Special Lagrangian Fibrations II: Geometry.

A Survey of Techniques in the Study of Special Lagrangian Fibrations

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§0. Introduction.

This paper is a progress report on work surrounding the Strominger-Yau-Zaslow mirror symmetry conjecture [28]. Roughly put, this conjecture suggests the following program for attacking an appropriate form of the mirror conjecture. Let X be a Calabi-Yau n-fold, with a large complex structure limit point p in a compactification of the complex moduli space of X. One expects mirrors of X to be associated to such boundary points of the complex moduli space of X. For complex structures on X in an open neighborhood of the boundary point p and suitable choice of a Ricci flat metric on X, one attempts to construct the mirror of X via the following program:

(1) There is an n-torus representing a homology class in $H_n(X, \mathbf{Z})$ which is invariant under all monodromy transformations about the discriminant locus passing through the point p. (See [14], §3 for details of this representative). The first task is to find an homologous n-torus which is special Lagrangian.

(2) Having found one special Lagrangian torus, show that it deforms to yield a fibration $f : X \rightarrow B$ all of whose fibres are special Lagrangian and whose general fibre is an n-torus.

(3) Construct the dual n-torus fibration as follows. Let $B_0 \subseteq B$ be the complement of the discriminant locus of f, $f_0 : X_0 \rightarrow B_0$ the restriction of f to $X_0 = f^{-1}(B_0)$. The dual n-torus fibration over B_0 is $\check{X}_0 = R^1 f_{0*}\mathbf{R}/R^1 f_{0*}\mathbf{Z} \rightarrow B_0$. Find a suitable compactification of $\check{X}_0$ to a manifold $\check{X}$ along with a fibration $\check{f} : \check{X} \rightarrow B$.

(4) Show that X and $\check{X}$ satisfy a topological form of mirror symmetry. It is not clear what this means in arbitrary dimension, but for threefolds, this will be

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an isomorphism $H^{even}(X,\mathbf{Q}) \cong H^{odd}(\check{X},\mathbf{Q})$ and $H^{odd}(X,\mathbf{Q}) \cong H^{even}(\check{X},\mathbf{Q})$. This in particular implies the usual interchange of Hodge numbers for threefolds. One might also dare to hope these isomorphisms also hold over $\mathbf{Z}$; we will see this will often be the case in Theorem 3.10.

(5) Put a complex and Kähler structure on $\check{X}$. The choice of such structures determines the mirror map. One expects that the complex structure on X should entirely determine the Kähler structure on $\check{X}$, while the Kähler structure on X along with a choice of the B-field, a cohomology class in $H^2(X,\mathbf{R}/\mathbf{Z})$, or a related cohomology group, will determine the complex structure on $\check{X}$. In [16], a somewhat more precise conjecture was given as to how this interchange of structures should look on the level of cohomology. Specifically, let ω, Ω be the Kähler form and holomorphic n-form on X with Ω normalised so that $\int_{X_b}\Omega = 1$. In addition, one is given a choice of B-field, which right now we'll take to be a cohomology class $\mathbf{B} \in H^1(B, R^1f_*\mathbf{R})$. (The B-field will always be denoted by a bold-face $\mathbf{B}$ to differentiate it typographically from the base B of the fibration.) The choice of Kähler and complex structures on the mirror, determined by forms $\check{\omega}$ and $\check{\Omega}$, should satisfy the following relationship: using the identifications $H^1(B,R^1f_*\mathbf{R}) \cong H^1(B,R^{n-1}\check{f}_*\mathbf{R})$ and $H^1(B,R^{n-1}f_*\mathbf{R}) \cong H^1(B,R^1\check{f}_*\mathbf{R})$ which conjecturally hold, the following identities should hold in these cohomology groups:

$$\begin{aligned}
[\check{\omega}] &= [\operatorname{Im}\Omega] \\
[\operatorname{Im}\check{\Omega}] &= [\omega] \\
[\operatorname{Re}\check{\Omega}] - [\sigma_0] &= \mathbf{B}
\end{aligned}$$

where σ_0 is a chosen zero-section of $\check{f}: \check{X} \to B$.

(6) Show that the above procedure yields the correct enumerative predictions for Gromov-Witten invariants of X and $\check{X}$.

This program is still a long way from completion, and this paper represents only one small step in this direction. Not much is known about items (1) and (2) yet; this may well prove to be the hardest part of the program. In [16], we gave examples of special Lagrangian T^3-fibrations with a degenerate metric. In [32], T^n-fibrations are constructed on Calabi-Yau hypersurfaces in smooth toric varieties. These fibrations are constructed as deformations of the natural T^n-fibration on the large complex structure limit given by the moment map. Unfortunately these tori are neither special nor Lagrangian, but this fibration may be sufficient for a purely topological version of mirror symmetry and provides evidence for the SYZ conjecture.

We will not address issues (1) and (2) further in this paper. Instead, throughout this paper, we assume the existence of a special Lagrangian fibration $f : X \to B$ on X, much as we did in [14]. However, we wish to delve more deeply into the properties of such fibrations, and to do so, we need to make reasonable guesses as to what kind of regularity properties such fibrations will possess. We discuss these issues and assumptions in §1. In [14], we did not make real use of the fact that $f : X \to B$ was special Lagrangian, but rather only used the topological fact that f was a torus fibration, along with a technical condition we called *simplicity* to control the cohomological contributions of the singular fibres. In this paper, we try to make more serious use of the special Lagrangian condition. We find that we make very heavy use of the Lagrangian condition, but as yet we do not use the additional special condition in a profound way. We do however use the fact that special Lagrangian submanifolds are in fact volume minimizing. This allows us to provide moral guidelines as to what we might expect of special Lagrangian fibrations, by drawing on the wealth of material known about volume minimizing rectifiable currents. Some of these ideas are discussed in §1.

Of course, the study of Lagrangian torus fibrations is a familiar one in the subject of completely integrable Hamiltonian systems. In §2 of this paper we review and generalise slightly for our purposes Duistermaat's work on global action-angle coordinates [12]. Simplifying a bit, if $f : X \to B$ is a Lagrangian T^n fibration with a Lagrangian section and if $X^\#$ is the complement of the critical locus of f in X, then one can canonically write $X^\#$ as a quotient of the cotangent bundle of B by a possibly degenerating family of lattices suitably embedded in $\mathcal{T}_B^*$. Furthermore, the canonical symplectic form on $\mathcal{T}_B^*$ descends to the symplectic form on $X^\#$. Thus if X is a Calabi-Yau manifold, the existence of a special Lagrangian fibration gives us coordinates on a large open subset of X on which it is easy to write the symplectic form. I think of these coordinates as *special Lagrangian coordinates.* These can be compared with traditional complex coordinates, where it is easy to write the holomorphic n-form $(dz_1 \wedge \cdots \wedge dz_n)$, but very difficult to write down the Kähler form of a Ricci flat metric. Thus we should expect that in the coordinates given by a special Lagrangian fibration, the difficulty will be to write down the holomorphic n-form, or equivalently, the complex structure.

Taking this analogy further, we note that complex coordinates give a filtration on de Rham cohomology, namely the Dolbeault cohomology groups. Similarly, special Lagrangian coordinates can be thought of as giving rise to a filtration on cohomology, namely that given by the Leray spectral sequence associated to the special Lagrangian fibration. Just as the Dolbeault cohomology groups give rise to the Hodge filtration, we saw in [14] that the Leray filtration should, modulo some conjectures about monodromy, give rise to the monodromy weight filtration associated to the large complex structure limit point.

In [14], we studied the Leray spectral sequence of f with coefficients in $\mathbf{Q}$. We now show in §3, modulo suitable regularity assumptions stated in §1, that three-dimensional special Lagrangian fibrations satisfy the condition of $\mathbf{Z}$-simplicity introduced in [14]. This is a stronger hypothesis then was used in [14], and as a result, we can analyse the Leray spectral sequence over $\mathbf{Z}$. This leads to some interesting results as to the role torsion in cohomology plays in mirror symmetry. In particular, it provides an explanation of the phenomenon proposed in [5] of the role of "discrete torsion" (torsion in $H^3(X, \mathbf{Z})$) in mirror symmetry. This is discussed in §3. In addition, the analysis of the Leray spectral sequence over $\mathbf{Z}$ sheds light on item (4) of the program proposed above.

Moving on, we next address the subject of putting symplectic and complex structures on the mirror $\check{X}$. This subject has already been discussed to some extent by Hitchin in [18]. Our approach is inspired by Hitchin's, and is a generalisation of that approach. In fact, one of the goals accomplished is to restate Hitchin's constructions in a more coordinate independent form, so as to allow us to understand the cohomological ramifications of these constructions.

One should in fact consider any special Lagrangian submanifold $M \subseteq X$, and consider the moduli space of deformations of this submanifold. Calling this moduli space B, the D-brane moduli space of M is then the set of pairs (M', α) where α is a flat $U(1)$ connection modulo gauge equivalence on M' a deformation of M. This is a T^s-bundle over B, where $s = b_1(M)$. Specifically, if $f : \mathcal{U} \to B$ is the universal family of special Lagrangian submanifolds parametrized by $B, \mathcal{U} \subseteq B \times X$, then the D-brane moduli space is $\mathcal{M} = R^1 f_* \mathbf{R}/R^1 f_* \mathbf{Z} \to B$. In addition, Mclean [22] gives us a canonical isomorphism between the tangent bundle of B, $\mathcal{T}_B$, and $R^1 f_* \mathbf{R} \otimes C^\infty(B)$. This gives a canonical embedding of $R^1 f_* \mathbf{Z}$ in $\mathcal{T}_B$. However, $\mathcal{T}_B$ of course does not carry a canonical symplectic form; rather, it is $\mathcal{T}_B^*$ which does. So to find a symplectic form on $\mathcal{M}$, we need to reembed $R^1 f_* \mathbf{Z}$ in $\mathcal{T}_B^*$. There are two ways of doing this. One is to use periods integrals related to $\operatorname{Im} \Omega$, the imaginary part of the holomorphic n-form on X; the other is to use a canonical metric introduced by McLean on $\mathcal{T}_B$ to identify $\mathcal{T}_B$ with $\mathcal{T}_B^*$. In fact these two methods give the same embedding of $R^1 f_* \mathbf{Z}$ in $\mathcal{T}_B^*$. This allows us to define a symplectic form on $\mathcal{M}$ by writing $\mathcal{M}$ as $\mathcal{T}_B^*/R^1 f_* \mathbf{Z}$ and taking the form on $\mathcal{M}$ induced by the canonical symplectic form on $\mathcal{T}_B^*$. This is the same method as proposed by Hitchin, recast in a slightly more invariant way, which makes it easy to see that the cohomology class of the symplectic form defined in this manner is as predicted by the conjecture of [16]. Thus, if we can understand how to extend this symplectic form to the compactification of the D-brane moduli space, we will solve the first half of item (5). This circle of ideas is discussed in §4.

Because it is easy to write the symplectic form in special Lagrangian coordinates, we expect it to be very difficult to write the complex structure. We take up this issue in §§5 and 6. First we explore what data is necessary to place a complex structure on a Lagrangian torus fibration in order to make the fibration special Lagrangian and the induced metric Ricci-flat. A moment's thought shows that given knowledge of ω, to specify an almost complex structure compatible with ω, it is enough to give for each point $b \in B$ a metric g on the fibre X_b, and for each point $x \in X_b$ the Lagrangian subspace $J(\mathcal{T}_{X_b,x}) \subseteq \mathcal{T}_{X,x}$. Then J is completely determined by the requirement that $g(v,w) = \omega(Jv, w)$. The collection of subspaces $J(\mathcal{T}_{X_b,x})$ can be thought of as the horizontal subspaces of an Ehresmann connection on $f : X^\# \to B$. We then need to ask when this data determines an integrable complex structure which induces a Ricci-flat metric in which the fibres of f are special Lagrangian.

Following [18], it is easier to determine this by describing the holomorphic n-form Ω. In local coordinates $y_1, \ldots, y_n$ on the base, and canonical coordinates $x_1, \ldots, x_n$ on the fibres of $\mathcal{T}_B^*$, we can write the general form of Ω as

$$\Omega = V \bigwedge_{i=1}^n (dx_i + \sum_{j=1}^n \beta_{ij} dy_j).$$

where β_{ij} is a complex-valued function on $\mathcal{T}_B^*$ and V is a real-valued function. The Ehresmann connection is in fact encoded in $\operatorname{Re} \beta$, while $\operatorname{Im} \beta$ is just the inverse of the metric on the fibres. The integrability and Ricci-flatness conditions are then easy to write down. In fact, we show that one needs the conditions:

(1) The matrix $\beta = (\beta_{ij})$ is symmetric, $\operatorname{Im}\beta$ is positive definite, and $V = 1/\sqrt{\det \operatorname{Im}\beta}$.

(2) $d\Omega = 0$.

The first condition is of course easily achieved, but the second condition is a quite subtle condition. The second condition is what requires real effort, and understanding it is really at the heart of the SYZ program. It turns out that (2) is equivalent to the following three conditions:

(1) The almost complex structure is integrable. (Note that $d\Omega = 0$ is a much stronger condition than integrability. Given any Lagrangian fibration on a Calabi-Yau manifold with a holomorphic n-form Ω, by replacing Ω by $e^{i\theta(x)}\Omega$ for a suitable function $\theta(x)$, one can ensure that $\operatorname{Im}\Omega$ restricted to each fibre is zero. However now Ω is no longer holomorphic.)

(2) $dx_1, \ldots, dx_n$ are harmonic 1-forms on X_b for each $b \in B$. (That this is a necessary condition follows from Mclean's results.)

(3) The volume form $V dx_1 \wedge \cdots \wedge dx_n$ on fibres is parallel under translation via the Ehresmann connection.

In [18], Hitchin constructed a complex structure on the D-brane moduli space by specifying the holomorphic n-form Ω. The form he constructed had very special properties: he used $\operatorname{Re}\beta = 0$ and $\operatorname{Im}\beta$ constant along fibres. As a result, the integrability condition $d\Omega = 0$ was much easier to check. In the general case, it requires a great deal more effort to analyse this equation, and the results of §5 and 6 represent only a beginning. The conditions (1)-(3) above should perhaps be thought of as mirror equations to the usual complex Monge-Ampère equations which arise in the study of Ricci-flat metrics. Understanding their solution should be one of the key steps in the Strominger-Yau-Zaslow program. We analyse the solution in several simple cases. For example, we show that if the Ehresmann connection is in fact flat, then the metric on each fibre must be flat. This explains why in Hitchin's situation, where the connection was trivial, it would be impossible to obtain any solutions which are not flat along the fibres. In particular, the connection clearly cannot be flat if the fibration possesses singular fibres. We also give a whole family of solutions related to Hitchin's, but which take the B-field into account. This gives some hint as to the role the B-field plays and leads us to a refined form of the mirror symmetry conjecture (Conjecture 6.6). We will argue that the B-field should not be thought of as an element of $H^2(X, \mathbf{R}/\mathbf{Z})$ but rather as an element of $H^1(B, R^1 f_* \mathbf{R}/\mathbf{Z})$. Thus the group the B-field takes values in should depend not just on X but on the fibration f. These two groups can in fact be different, so this suggestion is quite a serious modification of previous interpretations of the B-field (for example that of [5]).

In §7 we apply much of what we have done to the situation of K3 surfaces, by way of an extended example. Here we know that special Lagrangian fibrations do exist, and we know the precise construction of the mirror map. We are able to show that the various recipes and constructions given in earlier sections of the paper can be carried out completely for K3 surfaces. In particular, we obtain an almost purely differential geometric description of mirror symmetry for K3 surfaces.

Finally, in §8, we give a brief discussion of results about the Strominger-Yau-Zaslow conjecture obtained since the initial version of this paper was prepared.

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Convention. For a p-form α and tangent vectors $v_1,\ldots,v_q$, $\iota(v_1,\ldots,v_q)\alpha$ denotes the $p-q$-form $\alpha(v_1,\ldots,v_q,\cdot)$.

§1. Special Lagrangian Fibrations.

In what follows, X will denote a complex Calabi-Yau manifold with a Ricci-flat Kähler metric g, Kähler form ω, and holomorphic n-form Ω. The form Ω is always normalised to be of unit length, i.e.

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n\Omega\wedge\bar{\Omega}.$$

This only fixes Ω up to a phase factor. As the metric is Ricci-flat, we can take the metric to be real analytic.

Recall

Definition-Proposition 1.1. *[17]* $\mathrm{Re}\,\Omega$ *is a calibration, called the special Lagrangian calibration. An n-dimensional real submanifold $M\subseteq X$ is special Lagrangian if* $\mathrm{Re}\,\Omega|_M = Vol(M)$. *Modulo orientation, a submanifold $M\subseteq X$ of real dimension n is special Lagrangian if and only if $\omega|_M=0$ and $\mathrm{Im}\,\Omega|_M=0$.*

It is natural to extend the notion of special Lagrangian submanifolds to special Lagrangian integral currents, as done in [17]. For an introduction to integral and rectifiable currents and geometric measure theory, see [23], and for more rigourous treatments see [13] and [27]. We will not make much use of the language of geometric measure theory here except to justify some of the assumptions made on special Lagrangian fibrations. The reader unfamiliar with this language should just keep in mind that passing from submanifolds to integral currents means extending the class of submanifolds to subsets with integer multiplicities attached, over which one can still integrate forms. There are natural compactness theorems for integral currents, enabling one to easily construct volume minimizing currents. One can then try to control the singularities of such currents. The biggest regularity result of this nature is Almgren's monumental

Theorem 1.2. *[1,2] Suppose N is an $m+l$ dimensional submanifold of $\mathbf{R}^{m+n}$ of class $k+2$ and that T is an m-dimensional rectifiable current in $\mathbf{R}^{m+n}$ which is absolutely area minimizing with respect to N. Then there is an open subset U of $\mathbf{R}^{m+n}$ such that $Supp(T)\cap U$ is an m-dimensional minimal submanifold of N of class k and the Hausdorff dimension of $Supp(T)-(U\cup Supp(\partial T))$ does not exceed $m-2$.*

The proof [2] is unpublished and is over 1500 pages long in preprint form. Thus an urgent question is

Question 1.3. *Are there nice regularity theorems for special Lagrangian currents?*

For example, [19] shows that holomorphic integral currents are obtained by integration over complex analytic subvarieties, and of course the singular locus of such varieties is well-behaved. This is much stronger than Almgren's result, which doesn't even guarantee finite $m-2$ dimensional Hausdorff measure of the singular set. We do not want the theory of special Lagrangian currents to have to depend on Almgren's result.

Next we should consider what is a reasonable definition of a special Lagrangian fibration. It might be reasonable to say that

Definition 1.4. *If B is a topological space and $f : X \to B$ is a continuous map, we say f is a special Lagrangian fibration if for all $b \in B$, $X_b := f^{-1}(b)$ is the support of a special Lagrangian integral current T with $\partial T = 0$.*

Even this might be too strong; one might insist that the fibres be special Lagrangian on a dense subset of B. This would allow some fibres to jump dimension. Nevertheless, we do not expect fibres to decrease in dimension as this would suggest the cohomology class of the general fibre X_b was trivial, which contradicts $\int_{X_b} \Omega \neq 0$. I do not want to give a rigorous argument of this sort here as it requires being clearer about concepts such as the dimension of the fibre. However it is clear that special Lagrangian fibrations cannot behave as many completely integrable Hamiltonian systems do, in which some fibres are tori of smaller dimension. On the other hand, it is not as clear that we want to rule out the possibility of fibres jumping up in dimension, something which often happens in algebro-geometric contexts. Nevertheless, we will stick to Definition 1.4.

We are interested in very specific sorts of special Lagrangian T^n fibrations. As argued in [14], §3, we are looking for special Lagrangian fibrations on Calabi-Yau manifolds near a specific large complex structure limit point in the boundary of complex moduli space, and the homology class of a fibre should be represented by a specific vanishing cycle associated to the boundary point. It was argued in [14], Observation 3.4 that, in the three dimensional case, if this homology class is primitive, then for *general* choice of complex moduli near the large complex structure limit point, all fibres of $f : X \to B$ must be irreducible. Let us be more precise. There is a notion of *indecomposable integral current* ([13], 4.2.25) which is analagous to the notion of irreducibility in algebraic geometry, and we will say a fibre $f^{-1}(b)$ is irreducible if the current T_b obtained by integrating forms over $f^{-1}(b)$ (with orientation induced by Ω) is indecomposable.

Definition 1.5. *A special Lagrangian fibration is integral if each fibre is irreducible and the currents $\{T_b | b \in B\}$ all represent the same integral homology class.*

This last condition rules out fibres which need to be thought of as multiple fibres. However, it is difficult to define multiplicity of a fibre in the context of a continuous map. Here the term "integral" is being used in its algebro-geometric sense: each fibre is irreducible and "reduced".

Now a special Lagrangian fibration need not be integral any more than an elliptic fibration need only have integral fibres. But this is the generic behaviour of special Lagrangian fibrations in dimensions ≤ 3, and the assumption of integrality vastly decreases the range of potential singular fibres. Without assumptions of integrality, it would be very difficult to relate the cohomology of a T^n-fibration and its dual. But one should keep in mind that even in dimension 3, integrality should fail for special values of the complex structure. Furthermore, if the homology class of a fibre is not primitive, then there is still the possibility that integrality will fail, and perhaps even multiple fibres appear. I have no argument to rule this out, but as we shall see in later sections, it is special Lagrangian fibrations with sections which are most important, and for these the homology class of a fibre is primitive.

The next natural question is whether we expect B itself to be a manifold. I would like to give a rough argument that this is a natural expectation if the fibration $f : X \rightarrow B$ is integral. Indeed, given a fibre X_b, X_b will be smooth at a general point $x \in X_b$, and in a neighborhood of x, using the exponential map, the deformations of X_b can be identified with deformations of X_b inside its normal bundle near x. Thus locally the fibre of the normal bundle of X_b at x yields a natural local section for $f : X \rightarrow B$, and hence gives a manifold structure on B. This construction hopefully in addtion yields a C^∞ structure on B. Thus we will feel justified in making the following assumption, to be in force throughout the remainder of the paper.

Assumption 1.6. *All special Lagrangian fibrations $f : X \rightarrow B$ will be C^∞ maps of C^∞ manifolds. Furthermore, f will be assumed to have a local section at each point $b \in B$. In addition, if f is assumed to be integral, we will assume that for any point $x \in X_b - Sing(X_b)$, there is a local C^∞ section passing through x.*

Again, if the fibration is not integral, I would not be surprised if singular B can arise naturally. Also, since the metric on X is real analytic, we can hope that with suitable coordinates on B, f will in fact be real analytic. We will not assume that here, however.

Next, we pass to the nature of the discriminant locus. Given f integral as in Assumption 1.6, we can now consider the rank of the differential $f_* : \mathcal{T}_{X,x} \rightarrow \mathcal{T}_{B,f(x)}$ at various points $x \in X$. At points with a local C^∞ section, $\operatorname{rank} f_* = n$. It will be shown in Proposition 2.2 that if $\operatorname{rank} f_* = \nu$, then the fibre in fact contains a submanifold of dimension ν on which $\operatorname{rank} f_* = \nu$. Thus the existence of points $x \in X$ for which $\operatorname{rank} f_* = n-1$ contradicts Almgren's theorem. Even if one doesn't accept Almgren's theorem, it is not difficult to rule out the existence of such points in low dimension using regularity results for minimizing hypersurfaces. Thus it is quite safe to assume that $\operatorname{rank} f_* \neq n-1$ for any point $x \in X$. This gives a stratification of the discriminant locus

$$\Delta_\nu = f(\{x \in X \mid \operatorname{rank} f_* : \mathcal{T}_{X,x} \rightarrow \mathcal{T}_{B,f(x)} \leq \nu\},$$

with $\Delta_0 \subseteq \cdots \subseteq \Delta_{n-2} = \Delta$, the discriminant locus. Then Federer's generalization of Sard's Theorem, [13], 3.4.3, states that $\mathcal{H}^{\nu+(2n-\nu)/k}(\Delta_\nu) = 0$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. In particular, if $k = \infty$, $\mathcal{H}^{\nu+\epsilon}(\Delta_\nu) = 0$ for all $\epsilon > 0$, so Δ_ν is of Hausdorff dimension $\leq \nu$. We need however stronger information than Hausdorff dimension to reach any cohomological conclusions. Thus the following assumptions will be used at various points of this paper; unlike Assumption 1.6, we will assume these only when we need them, mentioning them specifically.

Assumption 1.7. *For an integral special Lagrangian fibration $f : X \to B$, $b \in \Delta_\nu - \Delta_{\nu-1}$, there is a dense subset $\mathcal{L}$ of the real Grassmannian of $n-\nu$ dimensional subspaces of $\mathcal{T}_{B,b}$ such that one can find locally, for each $L \in \mathcal{L}$, a submanifold B' of B passing through b such that $\mathcal{T}_{B',b} = L$ and $B' \cap \Delta_\nu = \{b\}$.*

This requires reasonable regularity results about the discriminant, which certainly hold if $f : X \to B$ is real analytic, so that Δ_ν is a sub-analytic set. A stronger form of this assumption which we will need in §3 and will comment on there is

Assumption 1.7′. *In addition in Assumption 1.7, B' can be found so that $f^{-1}(B')$ is a submanifold in a neighborhood of the fibre X_b.*

Finally, we will require some assumptions on $Sing(X_b)$. Almgren's theorem that the Hausdorff dimension of $Sing(X_b)$ is no more than $n-2$ is not sufficiently strong for most purposes, and at the very least, we will frequently need to use

Assumption 1.8. *If $f : X \to B$ is an integral special Lagrangian fibration then*

$$H^i(Sing(X_b), \mathbf{Z}) = 0$$

for $i > n-2$, for all $b \in B$.

This is a restriction on homological dimension. If f is real analytic, then this should hold given Almgren's result.

§2. Action-angle Coordinates.

There is a standard theory of global action-angle coordinates due to Duistermaat [12]. We will extend this slightly so as to include information about the smooth part of the singular fibres. In this section, $f : X \to B$ denotes any special Lagrangian fibration satisfying Assumption 1.6. However, many of the results in this section apply to C^k Lagrangian fibrations $f : X \to B$ with B a manifold and f having a local section in a neighborhood of each point $b \in B$.

The first observation is that there is an action of $\mathcal{T}_B^*$, the cotangent bundle of B, on X.

Proposition-Definition 2.1. *There is a C^∞ action of $\Gamma(U, \mathcal{T}_B^*)$ on $f^{-1}(U)$ for any $U \subseteq B$, which we write for any $\alpha \in \Gamma(U, \mathcal{T}_B^*)$ as a map $T_\alpha : f^{-1}(U) \to f^{-1}(U)$. This satisfies the following properties:*

(1) If $d\alpha = 0$, then T_α is a symplectomorphism.

(2) T_α acts fibrewise, and $T_\alpha|_{f^{-1}(b)} : f^{-1}(b) \to f^{-1}(b)$ only depends on the value of α at b.

(3) $T_\alpha \circ T_\beta = T_{\alpha+\beta}$.

Proof. This is standard: see for example [12], §1, or [3], Chap. 10. We review the definition of these maps however. If α is a compactly supported 1-form on Y, then $f^*\alpha$ is a compactly supported 1-form on $f^{-1}(U)$. There is then a vector field v_α on $f^{-1}(U)$ with $\iota(v_\alpha)\omega = f^*\alpha$. This generates a flow $\phi_t : f^{-1}(U) \to f^{-1}(U)$ for all t, and we take $T_\alpha = \phi_1$. It is then standard that if $d\alpha = 0$, v is locally Hamiltonian and ϕ_t then is a 1-parameter family of symplectomorphisms.

If locally $d\alpha = 0$, we can write $\alpha = dH$ for some function H on B, and if G is any other function on B, $\{G \circ f, H \circ f\} = 0$ since $f : X \to B$ is a Lagrangian fibration. But then ϕ_t is the Hamiltonian flow associated to H, and

$$0 = \{G \circ f, H \circ f\}(x) = \frac{d}{dt}\bigg|_{t=0} (G \circ f)(\phi_t(x))$$

so $G \circ f$ is a constant on $\phi_t(x)$ for any x. Thus ϕ_t acts on fibres. Clearly $v_\alpha|_{f^{-1}(b)}$ depends only on the value of α at b, so the action of ϕ_t on $f^{-1}(b)$ depends only on the value of α at b. In particular ϕ_t acts on fibres for arbitrary α, not just compactly supported α. •

Next, a standard analysis of the orbits of this action.

