get

$$\sum_{d \geq 0} q^d pf_d(\tilde{P}_d e^{\zeta\kappa}) = \sum_{d \geq 0} q^d e^{dg(q)} pf_d \tag{2.23}$$

$$\times \left(P_d e^{\kappa\zeta} e^{\kappa(g_+(qe^{\zeta\alpha}) - g_+(q))/\alpha} e^{-\kappa(g_-(qe^{\zeta\alpha}) + g_-(q))/\alpha} \right)$$

where $g = g_+ + g_-$ with $\bar{g}_\pm = \pm g_\pm$. Obviously for any $g(q) \in \mathcal{R}[[q]]$, $g_+(qe^{\zeta\alpha}) - g_+(q) \in \alpha \cdot \mathcal{R}[[q, \zeta]]$. Since the involution $\omega \mapsto \bar{\omega}$ on \mathcal{R} simply changes the sign of α , the fact that g_- is odd shows that $g_-(q) \in \alpha \cdot \mathcal{R}[[q]]$. Likewise for $g_-(qe^{\zeta\alpha})$. We know that $P_d \in \mathcal{RH}_G^*(N_d)$ (since P is Euler), and that pf_d maps $\mathcal{RH}_G^*(N_d)$ to \mathcal{R} . So the right hand side of (2.23) now clearly lies in $\mathcal{R}[[q, \zeta]]$. So, likewise for the left hand side of (2.23). It follows that

$$pf_d(\tilde{P}_d \kappa^s) \in \mathcal{R}, \quad s = 0, 1, 2, \dots \tag{2.24}$$

A priori $\tilde{P}_d \in \mathcal{R}^{-1}H_G^*(N_d)$ has the form

$$\tilde{P}_d = a_N \kappa^N + \dots + a_0, \quad a_i \in \mathcal{R}^{-1}, \quad N = (n + 1)d + n.$$

Since $pf_d(\kappa^N) = 1$, it follows from (2.24) that $a_N, \dots, a_0 \in \mathcal{R}$. Hence $\tilde{P}_d \in \mathcal{RH}_G^*(N_d)$ (rather than in $\mathcal{R}^{-1}H_G^*(N_d)$). By statement (A) above, \tilde{P} is an Euler data.

Second case:

$$HG[\tilde{B}](t) = e^{f/\alpha} HG[B](t).$$

Again applying (2.21) and writing $f \in e^t\mathcal{R}[[e^t]]$ as $f = f_+ + f_-$ with $\bar{f}_\pm = \pm f_\pm$, we get

$$\sum_{d \geq 0} q^d pf_d(\tilde{P}_d e^{\zeta\kappa}) = e^{-\bar{f}(e^t)/\alpha} e^{f(e^\tau)/\alpha} pf \left(\Omega^{-1} \overline{HG[B]}(t) HG[B](\tau) \right)$$

$$= e^{-(f_+(qe^{\zeta\alpha}) - f_+(q))/\alpha} e^{(f_-(qe^{\zeta\alpha}) + f_-(q))/\alpha}$$

$$\times \sum_{d \geq 0} q^d pf_d(P_d e^{\zeta\kappa}).$$

The right hand side lies in $\mathcal{R}[[q, \zeta]]$ as before, implying that \tilde{P} is Euler. \square

3. Applications

Definition 3.1. *A concave bundle V on \mathbf{P}^n is called a critical bundle if the induced bundle $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ has rank $\dim \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n) = (n+1)d+n-3$. We denote the nonequivariant Euler class by $c_{top}(U_d)$.*

Recall the notation that given a concave bundle $V = V^+ \oplus V^-$, we have

$$\Omega^V := e_T(V^+)/e_T(V^-)$$

$$Q : Q_d = \varphi!e_T(V_d), \quad d > 0.$$

By Theorem 2.8 the sequence Q is an Ω^V -Euler data. If V is a critical bundle, introduce

$$K_d := \int_{\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)} c_{top}(U_d), \quad \Phi := \sum_{d>0} K_d e^{dt}.$$

Theorem 3.2. *Let V be a concave bundle on \mathbf{P}^n .*

- (i) *The restrictions $I_d^*(Q_d) \in H_G^*(\mathbf{P}^n)$ has $\deg_\alpha I_d^*(Q_d) \leq (n+1)d-2$.*
- (ii) *If V is critical, then in the nonequivariant limit $\lambda \rightarrow 0$,*

$$\int_{\mathbf{P}^n} e^{-Ht/\alpha} \frac{I_d^*(Q_d)}{\prod_{m=1}^d (H - m\alpha)^{n+1}} = \alpha^{-3}(2 - dt)K_d$$

$$\int_{\mathbf{P}^n} (HG[\mathcal{I}(Q)](t) - e^{-Ht/\alpha}\Omega^V) = \alpha^{-3}(2\Phi - t\Phi').$$

Proof: The second equality in (ii) follows trivially from the first equality.

