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K3 Surfaces and String Duality

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ABSTRACT

The primary purpose of these lecture notes is to explore the moduli space of type IIA, type IIB, and heterotic string compactified on a K3 surface. The main tool which is invoked is that of string duality. K3 surfaces provide a fascinating arena for string compactification as they are not trivial spaces but are sufficiently simple for one to be able to analyze most of their properties in detail. They also make an almost ubiquitous appearance in the common statements concerning string duality. We review the necessary facts concerning the classical geometry of K3 surfaces that will be needed and then we review "old string theory" on K3 surfaces in terms of conformal field theory. The type IIA string, the type IIB string, the $E_8 \times E_8$ heterotic string, and $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string on a K3 surface are then each analyzed in turn. The discussion is biased in favour of purely geometric notions concerning the K3 surface itself.

CONTENTS

1. Introduction

The notion of "duality" has led to something of a revolution in string theory in the past year or two. Two theories are considered dual to each other if they ultimately describe exactly the same physics. In order for this to be a useful property, of course, it is best if the two theories appear, at first sight, to be completely unrelated. Different notions of duality abound depending on how the two theories differ. The canonical example is that of "S-duality" where the coupling of one theory is inversely related to that of the other so that one theory would be weakly coupled when the other is strongly coupled. "T-duality" can be defined by similarly considering length scales rather than coupling constants. Thus a theory at small distances can be "T-dual" to another theory at large distances.

In the quest for understanding duality many examples of dual pairs have been postulated. The general scenario is that one takes a string theory (or perhaps M-theory) and compactifies it on some space and then finds a dual partner in the form of another (or perhaps the same) string theory compactified on some other space. In this form duality has become a subject dominated by geometrical ideas since most of the work involved lies in analyzing the spaces on which the string theory is compactified.

One of the spaces which has become almost omnipresent in the study of string duality is that of the K3 surface. We will introduce the K3 surface in section 2 but let us make a few comments here. Mathematicians have been studying the geometry of the K3 surface as a real surface or a complex surface for over one hundred years [1]. In these lectures we will always be working with complex numbers and so "surface" will mean a space of complex dimension two, or real dimension four. A curve will be complex dimension one, etc. Physicists' interest in the K3 surface (for an early paper see, for example, [2]) was not sparked until Yau's proof [3] of Calabi's conjecture in 1977. Since then the K3 surface has become a commonly-used "toy model" for compactifications (see, for example, [4]) as it provides the second simplest example of a Ricci-flat compact manifold after the torus.

The study of duality is best started with toy models and so the K3 surface and the torus are bound to appear often. Another reason for the appearance of the K3 surface, as we shall see in these lectures, is that the mathematics of the heterotic string appears to be intrinsically bound to the geometry of the K3 surface. Thus, whenever the heterotic string appears on one side of a pair of dual theories, the K3 surface is likely to make an appearance in the analysis.

The original purpose of these lectures was to give a fairly complete account of the way K3 surfaces appear in the subject of string duality. For reasons outlined above, however, this is almost tantamount to covering the entire subject of string duality. In order to make the task manageable therefore we will have to omit some current areas of active research. Let us first then discuss what will *not* be covered in these lectures. Note that each of these subjects are covered excellently by the other lecture series anyway.

Firstly we are going to largely ignore M-theory. M-theory may well turn out to be an excellent model for understanding string theory or perhaps even replacing string theory. It also provides a simple interpretation for many of the effects we will discussing.

Secondly we are going to ignore open strings and D-branes. There is no doubt that D-branes offer a very good intuitive approach to many of the phenomena we are going to study. It may well also be that D-branes are absolutely necessarily for a complete understanding of the foundations of string theory.

Having said that M-theory and D-branes are very important we will now do our best to not mention them. One reason for this is to able to finish giving these lectures on time but another, perhaps more important reason is to avoid introducing unnecessary assumptions. We want to take a kind of "Occam's Razor" approach and only introduce constructions as necessary. To many people's tastes our arguments will become fairly cumbersome, especially when dealing with the heterotic string, and it will certainly be true that a simpler picture could be formulated using M-theory or D-branes in some instances. What is important however is that we produce a self-consistent framework in which one may analyze the questions we wish to pose in these lectures.

Thirdly we are going to try to avoid explicit references to solitons. Since non-perturbative physics is absolutely central to most of the later portions of these lectures one may view our attitude as perverse. Indeed, one really cannot claim to understand much of the physics in these lectures without considering the soliton effects. What we will be focusing on, however, is the structure of moduli spaces and we will be able to get away with ignoring solitons in this context quite effectively. The only time that solitons will be of interest is when they become massless.

Our main goal is to understand the type II string and the heterotic string compactified on a K3 surface and what such models are dual to. Of central importance will be the notion of the *moduli space* of a given theory.

In section 2 we will introduce the K3 surface itself and describe its geometry. The facts we require from both differential geometry and algebraic geometry are introduced. In section 3 we will review the "old" approach to a string theory on a K3 surface in terms of the world-sheet conformal field theory.

In section 4 we begin our discussion of full string theory on a K3 surface in terms of the type IIA and type IIB string. The start of this section includes some basic facts about target-space supergravity which are then exploited.

The heterotic string is studied in section 6 but before that we need to take a long detour into the study of string theories compactified down to four dimensions. This detour comprises section 5 and builds the techniques required for section 6. The heterotic string on a K3 surface is a very rich and complicated subject. The analysis is far from complete and section 6 is technically more difficult than the preceding sections.

Note that blocks of text beginning with a "☡" are rather technical and may be omitted on first reading.

2. Classical Geometry

In the mid 19th century the Karakorum range of mountains in Northern Kashmir acted as a natural protection to India, then under British rule, from the Chinese and Russians to the north. Accordingly, in 1856, Captain T. G. Montgomerie was sent out to make some attempt to map the region. From a distance of 128 miles he measured the peaks across the horizon and gave them the names K1, K2, K3, ... , where the "K" stood simply for "Karakorum" [**5**]. While it later transpired that most of these mountain peaks already had names known to the Survey of India, the second peak retained the name Montgomerie had assigned it.

It was not until almost a century later in 1954 that K2, the world's second highest peak, was climbed by Achille Compagnoni and Lino Lacedelli in an Italian expedition. This event led shortly afterwards to the naming of an object of a quite different character. The first occurrence of the name "K3" referring to an algebraic variety occurs in print in [**6**]. It is explained in [**7**] that the naming is after Kummer, Kähler and Kodaira and is inspired by K2. It was Kummer who did much of the earliest work to explore the geometry of the space in question.[1]

2.1. Definition. So what exactly is a K3 surface? Let us first introduce a few definitions. For a general guide to some of the basic principles used in these lectures we refer the reader to chapter 0 of [**8**]. First we define the Hodge numbers of a space X as the dimensions of the Dolbeault cohomology groups

$$h^{p,q}(X) = \dim(H^{p,q}(X)). \tag{1}$$

Next consider the *canonical class*, K, which we may be taken to be defined as

$$K = -c_1(T_X), \tag{2}$$

where T_X is the holomorphic tangent bundle of X.

A K3 surface, S, is defined as a compact complex Kähler manifold of complex dimension two, i.e., a surface, such that

$$\begin{gathered} h^{1,0}(S) = 0 \\ K = 0. \end{gathered} \tag{3}$$

Note that we will sometimes relax the requirement that S be a manifold.

The remarkable fact that makes K3 surfaces so special is the following

THEOREM 1. *Any two K3 surfaces are diffeomorphic to each other.*

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[1]This may explain the erroneous notion in some of the physics literature that the naming is K_3 after "Kummer's third surface" (whatever his first two surfaces may have been). To confuse the issue slightly there is a special kind of K3 surface known as a "Kummer surface" which is introduced in section 2.6.

Thus, if we can find one example of a K3 surface we may deduce all of the topological invariants. The simplest realization is to find a simple example as a complex surface embedded in a complex projective space, i.e., as an *algebraic variety.* The obvious way to do this is to consider the hypersurface defined by the equation

$$f = x_0^n + x_1^n + x_2^n + x_3^n = 0 \tag{4}$$

in the projective space $\mathbb{P}^3$ with homogeneous coordinates $[x_0, x_1, x_2, x_3]$.

It follows from the Lefschetz hyperplane theorem [8] that $h^{1,0}$ of such a hypersurface will be zero. Next we need to find if we can determine n such that $K = 0$. Associated to the canonical class is the *canonical line bundle.* This is simply the holomorphic line bundle, L, such that $c_1(L) = K$. From our definition of K it follows that the canonical line bundle for a manifold of dimension d can be regarded as the dth exterior power of the holomorphic cotangent bundle. Thus a section of the canonical line bundle can be regarded as a holomorphic d-form.

The fact that $K = 0$ for a K3 surface tells us that the canonical line bundle is trivial and thus has a holomorphic section which is nowhere zero. Consider two such sections, s_1 and s_2. The ratio s_1/s_2 is therefore a holomorphic function defined globally over the compact K3 surface. From basic complex analysis it follows that s_1/s_2 is a constant. We see that the K3 surface admits a globally defined, nowhere vanishing, holomorphic 2-form, Ω, which is unique up to a constant. It also follows that $h^{2,0}(S) = 1$.

Let us try to build Ω for our hypersurface of degree n in $\mathbb{P}^3$. First define affine coordinates in the patch $x_0 \neq 0$:

$$y_1 = \frac{x_1}{x_0}, \quad y_2 = \frac{x_2}{x_0}, \quad y_3 = \frac{x_3}{x_0}. \tag{5}$$

An obvious symmetric choice for Ω is then

$$\Omega = \frac{dy_1 \wedge dy_2}{\partial f/\partial y_3} = \frac{dy_2 \wedge dy_3}{\partial f/\partial y_1} = \frac{dy_3 \wedge dy_1}{\partial f/\partial y_2}. \tag{6}$$

This is clearly nonzero and holomorphic in our patch $x_0 \neq 0$. We can now consider another patch such as $x_1 \neq 0$. A straight forward but rather tedious calculation then shows that Ω will only extend into a holomorphic nonzero 2-form over this next patch if $n = 4$.

Our first example of a K3 surface is called the *quartic surface*, given by a hypersurface of degree 4 in $\mathbb{P}^3$. We could have arrived at this same conclusion in a somewhat more abstract way by using the *adjunction formula.* Consider the tangent bundle of S, which we denote T_S, together with the normal bundle N_S for the embedding $S \subset \mathbb{P}^3$. One can then see that

$$T_S \oplus N_S = T_{\mathbb{P}^3|S}, \tag{7}$$

where $T_{\mathbb{P}^3|S}$ is the restriction of the tangent bundle of the embedding $\mathbb{P}^3$ to the hypersurface S.

Introducing the formal sum of Chern classes of a bundle E (see, for example, [9])

$$c(E) = 1 + c_1(E) + c_2(E) + \dots, \tag{8}$$

we see that

$$c(T_{\mathbb{P}^3|S}) = c(T_S) \wedge c(N_S). \tag{9}$$

Treating a wedge product as usual multiplication from now on, it is known that [8]

$$c(T_{\mathbb{P}^k}) = (1+x)^{k+1}, \tag{10}$$

where x is the fundamental generator of $H^2(\mathbb{P}^k, \mathbb{Z})$. Since H^2 is dual to $H^{2(k-1)}$ which is dual to $H_{2(k-1)}$, we may also regard x as the homology class of a *hyperplane* $\mathbb{P}^{k-1} \subset \mathbb{P}^k$ embedded in the obvious way by setting one of the homogeneous coordinates to zero.[2] Stated in this way one can see that $c_1(N_S)$ is given by nx.

Thus we have that

$$\begin{aligned} c(T_S) &= \frac{(1+x)^4}{1+nx} \\ &= 1 + (4-n)x + (6-4n+n^2)x^2. \end{aligned} \tag{11}$$

Again we see that $n = 4$ is required to obtain $K = 0$. Now we also have c_2 which enables us to work out the Euler characteristic, $\chi(S)$, of a K3 surface:

$$\begin{aligned} \chi(S) &= \int_S c_2(T_S) \\ &= \int_{\mathbb{P}^3} l_S \wedge c_2(T_S) \\ &= \int_{\mathbb{P}^3} 4x.6x^2 \\ &= 24, \end{aligned} \tag{12}$$

where l_S is the 2-form which is the dual of the dual of the hypersurface S in the sense explained above. One may also show using the Lefschetz hyperplane theorem that

$$\pi_1(S) = 0. \tag{13}$$

We now have enough information to compute all the Hodge numbers, $h^{p,q}$. Since $\pi_1(S) = 0$, we have that the first Betti number $b_1(S) = \dim H^1(S) = h^{1,0} + h^{0,1}$ must be zero. The Euler characteristic then fixes $b_2(S)$ which then determines $h^{1,1}$ since we already know $h^{2,0} = 1$ from above. The result is

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1. \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{14}$$

[2] Throughout these lectures we will often use the same notation for the 2-form and the associated $2(k-1)$-cycle.

2.2. Holonomy. Before continuing our discussion of K3 surfaces, we will take a detour and discuss the subject of holonomy which will be of considerable use at many points in these lectures.

Holonomy is a natural concept in the differential geometry of a manifold, M, with a vector bundle, $\pi : E \to M$, with a connection. Consider taking a point, $p \in M$, and a vector in the fibre, $e_1 \in \pi^{-1}(p)$. Now, following a closed path, Γ, starting and ending at p, parallel transport this vector according to the connection. When you are done, you will have another vector $e_2 \in \pi^{-1}(p)$. Write $e_2 = g_\Gamma(e_1)$, where $g_\Gamma(e_1)$ is an element of the structure group of the bundle. The (global) *holonomy* group of $E \to M$ is defined as the group generated by all such $g_\Gamma(e_1)$ for all closed paths Γ. The holonomy, $\mathfrak{H}_M$, of a Riemannian manifold, M, is defined as the holonomy of the tangent bundle equipped with the Levi-Civita connection from the metric.

The holonomy of a Riemannian manifold of real dimension d is contained in $\mathrm{O}(d)$. If it is orientable this becomes $\mathrm{SO}(d)$. The study of which other holonomy groups are possible is a very interesting question and will be of some importance to us. We refer the reader to [**10, 11**] for a full discussion of the results and derivations. We require the following:

1. $\mathfrak{H}_M \subseteq \mathrm{U}(\frac{d}{2})$ if and only if M is a Kähler manifold.
2. $\mathfrak{H}_M \subseteq \mathrm{SU}(\frac{d}{2})$ if and only if M is a Ricci-flat Kähler manifold.
3. $\mathfrak{H}_M \subseteq \mathrm{Sp}(\frac{d}{4})$ if and only if M is a hyperkähler manifold.
4. $\mathfrak{H}_M \subseteq \mathrm{Sp}(\frac{d}{4}).\mathrm{Sp}(1)$ if and only if M is a quaternionic Kähler manifold.[3]
5. A "symmetric space" of the form G/H where G and H are Lie groups has holonomy H.

Actually in each case the specific representation of the group in which the fibre transforms is also fixed. A celebrated theorem due to Berger, with contributions from Simons [**11**], then states that *the only other possibilities not yet mentioned are $\mathfrak{H}_M \cong G_2$ where the fibre transforms as a* **7** *or $\mathfrak{H}_M \cong \mathrm{Spin}(7)$ where the fibre transforms as an* **8**.

There is a fairly clear relationship between the holonomy groups and the invariant forms of the natural metric on M in the first cases. For the most general case we have that the form

$$\sum_{i=1}^{d} dx^i \otimes dx^i \tag{15}$$

on $\mathbb{R}^d$ admits $\mathrm{O}(d)$ as the group of invariances. In the complex Kähler case we consider the Hermitian form

$$\sum_{i=1}^{d/2} d\bar{z}^i \otimes dz^i \tag{16}$$

on $\mathbb{C}^{d/2}$ which admits $\mathrm{U}(\frac{d}{2})$ as the invariance group.

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[3]The "." denotes that we take the direct product except that the $\mathbb{Z}_2$ centers of each group are identified.

For the next case we consider the quaternionic numbers, $\mathbb{H}$. In this case the natural form

$$\sum_{i=1}^{d/4} d\zeta^i \otimes d\bar{\zeta}^i \tag{17}$$

on $\mathbb{H}^{d/4}$ is preserved by $\mathrm{Sp}(\frac{d}{4})$. Note that writing quaternions as 2×2 matrices in the usual way gives an embedding $\mathrm{Sp}(\frac{d}{4}) \subset \mathrm{SU}(\frac{d}{2})$. Thus, a hyperkähler manifold is always a Ricci-flat Kähler manifold. In fact, one is free to choose one of a family of complex structures. Let us denote a quaternion by $q = a + bI + cJ + cK$, where $a, b, c, d \in \mathbb{R}$, $I^2 = J^2 = K^2 = -1$ and $IJ = K$, $JI = -K$, etc. Given a hyperkähler structure we may choose a complex structure given by q, where $q^2 = -1$. This implies $a = 0$ and

$$b^2 + c^2 + d^2 = 1. \tag{18}$$

Thus for a given hyperkähler structure we have a whole S^2 of possible complex structures. We will see that the K3 surface is hyperkähler when equipped with a Ricci-flat metric.

Because of the fact that quaternionic numbers are not commutative, we also have the notion of a quaternionic Kähler manifold in addition to that of the hyperkähler manifold. The space $\mathbb{H}^n$ admits an action of $\mathrm{Sp}(n).\mathrm{Sp}(1)$ by multiplication on the right by $n \times n$ quaternionic matrices in $\mathrm{Sp}(n)$ and by a quaternion of unit norm on the left. This also leads to the notion of a manifold with a kind of quaternionic structure — this time the "quaternionic Kähler manifold". The main difference between this and the hyperkähler manifold is that the extra $\mathrm{Sp}(1)$ can act on the S^2 of complex structures between patches and so destroy any global complex structure. All that remains is an S^2 bundle of almost complex structures which need have no global section. Thus a generic quaternionic Kähler manifold will *not* admit a complex structure. When this bundle is trivial the situation reduces to the hyperkähler case. As an example, the space $\mathbb{HP}^n$ is quaternionic Kähler. Note that the case $n = 1$ is somewhat redundant as this reduces to $\mathrm{Sp}(1).\mathrm{Sp}(1) \cong SO(4)$, which gives a generic orientable Riemannian manifold.

2.3. Moduli space of complex structures. We now want to construct the moduli space of all K3 surfaces. In order to determine the moduli space it is very important to specify exactly what data defines a particular K3 surface. By considering various data we will construct several different moduli spaces throughout these lectures. To begin with we want to consider the K3 surface purely as an object in algebraic geometry and, as such, we will find the moduli space of complex structures for K3 surfaces.

To "measure" the complex structure we need some relatively simple quantity which depends on the complex structure. This will be provided by "periods" which are simply integrals of the holomorphic 2-form, Ω, over integral 2-cycles within S. To analyze periods we first then require an understanding of the integral 2-cycles $H_2(S, \mathbb{Z})$.

Since $b_2(S) = 22$ from the previous section, we see that $H_2(S,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{22}$ as a group.[4] In addition to this group structure we may specify an inner product between any two elements, $\alpha_i \in H_2(S,\mathbb{Z})$, given by

$$\alpha_1.\alpha_2 = \#(\alpha_1 \cap \alpha_2), \tag{19}$$

where the notation on the right refers to the oriented intersection number, which is a natural operation on homology cycles [**9**]. This abelian group structure with an inner product gives $H_2(S,\mathbb{Z})$ the structure of a lattice.

The signature of this lattice can be determined from the index theorem for the *signature complex* [**9**]

$$\tau = \int_S \tfrac{1}{3}(c_1^2 - 2c_2) = -\tfrac{2}{3}\chi(S) = -16. \tag{20}$$

Thus the 22-dimensional lattice has signature $(3,19)$.

Poincaré duality tells us that given a basis $\{e_i\}$ of 2-cycles for $H_2(S,\mathbb{Z})$, for each e_i we may find an e_j^* such that

$$e_i.e_j^* = \delta_{ij}, \tag{21}$$

where the set $\{e_j^*\}$ also forms a basis for $H_2(S,\mathbb{Z})$. Thus $H_2(S,\mathbb{Z})$ is a *self-dual* (or *unimodular*) lattice. Note that this also means that the lattice of integral cohomology, $H^2(S,\mathbb{Z})$, is isomorphic to the lattice of integral homology, $H_2(S,\mathbb{Z})$.

The next fact we require is that the lattice $H_2(S,\mathbb{Z})$ is *even*. That is,

$$e.e \in 2\mathbb{Z}, \quad \forall e \in H_2(S,\mathbb{Z}). \tag{22}$$

This is a basic topology fact for any spin manifold, i.e., for $c_1(T_X) = 0 \pmod 2$. We will not attempt a proof of this as this is rather difficult (see, for example, Wu's formula in [**13**]).

The classification of even self-dual lattices is extremely restrictive. We will use the notation $\Gamma_{m,n}$ to refer to an even self-dual lattice of signature (m,n). It is known that m and n must satisfy

$$m - n = 0 \pmod 8 \tag{23}$$

and that if $m > 0$ and $n > 0$ then $\Gamma_{m,n}$ is unique up to isometries [**14, 15**]. An isometry is an automorphism of the lattice which preserves the inner product.

[4] Actually we need the result that $H_2(S,\mathbb{Z})$ is *torsion-free* to make this statement to avoid any finite subgroups appearing. This follows from $\pi_1(S) = 0$ and the various relations between homotopy and torsion in homology and cohomology [**12**].

In our case, one may chose a basis such that the inner product on the basis elements forms the matrix

$$\left(\begin{array}{c|c|c|c|c} -E_8 & & & & \\ \hline & -E_8 & & & \\ \hline & & U & & \\ \hline & & & U & \\ \hline & & & & U \end{array}\right), \tag{24}$$

where $-E_8$ denotes the 8×8 matrix given by minus the Cartan matrix of the Lie algebra E_8 and U represents the "hyperbolic plane"

$$U \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{25}$$

Now we may consider periods

$$\varpi_i = \int_{e_i} \Omega. \tag{26}$$

We wish to phrase these periods in terms of the lattice $\Gamma_{3,19}$ we have just discussed. First we will fix a specific embedding of a basis, $\{e_i\}$, of 2-cycles into the lattice $\Gamma_{3,19}$. That is, we make a specific choice of which periods we will determine. Such a choice is called a "marking" and a K3 surface, together with such a marking, is called a "marked K3 surface".

There is the natural embedding

$$\Gamma_{3,19} \cong H^2(S,\mathbb{Z}) \subset H^2(S,\mathbb{R}) \cong \mathbb{R}^{3,19}. \tag{27}$$

We may now divide $\Omega \in H^2(S,\mathbb{C})$ as

$$\Omega = x + iy, \tag{28}$$

where $x, y \in H^2(S,\mathbb{R})$. Now a (p,q)-form integrated over S is only nonzero if $p = q = 2$ [8] and so

$$\begin{aligned} \int_S \Omega \wedge \Omega &= (x+iy).(x+iy) = (x.x - y.y) + 2ix.y \\ &= 0. \end{aligned} \tag{29}$$

Thus, $x.x = y.y$ and $x.y = 0$. We also have

$$\begin{aligned} \int_S \Omega \wedge \overline{\Omega} &= (x+iy).(x-iy) = (x.x + y.y) \\ &= \int_S \|\Omega\|^2 > 0. \end{aligned} \tag{30}$$

The vectors x and y must be linearly independent over $H^2(S,\mathbb{R})$ and so span a 2-plane in $H^2(S,\mathbb{R})$ which we will also give the name Ω. The relations (29) and (30) determine that this 2-plane must be space-like, i.e., any vector, v, within it satisfies $v.v > 0$.

Note that the 2-plane is equipped with a natural orientation but that under complex conjugation one induces $(x,y) \to (x,-y)$ and this orientation is reversed.

We therefore have the following picture. The choice of a complex structure on a K3 surface determines a vector space $\mathbb{R}^{3,19}$ which contains an even self-dual lattice $\Gamma_{3,19}$ and an oriented 2-plane Ω. If we change the complex structure on the K3 surface we expect the periods to change and so the plane Ω will rotate with respect to the lattice $\Gamma_{3,19}$.

We almost have enough technology now to build our moduli space of complex structures on a K3 surface. Before we can give the result however we need to worry about special things that can happen within the moduli space. A K3 surface which gives a 2-plane, Ω, which very nearly contains a light-like direction, will have periods which are only just acceptable and so this K3 surface will be near the boundary of our moduli space. As we approach the boundary we expect the K3 surfaces to degenerate in some way. Aside from this obvious behaviour we need to worry that some points away from this natural boundary may also correspond to K3 surfaces which have degenerated in some way. It turns out that there are such points in the moduli space and these will be of particular interest to us in these lectures. They will correspond to *orbifolds*, as we will explain in detail in section 2.6. For now, however, we need to include the orbifolds in our moduli space to be able to state the form of the moduli space.

The last result we require is the following

THEOREM 2 (Torelli). *The moduli space of complex structures on a marked K3 surface (including orbifold points) is given by the space of possible periods.*

For an account of the origin of this theorem we refer to [**16**]. Thus, the moduli space of complex structures on a marked K3 surface is given by the space of all possible oriented 2-planes in $\mathbb{R}^{3,19}$ with respect to a fixed lattice $\Gamma_{3,19}$.

Consider this space of oriented 2-planes in $\mathbb{R}^{3,19}$. Such a space is called a Grassmannian, which we denote $\mathrm{Gr}^+(\Omega,\mathbb{R}^{3,19})$, and we have

$$\mathrm{Gr}^+(\Omega,\mathbb{R}^{3,19}) \cong \frac{\mathrm{O}^+(3,19)}{(\mathrm{O}(2)\times \mathrm{O}(1,19))^+}. \tag{31}$$

This may be deduced as follows. In order to build the Grassmannian of 2-planes in $\mathbb{R}^{3,19}$, first consider all rotations, $\mathrm{O}(3,19)$, of $\mathbb{R}^{3,19}$. Of these, we do not care about internal rotations within the 2-plane, $\mathrm{O}(2)$, or rotations normal to it, $\mathrm{O}(1,19)$. For oriented 2-planes we consider only the index 2 subgroup which preserves orientation on the space-like directions. We use the "+" superscripts to denote this.

This Grassmannian builds the moduli space of marked K3 surfaces. We now want to remove the effects of the marking. There are diffeomorphisms of the K3 surface, which we want to regard as trivial as far as our moduli space is concerned, but which have a nontrivial action on the lattice $H^2(S,\mathbb{Z})$. Clearly any diffeomorphism induces an isometry of $H^2(S,\mathbb{Z})$, preserving the inner product. We denote the full group of such isometries as $\mathrm{O}(\Gamma_{3,19})$.[5] Our moduli space of marked K3 surfaces can

[5]Sometimes the less precise notation $\mathrm{O}(3,19;\mathbb{Z})$ is used.

be viewed as a kind of Teichmüller space, and the image of the diffeomorphisms in $\mathrm{O}(\Gamma_{3,19})$ can be viewed as the modular group. The moduli space is the quotient of the Teichmüller space by the modular group.

What is this modular group? It was shown in [**17, 18**] that any element of $\mathrm{O}^+(\Gamma_{3,19})$ can be induced from a diffeomorphism of the K3 surface. It was shown further in [**19**] that any element of $\mathrm{O}(\Gamma_{3,19})$ which is not in $\mathrm{O}^+(\Gamma_{3,19})$ *cannot* be induced by a diffeomorphism. Thus our modular group is precisely $\mathrm{O}^+(\Gamma_{3,19})$.

Treating (31) as a right coset we will act on the left for the action of the modular group. The result is that the moduli space of complex structures on a K3 surface (including orbifold points) is

$$\mathscr{M}_c \cong \mathrm{O}^+(\Gamma_{3,19})\backslash \mathrm{O}^+(3,19)/(\mathrm{O}(2)\times \mathrm{O}(1,19))^+. \tag{32}$$

When dealing with $\mathscr{M}_c$ it is important to realize that $\mathrm{O}^+(\Gamma_{3,19})$ has an ergodic action on the Teichmüller space and thus $\mathscr{M}_c$ is actually not Hausdorff. Such unpleasant behaviour is sometimes seen in string theory in fairly pathological circumstances [**20**] but it seems reasonable to expect that under reasonable conditions we should see a fairly well-behaved moduli space. As we shall see, the moduli space $\mathscr{M}_c$ does not appear to make any natural appearance in string theory and the related moduli spaces which do appear will actually be Hausdorff.

2.4. Einstein metrics. The first modification we will consider is that of considering the moduli space of Einstein metrics on a K3 surface. We will always assume that the metric is Kähler.

An Einstein metric is a (pseudo-)Riemannian metric on a (pseudo-)Riemannian manifold whose Ricci curvature is proportional to the metric. Actually, for a K3 surface, this condition implies that the metric is Ricci-flat [**21**]. We may thus use the terms "Einstein" and "Ricci-flat" interchangeably in our discussion of K3 surfaces.

The Hodge star will play an essential rôle in the discussion of the desired moduli space. Recall that [**9, 10**]

$$\alpha \wedge *\beta = (\alpha,\beta)\omega_g, \tag{33}$$

where α and β are p-forms, ω_g is the volume form and (α,β) is given by

$$(\alpha,\beta) = p!\int \sum_{i_1 i_2 j_1 j_2\ldots} g^{i_1 i_2} g^{j_1 j_2} \alpha_{i_1 j_1\ldots}\beta_{i_2 j_2\ldots}\, dx_1 dx_2 \ldots\,, \tag{34}$$

in local coordinates. In particular, if α is self-dual, in the sense $\alpha = *\alpha$, then $\alpha.\alpha > 0$ in the notation of section 2.3. Similarly an anti-self-dual 2-form will obey $\alpha.\alpha < 0$. On our K3 surface S we may decompose

$$H^2(S,\mathbb{R}) = \mathscr{H}^+ \oplus \mathscr{H}^-, \tag{35}$$

where $\mathscr{H}^\pm$ represents the cohomology of the space of (anti-)self-dual 2-forms. We then see that

$$\dim\mathscr{H}^+ = 3, \quad \dim\mathscr{H}^- = 19, \tag{36}$$

from section 2.3.

The curvature acts naturally on the bundle of (anti)-self-dual 2-forms. By standard methods (see, for example, [**10**]) one may show that the curvature of the bundle of self-dual 2-forms is actually zero when the manifold in question is a K3 surface. This is one way of seeing directly the action of the SU(2) holonomy of section 2.2. Since a K3 is simply-connected, this shows that the bundle $\mathscr{H}^+$ is trivial and thus has 3 linearly independent sections.