Proposition 2.2. *If $b \in B$, $x \in f^{-1}(b)$, then the orbit $\{T_\alpha(x) | \alpha \in T^*_{B,b}\}$ is diffeomorphic to $\mathbf{R}^l \times T^s$ and $l + s$ coincides with $rank(f^* : \mathcal{T}^*_{B,b} \to \mathcal{T}^*_{X,x}) = rank(f_* : \mathcal{T}_{X,x} \to \mathcal{T}_{B,b})$.*

Proof. If $\alpha \in \mathcal{T}^*_{B,b}$ is in the kernel of f^*, then v_α is zero at x, so $T_\alpha(x) = x$. Thus for any $\alpha \in \mathcal{T}^*_{B,b}$, $T_\alpha(x)$ depends only on α modulo $\ker f^*$, and the orbit of x is homeomorphic to a quotient of $V = \mathcal{T}^*_{B,b}/\ker f^*$ by a subgroup Γ, via the map $\alpha \in V \mapsto T_\alpha(x)$. The differential of this map at $0 \in V$ is injective, and hence the map $V \to X$ is a diffeomorphism of an open neighborhood of $0 \in V$ with its image in X. Thus Γ is a discrete subgroup of V, and the orbit of x is diffeomorphic to V/Γ. •

Suppose first that $f : X \to B$ is equipped with a C^∞ section $\sigma_0 : B \to X$. We define a C^∞ map $\pi : \mathcal{T}^*_B \to X$ by, for $\alpha \in \mathcal{T}^*_{B,b}$, $\pi(\alpha) = T_\alpha(\sigma_0(b))$. The image of the zero section of $\mathcal{T}^*_B$ is $\sigma_0(B)$. Let $\Lambda \subseteq \mathcal{T}^*_B$ be given by $\Lambda = \pi^{-1}(\sigma_0(B))$. Then $\Lambda_b \subseteq \mathcal{T}^*_{B,b}$ is the discrete subgroup of the vector space $\mathcal{T}^*_{B,b}$ given by $\Lambda_b = \pi^{-1}(\sigma_0(b))$. Let $X_0^\#$ be the image of the map π, $f^\# : X_0^\# \to B$ the projection. Then clearly Λ_b is canonically isomorphic to $H_1((f^\#)^{-1}(b), \mathbf{Z})$, which is isomorphic to $H_c^{n-1}((f^\#)^{-1}(b), \mathbf{Z})$ by Poincaré duality. Here we use the fact that f is special Lagrangian to give a canonical orientation on the fibres of $f^\#$. Also, Λ, as a subset of the total space $\mathcal{T}^*_B$, is closed as π is continuous. Since the map π is a local isomorphism, Λ is also étale over B. Thus we can think of Λ as the éspace étalé of $R_c^{n-1} f_*^\# \mathbf{Z}$, and in particular, we obtain an exact sequence of sheaves of abelian groups

$$0 \to R_c^{n-1} f_*^\# \mathbf{Z} \to \mathcal{T}^*_B \to X_0^\# \to 0,$$

where this now defines the group structure on $X_0^\#$. Observe also that since Λ is étale over B, $R_c^{n-1} f_*^\# \mathbf{Z}$ has no sections with support in proper closed subsets of B. In particular, $\underline{H}^0_\Delta(B, R_c^{n-1} f_*^\# \mathbf{Z}) = 0$.

We recall the notion of canonical coordinates on the total space of the bundle $\mathcal{T}^*_B$. Given $U \subseteq B$ an open set with coordinates $y_1, \ldots, y_n$, *canonical coordinates* on $\mathcal{T}^*_U$ are $y_1, \ldots, y_n, x_1, \ldots, x_n$, where $(y_1, \ldots, y_n, x_1, \ldots, x_n)$ is the coordinate representation of the differential form $\sum x_i dy_i \in \mathcal{T}^*_{U,(y_1,\ldots,y_n)}$. We will use canonical coordinates consistently throughout this paper, whenever we use local coordinates to perform calculations. Note that $\mathcal{T}_B^*$ always carries a standard symplectic form, which in canonical coordinates can be written as $\sum_{i=1}^n dx_i \wedge dy_i$. (This is the opposite of some sign conventions).

Proposition 2.3. *With notation as above, let $y_1, \ldots, y_n$ be local coordinates on a neighborhood $U \subseteq B$. Then on $\mathcal{T}_U^*$,*

$$\pi^*\omega = \sum_i dx_i \wedge dy_i + \sum_{i,j} a_{ij} dy_i \wedge dy_j$$

where the a_{ij} are functions depending only on $y_1, \ldots, y_n$. Furthermore, if $\sigma_0 : B \rightarrow X$ is a Lagrangian section, then

$$\pi^*\omega = \sum_i dx_i \wedge dy_i$$

on U, and thus $\pi^*\omega$ is the standard symplectic form on $\mathcal{T}_B^*$. Finally, if $H^2(B, \mathbf{R}) = 0$, then every section of f is homotopic to a Lagrangian section.

Proof. Since the fibres of $\mathcal{T}_B^* \rightarrow B$ are Lagrangian with respect to $\pi^*\omega$, we can locally write

$$\pi^*\omega = \sum_i dx_i \wedge \theta_i + \sum_{i,j} a_{ij} dy_i \wedge dy_j$$

with the θ_i 1-forms not involving the dx_i's and a_{ij} functions on $\mathcal{T}_B^*$. Now the function y_i induces on X a Hamiltonian vector field which, by definition of the map π, must be $\pi_* \partial/\partial x_i$. Thus $\iota(\partial/\partial x_i)(\pi^*\omega) = dy_i$, from which we see that $\theta_i = dy_i$.

The condition $d(\pi^*\omega) = 0$ then implies that the functions a_{ij} are independent of $x_1, \ldots, x_n$.

If σ_0 is Lagrangian with respect to ω, then the zero section of $\mathcal{T}_B^*$ is Lagrangian, from which we see that $a_{ij} = 0$.

If σ_0 is not Lagrangian, let $\omega' = \sum_i dx_i \wedge dy_i$ (locally) be the standard symplectic form on $\mathcal{T}_B^*$. Then $\pi^*\omega - \omega'$ is a closed 2-form locally given by $\sum_{i,j} a_{ij} dy_i \wedge dy_j$, and hence is the pull-back of a closed 2-form on B. Thus if $H^2(B, \mathbf{R}) = 0$, there exists a one-form θ on B with $d\theta = \pi^*\omega - \omega'$. Then $-\theta$ defines a section of $\mathcal{T}_B^*$ which is Lagrangian with respect to $\pi^*\omega$, and this maps to a Lagrangian section of f homotopic to σ_0. •

This allows us to prove a result stated in [14]. Recall that $X^\#$ is the complement of the critical locus of f in X.

Theorem 2.4. *(Theorem 3.6 of [14]) Let X be a Calabi-Yau n-fold, B a smooth real n-dimensional manifold, with $f : X \rightarrow B$ a C^∞ special Lagrangian torus fibration such that $R^n f_* \mathbf{Q} = \mathbf{Q}_B$ and such that the singular locus of each singular fibre has cohomological dimension $\leq n-2$. Suppose furthermore that f has a C^∞ section σ_0. Then $X^\#$ has the structure of a fibre space of groups with σ_0 the zero section. In fact there is an exact sequence of sheaves of abelian groups*

$$0 \rightarrow R^{n-1} f_* \mathbf{Z} \rightarrow \mathcal{T}_B^* \rightarrow X^\# \rightarrow 0.$$

Given a section $\sigma \in \Gamma(U, X^{\#})$, one obtains a C^∞ diffeomorphism $T_\sigma : f^{-1}(U) \cap X^{\#} \to f^{-1}(U) \cap X^{\#}$ given by $x \mapsto x + \sigma(f(x))$, and this diffeomorphism extends to a diffeomorphism $T_\sigma : f^{-1}(U) \to f^{-1}(U)$.

Proof. This follows from the previous discussion if we can show that the hypotheses imply two things:
(1) $X^{\#} = X_0^{\#}$
(2) $R_c^{n-1} f_*^{\#}\mathbf{Z} \cong R^{n-1} f_*\mathbf{Z}$.

To show (1) and (2), let X_b be a fibre, and let $Z \subseteq X_b$ be the singular locus of $Supp(X_b)$. We then have an exact sequence

$$\begin{aligned} &H^{n-2}(Z, \mathbf{Q}) \to H_c^{n-1}(X_b - Z, \mathbf{Q}) \to H^{n-1}(X_b, \mathbf{Q}) \\ &\qquad \to H^{n-1}(Z, \mathbf{Q}) = 0 \to H_c^n(X_b - Z, \mathbf{Q}) \to H^n(X_b, \mathbf{Q}) \to 0. \end{aligned}$$

Here $H^{n-1}(Z, \mathbf{Q}) = 0$ by the assumption on the cohomological dimension of Z. Thus, since $H^n(X_b, \mathbf{Q}) = \mathbf{Q}$ by assumption, $X_b - Z$ can have only one connected component. Since the $\mathcal{T}^*_{B,b}$-orbit of $\sigma_0(b)$ is already one connected component of $X_b - Z$, we see that $X_b - Z = (X_0^{\#})_b$. But as $X_b^{\#} \subseteq X_b - Z$, we must have $X_0^{\#} = X^{\#}$.

We also see from the above exact sequence that there is a surjection

$$R_c^{n-1} f_*^{\#}\mathbf{Z} \to R^{n-1} f_*\mathbf{Z} \to 0.$$

This is an isomorphism outside of Δ. But since $\underline{H}^0_\Delta(B, R_c^{n-1} f_*^{\#}\mathbf{Z}) = 0$, we conclude this surjection is in fact an isomorphism. •

Note that the hypotheses of Theorem 2.4 hold if $f : X \to B$ is integral and satisfies Assumption 1.8. Another useful observation about the topology of singular fibres:

Lemma 2.5. *If $f : X \to B$ is integral and satisfies Assumption 1.8, and $b \in B$ with $Z = Sing(X_b)$, then Z is connected.*

Proof. Let $U = X_b - Z$. As argued above, $U \cong \mathbf{R}^k \times T^{n-k}$ for some k. Assuming Z is non-empty, we have an exact sequence

$$0 = H_c^0(U, \mathbf{Z}) \to H^0(X_b, \mathbf{Z}) \to H^0(Z, \mathbf{Z}) \to H_c^1(U, \mathbf{Z}) \to H^1(X_b, \mathbf{Z})$$

and we wish to show $H^0(Z, \mathbf{Z}) = \mathbf{Z}$, which is equivalent to the injectivity of $H_c^1(U, \mathbf{Z}) \to H^1(X_b, \mathbf{Z})$. If $H_c^1(U, \mathbf{Z}) = 0$, then there is no problem, so the only possiblity is that $k = 1$ and $H_c^1(U, \mathbf{Z}) = \mathbf{Z}$. In this case, $H_c^{n-1}(U, \mathbf{Z}) = \mathbf{Z}^{n-1}$ and thus in a small neighborhood $V \subseteq B$ of b, the sheaf $R_c^{n-1} f_*^{\#}\mathbf{Z}$ contains $\mathbf{Z}^{n-1}$ as a subsheaf. Thus over V, there is a T^{n-1} bundle $T \to B$, $T \subseteq X^{\#}$. The fundamental class of each fibre yields a non-zero section σ of $R_c^1 f_*^{\#}\mathbf{Z}$ over B, and under the map $R_c^1 f_*^{\#}\mathbf{Z} \to R^1 f_*\mathbf{Z}$, σ maps to a section of $R^1 f_*\mathbf{Z}$ which is non-zero on $V - \Delta$. But the map $H_c^1(U, \mathbf{Z}) \to H^1(X_b, \mathbf{Z})$ is the induced map on stalks, and hence is non-zero, thus injective. •

We now address the situation when $f : X \to B$ does not have a Lagrangian section but we assume f is integral. We recall though that Assumption 1.6 is always in force, giving the existence of a local section. This theory was developed in Duistermaat's paper [12], and is completely analagous to Kodaira's theory of elliptic surfaces, or to Ogg-Shafarevich theory.

Let $X^\# = X - Crit(f)$ as usual. We now obtain a map

$$\psi : R^{n-1}_c f^\#_* \mathbf{Z} \to \mathcal{T}^*_B$$

as follows. Given a fibre $X^\#_b$, $\gamma \in H_1(X^\#_b, \mathbf{Z}) \cong H^{n-1}_c(X^\#_b, \mathbf{Z})$, map γ to the differential

$$v \mapsto -\int_\gamma \iota(v)\omega$$

where we choose any lifting of $v \in \mathcal{T}_{B,b}$ to $X^\#$.

Now in the case f did have a section, we previously constructed an embedding $R^{n-1}_c f^\#_* \mathbf{Z} \hookrightarrow \mathcal{T}^*_B$. We compare these two constructions. Let $U \subseteq B$ be an open set where $f^{-1}(U) \to U$ possesses a section, which we can take to be Lagrangian. Using this section as the zero section we obtain an exact sequence

$$0 \to \Lambda = R^{n-1}_c f^\#_* \mathbf{Z}|_U \xrightarrow{\psi'} \mathcal{T}^*_U \xrightarrow{\pi} f^{-1}(U)^\# \to 0$$

as before. Now if $\lambda \in \Lambda_b = H^{n-1}_c(X^\#_b, \mathbf{Z}) = H_1(X^\#_b, \mathbf{Z}) \subseteq \mathcal{T}^*_{B,b}$ via the map ψ', then in local coordinates $(y_1, \ldots, y_n)$ on B, $\lambda = \sum \lambda_i dy_i$. Now

$$\begin{aligned} -\int_\lambda \iota(\partial/\partial y_i)\omega &= -\int_{(0,\ldots,0)}^{(\lambda_1,\ldots,\lambda_n)} \iota(\partial/\partial y_i)\pi^*\omega \\ &= \int_{(0,\ldots,0)}^{(\lambda_1,\ldots,\lambda_n)} dx_i \\ &= \lambda_i \end{aligned}$$

so the two maps $\psi, \psi' : R^{n-1}_c f^\#_* \mathbf{Z}|_U \to \mathcal{T}^*_U$ coincide. Since $\Lambda \subseteq \mathcal{T}^*_U$ is Lagrangian, we conclude that the image of $R^{n-1}_c f^\#_* \mathbf{Z}$ in $\mathcal{T}^*_B$ under ψ is Lagrangian. Thus there is an exact sequence

$$0 \to R^{n-1}_c f^\#_* \mathbf{Z} \to \mathcal{T}^*_B \to J^\# \to 0$$

defining $J^\#$ in which $J^\#$ inherits the standard symplectic form from $\mathcal{T}^*_B$. Since $f : X^\# \to B$ is locally isomorphic to $J^\# \to B$, a standard argument [12] shows that one can obtain $X^\# \to B$ from $J^\# \to B$ by regluing using a Čech 1-cocycle $\{(U_i, \sigma_i)\}$ where σ_i is a Lagrangian section of $J^\# \to B$ over U_i. We call $j : J^\# \to B$ the *Jacobian fibration* of $f : X^\# \to B$, in analogy with the theory of elliptic curves. This gives a one-to-one correspondence between the group $H^1(B, \Lambda(J^\#))$, where $\Lambda(J^\#)$ is the sheaf of Lagrangian sections of $J^\#$, and the set

$$\{f : Y^\# \to B \text{ a Lagrangian fibration with local section and Jacobian } j : J^\# \to B\}/\cong.$$

In fact, this can be extended to the compactifications. We phrase this more generally for Lagrangian fibrations.

Theorem 2.6. *Let $f : X \to B$ be a proper Lagrangian fibration with connected fibres, with a local section everywhere. Then there is a symplectic manifold J, called the Jacobian of f, and a (proper) Lagrangian fibration $j : J \to B$ which is locally isomorphic to $f : X \to B$, and which has a Lagrangian section. Furthermore, there is a one-to-one correspondence between the sets*

$$\{f : Y \to B \text{ a Lagrangian fibration with local section and Jacobian } j : J \to B\}/\cong$$

and $H^1(B, \Lambda(J^\#))$.

Proof. To construct J, choose an open covering $\{U_i\}$ such that $f^{-1}(U_i) \to U_i$ has a Lagrangian section σ_i for each i. Then $f^{-1}(U_i \cap U_j)$ has a symplectomorphism we will write as $T_{i,j}$ obtained by treating σ_i as the zero-section and then translating by σ_j, so that $T_{i,j}$ takes σ_i into σ_j. We construct J by identifying $f^{-1}(U_i)$ and $f^{-1}(U_j)$ along $f^{-1}(U_i \cap U_j)$, using $T_{i,j}$ to identify $f^{-1}(U_i \cap U_j) \subseteq f^{-1}(U_i)$ and $f^{-1}(U_i \cap U_j) \subseteq f^{-1}(U_j)$. These identifications are compatible over $U_i \cap U_j \cap U_k$ because $T_{j,k} \circ T_{i,j} = T_{i,k}$. Thus we obtain a Lagrangian fibration $j : J \to B$ with a section, as desired. The usual regluing construction gives the 1-1 correspondence. •

For $j : J \to B$ special Lagrangian and integral, one computes $H^1(B, \Lambda(J^\#))$ by using the exact sequence

$$0 \to R_c^{n-1} j_*^\# \mathbf{Z} \to \Lambda(\mathcal{T}_B^*) \to \Lambda(J^\#) \to 0.$$

$\Lambda(\mathcal{T}_B^*)$ is just the kernel of exterior differentiation acting on $\mathcal{T}_B^*$, so $H^i(B, \Lambda(\mathcal{T}_B^*)) = H^{i+1}(B, \mathbf{R})$ for $i \geq 1$. From this we obtain the sequence

$$H^2(B, \mathbf{R}) \to H^1(B, \Lambda(J^\#)) \to H^2(B, R_c^{n-1} j_*^\# \mathbf{Z}) \to H^3(B, \mathbf{R}).$$

In any dimension, if $H^2(B, \mathbf{R}) = 0$, then $H^2(B, R_c^{n-1} j_*^\# \mathbf{Z})_{tors} \subseteq H^1(B, \Lambda(J^\#))$. Duistermaat [12] observed that if an element $\alpha \in H^1(B, \Lambda(J^\#))$ comes from an element $[\alpha'] \in H^2(B, \mathbf{R})$, then the corresponding $f : X \to B$ can be obtained by choosing a 2-form α' on B representing $[\alpha']$, and taking $X = J$ with symplectic form $\omega + f^*\alpha'$. Any two choices of α' can be related by translation by a section.

Example 2.7. If $n = 3$ and $H^2(B, \mathbf{R}) = 0$, then we have

$$H^2(B, R_c^2 f_*^\# \mathbf{Z})_{tors} \subseteq H^1(B, \Lambda(J^\#)) \subseteq H^2(B, R_c^2 f_*^\# \mathbf{Z}).$$

If $n = 2$ and J is a K3 surface, then $H^2(B, R_c^1 f_*^\# \mathbf{Z}) = 0$ and so there is a sequence

$$H^1(B, R_c^1 f_*^\# \mathbf{Z}) \to H^2(B, \mathbf{R}) \to H^1(B, \Lambda(J^\#)) \to 0.$$

Remark 2.8. If $U \subseteq B - \Delta$ is a simply connected set, then there is no monodromy in the local system $(R^{n-1} f_* \mathbf{Z})|_U \subseteq \mathcal{T}_U^*$. Thus, if $\lambda_1, \ldots, \lambda_n$ are sections of $\mathcal{T}_U^*$ generating $(R^{n-1} f_* \mathbf{Z})|_U$, the fact that $d\lambda_i = 0$ shows there are functions u_i such that $du_i = \lambda_i$ on U. The $u_1, \ldots, u_n$ form local coordinates on U, since $du_1, \ldots, du_n$ are independent. This is the standard construction of action coordinates on U. We will say that $u_1, \ldots, u_n$ are *action coordinates* for $f : X \to B$ on U. Canonical coordinates $u_1, \ldots, u_n, x_1, \ldots, x_n$ are called action-angle coordinates. These are also the coordinates that Hitchin introduces after [18], Prop. 1. The advantage of working in this coordinate system is that the periods are now just the constant periods $du_1, \ldots, du_n$.

§3. Simplicity and the Leray spectral sequence revisited.

Recall from [14] that a special Lagrangian T^n-fibration $f : X \to B$ was said to be *G-simple* if

$$i_* R^p f_{0*} G = R^p f_* G$$

for all p, where $i : B - \Delta \to B$ is the inclusion, $f_0 = f|_{f^{-1}(B-\Delta)}$, and G is an abelian group. This condition was crucial for getting a handle on the topology of X and the relationship between the topology of X and its dual. In [14], we only made use of $\mathbf{Q}$-simplicity, while here, we will go further. We are interested in a broader range of groups G. In particular, $G = \mathbf{R}, \mathbf{Q}, \mathbf{Z}, \mathbf{Z}/m\mathbf{Z}$, or $\mathbf{R}/\mathbf{Z}$ will be of relevance for us. Clearly $\mathbf{Z}$-simplicity implies $\mathbf{Q}$-simplicity or $\mathbf{R}$-simplicity, but $\mathbf{Z}/m\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$-simplicity provides extra information about monodromy modulo m which may be valuable in trying to classify possible monodromy transformations about the discriminant locus.

If f is integral, then f has connected fibres, and $G = f_* G = i_* f_{0*} G$. We have seen in §2 that if Assumption 1.8 holds then $R^n f_* \mathbf{Z} = R^n_c f^\#_* \mathbf{Z} = \mathbf{Z}$, and thus by the universal coefficient theorem, $R^n f_* G = G$. Thus for integral fibrations satisfying Assumption 1.8, the simplicity condition holds for $p = 0$ and n and any abelian group G.

We now prove it holds for $p = 1$, under the additional Assumption 1.7′.

Definition 3.1. *We say a point $b \in B$ is a rank k point if*

$$k = \min_{x \in X_b} \operatorname{rank} f_* : \mathcal{T}_{X,x} \to \mathcal{T}_{B,b}.$$

In particular b is rank n if and only if f is smooth over b.

We first comment on when Assumption 1.7′ might hold.

Lemma 3.2. *If b is a rank k point, $k \neq 0$, then Assumption 1.7 implies Assumption 1.7′ at b if X_b contains only a finite number of orbits of the action of $\mathcal{T}^*_{B,b}$. Note Assumption 1.7 automatically implies Assumption 1.7′ at a rank 0 point.*

Proof. Consider the set $\mathcal{L}' = \{\operatorname{Im} f_* : \mathcal{T}_{X,x} \to \mathcal{T}_{B,b} | x \in X_b\}$ of subspaces of $\mathcal{T}_{B,b}$. Note that if $x, y \in X_b$ with $T_\sigma(x) = y$ for some σ, then $f_{*,x} = f_{*,y} \circ (T_\sigma)_*$ where $f_{*,x}$ denotes the pushforward of tangent vectors at x. Thus $\operatorname{Im} f_{*,x} = \operatorname{Im} f_{*,y}$. By the assumption that X_b is a union of a finite number of orbits, we see then that $\mathcal{L}'$ is finite, and $\min_{L \in \mathcal{L}'} \dim L = k$. Because $\mathcal{L}'$ is finite we can choose a subspace $T \subseteq \mathcal{T}_{B,b}$ of dimension $n - k$ in the set $\mathcal{L}$ given by Assumption 1.7 which intersects every element of $\mathcal{L}'$ in the expected dimension. It then follows that $\dim f_{*,x}^{-1}(T) = 2n - k$ for all $x \in X_b$. Now if B' is taken to be a submanifold of B with tangent space T at b, we see that the implicit function theorem implies that $f^{-1}(B')$ is a manifold in a neighborhood of X_b. •

This makes Assumption 1.7′ appear quite reasonable, at least in the three-dimensional case, as the finiteness of the number of orbits for rank 1 fibres would follow from finiteness of the $\mathcal{H}^1$-measure of the singular locus. (As mentioned in §1 however, such a result is not actually known.)

Theorem 3.3. *If $f : X \to B$ is an integral special Lagrangian fibration satisfying Assumption 1.7′, then $i_*R^1f_{0*}G = R^1f_*G$.*

Proof. Let

$$\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_{n-2} = \Delta \subseteq B$$

be the stratification of Δ given in §1. Let $i_k : B - \Delta_k \to B - \Delta_{k-1}$ be the inclusion. We will show using descending induction that

$$i_{k*}(R^1f_*G)|_{B-\Delta_k} = (R^1f_*G)|_{B-\Delta_{k-1}}$$

for each k. One always has a functorial map

$$(R^1f_*G)|_{B-\Delta_{k-1}} \to i_{k*}i_k^*(R^1f_*G)|_{B-\Delta_{k-1}} = i_{k*}(R^1f_*G)|_{B-\Delta_k},$$

and we just need to show this is an isomorphism on the level of stalks at each point $b \in \Delta_k$, $b \notin \Delta_{k-1}$.

We can choose a B' through the point b using Assumption 1.7′, so that $\Delta_k \cap B' = \{b\}$ and $X' = f^{-1}(B')$ is a manifold. This gives a diagram

$$\begin{array}{ccc} B' - \{b\} & \xrightarrow{i'} & B' \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ B - \Delta_k & \xrightarrow{i_k} & B - \Delta_{k-1} \end{array}$$

i', j', j the inclusions. Let $f' : X' \to B'$ be the restriction of f. Then $j^*R^1f_*G = R^1f'_*G$ and in particular the stalks of R^1f_*G and $R^1f'_*G$ at b are the same. On the other hand, there is a natural map $(i_{k*}i_k^*R^1f_*G)_b \to (i'_*i'^*R^1f'_*G)_b$. Indeed, an element of $(i_{k*}i_k^*R^1f_*G)_b$ represented by a germ (U, α), $\alpha \in \Gamma(U - \Delta_k, R^1f_*G)$, is mapped to $(U \cap B', \alpha|_{(U\cap B')-\{b\}})$, $\alpha|_{(U\cap B')-\{b\}} \in \Gamma((U \cap B') - \{b\}, R^1f'_*G)$. This map is in fact injective. Indeed, by descending induction, $(R^1f_*G)|_{B-\Delta_k}$ has no sections over any open subset of $B - \Delta_k$ supported on a proper closed subset. Thus the restriction maps of the sheaf $(R^1f_*G)|_{B-\Delta_k}$ are injective, and it follows that the map $(i_{k*}i_k^*R^1f_*G)_b \to (i'_*i'^*R^1f'_*G)_b$ is injective. We then have a diagram

$$\begin{array}{ccc} (R^1f_*G)_b & \xrightarrow{\cong} & (R^1f'_*G)_b \\ \downarrow & & \downarrow \\ (i_{k*}i_k^*R^1f_*G)_b & \hookrightarrow & (i'_*i'^*R^1f_*G)_b \end{array}.$$

so $(R^1f_*G)_b \to (i_{k*}i_k^*R^1f_*G)_b$ is an isomorphism if $(R^1f'_*G)_b \to (i'_*i'^*R^1f'_*G)_b$ is. Thus we need to show that

$$\varinjlim_{b\in U\subseteq B'} H^1(f'^{-1}(U), G) \to \varinjlim_{b\in U\subseteq B'} \Gamma(U - \{b\}, R^1f'_*G)$$

is an isomorphism. We can take the direct limit over contractible U.

We will need to know what $H^2_{X_b}(X', G)$ is. By the universal coefficient theorem, there is an exact sequence

$$0 \to H^i_{X_b}(X', \mathbf{Z}) \otimes_{\mathbf{Z}} G \to H^i_{X_b}(X', G) \to \mathrm{Tor}^1_{\mathbf{Z}}(H^{i+1}_{X_b}(X', \mathbf{Z}), G) \to 0.$$

Now by a suitable form of Poincaré duality ([10], V 9.3),

$$H^i_{X_b}(X', \mathbf{Z}) \cong H_{2n-k-i}(X_b, \mathbf{Z}),$$

where the latter group is Borel-Moore homology. This is computed via the exact sequence ([10] V, §3, (9))

$$0 \to \mathrm{Ext}^1(H^{2n-k-i+1}_c(X_b, \mathbf{Z}), \mathbf{Z}) \to H_{2n-k-i}(X_b, \mathbf{Z}) \to \mathrm{Hom}(H^{2n-k-i}_c(X_b, \mathbf{Z}), \mathbf{Z}) \to 0.$$

Given that $H^n_c(X_b, \mathbf{Z}) = \mathbf{Z}$ and $H^{n-1}_c(X_b, \mathbf{Z}) = H^{n-1}_c(X_b^\#, \mathbf{Z})$ is free, we see that

$$H^2_{X_b}(X', \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } k = n-2; \\ 0 & \text{if } k < n-2, \end{cases}$$

and $H^3_{X_b}(X', \mathbf{Z})$ is free, so that

$$H^2_{X_b}(X', G) = \begin{cases} G & \text{if } k = n-2; \\ 0 & \text{if } k < n-2. \end{cases}$$

Of course $H^i_{X_b}(X', G) = 0$ for $i < 2$.