By eqn. (2.8) in the proof of Theorem 2.8, we have

$$Q_d(\lambda_i) = \phi_{p_i}(\lambda_i)\phi_{\mathbf{P}^n}(\lambda_i) \sum_{F_d} \int_{F_d} \frac{\rho^*e_T(U_d)}{e_T(N(F_d))[\alpha(\alpha - c_1(L))]} \quad (3.1)$$

where $\phi_{p_i} := \prod_{j \neq i} (p - \lambda_j)$, $\phi_{\mathbf{P}^n} := \prod_{j=0}^n \prod_{m=1}^d (p - \lambda_j - m\alpha) \in H_T^*(\mathbf{P}^n)$. From the localization formula, we deduce that

$$\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))} ev^* \phi_{p_i} = \int_{\mathbf{P}^n} ev_! \left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))} \right) \phi_{p_i} = \iota_{p_i}^* ev_! \left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))} \right).$$
 Thus (3.1) can be written as

$$\iota_{p_i}^* [\phi_{\mathbf{P}^n} ev_! \left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))} \right)], \quad i = 0, \dots, n.$$
 It follows that $I_d^*(Q_d) = \phi_{\mathbf{P}^n} ev_! \left(\frac{\rho^* e_T(U_d)}{\alpha(\alpha - c_1(L))} \right)$. (3.2) This shows that $\deg_\alpha I_d^*(Q_d) \leq \deg_\alpha \phi_{\mathbf{P}^n} - 2 = (n + 1)d - 2$, proving (i).

Since $Q_d = \varphi_! e_T(V_d) \in H_G^*(N_d)$, their nonequivariant limit $\lambda \rightarrow 0$ exist. In this limit (3.2) gives

$$\begin{aligned}
 A &:= \int_{\mathbf{P}^n} e^{-Ht/\alpha} \frac{I_d^*(Q_d)}{\prod_{m=1}^d (H - m\alpha)^{n+1}} \\
 &= \int_{\bar{\mathcal{M}}_{0,1}(d, \mathbf{P}^n)} e^{-ev^* Ht/\alpha} \frac{\rho^* c_{top}(U_d)}{\alpha(\alpha - c_1(L))} \\
 &= \int_{\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)} c_{top}(U_d) \rho_! \left(\frac{e^{-ev^* Ht/\alpha}}{\alpha(\alpha - c_1(L))} \right).
 \end{aligned}$$

Now $c_{top}(U_d)$ has degree the same as the dimension of $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$. The second factor in the last integrand contributes a scalar factor given by integration over a generic fiber E (which is a \mathbf{P}^1) of ρ . So we pick out the degree 1 term in $\frac{e^{-ev^* Ht/\alpha}}{\alpha(\alpha - c_1(L))}$, which is just $\frac{-ev^* Ht}{\alpha^3} + \frac{c_1(L)}{\alpha^3}$. Restricting to the generic fiber E , say over $(f, C) \in \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$, the evaluation map ev is equal to f , which is a degree d map $E \cong \mathbf{P}^1 \rightarrow \mathbf{P}^n$. It follows that

$$\int_E ev^* H = d.$$

Moreover, since $c_1(L)$ restricted to E is just the first Chern class of the tangent bundle to E , it follows that

$$\int_E c_1(L) = 2.$$

So we have

$$A = \left(-\frac{d}{\alpha^3} t + \frac{2}{\alpha^3} \right) K_d. \quad \square$$

It is easy to work out the complete list of *critical* concavex vector bundles V on \mathbf{P}^n which are direct sums of line bundles. Such a V is of the form

$$\begin{aligned}
 V &= V^+ \oplus V^- \\
 V^+ &= \bigoplus_{a=1}^{N^+} \mathcal{O}(l_a) \\
 V^- &= \bigoplus_{b=1}^{N^-} \mathcal{O}(-k_b)
 \end{aligned}$$

where l_1, \dots, l_{N^+} , k_1, \dots, k_{N^-} are positive integers. By Riemann-Roch, the bundles U_d that V induces on $\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^n)$ has $\text{rank } U_d = d(\sum l_a + \sum k_b) + N^+ - N^-$, which must be $(n+1)d + n - 3$ for all d if V is critical. Thus we must have

$$\begin{aligned} \sum l_a + \sum k_b &= n + 1 \\ N^+ - N^- &= n - 3. \end{aligned} \tag{3.3}$$

The complete list of critical bundles that are also direct sums of line bundles is:

$$\begin{aligned} \mathbf{P}^1 &: \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \mathbf{P}^2 &: \mathcal{O}(-3) \\ \mathbf{P}^3 &: \mathcal{O}(2) \oplus \mathcal{O}(-2), \mathcal{O}(3) \oplus \mathcal{O}(-1) \\ \mathbf{P}^4 &: \mathcal{O}(5), \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(-1) \\ \mathbf{P}^5 &: \mathcal{O}(2) \oplus \mathcal{O}(4), \mathcal{O}(3) \oplus \mathcal{O}(3) \\ \mathbf{P}^6 &: \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3) \\ \mathbf{P}^7 &: \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2). \end{aligned} \tag{3.4}$$

Note that we have excluded the critical bundles in which the hyperplane bundle $\mathcal{O}(1)$ occurs because in the nonequivariant limit it only reduces a given case of \mathbf{P}^n to \mathbf{P}^{n-1} . For example, even though the bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on \mathbf{P}^2 is certainly critical, computing the K_d for the induced bundles is equivalent to doing the same with $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on \mathbf{P}^1 . It is curious to note that the numerical conditions (3.3) is rather similar to the condition for having a projective complete intersection Calabi-Yau threefold. In fact five of the examples on \mathbf{P}^4 through \mathbf{P}^7 above involving only positive bundles are exactly the cases in which each critical bundle cuts out a complete intersection Calabi-Yau threefold. We also note that, with the exception of the \mathbf{P}^1 case, the three examples which involve negative bundles in fact correspond to *noncompact Calabi-Yau threefolds*. The total space of $\mathcal{O}(-3) \rightarrow \mathbf{P}^2$, the total space of $\psi^*\mathcal{O}(-2) \rightarrow X$, where $\psi : X \hookrightarrow \mathbf{P}^3$ is a quadric, and the total space of $\psi^*\mathcal{O}(-1) \rightarrow \mathbf{P}^4$ where $\psi : X \hookrightarrow \mathbf{P}^4$ is the intersection of two quadrics, all three are noncompact Calabi-Yau. These examples arise in the so-called *local mirror symmetry*. In the next subsection, we shall compute the Euler classes of the induced bundles for the list above.