Consider a local orthonormal frame of the cotangent bundle $\{e_1, e_2, e_3, e_4\}$. We may write the three sections of $\mathscr{H}^+$ as

$$\begin{aligned} s_1 &= e_1 \wedge e_2 + e_3 \wedge e_4 \\ s_2 &= e_1 \wedge e_3 + e_4 \wedge e_2 \\ s_3 &= e_1 \wedge e_4 + e_2 \wedge e_3. \end{aligned} \tag{37}$$

Clearly an SO(4) rotation of the cotangent directions produces an SO(3) rotation of $\mathscr{H}^+$. That is, a rotation within $\mathscr{H}^+$ is induced by a reparametrization of the underlying K3 surface. One should note that the orientation of $\mathscr{H}^+$ is fixed.

Let us denote by Σ the space $\mathscr{H}^+$ viewed as a subspace of $H^2(S, \mathbb{R})$. Putting $dz_1 = e_1 + ie_2$ and $dz_2 = e_3 + ie_4$ we obtain a Kähler form equal to s_1, and the holomorphic 2-form $dz_1 \wedge dz_2$ is given by $s_2 + is_3$. This shows that Σ is spanned by the 2-plane Ω of section 2.3 together with the direction in $H^2(S, \mathbb{R})$ given by the Kähler form.

This fits in very nicely with Yau's theorem [**3**] which states that for any manifold M with $K = 0$, and a fixed complex structure, given a cohomology class of the Kähler form, there exists a unique Ricci-flat metric. Thus, we fix the complex structure by specifying the 2-plane, Ω, and then choose a Kähler form, J, by specifying another direction in $H^2(S, \mathbb{R})$. Clearly this third direction is space-like, since

$$\mathrm{Vol}(S) = \int_S J \wedge J > 0, \tag{38}$$

and it is perpendicular to Ω as the Kähler form is of type $(1,1)$. Thus Σ, spanned by Ω and J, is space-like.

The beauty of Yau's theorem is that we need never concern ourselves with the explicit form of the Einstein metric on the K3 surface. Once we have fixed Ω and J, we know that a unique metric exists. Traditionalists may find it rather unsatisfactory that we do not write the metric down — indeed no explicit metric on a K3 surface has ever been determined to date — but one of the lessons we appear to have learnt from the analysis of Calabi–Yau manifolds in string theory is that knowledge of the metric is relatively unimportant.

As far as our moduli space is concerned, one aspect of the above analysis which is important is that rotations within the 3-plane, Σ, may affect what we consider to be the Kähler form and complex structure but they do not affect the underlying Riemannian metric. We see that a K3 surface viewed as a Riemannian manifold may admit a whole family of complex structures. Actually this family is parametrized by the sphere, S^2, of ways in which Σ is divided into Ω and J.

This property comes from the fact that a K3 surface admits a hyperkähler structure. This is obvious from section 2.2 as a K3 is Ricci-flat and Kähler and thus has holonomy SU(2), and SU(2) $\cong$ Sp(1). The sphere (18) of possible complex structures is exactly the S^2 degree of freedom of rotating within the 3-plane Σ we found above.

Some care should be taken to show that the maps involved are surjective [**22, 23**] but we end up with the following [**24**]

THEOREM 3. *The moduli space of Einstein metrics, $\mathscr{M}_E$, for a K3 surface (including orbifold points) is given by the Grassmannian of oriented 3-planes within the space $\mathbb{R}^{3,19}$ modulo the effects of diffeomorphisms acting on the lattice $H^2(S,\mathbb{Z})$.*

In other words, we have a relation similar[6] to (32):

$$\mathscr{M}_E \cong \mathrm{O}^+(\Gamma_{3,19})\backslash \mathrm{O}^+(3,19)/(\mathrm{SO}(3)\times \mathrm{O}(19)) \times \mathbb{R}_+, \tag{39}$$

where the $\mathbb{R}_+$ factor denotes the volume of the K3 surface given by (38).

This is actually isomorphic to the space

$$\mathscr{M}_E \cong \mathrm{O}(\Gamma_{3,19})\backslash \mathrm{O}(3,19)/(\mathrm{O}(3)\times \mathrm{O}(19)) \times \mathbb{R}_+, \tag{40}$$

since the extra generator required, $-1 \in \mathrm{O}(3,19)$, to elevate $\mathrm{O}^+(3,19)$ to $\mathrm{O}(3,19)$, is in the center and so taking the left-right coset makes no difference.

Note that (40) is actually a Hausdorff space as the discrete group $\mathrm{O}(\Gamma_{3,19})$ has a properly discontinuous action [**25**]. Thus we see that the global description of this space of Einstein metrics on a K3 surface is much better behaved than the moduli space of complex structures discussed earlier.

2.5. Algebraic K3 Surfaces. In section 2.4 we enlarged the moduli space of complex structures of section 2.3 and we found a space with nice properties. In this section we are going to find another nice moduli space by going in the opposite direction. That is, we will restrict the moduli space of complex structures by putting constraints on the K3 surface. We are going to consider *algebraic* K3 surfaces, i.e., K3 surfaces that may be embedded by algebraic (holomorphic) equations in homogeneous coordinates into $\mathbb{P}^N$ for some N.

The analysis is done in terms of algebraic curves, that is, Riemann surfaces which have been holomorphically embedded into our K3 surface S. Clearly such a curve, C, may be regarded as a homology cycle and thus an element of $H_2(S,\mathbb{Z})$. By the "dual of the dual" construction of section 2.1 we may also regard it as an element $C \in H^2(S,\mathbb{Z})$. The fact that C is holomorphically embedded may also be used to show that $C \in H^{1,1}(S)$ [**8**]. We then have $C \in \mathrm{Pic}(S)$, where we define

$$\mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S), \tag{41}$$

which is called the "Picard group", or "Picard lattice", of S. We also define the "Picard number", $\rho(S)$, as the rank of the Picard lattice. Any element of the Picard group, $e \in \mathrm{Pic}(S)$, corresponds to a line bundle, L, such that $c_1(L) = e$ [**8**]. Thus the Picard group may be regarded as the group of line bundles on S, where the group composition is the Whitney product.

As the complex structure of S is varied, the Picard group changes. This is because an element of $H^2(S,\mathbb{Z})$ that was regarded as having type purely $(1,1)$ may pick up parts of type $(2,0)$ or $(0,2)$ as we vary the complex structure. A completely generic K3 surface will have completely trivial Picard group, i.e., $\rho = 0$.

The fact that S contains a curve C is therefore a restriction on the complex structure of S. An algebraic K3 surface similarly has its deformations restricted as

[6]Note that the orientation problem makes this look more unlike (32) than it needs to! We encourage the reader to not concern themselves with these orientation issues, at least on first reading.

the embedding in $\mathbb{P}^N$ will imply the existence of one or more curves. As an example let us return to the case where S is a quartic surface

$$f = x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0, \tag{42}$$

in $\mathbb{P}^3$. A hyperplane $\mathbb{P}^2 \subset \mathbb{P}^3$ will cut $f = 0$ along a curve and so shows the existence of an algebraic curve C. Taking various hyperplanes will produce various curves but they are all homologous as 2-cycles and thus define a unique element of the Picard lattice. Thus the Picard number of this quartic surface is at least one. Actually for the "Fermat" form of f given in (42) it turns out[7] that $\rho = 20$ (which is clearly the maximum value we may have since $\dim H^{1,1}(S) = 20$). A quartic surface need not be in the special form (42). We may consider a more general

$$f = \sum_{i+j+k+l=4} a_{ijkl}\, x_0^i x_1^j x_2^k x_3^l, \tag{43}$$

for arbitrary $a_{ijkl} \in \mathbb{C}$. Generically we expect no elements of the Picard lattice other than those generated by C and so $\rho(S) = 1$.

Now consider the moduli space of complex structures on a quartic surface. Since C is of type $(1,1)$, the 2-plane Ω of section 2.3 must remain orthogonal to this direction.

We may determine $C.C$ by taking two hyperplane sections and finding the number of points of intersection. The intersection of two hyperplanes in $\mathbb{P}^3$ is clearly $\mathbb{P}^1$ and so the intersection $C \cap C$ is given by a quartic in $\mathbb{P}^1$, which is 4 points. Thus $C.C = 4$ and C spans a space-like direction.

Our moduli space will be similar to that of (32) except that we may remove the direction generated by C from consideration. Thus our 2-plane is now embedded in $\mathbb{R}^{2,19}$ and the discrete group is generated by the lattice $\Lambda_C = \Gamma_{3,19} \cap C^{\perp}$. Note that we do not denote Λ_C as $\Gamma_{2,19}$ as it is *not* even-self-dual. The moduli space in question is then

$$\mathscr{M}_{\text{Quartic}} \cong \mathrm{O}(\Lambda_C)\backslash \mathrm{O}(2,19)/(\mathrm{O}(2) \times \mathrm{O}(19)). \tag{44}$$

This is Hausdorff. Note also that it is a space of complex dimension 19 and that a simple analysis of (43) shows that $f = 0$ has 19 deformations of a_{ijkl} modulo reparametrizations of the embedding $\mathbb{P}^3$. Thus embedding this K3 surface in $\mathbb{P}^3$ has brought about a better-behaved moduli space of complex structures but we have "lost" one deformation as (32) has complex dimension 20.

One may consider a more elaborate embedding such as a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ given by an equation of bidegree $(3,2)$, i.e.,

$$f = \sum_{\substack{a_0+a_1+a_2=3\\ b_0+b_1=2}} A_{a_0a_1a_2,b_0b_1}\, x_0^{a_0} x_1^{a_1} x_2^{a_2} y_0^{b_0} y_1^{b_1}. \tag{45}$$

This is an algebraic K3 since $\mathbb{P}^2 \times \mathbb{P}^1$ itself may be embedded in $\mathbb{P}^5$ (see, for example, [8]). Taking a hyperplane $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$ one cuts out a curve C_1. Taking the hyperplane $\mathbb{P}^2 \times \{p\}$ for some $p \in \mathbb{P}^1$ cuts out C_2. By the same method we used for the quartic above we find the intersection matrix

$$\begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}. \tag{46}$$

[7] I thank M. Gross for explaining this to me.

Thus, denoting $\Lambda_{C_1C_2}$ by the sublattice of $\Gamma_{3,19}$ orthogonal to C_1 and C_2 we have

$$\mathcal{M} \cong \mathrm{O}(\Lambda_{C_1C_2})\backslash \mathrm{O}(2,18)/(\mathrm{O}(2)\times \mathrm{O}(18)). \tag{47}$$

This moduli space has dimension 18 and this algebraic K3 surface has $\rho(S) = 2$ generically. In general it is easy to see that the dimension of the moduli space plus the generic Picard number will equal 20. Note that the Picard lattice will have signature $(1, \rho - 1)$. This follows as it is orthogonal to Ω inside $\mathbb{R}^{3,19}$ and thus has at most one space-like direction but the natural Kähler form inherited from the ambient $\mathbb{P}^N$ is itself in the Picard lattice and so there must be at least one space-like direction. Thus the moduli space of complex structures on an algebraic K3 surface will always be of the form

$$\mathcal{M}_{\mathrm{Alg}} \cong \mathrm{O}(\Lambda)\backslash \mathrm{O}(2,20-\rho)/(\mathrm{O}(2)\times \mathrm{O}(20-\rho)), \tag{48}$$

where Λ is the sublattice of $H^2(S,\mathbb{Z})$ orthogonal to the Picard lattice. This lattice is often referred to as the *transcendental lattice* of S. Note that this lattice is rarely self-dual.

2.6. Orbifolds and blow-ups. When discussing the moduli spaces above we have had to be careful to note that we may be including K3 surfaces which are not manifolds but, rather, *orbifolds*. The term "orbifold" was introduced many years ago by W. Thurston after their first appearance in the mathematics literature in **[26]** (where they were referred to as "V-manifolds"). The general idea is to slightly enlarge the concept of a manifold to objects which contain singularities produced by quotients. They have subsequently played a celebrated rôle in string theory after they made their entry into the subject in **[27]**. It is probably worthwhile noting that the definition of an orbifold is slightly different in mathematics and physics. We will adopt the mathematics definition which, for our purposes, we take to be defined as follows:

> *An orbifold is a space which admits an open covering, $\{U_i\}$, such that each patch is diffeomorphic to $\mathbb{R}^n/G_i$.*

The G_i's are discrete groups (which may be trivial) which can be taken to fix the origin of $\mathbb{R}^n$. The physics definition however is more global and defines an orbifold to be a space of the form M/G where M is a manifold and G is a discrete group. A physicist's orbifold is a special case of the orbifolds we consider here. Which definition is applicable to string theory is arguable. Most of the vast amount of analysis that has been done over the last 10 years on orbifolds has relied on the global form M/G. Having said that, one of the appeals of orbifolds is that string theory is (generically) well-behaved on such an object, and this behaviour only appears to require the local condition.

The definition of an orbifold can be extended to the notion of a complex orbifold where each patch is biholomorphic to $\mathbb{C}^n/G_i$ and the induced transition functions are holomorphic. Since we may define a metric on $\mathbb{C}^n/G_i$ by the natural inherited metric on $\mathbb{C}^n$, we have a notion of a metric on an orbifold. In fact, it is not hard to extend the notion of a Kähler-Einstein metric to include orbifolds **[28]**. Similarly, a complex orbifold may be embeddable in $\mathbb{P}^N$ and can thus be viewed as an algebraic variety. In this case the notion of canonical class is still valid. Thus, there is nothing to stop the definition of the K3 surface being extended to include orbifolds. As we will see in this section, such K3 surfaces lie naturally at the boundary of the moduli space of K3 manifolds.

An example of such a K3 orbifold is the following, which is often the first K3 surface that string theorists encounter. Take the 4-torus defined as a complex manifold of dimension two by dividing the complex plane $\mathbb{C}^2$ with affine coordinates (z_1, z_2) by the group $\mathbb{Z}^4$ generated by

$$z_k \mapsto z_k + 1, \quad z_k \mapsto z_k + i, \quad k = 1, 2. \tag{49}$$

Then consider the $\mathbb{Z}_2$ group of isometries generated by $(z_1, z_2) \mapsto (-z_1, -z_2)$. It is not hard to see that this $\mathbb{Z}_2$ generator fixes 16 points: $(0,0)$, $(0, \frac{1}{2})$, $(0, \frac{1}{2}i)$, $(0, \frac{1}{2} + \frac{1}{2}i)$, $(\frac{1}{2}, 0)$, $\ldots$, $(\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i)$. Thus we have an orbifold, which we will denote S_0.

Since the $\mathbb{Z}_2$-action respects the complex structure and leaves the Kähler form invariant we expect S_0 to be a complex Kähler orbifold. Also, a moment's thought shows that any of the non-contractable loops of the 4-torus may be shrunk to a point after the $\mathbb{Z}_2$-identification is made. Thus $\pi_1(S_0) = 0$. Also, the holomorphic 2-form $dz_1 \wedge dz_2$ is invariant. We thus expect $K = 0$ for S_0. All said, the orbifold S_0 has every right to be called a K3 surface.

We now want to see what the relation of this orbifold S_0 might be to the general class of K3 manifolds. To do this, we are going to modify S_0 to make it smooth. This process is known as "blowing-up". This procedure is completely local and so we may restrict attention to a patch within S_0. Clearly the patch of interest is $\mathbb{C}^2/\mathbb{Z}_2$.

The space $\mathbb{C}^2/\mathbb{Z}_2$ can be written algebraically by embedding it in $\mathbb{C}^3$ as follows. Let (x_0, x_1, x_2) denote the coordinates of $\mathbb{C}^3$ and consider the hypersurface A given by

$$f = x_0 x_1 - x_2^2 = 0. \tag{50}$$

A hypersurface is smooth if and only if $\partial f/\partial x_0 = \ldots = \partial f/\partial x_2 = f = 0$ has no solution. Thus $f = 0$ is smooth everywhere except at the origin where it is singular. We can parameterize $f = 0$ by putting $x_0 = \xi^2$, $x_1 = \eta^2$, and $x_2 = \xi\eta$. Clearly then (ξ, η) and $(-\xi, -\eta)$ denote the same point. This is the only identification and so $f = 0$ in $\mathbb{C}^3$ really is the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ we require.

Consider now the following subspace of $\mathbb{C}^3 \times \mathbb{P}^2$:

$$\{(x_0, x_1, x_2), [s_0, s_1, s_2] \in \mathbb{C}^3 \times \mathbb{P}^2;\ x_i s_j = x_j s_i,\ \forall i, j\}. \tag{51}$$

This space may be viewed in two ways — either by trying to project it onto the $\mathbb{C}^3$ or the $\mathbb{P}^2$. Fixing a point in $\mathbb{P}^2$ determines a line (i.e., $\mathbb{C}^1$) in $\mathbb{C}^3$. Thus, (51) determines a line bundle on $\mathbb{P}^2$. One may determine $c_1 = -H$ for this bundle, where H is the hyperplane class. We may thus denote this bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$. Alternatively one may fix a point in $\mathbb{C}^3$. If this is not the point $(0,0,0)$, this determines a point in $\mathbb{P}^2$. At $(0,0,0)$ however we have the entire $\mathbb{P}^2$. Thus $\mathcal{O}_{\mathbb{P}^2}(-1)$ can be identified pointwise with $\mathbb{C}^3$ except that the origin in $\mathbb{C}^3$ has been replaced by $\mathbb{P}^2$. The space (51) is thus referred to as a *blow-up of* $\mathbb{C}^3$ *at the origin*. The fact the $\mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathbb{C}^3$ are generically isomorphic as complex spaces in this way away from some subset means that these spaces are ***birationally equivalent*** [8]. A space X blown-up at a point will be denoted $\widetilde{X}$ and the birational map between them will usually be

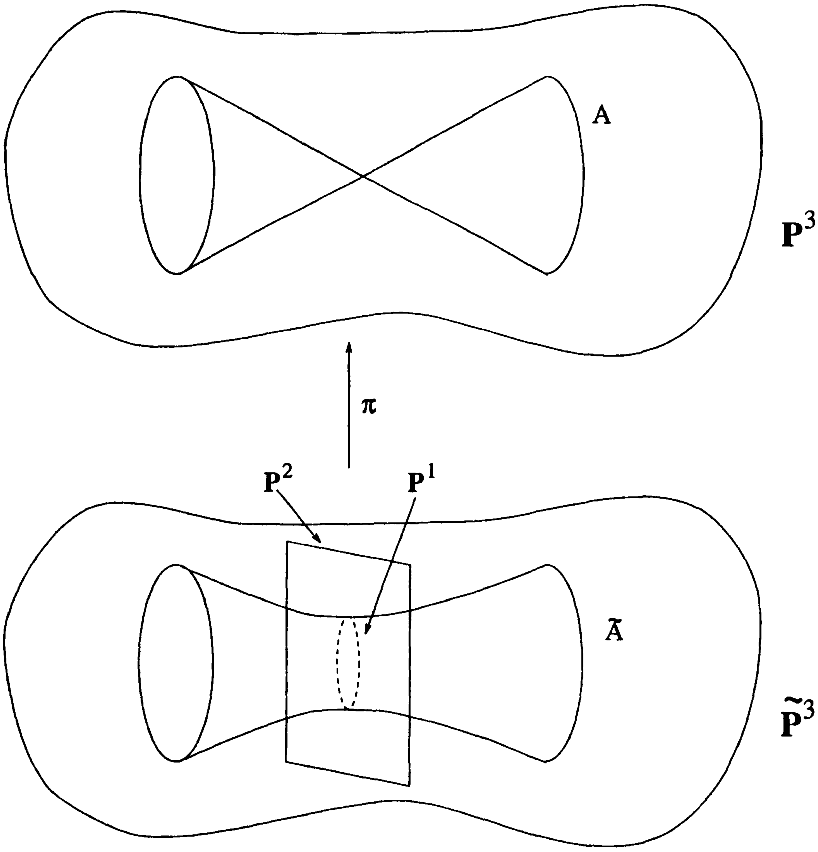

FIGURE 1. Blowing up $A \cong \mathbb{C}^2/\mathbb{Z}_2$.

written

$$\pi : \widetilde{X} \to X. \tag{52}$$

That is, π represents the *blow-down* of $\widetilde{X}$. The $\mathbb{P}^2$ which has grown out of the origin is called the *exceptional divisor.*

Now let us consider what happens to the hypersurface A given by (50) in $\mathbb{C}^3$ as we blow up the origin. We will consider the *proper transform,* $\widetilde{A} \subset \widetilde{X}$. If X is blown-up at the point $p \in X$ then $\widetilde{A}$ is defined as the closure of the point set $\pi^{-1}(A \setminus p)$ in $\widetilde{X}$.

Consider following a path in A towards the origin. In the blow-up, the point we land on in the exceptional $\mathbb{P}^2$ in the blow-up depends on the angle at which we approached the origin. Clearly the line given by (x_0t, x_1t, x_2t), $t \in \mathbb{C}$, $x_0x_1 - x_2^2 = 0$, will land on the point $[s_0, s_1, s_2] \in \mathbb{P}^2$ where again $s_0s_1 - s_2^2 = 0$. Thus the point set that provides the closure away from the origin is a quadric $s_0s_1 - s_2^2 = 0$ in $\mathbb{P}^2$. It is easy to show that this curve has $\chi = 2$ and is thus a sphere, or $\mathbb{P}^1$.

We have thus shown that when the origin is blown-up for $A \subset \mathbb{C}^3$, the proper transform of A replaces the old origin, i.e., the singularity, by a $\mathbb{P}^1$. Within the context of blowing up A, this $\mathbb{P}^1$ is viewed as the exceptional divisor and we denote it E. What is more, this resulting space, $\widetilde{A}$, is now smooth. We show this process in figure 1.

Carefully looking at the coordinate patches in $\widetilde{A}$ around E, we can work out the normal bundle for $E \subset \widetilde{A}$. The result is that this line bundle is equal to $\mathcal{O}_{\mathbb{P}^1}(-2)$. We will refer to such a rational (i.e., genus zero) curve in a complex surface as a "(-2)-curve".

Let us move our attention for a second to the general subject of complex surfaces with $K = 0$ and consider algebraic curves within them. Consider a curve C of genus g. The self-intersection of a curve may be found by deforming the curve to another one, homologically equivalent, and counting the numbers of points of intersection (with orientation giving signs). In other words, we count the number of points which remain fixed under the deformation. Suppose we may deform and keep the curve algebraic. Then an infinitesimal such deformation may be considered as a section of the normal bundle of C and the self-intersection is the number of zeros, i.e., the value of c_1 of the normal bundle integrated over C. Thus $c_1(N)$ gives the self-intersection, where N is the normal bundle. Note that two algebraic curves which intersect transversely always have positive intersection since the complex structure fixes the orientation. Thus this can only be carried out when $c_1(N) > 0$. We may extend the concept however when $c_1(N) < 0$ to the idea of negative self-intersection. In this case we see that C cannot be deformed to a nearby algebraic curve.

The adjunction formula tells us the sum $c_1(N) + c_1(T)$, where T is the tangent bundle of C, must give the first Chern class of the embedding surface restricted to C. Thus, if $K = 0$, we have

$$C.C = 2(g-1). \tag{53}$$

That is, any rational curve in a K3 surface must be a (-2)-curve. Note that (53) provides a proof of our assertion in section 2.3 that the self-intersection of a cycle is always an even number — at least in the case that the cycle is a smooth algebraic curve.

Actually, if we blow-up all 16 fixed points of our original orbifold S_0 we obtain a smooth K3 surface. To see this we need only show that the blow-up we have done does not affect $K = 0$. One can show that this is indeed the case so long as the exceptional divisor satisfies (53), i.e., it is a (-2)-curve [**8**]. A smooth K3 surface obtained as the blow-up of $T^4/\mathbb{Z}_2$ at all 16 fixed points is called a *Kummer surface.* Clearly the Picard number of a Kummer surface is at least 16. A Kummer surface need not be algebraic, just as the original T^4 need not be algebraic.

Now we have enough information to find how the orbifolds, such as S_0, fit into the moduli space of Einstein metrics on a K3 surface. If we blow down a (-2)-curve in a K3 surface we obtain, locally, a $\mathbb{C}^2/\mathbb{Z}_2$ quotient singularity. The above description appeared somewhat discontinuous but we may consider doing such a process gradually as follows. Denote the (-2)-curve as E. The size of E is given by the integral of the Kähler form over E, that is, $J.E$. Keeping the complex structure of the K3 surface fixed we may maintain E in the Picard lattice but we may move

J so that it becomes orthogonal to E. Thus E has shrunk down to a point — we have done the blow-down.

We have shown that any rational curve in a K3 surface is an element of the Picard lattice with $C.C = -2$. Actually the converse is true [16]. That is, given an element of the Picard lattice, e, such that $e.e = -2$, then either e or $-e$ gives the class of a rational curve in the K3 surface. This will help us prove the following. Let us define the *roots* of $\Gamma_{3,19}$ as $\{\alpha \in \Gamma_{3,19};\ \alpha.\alpha = -2\}$.

THEOREM 4. *A point in the moduli space of Einstein metrics on a K3 surface corresponds to an orbifold if and only if the 3-plane, Σ, is orthogonal to a root of $\Gamma_{3,19}$.*

If we take a root which is perpendicular to Σ, then it must be perpendicular to Ω and thus in the Picard lattice. It follows that this root (or minus the root, which is also perpendicular to Σ) gives a rational curve in the K3. Then, since J is also perpendicular to this root, the rational curve has zero size and the K3 surface must be singular. Note also that any higher genus curve cannot be shrunk down in this way as it would be a space-like or light-like direction in the Picard lattice and could thus not be orthogonal to Σ. What remains to be shown is that the resulting singular K3 surface is always an orbifold.

For a given point in the moduli space of Einstein metrics consider the set of roots orthogonal to Σ. Suppose this set can be divided into two mutually orthogonal sets. These would correspond to two sets of curves which did not intersect and thus would be blown down to two (or more) separate isolated points. Since the orbifold condition is local we may confine our attention to the case when this doesn't happen. The term "root" is borrowed from Lie group theory and we may analyze our situation in the corresponding way. We may choose the "simple roots" in our set which will span the root lattice in the usual way (see, for example, [29]). Now consider the intersection matrix of the simple roots. It must have -2's down the diagonal and be negative definite (as it is orthogonal to Σ). This is entirely analogous to the classification of simply-laced Lie algebras and immediately tells us that there is an *A-D-E* classification of such events.

We have already considered the A_1 case above. This was the case of a single isolated (-2)-curve which shrinks down to a point giving locally a $\mathbb{C}^2/\mathbb{Z}_2$ quotient singularity. To proceed further let us try another example. The next simplest situation is that of a $\mathbb{C}^2/\mathbb{Z}_3$ quotient singularity given by $(\xi,\eta) \mapsto (\omega\xi,\omega^2\eta)$, where ω is a cube root of unity. As before we may rewrite this as a subspace of $\mathbb{C}^3$ as

$$f = x_0x_1 - x_2^3 = 0. \tag{54}$$

The argument is very similar to the $\mathbb{C}^2/\mathbb{Z}_2$ case. The difference is that now as we follow a line into the blown-up singularity and consider the path (x_0t, x_1t, x_2t) within $x_0x_1 - x_2^3$, the x_2^3 term becomes irrelevant and the closure of the point set becomes $s_0s_1 = 0$ within $\mathbb{P}^2$. This consists of two rational curves ($s_0 = 0$ and $s_1 = 0$) intersecting transversely (at $[s_0, s_1, s_2] = [0,0,1]$). Thus, when we blow-up $\mathbb{C}^2/\mathbb{Z}_3$, we obtain as an exceptional divisor two (-2)-curves crossing at one point. Clearly this is the A_2 case.

Now consider the general case of a cyclic quotient $\mathbb{C}^2/\mathbb{Z}_n$. This is given by $f = x_0x_1 - x_2^n = 0$. At first sight the discussion above for the case $n = 3$ would appear to be exactly the same for any value of $n > 3$ but actually we need to be careful that after the blow-up we really have completely resolved the singularity. Consider the coordinate patch in $\mathcal{O}_{\mathbb{P}^2}(-1)$ written as (51) where $s_2 \neq 0$. We may use $y_0 = s_0/s_2$, $y_1 = s_1/s_2$ as good affine coordinates on the base $\mathbb{P}^2$ and $y_2 = x_2$ as a good coordinate in the fibre. Since $x_0 = y_0y_2$ and $x_1 = y_1y_2$, our hypersurface becomes

$$y_2^2(y_0y_1 - y_2^{n-2}) = 0. \tag{55}$$

Now $y_2 = 0$ is the equation for the exceptional divisor $\mathbb{P}^2 \subset \mathbb{P}^3$ in our patch. We are interested in the proper transform of our surface and thus we do not want to include the full $\mathbb{P}^2$ in our solution — just the intersection with our surface. Thus we throw this solution away and are left with

$$y_0y_1 - y_2^{n-2} = 0. \tag{56}$$

If $n = 2$ or 3 this is smooth. If $n > 3$ however we have a singularity at $y_0 = y_1 = y_2 = 0$. This point is at $s_0 = s_1 = 0$ which is precisely where the two $\mathbb{P}^1$'s produced by the blow-up intersect. One may check that the other patches contain no singularities.

What we have shown then is that starting with the space $\mathbb{C}^2/\mathbb{Z}_n$, $n > 2$, the blow-up replaces the singularity at the origin by two $\mathbb{P}^1$'s which intersect at a point but, in the case $n > 3$, this point of intersection is locally of the form $\mathbb{C}^2/\mathbb{Z}_{n-2}$. To resolve the space completely, the procedure is clear. We simply repeat the process until we are done. Note that when we blow up the point of intersection of two $\mathbb{P}^1$'s intersecting transversely, the fact that the $\mathbb{P}^1$'s approach the point of intersection at a different "angle" means that after the blow-up they pass through different points of the exceptional divisor and thus become disjoint. We show the process of blowing-up a $\mathbb{C}^2/\mathbb{Z}_6$ singularity in figure 2. In this process we produce a chain of 5 $\mathbb{P}^1$'s, $E_1, \ldots, E_5$, when completely resolving the singularity. We show the $\mathbb{P}^1$'s as lines.

Clearly we see that resolving the $\mathbb{C}^2/\mathbb{Z}_n$ singularity produces the A_{n-1} intersection matrix for the (-2)-curves. Thus we have deduced the form of the A-series. We now ponder the D- and E-series.