There are two cases:

Case 1: $k < n-2$. For $b \in U \subseteq B'$, $H^1(f'^{-1}(U), G) \cong H^1(f'^{-1}(U - \{b\}), G)$, by the relative cohomology long exact sequence. On the other hand, the Leray spectral sequence for $f' : X^* = f'^{-1}(U - \{b\}) \to B^* = U - \{b\}$ yields the exact sequence

$$0 = H^1(B^*, G) \to H^1(X^*, G) \to H^0(B^*, R^1 f'_* G) \to H^2(B^*, G) \to H^2(X^*, G).$$

This last map is injective as $X^* \to B^*$ can be assumed to have a section. Thus we obtain the isomorphism

$$H^1(f'^{-1}(U), G) \to H^0(B^*, R^1 f'_* G),$$

and taking direct limts gives the desired isomorphism.

Case 2. $k = n-2$. In this case, the relative cohomology exact sequence gives for $b \in U \subseteq B'$, $X^* = f'^{-1}(U - \{b\}), B^* = U - \{b\}$,

$$0 \to H^1(f'^{-1}(U), G) \to H^1(X^*, G) \to H^2_{X_b}(f'^{-1}(U), G) \to H^2(f'^{-1}(U), G).$$

In this case $H^2_{X_b}(f'^{-1}(U), G) = G$. We have a commutative square

$$\begin{array}{ccc} H^1(B^*, G) & \xrightarrow{\cong} & H^2_{\{b\}}(U, G) \\ \downarrow & & \downarrow \cong \\ H^1(X^*, G) & \longrightarrow & H^2_{X_b}(f'^{-1}(U), G) \end{array}$$

showing the map $H^1(X^*,G) \to H^2_{X_b}(f'^{-1}(U),G)$ is surjective, yielding

$$0 \to H^1(f'^{-1}(U),G) \to H^1(X^*,G) \to G \to 0.$$

On the other hand, the Leray spectral sequence for $f' : X^* \to B^*$ gives

$$0 \to H^1(B^*,G) \cong G \to H^1(X^*,G) \to H^0(B^*, R^1 f'_* G) \to 0.$$

Putting these two sequences together one finds that the map

$$H^1(f'^{-1}(U),G) \to H^0(B^*, R^1 f'_* G)$$

is an isomorphism, and hence

$$\varinjlim H^1(f'^{-1}(U),G) \to \varinjlim \Gamma(U - \{b\}, R^1 f'_* G)$$

is an isomorphism. •

We next try to understand $R^{n-1}f_*\mathbf{Z}$. In any event, in §2 we have seen that if $f : X \to B$ is integral and satisfies Assumption 1.8 then $R^{n-1}_c f^{\#}_* \mathbf{Z} \cong R^{n-1} f_* \mathbf{Z}$ and $\underline{H}^0_\Delta(B, R^{n-1}_c f^{\#}_* \mathbf{Z}) = 0$. Since there is an exact sequence

$$0 \to \underline{H}^0_\Delta(B, R^{n-1} f_* \mathbf{Z}) \to R^{n-1} f_* \mathbf{Z} \to i_* R^{n-1} f_{0*} \mathbf{Z} \to \underline{H}^1_\Delta(B, R^{n-1} f_* \mathbf{Z}) \to 0,$$

we have already shown at least that the natural map $R^{n-1} f_* \mathbf{Z} \to i_* R^{n-1} f_{0*} \mathbf{Z}$ is injective. However, to show surjectivity, we need a more delicate understanding of the inductive structure of the singular locus. If $b \in B$ is a rank k point, it would, for example, be sufficient to show that there is a fixed-point-free Hamiltonian T^k action in a neighborhood of X_b in order to achieve a sufficiently strong inductive description of the singular fibres. However failing to prove such a result, we will make an ad hoc argument in the $n = 3$ case. Nevertheless, in any dimension we have

Lemma 3.4. *If f is integral and satisfies Assumptions 1.7 and 1.8, Δ_0 the set of rank 0 points of B, $i_0 : B - \Delta_0 \hookrightarrow B$ the inclusion, then*

$$i_{0*} i_0^* R^{n-1} f_* \mathbf{Z} = R^{n-1} f_* \mathbf{Z}.$$

Proof. If U is a contractible open neighborhood of $b \in \Delta_0$, we have an exact sequence

$$0 \to H^{n-1}(f^{-1}(U),\mathbf{Z}) \to H^{n-1}(f^{-1}(U - \{b\}),\mathbf{Z}) \to H^n_{X_b}(f^{-1}(U),\mathbf{Z}) \cong \mathbf{Z} \to 0.$$

In addition, the map

$$\varinjlim_{b \in U} H^{n-1}(f^{-1}(U),\mathbf{Z}) \to \varinjlim_{b \in U} \Gamma(U - \{b\}, R^{n-1} f_* \mathbf{Z})$$

is injective. We just need to show this map is surjective. ¿From the Leray spectral sequence for $f : X^* = f^{-1}(U - \{b\}) \to B^* = U - \{b\}$, we obtain a map φ :

$H^{n-1}(X^*,\mathbf{Z}) \to \Gamma(B^*, R^{n-1}f_*\mathbf{Z})$. We first show this map is surjective. Indeed, given a section $\sigma \in \Gamma(B^*, R^{n-1}f_*\mathbf{Z}) \subseteq \Gamma(B^*, \mathcal{T}_B^*)$, let $M \subseteq X^*$ be the circle bundle over B^* whose fibre at $b \in B^*$ is $\mathbf{R}\sigma(b)/\mathbf{Z}\sigma(b) \subseteq X_b^\# = \mathcal{T}^*_{B,b}/(R^{n-1}f_*\mathbf{Z})_b$. Let $[M] \in H^{n-1}(X^*,\mathbf{Z})$ be the image of $1 \in H^{n-1}_M(X^*,\mathbf{Z}) \cong H^0(M,\mathbf{Z})$ in $H^{n-1}(X^*,\mathbf{Z})$. It is then clear that $\varphi([M]) = \sigma$. Thus φ is surjective.

We now have a diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \lim_\to H^{n-1}(f^{-1}(U),\mathbf{Z}) & \xrightarrow{\beta} & \lim_\to H^{n-1}(X^*,\mathbf{Z}) & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \longrightarrow & \operatorname{Im}\varphi\circ\beta & \xrightarrow{\alpha} & \lim_\to \Gamma(B^*, R^{n-1}f_*\mathbf{Z}) & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

and we wish to show that $\operatorname{coker}\alpha = 0$, so that $\alpha\circ\varphi' = \varphi\circ\beta$ is surjective as desired. By the snake lemma, $\ker\varphi \cong \ker\varphi''$. By the Leray spectral sequence for $f: X^* \to B^*$, the image of $H^{n-1}(B^*,\mathbf{Z}) \to H^{n-1}(X^*,\mathbf{Z})$ is contained in $\ker\varphi$. On the other hand, by the commutativity of

$$\begin{array}{ccc} H^{n-1}(B^*,\mathbf{Z}) & \xrightarrow{\cong} & H^n_{\{b\}}(U,\mathbf{Z}) \\ \downarrow & & \downarrow{\scriptstyle\cong} \\ H^{n-1}(X^*,\mathbf{Z}) & \longrightarrow & H^n_{X_b}(f^{-1}(U),\mathbf{Z}) \end{array}$$

it is then clear that $\ker\varphi$ surjects onto $H^n_{X_b}(f^{-1}(U),\mathbf{Z}) = \mathbf{Z}$, so $\ker\varphi'' = \mathbf{Z}$ and $\operatorname{coker}\alpha = 0$. •

Theorem 3.5. *If $\dim X = 3$, $f: X \to B$ an integral special Lagrangian fibration satisfying Assumptions 1.7 and 1.8, and if the fibres X_b for rank 1 points b are a union of a finite number of $\mathcal{T}^*_{B,b}$-orbits, then f is $\mathbf{Z}$-simple.*

Proof. The hypothesis of this theorem implies the hypothesis of Theorem 3.3, so the simplicity condition holds for $p = 0, 1$ and 3. In the notation of the proof of Theorem 3.3, we need to show $i_{1*}i_1^*(R^2f_*\mathbf{Z})|_{B-\Delta_0} = (R^2f_*\mathbf{Z})|_{B-\Delta_0}$, as Lemma 3.4 then allows us to complete the proof of simplicity. Choose $b \in \Delta_1$, $b \notin \Delta_0$, and choose a 2-dimensional disk B' passing through b as in the proof of Theorem 3.3, $f': X' = f^{-1}(B') \to B'$. As in that proof, we just need to show that

$$H^2(X_b,\mathbf{Z}) \cong \lim_{\substack{\to \\ b\in U\subseteq B'}} H^2(f'^{-1}(U),\mathbf{Z}) \to \lim_{\substack{\to \\ b\in U\subseteq B'}} \Gamma(U-\{b\}, R^2f'_*\mathbf{Z})$$

is an isomorphism, and we already know this map is injective. Note also that the cokernel of this map is torsion-free: since $R^2f'_*\mathbf{Z} \subseteq \mathcal{T}_B^*|_{B'}$, if an integer multiple of a section of $R^2f'_*\mathbf{Z}$ over $U-\{b\}$ extends to a section over U, the section itself extends. Now $B^* = B' - \{b\}$ is a punctured disk, and hence $(R^2f'_*\mathbf{Z})|_{B^*}$ is a local system determined by a single monodromy transformation T. By Poincaré duality, $(R^1f'_*\mathbf{Z})|_{B^*}$ has monodromy ${}^tT$. Since $\Gamma(B^*, R^2f'_*\mathbf{Z}) = \ker(T-I)$, we see that

$\operatorname{rank}\Gamma(B^*, R^2 f'_*\mathbf{Z}) = \operatorname{rank}\Gamma(B^*, R^1 f'_*\mathbf{Z}) = \operatorname{rank} H^1(X_b, \mathbf{Z})$ by simplicity for $p = 1$. Thus it will be sufficient to show that $\operatorname{rank} H^1(X_b, \mathbf{Z}) = \operatorname{rank} H^2(X_b, \mathbf{Z})$ to show the above map is an isomorphism.

Since b is a rank 1 point, the singular locus of X_b is a union of circles, each being a closed orbit of the action of $\mathcal{T}^*_{B,b}$ on X_b. Since there are assumed to be only a finite number of such orbits, each S^1 is a connected component of $Sing(X_b)$. But since $Sing(X_b)$ is connected by Corollary 2.5, $Z := Sing(X_b) \cong S^1$. Now we also have exact sequences

$$0 \to H^i_c(X_b - \mathbf{Z}, \mathbf{Z}) \to H^i(X_b, \mathbf{Z}) \to H^i(Z, \mathbf{Z}) \to 0$$

for all i, the exactness for $i = 1$ shown in the proof of Corollary 2.5 and for $i = 2$ shown in the proof of Theorem 2.4. From this we see that if $H^2_c(X_b - Z, \mathbf{Z}) = \mathbf{Z}$, then $H^1(X_b, \mathbf{Z}) = H^2(X_b, \mathbf{Z}) = \mathbf{Z}$, while if $H^2_c(X_b - Z, \mathbf{Z}) = \mathbf{Z}^2$, then $H^1(X_b, \mathbf{Z}) = H^2(X_b, \mathbf{Z}) = \mathbf{Z}^2$, completing the proof in these cases.

Finally, suppose $H^2_c(X_b - Z, \mathbf{Z}) = 0$. Then $H^2_c(X_b, \mathbf{Z}) = 0$ and it follows as in the proof of Theorem 3.3 that $H^3_{X_b}(f^{-1}(U), \mathbf{Z}) = 0$, so one obtains from the relative cohomology sequence a surjection

$$H^2(f'^{-1}(U), \mathbf{Z}) \to H^2(f'^{-1}(U - \{b\}), \mathbf{Z}) \to 0.$$

The argument of the proof of Lemma 3.4 shows that $H^2(f'^{-1}(U - \{b\}), \mathbf{Z}) \to \Gamma(U - \{b\}, R^2 f'_*\mathbf{Z})$ is surjective, and hence so is $H^2(f'^{-1}(U), \mathbf{Z}) \to \Gamma(U - \{b\}, R^2 f'_*\mathbf{Z})$. Taking direct limits, one concludes that $H^2(X_b, \mathbf{Z}) = \ker(T - I)$ as desired, showing simplicity. But notice in fact this case can't occur, since as $H^2(X_b, \mathbf{Z}) = 0$, we also have $H^1(X_b, \mathbf{Z}) = \ker({}^tT - I) = 0$, contradicting $H^1(X_b, \mathbf{Z}) = H^1(Z, \mathbf{Z}) = \mathbf{Z}$. •

Remark 3.6. If $f : X \to B$ is a $\mathbf{Z}$-simple special Lagrangian T^3 fibration, we obtain some restrictions on the cohomology of a singular fibre X_b. Clearly $H^0(X_b, \mathbf{Z}) = H^3(X_b, \mathbf{Z}) = \mathbf{Z}$, so if $b_i = \operatorname{rank} H^i(X_b, \mathbf{Z})$, we will say for the duration of this remark that X_b is of type (b_1, b_2). Clearly $b_1, b_2 \leq 3$, and if $b_2 = 3$, then X_b is non-singular. If $b_1 = 3$, then $b_2 = 3$, since $\bigwedge^2 H^1(X_b, \mathbf{Z}) \subseteq H^2(X_b, \mathbf{Z})$, and so X_b is non-singular. This also shows that if $b_1 = 2$ then $b_2 \geq 1$, and a similar argument shows that if $b_2 = 2$, then $b_1 \geq 1$. Thus the possible values for (b_1, b_2) are $(2, 2)$, $(2, 1)$, $(1, 2)$, $(1, 1)$, $(0, 1)$, $(1, 0)$ or $(0, 0)$. We describe the probable topology of an integral singular fibre with each of the above possible cohomology groups.

$(2, 2)$ Such cohomology is realised by a fibre of the form $I_1 \times S^1$, where I_1 denotes a Kodaira type I_1-fibre.

$(1, 2)$ $S^1 \times T^2/\{pt\} \times T^2$. This was seen in the example in §1 of [14].

$(2, 1)$ This was described in [14], Remark 1.4. Identify T^3 with the solid cube $[0, 1]^3$ with opposite sides identified. Then take $T^3/\sim$, where $(x_1, x_2, x_3) \sim (x'_1, x'_2, x'_3)$ if and only if $(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$ or $(x_1, x_2) = (x'_1, x'_2) \in \partial([0, 1])^2$. This fibre is singular along a figure eight.

$(1, 1)$ There are two possibilities here. The fibre could be $II \times S^1$, where II denotes a Kodaira type II fibre, or it could be $T^3/\sim$, where $(x_1, x_2, x_3) \sim (x'_1, x'_2, x'_3)$ if and only if $(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$ or $(x_1, x_2) = (x'_1, x'_2) \in \{0, 1\} \times [0, 1]$ or $(x_1, x_2), (x'_1, x'_2) \in [0, 1] \times \{0, 1\}$. (This is the same as contracting one loop of the singular figure eight in the $(2, 1)$ case to a point).

(0, 1) $T^3/\sim$, where $(x_1,x_2,x_3) \sim (x_1',x_2',x_3')$ if and only if $(x_1,x_2,x_3) = (x_1',x_2',x_3')$ or $(x_1,x_2),(x_1',x_2') \in \partial[0,1]^2$. (This is equivalent to contracting the singular figure eight of the $(1,2)$ case to a point).

(1, 0) $T^3/\sim$, where $(x_1,x_2,x_3) \sim (x_1',x_2',x_3')$ if and only if $(x_1,x_2,x_3) = (x_1',x_2',x_3')$ or $(x_1,x_2),(x_1',x_2') \in \partial[0,1]^2$ and $x_3 = x_3'$ or $x_3, x_3' \in \{0,1\}$.

(0, 0) $T^3/\sim$, where $(x_1,x_2,x_3) \sim (x_1',x_2',x_3')$ if and only if $(x_1,x_2,x_3) = (x_1',x_2',x_3')$ or $(x_1,x_2,x_3),(x_1',x_2',x_3') \in \partial[0,1]^3$. (We are contracting the boundary of $[0,1]^3$ to a point. This is topologically a sphere).

One notes that for each fibre of type (m,n), there is a fibre of type (n,m) which should then be its dual. (In particular, the fibres of type $(2,2)$ and $(1,1)$ should be self-dual).

I cannot prove yet that this provides a complete classfication of integral three-dimensional singular fibres, but it seems to be a reasonable conjecture. In addition, these types of examples extend to higher dimensions, and one finds a much wider range of possible topologies, which nonetheless exhibit the desired duality.

Having proven $\mathbf{Z}$-simplicity in some cases for special Lagrangian fibrations, we wish now to return to a more careful study of the Leray spectral sequence for special Lagrangian fibrations, with special consideration of the role torsion plays. First we make some observations on the Leray spectral sequence in any dimension.

Lemma 3.7. *If $f : X \to B$ is a $\mathbf{Z}$-simple special Lagrangian T^n-fibration with a section, then in the Leray spectral sequence for f, $E_{1,1}^2 = E_{1,1}^\infty$ and $E_{1,n-1}^2 = E_{1,n-1}^\infty$. In addition, the Leray filtration yields a surjection $H^n(X,\mathbf{Z})_{tors} \to (E_{1,n-1}^2)_{tors}$.*

Proof. The only possible non-zero differential to or from $E_{1,1}^2$ is $d_2 : E_{1,1}^2 \to E_{3,0}^2 = H^3(B,\mathbf{Z})$. But since f has a section, the map $H^3(B,\mathbf{Z}) \to H^3(X,\mathbf{Z})$ is injective, and thus $d_2 = 0$. Thus $E_{1,1}^2 = E_{1,1}^\infty$.

Next, $E_{1,n-1}^2 = H^1(B, R^{n-1}f_*\mathbf{Z})$, and recall from [14] that that $H^1(B,R^{n-1}f_*\mathbf{Z})$ is the group of sections of f modulo homotopy, with a fixed section, say σ_0, the zero section. Then for any section σ, the cohomology class $[\sigma] - [\sigma_0] \in H^n(X,\mathbf{Z})$ and the element $[\sigma] \in H^1(B,R^{n-1}f_*\mathbf{Z})$ representing the section coincide up to sign in $E_{1,n-1}^\infty$ by [14], Theorem 4.1 (which holds with $\mathbf{Z}$ coefficients if f is $\mathbf{Z}$-simple). Thus $E_{1,n-1}^\infty = E_{1,n-1}^2$, and if $H^n(X,\mathbf{Z}) = F^0 \supseteq F^1 \supseteq \cdots$ is the Leray filtration on $H^n(X,\mathbf{Z})$, then $F^1/F^2 \cong E_{1,n-1}^2$. Since $F^0/F^1 \cong \mathbf{Z}$, $H^n(X,\mathbf{Z})_{tors} = F_{tors}^1$ and thus there is a map $H^n(X,\mathbf{Z})_{tors} \to (E_{1,n-1}^2)_{tors}$. To see this map is surjective, suppose σ is a torsion section of f. Then σ must be disjoint from σ_0. Indeed, if $x \in \sigma \cap \sigma_0$, let $U \subseteq B$ be a small open neighborhood of $f(x) \in B$ in which σ is represented by a section $\tilde\sigma \in \Gamma(U, \mathcal{T}_B^*)$, such that $\tilde\sigma(f(x)) = 0$. Then if m is the order of the torsion section σ, $m\tilde\sigma$ is a non-zero section of $R_c^{n-1}f_*^{\#}\mathbf{Z}$ which is zero for at least one point, which is impossible. Thus σ and σ_0 are disjoint.

By [14], Theorem 4.1, $T_\sigma^* : H^*(X,\mathbf{Q}) \to H^*(X,\mathbf{Q})$ is a unipotent operator, but on the other hand $T_\sigma^m = I$ since σ is m-torsion. Thus $T_\sigma^* = I$, so $[\sigma] = [\sigma_0]$ in $H^n(X,\mathbf{Q})$ and $[\sigma] - [\sigma_0]$ is in fact a torsion element of $H^n(X,\mathbf{Z})$. This shows that the map $H^n(X,\mathbf{Z})_{tors} \to (E_{1,n-1}^2)_{tors}$ is surjective. •

Even if in general the Leray spectral sequence for f does not degenerate, the above result might be sufficient for many applications; as we will see in later sections, $H^1(B, R^1 f_*\mathbf{R})$ and $H^1(B, R^{n-1}f_*\mathbf{R})$ play important roles in mirror symmetry.

Note that if $\mathbf{Z}$-simplicity fails because f has reducible fibres, we expect the second part of the above result to fail.

We now focus on the three dimensional case.

Proposition 3.8. *Let $f : X \to B$ and $\check{f} : \check{X} \to B$ be dual $\mathbf{Z}$-simple special Lagrangian T^3 fibrations, and suppose that $H^1(B,\mathbf{Z}) = 0$. Then $H^1(X,\mathbf{Z}) = 0$ if and only if $H^1(\check{X},\mathbf{Z}) = 0$.*

Proof. Since $H^1(B,\mathbf{Z}) = 0$, $H^2(B,\mathbf{Z})$ is torsion, so

$$rank(H^1(X,\mathbf{Z})) = rank(H^0(B, R^1 f_*\mathbf{Z})).$$

But since X is Kähler, the first betti number of X is even, so if $H^1(X,\mathbf{Z}) \neq 0$ then $rank(H^0(B, R^1 f_*\mathbf{Z})) \geq 2$. The wedge of two independent sections of $R^1 f_*\mathbf{Z}$ yields a section of $R^2 f_*\mathbf{Z}$, and $R^2 f_*\mathbf{Z} \cong R^1 \check{f}_*\mathbf{Z}$, so $H^0(B, R^1\check{f}_*\mathbf{Z}) \neq 0$, hence $H^1(\check{X},\mathbf{Z}) \neq 0$. Repeating the same argument interchanging X and $\check{X}$ gives the result. •

Theorem 3.9. *Let $f : X \to B$, $\check{f} : \check{X} \to B$ be dual $\mathbf{Z}$-simple special Lagrangian T^3-fibrations with sections, and assume $H^1(X,\mathbf{Z}) = 0$. Then the Leray spectral sequences for f and $\check{f}$ with coefficients in $\mathbf{Z}$ degenerate at the E_2-term, and*

$$rank_{\mathbf{Z}} H^i(B, R^j f_*\mathbf{Z}) = rank_{\mathbf{Z}} H^{3-i}(B, R^{3-j} f_*\mathbf{Z}).$$

If in addition f and $\check{f}$ are $\mathbf{R}/\mathbf{Z}$-simple, then

$$Tors(H^i(B, R^j f_*\mathbf{Z})) \cong Tors(H^{4-i}(B, R^{3-j} f_*\mathbf{Z})).$$

Note that the additional assumption of $\mathbf{R}/\mathbf{Z}$-simplicity is not a particularly strong one. Indeed, we have seen that given suitable regularity hypotheses, we have only failed to show the G-simplicity condition for $p = 2$. But $\mathbf{Z}$-simplicity implies $R^2 f_*\mathbf{Z} \cong R^1\check{f}_*\mathbf{Z}$ and $R^2 f_*\mathbf{R} \cong R^1\check{f}_*\mathbf{R}$, whence $R^2 f_*\mathbf{R}/\mathbf{Z} \cong R^1\check{f}_*\mathbf{R}/\mathbf{Z}$. Hence the existence of the dual fibration $\check{f}$ and the $\mathbf{R}/\mathbf{Z}$-simplicity condition for $p = 1$ implies it for $p = 2$.

Proof. The E_2-term of the Leray spectral sequence, by the arguments of [14], Lemma 2.4, looks like

$$\begin{array}{ccccc}
 & \mathbf{Z} & 0 & T^{2,3} & \mathbf{Z} \\
(3.1) & 0 & \mathbf{Z}^{h^{1,2}} \oplus T^{1,2} & \mathbf{Z}^{h^{1,1}} \oplus T^{2,2} & T^{3,2} \\
 & 0 & \mathbf{Z}^{h^{1,1}} \oplus T^{1,1} & \mathbf{Z}^{h^{1,2}} \oplus T^{2,1} & T^{3,1} \\
 & \mathbf{Z} & 0 & T^{2,0} & \mathbf{Z}
\end{array}$$

with a similar diagram for $\check{f}$. Here $T^{i,j} = H^i(B, R^j f_*\mathbf{Z})_{tors}$. We are using $H^1(X,\mathbf{Z}) = H^1(\check{X},\mathbf{Z}) = 0$ to obtain the zeroes on the left column and the top and bottom rows. Clearly the desired statement on ranks follows.

Recalling from [14], §2 that $\mathbf{Z}$-simplicity implies $R^i f_*\mathbf{Z} \cong R^{3-i}\check{f}_*\mathbf{Z}$, it follows that if $\check{T}^{i,j} = H^i(B, R^j\check{f}_*\mathbf{Z})_{tors}$, then $\check{T}^{i,j} \cong T^{i,3-j}$.

Since f has a section, the argument of [14], Lemma 2.4, combined with the degeneration statement of Proposition 3.7, shows that the above spectral sequence degenerates. The same holds for $\check{f}$.

As for the statements about torsion, clearly $T^{2,3} \cong T^{2,0} \cong H^2(B,\mathbf{Z})$. To show the rest, we use Poincaré-Verdier duality (see for example [9]). In any dimension, applying duality to the map $s : B \to pt$, we obtain isomorphisms

$$\mathbf{R}\mathrm{Hom}(\mathbf{R}\Gamma(R^i f_*\mathbf{Z}),\mathbf{Z}) \cong \mathbf{R}\Gamma\mathbf{R}\underline{\mathrm{Hom}}(R^i f_*\mathbf{Z},\mathbf{Z}[n]).$$

Applying H^{-j} to both sides, we obtain

$$H^{-j}(\mathbf{R}\mathrm{Hom}(\mathbf{R}\Gamma(R^i f_*\mathbf{Z}),\mathbf{Z})) \cong \mathrm{Ext}^{n-j}(R^i f_*\mathbf{Z},\mathbf{Z}). \tag{3.2}$$

The left hand side is easily computed by choosing a complex of projective $\mathbf{Z}$-modules quasi-isomorphic to $\mathbf{R}\Gamma(R^i f_*\mathbf{Z})$ and applying the algebraic universal coefficient theorem, which yields exact sequences

$$0 \to \mathrm{Ext}^1(H^{j+1}(B,R^i f_*\mathbf{Z}),\mathbf{Z}) \to H^{-j}(\mathbf{R}\mathrm{Hom}(\mathbf{R}\Gamma(R^i f_*\mathbf{Z}),\mathbf{Z})) \to \mathrm{Hom}(H^j(B,R^i f_*\mathbf{Z}),\mathbf{Z}) \to 0.$$

The difficulty in applying Poincaré-Verdier duality for non-locally constant sheaves is the difficulty of comparing the Ext's and the cohomology groups. We will do this for $n=3, i=2$.

First on B_0 one has $\underline{\mathrm{Hom}}(R^2 f_{0*}\mathbf{Z},\mathbf{Z}) \cong R^1 f_{0*}\mathbf{Z}$ by Poincaré duality, and if $i : B_0 \hookrightarrow B$ is the inclusion, the natural map

$$i_*\underline{\mathrm{Hom}}(R^2 f_{0*}\mathbf{Z},\mathbf{Z}) \to \underline{\mathrm{Hom}}(i_*R^2 f_{0*}\mathbf{Z}, i_*\mathbf{Z})$$

is an isomorphism. Thus by $\mathbf{Z}$-simplicity, $\underline{\mathrm{Hom}}(R^2 f_*\mathbf{Z},\mathbf{Z}) \cong R^1 f_*\mathbf{Z}$. Also, $\mathbf{R}/\mathbf{Z}$-simplicity implies $\mathbf{Z}/m\mathbf{Z}$-simplicity for any m, and so a similar argument shows $\underline{\mathrm{Hom}}(R^2 f_*\mathbf{Z},\mathbf{Z}/m\mathbf{Z}) \cong R^1 f_*\mathbf{Z}/m\mathbf{Z}$.