3.1. The first convex example: The Mirror Conjecture

Throughout this subsection, we set $l = n + 1$, consider the convex bundle $V = \mathcal{O}(l)$ on \mathbf{P}^n , and fix $\Omega^V = lp$. P, Q shall denote the following two linked Euler data (cf. Theorems 2.8, 2.10.):

$$P : P_d = \prod_{m=0}^{ld} (l\kappa - m\alpha)$$

$$Q : Q_d = \varphi_{!}e_T(V_d).$$

Consider the hypergeometric differential equation

$$\left(\left(\frac{d}{dt} \right)^n - le^t \left(l \frac{d}{dt} + 1 \right) \cdots \left(l \frac{d}{dt} + n \right) \right) h(t) = 0.$$

We have seen that a basis $f_i, i = 0, \dots, n - 1$, of solutions can be read off from the hypergeometric series (cf. Introduction) in the limit $\lambda \rightarrow 0$:

$$\begin{aligned} HG[\mathcal{I}(P)](t) &= e^{-Ht/\alpha} \sum_{d \geq 0} \frac{\prod_{m=0}^{ld} (lH - m\alpha)}{\prod_{m=1}^d (H - m\alpha)^{n+1}} e^{dt} \\ &= lH \left(f_0 - f_1 \frac{H}{\alpha} + f_2 \frac{H^2}{\alpha^2} - \cdots \right). \end{aligned}$$

Recall that

$$T(t) := \frac{f_1}{f_0} = t + \frac{g_1}{f_0} \tag{3.5}$$

is the mirror map of Candelas et al, where

$$f_0 := \sum_{d \geq 0} \frac{(ld)!}{(d!)^{n+1}} e^{dt}, \quad g_1 := \sum_{d \geq 1} \frac{(ld)!}{(d!)^l} \sum_{m=d+1}^{ld} \frac{l}{m} e^{dt}.$$

Lemma 3.3. *In the limit $\lambda \rightarrow 0$, we have $HG[\mathcal{I}(Q)](T(t)) = \frac{1}{f_0} HG[\mathcal{I}(P)](t)$.*

Proof: By expanding in powers of α^{-1} and using the assumption that $l = n + 1$, we get

$$\begin{aligned} HG[\mathcal{I}(P)](t) &= e^{-pt/\alpha} \sum_{d \geq 0} \frac{\prod_{m=0}^{ld} (lp - m\alpha)}{\prod_{k=0}^n \prod_{m=1}^d (p - \lambda_k - m\alpha)} e^{dt} \\ &= lp \left[f_0 - \alpha^{-1} (p f_1 + g_2 \sum_{k=0}^n \lambda_k) + \cdots \right] \end{aligned} \tag{3.6}$$

where $g_2 = \sum_{d \geq 1} \frac{(ld)!}{(d!)^l} \sum_{m=1}^d \frac{1}{m} e^{dt}$.

Put $f := (\log f_0)\alpha + \frac{g_2}{f_0} \sum_{k=0}^n \lambda_k \in e^t \mathcal{R}[[e^t]]$. By Lemma 2.15, we have a mirror transformation μ such that

$$HG[\mathcal{I}(\tilde{P})](t) = e^{f/\alpha} HG[\mathcal{I}(P)](t) \tag{3.7}$$

where $\tilde{P} = \mu(P)$. Substituting (3.6) into (3.7), we get

$$\begin{aligned} HG[\mathcal{I}(\tilde{P})](t) &= (1 + \alpha^{-1} \frac{g_2}{f_0} \sum_{k=0}^n \lambda_k + \dots) \\ &\times f_0^{-1} lp \left[f_0 - \alpha^{-1} (p f_1 + g_2 \sum_{k=0}^n \lambda_k) + \dots \right] \\ &= lp - \alpha^{-1} lp^2 \frac{f_1}{f_0} + \dots \end{aligned} \tag{3.8}$$

By Lemma 2.15 again, we have a mirror transformation ν such that

$$HG[\mathcal{I}(\tilde{Q})](t) = HG[\mathcal{I}(Q)](t + \frac{g_1}{f_0})$$

where $\tilde{Q} = \nu(Q)$. Since, by Theorem 3.2 (i), $I_d^*(Q_d) = O(\alpha^{(n+1)d-2})$, it is straightforward to find that

$$HG[\mathcal{I}(\tilde{Q})](t) = e^{-p(t + \frac{g_1}{f_0})/\alpha} (lp + \dots) = lp - \alpha^{-1} lp^2 (t + \frac{g_1}{f_0}) + \dots \tag{3.9}$$