Consider the general form of the quotient $\mathbb{C}^2/G$. We are interested in the cases which occur locally in a K3 surface in which $K = 0$. This requires that G leaves the holomorphic 2-form $dz_1 \wedge dz_2$ invariant. This implies that G must be a discrete subgroup of SU(2). One may also obtain this result by noting that the holonomy of the orbifold near the quotient singularity can be viewed as being isomorphic to G. The subgroups of SU(2) are best understood from the well-known exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{SU}(2) \to \mathrm{SO}(3) \to 1. \tag{57}$$

Thus any subgroup of SU(2) can be projected into a subgroup of SO(3) and considered as a symmetry of a 3-dimensional solid. The cyclic groups $\mathbb{Z}_n$ may be thought of as, for example, the symmetries of cones over regular polygons. The other possibilities are the dihedral groups which are the symmetries of a prism over a regular polygon, and the symmetries of the tetrahedron, the octahedron (or cube), and the icosahedron (or dodecahedron). Each of these latter groups are nonabelian and correspond to a subgroup of SU(2) with twice as many elements as the subgroup of SO(3). They are thus called the binary dihedral, binary tetrahedral, binary octahedral, and binary icosahedral groups respectively. In each case the quotient $\mathbb{C}^2/G$

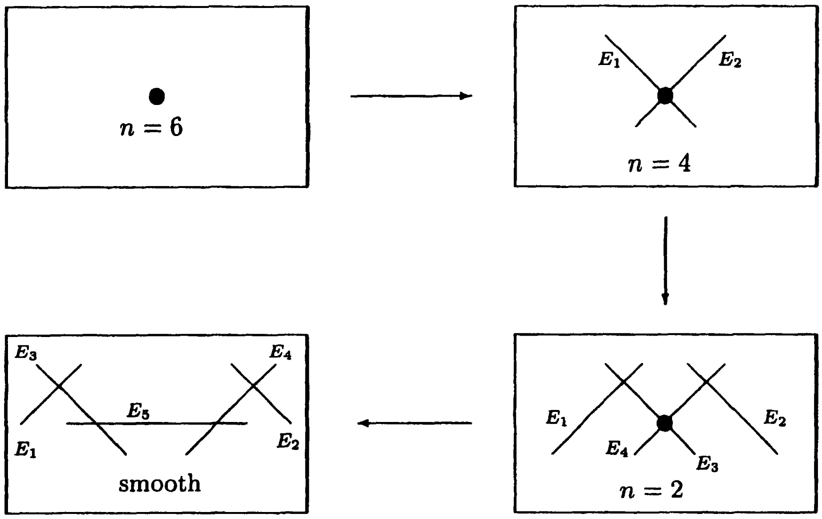

FIGURE 2. Blowing up $\mathbb{C}^2/\mathbb{Z}_6$.

can be embedded as a hypersurface in $\mathbb{C}^3$. This work was completed by Du Val (see [**30**] and references therein), after whom the singularities are sometimes named. The case of an icosahedron was done by Felix Klein last century [**31**] and so they are also often referred to as "Kleinian singularities".

Once we have a hypersurface in $\mathbb{C}^3$ we may blow-up as before until we have a smooth manifold. The intersection matrices of the resulting (-2)-curves can then be shown to be (minus) the Cartan matrix of D_n, E_6, E_7, or E_8 respectively. This process is laborious and is best approached using slightly more technology than we have introduced here. We refer the reader to [**16**] or [**32**] for more details. The results are summarized in table 1(see [**33**] for some of the details). We have thus shown that any degeneration of a K3 surface that may be achieved by blowing down (-2)-curves leads to an orbifold singularity.

One might also mention that in the case of the A_n blow-ups, explicit metrics are known which are asymptotically flat [**34, 35, 36**]. Unfortunately, since the blow-up inside a K3 surface is not actually flat asymptotically, such metrics represent only an approximation to the situation we desire. As we said earlier however, lack of an explicit metric will not represent much of a problem.

This miraculous correspondence between the A-D-E classification of discrete subgroups of SU(2) and Dynkin diagrams for simply-laced simple Lie groups must count as one of the most curious interrelations in mathematics. We refer to [**37**] or [**38**] for the flavour of this subject. We will see later in section 4.3 that string theory will provide another striking connection.

We have considered the resolution process from the point of view of blowing-up by changing the Kähler form. In terms of the moduli space of Einstein metrics on a K3 surface this is viewed as a rotation of the 3-plane Σ so that there are no longer

Group	Generators	Hypersurface	Resolution
Cyclic	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha = e^{\frac{2\pi i}{n}}$	$x^2 + y^2 + z^n = 0$	A_{n-1}
Binary Dihedral	$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\beta = e^{\frac{\pi i}{n}}$	$x^2 + y^2 z + z^{n+1} = 0$	D_{n+2}
Binary Tetrahedral	D_4, $\frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}$, $\varepsilon = e^{\frac{2\pi i}{8}}$	$x^2 + y^3 + z^4 = 0$	E_6
Binary Octahedral	E_6, $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}$, $\varepsilon = e^{\frac{2\pi i}{8}}$	$x^2 + y^3 + yz^3 = 0$	E_7
Binary Icosahedral	$-\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}$, $\frac{1}{\eta^2 - \eta^3} \begin{pmatrix} \eta + \eta^4 & 1 \\ 1 & -\eta - \eta^4 \end{pmatrix}$, $\eta = e^{\frac{2\pi i}{5}}$	$x^2 + y^3 + z^5 = 0$	E_8

TABLE 1. *A-D-E* Quotient Singularities.

any roots in the orthogonal complement. Since Σ is spanned by the Kähler form and Ω, which measures the complex structure, we may equally view this process in terms of changing the complex structure, rather than the Kähler form. In this language, the quotient singularity is *deformed* rather than blown-up. The process of resolving is now seen, not as giving a non-zero size to a shrunken rational curve, but rather changing the complex structure so that the rational curve no longer exists. This deformation process is actually very easy to understand in terms of the singularity as a hypersurface in $\mathbb{C}^3$. Consider the A_{n-1} singularity in table 1. A deformation of this to

$$x^2 + y^2 + z^n + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \ldots + a_0 = 0 \tag{58}$$

will produce a smooth hypersurface for generic values of the a_i's. (Note that the $a_{n-1}z^{n-1}$ term can be transformed away by a reparametrization.)

It is worth emphasizing that in general, when considering any algebraic variety, blow-ups and deformations are quite different things. We will discuss later how the difference between blowing up a singularity and deforming it away can lead to topology changing processes in complex dimension three, for example. It is the peculiar way in which the complex structure moduli and Kähler form get mixed up in the moduli space of Einstein metrics on a K3 surface that makes them amount to much the same thing in this context. The relationship between blowing up and deformations will be deepened shortly when we discuss mirror symmetry.

3. The World-Sheet Perspective

In this section we are going to embark on an analysis of string theory on K3 surfaces from what might be considered a rather old-fashioned point of view. That is, we are going to look at physics on the world-sheet. One point of view that was common more than a couple of years ago was that string theory could solve difficult problems by "pulling back" physics in the target space, which has a large number of dimensions and is hence difficult, to the world-sheet, which is two-dimensional and hence simple. Thus an understanding of two-dimensional physics on the world-sheet would suffice for understanding the universe.

More recently it has been realized that the world-sheet approach is probably inadequate as it misses aspects of the string theory which are nonperturbative in the string coupling expansion. Thus, attention has switched somewhat away from the world-sheet and back to the target space. One cannot forget the world-sheet however and, as we will see later in section 5, in some examples it would appear that the target space point of view appears on an equal footing with the world-sheet point of view. We must therefore first extract from the world-sheet as much as we can.

3.1. The Nonlinear Sigma Model. "Old" string theory is defined as a two-dimensional theory given by maps from a Riemann surface, Σ, into a target manifold X:

$$x : \Sigma \to X. \tag{59}$$

In the conformal gauge, the action is given by (see, for example, [**39**] for the basic ideas and [**40**] for conventions on normalizations)

$$S = \frac{i}{8\pi\alpha'} \int_\Sigma (g_{ij} - B_{ij})\, \partial x^i \bar{\partial} x^j \, d^2z - 2\pi \int \Phi R^{(2)} \, d^2z + \dots , \tag{60}$$

where we have ignored any terms which contain fermions. The terms are identified as follows. g_{ij} is a Riemannian metric on X and B_{ij} are the components of a real 2-form, B, on X. Φ is the "dilaton" and is a real number (which might depend on x) and $R^{(2)}$ is the scalar curvature of Σ. This two-dimensional theory is known as the "non-linear σ-model".

In order to obtain a valid string theory, we require that the resulting two-dimensional theory is conformally invariant with a specific value of the "central charge". (See [**41**] for basic notions in conformal field theory.) Conformal invariance puts constraints on the various parameters above [**42, 43**]. In general the result is in terms of a perturbation theory in the quantity α'/R^2, where R is some characteristic "radius" (coming from the metric) of X, assuming X to be compact.

The simplest way of demanding conformal invariance to leading order is to set the dilaton, Φ, to be a constant, and let B be closed and g_{ij} be Ricci-flat. There are other solutions, such as the one proposed in [**44**] and these do play a rôle in string duality as solitons (see, for example, [**45**]). It is probably safe to say however that the solution we will analyze, with the constant dilaton, is by far the best understood.

In many ways one may regard this conformal invariance calculation to be the string "derivation" of general relativity. To leading order in α'/R^2 we obtain Einstein's field equation and then perturbation theory "corrects" this to higher orders. Nonperturbative effects, i.e., "world-sheet instantons", should also modify notions in general relativity. Anyway, we see that in this simple case, a vacuum solution for string theory is the same as that for general relativity — namely a Ricci-flat manifold.

There is a simple and beautiful relationship between supersymmetry in this non-linear σ-model and the Kähler structure of the target space manifold X. We have neglected to include any fermions in the action (60) but they are of the form ψ^i and transform, as far as the target space is concerned, as sections of the cotangent bundle. A supersymmetry will be roughly of the form

$$\delta_\epsilon X^i = \bar{\epsilon}\, l^i_j \psi^j. \tag{61}$$

The object of interest here is l^i_j. With one supersymmetry ($N = 1$) on the world-sheet one may simply reparameterize to make it equal to δ^i_j. When the $N > 1$

however we have more structure. It was shown [46] that for $N = 2$ the second l^i_j acts as an almost complex structure and gives X the structure of a Kähler manifold. In the case $N = 4$, we have 3 almost complex structures, as in section 2.4, and this leads to a hyperkähler manifold. The converse also applies.

Note that this relationship between world-sheet supersymmetry and the complex differential geometry of the target space required no reference to conformal invariance. In the case that we have conformal invariance one may also divide the analysis into holomorphic and anti-holomorphic parts and study them separately. In this case we have separate supersymmetries in the left-moving (holomorphic) and right-moving (anti-holomorphic) sectors.

The case of interest to us, of course, is when X is a smooth K3 manifold. From what we have said, this will lead to an $N = (4,4)$ superconformal field theory, at least to leading order in α'/R^2. Actually for $N = 4$ non-linear σ-models, the perturbation theory becomes much simpler and it can be shown [47] that there are no further corrections to the Ricci-flat metric after the leading term. Additionally, one may present arguments that this is even true nonperturbatively [48]. Thus our Ricci-flat metric on the K3 surface is an exact solution. One must contrast this to the $N = 2$ case where there are both perturbative corrections and nonperturbative effects in general.

3.2. The Teichmüller space. The goal of this section is to find the moduli space of conformally invariant non-linear σ-models with K3 target space. This may be considered as an intermediate stage to that of the last section, where we considered classical geometry, and that of the following sections where we consider supposedly fully-fledged string theory. This will prove to be a very important step however.

Firstly note that there are three sets of parameters in (60) which may be varied to span the moduli space required. In each case we need to know which deformations will be effective in the sense that they really change the underlying conformal field theory. Here we will have to make some assumptions since a complete analysis of these conformal field theories has yet to be completed.

First consider the metric g_{ij}. We have seen that this must be Ricci-flat to obtain conformal invariance. We will assume that any generic deformation of this Ricci-flat metric to another inequivalent Ricci-flat metric will lead to an inequivalent conformal field theory. Since the dimension of the moduli space of Einstein metrics on a K3 surface given in (40) is 58, we see that the metric accounts for 58 parameters.

Next we have the 2-form, B. This appears in the action in the form

$$\int_{x(\Sigma)} B. \tag{62}$$

Thus, since the image of the world-sheet in X under the map x is a closed 2-cycle, any exact part of B is irrelevant. All we see of B is its cohomology class. As b_2 of a K3 surface is 22, this suggests we have 22 parameters from the B-field.

Lastly we have the dilaton, Φ. This plays a very peculiar rôle in our conformal field theory. Since Φ is a constant over X, by assumption, we may pull it outside the integral leaving a contribution of $2\Phi(g-1)$ to the action, where g is the genus of Σ. In this section we really only care about the conformal field theory for a fixed Σ and so this quantity remains constant. To be more complete we should sum over the genera of Σ. Taking the limit of $\alpha' \to \infty$, the world-sheet image in the target space will degenerate to a Feynman diagram and then g will count the number of loops. Thus we have an effective target space coupling of

$$\lambda = e^{\Phi}. \tag{63}$$

Anyway, since we want to ignore this summation of Σ for the time being we will ignore the dilaton. See, for example, [**49**] for a further discussion of world-sheet properties of the dilaton from a string field theory point of view.

All said then we have a moduli space of $58 + 22 = 80$ real dimensions. To proceed further we need to know some aspects about the holonomy of the moduli space. The local form of the moduli space was first presented in [**50**] but we follow more closely here the method of [**51**].

Let us return to the moduli space of Einstein metrics on a K3 surface. Part of the holonomy algebra of the symmetric space factor in (40) is $\mathfrak{so}(3) \cong \mathfrak{su}(2)$. This rotation in $\mathbb{R}^3$ comes from the choice of complex structures given a quaternionic structure as discussed in section 2.2. Thus, this part of the holonomy can be understood as arising from the symmetry produced by the S^2 of complex structures.

This $\mathfrak{su}(2)$ symmetry must therefore be present in the non-linear σ-model. In the case where we have a conformal field theory however, we may divide the analysis into separate left- and right-moving parts. Thus each sector must have an independent $\mathfrak{su}(2)$ symmetry. Indeed, it is known from conformal field theory that an $N = 4$ superconformal field theory contains an affine $\mathfrak{su}(2)$ algebra and so an $N = (4,4)$ superconformal field theory has an $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4)$ symmetry.

This symmetry acts on the tangent directions to a point in the moduli space (i.e., the "marginal operators") and so will be a subgroup of the holonomy. Thus the $\mathfrak{so}(3)$ appearing in the holonomy algebra of the moduli space of Einstein metrics is promoted to $\mathfrak{so}(4)$ for our moduli space of conformal field theories.

Now we are almost done. We need to find a space whose holonomy contains SO(4) as a factor and has dimension 80. One could suggest spaces such as $A \times B$, where A is a Riemannian manifold of dimension 4 and B is a Riemannian manifold of dimension 76. Such a factorization is incompatible with what we know about the conformal field theory, however. Analyzing the marginal operators in terms of superfields shows that each one is acted upon non-trivially by at least part of the SO(4). Given the work of Berger and Simons therefore leaves us with only one possibility.[8]

THEOREM 5. *Given the assumptions about the effectiveness of deformations on the underlying conformal field theory, any smooth neighbourhood of the moduli space of conformally invariant non-linear σ-models with a K3 target space is isomorphic to an open subset of*

$$\mathcal{T}_\sigma = \frac{\mathrm{O}(4,20)}{\mathrm{O}(4) \times \mathrm{O}(20)}. \tag{64}$$

We now want to know about the global form of the moduli space. Here we are forced to make assumptions about how reasonable our conformal field theories

[8]Actually we should rule out the compact symmetric space possibility. This is done by our completeness assumption, as we know we may make a K3 surface arbitrarily large.

can be. We will assume that from theorem 6 it follows that the moduli space of conformal field theories is given by $\mathscr{M}_\sigma \cong G_\sigma \backslash \mathscr{T}_\sigma$, where G_σ is some discrete group. That is, $\mathscr{T}_\sigma$ is the Teichmüller space. All we are doing here is assuming that our moduli space is "complete" in the sense that there are no pathological limit points in it possibly bounding some bizarre new region. While this assumption seems extremely reasonable I am not aware of any rigorous proof that this is the case.

3.3. The geometric symmetries. All that remains then is the determination of the modular group, G_σ. To begin with we should relate (64) to the Teichmüller spaces we are familiar with from section 2. This will allow us the incorporate the modular groups we have already encountered. This is a review of the work that first appeared in [**52, 53**].

The space $\mathscr{T}_\sigma$ is the Grassmannian of space-like 4-planes in $\mathbb{R}^{4,20}$. We saw that the moduli space of Einstein metrics on a K3 surface is given by the Grassmannian of space-like 3-planes in $\mathbb{R}^{3,19}$ and this must be a subspace of $\mathscr{T}_\sigma$ since the Einstein metric appears in the action (60). This gives us a clear way to proceed.

Let us introduce the even self-dual lattice $\Gamma_{4,20} \subset \mathbb{R}^{4,20}$. It would be nice if we could show that this played the same rôle as $\Gamma_{3,19}$ played in the moduli space of Einstein metrics. That is, we would like to show $G_\sigma \cong \mathrm{O}(\Gamma_{4,20})$. We will see this is indeed true.

First we want a natural way of choosing $\Gamma_{3,19} \subset \Gamma_{4,20}$. To do this fix a primitive element $w \in \Gamma_{4,20}$ such that $w.w = 0$. Now consider the space, $w^\perp \subset \mathbb{R}^{4,20}$, of all vectors x such that $x.w = 0$. Clearly w is itself contained in this space. Now project onto the codimension one subspace $w^\perp/w$ by modding out by the w direction. We now embed $w^\perp/w$ back into $\mathbb{R}^{4,20}$ such that

$$\frac{w^\perp}{w} \cap \Gamma_{4,20} \cong \Gamma_{3,19}. \tag{65}$$

It is important to be aware of the fact that all statements about $\Gamma_{3,19}$ are dictated by a choice of w. The embedding $w^\perp/w \subset \mathbb{R}^{4,20}$ can be regarded as a choice of a second lattice vector, w^*, such that w^* is orthogonal to $w^\perp/w$, $w^*.w^* = 0$ and $w^*.w = 1$. As we shall see, the choice of w^* is not as significant as the choice of w.

Now perform the same operation on the space-like 4-plane. We will denote this plane $\Pi \subset \mathbb{R}^{4,20}$. First define $\Sigma' = \Pi \cap w^\perp$ and then project this 3-plane into the space $w^\perp/w$ and embed back into $\mathbb{R}^{4,20}$ to give Σ. This Σ may now be identified with that of section 2.4 to give the Einstein metric on the K3 surface of some fixed volume.

Fixing Σ we may look at how we may vary Π to fill out the other deformations. Let Π be given by the span of Σ' and B', where B' is a vector, orthogonal to Σ', normalized by $B'.w = 1$. We may project B' into $w^\perp/w$ to give $B \in \mathbb{R}^{3,19}$. Note that $\mathbb{R}^{3,19}$ is the space $H^2(X, \mathbb{R})$ and so B is a 2-form as desired.

Lastly we require the volume of the K3 surface. Let us decompose B' as

$$B' = \alpha w + w^* + B, \tag{66}$$

We claim that α is the volume of the K3 surface. To see this we need to analyze the explicit form of the moduli space further.

We have effectively decomposed the Teichmüller space of conformal field theories as

$$\frac{\mathrm{O}(4,20)}{\mathrm{O}(4)\times\mathrm{O}(20)} \cong \frac{\mathrm{O}(3,19)}{\mathrm{O}(3)\times\mathrm{O}(19)} \times \mathbb{R}^{22} \times \mathbb{R}_{+}, \tag{67}$$

where the three factors on the right are identified as the Teichmüller spaces for the metric on the K3 surface (given by Σ), the B-field, and the volume respectively. Each of the spaces in the equation (67) has a natural metric, given by the invariant metric for the group action in the case of the symmetric spaces. This can be shown to coincide with the natural metric from conformal field theory — the Zamolodchikov metric — given the holonomy arguments above. The isomorphism (67) will respect this metric if "warping" factors are introduced as explained in [**54**]. It is these warping factors which determine the identification of the volume of the K3 surface as α in (66). This will be explained further in [**53**].

The part of G_σ we can understand directly is the part which fixes w but acts on $w^\perp/w$. This will affect the metric on the K3 surface of fixed volume and the B-field. We know that the modular group coming from global diffeomorphisms of the K3 surface is $\mathrm{O}^+(\Gamma_{3,19})$, which should be viewed as $\mathrm{O}^+(H^2(X,\mathbb{Z}))$. The discrete symmetries for the B-field meanwhile can be written as $B \cong B + e$, where $e \in H^2(X,\mathbb{Z})$. To see this note that shifting B by a integer element will shift the action (60) by a multiple of $2\pi i$ and hence will not effect the path integral. Thanks to our normalization of B', a shift of B by an element of $\Gamma_{3,19}$ amounts to a rotation of Π which is equivalent to an element of $\mathrm{O}(\Gamma_{4,20})$. Note that this can also be interpreted as a redefinition of w^*. That is, the freedom of choice of w^* is irrelevant once we take into account the B-field shifts.

We may also consider taking the complex conjugate of the non-linear σ-model action. This has the effect of reversing the sign of B while providing the extra element required to elevate $\mathrm{O}^+(\Gamma_{3,19})$ to $\mathrm{O}(\Gamma_{3,19})$.

The result is that the subgroup of G_σ that we see directly from the non-linear σ-model action is a subgroup of $\mathrm{O}(\Gamma_{4,20})$ consisting of rotations *and* translations of $\Gamma_{3,19} \subset \Gamma_{4,20}$. This may be viewed as the *space group* of $\Gamma_{3,19}$, or equivalently, the semi-direct product

$$G_\sigma \supset \mathrm{O}(\Gamma_{3,19}) \ltimes \Gamma_{3,19}. \tag{68}$$

This is as much as we can determine from $w^\perp/w$.

3.4. Mirror symmetry. To proceed any further in our analysis of G_σ we need to know about elements which do not correspond to a manifest symmetry of the non-linear σ-model action (60). This knowledge will be provided by *mirror symmetry*. Mirror symmetry is a much-studied phenomenon in Calabi–Yau threefolds (see, for example, [**55**] for a review). The basic idea in the subject of threefolds is that the notion of deformation of complex structure is exchanged with deformation of complexified Kähler form. The character of mirror symmetry in K3 surfaces is somewhat different since, as we have seen, the notion of what constitutes a deformation of complex structure and what constitutes a deformation of the Kähler form can be somewhat ambiguous. Also, we have yet to mention the possibility of complexifying the Kähler form, as that too is a somewhat ambiguous process.

Indeed mirror symmetry itself is somewhat meaningless when viewed in terms of the intrinsic geometry of a K3 surface and does not begin to make much sense until viewed in terms of *algebraic* K3 surfaces. The results for algebraic K3 surfaces were first explored by Martinec [**56**], whose analysis actually predates the discovery (and naming) of mirror symmetry in the Calabi–Yau threefold context. See also [**57, 58**] for a discussion of some of the issues we cover below.

Recall that the Picard lattice of section 2.5 was defined as the lattice of integral 2-cycles in $H^2(X,\mathbb{Z})$ which were of type $(1,1)$. The transcendental lattice, Λ, was defined as the orthogonal complement of the Picard lattice in $H^2(X,\mathbb{Z})$. The signature of Λ is $(2, 20-\rho)$, where ρ is the Picard number of X.

It is clear that $\Gamma_{4,20} \cong \Gamma_{3,19} \oplus U$, where U is the hyperbolic plane of (25). We will extend the Picard lattice to the "quantum Picard lattice", Υ, by defining

$$\Upsilon = \mathrm{Pic}(X) \oplus U, \tag{69}$$

as the orthogonal complement of Λ within $\Gamma_{4,20}$.

Given a K3 surface with an Einstein metric specified by Σ and a given algebraic structure, we know that the complex structure 2-plane, Ω, is given by $\Sigma \cap (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$.[9] The Kähler form direction, J is then given by the orthogonal complement of $\Omega \subset \Sigma$. Accordingly, J lies in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

We want to extend this to the non-linear σ-model picture. Keep Ω defined as above, or equivalently,

$$\Omega = \Pi \cap (\Lambda \otimes_{\mathbb{Z}} \mathbb{R}), \tag{70}$$

and introduce a new space-like 2-plane, $\mho$, as the orthogonal complement of $\Omega \subset \Pi$. Clearly we have

$$\mho = \Pi \cap (\Upsilon \otimes_{\mathbb{Z}} \mathbb{R}). \tag{71}$$

Our notion of a mirror map will be to exchange

$$\mu : (\Lambda, \Omega) \leftrightarrow (\Upsilon, \mho). \tag{72}$$

Note that as $\mho$ encodes the Kähler form and the value of the B-field we have the notion of exchange of complex structure data with that of Kähler form + B-field as befits a mirror map.

The moduli space of non-linear σ-models on an algebraic K3 surface will be the subspace of (64) which respects the division of Π into Ω and $\mho$. This is given by

$$\mathscr{T}_{\sigma,\mathrm{alg}} \cong \frac{\mathrm{O}(2,20-\rho)}{\mathrm{O}(2)\times \mathrm{O}(20-\rho)} \times \frac{\mathrm{O}(2,\rho)}{\mathrm{O}(2)\times\mathrm{O}(\rho)}, \tag{73}$$

where the first factor is the moduli space of complex structures and the second factor is moduli space of the Kähler form + B-field.

Given an algebraic K3 surface X with quantum Picard lattice $\Upsilon(X)$ we have a mirror K3 surface, Y, with quantum Picard lattice $\Upsilon(Y)$ such that $\Upsilon(Y)$ is the orthogonal complement of $\Upsilon(X) \subset \Gamma_{4,20}$. Translating this back into classical ideas, $\mathrm{Pic}(Y) \oplus U$ will be the orthogonal complement of $\mathrm{Pic}(X) \subset \Gamma_{3,19}$. If X is such that the orthogonal complement of $\mathrm{Pic}(X) \subset \Gamma_{3,19}$ has no U sublattice then X has no classical mirror.

[9] The notation $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ denotes the real vector space generated by the generators of Λ.

Such mirror pairs of K3 surfaces were first noticed some time ago by Arnold (see for example, [**59**]). The non-linear σ-model moduli space gives a nice setting for this pairing to appear.

Now we also have the mirror construction of Greene and Plesser [**60**] which takes an algebraic variety X as a hypersurface in a weighted projective space and produces a "mirror" Y as an quotient of X by some discrete group such that X and Y as target spaces produce completely identical conformal field theories.

An interesting case is when X is the hypersurface

$$f = x_0^2 + x_1^3 + x_2^7 + x_3^{42} = 0 \tag{74}$$

in $\mathbb{P}^3_{\{21,14,6,1\}}$. X is a K3 surface which happens to be an orbifold due to the quotient singularities in the weighted projective space. Blowing this up provides a K3 surface with Picard lattice isomorphic to the even self-dual lattice $\Gamma_{1,9}$. Thus in this case $\Lambda \cong \Upsilon \cong \Gamma_{2,10}$. What is interesting is that in this case, the Greene-Plesser construction says that the group by which X should be divided to obtain the mirror is trivial. This mirror map does not act trivially on the marginal operators however but one may show actually exchanges the factors in (73). This is little delicate and we refer the reader to [**53**] for details. Anyway, what this shows is that the Greene-Plesser mirror map, which is an honest symmetry in the sense that it produces an identical conformal field theory, really is given by the exchange, μ, in (72).

This implicitly provides us with another element of G_σ, namely an element of $\mathrm{O}(\Gamma_{4,20})$ which exchanges two orthogonal $\Gamma_{2,10}$ sublattices. It is now possible to show that this new element, together with the subgroup given in (68), is enough to generate $\mathrm{O}(\Gamma_{4,20})$. Thus G_σ at least contains $\mathrm{O}(\Gamma_{4,20})$.

To prove the required result that $G_\sigma \cong \mathrm{O}(\Gamma_{4,20})$ we use the result of [**25**] which states that any further generator will destroy the Hausdorff nature of the moduli space contrary to our assumption.

We thus have

THEOREM 6. *Given theorem 5 and the assumption that the moduli space is Hausdorff, we have that the moduli space of conformally invariant non-linear σ-models on a K3 surface is*

$$\mathscr{M}_\sigma = \mathrm{O}(\Gamma_{4,20})\backslash\,\mathrm{O}(4,20)/(\mathrm{O}(4)\times\mathrm{O}(20)) \tag{75}$$

Lastly we should discuss the meaning of $\Gamma_{4,20}$. The lattice $\Gamma_{3,19}$ was associated to the lattice $H^2(X,\mathbb{Z})$. Extend the notion of our inner product (19) to that of any p-cycle (or equivalently p-form) by saying that the product of two cycles is zero unless their dimensions add up to 4. Now one can see that the lattice of total cohomology, $H^*(X,\mathbb{Z})$, is isomorphic to $\Gamma_{4,20}$. This gives a tempting interpretation. We will see later that this really is the right one.

This tells us how to view w. A point in the moduli space $\mathscr{M}_\sigma$ determines a conformal field theory uniquely but we are required to make a choice of w before we can determine the geometry of the K3 surface in terms of metrics and B-fields. This amounts to deciding which direction in $\Gamma_{4,20}$ will be $H^0(X,\mathbb{Z})$. This choice is arbitrary and different choices will lead to potentially very different looking K3 surfaces which give the same conformal field theory. This may be viewed as a kind of T-duality.

3.5. Conformal field theory on a torus. Although our main interest in these lectures are K3 surfaces, it turns out that the idea of compactification of strings on a torus will be intimately related. We are thus required to be familiar with the situation when the σ-model has a torus as a target space.

The problem of the moduli space in this case was solved in [**61, 62**]. This subject is also covered by H. Ooguri's lectures in this volume and we refer the reader there for further information (see also [**63**]). Here we will quickly review the result in a language appropriate for the context in which we wish to use it.

To allow for the heterotic string case, we are going to allow for the possibility that the left-moving sector of the σ-model may live on a different target space to that of the right-moving sector. We thus consider the left sector to have a torus of n_L real dimensions as its target space and let the right-moving sector live on a n_R-torus. Of course, for the simple picture of a string propagating on a torus, we require $n_L = n_R$.

The moduli space of conformal field theories is then given by

$$\mathrm{O}(\Gamma_{n_L,n_R})\backslash \mathrm{O}(n_L,n_R)/(\mathrm{O}(n_L)\times \mathrm{O}(n_R)). \tag{76}$$

The even self-dual condition, required for world-sheet modular invariance [**61**], on the lattice Γ_{n_L,n_R} enforces $n_L - n_R \in 8\mathbb{Z}$. Note that whether we impose any supersymmetry requirements or not makes little difference. The only Ricci-flat metric on a torus is the flat one, which guarantees conformal invariance to all orders. The trivial holonomy then means that a torus may be regarded as Kähler, hyperkähler or whatever so long as the dimensions are right.