So by the local-global Ext spectral sequence one has a five-term sequence

$$\begin{aligned} 0 &\to H^1(B,R^1 f_*\mathbf{Z}) \to \mathrm{Ext}^1(R^2 f_*\mathbf{Z},\mathbf{Z}) \to H^0(B,\underline{\mathrm{Ext}}^1(R^2 f_*\mathbf{Z},\mathbf{Z})) \\ &\to H^2(B,R^1 f_*\mathbf{Z}) \to \mathrm{Ext}^2(R^2 f_*\mathbf{Z},\mathbf{Z}). \end{aligned} \tag{3.3}$$

I claim that $\underline{\mathrm{Ext}}^1(R^2 f_*\mathbf{Z},\mathbf{Z})$ is a torsion-free sheaf. Indeed, apply $\underline{\mathrm{Hom}}(R^2 f_*\mathbf{Z},\cdot)$ to the exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \to 0.$$

We obtain an exact sequence

$$0 \longrightarrow R^1 f_*\mathbf{Z} \xrightarrow{\cdot m} R^1 f_*\mathbf{Z} \longrightarrow R^1 f_*\mathbf{Z}/m\mathbf{Z} \longrightarrow \underline{\mathrm{Ext}}^1(R^2 f_*\mathbf{Z},\mathbf{Z}) \xrightarrow{\cdot m} \underline{\mathrm{Ext}}^1(R^2 f_*\mathbf{Z},\mathbf{Z}).$$

But in fact $R^1 f_*(\mathbf{Z}/m\mathbf{Z}) \cong R^1 f_*\mathbf{Z}/mR^1 f_*\mathbf{Z}$ since $R^2 f_*\mathbf{Z}$ is torsion-free, so we see the multiplication by m map is injective on $\underline{\mathrm{Ext}}^1(R^2 f_*\mathbf{Z},\mathbf{Z})$. Thus this sheaf is torsion free.

Now the left hand side of (3.2) is $\mathbf{Z}^{h^{1,2}} \oplus T^{2,2}$ for $j = 1$ and is $\mathbf{Z}^{h^{1,1}} \oplus T^{3,2}$ for $j = 2$. Thus by (3.2),

$$rank_{\mathbf{Z}}\mathrm{Ext}^1(R^2f_*\mathbf{Z},\mathbf{Z}) = h^{1,1} = rank_{\mathbf{Z}}H^1(B, R^1f_*\mathbf{Z}),$$

so in (3.3) the fact that $H^0(B, \underline{\mathrm{Ext}}^1(R^2f_*\mathbf{Z},\mathbf{Z}))$ is torsion-free shows that the map

$$\mathrm{Ext}^1(R^2f_*\mathbf{Z},\mathbf{Z}) \to H^0(B, \underline{\mathrm{Ext}}^1(R^2f_*\mathbf{Z},\mathbf{Z}))$$

is zero. Thus we have

$$H^1(B, R^1f_*\mathbf{Z}) \cong \mathrm{Ext}^1(R^2f_*\mathbf{Z},\mathbf{Z})$$

and

$$0 \to H^0(B, \underline{\mathrm{Ext}}^1(R^2f_*\mathbf{Z},\mathbf{Z})) \to H^2(B, R^1f_*\mathbf{Z}) \to \mathrm{Ext}^2(R^2f_*\mathbf{Z},\mathbf{Z})$$

exact. Thus (3.2) implies $T^{1,1} \cong T^{3,2}$, and by using the same argument for $\check{f}$, $T^{1,2} \cong T^{3,1}$. In addition, $T^{2,1} = H^2(B, R^1f_*\mathbf{Z})_{tors} \subseteq \mathrm{Ext}^2(R^2f_*\mathbf{Z},\mathbf{Z})_{tors} = T^{2,2}$. On the other hand, from (3.1) and Proposition 3.7 there are exact sequences (see the beginning of the proof of Theorem 3.10 for details of the first sequence)

$$0 \to T^{2,1} \to H^3(X,\mathbf{Z})_{tors} \to T^{1,2} \to 0,$$

$$0 \to T^{3,1} \to H^4(X,\mathbf{Z})_{tors} \to T^{2,2} \to 0,$$

and in addition for any oriented 6-manifold $H^3(X,\mathbf{Z})_{tors} \cong H^4(X,\mathbf{Z})_{tors}$ by Poincaré duality and the universal coefficient theorem. So $\#T^{2,2} = \#T^{2,1}$ and this implies $T^{2,2} \cong T^{2,1}$. •

Theorem 3.10. *Let $f : X \to B$, $\check{f} : \check{X} \to B$ be as in Theorem 3.9, and assume in addition that B is simply connected. Then there are non-canonical isomorphisms*

$$H^{even}(X, \mathbf{Z}[1/2]) \cong H^{odd}(\check{X}, \mathbf{Z}[1/2])$$
$$H^{odd}(X, \mathbf{Z}[1/2]) \cong H^{even}(\check{X}, \mathbf{Z}[1/2]).$$

In general, there are short exact sequences

$$0 \to H^2(B, R^1f_*\mathbf{Z})_{tors} \to H^3(X,\mathbf{Z})_{tors} \to H^1(B, R^2f_*\mathbf{Z})_{tors} \to 0$$

$$0 \to H^2(B, R^1\check{f}_*\mathbf{Z})_{tors} \to H^3(\check{X},\mathbf{Z})_{tors} \to H^1(B, R^2\check{f}_*\mathbf{Z})_{tors} \to 0$$

and if they split, the above isomorphisms hold over $\mathbf{Z}$. This happens, for example, if both X and $\check{X}$ are simply connected. In any event,

$$\#H^{even}(X,\mathbf{Z})_{tors} = \#H^{odd}(\check{X},\mathbf{Z})_{tors}$$

$$\#H^{odd}(X,\mathbf{Z})_{tors} = \#H^{even}(\check{X},\mathbf{Z})_{tors}.$$

Proof. The Leray filtration on $H^3(X,\mathbf{Z})$ is

$$0 \subseteq F_0 = \mathbf{Z}[T^3] \subseteq F_1 \subseteq F_2 \subseteq F_3 = H^3(X,\mathbf{Z}).$$

Since the cohomology class $[T^3]$ is primitive in $H^3(X,\mathbf{Z})$, $F_0 \subseteq F_1$ is a primitive embedding and $(F_1)_{tors} = (F_1/F_0)_{tors} = T^{2,1}$ in the notation of the proof of Theorem 3.9. It then follows from Proposition 3.7 that there is an exact sequence

$$0 \to T^{2,1} \to H^3(X,\mathbf{Z})_{tors} \to T^{1,2} \to 0.$$

First assume this sequence for X and $\check{X}$ splits, so $H^3(X,\mathbf{Z})_{tors} = T^{2,1} \oplus T^{1,2}$. Now $H^3(X,\mathbf{Z})_{tors} \cong H^4(X,\mathbf{Z})_{tors}$, so $H^4(X,\mathbf{Z})_{tors} = T^{2,1} \oplus T^{1,2}$ also. Putting this together we see that

$$H^{even}(X,\mathbf{Z})_{tors} = T^{1,1} \oplus T^{2,1} \oplus T^{1,2}$$

and

$$\begin{aligned} H^{odd}(X,\mathbf{Z})_{tors} =& T^{3,2} \oplus T^{2,1} \oplus T^{1,2} \\ =& T^{1,1} \oplus T^{2,1} \oplus T^{1,2}. \end{aligned}$$

On the other hand

$$\begin{aligned} H^{odd}(\check{X},\mathbf{Z})_{tors} =& \check{T}^{1,1} \oplus \check{T}^{2,1} \oplus \check{T}^{1,2} \\ =& T^{1,2} \oplus T^{2,2} \oplus T^{1,1}. \\ =& H^{even}(\check{X},\mathbf{Z})_{tors} \end{aligned}$$

Since $T^{2,2} \cong T^{2,1}$ by Theorem 3.9, we are done.

Note that if X and $\check{X}$ are simply connected, $0 = T^{1,1} \cong \check{T}^{1,2}$ and $0 = \check{T}^{1,1} \cong T^{1,2}$, so the sequences trivially split. If the sequences don't split, then it is still clear that the numerical equalities hold.

Finally, we finish the proof of the theorem by showing the sequences do split over $\mathbf{Z}[1/2]$. We define a map $\phi : T^{1,2} \otimes_{\mathbf{Z}} \mathbf{Z}[1/2] \to H^3(X,\mathbf{Z}[1/2])_{tors}$. Indeed, for σ a torsion section of $f : X \to B$, given by $\sigma \in H^1(B, R^2 f_*\mathbf{Z})_{tors}$, we can set $\phi(\sigma) = (\log T_\sigma^*)([\sigma_0]) \in H^3(X,\mathbf{Z}[1/6])$ where $[\sigma_0] \in H^3(X,\mathbf{Z})$ is the cohomology class of the zero section. Here $\log T_\sigma^* = (T_\sigma^* - I) - \frac{1}{2}(T_\sigma^* - I)^2 + \frac{1}{3}(T_\sigma^* - I)^3$, as $(T_\sigma^* - I)^4 = 0$ by [14] Theorem 4.1. As observed in the proof of Theorem 3.7, $(T_\sigma^* - I)([\sigma_0]) = [\sigma] - [\sigma_0]$ is torsion and represents the class $\sigma \in H^1(B, R^2 f_*\mathbf{Z})_{tors}$. Thus $(T_\sigma - I)^3([\sigma_0]) \in H^3(B,\mathbf{Z}) \subseteq H^3(X,\mathbf{Z})$ must be zero as this element is also torsion. So $(\log T_\sigma^*)([\sigma_0]) = ((T_\sigma^* - I) - \frac{1}{2}(T_\sigma^* - I))([\sigma_0]) \in H^3(X,\mathbf{Z}[1/2])$. Furthermore,

$$\begin{aligned} \phi(\sigma+\tau) &= (\log T_{\sigma+\tau}^*)([\sigma_0]) \\ &= (\log T_\sigma^* \circ T_\tau^*)([\sigma_0]) \\ &= (\log T_\sigma^* + \log T_\tau^*)([\sigma_0]) \\ &= \phi(\sigma) + \phi(\tau), \end{aligned}$$

so ϕ is a group homomorphism. Thus ϕ gives the desired splitting over $\mathbf{Z}[1/2]$. •

The problem that arises with two-torsion in the above theorem seems at the moment to be unavoidable, and does not make the statement very aesthetically pleasing. The heart of this issue is the following: given a two-torsion element in $H^2(\check{X},\mathbf{Z})$, is the square of this element non-zero in $H^4(\check{X},\mathbf{Z})$? If it is non-zero, then it follows from [14], Theorem 4.1, that if σ is the corresponding torsion section of f, then $[\sigma] - [\sigma_0]$ is not two-torsion, making it unlikely that the exact sequences in Theorem 3.10 split over $\mathbf{Z}$.

Example 3.11. The only example personally known to me of a Calabi-Yau threefold X with $H^3(X, \mathbf{Z})_{tors}$ non-zero is the "Enriques threefold", obtained by dividing out $K3 \times E$ with the involution $(\iota, -1)$, where ι is the Enriques involution on the K3 surface. (See [4] for calculations of the cohomology of this threefold.) This possesses a special Lagrangian T^3 fibration f in much the same way as the examples in [16]. In fact $H^3(X, \mathbf{Z})_{tors} = \mathbf{Z}/2\mathbf{Z}$, and the fibration $f : X \rightarrow B$ is seen to have a torsion section, so $H^1(B, R^2 f_* \mathbf{Z})_{tors} = \mathbf{Z}/2\mathbf{Z}$.

Remark 3.12. If $f : X \rightarrow B$ does not have a section, then $\check{f} : \check{X} \rightarrow B$ does, as well as the Jacobian $j : J \rightarrow B$ of f. Then j and $\check{f}$ are dual, and Theorem 3.11 applies to this pair. In addition, $R^i f_* \mathbf{Z} = R^i j_* \mathbf{Z}$, so the E^2 term of the Leray spectral sequence for f is still given by (3.1), and $H^2(B, R^1 f_* \mathbf{Z})_{tors} \cong H^2(B, R^2 f_* \mathbf{Z})_{tors}$. However now the spectral sequence won't degenerate. Since there is no class in $H^3(X, \mathbf{Z})$ restricting to the generator of $H^3(X_b, \mathbf{Z})$, one of the differentials from $H^0(B, R^3 f_* \mathbf{Z})$ must be non-zero.

This now gives an explanation for the speculations of [5] of the role that $H^3(X, \mathbf{Z})_{tors}$ should play in mirror symmetry. There it was argued on physical grounds that the Kähler moduli space of a Calabi-yau threefold was in fact $H^2(X, \mathbf{C}/\mathbf{Z})$, so in fact it had one component for each element of $H^3(X, \mathbf{Z})_{tors}$. Thus the complex moduli space of the mirror $\check{X}$ should have a similar number of components. It was not clear what this meant. But in our current context it is clear: if $H^2(B, R^1 f_* \mathbf{Z})_{tors} \neq 0$, $f : X \rightarrow B$ will have many dual fibrations, only one of which will have a section. All the other duals are obtained by twisting the one with a section. The set of such dual fibrations is classified by $H^2(B, R^1 f_* \mathbf{Z})_{tors} = H^2(B, R^2 \check{f}_* \mathbf{Z})_{tors}$ and we have seen that these groups are related to $H^3(X, \mathbf{Z})_{tors}$. However, they do not necessarily coincide with $H^3(X, \mathbf{Z})_{tors}$, and this will lead us to modify the definition of B-field in Conjecture 6.6.

We end this section with some comments concerning the de Rham realisation of the Leray spectral sequence, which we will need later. In general, let $f : X \rightarrow S$ be a smooth map (i.e., f_* always surjective) of differentiable manifolds. Then one has an exact sequence of vector bundles

$$0 \rightarrow f^* \Omega^1_S \rightarrow \Omega^1_X \rightarrow \Omega^1_{X/S} \rightarrow 0.$$

This gives rise to a filtration $F^p \Omega^{\cdot}_X$ on the de Rham complex,

$$\Omega^{\cdot}_X = F^0 \Omega^{\cdot}_X \supseteq F^1 \Omega^{\cdot}_X \supseteq \cdots$$

such that

$$F^p \Omega^{p+q}_X / F^{p+1} \Omega^{p+q}_X = \bigwedge^p f^* \Omega^1_S \otimes \bigwedge^q \Omega^1_{X/S}.$$

This filtration gives rise to a spectral sequence with

$$E^0_{pq} = \Gamma(X, \bigwedge^p f^* \Omega^1_S \otimes \bigwedge^q \Omega^1_{X/S})$$

with differential d^0 being exterior differentiation along fibres of the map f. Then

$$E^1_{pq} = \Gamma(S, \Omega^p_S \otimes R^q f_* \mathbf{R}),$$

and d^1 is the Gauss-Manin connection ∇_{GM}. Here, given a form $\alpha \in F^p\Omega^{p+q}_X$ with $d^0\alpha = 0$, the element represented in E^1_{pq} is given as follows. For $v_1, \ldots, v_p \in \mathcal{T}_{B,b}$, the form $(\iota(v_1, \ldots, v_p)\alpha)|_{X_b}$ defines a well-defined cohomology class in $H^q(X_b, \mathbf{R})$. This yields an $R^q f_* \mathbf{R}$-valued p-form in E^1_{pq}. Next

$$E^2_{pq} = H^p(S, R^q f_* \mathbf{R}),$$

which coincides with the E^2-term of the Leray spectral sequence for f.

Now let us specialise to the case that $f : X \to B$ is a special Lagrangian T^n-fibration, $f_0 : X_0 \to B_0 = B - \Delta$ the smooth part of the fibration. On B_0, McLean's result gives a natural isomorphism

$$\mathcal{T}_{B_0} \cong R^1 f_{0*} \mathbf{R} \otimes C^\infty(B_0).$$

Mclean also defines an n-form on the base. This is given by

$$\Theta(v_1, \ldots, v_n) = \int_{X_b} (-\iota(v_1)\omega) \wedge \cdots \wedge (-\iota(v_n)\omega).$$

In canonical coordinates, this is

$$\Theta(\partial/\partial y_1, \ldots, \partial/\partial y_n) = \int_{X_b} dx_1 \wedge \cdots \wedge dx_n.$$

Here X_b is oriented canonically by Ω as it is special Lagrangian. Of course, this form goes to infinity at singular fibres. Another way to think about this form is via integration along fibres of $\omega^n/n!$. Since

$$\omega^n/n! = (-1)^{n(n+1)/2} dy_1 \wedge \cdots \wedge dy_n \wedge dx_1 \wedge \cdots \wedge dx_n,$$

$f_*(\omega^n/n!) = (-1)^{n(n+1)/2}\Theta$. Thus in particular,

$$\int_X \omega^n/n! = (-1)^{n(n+1)/2} \int_B \Theta.$$

We can identify $\bigwedge^{n-q} \mathcal{T}_{B_0}$ with $\Omega^q_{B_0}$ by contracting with Θ, and so obtain isomorphisms

$$\Omega^q_{B_0} \cong R^{n-q} f_{0*} \mathbf{R} \otimes C^\infty(B_0).$$

Thus the $E^1_{p,n-q}$ term of the de Rham realisation of the Leray spectral sequence for f_0 is $\Gamma(B_0, \Omega^p_{B_0} \otimes \Omega^q_{B_0})$.

Definition 3.13. *A cohomology class $[\alpha_0] \in H^1(B_0, R^{n-1} f_{0*}\mathbf{R})$ is symmetric if there is a representative $\alpha_0 \in E^1_{1,n-1} = \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^1_{B_0})$ of $[\alpha_0]$ which is invariant under the involution of $\Omega^1_{B_0} \otimes \Omega^1_{B_0}$ given by $a \otimes b \mapsto b \otimes a$. In other words, $\alpha_0 \in \Gamma(B_0, S^2\Omega^1_{B_0}) \subseteq \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^1_{B_0})$. A cohomology class $\alpha \in H^1(B, R^{n-1} f_* \mathbf{R})$ is said to be symmetric if its restriction to $H^1(B_0, R^{n-1} f_{0*}\mathbf{R})$ is symmetric.*

The following result, which identifies the symmetric cohomology classes, will be useful in §6 in studying the role of the B-field.

Theorem 3.14. *A cohomology class $[\alpha_0] \in H^1(B_0, R^{n-1}f_{0*}\mathbf{R})$ is symmetric if and only if $[\alpha_0] \wedge [\omega] \in H^2(B_0, R^n f_{0*}\mathbf{R})$ is zero. In particular, if $H^2(B, \mathbf{R}) = 0$, then all elements of $H^1(B, R^{n-1}f_*\mathbf{R})$ are symmetric.*

Proof. We work in action-angle coordinates, on a neighborhood U, so that the Gauss-Manin connection is the trivial connection. Thus if $y_1, \ldots, y_n$ are the action coordinates,

$$\nabla_{GM}(\sum f_{IJ} dy_I \otimes dy_J) = \sum d(f_{IJ} dy_I) \otimes dy_J.$$

In addition, in suitably ordered action-angle coordinates $\Theta = dy_1 \wedge \cdots \wedge dy_n$, from which one sees easily that the cohomology class of $(-1)^{i-1} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$ on a fibre X_b is identified with $dy_i \in \Omega^1_{B,b}$.

Choose a representative $\alpha_0 \in \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^1_{B_0})$ for $[\alpha_0]$, so on U we write

$$\alpha_0 = \sum_{i,j} \alpha_{ij} dy_i \otimes dy_j, \tag{3.4}$$

with α_0 symmetric if and only if $\alpha_{ij} = \alpha_{ji}$. Locally, this class can also be represented by the n-form on $f^{-1}(U)$

$$\alpha_0 = \sum_{i,j} (-1)^{j-1} \alpha_{ij} dy_i \wedge dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n,$$

from which we see that

$$\begin{aligned} \alpha_0 \wedge \omega &= \sum_{i,j} (-1)^{j-1} \alpha_{ij} dy_i \wedge dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n \wedge dx_j \wedge dy_j \\ &= \sum_{i<j} (\alpha_{ji} - \alpha_{ij}) dy_i \wedge dy_j \wedge dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{3.5}$$

Thus we see α_0 is a symmetric representative for $[\alpha_0]$ if and only if $\alpha_0 \wedge \omega = 0$ in $\Gamma(B_0, \Omega^2_{B_0} \otimes \Omega^0_{B_0})$. Now $\alpha_0 \wedge \omega$ represents the cohomology class $[\alpha_0] \wedge [\omega] \in H^2(B_0, R^n f_{0*}\mathbf{R})$. This is zero if α_0 is symmetric. Conversely, suppose $[\alpha_0] \wedge [\omega]$ is the zero cohomology class. Then there exists a $\beta \in \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^0_{B_0})$ such that $\nabla_{GM}\beta = \alpha_0 \wedge \omega$. β also gives rise to an element $\beta' \in \Gamma(B_0, \Omega^0_{B_0} \otimes \Omega^1_{B_0})$ by using the map $a \otimes b \mapsto b \otimes a$. The following claim then proves the theorem.

Claim. $\alpha_0 + \nabla_{GM}\beta'$ is a symmetric representative for $[\alpha_0]$.

Proof. Locally write $\beta = \sum_i \beta_i dy_i \otimes 1$, so $\beta' = \sum_j \beta_j 1 \otimes dy_j$. Now $\nabla_{GM}(\beta) = \alpha_0 \wedge \omega$ means that

$$\frac{\partial \beta_j}{\partial y_i} - \frac{\partial \beta_i}{\partial y_j} = \alpha_{ji} - \alpha_{ij}.$$

Also,

$$\alpha_0 + \nabla_{GM}\beta' = \sum_{i,j} (\alpha_{ij} + \frac{\partial \beta_j}{\partial y_i}) dy_i \otimes dy_j.$$

But

$$\alpha_{ij} + \frac{\partial \beta_j}{\partial y_i} - (\alpha_{ji} + \frac{\partial \beta_i}{\partial y_j}) = 0,$$

so $\alpha_0 + \nabla_{GM}\beta'$ is a symmetric representative for $[\alpha_0]$. •

Remark 3.15. There is a related map which partly explains the interest in symmetric cohomology classes. If $f : X \to B$ is an integral, $\mathbf{Z}$-simple special Lagrangian fibration with a section, then there is a natural inclusion $R^{n-1}f_*\mathbf{R}/\mathbf{Z} \hookrightarrow \Lambda(X^\#)$. This is induced from the inclusion $R^{n-1}f_*\mathbf{Z} \hookrightarrow \Lambda(\mathcal{T}_B^*)$ and the inclusion obtained from this by tensoring with $\mathbf{R}$: $R^{n-1}f_*\mathbf{R} \hookrightarrow \Lambda(\mathcal{T}_B^*)$. Thus one obtains a natural map $H^1(B, R^{n-1}f_*\mathbf{R}/\mathbf{Z}) \to H^1(B, \Lambda(X^\#))$. To analyse this map, consider instead the map $H^1(B, R^{n-1}f_*\mathbf{R}) \to H^1(B, \Lambda(\mathcal{T}_B^*)) \cong H^2(B, \mathbf{R})$. This in fact coincides with the map $H^1(B, R^{n-1}f_*\mathbf{R}) \to H^2(B, R^n f_*\mathbf{R}) \cong H^2(B, \mathbf{R})$ obtained by cup-product with $-\omega$. It is easiest to see this over B_0: in this case, we have resolutions $0 \to R^{n-1}f_{0*}\mathbf{R} \to \Omega^{\cdot}_{B_0} \otimes \Omega^1_{B_0}$ and $0 \to \Lambda(\mathcal{T}_{B_0}^*) \to \Omega^{\cdot+1}_{B_0}$. The map $R^{n-1}f_{0*}\mathbf{R} \to \Lambda(\mathcal{T}_{B_0}^*)$ extends to a map of complexes $\Omega^{\cdot}_{B_0} \otimes \Omega^1_{B_0} \to \Omega^{\cdot+1}_{B_0}$ given by $\alpha \otimes \beta \mapsto \alpha \wedge \beta$. Thus α_0 as in (3.4) is mapped to $\sum_{i<j}(\alpha_{ij} - \alpha_{ji})dy_i \wedge dy_j$, which by (3.5) can be identified with $-\alpha_0 \wedge \omega$. Thus the map $H^1(B_0, R^{n-1}f_{0*}\mathbf{R}) \to H^1(B_0, \Lambda(\mathcal{T}_{B_0}^*))$ coincides with the map $\wedge(-\omega)$. It is not difficult to show this holds over B also, but we omit the cohomological argument.

Thus we see that the symmetric cohomology classes are those that map to zero in $H^1(B, \Lambda(X^\#))$.

§4. The symplectic form on $\mathcal{D}$-brane moduli space.

Let B be a moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold X of dimension n, along with a universal family

$$\begin{array}{ccc} \mathcal{U} & \subseteq & X \times B \\ \downarrow{\scriptstyle f} & & \\ B & & \end{array}$$

Let $p : \mathcal{U} \to X$ be the projection. For the moment we assume f is smooth, so that points in B are parametrizing only smooth special Lagrangian submanifolds. We do not assume these submanifolds are tori. Here $\dim B = \dim H^1(\mathcal{U}_b, \mathbf{R}) =: s$. The D-brane moduli space is the space of special Lagrangian submanifolds along with a choice of flat $U(1)$ connection modulo gauge equivalence, i.e. an element of $H^1(\mathcal{U}_b, \mathbf{R}/\mathbf{Z})$. Thus the D-brane moduli space $\mathcal{M}$ should be $R^1 f_*(\mathbf{R}/\mathbf{Z})$. The prediction from string theory is that $\mathcal{M}$ should be a complex Kähler manifold, so we need to understand how to put these structures on $\mathcal{M}$. As long as there are no singular fibres to deal with, Hitchin has described how to put a complex and Kähler structure on $\mathcal{M}$. Here, we will describe a more coordinate independent way of describing the same Kähler form (i.e. a symplectic form in the absence of a complex structure). This both allows us to compute the cohomology class represented by this symplectic form and in principle should allow one to extend this construction to singular fibres.

In what follows, we assume the fibres of f are special Lagrangian with respect to the symplectic form ω and holomorphic n-form Ω, with the standard normalization $\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n\Omega\wedge\bar{\Omega}$. We also use a holomorphic n-form Ω_n normalised by

$$\Omega_n = \frac{\Omega}{\int_{\mathcal{U}_b} p^*\Omega}.$$

We use Ω_n instead of Ω for several reasons. First, we do not want the symplectic structure on $\mathcal{M}$ to depend on ω, but only on Ω. If we multiply ω by a constant, we must also rescale Ω. If we rescale Ω and use Ω instead of Ω_n in the construction below, then the symplectic form $\check{\omega}$ we construct on $\mathcal{M}$ also changes. Secondly, this normalization fits with the usual form of the mirror map as described in item (5) of the introduction.

To obtain a symplectic form on $\mathcal{M}$, we define a map

$$R^1 f_*\mathbf{Z} \to \mathcal{T}_B^*$$

in such a way that the canonical symplectic form on $\mathcal{T}_B^*$ descends to a symplectic form on $\mathcal{T}_B^*/R^1 f_*\mathbf{Z}$. We follow Hitchin's suggestion of computing the periods of $\operatorname{Im} p^*\Omega_n$. Now $(R^1 f_*\mathbf{Z})_b \cong H^1(\mathcal{U}_b,\mathbf{Z}) \cong H_{n-1}(\mathcal{U}_b,\mathbf{Z})$, so for $\gamma \in H_{n-1}(\mathcal{U}_b,\mathbf{Z})$, map γ to the differential

$$v \mapsto -\int_\gamma \iota(v) \operatorname{Im} p^*\Omega_n$$

where again we choose an arbitrary lifting of v to $\mathcal{U}$.

Lemma 4.1. *The image of $R^1 f_*\mathbf{Z}$ in $\mathcal{T}_B^*$ is Lagrangian with respect to the standard symplectic form on $\mathcal{T}_B^*$. Thus $\mathcal{M} = \mathcal{T}_B^*/R^1 f_*\mathbf{Z}$ inherits this symplectic form, which we will call $\check{\omega}$.*

Proof. See Hitchin's paper [18]. His proof is as follows: in a small open set of B, choose $\Gamma \subseteq \mathcal{U}$ a family of $n-1$-dimensional submanifolds representing a section of $R^1 f_*\mathbf{Z}$ over U, $\pi : \Gamma \to B$ the projection. Then the section of $\mathcal{T}_B^*$ obtained by taking periods with respect to Γ is just the 1-form $-\pi_*((p^* \operatorname{Im}\Omega)|_\Gamma)$. Since Ω is a closed form, so is this push-down, and hence $-\pi_*((p^* \operatorname{Im}\Omega|_\Gamma)$ is a Lagrangian section of $\mathcal{T}_B^*$ with respect to the standard symplectic form. •

We will now clarify what the cohomology class of $\check{\omega}$ is. To do so, we will compare the Leray spectral sequences for f and $\check{f} : \mathcal{M} \to B$, but will use the de Rham realisation of these spectral sequences discussed in §3, which we can do as f and $\check{f}$ are smooth. Our construction yields a canonical isomorphism $H_{n-1}(\mathcal{U}_b,\mathbf{Z}) \cong H_1(\mathcal{M}_b,\mathbf{Z})$ and hence a canonical isomorphism $H^{n-1}(\mathcal{U}_b,\mathbf{Z}) \cong H^1(\mathcal{M}_b,\mathbf{Z})$, which yields a canonical isomorphism

$$R^{n-1} f_*\mathbf{R} \cong R^1 \check{f}_*\mathbf{R}. \tag{4.1}$$

Since $\operatorname{Im} p^*\Omega_n$ restricts to zero on the fibres of f, $\operatorname{Im} p^*\Omega_n \in F^1\Omega^n_{\mathcal{U}}$, and since $d\Omega_n = 0$, $\operatorname{Im} p^*\Omega_n$ in particular gives rise to a class $[\operatorname{Im} p^*\Omega_n]$ in $E^1_{1,n-1} = \Gamma(B, \Omega^1_B \otimes R^{n-1} f_*\mathbf{R})$. Now on $\mathcal{M}$, $\check{\omega} \in F^1(\Omega^2_{\mathcal{M}})$, as $\check{\omega}$ restricts to zero on the fibres of $\check{f}$, and thus $\check{\omega}$ determines an element $[\check{\omega}]$ of $\check{E}^1_{1,1} = \Gamma(B, \Omega^1_B \otimes R^1 \check{f}_*\mathbf{R})$. Via the isomorphism (4.1) we can identify $E^1_{1,n-1}$ and $\check{E}^1_{1,1}$.