From (3.8) and (3.9), we conclude that for $d > 0$,

$$\frac{I_d^*(\tilde{P}_d - \tilde{Q}_d)}{\prod_{k=0}^n \prod_{m=1}^d (p - \lambda_k - m\alpha)} \equiv 0$$

modulo order α^{-2} , and hence

$$\text{deg}_\alpha \iota_{p_i,0}^*(\tilde{P}_d - \tilde{Q}_d) \leq (n + 1)d - 2.$$

But $\tilde{P} = \mu(P)$ is linked to P , and $\tilde{Q} = \nu(Q)$ is linked to Q . Since P and Q are linked, it follows that \tilde{P} and \tilde{Q} are also linked. By Theorem 2.11, we have $\tilde{P} = \tilde{Q}$. In particular, we have

$$HG[\mathcal{I}(Q)](T(t)) = HG[\mathcal{I}(\tilde{Q})](t) = HG[\mathcal{I}(\tilde{P})](t) = e^{f/\alpha} HG[\mathcal{I}(P)](t).$$

Since both P_d, Q_d lie in $H_G^*(N_d)$, their nonequivariant limit exist. Taking $\lambda \rightarrow 0$ yields our assertion. \square

Throughout the rest of this subsection, we set $l = n + 1 = 5$ and consider the critical bundle $\mathcal{O}(5) \rightarrow \mathbf{P}^4$. We assume that we have taken the nonequivariant limit $\lambda \rightarrow 0$. Recall that

$$F(T) := \frac{5T^3}{6} + \sum_{d>0} K_d e^{dT}, \quad \mathcal{F}(T) := \frac{5}{2} \left(\frac{f_1 f_2}{f_0} - \frac{f_3}{f_0} \right).$$

Theorem 3.4. (The Mirror Conjecture) $F = \mathcal{F}$.

Proof: Since

$$HG[\mathcal{I}(P)](t) = 5H \left(f_0 - f_1 \frac{H}{\alpha} + f_2 \frac{H^2}{\alpha^2} - f_3 \frac{H^3}{\alpha^3} \right), \tag{3.10}$$

we will prove that (cf. [8])

$$HG[\mathcal{I}(Q)](T) = 5H \left(1 - T \frac{H}{\alpha} + \frac{F'}{5} \frac{H^2}{\alpha^2} - \frac{TF' - 2F}{5} \frac{H^3}{\alpha^3} \right). \tag{3.11}$$

Eqns. (3.10), (3.11) and the preceding lemma imply $F = \mathcal{F}$. Denote the right hand side of (3.11) by R . Then

$$e^{HT/\alpha} R = 5H \left(1 + \frac{\Phi'}{5} \frac{H^2}{\alpha^2} + \frac{2\Phi}{5} \frac{H^3}{\alpha^3} \right) = 5H + O(e^T)$$

where $\Phi := F - \frac{5T^3}{6} = O(e^T)$. Similarly $e^{HT/\alpha} HG[\mathcal{I}(Q)](T) = 5H + O(e^T)$, which also has no polynomial dependence on T . So (3.11) is equivalent to

$$\int_{\mathbf{P}^4} e^{-HT/\alpha} \left(e^{HT/\alpha} HG[\mathcal{I}(Q)] \right) = \int_{\mathbf{P}^4} e^{-HT/\alpha} \left(e^{HT/\alpha} R \right).$$

By Theorem 3.2(ii), this left hand side is

$$\int_{\mathbf{P}^4} HG[\mathcal{I}(Q)] = \alpha^{-3} (2\Phi - T\Phi') + \int_{\mathbf{P}^4} e^{-HT/\alpha} 5H = \alpha^{-3} (2F - TF'),$$

which coincides with $\int_{\mathbf{P}^4} R$. \square

It is straightforward to generalize Theorem 3.4 to all other critical *convex* bundles V in the list (3.4). In each case, Ω^V becomes $\prod_a l_a p$ and the Euler data P to be linked with $Q : Q_d := \varphi_! e_T(V_d)$ is given by $P_d = \prod_a \prod_{m=0}^{l_a d} (l_a \kappa - m\alpha)$. In the nonequivariant limit, the hypergeometric series $HG[\mathcal{I}(P)](t)$ will produce some hypergeometric functions f_0, \dots, f_3 defining the function \mathcal{F} . The generating function F for the K_d is modified by simply replacing the term $\frac{5T^3}{6}$ by $\frac{T^3}{6} \int_X \psi^* H^3$, where $\psi : X \rightarrow \mathbf{P}^n$ is the Calabi-Yau cut out by V . With these minor modifications in each case, Theorem 3.4 holds. We leave the details as an exercise for the reader.

3.2. *First concave example: multiple-cover formula*

Let V be the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on \mathbf{P}^1 . For $d > 1$, V induces a rank $2d - 2$ bundle $U_d \rightarrow \mathcal{M}_{0,0}(d, \mathbf{P}^1)$ whose fiber at (f, C) is the space $H^1(C, f^*V)$, thus V is a critical concave bundle on \mathbf{P}^1 . We set $\Omega^V = 1/e_T(V) = p^{-2}$. We shall compute the equivariant classes $Q_d := \varphi_! e_T(V_d)$, and the numbers K_d for this critical bundle. Note that by definition $Q_1 = 1$ and $K_1 = 1$. As a consequence of Theorem 2.10,

$$l_{p_i,0}^*(Q_d) = \prod_{m=1}^{d-1} (\lambda_i - m(\lambda_i - \lambda_j)/d)^2$$

at $\alpha = (\lambda_i - \lambda_j)/d, j \neq i$. Thus Q is linked to

$$P : P_d := \prod_{m=1}^{d-1} (\kappa - m\alpha)^2,$$

which is a p^{-2} -Euler data. Obviously $\text{deg}_\alpha I_d^*(P_d) = 2d - 2$. It follows from Theorem 3.2 (i) that $\text{deg}_\alpha I_d^*(P_d - Q_d) \leq 2d - 2$, implying $Q = P$ by Theorem 2.11.