Let us assume without loss of generality that $n_L \leq n_R$.[10] Our aim here is to interpret the moduli space (76) in terms of the Grassmannian of space-like n_L-planes in $\mathbb{R}^{n_L,n_R}$. Let us use Π to denote a space-like n_L-plane. Any vector in $\mathbb{R}^{n_L,n_R}$ may be written as a sum of two vectors, $p_L + p_R$, where

$$\begin{aligned} p_L &\in \Pi \\ p_R &\in \Pi^{\perp}, \end{aligned} \tag{77}$$

and $\Pi^{\perp}$ is the orthogonal complement of Π. Thus $p_L.p_L \geq 0$ and $p_R.p_R \leq 0$.

The winding and momenta modes of the string on the torus are now given by points in the lattice Γ_{n_L,n_R} according to Narain's construction [**61**]. The left and right conformal weights of such states are then given by $\frac{1}{2}p_L.p_L$ and $-\frac{1}{2}p_R.p_R$, thus allowing the mass to be determined in the usual way.

It is easy to see how the moduli space (76) arises from this point of view. The position of Π determines p_L and p_R for each mode. The modular group $\mathrm{O}(\Gamma_{n_L,n_R})$ merely rearranges the winding and momenta modes.

Let us try to make contact with the σ-model description of the string. We choose a null (i.e., light-like) n_L-plane, W, in $\Gamma_{n_L,n_R}\otimes_{\mathbb{Z}}\mathbb{R}\cong\mathbb{R}^{n_L,n_R}$ which is "aligned" along the lattice Γ_{n_L,n_R}. By this we mean that W is spanned by a subset of the generators of Γ_{n_L,n_R}. We then define W^* as the null n_L-plane dual to W,

[10] This means that the heterotic string will be chosen to be a superstring in the left-moving sector and a bosonic string in the right-moving sector. Unfortunately this differs from the usual convention. This is imposed on us, however, when we consider duality to a type IIA string and the natural orientation of a K3 surface.

and the time-like space $V \cong \mathbb{R}^{n_R - n_L}$ so that

$$\Gamma_{n_L,n_R} \otimes_{\mathbf{Z}} \mathbb{R} \cong W \oplus W^* \oplus V. \tag{78}$$

This decomposition is also aligned with the lattice so that $\Gamma_{n_L,n_R} \cap W$, $\Gamma_{n_L,n_R} \cap W^*$, and $\Gamma_{n_L,n_R} \cap V$ generate Γ_{n_L,n_R}.

We may write Π as given by

$$\Pi = \{x + \psi(x) + A(x);\ \forall x \in W\}, \tag{79}$$

where

$$\begin{aligned} \psi &: W \to W^* \\ A &: W \to V \end{aligned} \tag{80}$$

Viewing ψ as

$$\psi : W \times W \to \mathbb{R}, \tag{81}$$

we may divide ψ into symmetric, G, and anti-symmetric parts, B, such that $\psi = B + G$.

Physically W represents, under the metric G, the target space in which the string lives. To be precise, the target space n_L-torus is

$$\frac{W}{\Gamma_{n_L,n_R} \cap W}. \tag{82}$$

W^* is then the dual space in which the momenta live. Clearly B is the B-field. Lastly V represents the gauge group generated by the extra right-moving directions. To be precise, the group is given by

$$U(1)^{n_R - n_L} \cong \frac{V}{\Gamma_{n_L,n_R} \cap V}. \tag{83}$$

We will discuss in section 4.3 how this group can be enhanced to a nonabelian group for particular Π's. The quantity A then represents $\mathfrak{u}(1)$-valued 1-forms. We will follow convention and refer to these degrees of freedom within A as "Wilson Lines".

One may write G, B and A in terms of matrices to make contact with the expressions in [**62, 63**]. This is a somewhat cumbersome process but the reader should check it if they are unsure of this construction. Here we illustrate the procedure in the simplest case, namely that of a circle as target space. Here we have $n_L = n_R = 1$ and $\Gamma_{1,1} \cong U$ as in (25). An element of $\mathrm{O}(1,1)$ preserving the form U may be written in one of the forms

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} -t & 0 \\ 0 & -t^{-1} \end{pmatrix}, \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -t \\ -t^{-1} & 0 \end{pmatrix}, \tag{84}$$

where t is real and positive.

We divide this space by $\mathrm{O}(1) \times \mathrm{O}(1)$ from the right and by $\mathrm{O}(\Gamma_{1,1})$ from the left. Both these groups are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and are given by (84) with $t = 1$. The result is that the moduli space is given by the real positive line mod $\mathbb{Z}_2$ represented by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cong \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \tag{85}$$

where t is real and positive.

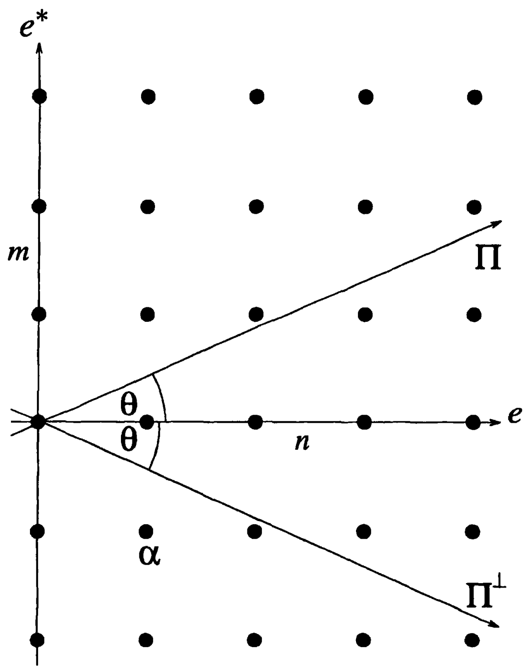

FIGURE 3. The moduli space picture for a string on a circle.

Let us chose a basis, $\{e, e^*\}$, for $\Gamma_{1,1}$ so that $e.e = e^*.e^* = 0$ and $e.e^* = 1$. Now let the space-like direction, Π, be given, as in figure 3, by

$$\Pi = \{xe + xe^* \tan\theta;\ x \in \mathbb{R}\}, \tag{86}$$

where $t = \tan\theta$. (Note that Π and $\Pi^\perp$ may not look particularly orthogonal in figure 3 but don't forget we are using the hyperbolic metric given by U!) Denoting a state in $\Gamma_{1,1}$ by $ne + me^*$, for $m, n \in \mathbb{Z}$, one can show

$$\begin{aligned} p_L.p_L &= \tfrac{1}{2}n^2 \tan\theta + nm + \frac{m^2}{2\tan\theta} \\ p_R.p_R &= -\tfrac{1}{2}n^2 \tan\theta + nm - \frac{m^2}{2\tan\theta}. \end{aligned} \tag{87}$$

Since G is given by $\tan\theta$, the radius of the circle is proportional to $\tan\theta$. The identification in (85) then gives the familiar "$R \leftrightarrow 1/R$" T-duality relation.

4. Type II String Theory

Now we have the required knowledge from classical geometry and "old" string theory on the world-sheet to tackle full string theory on a K3 surface. We begin with the most supersymmetry to make life simple. This is the type IIA or IIB superstring which has 2 supersymmetries when viewed as a ten-dimensional field theory. The heterotic string on a K3 surface constitutes a much more difficult problem and we won't be ready to tackle it until section 6.

d	N	Rep
11	1	$\mathbb{R}^{32}$
10	2	$\mathbb{R}^{16}$
9	2	$\mathbb{R}^{16}$
8	2	$\mathbb{C}^{8}$
7	2	$\mathbb{H}^{8}$
6	4	$\mathbb{H}^{4}$
5	4	$\mathbb{H}^{4}$
4	8	$\mathbb{C}^{2}$
3	16	$\mathbb{R}^{2}$

TABLE 2. Maximum numbers of supersymmetries.

4.1. Target space supergravity and compactification. We begin by switching our attention from the quantum field theory that lives in the world-sheet to the effective quantum field theory that lives in the target space in the limit that $\alpha'/R^2 \to 0$. Of particular interest will be theories with N supersymmetries in d dimensions (one of which is time).

A spinor is an irreducible representation of the algebra $\mathfrak{so}(1, d-1)$ and has dimension $2^{[\frac{d+1}{2}]-1}$, where the bracket means round down to the nearest integer. This is not the only information about the representation we require however. A spinor may be real ($\mathbb{R}$), complex ($\mathbb{C}$), or quaternionic[11] ($\mathbb{H}$) depending on d. The rule is

$$\begin{array}{l} \mathbb{R} \text{ if } d = 1,2,3 \pmod 8 \\ \mathbb{C} \text{ if } d = 0 \pmod 4 \\ \mathbb{H} \text{ if } d = 5,6,7 \pmod 8). \end{array}$$

A complex representation has twice as many degrees of freedom as a real representation. A quaternionic representation has the same number of degrees of freedom as a complex representation (as, even though one may define the representation naturally over the quaternions, one must subject it to constraints [**29**]). In terms of the language often used in the physics literature: in even numbers of dimensions we use Weyl spinors; real spinors are "Majorana" and quaternionic spinors are "symplectic Majorana".[12]

We may now list the maximal number of supersymmetries in each dimension subject to the constraint that no particle with spin > 2 appears. It was determined in [**64**] that $N = 1$ in $d = 11$ was maximal in this regard. To get the other values we simply maintain the number of degrees of freedom of the spinors. For example, in reducing to ten dimensions we go from 32 real degrees of freedom per spinor to 16 real degrees of freedom per spinor. We thus need two spinors in ten dimensions. The result is shown in table 2.

When $d \in 4\mathbb{Z} + 2$, the left and right spinors in the field theory are independent and the supersymmetries may be separated. As we did in section 3.1, we will denote a theory with N_L left supersymmetries and N_R right supersymmetries as having

[11] Sometimes the terminology "pseudo-real" is used.

[12] Note that two symplectic Majorana spinors make up one quaternionic spinor.

$N = (N_L, N_R)$ supersymmetry. For purposes of counting total supersymmetries, $N = N_L + N_R$.

Now we want to see what happens when we compactify such a theory down to a lower number of dimensions. That is, we replace the space $\mathbb{R}^{1,d_0-1}$ by $\mathbb{R}^{1,d_1-1} \times X$, for some compact manifold X of dimension $d_0 - d_1$. To see what happens to the supersymmetries we need to consider how a spinor of $\mathfrak{so}(1, d_0 - 1)$ decomposes under the maximal subalgebra $\mathfrak{so}(1, d_0 - 1) \supset \mathfrak{so}(1, d_1 - 1) \oplus \mathfrak{so}(d_0 - d_1)$. The holonomy of X will act upon representations of $\mathfrak{so}(d_0 - d_1)$. Any representation of $\mathfrak{so}(1, d_1 - 1) \oplus \mathfrak{so}(d_0 - d_1)$ which is invariant under this action will lead to representations of $\mathfrak{so}(1, d_1 - 1)$ in our new compactified target space. This tells us how to count supersymmetries.

As the first example, consider toroidal compactification. In this case the torus is flat and so the holonomy is trivial. Thus every representation survives. Compactifying $N = 1$ supergravity in eleven dimensions down to d dimensions results in reproducing the maximal supersymmetries in table 2. Note that for $d = 10$ the supersymmetry is $N = (1, 1)$ and for $d = 6$ the supersymmetry is $N = (2, 2)$. One simple way of deducing this immediately is from the general result [**65**] that compactification of eleven dimensional supergravity on any manifold results in a non-chiral theory.[13]

Naturally our next example will be compactification on a K3 surface. In this case the holonomy is SU(2). Let us consider the case of an $N = 1$ theory in ten dimensions compactified down to six dimensions on a smooth K3 surface. The required decomposition is

$$\mathfrak{so}(1, 9) \supset \mathfrak{so}(1, 5) \oplus \mathfrak{so}(4) \cong \mathfrak{so}(1, 5) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{88}$$

The spinor decomposes accordingly as $\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2})$. We may take the last $\mathfrak{su}(2)$ in (88) as the holonomy. Thus the $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ part is preserved. It may look like we have 2 spinors in 6 dimensions at this point but remember that our spinor in ten dimensions was real and spinors in six dimensions are quaternionic. Thus the degrees of freedom give only a single spinor in six dimensions. That is, $N = 1$ supergravity in ten dimensions compactified on a smooth K3 surface will give an $N = 1$ theory in six dimensions. The general rule is that compactification on a smooth K3 surface will preserve half as many supersymmetries as compactifying on a torus.

In the case that the number the supersymmetries or the number of dimensions is large, the form of the moduli space of possible supergravity theories becomes quite constrained. Holonomy is again the agent responsible for this. Let us write the notion of extended supersymmetry very roughly in the form

$$\{\overline{Q}^i, Q^j\} = \delta^{ij} P, \tag{89}$$

where Q are the supersymmetry generators and P is translation (we may ignore any central charge for purposes of our argument). Now such a relationship must clearly be preserved as we go around a loop in the moduli space. However, the supersymmetries may transform among themselves as we do this. This gives us a restriction on the holonomy of the bundle of supersymmetries over the moduli

[13]Fortunately recent progress in M-theory appears to tell us that compactification on spaces which are not manifolds can circumvent this statement (see, for example, [**66**]).

space. Comparing (89) to the analysis of invariant forms in section 2.2 tells us immediately what this restriction is. In the case that the spinor is of type $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the holonomy algebra will be (or contain) $\mathfrak{so}(N)$, $\mathfrak{u}(N)$, or $\mathfrak{sp}(N)$ respectively. Note that when we have chiral spinors in $4\mathbb{Z}+2$ dimensions we may factor the holonomy into separate left and right parts since these sectors will not mix.

Now the tangent directions in the moduli space are given by massless scalar fields which lie in supermultiplets. These multiplets have a definite transformation property under the holonomy group above. For a thorough account of this process we refer to [**67**]. This relates the holonomy of the bundle of supersymmetries to the holonomy of the tangent bundle of the moduli space. This knowledge can tell us a great deal about the form of the moduli space.

As an example consider $N = 4$ supergravity in five dimensions from table 2. We see immediately that the holonomy algebra of the moduli space is $\mathfrak{sp}(4)$. An analysis of the representation theory of the supergravity multiplet shows that the massless scalars transforms in a **42** of $\mathfrak{sp}(4)$. The only possibility from Berger and Simons result, and Cartan's classification of noncompact symmetric spaces, is that the moduli space is locally $E_{6(6)}/\,\mathrm{Sp}(4)_{\sim}$, where the tilde subscript denotes a quotient by the central $\mathbb{Z}_2$. The moduli spaces for all the entries in table 2 are given in [**68**].

4.2. The IIA string. The type IIA superstring in ten dimensions yields, in the low-energy limit, a theory of ten-dimensional supergravity with $N = (1,1)$ supersymmetry. If we compactify this theory on a K3 surface then each of these supersymmetries gives a supersymmetry in six dimensions and so the result is an $N = (1,1)$ theory in six dimensions.

The local holonomy algebra from above must therefore be $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4)$. There are two types of supermultiplet in six dimensions which contain moduli fields:

1. The supergravity multiplet contains the dilaton which is a real scalar.
2. Matter multiplets each contain 4 real massless scalars which transform as a **4** of $\mathfrak{so}(4)$.

Thus the moduli space must factorize (at least locally) into a product of a real line for the dilaton times the space parametrized by the moduli coming from the matter multiplets. Thus, given the assumption concerning completeness, the Teichmüller space is of the form

$$\frac{\mathrm{O}(4,m)}{\mathrm{O}(4)\times\mathrm{O}(m)} \times \mathbb{R}, \tag{90}$$

where m is the number of matter multiplets.

As well as the metric, B-field and dilaton, the type II string also contains "Ramond-Ramond" states (see, for example, [**39**]). For the type IIA string these may be regarded as a 1-form and a 3-form.[14] A p-form field can produce massless fields upon compactification by integrating it over nontrivial p-cycles in the compact manifold. As a K3 surface has no odd cycles, no moduli come from the R-R sector. Thus, comparing (90) to (75) we see that $m = 20$.

[14]In a sense all odd forms may be included [**69**].

The moduli space of conformal field theories may be considered as living at the boundary of the moduli space of string theories in the limit that $\Phi \to -\infty$, i.e., weak string-coupling. Thus $\mathrm{O}(\Gamma_{4,20})$ acts on this boundary. Given the factorization of the holonomy between the dilaton and the matter fields (as they transform in different representations of $\mathfrak{so}(4)$) the action of $\mathrm{O}(\Gamma_{4,20})$ also acts away from the boundary in a trivial way on the dilaton. It is believed that moduli space is given simply by

$$\mathscr{M}_{\mathrm{IIA}} \cong \mathscr{M}_{\sigma} \times \mathbb{R}. \tag{91}$$

That is, there are no identifications which incorporate the dilaton. Certainly any duality which mixed the factors would not respect the holonomy. Thus the only possibility remaining would be an action of the form $\Phi \to -\Phi$ which would be a strong-weak coupling duality (acting trivially on all other moduli). Instead of such an S-duality, a far more curious type of duality was suggested in [**70, 71**] as we now discuss.

The form of the moduli space of toroidal conformal field theories (76) in section 3.5 bares an uncanny resemblance to that of the moduli space of K3 conformal field theories, $\mathscr{M}_{\sigma}$. Indeed they are identical if $n_L = 4$ and $n_R = 20$. This is precisely the moduli space of conformal field theories associated to a heterotic string compactified on a 4-torus. Recall that the heterotic string consists of a superstring in the left-moving sector and a bosonic string in the right-moving sector. As such there are 16 extra dimensions in the right-moving sector which are compactified on a 16-torus and contribute towards the gauge group. In ten dimensions the heterotic string is an $N = (1,0)$ theory and thus yields an $N = (1,1)$ theory in six dimensions when compactified on a 4-torus.

The suggestion, which at first appears outrageous, is that the type IIA string compactified on a K3 surface is the same thing, physically, as the heterotic string compactified on a 4-torus. Although the moduli space of conformal field theories is identical for these models, the world-sheet formulation is so different that any notion that the two conformal field theories could be shown to be equivalent is doomed from the start. The claim however is not that the conformal field theories are equivalent, but that the full string theories are equivalent. The heterotic string has a dilaton, just like the type IIA string, and so the moduli space of heterotic string theories is also $\mathscr{M}_{\sigma} \times \mathbb{R}$. The only way we can map these two moduli spaces into each other without identifying the conformal field theories is thus to map Φ from one theory to $-\Phi$ from the other.

This then, is the alternative to S-duality for the type IIA string.

PROPOSITION 1. *The type IIA string compactified on a K3 surface is equivalent to the heterotic string compactified on a 4-torus. The moduli spaces are mapped to each other in the obvious way except that the strongly-coupled type IIA string maps to the weakly-coupled heterotic string and vica versa.*

It would be nice at this point if we could prove proposition 1. Unfortunately it appears that string theory is simply not sufficiently defined to allow this. We should check first that this proposition is consistent with what we do know about string theory.

The fact that the moduli spaces of the type IIA string on a K3 surface and the heterotic string on a 4-torus are identical is a good start. Next we may check that the effective six-dimensional field theory given by each is the same. This analysis was done in [**71**]. The result is affirmative but there is some new information. To achieve complete agreement the flat six-dimensional spaces given by the two string theories are not identical but instead the metrics are scaled with respect to each other. That is, let g_6 denote the six-dimensional space-time metric and Φ the dilaton, then

$$\begin{aligned}\Phi_{\text{Het}} &= -\Phi_{\text{IIA}} \\ g_{6,\text{Het}} &= e^{2\Phi_{\text{Het}}} g_{6,\text{IIA}}.\end{aligned} \tag{92}$$

Other checks may be performed too. Since the strings are related by a strong-weak coupling relations, the fundamental particles in one theory should map to solitons of the other theory. This has been analyzed to a great extent but we refer to J. Harvey's lectures for an account of this.

To date nothing has been discovered in known string theory to disprove proposition 1. Assuming there are no such inconsistencies one may wish to boldly assert that proposition 1 is true *by definition*. That is, whatever string theory may turn out to be, we will demand that it satisfies proposition 1. This is the point of view we will take from now on.

4.3. Enhanced gauge symmetries. One of the interesting questions we are now fully equipped to address is that of the gauge symmetry group of the six-dimensional theory resulting from either a compactification of a type IIA string on a K3 surface, or a heterotic string on a 4-torus.

Firstly we may consider this question from the point of view of conformal field theory. The type IIA string produces gauge fields from the R-R sector. The 1-form gives a U(1). The 3-form may be compactified down to a 1-form over 2-cycles in $b_2(S) = 22$ ways. Lastly, writing the 3-form as C_3, we have a dual field C_3^* given by

$$dC_3^* = *dC_3, \tag{93}$$

where C_3^* is a one form and also gives a U(1). Thus, all told, we have a gauge group of $\mathrm{U}(1)^{24}$. Note that it is not possible to obtain a nonabelian gauge theory, as far as the conformal field theory is concerned, because the R-R fields are so reluctant to couple to any other fields (see, for example, [**72**]).

Now consider the heterotic string picture. As explained in section 3.5, generically we obtain a $\mathrm{U}(1)^{16}$ contribution to the gauge group from the extra 16 right-moving degrees of freedom for the heterotic string. We also obtain 4 "Kaluza-Klein" U(1) factors from the metric from the 4 isometries of the torus. Lastly, the B-field is a 2-form and so contributes $b_1(T^4) = 4$ more U(1)'s. All told we have $\mathrm{U}(1)^{24}$ again as befits proposition 1.

Things become more interesting however when we realize that the heterotic string can exhibit larger gauge groups at particular points in the moduli space. What happens is that some winding/momentum modes of $\Gamma_{4,20}$ may happen to give physical massless vectors for special values of the moduli. Such states will be charged with respect to the generic $\mathrm{U}(1)^{24}$ and so a nonabelian group results.

The heterotic string is subject to a GSO projection to yield a supersymmetric field theory and this effectively prevents any new left-moving physical states from becoming massless. This may be thought of as a similar statement to the assertion that the type IIA string could never yield extra massless vectors. The right-moving sector of the heterotic string is not subject to such constraints however, and we may use our knowledge from section 3.5 to determine exactly when this gauge group enhancement occurs.

The vertex operator for one of our new massless states will be as follows. For the left-moving part we want simply ∂X^μ to give a vector index. For the right-moving part we require another operator with conformal weight one. This results in a requirement that we desire a state with $p_L = 0$ and $p_R.p_R = -2$. Thus we require an element $\alpha \in \Gamma_{4,20}$ which is orthogonal to Π and is such that $\alpha.\alpha = -2$.

The charge of such a state, α, with respect to the $\mathrm{U}(1)^{24}$ gauge group is simply given by the coordinates of α. This follows from the conformal field theory of free bosons and we refer to [41] for details. Comparing this to the standard way of building Lie algebras, we see that α looks like a root of a Lie algebra, whose Cartan subalgebra is $\mathfrak{u}(1)^{\oplus 24}$ (except that the Killing form is negative definite rather than positive definite).

In the simplest case, there will be only a single pair $\pm\alpha$ which satisfy the required condition. This then will add two generators to the gauge group charged with respect to one of the 24 U(1)'s. Thus the gauge group will be enhanced to $\mathrm{SU}(2) \times \mathrm{U}(1)^{23}$. Note that this enhancement to an SU(2) gauge group can also be seen in the simple case of a string on a circle as was pictured in figured 3. If $\theta = 45°$ then the lattice element marked α in the figure (together with $-\alpha$) will generate SU(2).

The general rule then should be that the set of all vectors given by

$$\mathscr{A} = \{\alpha \in \Gamma_{4,20} \cap \Pi^\perp;\ \alpha.\alpha = -2\}, \tag{94}$$

will form the *roots* of the nonabelian part of the gauge group. Note that the rank of the gauge group always remains 24. Note also that the roots all have the same length, i.e., the gauge group is always *simply-laced* and falls into the *A-D-E* classification.

This allows us to build large gauge groups. One thing one might do, for example, is to split $\Gamma_{4,20} \cong \Gamma_{4,4} \oplus \Gamma_8 \oplus \Gamma_8$, where Γ_8 is the Cartan matrix of E_8 (with a negative-definite signature). We may then consider the case where $\Pi \subset \Gamma_{4,4} \otimes_{\mathbb{Z}} \mathbb{R}$. This would mean that any element in $\Gamma_8 \oplus \Gamma_8$ is orthogonal to Π and thus the gauge group is at least $E_8 \times E_8$. Looking at (79) we see that this is equivalent to putting $A = 0$. Thus we reproduce the simple result that the $E_8 \times E_8$ heterotic string compactified on a torus has gauge group containing $E_8 \times E_8$ if no Wilson lines are switched on.

Proposition 1 now tells us something interesting. Despite the fact that the conformal field theory approach to the type IIA string insisted that it could never have any gauge group other than $\mathrm{U}(1)^{24}$, the dual picture in the heterotic string dictates otherwise. There must be some points in the moduli space of a type IIA string compactified on a K3 surface where the conformal field theory misses part of the story and we really do get an enhanced gauge group.

Since we know exactly where the enhanced gauge groups appear in the moduli space of the heterotic string and we know exactly how to map this to the moduli space of K3 surfaces, we should be able to see exactly when the conformal field theory goes awry.

We will determine just which K3 surfaces give rise to this behaviour for the type IIA string. To do this we are required to choose w as in section 3.3 so that we can find a geometric description. Let us first assume that we may choose w so that

$$\alpha \in \frac{w^\perp}{w}, \quad \forall \alpha \in \mathscr{A}. \tag{95}$$

Let w^* be the same vector as was introduced in equation (66). Any vector in Π can be written as a sum $x + bw^* + cw$, where $b, c \in \mathbb{R}$ and $x \in \Sigma$. Thus, the statement that α is orthogonal to Π implies that α is orthogonal to Σ. Now we use the results of section 2.6 which tell us that this implies that the K3 surface is an orbifold. To be precise, the set $\mathscr{A}$ corresponds to the root diagram of the A-D-E singularity given in table 1.

What we have just shown is a remarkable fact (and was first shown by Witten in [**71**]).

PROPOSITION 2. *If we have a K3 surface with an orbifold singularity then a type IIA string compactified on this surface can exhibit a nonabelian gauge group such that the A-D-E classification of orbifold singularities coincides perfectly with the A-D-E classification of simply-laced Lie groups.*

This proposition rests on proposition 1 and the assumptions that went into building the moduli spaces. As an example, we see that an $\mathrm{SU}(n)$ gauge group corresponds to a singularity locally of the form $\mathbb{C}^2/\mathbb{Z}_n$.

Note that the orbifold singularity is not a sufficient condition for an enhanced gauge symmetry. For Π to be orthogonal to α we also require that B', and hence B, is also perpendicular to α. One way of stating this is to say that *the component of B along the direction dual to α is zero.* Note that the volume of the K3 surface does not matter in this context.

It is probably worth emphasizing here that this statement about the B-field can be important [**73**]. Orbifolds are well-known as "good" target spaces for string theory in that they lead to finite conformal field theories. It might first appear then that we are saying that the conformal field theory picture is breaking down at a point in the moduli space where an enhanced gauge group appears but when the conformal field theory appears to be perfectly reasonable. This is not actually the case. The enhanced gauge symmetry appears when $B = 0$ along the relevant direction. Conformal field theory orbifolds however tend to give the value $B = \frac{1}{2}$. Thus, the point in the moduli space corresponding to the happy conformal field theory orbifold and the point where the enhanced gauge group appears are not the same. There is good reason to believe the conformal field theory at the point where the enhanced gauge group appears is not well-behaved [**74**].

What happens if we relax our condition on w given in (95)? Now the situation is not so clear. Σ need not be perpendicular to any α and so the K3 surface may be smooth. If this is the case, the volume of the K3 surface cannot be arbitrarily large. This is because in the large radius limit, Π is roughly the span of Σ and w, but α is not perpendicular to w. In fact, the volume of the K3 surface must be of order one in units of $(\alpha')^2$. Thus we see, assuming the B-field is tuned to the right value, that an enhanced gauge group arises when the K3 surface has orbifold singularities and is any size, or if the K3 surface is very small and is given just the right (possibly smooth) shape.

Note that to get a very large gauge group we probably require the K3 surface be singular *and* it to be very small. This is necessary, for example, if we want a gauge group $E_8 \times E_8 \times \mathrm{SU}(2)^4 \times \mathrm{U}(1)^4$. Note that this latter group is "maximal" in the sense that the nonabelian part is of rank 20, which is the maximal rank sublattice that can be orthogonal to Π.

We have proven the appearance of nonabelian gauge groups by assuming proposition 1. Instead one might like to attempt some direct justification. This is probably best seen by using Strominger's notion of "wrapping p-branes" [**75**]. The general idea is that the R-R solitons are associated with cycles in the target space and the mass of these states is given by the area (or volume) of these cycles. Thus, as the K3 surface acquires a quotient singularity an S^2 shrinks down and thus a soliton becomes massless. We will not pursue the details of this construction as solitons lie somewhat outside our intended focus of these lectures. We also refer the reader to [**76, 77**] for further discussion.

4.4. The IIB string. The type IIB superstring in ten dimensions yields, in the low-energy limit, a theory of ten-dimensional supergravity with $N = (2,0)$ supersymmetry. If we compactify this theory on a K3 surface then each of these supersymmetries gives a supersymmetry in six dimensions and so the result is an $N = (2,0)$ theory in six dimensions.

The local holonomy algebra from above must therefore be $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$. There is only one type of supermultiplet in six dimensions which contains massless scalars and that is the matter supermultiplet. Each such multiplet contains five scalars transforming as a **5** of $\mathfrak{so}(5)$. Thus, given the assumption concerning completeness, the Teichmüller space is of the form [**78**]

$$\frac{\mathrm{O}(5,m)}{\mathrm{O}(5) \times \mathrm{O}(m)}, \tag{96}$$

where m is the number of matter multiplets.

In addition to the metric, B-field and dilaton, moduli may also arise from the R-R fields. The type IIB string in ten dimensions has a 0-form, a 2-form and a self-dual 4-form. The 0-form gives $b_0(S) = 1$ modulus. The 2-form gives $b_2(S) = 22$ moduli. The 4-form gives $b_4(S) = 1$ modulus. One might also try to take the dual of the 4-form to give another modulus. This would be over-counting the degrees of freedom however, as the 4-form is self-dual. Thus, the number of moduli are given by

Metric	58
B-field	22
Dilaton	1
0-form	1
2-form	22
4-form	1
Total	105

To get the dimension of the Teichmüller space correct we require $m = 21$.

To find the discrete group acting on the Teichmüller space we go through a procedure remarkably similar to that of section 3.4 where we found the moduli space of conformal field theories. Firstly we may consider the Teichmüller space as the Grassmannian of space-like 5-planes in $\mathbb{R}^{5,21}$. Denote the 5-plane by Θ. Then we choose a primitive null element, u, of $\Gamma_{5,21}$. Now we have $u^{\perp}/u \cong \mathbb{R}^{4,20}$. We may define $\Pi' = \Theta \cap u^{\perp}$. Define the vector R' by demanding that Π' and R' span Θ, R' be orthogonal to Π' and $u.R' = 1$. Then project Π' and R' into $u^{\perp}/u$ to obtain the space-like 4-plane, Π, and the vector, R, respectively.