Proposition 4.2. *Under this identification, $[\check{\omega}] = [\operatorname{Im} p^*\Omega_n]$. Thus in particular, they represent the same class in $H^1(B, R^1 \check{f}_*\mathbf{R}) \cong H^1(B, R^{n-1} f_*\mathbf{R})$.*

Proof. A section of $\Gamma(B, \Omega^1_B \otimes R^{n-1} f_*\mathbf{R})$ associates, to any tangent vector $v \in \mathcal{T}_{B,b}$, an element of $H^{n-1}(\mathcal{U}_b, \mathbf{R})$. Specifically, $[\operatorname{Im} p^*\Omega_n]$ associates to a tangent vector $v \in \mathcal{T}_{B,b}$ the cohomology class represented by $\iota(v)(\operatorname{Im} p^*\Omega_n)$. To determine what cohomology class this is, we choose a basis $\gamma_1, \ldots, \gamma_s$ of $H_{n-1}(\mathcal{U}_b, \mathbf{Z})$ and calculate the periods

$$\int_{\gamma_i} \iota(v) \operatorname{Im} p^*\Omega_n.$$

On $\mathcal{M}$, $[\check{\omega}] \in \Gamma(B, \Omega_B^1 \otimes R^1\check{f}_*\mathbf{R})$ is similarly represented by

$$v \mapsto \iota(v)\check{\omega},$$

and $\gamma_1, \ldots, \gamma_s$ also form a basis for $H_1(\mathcal{M}_b, \mathbf{Z})$ by construction. Recall that we embedded $H_{n-1}(\mathcal{U}_b, \mathbf{Z}) = H_1(\mathcal{M}_b, \mathbf{Z})$ in $\mathcal{T}^*_{B,b}$ by mapping γ_i to the differential

$$v \mapsto -\int_{\gamma_i} \iota(v) \operatorname{Im} p^*\Omega_n.$$

Now choosing local coordinates $y_1, \ldots, y_s$ on the base, $x_1, \ldots, x_s, y_1, \ldots, y_s$ canonical coordinates on $\mathcal{T}^*_B$,

$$\int_{\gamma_i} \iota(\partial/\partial y_j)\check{\omega} = -\int_{\gamma_i} dx_j = \int_{\gamma_i} \iota(\partial/\partial y_j) \operatorname{Im} p^*\Omega_n$$

by construction. Thus $[\check{\omega}] = [\operatorname{Im} p^*\Omega_n]$. •

We recall here for future use:

Observation 4.3. *(McLean [22]) Because $\mathcal{T}_{B,b}$ is naturally isomorphic to the space of harmonic 1-forms on $\mathcal{U}_b$, there is a metric h on $\mathcal{T}_B$ coming from the Hodge metric. Precisely, for $v \in \mathcal{T}_{B,b}$, $-(\iota(v)p^*\omega)|_{X_b}$ is the corresponding harmonic one-form, and $-*(\iota(v)p^*(\omega))|_{X_b} = (\iota(v)\operatorname{Im} p^*(\Omega))|_{X_b}$, and we define, for $v, w \in \mathcal{T}_{B,b}$,*

$$h(v,w) = -\int_{X_b} (\iota(v)p^*\omega) \wedge (\iota(v) \operatorname{Im} p^*\Omega).$$

This is a Riemannian metric on B.

Specialising down to the case that $f : X \to B$ is a special Lagrangian T^n-fibration with possible degenerate fibres, the above method gives us a way of constructing an open subset of the dual fibration along with a symplectic form on that open set. On $B_0 = B - \Delta$, we have defined an embedding $R^1f_{0*}\mathbf{Z} \hookrightarrow \mathcal{T}^*_{B_0}$. This allows us to define $\check{X}_0$ via the exact sequence

$$0 \to R^1f_{0*}\mathbf{Z} \to \mathcal{T}^*_{B_0} \to \check{X}_0 \to 0$$

and $\check{X}_0$ acquires a symplectic form $\check{\omega}$ inherited from the canonical symplectic form on $\mathcal{T}^*_{B_0}$.

Next we need to prove

Conjecture 4.4. *The embedding $R^1 f_{0*}\mathbf{Z} \hookrightarrow \mathcal{T}^*_{B_0}$ extends to an embedding*

$$R^1 f_*\mathbf{Z} \hookrightarrow \mathcal{T}^*_B.$$

If $\check{X}^\#$ is defined as $\mathcal{T}^*_B/R^1 f_*\mathbf{Z}$, then $\check{X}^\#$ is a manifold with symplectic form $\check{\omega}$ inherited from the standard symplectic form on $\mathcal{T}^*_B$. Furthermore, $\check{X}^\#$ can be compactified to a manifold $\check{X}$ with a map $\check{f} : \check{X} \to B$ extending $\check{f}^\#$, on which $\check{\omega}$ extends to a symplectic form on $\check{X}$.

This involves first understanding the asymptotic behaviour of the periods as one approaches Δ, as well as understanding the issue of compactification.

If this conjecture holds and f and $\check{f}$ are both $\mathbf{R}$-simple, then it is easy to see that $[\check{\omega}] \in H^1(B, R^1\check{f}_*\mathbf{R})$ coincides with $[\mathrm{Im}\,\Omega_n] \in H^1(B, R^{n-1}f_*\mathbf{R})$. Indeed, by Proposition 4.2, these classes agree in $H^1(B_0, R^1\check{f}_*\mathbf{R}) \cong H^1(B_0, R^{n-1}f_*\mathbf{R})$. However, since f and $\check{f}$ were assumed to be simple, $H^i_\Delta(B, R^j f_*\mathbf{R}) = 0$ for $i = 0$ and 1, and ditto for $\check{f}$. Thus there is an injection $H^1(B, R^i f_*\mathbf{R}) \hookrightarrow H^1(B - \Delta, R^i f_*\mathbf{R})$, and so the classes $[\check{\omega}]$ and $[\mathrm{Im}\,\Omega]$ agree also in $H^1(B, R^1\check{f}_*\mathbf{R}) \cong H^1(B, R^{n-1}f_*\mathbf{R})$.

$\check{f} : \check{X} \to B$ is not the only possible Lagrangian fibration we might construct. This fibration possesses a Lagrangian section by construction. By Theorem 2.6, any element of $H^1(B, \Lambda(\check{X}^\#))$ gives rise to another, locally isomorphic Lagrangian fibration $\check{g} : \check{Y} \to B$. See also Remark 3.12 and Conjecture 6.6.

Remark 4.5. Having constructed a symplectic form $\check{\omega}$ on $\check{X}$, or on an open subset of $\check{X}$, $\check{\omega}^n/n!$ defines an orientation on $\check{X}$. Thus we can check that this agrees with the choice of orientation on $\check{X}$ made in [14], Convention 4.3. First, note that in having fixed Ω, we have fixed an orientation on the fibres of $f : X \to B$. If we have fixed canonical coordinates $y_1, \ldots, y_n, x_1, \ldots, x_n$, then $\Omega|_{X_b} = V dx_1 \wedge \cdots \wedge dx_n$ with V a real function and either $V > 0$ or $V < 0$. By changing the order of the variables y_i, we can ensure $V > 0$, and then $dy_1 \wedge \cdots \wedge dy_n$ yields a canonical orientation on B. We will always assume our coordinates are so oriented. Note that this orientation on B is the same as that induced by the n-form Θ on B_0.

Now let us check Convention 4.3 of [14] is correct. Recall that the convention of [14] for the cohomology class of a fibre, $[X_b]$, was that

$$\int_{X_b} \alpha = \int_X \alpha \wedge [X_b].$$

With $\alpha = [\Omega]$, we then have

$$0 < \int_{X_b} \Omega = \int_X \Omega \wedge [X_b].$$

We can take $[X_b]$ to be the pull-back of a nowhere zero n-form on B, locally $f dy_1 \wedge \cdots \wedge dy_n$. Then $\Omega \wedge [X_b] = V f dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$, while $\omega^n/n! = (-1)^{n(n-1)/2} dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$. Since $V > 0$, we need $sign(f) = (-1)^{n(n-1)/2}$. Thus we take $[X_b]$ locally to be of the form $(-1)^{n(n-1)/2}|f| dy_1 \wedge \cdots \wedge dy_n$. Now the dual class of $(-1)^n[X_b]$ in $H^n(B, R^n\check{f}_*\mathbf{R})$ will be locally represented by something like $(-1)^{n(n+1)/2} g dy_1 \wedge \cdots \wedge dy_n \wedge dx_1 \wedge \cdots \wedge dx_n$, $g > 0$. This is the same sign as $\check{\omega}^n/n! = (-1)^{n(n+1)/2} dy_1 \wedge \cdots \wedge dy_n \wedge dx_1 \wedge \cdots \wedge dx_n$, so the orientations agree.

In the case of torus fibrations, we now describe an alternative way of putting a symplectic form on $\check{X}_0$. We do this by providing an alternative description of the embedding $R^1 f_{0*}\mathbf{Z} \hookrightarrow \mathcal{T}^*_{B_0}$, using the Riemannian metric h on the base defined in Observation 4.3. Since we obtained in §2 an embedding $R^{n-1} f_{0*}\mathbf{Z} \to \mathcal{T}^*_{B_0}$, we obtain dually an embedding $R^1 f_{0*}\mathbf{Z} \to \mathcal{T}_{B_0}$. We will then use a normalised form of h to identify $\mathcal{T}_{B_0}$ and $\mathcal{T}^*_{B_0}$.

First, given that there is an identification between $H^1(X_b, \mathbf{R})$ and $\mathcal{T}_{B,b}$, via $v \in \mathcal{T}_{B,b} \mapsto -\iota(v)\omega$, we have also a canonical identification of $H_{n-1}(X_b, \mathbf{R})$ with $\mathcal{T}_{B,b}$, via Poincaré duality. In fact, write $X_b = V/\Lambda$, $V = \mathcal{T}^*_{B,b}$, $\Lambda = H_1(X_b, \mathbf{Z}) \subseteq V$, $V^\vee = \mathcal{T}_{B,b}$, $\Lambda^\vee = \{\varphi \in V^\vee | \varphi(\Lambda) \subseteq \mathbf{Z}\} \subseteq V^\vee$. There is a canonical identification of $\Lambda^\vee$ with $H^1(X_b, \mathbf{Z})$. On the other hand, the identification $\bigwedge^n \Lambda \cong \mathbf{Z}$ determined by the orientation on X_b gives us a natural identification of $H_{n-1}(X_b, \mathbf{Z}) = \bigwedge^{n-1} \Lambda$ with $\Lambda^\vee$ via the perfect pairing

$$\Lambda \times \bigwedge^{n-1} \Lambda \to \bigwedge^{n} \Lambda \xrightarrow{\cong} \mathbf{Z}.$$

(Note: Whenever we use Poincaré duality, there is an arbitrary choice of order in this pairing which may affect the signs of the isomorphisms. This was seen in [14], where certain conventions were chosen. Here we also make a choice, and keep in mind that we could just as well have chosen the pairing $\bigwedge^{n-1} \Lambda \times \Lambda \to \mathbf{Z}$.)

Proposition 4.6. *If $\alpha \in H^{n-1}(X_b, \mathbf{R})$, $\gamma \in H_{n-1}(X_b, \mathbf{Z}) \cong \Lambda^\vee \subseteq \mathcal{T}_{B,b}$ via the above identification, then*

$$\int_\gamma \alpha = -\int_{X_b} \iota(\gamma)\omega \wedge \alpha.$$

Proof. We compute both sides using local action-angle coordinates. Let $y_1, \cdots, y_n$ be action coordinates as in Remark 2.8. Then the lattice $\Lambda \subseteq V = \mathcal{T}^*_{B,b}$ is generated by $e_1, \ldots, e_n$, $e_i = dy_i$, and then $e_1^*, \ldots, e_n^*$ form a dual basis for $\Lambda^\vee \subseteq \mathcal{T}_{B,b}$, $e_i^* = \partial/\partial y_i$. Suppose $\alpha = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$. Then

$$\begin{aligned} -\int_{X_b} \iota(e_i^*)\omega \wedge \alpha &= \int_{X_b} dx_i \wedge \alpha \\ &= (-1)^{j-1}\delta_{ij}. \end{aligned}$$

On the other hand, the isomorphism between $\Lambda^\vee$ and $\bigwedge^{n-1} \Lambda$ identifies e_i^* with $(-1)^{i-1} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$. The latter defines an oriented $n-1$-torus in X_b, namely the quotient of the subspace of V spanned by $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$ by the lattice generated by these vectors, and $\int_{e_i^*} \alpha$ is just the integral of α over this torus, which is clearly $(-1)^{j-1}\delta_{ij}$. Thus

$$\int_{e_i^*} \alpha = -\int_{X_b} \iota(e_i^*)\omega \wedge \alpha$$

and the result follows from linearity. •

This now gives us the opportunity to rephrase the embedding $H^1(X_b, \mathbf{Z}) \hookrightarrow \mathcal{T}^*_{B,b}$. Let h_n be the normalised metric on B_0 given by

$$h_n(v, w) = \frac{h(v, w)}{\int_{X_b} \Omega}.$$

Then we have

Proposition 4.7. *For $\gamma \in H^1(X_b, \mathbf{Z}) \cong H_{n-1}(X_b, \mathbf{Z}) = \Lambda^\vee \subseteq \mathcal{T}_{B,b}$, the 1-form*

$$v \mapsto -\int_\gamma \iota(v) \operatorname{Im} \Omega_n$$

coincides with the 1-form $-h_n(\gamma, \cdot)$. Thus the embedding $R^1 f_{0*}\mathbf{Z} \to \mathcal{T}^*_{B_0}$ previously defined coincides up to sign with the embedding $R^1 f_{0*}\mathbf{Z} \to \mathcal{T}_{B_0}$ composed with the isomorphism $\mathcal{T}_{B_0} \cong \mathcal{T}^*_{B_0}$ induced by the Riemannian metric h_n.

Proof. By Proposition 4.6,

$$\begin{aligned}
-\int_\gamma \iota(v) \operatorname{Im} \Omega_n &= \int_{X_b} \iota(\gamma)\omega \wedge \iota(v) \operatorname{Im} \Omega_n \\
&= \frac{1}{\int_{X_b} \Omega} \int_{X_b} \iota(\gamma)\omega \wedge \iota(v) \operatorname{Im} \Omega \\
&= -\frac{h(\gamma, v)}{\int_{X_b} \Omega}. \bullet
\end{aligned}$$

In fact, we see that h_n also describes the class $[\operatorname{Im} \Omega_n] \in E^1_{1,n-1} = \Gamma(B_0, \Omega^1_{B_0} \otimes R^{n-1} f_{0*}\mathbf{R})$:

Proposition 4.8. *Under the isomorphism*

$$E^1_{1,n-1} = \Gamma(B_0, \Omega^1_{B_0} \otimes R^{n-1} f_{0*}\mathbf{R}) \cong \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^1_{B_0})$$

given in §3, $[\operatorname{Im} \Omega_n]$ coincides with $h_n \in \Gamma(B_0, S^2\Omega^1_{B_0}) \subseteq \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^1_{B_0})$.

Proof. First note that for a point $b \in B$, $e_1^*, \ldots, e_n^*$ a basis of $H_{n-1}(X_b, \mathbf{Z}) \cong \Lambda^\vee \subseteq \mathcal{T}_{B,b}$, $[\operatorname{Im} \Omega_n]$ associates to a vector $v \in \mathcal{T}_{B,b}$ the class of $\iota(v) \operatorname{Im} \Omega_n \in H^{n-1}(X_b, \mathbf{R})$, and in terms of the periods,

$$\begin{aligned}
\int_{e_i^*} \iota(v) \operatorname{Im} \Omega_n &= -\int_{X_b} \iota(e_i^*)\omega \wedge \iota(v) \operatorname{Im} \Omega_n \\
&= h_n(e_i^*, v).
\end{aligned}$$

Thus, in action-angle coordinates as used in the proof of Theorem 3.14, $[\operatorname{Im} \Omega_n]$ corresponds to the element of $\Gamma(B_0, \Omega^1_{B_0} \otimes R^{n-1} f_{0*}\mathbf{R})$ given as

$$\sum_{i,j} h_n(\partial/\partial y_i, \partial/\partial y_j) dy_i \otimes (-1)^{j-1} dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n$$

which coincides with

$$\sum_{i,j} h_n(\partial/\partial y_i, \partial/\partial y_j)dy_i \otimes dy_j \in \Gamma(B_0, S^2\Omega^1_{B_0})$$

as desired. •

§5. Complex structures on special Lagrangian torus fibrations.

Recall from [18] the following: (for K3 surfaces, this was noticed in [31]; see also [30].)

Theorem 5.1. *Let X be a real $2n$-dimensional manifold. If Ω is a complex-valued C^∞ n-form on X satisfying the three properties*
(1) $d\Omega = 0$;
(2) Ω is locally decomposable (i.e. can be written locally as $\theta_1 \wedge \cdots \wedge \theta_n$ where $\theta_1, \ldots, \theta_n$ are 1-forms);
(3) $(-1)^{n(n-1)/2}(i/2)^n\Omega \wedge \bar{\Omega} > 0$ everywhere on X,

then Ω determines a complex structure on X for which Ω is a holomorphic n-form.

Theorem 5.2. *Let X be a real $2n$-dimensional manifold. Suppose ω is a symplectic form on X and Ω is a complex-valued n-form on X such that*
(1) Ω satisfies the conditions of Theorem 5.1;
(2) ω is a positive $(1,1)$ form in the complex structure of Theorem 5.1;
(3) $(-1)^{n(n-1)/2}(i/2)^n\Omega \wedge \bar{\Omega} = \omega^n/n!$.

Then Ω induces a complex structure on X such that ω is a Kähler form on X whose corresponding metric is Ricci-flat.

Now let $f : X \to B$ be an integral special Lagrangian fibration with a Lagrangian section, so that in local coordinates, ω takes the standard form. Let Ω be the holomorphic n-form on X normalised so that $\operatorname{Im}\Omega|_{X_b} = 0$ for all $b \in B$, $\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n\Omega \wedge \bar{\Omega}$, and $\int_{X_b} \Omega = Vol(X_b)$ with respect to the metric induced by the Kähler form ω. As before, we set $\Omega_n = \Omega / \int_{X_b} \Omega$.

Following the suggestion of [18], since Ω is locally decomposable, we can write, for local coordinates $y_1, \ldots, y_n$ on B as usual,

$$\Omega = V \bigwedge_i (dx_i + \sum_j \beta_{ij} dy_j),$$

where V is a real function of $y_1, \ldots, x_n$ and $V|_{X_b} dx_1 \wedge \cdots \wedge dx_n$ is the volume form on X_b, while β_{ij} is a complex valued function. Using Remark 4.5, we will always assume that $V > 0$. The forms of type $(1,0)$ are spanned by the 1-forms $\theta_i := dx_i + \sum_{i,j} \beta_{ij} dy_j$. Thus the entire complex structure is encoded in the matrix (β_{ij}). We now look to see how the conditions of Theorems 5.1 and 5.2 translate into conditions for the functions V and β_{ij}:

Calculation 5.3.

$$(-1)^{n(n-1)/2}(i/2)^n\Omega \wedge \bar{\Omega} = V^2 \det(\operatorname{Im}\beta)\omega^n/n!.$$

Proof. Write

$$\Omega = V\theta_1 \wedge \cdots \wedge \theta_n.$$

Now

$$\begin{aligned}\theta_i \wedge \bar{\theta}_i &= \sum_j (\bar{\beta}_{ij} - \beta_{ij}) dx_i \wedge dy_j + \cdots \\ &= -2i dx_i \wedge \left(\sum_j \operatorname{Im}\beta_{ij} dy_j\right) + \cdots.\end{aligned}$$

Thus

$$\Omega \wedge \bar{\Omega} = V^2(-2)^n i^n \det(\operatorname{Im}\beta) dx_1 \wedge dx_2 \wedge \ldots \wedge dy_n$$

so

$$(-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega} = (-1)^{n(n-1)/2} V^2 \det(\operatorname{Im}\beta) dx_1 \wedge dx_2 \wedge \ldots \wedge dy_n.$$

On the other hand,

$$\omega^n/n! = (-1)^{n(n-1)/2} dx_1 \wedge \cdots \wedge dy_n,$$

hence the result. •

Calculation 5.4. *ω is a positive form of type* $(1,1)$ *if and only if β is symmetric and* $\operatorname{Im}\beta$ *is positive definite.*

Proof. We first examine the condition that ω is of type $(1,1)$, i.e. we can write

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij}\theta_i \wedge \bar{\theta}_j.$$

Now

$$\begin{aligned}\theta_i \wedge \bar{\theta}_j =& dx_i \wedge dx_j + dx_i \wedge \left(\sum_k \bar{\beta}_{jk} dy_k\right) - dx_j \wedge \left(\sum_k \beta_{ik} dy_k\right) \\ &+ \sum_{k,l} \beta_{ik}\bar{\beta}_{jl} dy_k \wedge dy_l\end{aligned}$$

and

$$\begin{aligned}\sum h_{ij}\theta_i \wedge \bar{\theta}_j =& \sum_{i<j}(h_{ij} - h_{ji}) dx_i \wedge dx_j + \sum_i dx_i \wedge \left(\sum_{j,k} h_{ij}\bar{\beta}_{jk} dy_k - \sum_{j,k} h_{ji}\beta_{jk} dy_k\right) \\ &+ \sum_{i,j,k,l} h_{ij}\beta_{ik}\bar{\beta}_{jl} dy_k \wedge dy_l.\end{aligned}$$

Thus in order for $\omega = \sum dx_i \wedge dy_i = \frac{i}{2}\sum h_{ij}\theta_i \wedge \bar{\theta}_j$, we must have, in particular, $h_{ij} = h_{ji}$, i.e. h is symmetric. On the other hand, since ω is real, $h_{ij} = \bar{h}_{ji}$, and thus the matrix h is real. Also,

$$\begin{aligned}I &= \frac{i}{2}(h\bar{\beta} - h\beta) \\ &= h \operatorname{Im}\beta.\end{aligned}$$

Thus we have $h = (\operatorname{Im}\beta)^{-1}$, and $\operatorname{Im}\beta$ is symmetric.

To ensure the last term vanishes, we need

$$\sum_{i,j} h_{ij}\beta_{ik}\bar{\beta}_{jl} = \sum_{i,j} h_{ij}\beta_{il}\bar{\beta}_{jk},$$

or equivalently, the matrix ${}^t\beta h\bar{\beta}$ is symmetric.

Now

$$\begin{aligned} {}^t\beta h\bar{\beta} &= (\operatorname{Re}{}^t\beta + i\operatorname{Im}\beta)(\operatorname{Im}\beta)^{-1}(\operatorname{Re}\beta - i\operatorname{Im}\beta) \\ &= (\operatorname{Re}{}^t\beta)(\operatorname{Im}\beta)^{-1}(\operatorname{Re}\beta) + i\operatorname{Re}\beta - i\operatorname{Re}{}^t\beta + \operatorname{Im}\beta, \end{aligned}$$

while

$$\begin{aligned} {}^t\bar{\beta} h\beta &= (\operatorname{Re}{}^t\beta - i\operatorname{Im}\beta)(\operatorname{Im}\beta)^{-1}(\operatorname{Re}\beta + i\operatorname{Im}\beta) \\ &= (\operatorname{Re}{}^t\beta)(\operatorname{Im}\beta)^{-1}(\operatorname{Re}\beta) + i\operatorname{Re}{}^t\beta - i\operatorname{Re}\beta + \operatorname{Im}\beta, \end{aligned}$$

so symmetry of ${}^t\beta h\bar{\beta}$ is equivalent to $\operatorname{Re}{}^t\beta = \operatorname{Re}\beta$.

Thus ω is of type $(1,1)$ if and only if β is symmetric. In addition, to ensure ω is a positive $(1,1)$ form, $h = (\operatorname{Im}\beta)^{-1}$ must be positive definite, so $\operatorname{Im}\beta$ must be positive definite. •

The real problem is understanding the condition $d\Omega = 0$. This is the heart of the difficulty, and we will return to this shortly.

We first connect Ω to the description of the choice of almost complex structure given in the introduction, namely as a choice of horizontal subspaces of an Ehresmann connection and the choice of a metric on the fibres.

Proposition 5.5. *The matrix* $(\operatorname{Im}\beta)^{-1}$ *is the matrix of the metric* (g_{ij}) *on the fibres of* f. *For a point* $x \in X_b^{\#}$, $J(\mathcal{T}_{X_b,x})$ *is spanned by the tangent vectors*

$$\{\partial/\partial y_j - \sum_i \operatorname{Re}\beta_{ij}\partial/\partial x_i | 1 \le j \le n\},$$

where $J : \mathcal{T}_{X^\#} \to \mathcal{T}_{X^\#}$ *is the almost complex structure induced by* Ω.

Proof. Since $\omega = \frac{i}{2}\sum_{i,j} h_{ij}\theta_i \wedge \bar{\theta}_j$ with $h = (\operatorname{Im}\beta)^{-1}$ is the Kähler form of the metric, the Kähler metric itself is $g = \sum_{i,j} h_{ij}\theta_i \otimes \bar{\theta}_j$. Thus $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j) = h_{ij}$, giving the interpretation of $\operatorname{Im}\beta$.

Next, let J be the almost complex structure on $\mathcal{T}_{X^\#}$ induced by Ω, and ${}^tJ$ the almost complex structure on $\mathcal{T}^*_{X^\#}$. Since the space spanned by $\theta_1, \ldots, \theta_n$ is the $+i$ eigenspace of ${}^tJ$ at a point $x \in X^\#$, the cotangent space $\mathcal{T}^*_{X,x}$ decomposes as $V_1 \oplus V_2$ with

$$\begin{aligned} V_1 &= span(\operatorname{Re}\theta_1, \ldots, \operatorname{Re}\theta_n) \\ V_2 &= span(\operatorname{Im}\theta_1, \ldots, \operatorname{Im}\theta_n), \end{aligned}$$

with ${}^tJ(V_1) = V_2$ and ${}^tJ(V_2) = V_1$. Note that $V_2 = span(dy_1, \ldots, dy_n)$ since $\operatorname{Im}\beta$ is invertible. Thus also $\mathcal{T}_{X,x} = V_1^\circ \oplus V_2^\circ$, V_i° the annihilator of V_i, with J interchanging

V_1° and V_2°. Now

$$V_2^\circ = span(\partial/\partial x_1, \ldots, \partial/\partial x_n)$$
$$V_1 = span(\{dx_i + \sum_j \mathrm{Re}\,\beta_{ij} dy_j\})$$
$$V_1^\circ = span(\{\partial/\partial y_j - \sum_i \mathrm{Re}\,\beta_{ij} \partial/\partial x_i\}).$$

Thus we see that $\mathrm{Re}\,\beta$ determines $J(\mathcal{T}_{X_b,x}) = V_1^\circ$ as claimed. •

Summarizing, we now have

Theorem 5.6. *Specifying an n-form Ω on $X^\#$ satisfying properties (2) and (3) of Theorem 5.1 and properties (2) and (3) of Theorem 5.2 and such that $f^\#$ is a special Lagrangian fibration with respect to Ω is equivalent to specifying*

(1) A metric (g_{ij}) on each fibre of $f^\# : X^\# \to B$.