Corollary 3.5. $K_d = d^{-3}$.

Proof: By Theorem 3.2(ii) in the limit $\lambda \rightarrow 0$, we have

$$\int_{\mathbf{P}^1} e^{-Ht/\alpha} \frac{I_d^*(Q_d)}{\prod_{m=1}^d (H - m\alpha)^2} = \alpha^{-3} (2 - d t) K_d.$$

Since $Q = P$, we have $I_d^*(Q_d) = I_d^*(P_d) = \prod_{m=1}^{d-1} (H - m\alpha)^2$, giving

$$\int_{\mathbf{P}^1} e^{-Ht/\alpha} \frac{I_d^*(Q_d)}{\prod_{m=1}^d (H - m\alpha)^2} = \alpha^{-3} d^{-3} (2 - d t). \quad \square$$

3.3. Second concave example: $K_{\mathbf{P}^2}$

Let V be the canonical bundle $\mathcal{O}(-3) \rightarrow \mathbf{P}^2$. For $d > 0$, this bundle induces a rank $3d - 1$ bundle $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^2)$. Thus V is a critical concave bundle. We set $\Omega^V = 1/e_T(V) = (-3p)^{-1}$. We shall compute the equivariant classes $Q_d := \varphi_! e_T(V_d)$, and the numbers K_d for this critical bundle. As a consequence of Theorem 2.10, we have

$$\iota_{p_i,0}^*(Q_d) = \prod_{m=1}^{3d-1} (-3\lambda_i + m(\lambda_i - \lambda_j)/d).$$

at $\alpha = (\lambda_i - \lambda_j)/d, j \neq i$. Thus Q is linked to

$$P : P_d := \prod_{m=1}^{3d-1} (-3\kappa + m\alpha).$$

which is a $(-3p)^{-1}$ -Euler data.

Corollary 3.6. $HG[\mathcal{I}(Q)](t + g) = HG[\mathcal{I}(P)](t)$ where $g := \sum_{d>0} \frac{(-1)^d}{d} \frac{(3d)!}{d!^3} e^{dt}$.

Proof: By expanding in powers of α^{-1} , we get

$$\begin{aligned} HG[\mathcal{I}(P)](t) &= e^{-pt/\alpha} \\ &\times \left((-3p)^{-1} + \sum_{d>0} \frac{\prod_{m=1}^{3d-1} (-3p + m\alpha)}{\prod_{k=0}^2 \prod_{m=1}^d (p - \lambda_k - m\alpha)} e^{dt} \right) \\ &= (-3p)^{-1} + \alpha^{-1} \frac{t + g}{3} + \dots \end{aligned} \tag{3.12}$$

As before, it is now straightforward to show that

$$HG[\mathcal{I}(Q)](t + g) \equiv HG[\mathcal{I}(P)](t)$$

modulo order α^{-2} . Once again by Theorem 2.11, the two sides are equal identically. \square

Using Theorem 3.2(ii) and the preceding corollary, we obtain the

K_d , for $d = 1, \dots, 10$:

d	K_d
1	3
2	$-\frac{45}{8}$
3	$\frac{244}{9}$
4	$-\frac{12333}{64}$
5	$\frac{211878}{125}$
6	$-\frac{102365}{6}$
7	$\frac{64639725}{343}$
8	$-\frac{1140830253}{512}$
9	$\frac{6742982701}{243}$
10	$-\frac{36001193817}{100}$

3.4. A concave bundle on \mathbf{P}^3 .

Let $V = \mathcal{O}(2) \oplus \mathcal{O}(-2)$ on \mathbf{P}^3 . This is a direct sum of a convex and a concave bundle. The induced bundle $U_d \rightarrow \bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^3)$, with fiber at (f, C) being $H^0(C, f^*\mathcal{O}(2)) \oplus H^1(C, f^*\mathcal{O}(-2))$, has rank $4d$. We set $\Omega^V = e_T(\mathcal{O}(2))/e_T(\mathcal{O}(-2)) = -1$. We shall compute the equivariant classes $Q_d := \varphi_! e_T(V_d)$, and the numbers K_d for this critical bundle. As a consequence of Theorem 2.10, we have

$$\iota_{\mathbf{P}^3,0}^*(Q_d) = \prod_{m=0}^{2d} (2\lambda_i - m(\lambda_i - \lambda_j)/d) \times \prod_{m=1}^{2d-1} (-2\lambda_i + m(\lambda_i - \lambda_j)/d).$$

at $\alpha = (\lambda_i - \lambda_j)/d, j \neq i$. Thus Q is linked to

$$P : P_d := \prod_{m=0}^{2d} (2\kappa - m\alpha) \times \prod_{m=1}^{2d-1} (-2\kappa + m\alpha).$$

which is a -1 -Euler data.