Clearly we now interpret Π in terms of the underlying conformal field theory on the K3 surface. R represents the degrees of freedom coming from the R-R sector. The six-dimensional dilaton may be deduced from $R' - R$ just as the volume of the K3 surface was deduced from $B' - B$ in section 3.4.

It is important to note that this construction provides considerable evidence for our identification of $\mathbb{R}^{4,20}$ with $H^*(S,\mathbb{Z})$ in section 3.4. This is because R, which represents the R-R degrees of freedom, is a vector in $\mathbb{R}^{4,20}$ — the same space in which Π lives. We also saw how these moduli arise from $H^0 \oplus H^2 \oplus H^4 \cong H^*$.

Generating the discrete modular group, G_{IIB} (which, as the reader will have guessed, will turn out to be $\mathrm{O}(\Gamma_{5,21})$) is slightly different to the way we built $\mathrm{O}(\Gamma_{4,20})$ in section 3.4, but we show here that we can reduce it to the same problem. First note that we have $\mathrm{O}(\Gamma_{4,20}) \subset G_{\text{IIB}}$. That is, any symmetry of the conformal field theory will be a symmetry of the string theory (just as any symmetry of the classical geometry is a symmetry of the conformal field theory).

The next ingredient we use will be that of S-duality of the type IIB string in ten dimensions **[70, 71]**. This asserts that there is an $\mathrm{SL}(2,\mathbb{Z})$ symmetry acting on the ten-dimensional dilaton and the axion (i.e., the R-R 0-form). This group is generated by a strong-weak coupling interchange of the form $\Phi_{10,\text{IIB}} \to -\Phi_{10,\text{IIB}}$, and a translation of the axion by one. While one might assert this S-duality statement as a distinct conjecture, it is certainly intimately related to other duality statements. One simple way of "deriving" it is to see that it is almost an inevitable consequence of M-theory **[79, 80]** (see also J. Schwarz's lectures). We will also see in section 5.1 that it follows from proposition 1 and mirror symmetry.

To embed $\mathrm{SL}(2,\mathbb{Z})$ into $\Gamma_{5,21}$ we note that $\mathrm{SL}(2,\mathbb{Z})$ is the group of automorphisms of $\mathbb{Z}^2$, which means that it is the group of isometries of a null lattice of rank two. We may split $\Gamma_{5,21} \cong \Gamma_{3,19} \oplus \Gamma_{2,2}$ and then let the $\mathrm{SL}(2,\mathbb{Z})$ act on a null 2-plane in the $\Gamma_{2,2}$ part. If we identify $\Gamma_{3,19}$ with $H^2(S,\mathbb{Z})$ then we see that the group of classical symmetries of the K3 surface (i.e., $\mathrm{O}(\Gamma_{3,19})$) commutes with S-duality. This is exactly what we desire from the effective target space theory.

Now the shift of the axion is a shift in the $H^0(S,\mathbb{Z})$ direction. Since $\mathrm{O}(\Gamma_{4,20})$ acts transitively on primitive null vectors, we immediately see that any shift by an element of $\Gamma_{4,20}$ is a symmetry of the string theory. This is an analogue of the "integral B-field shift". That is, *shifting any R-R modulus by an element of $H^*(S,\mathbb{Z})$ is a symmetry of the string theory.*

The other generator of $\mathrm{SL}(2,\mathbb{Z})$ may be taken as one which exchanges the two null vectors generating the null 2-plane in $\Gamma_{2,2}$. That is, we swap the axion direction, $H^0(S,\mathbb{Z})$, with u, a null vector outside $\mathbb{R}^{4,20}$. This is the exact analogue of mirror symmetry (which exchanged an element of $H^2(S)$ with $H^0(S)$). We have reduced the problem to one completely analogous to finding the modular group for conformal field theories on a K3 surface. Thus we deduce that we can generate all of $\mathrm{O}(\Gamma_{5,21})$ (as asserted first in **[71]**). Assuming the moduli space is Hausdorff we have

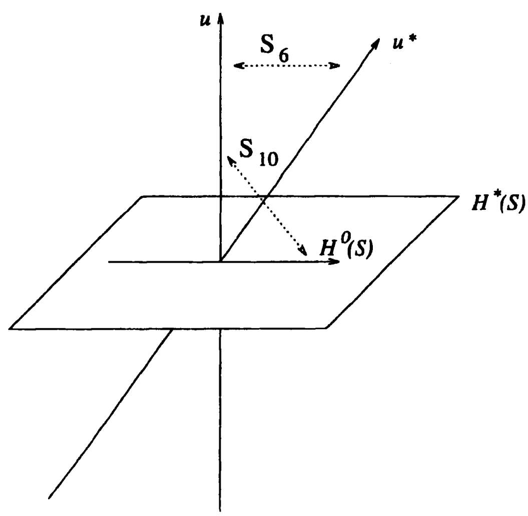

FIGURE 4. The S-dualities of the type IIB string.

PROPOSITION 3. *The moduli space of type IIB string theories compactified on a K3 surface is*

$$\mathcal{M}_{\mathrm{IIB}} = \mathrm{O}(\Gamma_{5,21})\backslash\, \mathrm{O}(5,21)/(\mathrm{O}(5)\times \mathrm{O}(21)). \tag{97}$$

This proposition depends on the S-duality conjecture for the type IIB string in ten dimensions, theorem 6, and the completeness and Hausdorff constraints.

We should mention that there is a strong-weak coupling duality in the resulting six-dimensional theory. Let u^* be a null vector in $\Gamma_{5,21}$ dual to u such that u and u^* span the $\Gamma_{1,1}$ sublattice orthogonal to $H^*(S,\mathbb{Z}) \cong \Gamma_{4,20}$. One of the elements in $\mathrm{O}(\Gamma_{5,21})$ exchanges u and u^* and has the effect of reversing the sign of the six-dimensional dilaton. Note that this is not at all the same as the element of $\mathrm{O}(\Gamma_{5,21})$ which changed the sign of the *ten*-dimensional dilaton as the latter exchanged u with $H^0(S,\mathbb{Z})$ as shown in figure 4. In general an S-duality in a given number of dimensions will not give rise to an S-duality in a lower number of dimensions upon compactification. In the type IIB string however, we see S-duality in both six and ten dimensions.

The behaviour of the type IIA and type IIB string can be contrasted. The strongly-coupled type IIA string on a K3 surface is dual to a different string theory (the heterotic string) which is weakly-coupled. The type IIB strongly-coupled string on a K3 surface is dual to a weakly-coupled version of itself.

Finally in this section let us mention some strange properties of the type IIB string on a K3 surface. We know that when the type IIA string is compactified on an orbifold with B=0 then an enhanced gauge symmetry appears. This indicated some divergence within the underlying conformal field theory. It should be true therefore that a type IIB string compactified on the same space must have some interesting nonperturbative physics since it is associated with the same divergent conformal field theory. It was explained in [**74**] that these new theories are associated with massless string-like solitons which appear at these points in the moduli space.

5. Four-Dimensional Theories

Now let us explore what happens when we compactify string theories down to four dimensions. This process need not involve a K3 surface in general but we will find that in all the easy cases K3 surfaces will be present in abundance!

5.1. $N = 4$ theories. Supersymmetries are not chiral in four dimensions and so, in contrast to the six-dimensional case, there is only one kind of $N = 4$ theory.

The local holonomy algebra must be $\mathfrak{u}(4) \cong \mathfrak{u}(1) \oplus \mathfrak{su}(4) \cong \mathfrak{u}(1) \oplus \mathfrak{so}(6)$. There are two types of $N = 4$ supermultiplet in four dimensions which contain moduli fields:

1. The supergravity multiplet contains the dilaton-axion field which is a complex object under the $\mathfrak{u}(1)$ holonomy.
2. Matter multiplets each contain 6 real massless scalars which transform as a **6** of $\mathfrak{so}(6)$.

Thus the moduli space must factorize into a product of a complex plane, for the dilaton-axion, times the space parametrized by the moduli coming from the matter multiplets. Thus, given the assumption concerning completeness, the Teichmüller space may be written in the form

$$\frac{\mathrm{O}(6,m)}{\mathrm{O}(6)\times\mathrm{O}(m)}\times\frac{\mathrm{SL}(2)}{\mathrm{U}(1)}, \tag{98}$$

where m is the number of matter multiplets.

A very simple way to arrive at this theory is to compactify a heterotic string on a 6-torus. The first factor of (98) is then clearly the moduli space of the conformal field theories on the torus with 6 left-moving dimensions and 22 right-moving dimensions — that is, $m = 22$. The $\mathrm{SL}(2)/\mathrm{U}(1)$ term then comes from the dilaton-axion system: the dilaton being the string dilaton as usual and the axion from dualizing the B-field to obtain a scalar.

The type IIA string may be compactified on $\mathrm{K3}\times T^2$ to obtain an N=4 theory too. The conformal field theory on a K3 has 80 real deformations and for T^2 it has 4 deformations. The 1-form R-R field gives $b_1(\mathrm{K3}\times T^2) = 2$ moduli and the 1-form R-R field gives $b_3(\mathrm{K3}\times T^2) = 44$ moduli. The 3-form may also be compactified down to a 2-form in $b_1(\mathrm{K3}\times T^2) = 2$ ways and then dualized to give 2 more scalars. Adding the dilaton-axion we have $80 + 4 + 2 + 44 + 2 + 2 = 134$. This implies $m = 22$ again. Actually proposition 1 tells us that this must be the same theory as the heterotic string on a 6-torus.

Let us examine the moduli space of conformal field theories on a 2-torus. As far as the Teichmüller space is concerned we have

$$\frac{\mathrm{O}(2,2)}{\mathrm{O}(2)\times\mathrm{O}(2)}\cong\frac{\mathrm{SL}(2)}{\mathrm{U}(1)}\times\frac{\mathrm{SL}(2)}{\mathrm{U}(1)}, \tag{99}$$

up to $\mathbb{Z}_2$ identifications. One of the $\mathrm{SL}(2)/\mathrm{U}(1)$ factors may be regarded as the complex structure of the torus and the other $\mathrm{SL}(2)/\mathrm{U}(1)$ factor represents the Kähler form and B-field on the torus. We refer to [**63**] for a review of this. What does the second factor in (98) represent? There are many ways of approaching this problem. Here we use a trick following [**81**] that will come in use later on.

Begin with a six-dimensional field theory given by the heterotic string compactified on a 4-torus. The effective field theory in six dimensions will be roughly of the form

$$S = \int d^6X\,\sqrt{g_6}e^{-2\Phi_{6,\mathrm{Het}}}(R+\dots) \tag{100}$$

Now compactify over a 2-torus of area A_{Het} (as measured by the heterotic string).

$$\begin{aligned} S &= \int d^4X\,\sqrt{g_4}\,e^{-2\Phi_{6,\mathrm{Het}}}A_{\mathrm{Het}}(R+\dots) \\ &= \int d^4X\,\sqrt{g_4}\,e^{-2\Phi_{4,\mathrm{Het}}}(R+\dots), \\ &= \int d^4X\,\sqrt{g_4}\,A_{\mathrm{IIA}}(R+\dots), \end{aligned} \tag{101}$$

where A_{IIA} is the area of the 2-torus when we consider building the same theory by compactifying the type IIA string on K3 $\times$ T^2. We have made use of (92) to derive this. Looking at the second line of (101) we see that the area of the T^2 is actually playing the rôle of the coupling constant. Thus, in going from the heterotic string description of the situation to the type IIA description of the same situation, the dilaton of the heterotic string has been replaced by the area of the T^2 of the type IIA string. Thus, *the* $\mathrm{SL}(2)/\mathrm{U}(1)$ *factor in (98) must represent the Kähler form and* B*-field of the 2-torus in the type IIA picture.*

We know from conformal field theory that $\mathrm{SL}(2,\mathbb{Z})$ acts on the $\mathrm{SL}(2)/\mathrm{U}(1)$ part of the Teichmüller space giving the Kähler form and B-field of the 2-torus. Combining this knowledge with what we found from the heterotic string, we see that $\mathrm{O}(\Gamma_{6,22}) \times \mathrm{SL}(2,\mathbb{Z})$ acts as the modular group for our $N = 4$ theory.

PROPOSITION 4. *The type IIA string compactified on K3* $\times$ T^2 *is equivalent to the heterotic string compactified on a 6-torus and they form the moduli space*

$$\mathscr{M}_{N=4} = (\mathrm{O}(\Gamma_{6,22})\backslash \mathrm{O}(6,22)/(\mathrm{O}(6)\times \mathrm{O}(22))) \times (\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2)/\mathrm{U}(1)). \tag{102}$$

This rests on the same assumptions as proposition 3.[15]

The $\mathrm{SL}(2,\mathbb{Z})$ factor of the modular group acts as an S-duality in the effective four-dimensional theory. Thus we have *derived*, from proposition 1 the existence of Montonen-Olive S-duality [**83, 84, 85**] for $N = 4$ theories in four dimensions. (Again we are going to neglect to discuss solitons — see [**82**] for such analysis.)

Lastly we may consider the type IIB string compactified on K3 $\times$ T^2. Again there are a multitude of ways of arriving at the desired result. One of the easiest ways is to take following proposition from [**86, 87**]:

[15]Depending on one's tastes, in the case of the moduli space of the heterotic string on a 6-torus one may wish to consider this statement as more fundamental than proposition 1 as it may be analyzed directly in terms of solitons [**82**].

PROPOSITION 5. *The type IIA superstring and type IIB superstring compactified down to nine dimensions on a circle are equivalent except that the radii of the circles are inversely related.*

One also needs to shift the dilaton of one theory relative to the other to achieve the same target space effective field theory for the two string theories.

Thus, since the T^2 of $\mathrm{K3} \times T^2$ contains a circle, the type IIB theory compactified on $\mathrm{K3} \times T^2$ can also be bundled into proposition 4. Note that an $R \leftrightarrow 1/R$ transformation on one of the circles in the T^2 is a mirror map in the sense that the notions of deformation of complex structure and complexified Kähler form are interchanged. Thus, the $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2)/\mathrm{U}(1)$ factor in the moduli space in (102) represents the complex structure moduli space of the 2-torus in the case of the type IIB string.

Thus the $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2)/\mathrm{U}(1)$ factor in the moduli space in (102) can play 3 rôles [**88, 89**]:

1. The dilaton-axion variable in the case of the heterotic string.
2. The area and B-field of the 2-torus in the case of the type IIA string.
3. The complex structure of the 2-torus in the case of the type IIB string.

Comparing the heterotic string to the type IIA string in this setup may be regarded as fairly profound. The dilaton of the heterotic string, i.e., the coupling of the space-time field theory, is mapped to an area in the type IIA theory, i.e., the coupling of the world-sheet field theory. Thus, in a sense *we are mapping the space-time field theory associated to one string theory to the world-sheet field theory associated to another*. One might take this as evidence that neither the target space point of view nor the world-sheet point of view of string theory may be regarded as more fundamental than the other since they may be exchanged.

The $\mathrm{SL}(2,\mathbb{Z})$ S-duality of the type IIB string in ten dimensions is now sitting in the group $\mathrm{O}(\Gamma_{6,22})$. In fact, the above analysis can be used to "prove" the existence of this S-duality group. This can be viewed as an analogue of the deduction of this same S-duality group from M-theory as was done in [**79, 80**].

Note that we can play the same game as in section 4.3 to find the enhanced gauge groups. In this case we have a space-like 6-plane in $\Gamma_{6,22} \otimes_{\mathbb{Z}} \mathbb{R}$ and we look for roots perpendicular to this plane. The gauge group is always of rank 28.

5.2. More $N = 4$ theories. Does the moduli space (102) represent all possible $N = 4$ theories in four dimensions? It seems unlikely as one expects to be able to build theories with a gauge group of rank < 28. Consider compactification of the type II string. All we demand to obtain the desired theory in four dimensions is that the manifold on which we compactify have $\mathrm{SU}(2)$ holonomy. The only complex surface with $\mathrm{SU}(2)$ holonomy is a K3 surface. We have more possibilities in complex dimension three however. Thus we expect that the moduli spaces we discussed in section 4 to give the complete story for $N = 2$ theories in six dimensions but we are not done yet for $N = 4$ theories in four dimensions.

Note that this is a similar statement to the one that Seiberg gave using anomalies [**50**]. When using a conformal field theory to compactify a ten-dimensional theory to six dimensions one may consider the case of a type IIB string compactified to a chiral $N = (2,0)$ six-dimensional theory and analyze the anomalies. The Hodge numbers of a K3 surface are found to be necessary for a consistent theory.

We should add that one can find further theories in six dimensions if one is willing to drop the requirement that the compactification has some conformal field theory description (and switching to something like M-theory instead). See [**90**] for an example.

Let us consider how to build a complex threefold with holonomy SU(2). First we note the existence of a covariantly-constant holomorphic 2-form and thus $h^{2,0} = 1$. The Dolbeault index [**9**] may then be used to establish $h^{1,0} = 1$. Thus our manifold cannot be simply-connected. Now we may use the Cheeger–Gromoll theorem [**91**] which tells us that the universal cover of the manifold is isometric to $M \times \mathbb{R}^n$ for some compact simply-connected manifold, M. It is clear that we require that the universal cover to be $\mathrm{K3} \times \mathbb{R}^2$. In other words, *any complex threefold with holonomy* SU(2) *is isometric to K3* $\times$ T^2 *or some quotient thereof.*

To build more $N = 4$ theories we will consider compactifying type II strings on a quotient of $\mathrm{K3} \times T^2$. This quotient must of course preserve the global SU(2) holonomy and thus any element of the quotienting group must preserve the holomorphic 2-form on the K3 surface. Any such action has fixed points on the K3 surface. Thus, to avoid getting a quotient singularity, any such action on the K3 surface must be accompanied by a translation on the T^2. That is, the quotienting group must have translations in $\mathbb{R}^2$ as a faithful representation. An immediate consequence is that the quotienting group must be abelian.

The classification of such groups, G, has been done by Nikulin [**92**] and we list the results in table 3. M is the rank of the maximal sublattice of $H^2(\mathrm{K3}, \mathbb{Z})$ that transforms nontrivially under G.

This action of G on $\Gamma_{6,22} \cong \Gamma_{4,20} \oplus \Gamma_{2,2}$ is now determined. Firstly, the action on K3 is a geometric symmetry and so must preserve w and w^*. The K3 part of the action is then determined by the action on $H^2(\mathrm{K3}, \mathbb{Z}) \cong \Gamma_{3,19} \subset \Gamma_{4,20}$ For the explicit form of the action on the lattice $H^2(\mathrm{K3}, \mathbb{Z})$ we refer the reader to [**93**]. Lastly we need the action on $\Gamma_{2,2}$. This encodes the action of G on T^2 which we require to be a translation. We also want this action to be geometric and therefore left-right symmetric. This forces the shift to be a *null* direction. This is sufficient to determine the shift up to isomorphism.

Now that we know the action of G on $\Gamma_{6,22}$ we may copy the description of the quotienting procedure over into the heterotic string picture. The result is that we are now describing an *asymmetric orbifold* of a heterotic string on T^6. Such objects were first analyzed in [**94**]. An asymmetric orbifold is an string-theory orbifold in which the left-movers and right-movers of the conformal field theory are not treated identically. Because of this the geometric description of the quotienting process in terms of target space geometry is obscure. Also the chiral nature of quotienting can produce anomalies. One manifestation of this can be lack of modular invariance of the resulting conformal field theory.

Let us consider an asymmetric orbifold of a toroidal theory built on a lattice Λ. Consider an element of the quotienting group, $g \in G$, and represent it as a rotation

G	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	$\mathbb{Z}_7$	$\mathbb{Z}_8$
M	8	12	16	18	12	16	14	18	16	16	18	18

TABLE 3. Nikulin's K3 quotienting groups.

of the lattice followed by a shift, $\delta \in \Lambda \otimes_{\mathbf{Z}} \mathbb{R}$. Let the eigenvalues of the rotation be of the form $\exp(2\pi i.r_j)$. A necessary condition for modular invariance is that [**94, 95**]

$$|g|\left(\tfrac{1}{4}\sum_j r_j(1-r_j)+\tfrac{1}{2}\delta.\delta\right)\in\mathbb{Z};\quad \forall g\in G, \tag{103}$$

where $|g|$ is the order of g. One may check the groups in table 3 and show that this condition is indeed satisfied in every case.

Did we have the right to expect that (103) should be satisfied for all the groups in table 3? To this author the result seems a little mysterious. Indeed, it is the case that if one considers quotients which destroy the $N = 4$ supersymmetry then one need not be so lucky [**96, 97**]. For now though, since we are concerned with $N = 4$ theories at this point, we may content ourselves with the knowledge that the anomalies appear to be looking after themselves and press on.

Now let us determine the moduli space. Firstly any deformation of the original theory of the type II string on K3 $\times\, T^2$ or the heterotic string on T^6 which is invariant under G will be a deformation of the resulting quotient. Secondly we need to worry in string theory that we may introduce some "twisted marginal operators" — that is, massless modes associated with fixed points. Since there are no fixed points of G (at least in the type II picture) we may ignore the latter possibility. To obtain the first type of deformation we may simply restrict attention to the invariant sublattice $\Lambda_G \subset \Gamma_{6,22}$ under (the rotation part of) G. Note that all of the space-like directions of $\Gamma_{6,22}$ are not rotated by G and so the resulting invariant sublattice, Λ_G, will have signature $(6, m)$, where $m = 22 - M$ from table 3.

What we have done is to build a moduli space of the form

$$\mathrm{O}(\Lambda_G)\backslash\,\mathrm{O}(6,m)/(\mathrm{O}(6)\times\mathrm{O}(m)), \tag{104}$$

of an $N = 4$ theory in four dimensions which we viewed either as a type IIA string on a freely-acting quotient of K3$\times T^2$ or an asymmetric orbifold of a heterotic string on T^6.

Are there any further possibilities beyond those listed in table 3? We restricted ourselves to classical symmetries of the K3 surface. There should certainly be more symmetries from the stringy geometry of the K3 surface when it is Planck-sized. One may also look at M-theory to provide more possibilities. We refer the reader to [**93, 98**] for further discussion.

Although (104) looks suspiciously like a Narain moduli space for a heterotic string on a torus, it is important to notice that Λ_G is, in general, not self-dual. Any attempt to describe this theory in a straight forward way as a toroidal compactification is doomed as it would imply that the theory is not modular invariant.

On a similar point we have to be careful when considering which enhanced gauge groups can appear. One can view the moduli space as a Grassmannian of space-like 6-planes in $\mathbb{R}^{6,m}$ but it is no longer the case that only elements, $v \in \Lambda_G$, perpendicular to this 6-plane with $v.v = -2$ will give massless vector fields. The simplest way to approach the question of enhanced gauge groups is as follows (see [**99, 100**] for the original analysis in terms of heterotic strings). Consider the original theory before we divide by G. In this case, we know what the roots of the enhanced gauge group are, given the space-like 6-plane. As G has a nontrivial

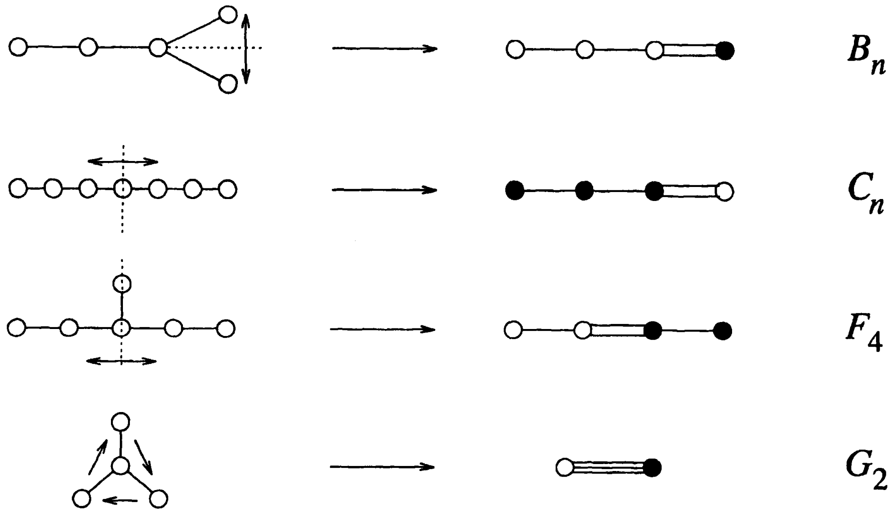

FIGURE 5. Quotients of simply-laced groups.

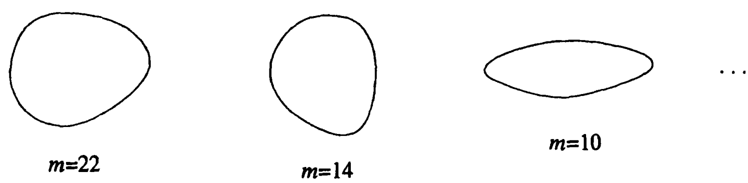

FIGURE 6. The moduli space of $N = 4$ theories.

action on the lattice $\Gamma_{6,22}$, it may also act on the roots of the gauge group. Since our desired theory is the invariant part of the original theory under the quotient by G, the resulting gauge group will be the invariant part of the original gauge group under the action of the discrete group G.

The problem we have therefore is as follows. Given a simply-laced gauge group and an action of a discrete group, G, on the roots of this gauge group, find the subgroup of the gauge group which is invariant under this action. This will be the enhanced gauge group of the desired quotient theory. Fortunately this is a well-known problem in Lie group theory (see, for example, exercise 22.24 in [**29**]). The outer automorphism of the group, given by an action on the roots can be written as a symmetry of the Dynkin diagram in the obvious way.[16] The results are shown in figure 5 and show that non-simply-laced Lie groups can result. In particular one may show that *any* Lie group (of sufficiently small rank) can appear as an enhanced gauge symmetry.

Lastly let us note an important point about the $N = 4$ moduli spaces. This is that they are *disconnected* from each other as shown in figure 6. If components of the moduli space with different values of m were to touch each other then, at such

[16]Except for the case of $\mathfrak{su}(2n+1)$ in which case the outer automorphism yields $\mathfrak{so}(2n+1)$ as the invariant subalgebra.

a point of contact, we would have a theory with very special properties. As one approached such a theory from within the interior of one of the regions, extra states would become massless to furnish the deformations into the other region. This does not happen according to the conformal field description of either the heterotic string or the type II strings. Thus it would appear unreasonable to expect it to occur in the full string theory. This is to be contrasted with the behaviour of $N = 2$ theories in four dimensions, as we discuss in section 5.7.

5.3. Generalities for $N = 2$ theories. Now that we have understood the main features of the moduli space of $N = 4$ theories in four dimensions we are ready to embark on a study of the much richer field of $N = 2$ theories. Much of the recent interest in duality was sparked by Seiberg and Witten's work on $N = 2$ Yang-Mills field theory [**101, 102**]. Here we are hoping to analyze full string theory in the same context. Thus we expect the subject to be at least as rich as Seiberg-Witten theory. In the short period that $N = 2$ theories have been studied in the context of string duality, the subject is already vast and it will be difficult to do justice to it here. As in the rest of these lectures, we will attempt to confine our attention to matters related to the moduli space of theories.

How can we obtain an $N = 2$ theory in four dimensions from string theory? Two answers immediately appear given the usual holonomy argument. Firstly one may take a heterotic string theory in ten dimensions and compactify it on a complex threefold with SU(2) holonomy. We have already discussed such manifolds in section 5.2 and found that they are of the form K3 $\times T^2$, or some free quotient thereof. Secondly one might take a type II string and compactify it on a complex threefold of SU(3) holonomy, i.e., a Calabi–Yau manifold. Given the story for $N = 4$ theories above it is tempting to conjecture that there may be dual pairs of such theories. That is, we wish that a heterotic string when compactified in a specific way on K3 $\times T^2$ be physically equivalent to a type II string compactified on a Calabi–Yau threefold. This story began with the papers of [**103, 96**] and, as we shall see, the full picture is still to be uncovered.

There is one immediately apparent curiosity which is associated with such a conjecture. This is that there are a very large number of topological classes of Calabi–Yau threefolds. The exact number is not known since a classification remains elusive. Indeed, one cannot rule out the possibility that the number is infinite. Contrasted to this are the few manifolds of K3 $\times T^2$ and its quotients. At first it might appear that only a tiny fraction of the type II compactifications can have heterotic partners. This argument is flawed, however, as the heterotic string requires more data to specify its class than just the topology of the space on which it is compactified.

The heterotic string in ten dimensions has a gauge group which is either $E_8 \times E_8$ or Spin(32)$/\mathbb{Z}_2$. Let us consider the $E_8 \times E_8$ string for purposes of discussion. This "primordial" gauge group must be compactified in addition to the extra dimensions. The generally accepted way to do this is to take a vector bundle $E \to X$ over the compactification manifold, X, with a structure group contained in $E_8 \times E_8$. The embedding of this structure group into the heterotic string's gauge group then gives a recipe for compactifying the heterotic string on X including the gauge degrees of freedom.

This is exactly what we were doing in section 3.5. In the case of a heterotic string we considered a rank 16 principal $U(1)^{16}$-bundle over a torus. The structure group was embedded as the Cartan subgroup of $E_8 \times E_8$ and the connection on the bundle was specified by the parameters of the matrix A. The equations of motion demand that A be a constant and so the bundle is flat. The Narain moduli space then gives the full moduli space of such flat vector bundles on a torus.

In the case of compactification over a more general manifold, one way of solving the equations of motion [**104**] is to demand that the vector bundle be *holomorphic* and that the curvature satisfies

$$g^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0. \tag{105}$$

One also requires $c_1(E) \in H^2(X, 2\mathbb{Z})$ and that

$$c_2(E) = c_2(T_X), \tag{106}$$

for anomaly cancelation. Clearly the torus fits into this picture if we replace the principal $U(1)^{16}$-bundle by the associated sum of holomorphic line bundles.

The analysis of this bundle for the case of a heterotic string on a K3 surface is going to be much harder than the toroidal case because now the bundle *cannot* be flat as it must satisfy $c_2(E) = 24$.[17] We can see hope then that the large number of choices of possible Calabi–Yau manifolds for compactification of the type II string might be matched by the large number of choices of suitable bundles over K3 $\times T^2$ for the heterotic string compactification.