(2) A splitting $\mathcal{T}_{X^\#} = \mathcal{T}_{X^\#/B} \oplus \mathcal{F}$, where $\mathcal{T}_{X^\#/B}$ is the subbundle of $\mathcal{T}_{X^\#}$ with $\mathcal{T}_{X^\#/B,x} = \mathcal{T}_{X_b^\#,x}$, and $\mathcal{F}$ is a Lagrangian subbundle of $\mathcal{T}_{X^\#}$.

Proof. Proposition 5.5 shows that Ω specifies the metric on the fibres and a splitting as desired with $\mathcal{F} = J(\mathcal{T}_{X^\#/B})$. This is clearly a Lagrangian subbundle since $\mathcal{T}_{X^\#/B}$ is.

Conversely, giving a splitting of the exact sequence

$$0 \longrightarrow \mathcal{T}_{X^\#/B} \longrightarrow \mathcal{T}_{X^\#} \xrightarrow{p} \mathcal{F} \longrightarrow 0$$

determines $\mathrm{Re}\,\beta$ for us. Indeed, at a point $x \in X^\#$, with local coordinates as usual, such a splitting gives a map $s : \mathcal{F}_x \to \mathcal{T}_{X^\#,x}$, and there is a matrix (b_{ij}) such that

$$s(p(\partial/\partial y_j)) = \partial/\partial y_j - \sum b_{ij} \partial/\partial x_i.$$

We take $\mathrm{Re}\,\beta_{ij} = b_{ij}$. Note that the symmetry of the matrix b_{ij} is equivalent to $s(\mathcal{F}_x)$ being Lagrangian. Thus specifying (1) and (2) is equivalent, in local coordinates, to giving $\beta_{ij} = b_{ij} + ig^{ij}$, where g_{ij} is the metric on the fibre. Then we must have, with

$$\theta_i = dx_i + \sum_j \beta_{ij} dy_j,$$
$$\Omega = V\theta_1 \wedge \cdots \wedge \theta_n$$

for some real function V. Calculation 5.4 then tells us that ω is a positive $(1,1)$-form in the corresponding almost complex structure, since by construction β is symmetric and $\mathrm{Im}\,\beta$ is positive definite. Calculation 5.3 shows that $\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n \Omega \wedge \bar{\Omega}$ if and only if $V = \sqrt{\det(g_{ij})}$ since $(\sqrt{\det(g_{ij})})^2 \det(g^{ij}) = 1$. •

We have now seen how the n-form Ω can be determined by choosing a matrix $\beta = (\beta_{ij}) = (b_{ij} + ig^{ij})$, so that

$$\Omega = \sqrt{\det(g_{ij})} \bigwedge_i (dx_i + \sum_j \beta_{ij} dy_j).$$

If β is chosen to be symmetric and $\operatorname{Im}\beta$ positive definite, then we have seen that $d\Omega = 0$ implies both the integrability of the almost complex structure induced by Ω and the Ricci-flatness of the metric induced by this complex structure and the standard symplectic form ω. At first sight, the condition $d\Omega = 0$ looks very complicated if one proceeds by brute force and tries to compute the exterior derivative of Ω. In fact, this condition can be simplified, and we wish to examine this here. We note that the calculation below of $d\Omega$ is quite similar to calculations carried out in [29] and [30] for an analagous situation in the study of deformations of complex structures on Calabi-Yau manifolds. However, we will introduce a formalism using differential operators to make the calculation easier.

To begin, first note that the integrability of the almost complex structure determined by Ω is a weaker condition than $d\Omega = 0$. Let us first understand this weaker condition.

Theorem 5.7. *The almost complex structure induced by Ω is integrable if and only if*

$$\sum_i \left(\frac{\partial \beta_{lk}}{\partial x_i}\beta_{ij} - \frac{\partial \beta_{lj}}{\partial x_i}\beta_{ik} \right) = \frac{\partial \beta_{lk}}{\partial y_j} - \frac{\partial \beta_{lj}}{\partial y_k}.$$

for all j, k and l.

Proof. Writing as before

$$\Omega = V\theta_1 \wedge \cdots \wedge \theta_n,$$

the almost complex structure is determined by the fact that $\theta_1, \ldots, \theta_n$ should span the space of $(1,0)$ forms. To show that the almost complex structure induced by $\theta_1, \ldots, \theta_n$ is integrable we need to show that $d\theta_i$ is of type $(2,0) + (1,1)$ for all i. Since

$$\theta_i = dx_i + \sum_{j=1}^n \beta_{ij} dy_j,$$

$\theta_1, \ldots, \theta_n, dy_1, \ldots, dy_n$ form a basis for the space of 1-forms, and thus we can write

$$d\theta_l = \sum_{i,j} \frac{1}{2} A^l_{ij}\theta_i \wedge \theta_j + \sum_{ij} B^l_{ij}\theta_i \wedge dy_j + \sum_{i,j} \frac{1}{2} C^l_{ij} dy_i \wedge dy_j.$$

The almost complex structure is integrable if and only if $C^l_{ij} = 0$ for all i, j and l. Here A^l and C^l are skew-symmetric matrices. Note

$$d\theta_l = \sum_{i,j} \frac{\partial \beta_{lj}}{\partial x_i} dx_i \wedge dy_j + \sum_{i,j} \frac{\partial \beta_{lj}}{\partial y_i} dy_i \wedge dy_j.$$

Since $d\theta_l$ contains no $dx_i \wedge dx_j$ terms, we must have $A^l_{ij} = 0$, and then $B^l_{ij} = \partial\beta_{lj}/\partial x_i$. Then the almost complex structure is integrable if and only if

$$\begin{aligned} d\theta_l &= \sum_{i,j} B^l_{ij}\theta_i \wedge dy_j \\ &= \sum_{i,j} \frac{\partial \beta_{lj}}{\partial x_i} dx_i \wedge dy_j + \sum_{i,j,k} \frac{\partial \beta_{lj}}{\partial x_i}\beta_{ik} dy_k \wedge dy_j \end{aligned}$$

which holds if and only

$$\sum_i \left(\frac{\partial \beta_{lk}}{\partial x_i}\beta_{ij} - \frac{\partial \beta_{lj}}{\partial x_i}\beta_{ik}\right) = \frac{\partial \beta_{lk}}{\partial y_j} - \frac{\partial \beta_{lj}}{\partial y_k}.$$

This is the desired condition. •

We wish to rephrase this condition. We are going work locally for the moment, fixing coordinates $y_1, \ldots, y_n$ on an open subset $U \subseteq B$, and consider all forms as living on $U \times \mathbf{R}^n$ with coordinates $y_1, \ldots, y_n, x_1, \ldots, x_n$. We then write

$$\beta = \sum_{i,j} \beta_{ij} dy_j \otimes \frac{\partial}{\partial x_i} \in \Gamma(U \times \mathbf{R}^n, f^*\Omega^1_B \otimes \mathcal{T}_{X/B}).$$

(This is not a coordinate independent expression. The correct coordinate independent expression would be

$$\sum_i dx_i \otimes \frac{\partial}{\partial x_i} + \beta \in \Gamma(f^{-1}(U), \Omega^1_X \otimes \mathcal{T}_{X/B}),$$

but for practical purposes it is more convenient to work with the above expression.)

Definition 5.8. *For expressions of the type $v = \sum_j dy_j \otimes v_j$, $w = \sum_j dy_j \otimes w_j$ with v_j, w_j vector fields, $v_j = \sum_i v_{ij}\partial/\partial x_i$, $w_j = \sum_i w_{ij}\partial/\partial x_i$, we define*

$$[v, w] = \sum_{l,m} [v_l, w_m] dy_l \wedge dy_m$$

where

$$[v_l, w_m] = \sum_{i,j} \left(v_{il}\frac{\partial w_{jm}}{\partial x_i} - w_{im}\frac{\partial v_{jl}}{\partial x_i}\right)\frac{\partial}{\partial x_j}$$

is the usual Lie bracket of vector fields. We also set

$$d_y v = \sum_{i,j,k} \frac{\partial v_{ij}}{\partial y_k} dy_k \wedge dy_j \otimes \frac{\partial}{\partial x_i}.$$

We then obtain

Theorem 5.9. *The almost complex structure induced by Ω is integrable if and only if*

$$d_y\beta - \frac{1}{2}[\beta, \beta] = 0.$$

Proof.

$$d_y\beta = \sum_{\substack{l \\ i<j}} \left(\frac{\partial \beta_{lj}}{\partial y_i} - \frac{\partial \beta_{li}}{\partial y_j}\right) dy_i \wedge dy_j \frac{\partial}{\partial x_l}$$

while

$$[\beta, \beta] = \sum_{\substack{i,j \\ l<m}} 2\left(\beta_{il}\frac{\partial \beta_{jm}}{\partial x_i} - \beta_{im}\frac{\partial \beta_{jl}}{\partial x_i}\right) dy_l \wedge dy_m \frac{\partial}{\partial x_j}.$$

Comparing with the formula of Theorem 5.7, we see that the two formulae are equivalent. •

Our next goal is to prove

Theorem 5.10. *$d\Omega = 0$ if and only if the almost complex structure induced by Ω is integrable and $d\Omega \subseteq F^2\Omega_X^{n+1}$.*

Thus, locally, one just needs to check that the coefficients of $dy_i \wedge dx_1 \wedge \cdots \wedge dx_n$, $1 \le i \le n$ in $d\Omega$ are zero, and also check the integrability condition.

To prove this theorem, we must introduce some additional algebraic structure to accomplish the calculation. We continue to use a choice of local coordinates, and work on the space $U \times \mathbf{R}^n$, $f : U \times \mathbf{R}^n \to U$ the projection. Let $\mathcal{T}_x$ be the subbundle of the tangent bundle of $U \times \mathbf{R}^n$ generated by $\partial/\partial x_1, \ldots, \partial/\partial x_n$, and let Ω_y be the subbundle of the cotangent bundle generated by $dy_1, \ldots, dy_n$. For $q \ge 0, p \le 0$, set

$$\Omega_U^{p,q} = \Gamma(\bigwedge^q \Omega_y \otimes \bigwedge^{-p} \mathcal{T}_x)$$

and set $\Omega_0 = dx_1 \wedge \cdots \wedge dx_n$. Then there is an isomorphism

$$\bigoplus_{i=0}^{p} \Omega_U^{p-n-i,i} \xrightarrow{\cong} \Gamma(\Omega^p_{U\times\mathbf{R}^n}) \tag{5.1}$$

where $\Omega^p_{U\times\mathbf{R}^n}$ denotes the sheaf of C^∞ p-forms on $U \times \mathbf{R}^n$. This isomorphism sends $\theta \otimes v$ to $\theta \wedge \iota(v)\Omega_0$. In particular, $\Gamma(\Omega^n_{U\times\mathbf{R}^n}) \cong \bigoplus_{p=0}^n \Omega_U^{-p,p}$.

For $\alpha \otimes \beta \in \Omega_U^{p,q}$, $\alpha' \otimes \beta' \in \Omega_U^{p',q'}$, we can define the product

$$(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') := (\alpha \wedge \alpha') \otimes (\beta \wedge \beta') \in \Omega_U^{p+p',q+q'}.$$

This satisfies the commutation relations

$$(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (-1)^{pp'+qq'}(\alpha' \otimes \beta') \cdot (\alpha \otimes \beta).$$

This gives us a bigraded ring structure on $\bigoplus \Omega_U^{p,q}$. Note that the subring $\bigoplus_{p=0}^n \Omega_U^{-p,p} \cong \Gamma(\Omega^n_{U\times\mathbf{R}^n})$ is in fact a commutative ring with 1, and $1 \in \Omega_U^{0,0}$ corresponds to Ω_0 under this isomorphism.

Lemma 5.11. *Under the isomorphism (5.1),*

$$\bigwedge_{i=1}^{n}\left(dx_i + \sum_j \beta_{ij}dy_j\right) = exp(\beta) := \sum_{p=0}^{\infty} \beta^p/p!$$

where

$$\beta = \sum_{i,j} \beta_{ij} dy_j \otimes \frac{\partial}{\partial x_i}$$

in the notation introduced above.

Proof. This is a straightforward though slightly tedious calculation. Here is where all the signs must be dealt with correctly. For this and subsequent calculations it is convenient to keep in mind that $\iota(\partial/\partial x_I)\Omega_0 = (-1)^M dx_{I^*}$, where $I^* = \{1, \ldots, n\} - I$ and $M = \#\{(i,j) | i \in I, j \in I^*, i > j\}$. •

The next step is to turn $\Omega_X^{\cdot\cdot}$ into a double complex. We have exterior differentiation

$$d : \Gamma(\Omega^i_{U\times\mathbf{R}^n}) \to \Gamma(\Omega^{i+1}_{U\times\mathbf{R}^n}),$$

and under the isomorphism (5.1), it is clear that $d(\Omega_U^{p,q}) \subseteq \Omega_U^{p+1,q} \oplus \Omega_U^{p,q+1}$. Thus we can write $d = d_x + d_y$, so that $(\Omega_U^{\cdot\cdot}, d_x, d_y)$ defines a bicomplex, with

$$d_x : \Omega_U^{p,q} \to \Omega_U^{p+1,q},$$

$$d_y : \Omega_U^{p,q} \to \Omega_U^{p,q+1}.$$

One checks that

$$d_x(dy_J \otimes \alpha) = (-1)^q \sum_{i=1}^n dy_J \otimes (\frac{\partial\alpha}{\partial x_i} \llcorner dx_i). \tag{5.2}$$

(Here, $(\partial/\partial x_I \wedge \partial/\partial x_i)\llcorner dx_i = \partial/\partial x_I$, and $\partial/\partial x_I \llcorner dx_i = 0$ if $i \notin I$.)

Let $D : \Omega_U \to \Omega_U$ be a graded endomorphism of $\Omega_U := \Omega_U^{\cdot\cdot}$. We now recall what it means for D to be a differential operator of order $\leq r$. Put on $\Omega_U \otimes \Omega_U$ the anti-commutative algebra structure given by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{(\deg a')\cdot(\deg b)} aa' \otimes bb'.$$

Here the dot is the standard dot product, keeping in mind Ω_U is bigraded. This turns $\Omega_U \otimes \Omega_U$ into a bigraded anti-commutative algebra. Let $\lambda : \Omega_U \to \Omega_U \otimes \Omega_U$ be given by $\lambda(a) = a \otimes 1 - 1 \otimes a$. We define

$$\Phi_D^r : \Omega_U^r \to \Omega_U$$

by

$$\Phi_D^r(a_1, \ldots, a_r) = m \circ (D \otimes id_{\Omega_U})(\prod_{i=1}^r \lambda(a_i)).$$

Here $m(a \otimes b) = ab$. We say D is a differential operator of order $\leq r$ if Φ_D^{r+1} is identically zero. Note that

$$\Phi_D^2(a, b) = D(ab) - D(a)b - (-1)^{(\deg a)\cdot(\deg b)} D(b)a + D(1)ab$$

and

$$\Phi_D^3(a, b, c) = \Phi_D^2(a, bc) - \Phi_D^2(a, b)c - (-1)^{(\deg b)\cdot(\deg c)} \Phi_D^2(a, c)b.$$

(See [20], §1.) As usual, the composition of differential operators of orders $\leq r$ and s is order $\leq r + s$.

Definition 5.12. *We set $d_x' : \Omega_U \to \Omega_U$ to be the operator acting on $\Omega_U^{p,q}$ by $(-1)^{p+q+1} d_x$.*

Lemma 5.13. *d_x is a differential operator of order ≤ 2 and d_y is a differential operator of order ≤ 1.*

Proof. That d_y is a differential operator of order ≤ 1 follows immediately from the definition, while (5.2) shows that d_x' can be written as a sum of a composition of two operators: differentiation in the direction $\partial/\partial x_i$ and $\alpha\otimes\beta \mapsto \alpha\otimes(-1)^{p+1}\beta\llcorner dx_i$. One checks easily that these are each first order operators, and hence d_x is second order. •

Now define, for $\alpha, \beta \in \Omega_X$,

$$[\alpha,\beta] := \Phi^2_{d'_x}(\alpha,\beta),$$

so

$$d'_x(\alpha\beta) = [\alpha,\beta] + d'_x(\alpha)\beta + (-1)^{(\deg\alpha)\cdot(\deg\beta)}d'_x(\beta)\alpha. \tag{5.3}$$

Since $\Phi^3_{d'_x} = 0$, we obtain

$$[\alpha,\beta\gamma] = [\alpha,\beta]\gamma + (-1)^{(\deg\beta)\cdot(\deg\gamma)}[\alpha,\gamma]\beta, \tag{5.4}$$

and from Lemma 5.13,

$$d_y(\alpha\beta) = d_y(\alpha)\beta + (-1)^{(\deg\alpha)\cdot(\deg\beta)}d_y(\beta)\alpha. \tag{5.5}$$

We note that this definition of bracket is an extension of Definition 5.8:

Proposition 5.14. *If $\alpha, \beta \in \Omega^{-1,1}$, then $[\alpha,\beta]$ as defined above agrees with the earlier definition.*

Proof. By linearity, we can assume that $\alpha = f dy_j \partial/\partial x_i$ and $\beta = g dy_l \partial/\partial x_k$. Then

$$\begin{aligned}
[\alpha,\beta] =& - d_x(\alpha\beta) + d_x(\alpha)\beta + d_x(\beta)\alpha\\
=& - d_x(fg dy_j \wedge dy_l \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_k}) + d_x(f dy_j \frac{\partial}{\partial x_i}) g dy_l \frac{\partial}{\partial x_k}\\
&+ d_x(g dy_l \frac{\partial}{\partial x_k}) f dy_j \frac{\partial}{\partial x_i}\\
=& - \left(\frac{\partial f}{\partial x_k} g + f\frac{\partial g}{\partial x_k}\right) dy_j \wedge dy_l \frac{\partial}{\partial x_i} + \left(\frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i}\right) dy_j \wedge dy_l \frac{\partial}{\partial x_k}\\
&- g\frac{\partial f}{\partial x_i} dy_j \wedge dy_l \frac{\partial}{\partial x_k} + f\frac{\partial g}{\partial x_k} dy_j \wedge dy_l \frac{\partial}{\partial x_i}\\
=& f\frac{\partial g}{\partial x_i} dy_j \wedge dy_l \frac{\partial}{\partial x_k} - \frac{\partial f}{\partial x_k} g dy_j \wedge dy_l \frac{\partial}{\partial x_i},
\end{aligned}$$

proving the desired equality. •

Proof of Theorem 5.10. First suppose that $d\Omega = 0$. Then the almost complex structure is integrable by Theorem 5.1, and obviously $d\Omega \subseteq F^2\Omega^{n+1}_X$.

Conversely, suppose $d\Omega \subseteq F^2\Omega^{n+1}_X$ and the almost complex structure is integrable. We use the isomorphism (5.1) and Lemma 5.11 to write $\Omega = V\exp(\beta)$ for some $\beta \in \Omega^{-1,1}$, $V \in \Omega^{0,0}$. To show that $d\Omega = 0$, we need to show that each graded piece of $d\Omega$ is zero, i.e.

$$\frac{1}{n!}d_y(V\beta^n) + \frac{1}{(n+1)!}d_x(V\beta^{n+1}) = 0, \quad \text{for all } n \geq 0.$$

This is equivalent to

$$d_y(V\beta^n) = \frac{1}{n+1}d'_x(V\beta^{n+1}). \tag{5.6}$$

The fact that $d\Omega \subseteq F^2\Omega_X^{n+1}$ is equivalent to (5.6) for $n = 0$. Note that this states

$$d_y(V) = d'_x(V\beta) = [\beta, V] + d'_x(\beta)V \tag{5.7}$$

by (5.3), since $d'_x(V) = 0$. Observe that since $[\beta, \cdot]$ acts as an ordinary (non-graded) derivation on the commutative ring $\bigoplus_p \Omega^{-p,p}$, we have $[\beta, \beta^n] = n[\beta,\beta]\beta^{n-1}$.

We prove by induction that (5.7) implies

$$d'_x(V\beta^{n+1}) = (n+1)d_y(V)\beta^n + \frac{n(n+1)}{2}[\beta,\beta]V\beta^{n-1}. \tag{5.8}$$

Indeed this is true for $n = 0$, by (5.7). Then

$$\begin{aligned} d'_x(V\beta^{n+1}) &= d'_x(\beta \cdot V\beta^n) \\ &= [\beta, V\beta^n] + d'_x(\beta)V\beta^n + d'_x(V\beta^n)\beta \\ &= [\beta, V]\beta^n + n[\beta,\beta]V\beta^{n-1} + d'_x(\beta)V\beta^n + d'_x(V\beta^n)\beta \\ &= d_y(V)\beta^n + n[\beta,\beta]V\beta^{n-1} + d'_x(V\beta^n)\beta \end{aligned}$$

by (5.7), so by induction the desired result holds.

Next, we note using the integrability condition $d_y\beta = [\beta,\beta]/2$ and (5.8) that

$$\begin{aligned} d_y(V\beta^n) &= d_y(V)\beta^n + d_y(\beta^n)V \\ &= d_y(V)\beta^n + nd_y(\beta)V\beta^{n-1} \\ &= d_y(V)\beta^n + \frac{n}{2}[\beta,\beta]V\beta^{n-1} \\ &= \frac{1}{n+1}d'_x(V\beta^{n+1}). \end{aligned}$$

This proves (5.6), and hence the theorem. •

Next I would like to reinterpret the equations we've seen above so that they may look more natural. As we have seen earlier, $b = \operatorname{Re}\beta$ defines an Ehresmann connection whose horizontal subspaces, given by the subbundle $\mathcal{F}$, are Lagrangian. In local coordinates, this connection is determined by

$$b = \sum_{i,j} b_{ij}dy_j \otimes \frac{\partial}{\partial x_i} = \operatorname{Re}\beta.$$

It makes sense to define the covariant derivative with respect to this connection. This will be an operator

$$\nabla_b : \Gamma(f^*\Omega^q_B \otimes \mathcal{T}_{X^\#/B}) \to \Gamma(f^*\Omega^{q+1}_B \otimes \mathcal{T}_{X^\#/B})$$

defined by

$$\nabla_b \alpha := d_y\alpha - [b, \alpha].$$

It is easy to check that this definition is now independent of the choice of coordinates. The curvature tensor of ∇_b is then $F_b \in \Gamma(f^*\Omega^2_B \otimes \mathcal{T}_{X^\#/B})$ given by

$$F_b := d_y b - \frac{1}{2}[b, b].$$

It is easy to check that $F_b = 0$ if and only if the horizontal distribution $\mathcal{F}$ is integrable. Of course, an Ehresmann connection gives rise to parallel transport along a path contained in B_0; we say a family of p-forms on the fibres of $f_0 : X_0 \to B_0$ is *parallel* if it is invariant under parallel transport. If in local coordinates over $U \subseteq B_0$ this family of forms is written as $\alpha = \sum_I f_I dx_I$, f_I a function on $f^{-1}(U)$, α is parallel if

$$d_y\alpha - \mathcal{L}_b\alpha = 0;$$

by this we mean

$$\frac{\partial\alpha}{\partial y_j} - \mathcal{L}_{\sum_i b_{ij}\partial/\partial x_i}\alpha = 0$$

for each j. (For a similar treatment of Ehresmann connections, see [21]).

We can now rephrase the integrability conditions in a more invariant way.

Corollary 5.15. *Let $(\beta_{ij}) = (b_{ij} + ig^{ij})$, so we write $\beta = b + ig^{-1}$, $V = \sqrt{\det g}$. Then $d\Omega = 0$ if and only if*

(5.9) $$F_b + \frac{1}{2}[g^{-1}, g^{-1}] = 0$$

(5.10) $$\nabla_b g^{-1} = 0$$

(5.11) $dx_1, \ldots, dx_n$ *are harmonic forms on each fibre*

(5.12) $V dx_1 \wedge \cdots \wedge dx_n$ *is parallel.*

Proof. The first two equations are the real and imaginary parts of $d_y\beta - \frac{1}{2}[\beta, \beta] = 0$. The last two are (5.7) broken up again into its real and imaginary parts. •

We remark here that in the study of Ricci curvature in the context of Riemannian submersions, some similar structures arise. See [8], Chapter 9.

Corollary 5.16. *Suppose $d\Omega = 0$ and ∇_b is flat. Then the metric g is flat along the fibres.*

Proof. By (5.9) we have $[g^{-1}, g^{-1}] = 0$. We work on one fixed fibre X_b with coordinates $x_1, \ldots, x_n$. Let g^j be the vector field $\sum_i g^{ij}\partial/\partial x_i$, so $g^1, \ldots, g^n$ form a basis for $\mathcal{T}_{X_b}$ at each point of X_b, and $[g^i, g^j] = 0$. Let $\omega_1, \ldots, \omega_n$ be the dual basis of one-forms. Then

$$\begin{aligned} d\omega_i(g^j, g^k) &= g^j(\omega_i(g^k)) - g^k(\omega_i(g^j)) - \omega_i([g^j, g^k]) \\ &= 0 \end{aligned}$$

since $\omega_i(g^k) = \delta_{ik}$ is constant. Thus $\omega_1, \ldots, \omega_n$ are closed 1-forms, and so $\omega_i = d\varphi_i$ for a function φ_i on $\mathbf{R}^n$, the universal cover of X_b. φ_i will be a linear function plus a periodic function. Since $\omega_i = \sum_i g_{ij}dx_j$, we see that $g_{ij} = \partial\varphi_i/\partial x_j = g_{ji} = \partial\varphi_j/\partial x_i$, so there exists a function φ on $\mathbf{R}^n$ such that $\varphi_i = \partial\varphi/\partial x_i$, and $g_{ij} = \partial^2\varphi/\partial x_i\partial x_j$. φ satisfies the real inhomogeneous Monge-Ampère equation $\det(\partial^2\varphi/\partial x_i\partial x_j) = V^2$.

Next we prove V is constant. Note that $V^{-1}\omega_1\wedge\cdots\wedge\omega_n = V dx_1\wedge\cdots\wedge dx_n$, the volume form on X_b, so $*dx_i = \iota(g^i)(V^{-1}\omega_1\wedge\cdots\wedge\omega_n) = \pm V^{-1}\omega_1\wedge\cdots\wedge\hat{\omega}_i\wedge\cdots\wedge\omega_n$, and so $d(*dx_i) = \pm g^i(V^{-1})\omega_1\wedge\cdots\wedge\omega_n$. Thus, by (5.11), we must have $g^i(V^{-1}) = 0$ for all i so V^{-1}, hence V, is constant.

Thus φ is a solution to the equation

$$\det\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right) = C,$$

C a constant, and φ is a function on $\mathbf{R}^n$, with $\varphi = \varphi_{quad} + \varphi_{lin} + \varphi_{per}$, the decomposition into a quadratic, linear, and periodic part. We can of course assume that $\varphi_{lin} = 0$ and $\int_{X_b}\varphi_{per}dx_1\wedge\cdots\wedge dx_n = 0$. Then I claim $\varphi = \varphi_{quad}$, and so $g_{ij} = \partial^2\varphi/\partial x_i\partial x_j$ is constant. To see this, one applies a standard technique for non-linear elliptic partial differential equations. Let $\varphi_t = t\varphi + (1-t)\varphi_{quad}$. Let $m_{ij}(x_{kl})$ denote the ijth cofactor of the matrix (x_{kl}), so in particular

$$m_{ij}(x_{kl}) = \frac{\partial\det(x_{kl})}{\partial x_{ij}}.$$

Put

$$a_{ij} = \int_0^1 m_{ij}(\partial^2\varphi_t/\partial x_k\partial x_l)dt.$$

Now $(\partial^2\varphi/\partial x_i\partial x_j)$ is positive definite. So in fact is $(\partial^2\varphi_{quad}/\partial x_i\partial x_j)$. To see this, note that if we put $h_{ij} = h(\partial/\partial y_i, \partial/\partial y_j)$, we calculate

$$\begin{aligned} h_{ij} &= -\int_{X_b}\iota(\partial/\partial y_i)\omega\wedge\iota(\partial/\partial y_j)\operatorname{Im}\Omega \\ &= \int_{X_b} dx_i\wedge\left(V\sum_k(-1)^{k-1}g^{kj}dx_1\wedge\cdots\wedge\hat{dx_k}\wedge\cdots\wedge dx_n\right) \\ &= \int_{X_b} V g^{ij}dx_1\wedge\cdots\wedge dx_n. \end{aligned}$$

Since V is constant, $h_{ij} = V\frac{\partial^2\varphi_{quad}}{\partial x_i\partial x_j}\int_{X_b} dx_1\wedge\cdots\wedge dx_n$. Thus since (h_{ij}) is positive definite, so is $(\partial^2\varphi_{quad}/\partial x_i\partial x_j)$. Thus $(\partial^2\varphi_t/\partial x_i\partial x_j)$ is positive definite for all $0<t<1$, so the matrix (a_{ij}) is also positive definite.