Corollary 3.7. $HG[\mathcal{I}(Q)](t + g) = HG[\mathcal{I}(P)](t)$ where $g :=$

$$\sum_{d>0} \frac{1}{d} \frac{(2d)!^2}{d!^4} e^{dt}.$$

Proof: By expanding in powers of α^{-1} , we get

$$\begin{aligned} HG[\mathcal{I}(P)](t) &= e^{-pt/\alpha} \\ &\times \left(-1 + \sum_{d>0} \frac{\prod_{m=0}^{2d} (2p - m\alpha) \times \prod_{m=1}^{2d-1} (-2p + m\alpha)}{\prod_{k=0}^3 \prod_{m=1}^d (p - \lambda_k - m\alpha)} e^{dt} \right) \\ &= -1 + \alpha^{-1} p(t + g) + \dots, \end{aligned} \tag{3.13}$$

which, as in the previous examples, agrees with $HG[\mathcal{I}(Q)](t + g)$ up to order α^{-2} . Hence Theorem 2.11 yields our assertion. \square

Using Theorem 3.2(ii) and the preceding corollary, we obtain the K_d , for $d = 1, \dots, 10$:

d	K_d
1	-4
2	$-\frac{9}{2}$
3	$-\frac{328}{27}$
4	$-\frac{777}{16}$
5	$-\frac{30004}{125}$
6	$-\frac{4073}{3}$
7	$-\frac{2890808}{343}$
8	$-\frac{7168777}{128}$
9	$-\frac{285797488}{729}$
10	$-\frac{714787509}{250}$

3.5. A concave bundle on \mathbf{P}^4 .

Consider now the critical bundle $V = \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(-1)$ on \mathbf{P}^4 . The induced bundle $U_d \rightarrow \mathcal{M}_{0,0}(d, \mathbf{P}^3)$, has rank $5d + 1$. We set $\Omega^V = e_T(\mathcal{O}(2))^2 / e_T(\mathcal{O}(-1)) = -4p$. We shall compute the equivariant classes $Q_d := \varphi_! e_T(V_d)$, and the numbers K_d for this critical bundle. As a consequence of Theorem 2.10, we have

$$i_{p_i,0}^*(Q_d) = \prod_{m=0}^{2d} (2\lambda_i - m(\lambda_i - \lambda_j)/d)^2 \times \prod_{m=1}^{d-1} (-\lambda_i + m(\lambda_i - \lambda_j)/d).$$

at $\alpha = (\lambda_i - \lambda_j)/d, j \neq i$. Thus Q is linked to

$$P : P_d := \prod_{m=0}^{2d} (2\kappa - m\alpha)^2 \times \prod_{m=1}^{d-1} (-\kappa + m\alpha).$$

which is a $-4p$ -Euler data.

Corollary 3.8. $HG[\mathcal{I}(Q)](t + g) = HG[\mathcal{I}(P)](t)$ where $g := \sum_{d>0} \frac{(-1)^d}{d} \frac{(2d)!^2}{d!^4} e^{dt}$.

Proof: By expanding in powers of α^{-1} , we get

$$\begin{aligned} HG[\mathcal{I}(P)](t) &= e^{-pt/\alpha} \left(-4p + \sum_{d>0} \frac{\prod_{m=0}^{2d} (2p - m\alpha)^2 \times \prod_{m=1}^{d-1} (-p + m\alpha)}{\prod_{k=0}^d \prod_{m=1}^d (p - \lambda_k - m\alpha)} e^{dt} \right) \\ &= -4p + \alpha^{-1} 4p^2(t + g) + \dots, \end{aligned} \tag{3.14}$$

which, as in the previous examples, agrees with $HG[\mathcal{I}(Q)](t + g)$ up to order α^{-2} . Hence Theorem 2.11 yields our assertion. \square

We can work out the K_d here as we did before. The K_d here can be obtained by taking K_d from the preceding example on \mathbf{P}^3 , and multiply it by $4(-1)^d$. This is so because in the nonequivariant limit, the hypergeometric series $HG[\mathcal{I}(P)](t)$ (cf. (3.13) and (3.14)) in this example on \mathbf{P}^4 and the preceding example on \mathbf{P}^3 are related by first a multiplication of $4p$ followed by a change of variable $e^{dt} \mapsto (-1)^d e^{dt}$.

3.6. General concavex bundles

In fact the examples above are representative of the most general concavex bundle. Let $V = V^+ \oplus V^-$ be a concavex bundle on \mathbf{P}^n , and let Q, P be as defined in Theorem 2.10, and assume that V has splitting type $(l_1, \dots, l_{N^+}; k_1, \dots, k_{N^-})$. Note that $\sum l_a + \sum k_b$ is the value of the class $c_1(V^+) - c_1(V^-)$ on a T -invariant \mathbf{P}^1 in \mathbf{P}^n .

Theorem 3.9. *If $d(\sum l_a + \sum k_b) - N^- \leq (n+1)d - 2$ for all $d > 0$, then $Q = P$. If $d(\sum l_a + \sum k_b) - N^- \leq (n+1)d$ for all $d > 0$, then there exists a mirror transformation μ , depending only on the l_a, k_b , such that $Q = \mu(P)$.*

Proof: By definition of P in Theorem 2.10,

$$\text{deg}_\alpha I_d^*(P_d) = d \left(\sum l_a + \sum k_b \right) - N^-.$$

Consider the first case, where this is bounded above by $(n + 1)d - 2$ for all d . Then by Theorem 3.2,

$$\text{deg}_\alpha(I_d^*(P_d) - I_d^*(Q_d)) \leq (n + 1)d - 2,$$

implying $Q = P$ by Theorem 2.11.