It will take a fairly long argument before we are able to give an explicit example of such a pair so we will discuss the situation in general first. Let us start in our usual way by thinking about the holonomy of the moduli space. For an $N = 2$ theory in four dimensions the holonomy algebra is $\mathfrak{u}(2) \cong \mathfrak{u}(1) \oplus \mathfrak{sp}(1)$. There are two types of $N = 2$ supermultiplet which contain massless scalars:

1. The vector multiplets each contain 2 real fields which form a complex object under the $\mathfrak{u}(1)$ holonomy.
2. The hypermultiplets each contain 4 real massless scalars which transform as a quaternionic object under the $\mathfrak{sp}(1)$ holonomy.

Thus, at least away from points where the manifold structure may break down, we expect the moduli space to be in the form of a product

$$\mathcal{M}_{N=2} = \mathcal{M}_V \times \mathcal{M}_H, \tag{107}$$

where $\mathcal{M}_V$ is a Kähler manifold spanned by moduli in vector supermultiplets and $\mathcal{M}_H$ is a quaternionic Kähler manifold spanned by moduli in hypermultiplets. It can be shown [**105**] that $\mathcal{M}_H$ is not hyperkähler.

The effective four-dimensional theory always contains a dilaton-axion system. This will govern the string-coupling. As it plays such an important rôle we will look first at whether the dilaton-axion lives in a hypermultiplet or a vector multiplet. One way to do this is simply to count the dimensions of the moduli space and use the fact that $\mathcal{M}_V$ has an even number of real dimensions and $\mathcal{M}_H$ has a multiple of four dimensions. For a more direct way of justifying which kind of supermultiplet the dilaton-axion lives in see [**97**].

[17] As is common, we assume integration over the base K3 surface in this notation.

Consider first the type IIA superstring on a Calabi–Yau manifold X. First consider the moduli space of underlying conformal field theories (see, for example, [**40**] for a full discussion). We have $h^{1,1}(X)$ complex dimensions of moduli space coming from the deformations of Kähler form and B-field and we have $h^{2,1}(X)$ complex dimensions coming from deformations of complex structure. The R-R sector moduli come from a 3-form giving $b_3(X) = 2(h^{2,1}(X)+1)$ real deformations. Finally we have 2 real deformations given by the dilaton-axion. Given that $h^{1,1}(X)$ and $h^{2,1}(X)$ can be even or odd, the only way to arrange these deformations in a way consistent with the dimensionality of the moduli space is to arrange:

- There are $h^{1,1}(X)$ vector supermultiplets.
- There are $h^{2,1}(X)+1$ hypermultiplets, one of which contains the dilaton-axion.

Next consider the type IIB superstring compactified on a Calabi–Yau manifold, Y. The moduli space of conformal field theories is as for the type IIA string with $h^{1,1}(Y)$ complex deformations of complexified Kähler form and $h^{2,1}$ complex deformations of complex structure. The R-R moduli consist of one from the 0-form, $b_2(Y) = h^{1,1}(Y)$ from the 2-form, $b_4(Y) = h^{1,1}(Y)$ from the self-dual 4-form and one more from dualizing the 2-form. Lastly we have two more moduli from the dilaton-axion. Now we are forced to arrange as follows.

- There are $h^{2,1}(Y)$ vector supermultiplets.
- There are $h^{1,1}(Y)+1$ hypermultiplets, one of which contains the dilaton-axion.

Note that the type IIA picture and the type IIB picture are related by an exchange

$$
\begin{aligned}
h^{1,1}(X) &\leftrightarrow h^{2,1}(Y)\\
h^{2,1}(X) &\leftrightarrow h^{1,1}(Y).
\end{aligned}
\tag{108}
$$

If we have a pair of Calabi–Yau varieties, X and Y, such that type IIA string theory compactified on X is equivalent to type IIB string theory on Y then we may use this as definition of the statement that "X and Y are a mirror pair". See [**106**] for further discussion of this point.

Lastly we consider the heterotic string compactified on a product of a K3 surface and a 2-torus. First let us assume that there are no nasty obstructions in the moduli space and we can count the number of deformations of the K3 surface, the bundle over the K3 surface, the 2-torus, and the bundle over the 2-torus and simply sum the result. There are 80 deformations of the K3 surface as far as conformal field theory is concerned and the resulting moduli space is a quaternionic Kähler manifold. It was shown by Mukai [**107**] that the moduli space of holomorphic vector bundles over the K3 will be hyperkähler. Thus it appears very reasonable to expect that the complete moduli space coming from the K3 surface will be a quaternionic Kähler manifold. Certainly its dimension is a multiple of four assuming unobstructedness. The moduli space coming from the torus will be locally of the form $\mathrm{O}(2,m)/(\mathrm{O}(2)\times\mathrm{O}(m))$ and has an even number of dimensions. Thus we assemble the supermultiplets as follows

- The vector multiplets come from the 2-torus, together with its bundle, and the dilaton-axion.

- The hypermultiplets are associated to the K3 surface, together with its bundle.

Note that we are assuming we can give simple geometric interpretations to all the moduli. We will see later that there may be other vector or hypermultiplet moduli we have not included in the above lists.

5.4. K3 fibrations. Suppose we are able to find a pair of theories, one a type II string on a Calabi–Yau manifold and the other a heterotic string theory compactified over some bundle on K3 $\times T^2$. The first thing one would do would be to line up the moduli spaces of the two theories so that the parameters of one theory could be understood in terms of the other. We begin by analyzing what would happen in the case of the vector multiplet moduli space. The is the first step required before we can actually propose a dual pair of such theories. We now follow an argument first presented in [**108**].

For definiteness we choose a type IIA string rather than a type IIB. The reason for this is that we will ultimately be able to tie our analysis to proposition 1, which was also phrased in terms of the type IIA string.

The holonomy argument leads to a factorization of the moduli space, which in turn has another consequence given the fact that all the matter fields are related to the moduli by supersymmetry. This is that the couplings between fields in the vector multiplets can only depend on the moduli from the vector multiplets and the couplings between fields in the hypermultiplets can only be affected by the moduli from the hypermultiplets. This may also be deduced directly from the supergravity Lagrangian [**109**].

Let us consider just the moduli space, $\mathscr{M}_V$, coming from the scalars in the vector multiplets. We know this is a complex Kähler manifold. Further analysis of the supergravity Lagrangian puts more constraints on the geometry of the moduli space [**110**]. A manifold satisfying these extra conditions is called "special Kähler".[18] The main importance of special Kähler geometry is the fact that all the information we require about the theory is encoded in a single holomorphic function $\mathscr{F}$ on the moduli space. If we use specific complex coordinates, the "special coordinates", on the moduli space, the metric is of the form

$$\begin{aligned} K &= -\log\left(2(\mathscr{F}+\overline{\mathscr{F}}) - (q^i - \bar{q}^{\bar{\imath}})\left(\frac{\partial\mathscr{F}}{\partial q^i} - \frac{\partial\overline{\mathscr{F}}}{\partial\bar{q}^{\bar{\imath}}}\right)\right) \\ g_{i\bar{\jmath}} &= \frac{\partial K}{\partial q^i \partial\bar{q}^{\bar{\jmath}}}. \end{aligned} \tag{109}$$

When viewed from the point of view of the heterotic string, we expect the dilaton-axion to be contained in this moduli space. Let us suppose for the time being that all the other moduli can be understood from the world-sheet perspective of the heterotic string. This should mean that we have a moduli space of conformal field theories spanning all but one of the complex directions in the moduli space with the extra dimension being given by the dilaton-axion system. In the limit that the string coupling becomes very small, i.e., the dilaton approaches $-\infty$, we expect

[18]The quaternionic Kähler manifold parametrized by the hypermultiplets is also subject to extra constraints [**111**].

that the moduli space as described by the conformal field theory becomes exact. Thus, in the limit of small dilaton, the moduli space should factorize into a product of the moduli space of conformal field theories, and the extra bit spanned by the dilaton-axion. In **[112]** precisely this problem was analyzed. It was discovered that the only way a special Kähler manifold could factorize was if it became locally a product of the form

$$\frac{\mathrm{O}(2,m)}{\mathrm{O}(2)\times\mathrm{O}(m)}\times\frac{\mathrm{SL}(2)}{\mathrm{U}(1)}. \tag{110}$$

This of course is excellent news. The first term in (110) looks suspiciously like the Narain moduli space for a 2-torus and the second term looks like a dilaton-axion. This is exactly what we wanted. Note that the form (110) is only expected in the limit that the dilaton approaches $-\infty$. Away from this limit we expect the two factors to begin to interfere with each other.

Now we want to carry this information over to the type IIA compactification on the Calabi–Yau manifold, X. All of the vector multiplet moduli are expected to be associated to deformations of the Kähler form and B-field on X. Let us fix some notation. In contrast to the K3 surface, degrees of freedom of the Kähler form and the B-field for the Calabi–Yau threefold, X, can be nicely paired-up. Introduce a basis of divisors, or 4-cycles, $\{D_k\}$, spanning $H_4(X,\mathbb{Z})$, where $k=0,\ldots,h^{1,1}(X)-1$. Dual to the dual of these divisors we have a basis of 2-forms, $\{e_k\}$, generating $H^2(X,\mathbb{Z})$.[19] Expand out the Kähler form and B-field as

$$B+iJ=\sum_{k=0}^{h^{1,1}-1}(B_k+iJ_k)e_k, \tag{111}$$

for real numbers B_k, J_k. We will take e_0 to correspond to the generator associated to the direction in moduli space given by the heterotic dilaton-axion.

Our information for the heterotic side is in terms of the local form of the moduli space. Thanks to special Kähler geometry we can translate this into information concerning couplings between certain fields. The fields we are interested in are the superpartners of the moduli of the vector superfields — i.e., the gauginos, the vector bosons and the moduli themselves. One may consider couplings in the effective action of the form of "Yukawa couplings", i.e., $\kappa_{ijk}=\langle a_i\psi_j\psi_k\rangle$, or other terms equivalent by supersymmetry. It was shown in **[113]** that, to leading order in the non-linear σ-model, the coupling between three fields is given by

$$\kappa_{ijk}=\#(D_i\cap D_j\cap D_k), \tag{112}$$

where the D's are the divisors in X associated to the fields. The reader is also referred to **[114]** for an account of this.[20] It is also known from special Kähler geometry that

$$\kappa_{ijk}=\frac{\partial\mathscr{F}}{\partial q_i\partial q_j\partial q_k}. \tag{113}$$

We now have some approximate knowledge about both the heterotic string and the type IIA string. In the case of the heterotic string we know that, in the small

[19] For simplicity let us assume all cohomology is torsion-free.

[20] Note that **[114]** explicitly refers to a heterotic string compactified on a Calabi–Yau manifold whereas we are considering a type IIA string. Most of the calculations are unaffected however.

dilaton limit, the moduli space factorizes in the form (110) and in the case of the type IIA string, we know that, in the small α'/R^2 limit, the couplings are of the form (112). We can make a useful statement about a heterotic-type II dual pair if both of these approximations happen to be simultaneously true. Note that, for the heterotic string, the dilaton lies in a vector multiplet and that, in the type IIA string, the size parameters lie in vector multiplets. Thus we wish to assert that the moduli spaces of the theories are aligned in the right way so that as the dilaton in the heterotic string approaches $-\infty$, some size in the Calabi–Yau space on which the type IIA string is compactified is becoming very large.

To picture this presumed aligning of the moduli spaces it is best to picture exactly what makes corrections to the approximations we are considering. In the case of the heterotic string, corrections arise from instantons in the Seiberg-Witten theory [**101**]. The action of such an instanton becomes very large, and hence the contribution to any physical quantity becomes very small, when the dilaton becomes close to $-\infty$. In the case of the non-linear σ-model, corrections come from world-sheet instantons [**115**]. A world-sheet instanton takes the form of a holomorphic map from the world-sheet into the target space. We will be interested only in tree-level effects and so as far as we are considered *world-sheet instantons are rational curves.* The action for such an instanton is simply the area of the curve. Thus the effect of an instanton gets weaker as the Kähler form is varied so as to make the rational curve bigger.

The picture one should have mind therefore is that as the heterotic dilaton is decreased down to $-\infty$, some rational curve (or some set or family of rational curves) is getting bigger. The important thing is that no curve should shrink down during this process.

Let us assume we are now at the edge of our moduli space where, in the type IIA interpretation, all of the rational curves are very large compared to α'. Thus, by assumption, the heterotic string's dilaton is close to $-\infty$. We can take the moduli space given by (110) and deduce the form of $\mathscr{F}$. We may then translate this into a statement about κ_{ijk} and thus about the topology of X from (112). The result is that [**112**]

$$\begin{aligned}
&\#(D_0 \cap D_0 \cap D_0) = 0\\
&\#(D_0 \cap D_0 \cap D_i) = 0, \quad i = 1, \ldots, h^{1,1} - 1 \\
&\#(D_0 \cap D_i \cap D_j) = \eta_{ij}, \quad i, j = 1 \ldots, h^{1,1} - 1,
\end{aligned} \tag{114}$$

where η_{ij} is a matrix of nonzero determinant and signature $(+, -, -, \ldots, -)$.

Suppose that X is such that there is some smooth complex surface embedded in X whose class, as a divisor, is D_0. From (114) we see that $D_0 \cap D_0 = 0$ and so the normal bundle for this surface is trivial. It then follows from the adjunction formula, and the fact that X is a Calabi–Yau space, that the tangent bundle for this surface has trivial c_1. Thus, the surface representing D_0 must either be a K3 surface or an algebraic 4-torus (also known as an "abelian surface"). The fact that the normal bundle is trivial also suggests that the K3 or abelian surface can be "moved" parallel to itself to sweep out the entire space X. That is, X is a fibration where the generic fibre is either a K3 surface or an abelian surface.

We can make the above more rigorous by appealing to a theorem by Oguiso [**116**] which states that

THEOREM 7. *Let X be a minimal Calabi–Yau threefold. Let D be a nef divisor on X. If the numerical D-dimension of D equals one then there is a fibration $\Phi : X \to W$, where W is $\mathbb{P}^1$ and the generic fibre is either a K3 surface or an abelian surface.*

The numerical D-dimension of a divisor is the maximal number of times it may be intersected with itself to produce something nontrivial. We see above that D_0 has D-dimension equal to one. The statement that D is "nef" is the assertion that for any algebraic curve, $C \subset X$, we have that

$$\#(D \cap C) \geq 0, \quad \forall C \subset X. \tag{115}$$

We can come very close to proving that D_0 is nef following our assumption about the way that moduli spaces are aligned. The special coordinates on a our moduli space of type IIA string theories may be written in the form

$$q_k = e^{2\pi i(B_k + iJ_k)}. \tag{116}$$

We are in an area of moduli space where all $q_k \ll 1$. The contribution of a curve, C, to the instanton sum will then scale roughly as

$$\prod_{k=1}^{h^{1,1}} q_k^{\#(D_k \cap C)}. \tag{117}$$

Clearly, if C is not nef then negative powers of q_k appear and the instanton sum will fail to converge, in contradiction to expectation.

The above argument that C is nef is not actually complete. When one computes the instanton sum, one also has to compute the coefficient in front of the monomial of the form (117). This can be done using the methods discussed in [**117, 118**]. In the simple case of an isolated curve, the coefficient is simply one (although extra contributions arise from multiple covers). C may not always be isolated however. One may be able to deform C into a whole family of rational curves. In this case the coefficient might be zero. One cannot then rule out that $\#(D \cap C) < 0$ as the field theory is simply unaffected by C.

An example of a case where rational curves don't count in this way is that of $N = 4$ theories. Compactifying the type IIA string on K3 $\times T^2$, there are no instanton corrections but the K3 surface may contain rational curves. Each rational curve will clearly be in the form of a family $C \times T^2$ inside K3 $\times T^2$. An algebraic surface of the form $\mathbb{P}^1 \times T^2$ is known as an "elliptic scroll".[21] Since the calculation in [**117, 118**] is essentially local we may generalize this result to any $N = 2$ Calabi–Yau compactification. That is we claim that *rational curves in an elliptic scroll do not contribute to the instanton sum.* Note that rational curves in a K3 surface are unstable in the sense that a generic deformation of complex structure of the K3 surface will kill them. The work of Wilson [**119**] showed that essentially the same is true for elliptic scrolls in Calabi–Yau manifolds. Thus, if we choose a generic complex structure on the Calabi–Yau manifold then there will be no curves lying in an elliptic scroll.

If we assume that the only curves contributing zero to the instanton sum lie in an elliptic scroll then we may complete our proof that D_0 is nef by choosing a generic complex structure. We do not know if this assumption is valid but it seems reasonable.

Note that the fibration $\Phi : X \to W$ is allowed to "degenerate" at a finite number of points in W. Indeed, if this were not the case then X could not be a Calabi–Yau variety. Note that the degenerate fibre need not be some degenerate

[21] An "elliptic curve" is the algebraic geometer's name for an algebraic curve of genus 1.

limit of a K3 surface. It could be a perfectly smooth manifold (which is neither a K3 or abelian surface).

Finally we would like to know if the generic fibre is a K3 surface or an abelian surface. We can determine this by finding c_2, and hence the Euler characteristic, of the fibre. This may be determined by using the *holomorphic anomaly* of [**120**]. The result is that the Euler characteristic of the generic fibre is 24 and so the fibration is of the K3 type. We refer the reader to [**108**] for details of this calculation.

We thus arrive at the following:

PROPOSITION 6. *Given a dual pair of theories, one of which is a heterotic string compactified on $K3 \times T^2$ (or some free quotient thereof) and the other is a type IIA string compactified on a Calabi–Yau manifold X, then if there is a region of moduli space in which both the respective perturbation theories converge, then X must be of the form of a K3-fibration over $\mathbb{P}^1$.*

This proposition depends upon our statement about zero contributions from rational curves only arising from elliptic scrolls. The fact that the base of the fibration is a $\mathbb{P}^1$ may be deduced from the fact that H^1 of the base injects[22] into the total cohomology of X and that X has $b_1 = 0$.

Not only do we know that X is a K3 fibration, but we also know that the divisor D_0 is the generic K3 fibre. This tells us immediately which deformation of X corresponds to a deformation of the heterotic dilaton — it is the component of the Kähler form that affects the area of the curve dual to D_0.

Let us first assume that the K3-fibration of X has a global section. This means that as well as the fibration map:

$$\Phi : X \to W, \tag{118}$$

we also have a holomorphic embedding

$$\gamma : W \to X. \tag{119}$$

The image of this embedding is a rational curve in X. Clearly this curve is dual to D_0. Thus the value of the dilaton in the heterotic string is given by the area of this rational curve. The instanton sums converge as the dilaton approaches $-\infty$ and as this rational curve gets very large. Thus the weakly-coupled limit of the heterotic string is given by the limit of X in which the base $\mathbb{P}^1$ swells up to infinite size.

Note that if the K3 fibration does not have a global section, but rather a multi-valued section, we may play a similar game. In this case the multi-section will not define an embedding of the base $\mathbb{P}^1$ into X but rather an embedding of a multiple cover of X. Thus, the algebraic curve within X whose area gives the heterotic dilaton will be of genus greater than zero.

One should note that what we have discussed here for the $N = 2$ dual pairs is actually very similar to the $N = 4$ case discussed above. In that case we had a type IIA string compactified on K3 $\times$ T^2 or a quotient thereof. This space may be viewed as a smooth K3-fibration over T^2 (that is, an elliptic curve). This fibration is trivial in the case that the target space is K3 $\times$ T^2 but becomes nontrivial when we take a free quotient. We saw in section 5.1 that the dilaton for the heterotic string is given by the area of the T^2 — i.e., the size of the base of the fibration.

[22]This follows from the Leray spectral sequence for a fibration.

The general picture we see is the following. If we have a heterotic-type IIA dual pair in four dimensions, we may expect that the type IIA string is compactified on

- a K3-fibration over a 2-torus in the case of $N = 4$ supersymmetry, or,
- a K3-fibration over a $\mathbb{P}^1$ in the case of $N = 2$ supersymmetry.

In either case, the area of the base of the fibration gives the value of the heterotic dilaton.

The link to proposition 1 is clear in the $N = 4$ case. For the $N = 2$ case we should note that the K3 surface on which the heterotic string is compactified might be viewable as an elliptic fibration itself — that is, a fibration over $\mathbb{P}^1$ with generic fibre given by a T^2. Thus the K3 $\times$ T^2 (or quotient thereof) upon which the heterotic string is compactified may be viewed as an "abelian fibration" over $\mathbb{P}^1$ — i.e., a fibration over $\mathbb{P}^1$ with generic fibre given by a T^4. When viewed in this way the relationship between the $N = 2$ theories in four dimensions given by the heterotic string and the type IIA string may be viewed as a *fibre-wise* application of the duality of proposition 1.

Such "fibre-wise duality", which was first suggested in [**97**], is a potentially very powerful tool. It has been extended to fibre-wise mirror symmetry in [**121**] and has recently been applied to the problem of mirror symmetry itself in [**122**]. Both of these developments deserve to be covered here in some detail since they both have direct relevance to K3 surfaces, but we do not have time to do so. We refer the reader to [**123, 124**] for more details of the latter in the context of K3 surfaces.

Let us note that the general appearance of K3-fibrations in the area of heterotic-type II duality was first noted in [**125**] where some naturalness arguments based on the work of [**126**] were presented. It is interesting to note that in this context it was really the type IIB string that being studied rather than the type IIA. This raises the question as to whether the mirror of a K3 fibration is another K3 fibration, which again raises the possibility of some fibre-wise duality argument.

5.5. More enhanced gauge symmetries. So far we have identified one of the moduli lying in a vector multiplet. This is the dilaton-axion in the heterotic string, or the component of the complexified Kähler form associated to the base of the K3-fibration on which the type IIA string is compactified. Now we wish to turn our attention to some of the other vector multiplet moduli.

Let us first look at things from the type IIA perspective. We know that our Calabi–Yau space, X, is a K3-fibration and we wish to analyze elements of $H^2(X)$, or equivalently, $H_4(X)$. In general, given a fibration $X \to W$, the cohomology of X may be determined from the cohomology of the base together with the cohomology of the fibre. The mechanism by which this happens is called a "spectral sequence". We do not wish to get involved with the technicalities of spectral sequences here and refer the reader to [**127**] for the general idea.

The result is that the contributions to $H_4(X)$ are as follows (see figure 7):

1. The generic fibre, D_0, will generate an element of $H_4(X)$.
2. Take an algebraic 2-cycle in the fibre, i.e., an element of $H_2(D_0)$ and use it to "sweep out" a 4-cycle in X by transporting it around the base, W. Note that this 2-cycle needs to be monodromy invariant for this to make sense. Thus the 2-cycle might be an irreducible curve in D_0 which is monodromy

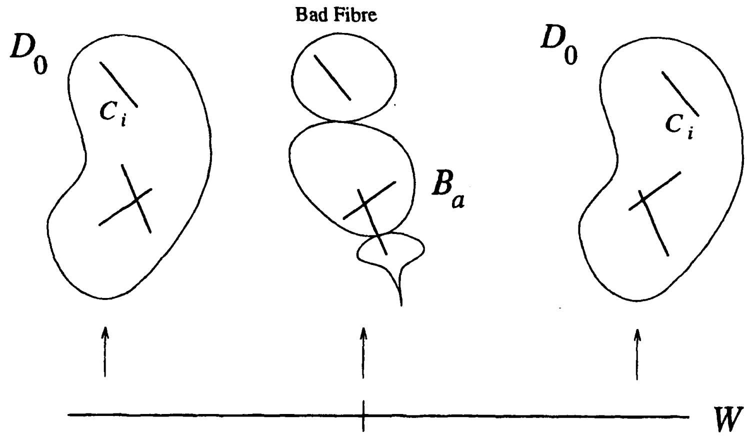

FIGURE 7. X as a K3 fibration over W.

invariant or it may be the sum of two curves which are interchanged under monodromy, etc.

3. When we have a bad fibre which is a reducible divisor in X, we may vary the volumes of the components of this bad fibre independently. Thus such fibres will contribute extra pieces to $H_4(X)$.

The second class above is clearly generated by elements of the Picard group of the generic fibre. A monodromy-invariant element of the group C_i will sweep out a divisor D_i, where $C_i = D_0 \cap D_i$. Thus

$$\begin{aligned} \#(D_0 \cap D_i \cap D_j)_X &= \#((D_0 \cap D_i) \cap (D_0 \cap D_j))_{D_0} \\ &= \#(C_i \cap C_j)_{D_0}, \end{aligned} \tag{120}$$

where the subscript denotes the space within which we are considering the intersection theory. This agrees very nicely with our earlier result of (114). We see that η_{ij} is simply the natural inner product of the monodromy-invariant part of the Picard lattice of the generic fibre. As we mentioned in section 2.5, the Picard lattice has signature $(+,-,-,\dots,-)$. When we take the monodromy-invariant part we retain the positive eigenvalue as the Kähler form restricted to the generic K3 fibre must be monodromy invariant and the generic fibre has positive volume. Thus η_{ij} has the correct signature.

Now consider the third class. We denote such an element by B_a. Since this class is supported away from the generic fibre, we have $D_0 \cap B_a = 0$. Now comparing to (114) we are in trouble. The moduli coming from such vector supermultiplets cannot live in the space (110) we expected from the heterotic string. What can have gone wrong? The only place our argument was flawed was in the *perturbative* analysis of the heterotic string. It turns out that the classes B_a cannot be understood perturbatively from the perspective of the heterotic string. We will encounter such objects later in section 6.2 but in this section we will restrict our attention to perturbative questions.

We now know exactly how to interpret the Teichmüller space

$$\frac{\mathrm{O}(2,m)}{\mathrm{O}(2)\times\mathrm{O}(m)}\times\frac{\mathrm{SL}(2)}{\mathrm{U}(1)}, \tag{121}$$

both in terms of the heterotic string compactified on $\mathrm{K3}\times T^2$ and its supposed type IIA dual compactified on X. The second term comes from the heterotic dilaton-axion and the complexified Kähler form associated to the class $D_0 \in H_4(X)$. The first term is considered to be associated to the Narain moduli space of the heterotic string on T^2, or, from what we have just said, the complexified Kähler form on the (monodromy-invariant part of) the K3 fibre. We computed the stringy moduli space of the Kähler-form and B-field on an algebraic K3 surface earlier and obtained the second term of (73). Fortunately it turns out to have just the right form!

Note that we have reduced our problem once again to a comparison of a heterotic string on a torus (this time a 2-torus) with a type IIA string on a K3 surface (this time an algebraic fibre in X). We should be able to use the same old arguments we used in sections 4.3 and 5.2 to obtain the enhanced gauge group.

Let us first assume there is no monodromy acting on the Picard group of the K3 fibre for X. We expect the slice of the moduli space coming from varying the complexified Kähler form on the generic K3 fibre to be of the form

$$\mathrm{O}(\Upsilon)\backslash\,\mathrm{O}(2,\rho)/(\mathrm{O}(2)\times\mathrm{O}(\rho)), \tag{122}$$

where Υ is the quantum Picard lattice introduced in (69). Identifying this with the Narain moduli space of T^2 that we expect to see in the weak-coupling limit of the heterotic string, we see that the Narain lattice for the T^2 is isomorphic to Υ.[23] That is, we have a gauge bundle of rank $\rho-2$ compactified over T^2.

To obtain the enhanced gauge groups we take (122) to be the Grassmannian of space-like 2-planes in $\mathbb{R}^{2,\rho}$ and look for points where the 2-plane becomes orthogonal to roots in the lattice Υ. The roots will give the root diagram of the enhanced gauge group in the usual way.

Let us clarify all this general discussion with an example taken from [**103**]. Let us take X_0 to be the hypersurface

$$z_1^2+z_2^3+z_3^{12}+z_4^{24}+z_5^{24}=0, \tag{123}$$

in the weighted projective space $\mathbb{P}^4_{\{12,8,2,1,1\}}$. The weighted projective space contains various quotient singularities. We will blow these up and take the proper transform of X_0 to be X. There is a $\mathbb{Z}_2$-quotient singularity along the locus $[z_1,z_2,z_3,0,0]$. We may blow this up, replacing each point in the locus by $\mathbb{P}^1$. The projection of X onto to this $\mathbb{P}^1$ will be the K3-fibration. That is, roughly speaking, we view $[z_4,z_5]$ as the homogeneous coordinates of the base $W\cong\mathbb{P}^1$. To find the fibre fix a point in the base by fixing z_4/z_5. This projects $\mathbb{P}^4_{\{12,8,2,1,1\}}$ onto the subspace $\mathbb{P}^3_{\{12,8,2,1\}}$. Now $\mathbb{P}^3_{\{12,8,2,1\}}$ may be viewed as a $\mathbb{Z}_2$-quotient of $\mathbb{P}^3_{\{6,4,1,1\}}$ by taking $z_4\mapsto -z_4$. Such

[23]Note that there is no reason to expect that Υ should be self-dual. In many proposed dual pairs (e.g., some of those of [**103**]) it is not self-dual. On the heterotic side this says that the heterotic string on T^2 is not modular invariant. We shall assume that modular invariance is satisfied once the K3 factor is taken into account too. In general one might worry that strange effects such as those encountered in [**96**] might cause problems with this. We will assume here that modular invariance looks after itself.

codimension quotients are equivalent to reparametrizations in complex geometry and so the fibre may be taken to be the hypersurface

$$z_1^2 + z_2^3 + z_3^{12} + z_4^{12} = 0, \tag{124}$$

in the weighted projective space $\mathbb{P}^3_{\{6,4,1,1\}}$. This is a K3 surface as expected. Note that we still have a $\mathbb{Z}_2$-quotient singularity in the fibre along $[z_1, z_2, 0, 0]$. This intersects the generic K3 fibre at a single point. Thus for a smooth X we blow-up again to introduce a single (-2)-curve into each generic fibre.

Now let us work out the Picard lattice of the generic fibre. A generic hyperplane in $\mathbb{P}^3_{\{6,4,1,1\}}$ may be written as $az_3 + bz_4 = 0$ for some $a, b \in \mathbb{C}$. Any two such hyperplanes will intersect at the point $[z_1, z_2, 0, 0]$ but this is exactly where we are blowing up. Thus the hyperplane doesn't intersect itself at all but will intersect the (-2)-curve (i.e., the exceptional divisor) once. The Picard lattice, for a generic value of complex structure of X, will be generated by the hyperplane and the single (-2)-curve and has intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \tag{125}$$

A simple change of basis shows that this is $\Gamma_{1,1} \cong U$. Note that neither the hyperplane nor the exceptional (-2)-curve are effected by monodromy around W and so we needn't worry about monodromy invariance in this case.

The bad fibres occur when we fix a point on the base W such that $z_4^{24} + z_5^{24} = 0$. Thus, at 24 points, we have fibres

$$z_1^2 + z_2^3 + z_3^{12} = 0, \tag{126}$$

in $\mathbb{P}^3_{\{6,4,1,1\}}$. Although singular, this equation does not factorize and so the bad fibres are irreducible. Thus there are no contributions to $h^{1,1}$ from bad fibres. Thus, $h^{1,1} = 1 + 2 = 3$ since we have the generic fibre itself together with $\rho = 2$.