Claim:

$$\sum_{i,j} a_{ij}\frac{\partial^2(\varphi-\varphi_{quad})}{\partial x_i\partial x_j} = constant.$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\det\left(\frac{\partial^2\varphi_t}{\partial x_k\partial x_l}\right)\right) &= \sum_{i,j} m_{ij}\left(\frac{\partial^2\varphi_t}{\partial x_k\partial x_l}\right)\frac{\partial^3\varphi_t}{\partial x_i\partial x_j\partial t}\\ &= \sum_{i,j} m_{ij}\left(\frac{\partial^2\varphi_t}{\partial x_k\partial x_l}\right)\frac{\partial^2(\varphi-\varphi_{quad})}{\partial x_i\partial x_j}.\end{aligned}$$

Integrating with respect to t gives the desired result. •

Now $\varphi-\varphi_{quad}$ is a periodic function, so applying the maximum principal (or minimum principal, depending on the sign of the constant), $\varphi-\varphi_{quad}$ is constant. Hence $\varphi=\varphi_{quad}$. •

This explains why in [18], the integrable complex structures constructed on torus fibrations had to have flat metric on the fibres given that $\mathrm{Re}\,\beta$ was taken to be zero in that paper.

§6. The complex structure on the mirror.

Having understood, at least to some extent, how one describes complex structures on torus fibrations, we now wish to explain how one should put a complex structure on the D-brane moduli space. We cannot solve this problem at present due to the complexity of the equation $d\Omega=0$; however, here we will give guidelines as to where to look for the correct solutions.

We continue with an integral special Lagrangian torus fibration $f: X\to B$ with a Lagrangian section, along with forms ω and Ω. In §4, we have seen how to put a symplectic form $\check\omega$ on $\check f:\check X_0\to B$ which has the property that $[\check\omega]$ and $[\mathrm{Im}\,\Omega_n]$ agree in $\Gamma(B_0,\Omega^1_{B_0}\otimes R^1\check f_{0*}\mathbf{R})\cong\Gamma(B_0,\Omega^1_{B_0}\otimes R^{n-1}f_{0*}\mathbf{R})$. To specify the complex structure on $\check X$, we need to construct the form $\check\Omega$. This should, according to the appropriate conjectures, be determined by the B-field, i.e. something like an element $\mathbf{B}\in H^2(X,\mathbf{R}/\mathbf{Z})$, and ω the Kähler form on X. Certainly the first requirement for $\check\Omega$ should be that $[\omega]$ and $[\mathrm{Im}\,\check\Omega_n]$ should agree on $\Gamma(B_0,\Omega^1_{B_0}\otimes R^1f_{0*}\mathbf{R})\cong\Gamma(B_0,\Omega^1_{B_0}\otimes R^{n-1}\check f_{0*}\mathbf{R})$, so that the double dual brings us back to X. The second requirement should involve $\mathrm{Re}\,\check\Omega_n$ and is much less precise at this point. We can only be guided by item (5) of the introduction, but will try to be more precise later.

Proposition 6.1. *If $\check\Omega_n$ is a normalised holomorphic n-form on $\check X_0$ making $\check f_0:\check X_0\to B$ special Lagrangian and if $\check h_n$ is the induced normalised Riemannian metric on the base, then $[\omega]=[\mathrm{Im}\,\check\Omega_n]$ in $\Gamma(B_0,\Omega^1_{B_0}\otimes R^1f_{0*}\mathbf{R})\cong\Gamma(B_0,\Omega^1_{B_0}\otimes R^{n-1}\check f_{0*}\mathbf{R})$ if and only if $h_n=\check h_n$.*

Proof. Let $y_1, \ldots, y_n$ be action coordinates for f. (We now have two different special Lagrangian fibrations, f and $\check{f}$, hence two different sets of action coordinates.) Fixing $b \in B$, $X_b = \mathcal{T}^*_{B,b}/\Lambda$, Λ is generated by $e_1, \ldots, e_n$ with $e_i = dy_i$ and $\Lambda^\vee \subseteq \mathcal{T}_{B,b}$ is generated by $e_1^*, \ldots, e_n^*$, $e_i^* = \partial/\partial y_i$. Now $\check{X}_b$ is identified with $\mathcal{T}^*_{B,b}/\Lambda^\vee$, where e_i^* is identified with the 1-form $-h_n(e_i^*, \cdot) = -\sum_j (h_n)_{ij} dy_j$, and thus $e_i \in \Lambda$ is identified with $-\sum_j h_n^{ij} \partial/\partial y_j \in \mathcal{T}_{B,b}$. Thus

$$\begin{aligned}\check{h}_n(\sum_j h_n^{ij}\partial/\partial y_j, \partial/\partial y_k) &= -\int_{\check{X}_b} \iota(\sum_j h_n^{ij}\partial/\partial y_j)\check{\omega} \wedge \iota(\partial/\partial y_k)\,\mathrm{Im}\,\check{\Omega}_n \\ &= -\int_{e_i} \iota(\partial/\partial y_k)\,\mathrm{Im}\,\check{\Omega}_n\end{aligned}$$

by Proposition 4.6, where $e_i \in \Lambda = H_{n-1}(\check{X}_b, \mathbf{Z}) \cong H_1(X_b, \mathbf{Z})$. If $[\omega] = [\mathrm{Im}\,\check{\Omega}_n]$, this latter integral coincides with

$$-\int_{e_i} \iota(\partial/\partial y_k)\omega = \delta_{ik}.$$

Thus $\check{h}_n(\partial/\partial y_i, \partial/\partial y_j) = (h_n)_{ij}$, so $\check{h}_n = h_n$. The argument reverses to prove the converse. •

Moral 6.2. *$\check{\Omega}_n$ must be chosen on $\check{X}$ so that $\check{h}_n = h_n$.*

Remark 6.3. While this moral was deduced by beginning with a special Lagrangian torus fibration and applying the principal that double dualising should bring one back to the initial fibration, there is no reason this can't then be generalised to provide a guide for putting complex structures on more general D-brane moduli spaces. In the situation of §4, given a family $\mathcal{U} \to B$, one has the metric h_n on B. Then a holomorphic n-form should be chosen on $\mathcal{M}$ so that $\mathcal{M} \to B$ is special Lagrangian and the induced metric on B is h_n.

It is more difficult to say exactly what role the B-field plays. According to the conjecture originally stated in [16], and restated in the introduction, $\check{\Omega}_n$ should be chosen so that $[\check{\Omega}_n] - [\sigma_0] \in H^1(B, R^{n-1}\check{f}_*\mathbf{R})$ should coincide with the choice of the B-field $\mathbf{B} \in H^1(B, R^1 f_*\mathbf{R})$. This provides little guidance, but we will see an example of this below which may point in the correct direction for interpreting the B-field.

Example 6.4. (The Hitchin solution.) Hitchin [18] gives a choice of $\check{\Omega}_n$ which under certain assumptions about the metric h_n satisfies $d\check{\Omega}_n = 0$. He expresses it locally in terms of action-angle coordinates, but it can be written down in arbitrary coordinates on the base in a natural way. One takes for $\check{\nabla}_b$ the Gauss-Manin connection. This is a linear connection $\check{\nabla}_{GM}$ on $\mathcal{T}^*_{B_0} = (R^{n-1}\check{f}_{0*}\mathbf{R}) \otimes C^\infty(B_0)$, whose flat sections are the sections of $R^{n-1}f_{0*}\mathbf{R}$. Taking the horizontal subspaces of this connection, it is easy to see that these descend to give a flat Ehresmann connection on $\check{X}_0$, which one takes to define $\check{\nabla}_b$. Note that in action coordinates $v_1, \ldots, v_n$ for $\check{f}_0$, the Gauss-Manin connection is trivial, so in these coordinates one takes $b = 0$. Since now $\check{\nabla}_b$ is flat, we must have $\check{g}$ constant along fibres by Corollary 5.16. Thus

$$\begin{aligned}\check{h}_n(\partial/\partial y_i, \partial/\partial y_j) &= \int_{\check{X}_b} \check{V}\check{g}^{ij} dx_1 \wedge \cdots \wedge dx_n / \int_{\check{X}_b} \check{V} dx_1 \wedge \cdots \wedge dx_n \\ &= \check{g}^{ij},\end{aligned}$$

and since we want $\check{h}_n = h_n$, we have no choice but to take $\check{g}^{ij} = (h_n)_{ij}$, giving rise to a choice of $\check{\Omega}_n$. To check to see if $d\check{\Omega}_n = 0$, one checks conditions (5.9)-(5.12). (5.9) and (5.11) are immediate, while (5.10) can be checked. (5.12) is then equivalent, in this case, to $\int_{\check{X}_b} \check{\Omega}_n$ being independent of b. But

$$\begin{aligned}\int_{\check{X}_b} \check{\Omega}_n &= \int_{\check{X}_b} \sqrt{\det \check{g}_{ij}} dx_1 \wedge \cdots \wedge dx_n \\ &= \frac{1}{\sqrt{\det(\check{h}_n)}} \int_{\check{X}_b} dx_1 \wedge \cdots \wedge dx_n.\end{aligned}$$

Hence this quantity must be independent of b for (5.12) to be satisfied. In particular, if $u_1, \ldots, u_n$ are action coordinates for f_0 and $v_1, \ldots, v_n$ are action coordinates on the same open subset for $\check{f}_0$, and $v_1, \ldots, v_n, x_1, \ldots, x_n$ are canonical coordinates, then $\int_{\check{X}_b} dx_1 \wedge \cdots \wedge dx_n$ is a constant independent of b, so in these coordinates, $d\check{\Omega}_n = 0$ is equivalent to

$$\det(\check{h}_n) = \textit{constant}.$$

A simple calculation now shows that if this is the case, then $\check{\Omega}_n$ coincides with Hitchin's $\tilde{\Omega}^c$ (up to a constant factor) in [18], §6, where

$$\tilde{\Omega}^c = \bigwedge_i (dx_i + \sqrt{-1} du_i).$$

Thus one recovers [18], Proposition 5. Of course, $\check{\omega} = \sum dx_i \wedge dv_i$, and Hitchin views mirror symmetry as an exchanging of the roles of the two sets of action coordinates $\{u_i\}$ and $\{v_i\}$. By [18], Proposition 3, the condition $\det(\check{h}_n) = \textit{constant}$ (in coordinates $v_1, \ldots, v_n$) is equivalent to the condition $\det(h_n) = \textit{constant}$ (in coordinates $u_1, \ldots, u_n$). This holds in particular if the metric g_{ij} is constant on fibres.

Example 6.5. (The Hitchin solution twisted by the B-field.) Continuing with the above example, assume $d\check{\Omega}_n = 0$. Now choose a symmetric cohomology class $\mathbf{B} \in H^1(B, R^{n-1}\check{f}_*\mathbf{R})$, with a symmetric representative $b \in \Gamma(B_0, \Omega^1_{B_0} \otimes \Omega^1_{B_0})$; in action-angle coordinates for $\check{f}$, this will be of the form $\sum b_{ij} dv_i \otimes dv_j$. Now take, in this coordinate system, the n-form $\check{\Omega}_{n,b}$ to be given by the matrix $(\beta_{ij}) = (b_{ij} + \sqrt{-1}(h_n)_{ij})$. This in fact gives a well-defined n-form on all of $\check{X}_0$. Note that since (b_{ij}) is symmetric, so is β, and thus we just need to show that $d\check{\Omega}_{n,b} = 0$ in order to show that $\check{\Omega}_{n,b}$ induces a complex structure with a Ricci-flat metric. This closedness can be seen to be true as follows. If $U \subseteq B_0$ is a sufficiently small open set with action-angle coordinates $v_1, \ldots, v_n, x_1, \ldots, x_n$ for $\check{f}$, we can find an element $a \in \Gamma(U, \Omega^0_U \otimes \Omega^1_U)$ such that $\check{\nabla}_{GM}(a) = b$, since $\check{\nabla}_{GM}(b) = 0$. Here $a = \sum_i a_i 1 \otimes dv_i$, with $\partial a_i/\partial v_j = b_{ij}$. Thus by symmetry of b_{ij}, $(v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_n, a_1, \ldots, a_n)$ gives a Lagrangian section σ of $\check{f}^{-1}(U) \to U$, and it is easy to see that

$$\check{\Omega}_{n,b} = T_\sigma^* \check{\Omega}_n,$$

where $\check{\Omega}_n$ is the Hitchin solution of Example 6.4. Since $d\check{\Omega}_n = 0$, $d\check{\Omega}_{n,b} = 0$ also.

Note also that if $b' = \sum b'_{ij} dy_i \otimes dy_j$ is a different symmetric representative for $\mathbf{B}$, then there exists an $a \in \Gamma(B_0, \Omega^0_{B_0} \otimes \Omega^1_{B_0})$ such that $\nabla_{GM}(a) = b' - b$. As before, a gives a Lagrangian section σ of $\check{f}_0$, and

$$T_\sigma^* \check{\Omega}_{n,b} = \check{\Omega}_{n,b'}.$$

Finally, one sees that $\check{\Omega}_{n,b} - \check{\Omega}_n$ represents the class $\mathbf{B} \in H^1(B, R^{n-1}\check{f}_*\mathbf{R})$, and $\check{\Omega}_{n,b}$ and $\check{\Omega}_n$ yield the same complex structure if $\mathbf{B} \in H^1(B, R^{n-1}\check{f}_*\mathbf{Z})$, for then there is a global section σ of $f : X \to B$ with $T_\sigma^* \check{\Omega}_n = \check{\Omega}_{n,b}$.

One can in fact go further. We have observed that for small open sets $U \subseteq B_0$, the complex and Kähler structures on $\check{X}_0$ induced by $\check{\Omega}_n$ and $\check{\Omega}_{n,b}$ on $\check{f}^{-1}(U)$ are isomorphic, so one should think of the Kähler structure induced by $\check{\Omega}_{n,b}$ as a torseur over that induced by $\check{\Omega}_n$. Specifically, fixing the complex structure $\check{\Omega}_n$ on $\check{X}_0$, note that the sheaf $\mathcal{A}$ on B_0 defined by

$$\mathcal{A}(U) = \{\text{sections } \sigma : U \to \check{f}^{-1}(U) \text{ such that } T_\sigma^* \check{\omega} = \check{\omega} \text{ and } T_\sigma^* \check{\Omega}_n = \check{\Omega}_n\}$$

coincides with $R^{n-1}f_*\mathbf{R}/\mathbf{Z}$. In fact, writing these conditions in the coordinates u_i and x_i of Example 6.4, one sees that the condition $T_\sigma^*\check{\Omega}_n = \check{\Omega}_n$ guarantees that the section σ is constant with respect to the Gauss-Manin connection. Thus the set of all special Lagrangian fibrations over B_0 obtained from $\check{f} : \check{X}_0 \to B_0$ by regluing using these translations is $H^1(B_0, R^{n-1}\check{f}_{0*}\mathbf{R}/\mathbf{Z})$. Because $\check{\omega}$ and $\check{\Omega}_n$ are preserved by these translations, they glue to give forms on the twisted fibrations. Thus each element $\mathbf{B} \in H^1(B_0, R^{n-1}\check{f}_{0*}\mathbf{R}/\mathbf{Z})$ gives rise to a fibration $\check{f}_{\mathbf{B}} : \check{X}_{0,\mathbf{B}} \to B_0$ with symplectic form $\check{\omega}_{\mathbf{B}}$ and holomorphic n-form $\check{\Omega}_{n,\mathbf{B}}$. This is a potentially wider class of examples than were constructed above using symmetric cohomology classes in $H^1(B, R^{n-1}\check{f}_*\mathbf{R})$; if $\mathbf{B}$ does not come from a symmetric class, then $\check{f}_{\mathbf{B}}$ will not possess a Lagrangian section, and may not even possess a topological section.

Note also that if f and $\check{f}$ are $\mathbf{R}/\mathbf{Z}$-simple, then

$$H^1(B, R^{n-1}\check{f}_*\mathbf{R}/\mathbf{Z}) \cong H^1(B, R^1 f_*\mathbf{R}/\mathbf{Z}).$$

This leads us to conjecture that the correct group for the B-field to live in is $H^1(B, R^1 f_*\mathbf{R}/\mathbf{Z})$. This new proposed definition for the B-field is dependent not just on X but on the topology of the fibration, and even in the threefold case does not necessarily coincide with $H^2(X, \mathbf{R}/\mathbf{Z})$, as we saw in Example 3.11. Nevertheless, I believe this is the correct interpretation of the B-field.

This now leads us to a refined mirror symmetry conjecture.

(1) for each open set $U \subseteq B_0 = B - \Delta$ on which both f and $\check{f}$ have sections, $f^{-1}(U) \to U$ and $\check{f}^{-1}(U) \to U$ are topologically dual fibrations.

(2) For $b \in B_0$, $\gamma \in H_1(\check{X}_b, \mathbf{Z}) \cong H_{n-1}(X_b, \mathbf{Z})$ and $v \in \mathcal{T}_{B,b}$,

$$\int_\gamma \iota(v)\check{\omega} = \int_\gamma \iota(v) \operatorname{Im} \Omega_n.$$

(3) For $\gamma \in H_{n-1}(\check{X}_b, \mathbf{Z}) \cong H_1(X_b, \mathbf{Z})$,

$$\int_\gamma \iota(v)\omega = \int_\gamma \iota(v) \operatorname{Im} \check{\Omega}_n.$$

(4) $\check{f}$ possesses a topological section if

$$\mathbf{B} \in H^1(B, R^1 f_*\mathbf{R})/H^1(B, R^1 f_*\mathbf{Z}) \subseteq H^1(B, R^1 f_*\mathbf{R}/\mathbf{Z}).$$

In this case, if $\check{\sigma}$ is a topological section, then $[\operatorname{Re} \check{\Omega}_n] - [\check{\sigma}]$ defines a class in $H^1(B, R^{n-1}\check{f}_*\mathbf{R})$ which is well-defined modulo $H^1(B, R^{n-1}\check{f}_*\mathbf{Z})$, and agrees with $\mathbf{B}$ in

$$H^1(B, R^1 f_*\mathbf{R})/H^1(B, R^1 f_*\mathbf{Z}).$$

(5) If $\check{J}$ is the Jacobian of $\check{X}$, then $\check{X}$ is obtained from $\check{J}$ as a symplectic manifold via the image of $\mathbf{B}$ under the composed map $H^1(B, R^1 f_*\mathbf{R}/\mathbf{Z}) \cong H^1(B, R^{n-1}\check{f}_*\mathbf{R}/\mathbf{Z}) \to H^1(B, \Lambda(\check{J}^\#))$ of Remark 3.15.

(6) Once $\check{X}$ and $\check{\omega}$ are fixed, $\check{\Omega}_n$ is unique up to translation by a Lagrangian section of $\check{J}$ acting on $\check{X}$.

I do not however suggest that $\check{f} : \check{X} \to B$ is obtained as a *Kähler* manifold as a torseur over some basic $\check{J} \to B$. While this occurred in Example 6.5, there is no reason to suspect this works when the metric on the fibres is not flat. We simply don't expect there to be isometries given by translation by a section. However, in some sense this might provide an initial approximation to the correct answer.

Remark 6.7. We can show a local form of the conjectured uniqueness. Suppose Ω_t is a family of holomorphic n-forms on X with respect to which a fixed symplectic form ω induces a Ricci-flat metric and $f : X \to B$ is special Lagrangian. In addition assume $[\Omega_t] \in H^n(X, \mathbf{C})$ is a fixed cohomology class. Then by local Torelli all Ω_t induce the same complex structure, so there exists diffeomorphisms $\phi_t : X \to X$ such that $\phi_t^*\Omega_t = \Omega_0$, $\phi_0 = id$. Now $\phi_t^*\omega$ is a symplectic form on X inducing a Ricci-flat metric in the complex structure induced by $\phi_t^*\Omega_t = \Omega_0$, and represents the same cohomology class as ω, so by uniqueness of Ricci-flat metrics, $\phi_t^*\omega = \omega$. Thus ϕ_t is a family of symplectomorphisms. Assuming $H^1(X, \mathbf{R}) = 0$, differentiating this family of diffeomorphisms at $t = 0$ yields a Hamiltonian vector field v induced by a Hamiltonian function H on X: $\iota(v)\omega = dH$. Then $\operatorname{Im} \Omega_t|_{X_b} = 0$ for all b implies that $(\mathcal{L}_v \operatorname{Im} \Omega_0)|_{X_b} = 0$ for all b. But $\mathcal{L}_v \operatorname{Im} \Omega_0 = d(\iota(v) \operatorname{Im} \Omega_0)$, and if Ω_0 is given as usual by a matrix $\beta = (\beta_{ij})$, $\beta_{ij} = b_{ij} + ig^{ij}$, then a simple calculation shows that

$$(d(\iota(v) \operatorname{Im} \Omega_0))|_{X_b} = -\sum_{i,j} V g^{ij} \frac{\partial^2 H}{\partial x_i \partial x_j} dx_1 \wedge \cdots \wedge dx_n.$$

Here we have used the fact that $\iota(\partial/\partial y_i)\,\mathrm{Im}\,\Omega_0$ is closed. Thus we see that on each fibre, H satisfies the second order elliptic partial differential equation

$$\sum_{i,j} g^{ij}\frac{\partial^2 H}{\partial x_i \partial x_j} = 0.$$

By the maximum principal, H cannot have a local maximum on each non-singular fibre unless H is constant on the fibre. Since the set of non-singular fibres is dense, we conclude that H is the pullback of a function on B. In particular, it follows from §2 that ϕ_t must be translation by a Lagrangian section.

Remark 6.8. One natural question is to determine the relationship between $Vol(X_b) = \int_{X_b}\Omega$ and $Vol(\check{X}_b) = \int_{\check{X}_b}\check{\Omega}$, since knowledge of the latter allows us to reconstruct $\check{\Omega}$ from $\check{\Omega}_n$. We can describe this relationship if the metric is constant along the fibres of $f : X \to B$ and $\check{f} : \check{X} \to B$. Let $y_1, \ldots, y_n$ be action coordinates for f. As in Example 6.4,

$$h_n^{ij} = g_{ij}.$$

Then

$$\begin{aligned} Vol(X_b) &= \int_{X_b} \sqrt{\det(g_{ij})}dx_1 \wedge \cdots \wedge dx_n \\ &= \sqrt{\det(h_n^{ij})}. \end{aligned}$$

On the other hand, for a fixed $b \in B$, $\check{X}_b$ can be written canonically as $\mathcal{T}^*_{B,b}/\Lambda^\vee$ as in Proposition 4.7, where $\Lambda^\vee$ is generated by the one-forms $h_n(\partial/\partial y_i, \cdot)$, i.e. the one-forms $\sum_j (h_n)_{ij} dy_j$. The metric on the fibre $\check{X}_b$ is still given by $\check{g}_{ij} = (h_n)^{ij}$, since $h_n = \check{h}_n$. Thus

$$\begin{aligned} Vol(\check{X}_b) &= \int_{\check{X}_b} \sqrt{\det(h_n^{ij})}dx_1 \wedge \cdots \wedge dx_n \\ &= \sqrt{\det(h_n^{ij})\det((h_n)_{ij})} \\ &= \frac{1}{Vol(X_b)}. \end{aligned}$$

This is the familiar "$R \mapsto 1/R$" relationship of T-duality. If the metric is not flat, we expect some corrections to the volume, and this may affect this relationship. However, as we shall see in §7, this relationship continues to hold for K3 surfaces.

We end this section with a brief discussion of the Yukawa coupling. Mirror symmetry instructs us that given a Calabi-Yau X, the Yukawa coupling on $\check{X}$ contains information about the genus 0 Gromov-Witten invariants on X, and in particular in the three dimensional case, these Gromov-Witten invariants can be completely recovered from the Yukawa coupling on $\check{X}$. How do we see the Yukawa coupling in the context of special Lagrangian fibrations?

Suppose that Conjecture 6.6 holds. Assume for simplicity that we only consider values of the B-field $\mathbf{B} \in H^1(B, R^{n-1}\check{f}_*\mathbf{R}/\mathbf{Z})$ which come from symmetric classes in $H^1(B, R^{n-1}\check{f}_*\mathbf{R})$, so that although $\mathbf{B}$ varies, we can fix the underlying manifold $\check{X}$

and symplectic form $\check{\omega}$ and simply let $\check{\Omega}_n$ vary; we denote the dependence on $\mathbf{B}$ by writing $\check{\Omega}_{n,\mathbf{B}}$. Of course, we will not have $\check{\Omega}_{n,\mathbf{B}+\alpha} = \check{\Omega}_{n,\mathbf{B}}$ for $\alpha \in H^1(B, R^{n-1}\check{f}_*\mathbf{Z})$, but merely expect that there exists a Lagrangian section σ of $\check{f} : \check{X} \to B$ with $T_\sigma^*\check{\Omega}_{n,\mathbf{B}} = \check{\Omega}_{n,\mathbf{B}+\alpha}$. The $(1, n-1)$-Yukawa coupling of interest is then, for tangent directions $\partial/\partial b_1, \ldots, \partial/\partial b_n \in H^1(B, R^{n-1}\check{f}_*\mathbf{R})$, the tangent space of the torus $H^1(B, R^{n-1}\check{f}_*\mathbf{R}/\mathbf{Z})$,

$$\left\langle \frac{\partial}{\partial b_1}, \ldots, \frac{\partial}{\partial b_n} \right\rangle = \int_{\check{X}} \check{\Omega}_{n,\mathbf{B}} \wedge \frac{\partial^n}{\partial b_1 \ldots \partial b_n} \check{\Omega}_{n,\mathbf{B}}.$$

In local coordinates, we write

$$\check{\Omega}_{n,\mathbf{B}} = V_{n,\mathbf{B}} \theta_1(\mathbf{B}) \wedge \cdots \wedge \theta_n(\mathbf{B}).$$

Note that in taking the n derivatives of $\check{\Omega}_{n,\mathbf{B}}$ by using the product rule, all terms still containing any undifferentiated θ_i will disappear after we wedge with $\check{\Omega}_{n,\mathbf{B}}$. Thus

$$\int_{\check{X}} \check{\Omega}_{n,\mathbf{B}} \wedge \frac{\partial^n}{\partial b_1 \ldots \partial b_n} \check{\Omega}_{n,\mathbf{B}} = \sum_{\sigma \in S_n} \int_{\check{X}} V_{n,\mathbf{B}}^2 \theta_1 \wedge \ldots \wedge \theta_n \wedge \frac{\partial \theta_1}{\partial b_{\sigma(1)}} \wedge \cdots \wedge \frac{\partial \theta_n}{\partial b_{\sigma(n)}}.$$

Writing $\theta_i = dx_i + \sum_j \check{\beta}_{ij}(\mathbf{B}) dy_j$,

$$\frac{\partial \theta_i}{\partial b_{\sigma(i)}} = \sum \frac{\partial \check{\beta}_{ij}}{\partial b_{\sigma(i)}} dy_j,$$

so the above integral is

$$\int_{\check{X}} V_{n,\mathbf{B}}^2 \sum_{\sigma \in S_n} \det \left(\frac{\partial \check{\beta}_{ij}}{\partial b_{\sigma(i)}} \right) dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n.$$

So far, this is not particularly illuminating in the general case. However, for the twisted Hitchin solutions, it is an elementary calculation to show this integral can be evaluated in terms of the topological coupling on X, as expected. This comes from observing that if $\partial \theta_j / \partial b_i = \sum_k b_{jk}^i dy_k$, then in action-angle coordinates, where $V_n = 1$, the integrand above coincides with

$$(-1)^{n(n-1)/2} \bigwedge_{i=1}^n \left(\sum_{j,k} b_{jk}^i dx_j \wedge dy_k \right).$$

§7. K3 Surfaces.

Mirror symmetry for K3 surfaces has been completely understood using Torelli theorems for K3 surfaces. We will now show that the previous material of this paper gives us a differential geometric construction of mirror symmetry for K3 surfaces, and in doing so, we will show Conjecture 6.6 holds in two dimensions. In other words, given a special Lagrangian T^2-fibration on a K3 surface, and a choice of

B-field, we will construct the mirror in the sense made explicit in Conjecture 6.6. This will prove to be a variant of the mirror symmetry for K3 surfaces described in [6] and [11].