Consider now the second case. Obviously our assumption implies that $\sum l_a + \sum k_b \leq (n + 1)$. It is trivial to show that the only possibilities not covered by the first case are: (1) $N^- = 0$ and $\sum l_a = n + 1$; (2) $N^- = 1$ and $\sum l_a + k_1 = n + 1$. (3) $N^- = 0$ and $\sum l_a = n$; In each of these cases, a mirror transformation can be constructed by imitating the previous examples in a straightforward way. Case (1) imitates the example $\mathcal{O}(5) \rightarrow \mathbf{P}^4$, while cases (2), (3) imitate the example $\mathcal{O}(-3) \rightarrow \mathbf{P}^2$. It is obvious that in each case, the mirror transformation depends only on the data l_a, k_b . \square

Corollary 3.10. *Under the same hypotheses as in the preceding theorem, the Euler data $Q : Q_d = \varphi_{!e_T}(V_d)$ depends only on the splitting type, ie. the numbers l_a, k_b , of the concave bundle V on \mathbf{P}^n .*

Note that not every concave bundle on \mathbf{P}^n is a direct sum of line bundles. For example the tangent bundle is convex, but is not a direct sum of line bundles.

3.7. Equivariant total Chern class

For simplicity, we restrict to convex bundles. Let V be a rank r convex bundle on \mathbf{P}^n , and let

$$c_T(V) = x^r + x^{r-1}c_1(V) + \dots + c_r(V)$$

be the T -equivariant Chern polynomial of V . Similarly we denote by $c_T(U_d)$ the equivariant Chern polynomial for U_d . As explained in Example 9, we can extend the notion of Euler data Q allowing Q_d to depend on x polynomially, simply by replacing the ground field \mathbf{Q} by $\mathbf{Q}(x)$. Then a similar argument as in Theorem 2.8 shows that the sequence $Q_d := \varphi_{!}(\pi^*c_T(U_d))$ is also an Euler data in the generalized sense. Moreover, the analogue of Theorem 2.10 holds, ie. at $\alpha = (\lambda_j - \lambda_i)/d$, we have the form

$$\iota_{p_j,0}^*(Q_d) = \prod_a \prod_{m=0}^{l_a d} (x + l_a \lambda_j - m(\lambda_j - \lambda_i)/d).$$

Hence Q is linked to the Euler data

$$P : P_d = \prod_a \prod_{m=0}^{l_a d} (x + l_a \kappa - m\alpha).$$

Again, under a suitable bound on $c_1(V)$, one can easily relate P, Q by a generalized (depending on x) mirror transformation.

For example, by taking the bundle $\mathcal{O}(4)$ on \mathbf{P}^3 , and applying the above result, we can compute the nonequivariant limits of all $\int_{\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^3)} c_{4d}(U_d)$. They are expected to count rational curves in a \mathbf{P}^1 -family of K3 hypersurfaces in \mathbf{P}^3 . Similarly we can take $\mathcal{O}(3)$ on \mathbf{P}^2 and compute $\int_{\bar{\mathcal{M}}_{0,0}(d, \mathbf{P}^2)} c_{3d-1}(U_d)$ which should count the number of rational curves in a \mathbf{P}^2 -family of elliptic curves in \mathbf{P}^2 . Details will be reported in full in our forth-coming paper [37].

3.8. Concluding remarks

The most important result we established in this paper is the Mirror Principle. For simplicity we have restricted our examples in this paper, to studying only Euler classes and total Chern classes. As mentioned in the Introduction, the Mirror Principle works well for any multiplicative equivariant characteristic classes. We shall study in details more examples of the total Chern class in our forthcoming paper [37]. Generalization to manifolds with torus action will also be dealt with in details there.

Finally, we make a tantalizing observation which might be of both physical and mathematical significance. As we have seen, the set of linked Euler data has an infinite dimensional transformation group – the mirror group. For suitable concavex bundle $V \rightarrow \mathbf{P}^n$, two special linked Euler data (cf. Theorem 2.10) $Q : Q_d = \varphi_{!} e_T(V_d)$ arising from the nonlinear sigma model (the stable map moduli), and P the corresponding Euler data of hypergeometric type, are related by a mirror transformation. Since the mirror group is so big, there are many other Euler data which are linked to P and can be obtained simply by acting on P by the mirror group. From the physical point of view, P arises from type IIB string theory while Q arises from type IIA string theory, and mirror symmetry is a duality between the two. This relationship manifests itself on the linear sigma model as a duality transformation. This suggests that some other Euler data linked to P may arise from some other string theories which are dual to type IIA and IIB, via more general mirror transformations. From the point of view of moduli theory, P is associated to the linear sigma model compactification N_d of the moduli space M_d^0 we discussed in

Example 10. Whereas Q is associated to the nonlinear sigma model M_d , which is the stable map compactification of M_d^0 . This suggests that some other Euler data linked to P may correspond to other compactifications of M_d^0 . If true, we will have an association between string theories, linked Euler data, and compactifications of moduli space of maps, all in the same picture, whereby there is a duality in each kind which one sees in the linear sigma model. It would be interesting to understand this duality more precisely.