Now $\Upsilon = U \oplus U \cong \Gamma_{2,2}$ and the moduli space of vector multiplets coming from the quantum Picard lattice will be

$$\mathrm{O}(\Gamma_{2,2})\backslash \mathrm{O}(2,2)/(\mathrm{O}(2) \times \mathrm{O}(2)). \tag{127}$$

This is exactly the moduli space for a string on T^2. Thus if the type IIA string compactified on X is dual to a heterotic string on K3 $\times$ T^2 then the T^2 part of the latter has none of the gauge group from the ten-dimensional string wound around it. All must be wound around the K3 factor.

The lattice $\Gamma_{2,2}$ contains root diagrams for $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{su}(3)$ and so we should be able to obtain these gauge symmetries for suitable choices of vector moduli. Clearly the (-2)-curve in the generic K3 fibre may be shrunk down to a point to obtain SU(2). To obtain more gauge symmetry one must shrink the K3 fibre itself down to a size of order $(\alpha')^2$ as discussed in section 4.3.

There is an important point we should note in general about enhanced gauge groups. It is known that in $N = 2$ theories quantum effects may break the gauge group down to a Cartan subgroup of the classical gauge group. Exactly how this happens depends on whether any extra hypermultiplets are becoming massive when the point of classical enhanced gauge symmetry occurs. We don't want to discuss hypermultiplets yet but, at least in simple cases, there are usually a small number, if any, of such massless particles and the theory will be "asymptotically free". In this case the gauge group will be broken.

Thus we only really expect the nonabelian gauge group to appear in the heterotic string for the case that the coupling tends to zero. In the type IIA picture this corresponds to the base $\mathbb{P}^1$ blowing up to infinite size. This means that we are "decompactifying" the type IIA picture so that it is compactified, not on X, but on the generic fibre — a K3 surface. In this respect we really are saying little more than we already said in section 4.3. We can obtain enhanced gauge symmetries when we have a theory in six dimensions from a type IIA string compactified on a certain K3 surface. This is approximately true in four dimensions so long as the base $\mathbb{P}^1$ is very large.

For the heterotic string, the gauge group is broken in the quantum theory by Yang-Mills instantons. For the type IIA string we note that world-sheet instantons wrapping around the base $\mathbb{P}^1$ presumably play an analogous rôle. This is another example of one string's target space field theory being another string's world-sheet field theory as we saw in section 5.1. It would be interesting to see if these instantons can be explicitly mapped to each other.

To determine the gauge group we should say what happens when there is monodromy in the Picard lattice of the generic K3 fibre as we move about W. There really is no difference between this and the $N = 4$ analogue we discussed in section 5.2. The monodromy of the Picard lattice should be translated into an action on the heterotic string on T^2 and divided out. Thus we expect an *asymmetric orbifold* of T^2 for the heterotic dual. When finding the enhanced gauge group we should take the monodromy-invariant subdiagram of the root diagram. This may lead to non-simply-laced gauge groups again.

One should also worry about the global form of the gauge group. That is, one may have the simply connected form of the group or one may have to mod out by part of the center, e.g., the gauge group might be SU(2) or SO(3). All we have said above is only really enough to determine the *algebra* of the gauge symmetry. We will evade this issue where possible in these lectures by only specifying gauge algebras rather than gauge groups. There will be times later on in these lectures when we have to confront this problem, however.

Let us mention here that compactifying the type IIB string, rather than the type IIA string, on a K3 fibration, can lead to a very direct link between the geometry of the Calabi–Yau threefold and Seiberg-Witten theory as explored in [**128, 129**]. Therefore an understanding of mirror symmetry within K3 fibrations may shed considerable light on some of the details of string duality.

Finally let us note that further analysis may be done on the moduli space of vector multiplets to check that string duality is working as expected. We refer the reader to [**130, 131**] for examples and especially to [**132**] where further direct links to geometry were established.

5.6. Heterotic-heterotic duality. We have seen how a type IIA string compactified on a Calabi–Yau space with a K3 fibration may have as a dual partner a heterotic string compactified on $\mathrm{K3} \times T^2$. An obvious question springs immediately to mind. What happens if X can admit more than one K3 fibration? One might expect it may be dual to more than one heterotic string. This implies that we can find pairs of heterotic strings that are dual to each other. We will follow this line of

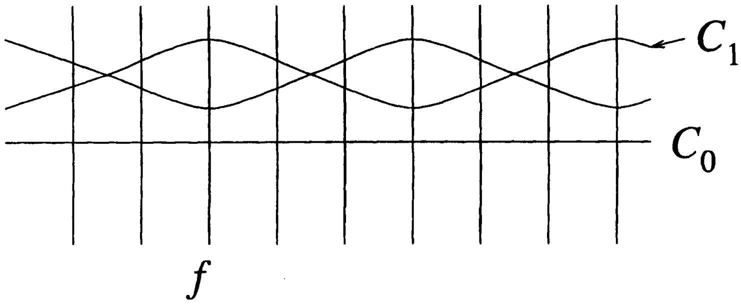

FIGURE 8. The Hirzebruch surface $\mathbf{F}_n$.

logic to analyze a case introduced in [**133**]. This geometric approach was discussed originally in [**134, 135**].

We are going to consider the example we introduced in the last section based on the hypersurface in $\mathbb{P}^4_{\{12,8,2,1,1\}}$ given by (123). In the last section we projected onto the last 2 coordinates of the $\mathbb{P}^4_{\{12,8,2,1,1\}}$ to obtain a K3 fibration over $\mathbb{P}^1$. Now let us project into the last 3 coordinates. Our base space will now be $\mathbb{P}^2_{\{2,1,1\}}$. The fibre will be an algebraic 2-torus, that is, an elliptic curve. Thus X may be considered as an elliptic fibration as well as a K3 fibration.

The space $\mathbb{P}^2_{\{2,1,1\}}$ is singular. Writing the homogeneous coordinates as $[x_0, x_1, x_2]$, there is a $\mathbb{Z}_2$ quotient singularity at the point $[1,0,0]$. This may be blown up, introducing a (-2)-curve. This exceptional curve provided the base of the K3 fibration in the last section. We may also use it to write the blow-up of $\mathbb{P}^2_{\{2,1,1\}}$ as a fibration. From the same argument as we used in the last section, it is straight forward to see that the fibre will be $\mathbb{P}^1$. Thus our blown-up $\mathbb{P}^2_{\{2,1,1\}}$ is a fibration with base space given by $\mathbb{P}^1$ and with fibre given by $\mathbb{P}^1$. Note that the fibre never degenerates. Such complex surfaces are called "Hirzebruch surfaces". These objects will turn out to be important for the analysis of the heterotic string on a K3 surface so we discuss the geometry here in some detail.

A $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ may be regarded as the compactification of a complex line bundle over $\mathbb{P}^1$ by adding a point to each fibre "at infinity". Such line bundles are classified by an integer — the first Chern class of the bundle integrated over the base $\mathbb{P}^1$. We use the notation $\mathbf{F}_n$, to denote the Hirzebruch surface built from the line bundle with $c_1 = -n$. Assume first that $n \geq 0$. Denote the base rational curve by C_0. The line bundle with $c_1 = -n$ represents the normal bundle to C_0 and so the self-intersection of C_0 equals $-n$. Thus C_0 is isolated (assuming $n > 0$). Denote the fibre by f. Clearly $\#(f \cap f) = 0$ and $\#(f \cap C_0) = 1$. We may introduce a class $C_1 = C_0 + nf$. This intersects f once and C_0 not at all. C_1 is a section of the bundle, and hence a rational curve, away from the isolated section C_0. The self intersection of C_1 is $+n$. Note that C_1 can be deformed. Thus we view $\mathbf{F}_n$ as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ with one isolated section, of self-intersection $-n$, and a family of sections, disjoint from the isolated section of self-intersection $+n$. This picture of a Hirzebruch surface, with two sections in the class C_1 shown, is drawn in figure 8. If $n = 0$ this picture degenerates and we simply have $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Note that if $n < 0$ we may exchange the rôles of C_0 and C_1 and recover the surface $\mathbf{F}_{-n}$. Thus we may assume $n \geq 0$.

For the case we are concerned with in this section C_0 is the exceptional divisor with self-intersection -2 and so we have $\mathbf{F}_2$ as the base of X as an elliptic fibration. It turns out however that $\mathbf{F}_2$ is somewhat "unstable" in this context. Note that one may embed $\mathbb{P}^2_{\{2,1,1\}}$ into $\mathbb{P}^3$ as follows. Denote the homogeneous coordinates by $[x_0, x_1, x_2]$ and $[y_0, y_1, y_2, y_3]$ respectively. By putting

$$\begin{aligned} y_0 &= x_0 \\ y_1 &= x_1^2 \\ y_2 &= x_2^2 \\ y_3 &= x_1 x_2, \end{aligned} \tag{128}$$

we embed $\mathbb{P}^2_{\{2,1,1\}}$ as the hypersurface

$$y_1 y_2 - y_3^2 = 0. \tag{129}$$

This is singular, as expected, but we may deform this hypersurface to a generic quadric. It is well-known that a generic quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbf{F}_0$ (see, for example, [**8**]). Thus $\mathbb{P}^2_{\{2,1,1\}}$ may be blown-up to give $\mathbf{F}_2$ but deformed to give $\mathbf{F}_0$. A generic point in the moduli space of X is actually an elliptic fibration over $\mathbf{F}_0$ rather than $\mathbf{F}_2$.[24]

We have arrived at the result that X is an elliptic fibration over $\mathbf{F}_0$. This may be viewed as a two-stage fibration. X may be projected onto a $\mathbb{P}^1$ to produce a K3 fibration. The fibre of this map may be projected onto the other $\mathbb{P}^1$ to write the K3 as an elliptic fibration.

Since the base space of the elliptic fibration is $\mathbb{P}^1 \times \mathbb{P}^1$, if one of the $\mathbb{P}^1$'s in the base may be viewed as the base for X as a K3 fibration, then so may the other $\mathbb{P}^1$. Thus X can be written as a K3 fibration in two different ways. This suggests that there should be two heterotic strings dual to the type IIA theory on X and hence dual to each other. What is the relationship between these two heterotic strings?

One of the heterotic strings will be compactified on $S_1 \times T_1^2$ and the other on $S_2 \times T_2^2$, where S_1 and S_2 are K3 surfaces. We examined the moduli space coming from the vector multiplets in section 5.5 and saw that the 3 moduli described the dilaton-axion and the Narain moduli of the T^2 (without any Wilson lines switched on). The dilaton is given by the area of the base $\mathbb{P}^1$ in X. Let us determine the area of the heterotic T^2 in terms of the moduli of the K3 fibre within X.

The generic K3 fibre within X has Picard number 2 and has a moduli space of Kähler form and B-field given by (127). Let $\Gamma_{2,2}$ be generated by the null vectors w and v together with their duals w^* and v^*. For simplicity we will avoid switching on any B-fields. Thus we consider a point in the moduli space (127) to be given by a space-like 2-plane, $\mho$, spanned by $w^* + \alpha w$ and $v^* + \beta v$, for $\alpha, \beta > 0$. Following our construction to determine the Kähler form, we have that $\mho \cap w^\perp$ is spanned

[24]To see this note that when we blow-up the fibration of X over $\mathbb{P}^2_{\{2,1,1\}}$ to a fibration over $\mathbf{F}_2$ we introduce an elliptic scroll just as in section 5.4. The results of [**119**] tell us that the (-2)-curve will therefore vanish for a generic complex structure.

by $v^* + \beta v$ which is contained in $w^\perp/w$. Thus, as promised, B is zero. From our analysis in section 3.3 we see that the Kähler form determines the volume to be

$$J.J = \alpha. \tag{130}$$

We also have that the direction of J is given by $v^* + \beta v$. Thus the Kähler form is

$$J = \sqrt{\frac{\alpha}{2\beta}}v^* + \sqrt{\frac{\alpha\beta}{2}}v. \tag{131}$$

We have seen that the generic K3 fibre is itself an elliptic fibration over $\mathbb{P}^1$. Knowing the intersection numbers of the curves within this fibration together with the positivity of the Kähler class determines the class of the base $\mathbb{P}^1$ to be $v^* - v$ and that of the elliptic fibre to be v. Thus, the area of the $\mathbb{P}^1$ within the K3 is given by

$$J.C = \sqrt{\frac{\alpha\beta}{2}} - \sqrt{\frac{\alpha}{2\beta}}. \tag{132}$$

Now let us reinterpret $\mho$ in terms of the moduli of the T^2. From section 3.5 we obtain a map from W into W^* given by[25]

$$\psi = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}. \tag{133}$$

This is symmetric and thus gives the metric. Therefore, the area of the T^2 is given by $\sqrt{\det(\psi)} = \sqrt{\alpha\beta}$.

We may now obtain an interesting result confirming the analysis of [**133**] by going to the limit where α and β are taken to be very large. In this case, our heterotic string is compactified on K3 $\times\, T^2$ where the 2-torus is very large. In the type IIA string, the generic K3 fibre contains a rational curve which becomes very large. From the equation (132), the area of this rational curve is proportional to the heterotic string's 2-torus in this limit.

This limit as $\alpha, \beta \to \infty$ may be viewed as a decompactification of the model to a six-dimensional theory given by the heterotic string compactified on a K3 surface. Let us consider the coupling of this six-dimensional theory, λ_6. This is given by the four-dimensional coupling prior to decompactification and the area of the T^2. The former is given by the size of the base $\mathbb{P}^1$ of the K3 fibration, which we refer to as $\mathbb{P}^1_1$. The latter is given by the area of the base of the K3 fibre itself written as an elliptic fibration, which we will refer to as $\mathbb{P}^1_2$. We have that

$$\lambda_6^2 = \lambda_4^2.\,\mathrm{Area}(T^2) \sim \frac{\mathrm{Area}(\mathbb{P}^1_2)}{\mathrm{Area}(\mathbb{P}^1_1)}. \tag{134}$$

The other heterotic string is obtained by exchanging the rôles of $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$. Thus we have

PROPOSITION 7. *Let X be the Calabi–Yau manifold given by a resolution of the degree 24 hypersurface in $\mathbb{P}^4_{\{12,8,2,1,1\}}$. The two heterotic string theories dual to the type IIA string theory on X decompactify to two heterotic string theories compactified on K3 surfaces. From (134) these two six-dimensional theories are*

[25] Note that W is spanned by w^* and v^*; and W^* is spanned by w and v. There is no simple choice of conventions which would have circumvented this notational irritation!

S-dual in the sense that the coupling of one theory is inversely proportional to the coupling of the other theory.

The main assumption underlying this proposal is that these heterotic string theories actually exist. We will get closer to identifying these theories in section 6.3.

We may vary the complex structure of X and this should correspond to deformations of the K3 together with its vector bundle on the heterotic side. Note that the K3 of one of the heterotic strings will, in general, have quite different moduli than the K3 of the other heterotic string since exchanging the rôles of $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$ need not be a geometrical symmetry of X — only a topological symmetry. Before we can explicitly give the map between these K3 surfaces and their vector bundles we need to map out the moduli space of the moduli from the hypermultiplets. This remains to be done.

There are many examples of Calabi–Yau manifolds which admit more than one K3 fibration. This will lead to many examples of heterotic-heterotic duality. In most cases however the result will be a rather tortuous mapping between four-dimensional theories and will not be as simple as the above example.

5.7. Extremal transitions and phase transitions. We will take our first tentative steps into the moduli space of hypermultiplets in this section. This will deal with the simplest aspects of the bundle over K3 — namely when this bundle, or part of this bundle, becomes trivial.

Our heterotic string theory is compactified on a bundle over K3 $\times$ T^2 which we view as the product of a bundle over K3 and a bundle over T^2. Generically one would expect the deformation space of this bundle structure to be smooth. Where this can break down however is when part of the bundle becomes trivial. To deform away from such a bundle we may wrap the trivial part around either the K3 surface or the T^2. Thus we obtain branches in the moduli space. A deformation in the K3 part will be a hypermultiplet modulus while a deformation in the T^2 part will be a vector modulus. Our picture therefore for a transition across this branch will consist of moving in the moduli space of hypermultiplets until suddenly a vector scalar becomes massless and can be used as a marginal operator to move off into a new branch of the moduli space, at which point some of the moduli in the hypermultiplets in the original theory may acquire mass.

When we compactify the $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string on a vector bundle, the original gauge group is broken by the holonomy of the bundle. Thus, when we move to a transition point where part of the vector bundle becomes trivial we may well expect the holonomy group to shrink and thus the observed gauge group is enhanced. Since we have only $N = 2$ supersymmetry this observed gauge group enhancement may get killed by quantum effects but should be present in the zero-coupling limit.

The picture is the geometric version of the "Higgs" transitions explored in, for example, **[102]**. We wish to see how this transition appears from the type IIA picture.

We have already seen what the type IIA picture is for moving within the moduli space of vector scalars to a point at which an enhanced gauge symmetry appears. This is where we vary the Kähler form on the generic K3 fibre to shrink down some rational curves (or go to Planck scale effects). Thus, every K3 fibre becomes singular. That is, we have a curve of singularities in X. Now given such a singular Calabi–Yau manifold, it may be possible to deform this space by a deformation of complex structure to obtain a new smooth Calabi–Yau manifold. This would correspond to a deformation of each singular K3 fibre to obtain a smooth K3 fibre. This process will decrease the Picard number of the generic fibre (as we have lost the class of rational curves we shrank down) and will change the topology of the underlying Calabi–Yau threefold.

In the type IIA language then, this Higgs transition consists of deforming the Kähler form on X to obtain a singular space and then smoothing by a deformation to another smooth manifold. Such a topology-changing process is called an "extremal transition".

One example of an extremal transition is the "conifold" transition of [**136**]. A conifold transition consists of shrinking down isolated rational curves and then deforming away the singularities. These were explored in the context of full string theory in [**75, 137**] (see also B. Greene's lectures). In our case however we are not shrinking down isolated curves but whole curves of curves and so we are not discussing a conifold transition.

There has been speculation [**138, 137**] that the moduli space of all Calabi–Yau threefolds is connected because of extremal transitions (based on an older, much weaker, statement by Reid [**139**]). See [**140, 141**] for recent results in this direction. Certainly no counter example is yet known to this hypothesis. The heterotic picture of these extremal transitions is simple to understand in the case of singularities developing within the generic fibre. Such specific extremal transitions are certainly not sufficient to connect the moduli space and we will require an understanding of nonperturbative heterotic string theory to complete the picture. One obvious example to worry about is when the Calabi–Yau threefold on the type IIA side goes through an extremal transition from something that is a K3 fibration to something that is not. It is not difficult to see that such a transition must involve shrinking down the base, W, of the fibration and thus going to a strongly-coupled heterotic string. It is not surprising therefore that we cannot understand such a transition perturbatively in the heterotic picture.

Given a dual pair of a type IIA string compactified on a Calabi–Yau manifold and a heterotic string compactified on K3 $\times$ T^2 we may generate more dual pairs by following each through these phase transitions we do understand perturbatively. Such "chains" of dual pairs were first identified in [**142**] and many examples have been given in [**143**]. In order to understand where these chains come from, and indeed the original Kachru–Vafa examples of dual pairs in [**103**], we need to confront the issue of compactifying the heterotic string on a K3 surface, a subject we have done our best to avoid up to this point!

6. The Heterotic String

This section will be concerned with the heterotic string compactified on a K3 surface. In particular we would like to find a string theory dual to this. An obvious answer is another heterotic string compactified on another K3 surface, as we saw in section 5.6. We will endeavor to find a type II dual. It turns out the we will not be able to find a type II dual directly but will have to go via the construction of section 5. This process (in its various manifestations) is often called "F-theory". A great deal of the following analysis is based on work by Morrison and Vafa [**144, 135, 145**].

6.1. $N = 1$ **theories.** The heterotic string compactified on a K3 surface gives a theory of $N = 1$ supergravity in six dimensions. From section 4.1 the holonomy algebra of the moduli space coming from supersymmetry will be $\mathfrak{sp}(1)$. There are two types of supermultiplet in six dimensions which contain moduli fields:

1. The hypermultiplets each contain 4 real massless scalars which transform as quaternionic objects under the $\mathfrak{sp}(1)$ holonomy.
2. The "tensor multiplets" each contain one real massless scalar.

Thus, at least away from points where the manifold structure may break down, we expect the moduli space to be in the form of a product

$$\mathscr{M}_{N=1} = \mathscr{M}_T \times \mathscr{M}_H, \tag{135}$$

(possibly divided by a discrete group) where $\mathscr{M}_T$ is a generic Riemannian manifold spanned by moduli in tensor supermultiplets and $\mathscr{M}_H$ is a quaternionic Kähler manifold spanned by moduli in hypermultiplets.

For a review of some of the aspects of these theories we refer to [**146**]. The six-dimensional dilaton lives in a tensor multiplet. An interesting feature of these theories is that it appears to be impossible to write down an action for these theories unless there is exactly one tensor multiplet.[26] Thus moduli in tensor multiplets other than the dilaton should be regarded as fairly peculiar objects.

It is useful to compare $N = 1$ theories in six dimensions with the resulting $N = 2$ theory in four dimensions obtained by compactifying the six-dimensional theory on a 2-torus. To explain the moduli which appear in four dimensions we need to consider another supermultiplet of the $N = 1$ theory in six dimensions. This is the vector multiplet which contains a real vector degree of freedom but no scalars. Each supermultiplet in six dimensions produces moduli in four dimensions as follows:

1. The 4 real scalars of a hypermultiplet in six dimensions simply produce the 4 real scalars of a hypermultiplet in four dimensions.
2. The 1 real scalar of a tensor multiplet together with the anti-self-dual two-form compactified on T^2 produce the two real scalars of a vector multiplet in four dimensions.
3. The vector field of a vector multiplet compactified on the two 1-cycles of T^2 of produces the two real scalars of a vector multiplet in four dimensions.

That is, hypermultiplets in six dimensions map to hypermultiplets in four dimension but both tensor multiplets and vector multiplets in six dimensions map to vector multiplets in four dimensions.

We should emphasize that quantum field theories in four and six dimensions are quite different. In particular, conventional arguments imply that six-dimensional quantum field theories should always be infra-red free and therefore rather boring.

[26]This is because the gravity supermultiplet (of which there is always exactly one) contains a self-dual two-form and the tensor multiplets contain anti-self-dual two-forms. It is problematic to write down Lorentz-invariant actions for theories with net (anti-)self-dual degrees of freedom [**147**].

This notion has been revised recently in light of many of the results coming from string duality [**146, 148**] where it is now believed that nontrivial theories can occur in six dimensions as a result of "tensionless string-like solitons" appearing. Such theories potentially appear when one goes through extremal transitions between tensor multiplet moduli and hypermultiplet moduli.

Despite the strange properties of these exotic six-dimensional field theories we will be able to avoid having to explain them here. This is because all of our discussion will really happen in four dimensions — the six-dimensional picture is only considered as a large T^2 limit. It is important to realize however that some of the things we will say, based on four-dimensional physics, may well be rather subtle in six dimensions. An example of this will be when certain enhanced gauge symmetries are said to appear when the hypermultiplet moduli are tuned to a certain value. If massless tensor moduli also happen to appear at the same time (which will happen for E_8 as we shall see) then any conventional description of the resulting six-dimensional field theory is troublesome. Declaring what the massless spectrum of such a theory is not an entirely well-defined question and one should move the theory slightly by perturbing either a massless hypermultiplet modulus or a tensor modulus before asking such a question.

6.2. Elliptic fibrations. Our method of approach will be to consider a type IIA string compactified on a Calabi–Yau threefold, X, dual to a heterotic string compactified on K3 $\times$ T^2 and then take the volume of the 2-torus to infinity thus decompactifying our theory to a heterotic string on a K3 surface. Actually, decompactification is a rather delicate process and perhaps we should be more pragmatic and say that we will consider a heterotic string on K3 $\times$ T^2 and try to systematically ignore all aspects of the type IIA string on X coming from the T^2 part of the heterotic string.

Much of the analysis we require we have already covered in section 5.6. There, for a specific example, we did precisely the decompactification we require. We need to consider how general we can make this process. Clearly we should insist that our moduli space of vector multiplets in the four-dimensional theory corresponding to the heterotic string on K3 $\times$ T^2 contain the space

$$\mathrm{O}(\Gamma_{2,2})\backslash\,\mathrm{O}(2,2)/(\mathrm{O}(2)\times\mathrm{O}(2)). \qquad (136)$$

That is, we have the moduli space of the string on T^2. If this were not the case, we could not claim that we had really compactified on a true product of K3$\times T^2$. From the analysis of section 5.4 we expect X to be a K3 fibration. Given the appearance of $\Gamma_{2,2}$ in (136) we expect from section 5.5 that the Picard lattice of the generic fibre of X must contain the lattice $\Gamma_{1,1}$.

A K3 surface with a Picard lattice containing $\Gamma_{1,1}$ is an elliptic fibration. To see this, let v and v^* be a basis for $\Gamma_{1,1}$. The class $v - v^*$ is a primitive element of self-intersection -2 and thus (see, for example, [**16**]) either this class, or minus this class, corresponds to a rational curve within the K3. Now either v or v^* is a nef curve of zero self-intersection. This will be the class of the elliptic fibre. Note that we have a rational curve in the K3 surface intersecting each fibre once — our elliptic fibration has a global section. What's more, it is easy to see that this section is unique as there is no other (-2)-curve.[27]

We thus claim that X is a K3 fibration and that each K3 fibre can be written as an elliptic fibration with a unique section. Together these give [**149**]

PROPOSITION 8. *If a heterotic string compactified on $K3 \times T^2$ is dual to a type IIA string on X and there are no obstructions in the moduli space to taking the size of T^2 to infinity and thereby ignoring the effects associated to it, then X is an elliptic fibration over a complex surface with a section.*

This proposition is subject to the same conditions as proposition 6 and to the caveat in the footnote above.

Let us denote this elliptic fibration as $p: X \to \Theta$. Θ itself is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$, $\Theta \to W$. The simplest possibility is that Θ is the Hirzebruch surface $\mathbf{F}_n$. This need not be the case however — there may be some bad fibres over some points in W.

The subject of elliptic fibrations has been studied intensively by algebraic geometers, thanks largely to Kodaira [**150**]. This can be contrasted to the subject of K3 fibrations, about which relatively little is known.

Let us fix our notation. Let W, the base of X as a K3 fibration, be $\mathbb{P}^1$ with homogeneous coordinates $[t_0, t_1]$. We will also use the affine coordinate $t = t_1/t_0$. Θ is a $\mathbb{P}^1$ fibration over W. Over a generic point in W, the $\mathbb{P}^1$ fibre will have homogeneous coordinates $[s_0, s_1]$ and affine coordinate $s = s_1/s_0$. Now we want to write down the fibre of X as an elliptic fibration over Θ. Any elliptic curve may be written as a cubic hypersurface in $\mathbb{P}^2$. Writing this in affine coordinates we may put the fibration in "Weierstrass form" for a generic point $(s,t) \in \Theta$:

$$y^2 = x^3 + a(s,t)x + b(s,t), \tag{137}$$

where x and y are affine coordinates in a patch of $\mathbb{P}^2$ and a and b are arbitrary polynomials. If Θ itself has bad fibres — that is, it is not $\mathbf{F}_n$ — then more coordinate patches need to be introduced to give a global description of the elliptic fibration.

Note that not any elliptic fibration can be written in Weierstrass form. Homogenizing the coordinates of $\mathbb{P}^2$ putting $x = x_1/x_0$ and $y = x_2/x_0$, we see that $[0,0,1]$ always lies in (137), which gives the fibration a global section. Thus only elliptic fibrations with a section can be written in this form. Luckily this is the case we are interested in.

The j-invariant of the elliptic fibre (137) is $4a^3/\delta$, where δ is the discriminant of (137) given by

$$\delta = 4a^3 + 27b^2. \tag{138}$$

If $\delta = 0$ then the elliptic fibre is singular. The hypersurface, or divisor, $\delta(w,z) = 0$ in Θ thus gives the locus of bad fibres. It is commonly called the discriminant locus.

[27] Actually we have cheated here. We have assumed that the bundle over K3 may be chosen so that it breaks the $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbf{Z}_2$ gauge group. Then the Narain moduli space for the T^2 really is given by (136). There can be times however, as we shall see soon, when this is not possible and then the Narain moduli space for the 2-torus becomes larger. This allows for more than one section of the elliptic fibration. We claim this is not important for the examples we discuss in these lectures, however, as we may begin in a case where the gauge group is broken by the K3 bundle and then proceed via extremal transitions to the case we desire, preserving the section of the elliptic fibration if X.

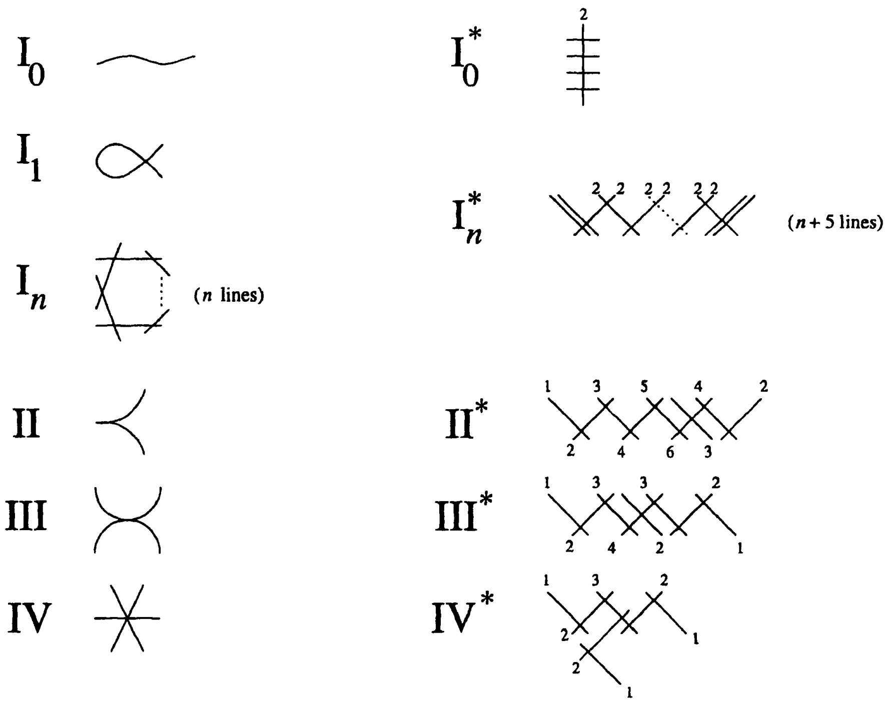

FIGURE 9. Classification of elliptic fibres.