To begin, let S be a K3 surface with holomorphic 2-form Ω and Kähler form ω corresponding to a Ricci-flat metric. We insist on the usual normalisation and this implies in particular that $(\mathrm{Re}\,\Omega)^2 = (\mathrm{Im}\,\Omega)^2 = \omega^2 > 0$ and $(\mathrm{Re}\,\Omega)\wedge(\mathrm{Im}\,\Omega) = \omega\wedge(\mathrm{Re}\,\Omega) = \omega\wedge(\mathrm{Im}\,\Omega) = 0$. In order for S to possess a special Lagrangian fibration there must be a cohomology class $E \in H^2(S,\mathbf{Z})$ such that $E^2 = 0$ and $\omega.E = 0$. We assume such a class exists, and we fix it. We take E to be primitive. Now one constructs a special Lagrangian fibration on S by the usual hyperkähler trick, as originally suggested in [28]. First, multiply Ω by a phase $e^{i\theta}$ to ensure $\mathrm{Im}\,\Omega.E = 0$. Following the notation of [16], there is a complex structure K compatible with the Ricci-flat metric in which $\Omega_K = \mathrm{Im}\,\Omega + i\omega$ and $\omega_K = \mathrm{Re}\,\Omega$. Then special Lagrangian submanifolds on S are complex submanifolds in the K complex structure. Denote the K3 surface in the K complex structure by S_K. Then by construction $E.\Omega_K = 0$ so $E \in Pic(S_K)$. It is then standard that S_K possesses an elliptic fibration. However, the class of the fibre need not be E; E might be represented by a fibre plus a sum of -2 divisors. In any event, replace E by the class of the fibre, positively oriented. E will now remain fixed, and we have an elliptic fibration $f : S_K \to \mathbf{P}^1$, which we identify with a special Lagrangian T^2 fibration $f : S \to B = S^2$. We will assume f is integral, which is true if and only if there is no $\delta \in Pic(S_K)$ with $\delta^2 = -2$ and $\delta.E = 0$. This will certainly hold for general S.

We now consider the spectral sequence for f over $\mathbf{Z}$. Since there exists a class σ such that $\sigma.E = 1$, the sequence in fact degenerates and takes the form

$$\begin{array}{ccc} H^0(B,\mathbf{Z}) & 0 & H^2(B,\mathbf{Z}) \\ 0 & H^1(B,R^1f_*\mathbf{Z}) & 0 \\ H^0(B,\mathbf{Z}) & 0 & H^2(B,\mathbf{Z}) \end{array}$$

It is also clear that $H^1(B,R^1f_*\mathbf{Z})$ is canonically isomorphic to $E^\perp/E$.

We now wish to construct a mirror fibration given the data

$f : S \to B$ as above and a choice of B-field $\mathbf{B} \in E^\perp/E \otimes \mathbf{R}/\mathbf{Z} \cong H^1(B,R^1f_*\mathbf{R}/\mathbf{Z})$.

We first recall some facts about elliptic fibrations. See [7] for general facts about the analytic theory of elliptic surfaces.

The fibration $f : S_K \to \mathbf{P}^1$ in general does not possess a holomorphic section; in fact for general choice of S, $\mathrm{Pic}S_K = \mathbf{Z}E$. However, there is a Jacobian fibration $j : J_K \to \mathbf{P}^1$ of f which is locally isomorphic to f, and which does possess a holomorphic section.

Proposition 7.1. *There is a diffeomorphism $\phi : J_K \to S_K$ over $\mathbf{P}^1$ which is holomorphic when restricted to each fibre. Furthermore, if $U \subseteq \mathbf{P}^1$ is an open subset on which there exists a biholomorphic map $\xi : j^{-1}(U) \to f^{-1}(U)$ over U, then $\phi^{-1}\circ\xi : j^{-1}(U) \to j^{-1}(U)$ is given by translation by a (not necessarily holomorphic) section of $j^{-1}(U)$.*

Proof. This follows easily from the fact that the "C^∞ Tate-Shafarevich group," i.e. the first cohomology group of the sheaf of C^∞ sections of $j : J_K^\# \to \mathbf{P}^1$, is zero.

Thus $f : S_K \to \mathbf{P}^1$ possesses a C^∞ section, and ϕ can be taken to identify this C^∞ section of f with a holomorphic section of j, such that ϕ is holomorphic on each fibre. •

We fix one holomorphic section σ_0 of $j : J \to \mathbf{P}^1$, and identify σ_0 with the topological section $\phi(\sigma_0)$ of $f : S \to B$. Having chosen this section, we can take it to be the zero section of $j : J_K \to \mathbf{P}^1$ and obtain a standard exact sequence

$$0 \to R^1 f_* \mathbf{Z} \to R^1 f_* \mathcal{O}_{S_K} \xrightarrow{\psi} J_K^\# \to 0$$

where $J_K^\#$ denotes the sheaf of holomorphic sections of $j : J_K \to \mathbf{P}^1$. Here $R^1 f_* \mathcal{O}_{S_K}$ can be identified with the normal bundle of the zero section, and the map ψ is just the fibre-wise exponential map. For K3 surfaces, $R^1 f_* \mathcal{O}_{S_K} \cong \omega_{\mathbf{P}^1}$. The underlying real bundle is $\mathcal{T}^*_{S^2}$. This also gives a map $\pi : \mathcal{T}^*_{S^2} \to S_K^\#$ with $\pi = \phi \circ \psi$.

Just as in the real case, the total space of $\omega_{\mathbf{P}^1}$, the holomorphic cotangent bundle of $\mathbf{P}^1$, has a canonical holomorphic symplectic form Ω_c. In local coordinates, if z is a coordinate on $\mathbf{P}^1$ and w the canonical coordinate on the cotangent bundle, then $\Omega_c = dw \wedge dz$. Furthermore, any holomorphic symplectic form on the cotangent bundle of $\mathbf{P}^1$ is proportional to Ω_c.

Proposition 7.2. *There is a map $\chi : \mathcal{T}^*_{S^2} \to \mathcal{T}^*_{S^2}$ given by fibrewise multiplication by a complex constant so that, for $\pi' = \pi \circ \chi$, $\pi'^*(\Omega_K) = \Omega_c + f^*\alpha$, where α is a 2-form on S^2.*

Proof. First note that there are two different complex structures on the total space of $\mathcal{T}^*_{S^2}$ in this picture: one is the standard complex structure coming from being the holomorphic line bundle $\mathcal{O}_{\mathbf{P}^1}(-2)$, while the other is induced by $\pi^*\Omega_K$. To distinguish between these two complex structures, let $\tilde{J}$ be the total space of $\mathcal{T}^*_{S^2}$ with the standard complex structure, and let $\tilde{S}$ denote the total space of $\mathcal{T}^*_{S^2}$ with the complex structure induced by $\pi^*\Omega_K$. Let $\tilde{j} : \tilde{J} \to S^2$, $\tilde{f} : \tilde{S} \to S^2$ be the projections.

Now let $U \subseteq S^2$ be a sufficiently small open set so that there exists a biholomorphic map $\xi : j^{-1}(U) \to f^{-1}(U)$ over U. By Proposition 7.1, $\phi^{-1} \circ \xi = T_\sigma$ for some section σ of $j^{-1}(U) \to U$, and if U is small enough, σ can be lifted to a section $\tilde{\sigma}$ of $\tilde{j}^{-1}(U) \to U$. We let $\tilde{\xi}$ denote translation by the section $\tilde{\sigma}$ (with the zero-section of $\mathcal{T}^*_{S^2}$ taken to be the origin). We then have a commutative diagram

$$\begin{array}{ccc} \tilde{j}^{-1}(U) & \xrightarrow{\tilde{\xi}} & \tilde{f}^{-1}(U) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\pi} \\ j^{-1}(U) & \xrightarrow{\xi} & f^{-1}(U) \end{array}$$

since $\pi \circ \tilde{\xi} = \phi \circ \psi \circ T_{\tilde{\sigma}} = \phi \circ T_\sigma \circ \psi = \xi \circ \psi$. In particular, since Ω_K is a holomorphic 2-form on $\tilde{f}^{-1}(U)$, $\tilde{\xi}^*\pi^*\Omega_K$ is a holomorphic 2-form on $\tilde{j}^{-1}(U)$, and hence can be written as $g dw \wedge dz$ in local coordinates, for some holomorphic function g. This function g must be constant along the fibres of $\tilde{j}$ since g must descend to a holomorphic function on the compact fibres of j. Thus on $\tilde{f}^{-1}(U)$,

$$\begin{aligned} \pi^*\Omega_K &= (\tilde{\xi}^{-1})^*(g dw \wedge dz) \\ &= T^*_{-\tilde{\sigma}}(g dw \wedge dz) \\ &= (g \circ T_{-\tilde{\sigma}}) dw \wedge dz + h dz \wedge d\bar{z} \end{aligned}$$

for some function h. Of course, $g \circ T_{-\check{\sigma}} = g$ since g is constant on fibres.

Let $i : \check{J} \to \check{S}$ be the identity map; this is of course non-holomorphic. The above equation shows that the $(2,0)$ part of $i^*\pi^*\Omega_K$, being locally of the form $g dz \wedge dw$, is in fact a holomorphic 2-form. In addition, this holomorphic 2-form is nowhere vanishing: if g vanishes then $\Omega_K \wedge \bar{\Omega}_K = 0$ at that point. Thus the $(2,0)$ part of $i^*\pi^*\Omega_K$ is proportional to Ω_c, say $C\Omega_c$. In addition, we then see from $d\Omega_K = 0$ that h must be constant along fibres and hence $\pi^*\Omega_K - C\Omega_c$ is the pullback of a $(1,1)$ form α on $\mathbf{P}^1$, i.e.

$$\pi^*\Omega_K = C\Omega_c + \check{f}^*\alpha.$$

Now let $\chi : \mathcal{T}^*_{S^2} \to \mathcal{T}^*_{S^2}$ be given by $w \mapsto C^{-1}w$. Then $\chi^*\pi^*\Omega_K = \Omega_c + \check{f}^*\alpha$ as desired. •

To sum up, replacing π by π', we now have a map $\pi : \mathcal{T}^*_{S^2} \to S^\#_K$ with kernel $R^1 f_*\mathbf{Z}$ and such that $\pi^*\Omega_K = \Omega_c + \check{f}^*\alpha$. This gives, identifying the underlying topological spaces $S^\#_K$ and $S^\#$, $\pi^*\omega = \mathrm{Im}(\Omega_c + \check{f}^*\alpha)$ and $\pi^*(\mathrm{Im}\,\Omega) = \mathrm{Re}(\Omega_c + \check{f}^*\alpha)$.

A local description of these forms are as follows: given complex canonical coordinates z, w on $\mathcal{T}^*_U$, z a coordinate on U, write $z = y_1 - iy_2$, $w = x_1 + ix_2$. The signs are chosen so that with real coordinates y_1, y_2 on the base, y_1, y_2, x_1, x_2 are canonical coordinates on $\mathcal{T}^*_{S^2}$. Then

$$\begin{aligned} \pi^*\omega &= \mathrm{Im}(dw \wedge dz + \check{f}^*\alpha) \\ &= dy_2 \wedge dx_1 + dx_2 \wedge dy_1 + \check{f}^*\,\mathrm{Im}\,\alpha \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} \pi^*\,\mathrm{Im}\,\Omega &= \mathrm{Re}(dw \wedge dz + \check{f}^*\alpha) \\ &= dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \check{f}^*\,\mathrm{Re}\,\alpha. \end{aligned} \tag{7.2}$$

This completes our first goal of finding coordinates on $S^\#$ in which ω and $\mathrm{Im}\,\Omega$ have simple forms. Note that our map $\pi : \mathcal{T}^*_{S^2} \to S^\#$ is not the same as defined in §2 because the symplectic form is not the standard one. We will see why we have made this choice in the proof of the following theorem.

Theorem 7.3. *Conjecture 6.6 holds for the general integral special Lagrangian fibration $f : S \to B = S^2$.*

Proof. Choose a B-field $\mathbf{B} \in (E^\perp/E) \otimes \mathbf{R}/\mathbf{Z}$. Lift this to a representative $\mathbf{B} \in E^\perp/E \otimes \mathbf{R}$. At times, we will also further lift $\mathbf{B}$ to an element $\mathbf{B} \in E^\perp \otimes \mathbf{R}$ chosen so that $\mathbf{B}.[\sigma_0] = 0$, where σ_0 is the fixed topological section of $f : S \to B$ chosen previously. We are now trying to construct a special Lagrangian fibration $\check{f} : \check{S} \to B$ satisfying the properties of Conjecture 6.6. Because this fibration may not have a Lagrangian section, we first construct the Jacobian $\check{j} : \check{J} \to B$ (in the sense of §2) as a symplectic manifold. This Jacobian should be the dual fibration with a symplectic form $\check{\omega}_J$ as constructed in §4.

To do so, we reembed $R^1 f_*\mathbf{Z} \hookrightarrow \mathcal{T}^*_{S^2}$ using the periods given by $\mathrm{Im}\,\Omega_n$. Now $\Omega_n = \frac{1}{Vol(S_b)}\Omega$, and this embedding takes a cycle $\gamma \in H_1(S_b, \mathbf{Z}) \cong H^1(S_b, \mathbf{Z}) \subseteq \mathcal{T}^*_{S^2}$

to the one-form

$$\begin{aligned} v \mapsto & -\int_\gamma \iota(v)\operatorname{Im}\Omega_n \\ &= -\frac{1}{Vol(S_b)}\int_\gamma \iota(v)(dx_1\wedge dy_1 + dx_2\wedge dy_2) \end{aligned}$$

which yields the 1-form $\gamma/Vol(S_b)$.

The moral is: *The dual lattice is simply the original lattice scaled by a factor of* $1/Vol(S_b)$.

We have in fact chosen the map π so that this would happen and so make it transparent that dualising does not change the topology of the fibration.

Instead of rescaling the lattice, it is easier to identify $\check{J}$ with S topologically, and rescale the symplectic form. Since in local coordinates we want $\check{\omega}_{\check{J}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, by (7.2) we take on $\check{J} = S$

$$\check{\omega}_{\check{J}} = (\operatorname{Im}\Omega - f^*\operatorname{Re}\alpha)/Vol(S_b).$$

How do we obtain $\check{S}$ as a symplectic manifold? Item (5) of Conjecture 6.6 instructs us to proceed as follows. $\mathbf{B} \in H^1(B, R^1f_*\mathbf{R}/\mathbf{Z}) \cong H^1(B, R^1\check{j}_*\mathbf{R}/\mathbf{Z})$ maps to an element of $H^1(B, \Lambda(\check{J}^\#))$. Having lifted $\mathbf{B}$ to an element of $H^1(B, R^1f_*\mathbf{R})$, we obtain in this way an element $-\mathbf{B}\wedge[\check{\omega}_{\check{J}}] \in H^2(B,\mathbf{R})$, via Remark 3.15, which then maps to the appropriate element of $H^1(B, \Lambda(\check{J}^\#))$. As remarked before Example 2.7, this means $\check{\omega} = \check{\omega}_{\check{J}} + \check{j}^*\alpha_1$, where α_1 is a form on B such that

$$\int_B \alpha_1 = -\mathbf{B}.[\check{\omega}_{\check{J}}].$$

Now $[\check{\omega}_{\check{J}}] = [\operatorname{Im}\Omega_n]$ by construction, so it would appear that we take $\int_B \alpha_1 = -\mathbf{B}.[\operatorname{Im}\Omega_n]$. There is a slight subtlety in this: here we are representing $\mathbf{B} \in H^1(B, R^1f_*\mathbf{R})$ as an element of $E^\perp/E\otimes\mathbf{R}$, but that does not mean that if we reinterpret $\mathbf{B}$ as a class in $H^1(B, R^1\check{j}_*\mathbf{R})$, this class will coincide with the original $\mathbf{B}$ in S under our identification of S with $\check{J}$. In fact, the correct choice is

$$\int_B \alpha_1 = \mathbf{B}.[\operatorname{Im}\Omega_n].$$

We will be vague about this here, but will see this sign change more explicitly shortly. We then set

$$\check{\omega} = \check{\omega}_{\check{J}} + \check{j}^*\alpha_1.$$

Note that the choice of α_1 is not important; any two choices representing the same cohomology class can be identified by translation by a section. Let $\check{S}$ be the symplectic manifold obtained with underlying manifold S and symplectic form $\check{\omega}$, and let $\check{f} : \check{S} \to B$ be the same map as $f : S \to B$. We note that the cohomology class of $\check{\omega}$ satisfies the relation

$$[\check{\omega}] = [\operatorname{Im}\Omega_n] + (\operatorname{Im}\Omega_n.(\mathbf{B} - [\sigma_0]))E.$$

Next we construct the form $\operatorname{Im}\check{\Omega}_n$. The first observation is that $\iota(v)\operatorname{Im}\check{\Omega}_n$ must be harmonic for any $v \in \mathcal{T}_{B,b}$. On the other hand, the same is true of $\iota(v)\check{\omega}$, and $\iota(v)\check{\omega} = adx_1 + bdx_2$ for a and b constant. Thus we already know, in the 2-dimensional case, the harmonic $n-1$ forms. This is a crucial point in dimension 2 which fails in higher dimensions. Applying item (3) of Conjecture 6.6, in the form given in Proposition 6.1, we can now determine $\check{\Omega}_n$ as follows. First

$$h_n(\partial/\partial y_i, \partial/\partial y_j) = \frac{\delta_{ij}}{Vol(S_b)} \int_{S_b} dx_1 \wedge dx_2$$

at a point $b \in B$, as is easily computed from the definition and (7.1), (7.2). Then in order for $\check{h}_n = h_n$, we must have

$$-\int_{S_b} \iota(\partial/\partial y_i)\check{\omega} \wedge \iota(\partial/\partial y_j) \operatorname{Im}\check{\Omega}_n = \frac{\delta_{ij}}{Vol(S_b)} \int_{S_b} dx_1 \wedge dx_2,$$

and a quick calculation shows this implies we locally can write

$$\operatorname{Im}\check{\Omega}_n = -\omega + hdy_1 \wedge dy_2,$$

where h is a function. But the condition that $d\operatorname{Im}\check{\Omega}_n = 0$ implies h is constant on fibres, so

$$\operatorname{Im}\check{\Omega}_n = -\omega + f^*\alpha_2$$

for some form α_2 on the base.

Here we see the sign reversal explicitly. It might be a bit surprising that we have obtained $-\omega$ instead of ω. But this is the fault of the identification we have chosen. We want $[\operatorname{Im}\check{\Omega}_n] = [\omega]$ as classes in $H^1(B, R^1\check{f}_*\mathbf{R}) \cong H^1(B, R^1 f_*\mathbf{R})$; it is only an accident of dimension that we have been able to identify S and $\check{S}$ as manifolds and then compare cohomology classes directly. In fact, this sign change *must* occur if we want to identify S and $\check{S}$ without changing the orientation of the fibres.

Condition (3) of Conjecture 6.6 does not tell us what α_2 must be; α_2 is not determined until one knows something about $\operatorname{Re}\check{\Omega}_n$. In fact, condition (4) tells us that we require

$$[\operatorname{Re}\check{\Omega}_n] = [\sigma_0] - \mathbf{B} \bmod E,$$

where again we are making use of the sign reversal observed above. Now our form $\operatorname{Im}\check{\Omega}_n$ constructed above satisfies

$$[\operatorname{Im}\check{\Omega}_n] = [-\omega] \bmod E,$$

which tells us that in order for $\check{\Omega}_n^2 = 0$, we must have $[\check{\Omega}_n]$ satisfying

$$[\check{\Omega}_n] = [\sigma_0] - (\mathbf{B} + i\omega) + (1 - (\mathbf{B} + i\omega)^2/2 + i(\omega.\sigma_0))E. \tag{7.3}$$

Here we have chosen a representative of $\mathbf{B} \in E^\perp$ such that $\mathbf{B}.\sigma_0 = 0$. Thus, in particular, we need to choose α_2 so that

$$\int_B \alpha_2 = -\mathbf{B}.\omega + \sigma_0.\omega.$$

Again, we need to ask how much freedom we have to choose α_2, given that we have fixed α_1. We had seen that α_1 could be chosen to be any representative of its cohomology class, as any choice could be obtained from any other by translating $\check{\omega}$ by a section of $\mathcal{T}^*_{S^2}$. Once we have fixed the form α_1, however, we can only translate by sections corresponding to 1-forms σ with $d\sigma = 0$. Let $\sigma = \sigma_1 dy_1 + \sigma_2 dy_2$. Then

$$T^*_\sigma(dy_2 \wedge dx_1 + dx_2 \wedge dy_1) = dy_2 \wedge dx_1 + dx_2 \wedge dy_1 - \left(\frac{\partial \sigma_1}{\partial y_1} + \frac{\partial \sigma_2}{\partial y_2}\right) dy_1 \wedge dy_2.$$

Thus $T^*_\sigma(\omega) - \omega = -f^*(d * \sigma)$, where $*$ denotes the Hodge $*$-operator in, say, the Fubini-Study metric on $B = \mathbf{P}^1$. Thus if α_2, α_2' are two 2-forms on B representing the same cohomology class, we just need to find a 1-form σ on B such that $d\sigma = 0$ and $d * \sigma = \alpha_2 - \alpha_2'$, and then $T^*_\sigma(\omega + f^*\alpha_2) = \omega + f^*\alpha_2'$. By the Hodge theorem, such a σ can always be found. Thus we have complete freedom to choose α_2, and any two choices are related by translation by a Lagrangian section.

This will be the last remaining choice in the construction which is not forced on us by any items in Conjecture 6.6; thus the uniqueness of item (6) of Conjecture 6.6 will hold, given that the lack of uniqueness in the lifting of $\mathbf{B}$ can be rectified by changing the choice of the zero section σ_0.

Finally, we set

$$\operatorname{Im} \check{\Omega} = (\operatorname{Im} \check{\Omega}_n)/Vol(S_b).$$

It now follows immediately from (7.1) and (7.2) that as forms,

$$(\operatorname{Im} \check{\Omega}) \wedge (\operatorname{Im} \check{\Omega}) = \check{\omega} \wedge \check{\omega} > 0,$$

and

$$(\operatorname{Im} \check{\Omega}) \wedge \check{\omega} = 0.$$

Thus $\check{\Omega}_K = \operatorname{Im} \check{\Omega} + i\check{\omega}$ is a 2-form which satisfies the conditions of Theorem 5.1 and hence determines a complex structure on $\check{S}$. We call $\check{S}$ with this complex structure $\check{S}_K$. As observed above, $[\check{\Omega}_n]$ must satisfy (7.3), so we need to look for a form $\operatorname{Re} \check{\Omega}$ such that

$$[\operatorname{Re} \check{\Omega}] = \frac{1}{Vol(S_b)}([\sigma_0] - \mathbf{B} - (\mathbf{B}^2 - \omega^2 - 2)E/2).$$

If $[\operatorname{Re} \check{\Omega}]$ is a Kähler class on $\check{S}_K$, then by Yau's theorem, there exists a unique Kähler form $\operatorname{Re} \check{\Omega}$ whose metric is Ricci-flat, so we only need to ensure $[\operatorname{Re} \check{\Omega}]$ is a Kähler class. We first note that $[\operatorname{Re} \check{\Omega}].[\operatorname{Im} \check{\Omega}] = [\operatorname{Re} \check{\Omega}].[\check{\omega}] = 0$, so $[\operatorname{Re} \check{\Omega}]$ is a $(1,1)$ class. Next we observe it is a positive class. Indeed, $\check{f} : \check{S}_K \to B$ is still a holomorphic elliptic fibration, and the complex structure on each fibre $\check{S}_{K,b}$ is the same as that of $S_{K,b}$. Since $[\operatorname{Re} \check{\Omega}].E > 0$, this shows then that $[\operatorname{Re} \check{\Omega}]$ is in fact positive on E. As long as $Pic(\check{S}_K)$ contains no -2 classes, this shows that $[\operatorname{Re} \check{\Omega}]$ is a Kähler class. Hence we obtain a Kähler form $\operatorname{Re} \check{\Omega}$ as desired, and set $\check{\Omega} = \operatorname{Re} \check{\Omega} + i \operatorname{Im} \check{\Omega}$. •

We make a few closing comments. First, in the above proof we can write

$$[\Omega] = Vol(S_b)([\sigma_0] + \check{\mathbf{B}} + i[\check{\omega}]) \bmod E$$

and

$$[\check{\Omega}] = \frac{1}{Vol(S_b)}([\sigma_0] - (\mathbf{B} + i[\omega])) \bmod E.$$

Thus we see that $Vol(\check{S}_b) = 1/Vol(S_b)$, conforming with Remark 6.8. However this does not quite look like mirror symmetry: if we repeat the process we appear to get

$$[\check{\check{\Omega}}] = Vol(S_b)([\sigma_0] - (\check{\mathbf{B}} + i[\check{\omega}])) \bmod E.$$

This again is the fault of our identifications, essentially having put in a 90° twist in $\mathcal{T}_B^*$ during each dualising. This is rectified on the double mirror by pulling back all forms by the fibrewise negation map on S. This acts trivially on $H^0(B, R^2 f_*\mathbf{R})$ and $H^2(B, f_*\mathbf{R})$, but by negation on $H^1(B, R^1 f_*\mathbf{R})$.

Finally we note that the construction of the mirror K3 surface given in this proof is of a different nature from previous constructions of mirror symmetry for K3 surfaces. Normally one appeals to the Torelli theorem to construct the mirror. Here, once we have produced a special Lagrangian fibration on S, we produce the mirror without an appeal to Torelli. Instead, we are essentially applying Yau's Theorem to solve the equations of Corollary 5.15. However we are still aided by some key points which don't hold in higher dimensions. These are:

(1) We know the harmonic $n-1$-forms on fibres, since $n - 1 = 1$.

(2) We know the cohomology class of a holomorphic 2-form Ω if we know its class modulo E; this is completely determined by the requirement $[\Omega]^2 = 0$.

(3) We can use the hyperkähler trick.

§8. Postscript.

Since the initial version of this paper was prepared, there have been a number of new results which have greatly increased the evidence for the Strominger-Yau-Zaslow conjecture. The first is a result of W.-D. Ruan [25] constructing a Lagrangian T^3-fibration on the quintic threefold. The basic idea is to deform the Lagrangian T^3-fibration on the degenerate Calabi-Yau threefold $z_0z_1z_2z_3z_4 = 0$ in $\mathbf{P}^4$ induced by the moment map $\mu : \mathbf{P}^4 \to \Xi$ (where Ξ is the polytope corresponding to $\mathbf{P}^4$) as suggested in [16]. Such a procedure was also carried out in [32]. But Ruan introduces a new technique, using a symplectic flow, to move Lagrangian tori from the singular Calabi-Yau to the non-singular ones. Of course, this gives tori which are Lagrangian with respect to the Fubini-Study Kähler form, but by Moser's theorem, such a Calabi-Yau is symplectomorphic to the one whose Kähler form is given by the Ricci-flat metric.

Because this fibration constructed is not differentiable, the structure of Lagrangian fibrations studied in §2 does not apply, and in particular these fibrations look quite bad from the point of view of mirror symmetry. The discriminant locus is codimension one, which rules out the possibility of G-simplicity, and this construction has to be modified to obtain a fibration suitable for dualizing.

In my own paper [15], I addressed this issue in a completely different way. I worked in a completely topological setting, dealing with T^3-fibrations whose topological properties resemble those one expects of special Lagrangian fibrations as discussed in §1 and 2 here. I then showed how to use $\mathbf{Z}$ and $\mathbf{Z}/n\mathbf{Z}$-simplicity to classify monodromy near *semi-stable* fibres, i.e. fibres where the local monodromy group is unipotent. By construction of examples of such topological fibrations, one in fact proves this gives a complete classification of possible monodromy. In particular, using this classification, one accomplishes step (3) of the program outlined in the introduction for those topological T^3-fibrations with only semistable fibres. This gives one (but not necessarily the only!) solution to the compactification problem.

Next, I applied these ideas to an explicit example. In [25], Ruan had calculated the monodromy around his codimension one discriminant locus. Using this, one can explictly construct a torus fibre bundle with the same monodromy, and then glue in a fibration with suitable discriminant locus to match the same monodromy. This yields a fibration $f : X \to S^3$, and one can then prove that X is diffeomorphic to the quintic.

The best test of the Strominger-Yau-Zaslow conjecture is then to dualize this fibration using the general compactification method that we have developed, to obtain $\check{f} : \check{X} \to S^3$. One can then show that $\check{X}$ is indeed diffeomorphic to the mirror quintic, or rather to one particular minimal model of the mirror quintic. Different choices made for the fibration $f : X \to B$ yield different minimal models of the mirror. This now shows that at least on the topological level, the Strominger-Yau-Zaslow conjecture gives a satisfactory explanation of mirror symmetry.

More recently, Ruan has released another paper [26], announcing a construction of Lagrangian fibrations on the quintic and its mirror with the correct properties (in particular, these fibrations will be simple), and he shows that these fibrations are dual. The method is via a modification of his symplectic flow technique, though details of the construction are postponed until future papers. The fibrations constructed are, on the topological level, the same as those constructed in [26].

There still remains the hardest question: that of obtaining special Lagrangian fibrations. This remains a very difficult problem, despite this recent progress.

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