References

- [1] P. Aspinwall and D. Morrison, *Topological field theory and rational curves*, Commun. Math. Phys. 151 (1993), 245-262.
- [2] J. Bryan and N. Leung, *The enumerative geometry of K3 surfaces and modular forms*, alg-geom/9711031.
- [3] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology 23 (1984) 1-28.
- [4] V. Batyrev and L. Borisov, *On Calabi-Yau complete intersections in toric varieties*, alg-geom/9412017.
- [5] V. Batyrev and D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties*, Comm. Math. Phys. 168 (1995) 495-533.
- [6] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitude*, Commun. Math. Phys. 165 (1994) 311-428.
- [7] R. Bott, *A residue formula for holomorphic vector fields*, J. Diff. Geom. 1 (1967) 311-330.
- [8] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B359 (1991) 21-74.
- [9] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, *Mirror symmetry for two parameter models I*, hep-th/9308083.
- [10] L. Caporaso and J. Harris, *Parameter spaces for curves on surfaces and enumeration of rational curves*, alg-geom/9608024.
- [11] B. Crauder and R. Miranda, *Quantum cohomology of rational surfaces*, Progress in Mathematics 129, Birkhauser (1995) 33.
- [12] P. DiFrancesco and C. Itzykson, *Quantum intersection rings*, Progress in Mathematics 129, Birkhauser (1995) 81.
- [13] R. Dijkgraaf, *Mirror symmetry and elliptic curves*, Progress in Mathematics 129, Birkhauser (1995) 149.
- [14] *Essays on Mirror Manifolds*, ed. S.T. Yau, International Press.
- [15] G. Ellingsrud and S.A. Stromme, *Bott's formula and enumerative geometry*, JAMS 9 (1996), no. 1, 175-193.

- [16] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, alg-geom/9608011.
- [17] E. Getzler, *The elliptic Gromov-Witten invariants on CP^3* , alg-geom/9612009.
- [18] A. Givental, *Equivariant Gromov-Witten invariants*, alg-geom/9603021.
- [19] A. Givental, *A mirror theorem for toric complete intersections*, alg-geom/9701016.
- [20] T. Graber and R. Pandharipande, *Localization of Virtual classes*, alg-geom/9708001.
- [21] B. Greene and R. Plesser, *Duality in Calabi-Yau moduli space*, Nucl. Phys. B338 (1990) 15-37.
- [22] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics 52. Berlin, Heidelberg, New York Springer 1977
- [23] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag, Berlin 1995, 3rd Ed.
- [24] Y. Hu, *Moduli spaces of stable polygons and symplectic structures on $\bar{\mathcal{M}}_{0,n}$* , Comp. Math. (to appear), alg-geom/9701011.
- [25] M. Jinzenji and M. Nagura, *Mirror symmetry and exact calculation of $N - 2$ point correlation function on Calabi-Yau manifold embedded in CP^{N-1}* , hep-th/9409029.
- [26] M. Kapranov, *Chow quotients of Grassmannian*, I.M. Gel'fand Seminar, AMS (1993).
- [27] B. Kim, *On equivariant quantum cohomology*, IMRN 17 (1996) 841.
- [28] F. Knudsen, and D. Mumford, *The projectivity of the moduli space of stable curves I*, preliminaries on 'det' and 'div'. Math. Scand. 39 (1976) 19-55.
- [29] D. Morrison and R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, alg-geom/9412236.
- [30] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, hep-th/9406055.

- [31] S. Hosono, B.H. Lian and S.T. Yau, *GKZ-generalized hypergeometric systems and mirror symmetry of Calabi-Yau hypersurfaces*, alg-geom/9511001, Commun. Math. Phys. 182 (1996) 535-578.
- [32] W.Y. Hsiang, *On characteristic classes and the topological Schur lemma from the topological transformation groups viewpoint*, Proc. Symp. Pure Math. XXII. (1971) 105-112.
- [33] S. Katz, *On the finiteness of rational curves on quintic threefolds*, Comp. Math. 60 (1986) 151-162.
- [34] M. Kontsevich, *Enumeration of rational curves via torus actions*. In: The Moduli Space of Curves, ed. by R.Dijkgraaf, C.Faber, G.van der Geer, Progress in Math. vol. 29, Birkhäuser, 1995, 335-368.
- [35] J. Li, *Algebraic geometric interpretation of Donaldson's polynomial invariants*, Journ. Diff. Geom. 37 (1993) 417-466.
- [36] A. Libgober and J. Teitelboim, *Duke Math. Journ., Int. Res. Notices* 1 (1993) 29.
- [37] B. Lian, K. Liu, and S.T. Yau, *Mirror Principle II*, in preparation.
- [38] Yu.I. Manin, *Generating functions in algebraic geometry and sums over trees*. In: The Moduli Space of Curves, ed. by R.Dijkgraaf, C.Faber, G.van der Geer, Progress in Math. vol. 29, Birkhäuser, 1995, 401-418.
- [39] Yu.I. Manin, *Frobenius Manifolds, quantum cohomology, and moduli spaces (Chapter I,II,III)*., Max-Planck Inst. preprint MPI 96-113.
- [40] D. Morrison, *Picard-Fuchs Equations and Mirror Maps for Hypersurfaces*, in [14].
- [41] C. Okonek, M. Schneider and H. Spindler *Vector bundles on complex projective spaces*, Progress in Math., Birkhauser.
- [42] Y.B. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, Journ. Diff. Geom. Vol. 42, No. 2 (1995) 259-367.
- [43] G. Tian, *private communication*, November 1995.
- [44] C. Voisin, *A mathematical proof of a formula of Aspinwall-Morrison*, Comp. Math. 104 (1996) 135-151.

- [45] E. Witten, *Phases of $N=2$ theories in two dimension*, hep-th/9301042.