There are only so many things that can happen at a generic bad fibre of an elliptic K3 surface and these have been classified (see, for example, [**16**]). Let us take a small disc $D \subset \mathbb{C}$, embedded in Θ, with coordinate z which cuts a generic point on the discriminant locus transversely. Let $z = 0$ be the location of the discriminant locus. Let us analyze the restriction of the fibration of X to the part which is fibred over D, We are thus considering an open set of a fibration of a complex *surface*. Assuming that the total space of the fibration is smooth and there are no (-1)-curves, which will certainly be true if we are talking about a Calabi–Yau manifold, the possibilities for what happens to the fibre when $z = 0$ is shown in figure 9. The case I_0 is when there is no zero of δ and the elliptic fibre is smooth. In all other cases the lines and curves in figure 9 represent rational curves. Case I_1 is a rational curve with a "double point", i.e., it looks locally like $y^2 = x^2$ at one point and case II is a rational curve with a "cusp", i.e., it looks locally like $y^2 = x^3$ at one point. All the other cases consist of multiple rational curves. All of the singular fibres should be homologous to the smooth fibre. To achieve this, some of the rational curves in the bad fibre must be counted more than once to obtain the correct homology. The multiplicity of the curves are shown as the small numbers in figure 9. If omitted, multiplicity one is assumed.[28]

[28]There is also a possibility that I_n fibres can appear with multiplicity greater than one. We ignore this as the canonical class of such a fibration cannot be trivial.

L	K	N	Fibre	Υ'
≥ 0	≥ 0	0	I_0	
0	0	>0	I_N	A_{N-1}
≥ 1	1	2	II	
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
≥ 2	≥ 3	6	I_0^*	D_4
2	3	≥ 7	I_{N-6}^*	D_{N-2}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

TABLE 4. Weierstrass classification of fibres.

The possibilities listed in figure 9 can also be classified according to the Weierstrass form. On our disc, D, the polynomials, a and b, in (137) will be polynomials in z. Let us define the non-negative integers (L, K, N) by

$$\begin{aligned} a(z) &= z^L a_0(z) \\ b(z) &= z^K b_0(z) \\ \delta(z) &= z^N \delta_0(z), \end{aligned} \tag{139}$$

where $a_0(z)$, $b_0(z)$, and $\delta_0(z)$ are all nonzero at $z = 0$. The triple (L, K, N) then determines which fibre we have according to table 4. See [**32**] for an explanation of this. To be precise, the Weierstrass form of the fibration in x, y and z will produce a surface singularity which, when blown-up, will have the fibres in figure 9. This is closely linked to the results in table 1.

It is important to note that this classification procedure only applies to a smooth point on the discriminant locus. Only in this case can we characterize the bad fibre in terms of the family of elliptic curves over our small disc, D. When the discriminant is singular the nature of the bad fibre need not be expressible in terms of the geometry of a complex surface — it will be higher-dimensional in character. In general, for Calabi–Yau threefolds, we should expect to encounter some singular fibres not listed above. Such exotic fibres are important in string theory but we will try to avoid such examples here as it makes the analysis somewhat harder.

For our elliptic fibration, $p : X \to \Theta$, a knowledge of the explicit Weierstrass form is enough to calculate the canonical class, K_X. This may be done as follows. In homogeneous coordinates, the Weierstrass form is

$$x_0 x_2^2 = x_1^3 + a x_0^2 x_1 + b x_0^3, \tag{140}$$

giving a cubic curve in $\mathbb{P}^2$. Now since the elliptic fibration is not trivial, this $\mathbb{P}^2$ will vary nontrivially as we move over Θ. We may describe such a $\mathbb{P}^2$ as the projectivization of a sum of three line bundles over Θ. We are free to declare that x_0 is a section of a trivial line bundle. We may then find a line bundle, $\mathscr{L}$, such that x_1 is a section of $\mathscr{L}^2$ and x_2 is a section of $\mathscr{L}^3$ in order to be compatible with (140). It also follows that a is a section of $\mathscr{L}^4$ and b is a section of $\mathscr{L}^6$.

Now let us consider the normal bundle of the section, σ, given by $[x_0, x_1, x_2] = [0, 0, 1]$, embedded in X. We may use affine coordinates $\xi_1 = x_1/x_2$ and $\xi_2 = x_0/x_2$ whose origin gives the section σ. Note that ξ_1 is a section of $\mathscr{L}^{-1}$ and ξ_2 is a section of $\mathscr{L}^{-3}$. Near σ, (140) becomes

$$\xi_2 = \xi_1^3. \tag{141}$$

Thus, ξ_1 is a good coordinate to describe the fibre of the normal bundle of σ in X. This implies that this normal bundle is given by $\mathscr{L}^{-1}$.

We may now use the adjunction formula for $\sigma \subset X$ to give

$$K_X|_\sigma = K_\sigma + \mathscr{L}. \tag{142}$$

Actually, this is the only contribution towards the canonical class of X. That is,

$$K_X = p^*(K_\Theta + \mathscr{L}). \tag{143}$$

For the case we will be interested in, we want $K_X = 0$ and so $\mathscr{L} = -K_\Theta$. We will give examples of such constructions later.

Note that if there is a divisor within Θ over which a vanishes to order ≥ 4 and b vanishes to order ≥ 6, we may redefine $\mathscr{L}$ to "absorb" this divisor and lower the degrees of a and b accordingly. This is why no such fibres appear in table 4. As this will change K_X, such occurrences cannot happen in a Calabi–Yau variety.

Let us consider a K3 surface written as an elliptic fibration with a section. The Picard number of the K3 is at least two — we have the section and a generic fibre as algebraic curves. If we have any of the fibres I_n, for $n > 2$, or I_n^*, III, IV, II*, III*, or IV* we will also have a contribution to the Picard group from reducible fibres. Each of these fibres contains rational curves in the form of a root lattice of a simply-laced group. Let us denote this lattice Υ'. The possibilities are listed in table 4. Thus, shrinking down these rational curves will induce the corresponding gauge group for a type IIA string.

We know that for a Calabi–Yau manifold compactified on a K3 fibration, the moduli coming from varying the Kähler form on the K3 fibre map to the T^2 part of the heterotic string compactified on K3 $\times$ T^2. In particular, the act of blowing-up rational curves in the K3 to resolve singularities, and hence break potential gauge groups, is identified with switching on Wilson lines on T^2. Thus, to ignore Wilson lines, these rational curves must all be blown down and held at zero area. That is, *any of the fibres* I_n, *for* $n > 2$, *or* I_n^*, *III, IV, II*, III*, or IV* appearing in the elliptic fibration will produce an enhanced gauge symmetry in the theory.*

From section 5.6 and, in particular (131), the size of an elliptic fibre within this K3 will be fixed to some constant $\sqrt{\alpha/2\beta}$ as $\alpha, \beta \to \infty$ to make the T^2 infinite area. Thus this size is "frozen out" as a degree of freedom. To ignore the 2-torus degrees of freedom for the type IIA string compactified on X we should take the K3 fibre within X, consider it as an elliptic fibration with an elliptic fibre of frozen area and blow down any rational curves which may take the Picard number of the K3 fibre beyond 2. In summary, any degrees of freedom coming from sizes within the elliptic fibre structure are ignored.

Consider the base, Θ, as a $\mathbb{P}^1$ bundle over W. Suppose we have bad fibres in this case. These must correspond to reducible fibres. Now, when we build X as a K3 fibration over W, these reducible fibres in Θ will build reducible fibres in X. This is exactly case 3 listed at the beginning of section 5.5. That is, varying the size of irreducible parts of these reducible fibres will give moduli in vector supermultiplets which cannot be understood perturbatively on the heterotic side. The important point to note here is that these degrees of freedom will not go away when we unwind any Wilson lines around T^2 and take its area to infinity. Thus, these degrees of freedom must be associated to the six-dimensional theory of the heterotic string compactified on a K3. From section 6.1 these moduli must therefore come from tensor multiplets in the six-dimensional theory. It is the peculiar nature of tensor moduli which prevented us from having a perturbative understanding of these moduli in section 5.5.

We saw in section 5.6 how the size of W and the size of the fibre of Θ were used to produce the area of the heterotic string's T^2 and the size of its dilaton. The area of the T^2 is lost as a degree of freedom in our six-dimensional theory. We see then *that the number of tensor multiplets in our theory will be the Picard number of* Θ *minus one.* In the case that Θ is $\mathbf{F}_n$, that is, there are no bad fibres, the number of tensor multiplets will be one and this single multiplet contains the heterotic dilaton as a modulus.

Questions concerning hypermultiplets between the four-dimensional theory and the six-dimensional theory are unchanged. In particular, we retain the relationship from section 5.3 that the number of hypermultiplets is given by $h^{2,1}(X)+1$.

Now we have counted massless tensor multiplets and hypermultiplets, let us count massless vector multiplets. We know that any vector multiplet in the four-dimensional theory must have its origin in either a vector multiplet or a tensor multiplet in six dimensions. Thus we can count the number of six-dimensional vectors by subtracting $h^{1,1}(\Theta)-1$ from the number of four-dimensional vectors.

Many of these vectors can be seen directly in terms of the enhanced nonabelian gauge symmetry but there is an additional contribution. In section 5.5 we saw how $H^2(X)$, where X is a K3 fibration, could be built from elements from the base, from the generic fibre and from the bad fibres in a fairly obvious way. Now we want to consider if the same thing is true for an elliptic fibration. Analyzing the spectral sequence one can see that $H^2(\Theta,\mathbb{Z})$ contributes, for the base, to give the tensor multiplets and $H^0(\Theta,R^2p_*\mathbb{Z})$ contributes for the fibres. This latter piece accounts for the enhanced nonabelian symmetry discussed above. The object of interest will be the term $H^1(\Theta,R^1p_*\mathbb{Z})$. In the case of K3 fibrations, this is trivial since $H^1(\mathrm{K3})=0$. This ceases to be true for an elliptic fibration however. This contribution may be associated to the group of sections of the bundle as seen in [151]. Later on, in section 6.5, we will consider a case where this is a finite group, but here we note that if this group is infinite, then its rank will contribute to the dimension of $H^2(X)$.

We see then that if the elliptic fibration has an infinite number of sections, there will be massless vector fields beyond those accounted for from the nonabelian gauge symmetry from bad fibres. These may contribute extra $\mathfrak{u}(1)$ terms to the gauge symmetry [145] and, conceivably, more nonabelian parts. As this part of the gauge group is rather difficult to analyze we will restrict ourselves to examples in these lectures where there is no such contribution.

6.3. Small instantons. Our goal in this section will be to find the map from at least part of the moduli space of hypermultiplets for a type IIA string on a Calabi–Yau manifold to the moduli space of hypermultiplets of the heterotic string compactified on a K3$\times T^2$. As we have seen, the T^2 factor is irrelevant for the hypermultiplet moduli space and we are free to consider the latter as a heterotic string on a K3 surface so long as the Calabi–Yau is an elliptic fibration with a section. We will be able to go some way to determining "which" heterotic string a given elliptic threefold is dual to.

The best policy when finding a map between two moduli spaces is to start with a particularly special point in the moduli space of one theory with hopefully unique properties which allow it to be identified with a correspondingly special point in the other theory's moduli space. This special point will usually be very symmetric in some sense. The trick, invented in [**135**] and inspired by the work of [**152**], is to look for very large gauge symmetries resulting from collapsed instantons in the heterotic string.

Recall that the K3 surface, S, on which the heterotic string is compactified comes equipped with a bundle, $E \to S$, with $c_2(E) = 24$. Fixing S, the bundle E will have moduli. A useful trick when visualizing the moduli space of bundles is to try to flatten out as much of the connection on the bundle as possible. Of course, the fact that $c_2(E) = 24$ makes it impossible to completely flatten out E but we can concentrate the parts of the bundle with significant curvature into small regions isolated from each other. In this picture, an approximate point of view of at least part of the moduli space of the bundle can be viewed as 24 "instantons" localized in small regions over S each of which contributes one to $c_2(E)$.

As well as its position, each instanton will have a degree of freedom associated to its size — that is, the characteristic length away from the centre of the instanton where the curvature becomes small. At least from a classical point of view, there is nothing to stop one shrinking this length scale down to zero.[29] Such a process naturally takes one to the boundary of the moduli space of instantons.

Let us consider the $E_8 \times E_8$ heterotic string. The observed gauge group of a heterotic string theory compactified on S will be the part of the original $E_8 \times E_8$ which is not killed by the holonomy of E. It is the "centralizer" of the embedding of the holonomy in $E_8 \times E_8$ — i.e., all the elements of $E_8 \times E_8$ which commute with the holonomy. What is the holonomy around a collapsed instanton? In general the global holonomy is generated by contractable loops due to the curvature of the bundle, and from non-contractable loops in the base. The curvature is zero everywhere when the instanton has become a point. Also, if S is smooth we may look at a 3-sphere surrounding the instanton to look for holonomy effects. Since $\pi_1(S^3) = 0$ we cannot generate non-contractable loops. Thus, the holonomy of a point-like instanton is trivial.[30]

One possibility is to shrink all 24 instantons down to zero size. When we do this, E will have no holonomy whatsoever and the resulting heterotic string theory will retain its full $E_8 \times E_8$ gauge symmetry. As this is such a big gauge group it is a good place to start analyzing our duality map.

We thus want to find a Calabi–Yau space on which we may compactify the type IIA string to give an $E_8 \times E_8$ gauge symmetry. That is, we want an elliptic fibration over $\mathbf{F}_n$ such that when viewed as a K3 fibration, the generic K3 fibre has two E_8 singularities. Let us discuss what this implies about the discriminant locus within $\Theta \cong \mathbf{F}_n$.

First, let us be a little sloppy with notation and not distinguish between curves and their divisor class (roughly speaking, homology class) in Θ. Thus we use C_0 to

[29] Although there is no reason to suppose that one may do this independently for all 24 instantons.
[30] Note that this need not be the case if S has an orbifold singularity and the instanton sits at this point. Then we can only surround the instanton by a lens space which is not simply connected. One should note that we have conveniently ignored those loops which happen to go through the point where the point-like instanton sits.

denote the class of base, i.e., the $(-n)$-curve within Θ, and f to denote the class of the generic $\mathbb{P}^1$ fibre. Let us determine K_Θ in terms of these classes. Consider the adjunction formula for a curve $C \in \Theta$. Integrating this over C we obtain its Euler characteristic

$$\chi(C) = -C.(C + K_\Theta). \tag{144}$$

Knowing that C_0 and f are spheres is enough to determine

$$K_{\mathbf{F}_n} = -2C_0 - (2+n)f. \tag{145}$$

Let us use the letters A, B, and Δ to denote the divisors associated to the equations $a = 0$, $b = 0$, and $\delta = 0$ respectively. From (143), the classes of these divisors will be

$$\begin{aligned} A &= 8C_0 + (8+4n)f \\ B &= 12C_0 + (12+6n)f \\ \Delta &= 24C_0 + (24+12n)f, \end{aligned} \tag{146}$$

in order that X be Calabi–Yau.

The locus of E_8 singularities in X will map to curves in Θ. As these E_8's are independent, we want to make these curves disjoint sections of Θ, that is, one curve will be in the class C_0 (the isolated zero section of the Hirzebruch surface as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$) and the other in the class $C_\infty = C_0 + nf$ (a section in the non-isolated class which we view as a section "at infinity" of the Hirzebruch surface). From table 4 we see that we want II^* fibres over these curves.

These two curves of II^* fibres will account for a large portion of the A, B, and Δ divisors. Let us write A', B', and Δ' for the remaining parts of the divisors not contained in the curves C_0 and C_∞. From table 4 we have

$$\begin{aligned} A' &= A - 4(C_0) - 4(C_0+nf) = 8f \\ B' &= B - 5(C_0) - 5(C_0+nf) = 2C_0 + (12+n)f \\ \Delta' &= \Delta - 10(C_0) - 10(C_0+nf) = 4C_0 + (24+2n)f. \end{aligned} \tag{147}$$

This means that what is left of the discriminant, Δ', will collide with the curve C_0 a total number of $C_0.\Delta' = 2(12-n)$ times and with the curve C_∞ a total number of $2(12+n)$ times. These 48 points of intersection are not independent. The reason that Δ' collides with the curves of II^* fibres is precisely because B' also collides with these curves. Each time B' hits these curves, the degree of the discriminant will rise by 2 and hence Δ' hits them twice in the same place. To see exactly what shape this intersection is one may explicitly write out the equations. The result is that Δ' crosses itself transversely at these points as well as hitting C_0 or C_∞. We see then that, generically, Δ' collides with C_0 at $12-n$ points and with C_∞ at $12+n$ points. Within Δ', away from these collisions, we expect the discriminant to behave reasonably nicely.[31] The result is shown in the upper part of figure 10.

We know how to deal with all of the points on the discriminant from Kodaira's list in figure 9 except for the 24 points of collision of Δ' with the two lines of II^* fibres. Here we have to work harder to obtain a smooth model for X.

[31] Although it will have cusps.

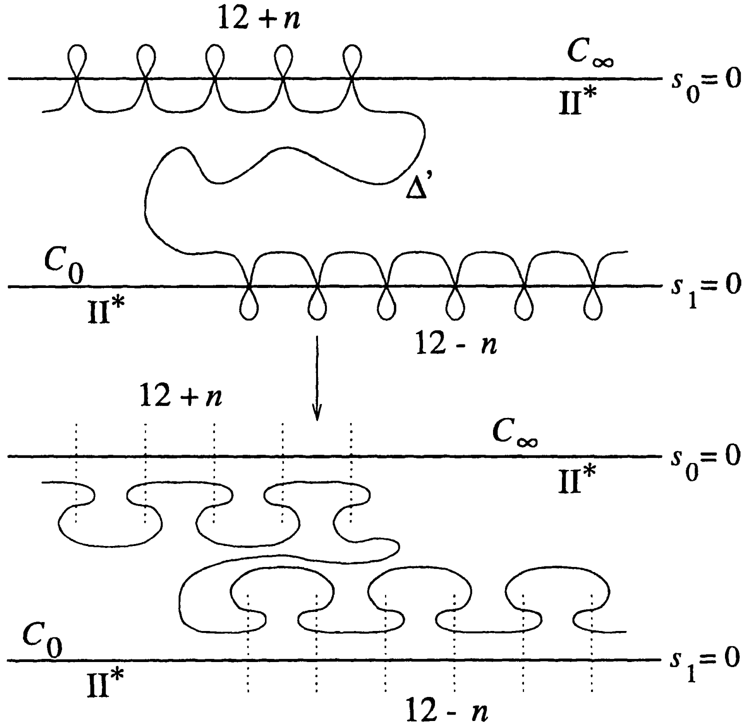

FIGURE 10. 24 small instantons in the $E_8 \times E_8$ string.

Let us focus on one of these points where either C_0 or C_∞ hits Δ' twice within Θ. Blow up this point of intersection, $\pi : \widetilde{\Theta} \to \Theta$. This will introduce a new rational curve, E, in the blown-up surface $\widetilde{\Theta}$ such that $K_{\widetilde{\Theta}} = \pi^* K_\Theta + E$. Pulling back $\mathscr{L}$ onto $\widetilde{\Theta}$ will show that a now vanishes to degree 4 on E and b vanishes to degree 6 on E. Introduce $\mathscr{L}' = \pi^*\mathscr{L} - E$ and write a and b in terms of $\mathscr{L}'$ instead of $\mathscr{L}$. Now a and b will not vanish at all on a generic point on E. Note that the effect on K_X of blowing up the base, Θ, and then subtracting E from $\mathscr{L}$ nicely cancels out and so X is still Calabi–Yau.

Thus, to obtain a smooth X we need to blow up all 24 points of collision. This process is shown in the lower part of figure 10. The dotted lines represent the 24 new $\mathbb{P}^1$'s introduced into the base. Note that they are not part of the discriminant.

From our interpretation of moduli in section 6.2 we see that in terms of the underlying six-dimensional field theory we have 24 new moduli from tensor multiplets as soon as we try to enhance the gauge group to $E_8 \times E_8$ in this way. Remember that in the language of the heterotic string this large gauge group was meant to be the result of shrinking down 24 instantons to zero size. What we have therefore shown here is that shrinking down 24 instantons to zero size results in the appearance of 24 new tensor moduli.

Given that the appearance of tensor multiplets should be a nonperturbative phenomenon in the heterotic string, it would seem unreasonable to expect them to appear when the target space and vector bundle is smooth. At least in the case that the underlying K3 surface is large, it is then clear that each tensor modulus can be tied to each shrunken instanton in this picture. That is, one point-like instanton will result in one tensor modulus appearing.

Suppose we try to give size to some of the instantons. This should correspond to a deformation of the complex structure of the Calabi–Yau threefold. We know that this should result in the disappearance of (at least) one of the tensor multiplets and thus (at least) one of the blow-ups in Θ. Thus the deformation has to disturb one of the curves of II* and thus lower the size of the effective gauge group. What is important to notice is that $12+n$ of the small instantons are thus embedded in the E_8 factor associated to C_∞ and $12-n$ of the instantons are embedded in the E_8 associated to C_0. To put this another way, when we smooth everything out to obtain a theory with no extra tensor moduli we expect to have an $(E_8 \times E_8)$-vector bundle which is a sum of two E_8-bundles, one of which has $c_2 = 12+n$ and the other with $c_2 = 12-n$.

This pretty well specifies exactly which heterotic string our type IIA string is dual to. For a G-bundle on a K3 surface, where G is semi-simple and simply-connected, the topological class of this bundle is specified by a map $H_4(\mathrm{K3}) \to \pi_3(G)$ (for a clear explanation of this see **[153]**). This may be viewed as given by the second Chern class of each sub-bundle associated to each factor of G. In our case we fix the total second Chern class and so the only freedom remaining is specified by how c_2 of the bundle is split between the factors of G. Thus n determines the class of our $E_8 \times E_8$ bundle. We have arrived at the following:

PROPOSITION 9. *Let E be a sum, $E_1 \oplus E_2$, of two E_8-bundles on a smooth K3 surface such that $c_2(E_1) = 12+n$ and $c_2(E_2) = 12-n$. Then a heterotic string compactified on this bundle on K3 is dual to (a limit of) a type IIA string compactified on a Calabi–Yau threefold which is an elliptic fibration with a section over the Hirzebruch surface $\mathbf{F}_n$.*

Our main assumption here is that the type IIA string on the Calabi–Yau manifold really is dual to a heterotic string. If it is, then we have certainly identified the correct one subject to the provisos of proposition 6.

This proposition first appeared in **[135]**. Although virtually all of the mathematics above has been copied from that paper our presentation has been slightly different. Rather than take the limit of decompactifying a T^2 in a heterotic string on $\mathrm{K3} \times T^2$, the line of attack in **[135]** was effectively to decompactify the dual type IIA string (or its mirror partner, the type IIB string) to a twelve-dimensional theory compactified on a Calabi–Yau manifold. It is not clear whether this twelve-dimensional "F-theory" exists in the usual sense of ten-dimensional string theory or eleven-dimensional M-theory or whether it serves simply as useful mnemonic for the above analysis. Another point of view of F-theory is to think of it as the type IIB string compactified down to six dimensions on Θ (see, for example, **[154]** for some nice results along these lines). The fact that Θ is not a Calabi–Yau space is corrected for by placing fixed D-branes within it. Again this is essentially equivalent to the above. The key ingredient to associate to the term "F-theory" is the elliptic fibration structure. Whether one wishes to think of this in terms of a mysterious twelve-dimensional theory or a type IIA string or a type IIB string is up to the reader.

The above proposition establishing the link between how the 24 of the second Chern class is divided between the two E_8's, and over which Hirzebruch surface X is elliptically fibred, is in agreement with all the relevant conjectured dual pairs of [**103**] and [**142**] for example. The case of $n = 12$ was established in [**132**].

The appearance of tensor moduli for small instantons was first noted in [**146**]. Since we are not allowing ourselves to appeal to M-theory or D-branes in these lectures we will not reproduce the argument here but just note that all necessary information appears to be contained in the type IIA approach we use here.

6.4. Aspects of the $E_8 \times E_8$ string. Let us now follow the analysis of [**135, 145**] and continue to explore the duality between the type IIA string on X and the $E_8 \times E_8$ heterotic string.

In the previous section we looked at the case of fixing hypermultiplet moduli in order to break none of the $E_8 \times E_8$ gauge group. We should ask the opposite question of what the gauge group is at a generic point in the hypermultiplet moduli space. We shall do this as follows. If any of the $E_8 \times E_8$ gauge group remains unbroken we expect either of our curves C_0 or C_∞ to contain part of the discriminant locus, Δ. Let us focus on C_0. Split off the part of the discriminant not contained in C_0 by putting

$$\Delta = NC_0 + \Delta', \tag{148}$$

where $N \geq 0$ and Δ' does not contain C_0. Since the only way that the intersection number of two algebraic curves in an algebraic surface can be negative is if one of the curves contains the other, we have

$$\Delta'.C_0 \geq 0. \tag{149}$$

Following (146) we have, for $n \geq 0$,

$$N \geq 12 - \frac{24}{n}. \tag{150}$$

Similarly we may analyze the divisors A and B to obtain the respective orders, L and K, to which a and b vanish on C_0. This gives

$$\begin{aligned} L &\geq 4 - \frac{8}{n} \\ K &\geq 6 - \frac{12}{n}. \end{aligned} \tag{151}$$

We may now use table 4 to determine the fibre over a generic point in C_0. Repeating this procedure for the other "primordial" E_8 along C_∞ shows that no singular fibres are required there for $n \geq 0$.

We see that in the case $n > 2$, we will have singular fibres over C_0 generating a curve of singularities within X. Thus we expect an enhanced gauge group. Loosely speaking, the gauge group can be read from the last column of table 4. The only thing we have to worry about is the monodromy of section 5.2 — it may be that there is monodromy on the singular fibres as we move about C_0. If $\Delta'.C_0 = 0$ then there can be no monodromy since the fibre is the same over every point of C_0. This occurs for $n = 2, 3, 4, 6, 8, 12$. When $n = 7, 9, 10, 11$, the fibre admits no symmetry

n	L	K	N	Fibre	Mon.	G	H_0
≤ 2	0	0	0	I_0			E_8
3	2	2	4	IV		$\mathfrak{su}(3)$	E_6
4	2	3	6	I_0^*		$\mathfrak{so}(8)$	$\mathfrak{so}(8)$
5	3	4	8	IV^*	$\mathbb{Z}_2$	F_4	G_2
6	3	4	8	IV^*		E_6	$\mathfrak{su}(3)$
7	3	5	9	III^*		E_7	$\mathfrak{su}(2)$
8	3	5	9	III^*		E_7	$\mathfrak{su}(2)$
≥ 9	4	5	10	II^*		E_8	

TABLE 5. Generic gauge symmetries, G.

and thus there cannot be any monodromy. Therefore, the only time we have to worry about monodromy is for the IV^* fibre in the case $n = 5$. There is indeed monodromy in this case [**149**]. (See [**155**] for an account of this in terms of "Tate's algorithm" or [**156**] for an alternative approach.) Thus, whereas one associates E_6 with a type IV^* fibre, this becomes F_4 from figure 5 when $n = 5$. The gauge algebras for generic moduli are summarized in table 5.

This agrees nicely with the heterotic picture. For a given value of c_2 of a bundle, find the largest possible structure group, H, of a vector bundle. Then the desired gauge group, G, will be the centralizer of H within $E_8 \times E_8$. One "rough and ready" approach to this question is as follows. Consider an H-bundle E with fibre in an irreducible representation, R, of the structure group. The Dolbeault index theorem on the K3 surface then gives [**9**]

$$\begin{aligned}\dim H^0(E) - \dim H^1(E) + \dim H^2(E) &= \int_S td(T) \wedge ch(E) \\ &= 2\,\mathrm{rank}(R) - l(R)c_2(E),\end{aligned} \tag{152}$$

where $l(R)$ is the index of R using the conventions of [**157**]. If E really is a strict H-bundle with fibre R, it should have no nonzero global sections, otherwise the structure group would be a strict subgroup of H. Thus $H^0(E)$ is trivial. Similarly, by Serre duality, we expect the same for $H^2(E)$. Thus, the right hand side of (152) must be non-positive. That is,

$$c_2(E) \geq \frac{2\,\mathrm{rank}(R)}{l(R)}, \tag{153}$$

for any irreducible representation, R. The bundle with $c_2 = 12 + n$ may have the full E_8 as structure group so one E_8 of the $E_8 \times E_8$ will be broken generically for any $n \geq 0$. Table 5 then shows how the other E_8 is broken down to G by a bundle with structure group H_0 and $c_2 = 12 - n$. For example, the $\mathbf{3}$ of $\mathfrak{su}(3)$ has $l(\mathbf{3}) = 1$ and so c_2 of a generic $\mathfrak{su}(3)$-bundle with no global sections is at least 6. This is why it appears on the row for $n = 6$ in the table.

Note that there need not exist a bundle that saturates the bound in (153) and so we cannot reproduce all of the rows in table 5. For example, in the case $n = 3$ one sees that (153) does not rule out a bundle with the full E_8 structure group but we see that only an E_6-bundle is expected. See [**133**] for a discussion of this. It is interesting to note that proposition 9 implies that $\mathfrak{su}(3)$ *must* appear as the gauge symmetry in the case $n = 3$ and so the E_8-bundle must not exist. If a smooth E_8-bundle on a K3 surface with $c_2 = 9$ is discovered it will violate proposition 9.

The cases $n = 9, 10, 11$ are somewhat peculiar since we appear to be suggesting that we have a bundle with trivial structure group and yet $c_2 > 0$. Clearly this is not possible classically. The fact that classical reasoning is breaking down somewhat can be seen from the fact that the reasoning of section 6.3 applies to this case and we have $12 - n$ tensor multiplets. That is, we have $12 - n$ point-like instantons which cannot be given nonzero size.

The fact that $n \leq 12$ can be understood from both the type IIA side and the heterotic side. In the case of our elliptic fibration over $\mathbf{F}_n$, if $n > 12$ then (L, K, N) as determined from (150) and (151) is at least $(4, 6, 12)$. As discussed in section 6.2, this means that we may redefine $\mathscr{L}$ to absorb C_0 to reduce the fibre to something in the list in figure 9. This kills $K_X = 0$ however and so we do not have a Calabi–Yau space. On the heterotic side, $c_2 < 0$ would be a clear violation of (153). One might worry about "point-like anti-instantons" but such objects would break supersymmetry and as such do not solve the equations of motion.

In section 5.7 we discussed extremal transitions between Calabi–Yau manifolds which, in the heterotic language, corresponded to unwrapping part of the gauge bundle around the K3 surface and rewrapping it around the T^2. Such transitions are not of much interest to us in this section as we are concerned only with the K3 part of the story. There are other possible extremal transitions which will effect us though. One kind which is of interest are ones which will take us from an elliptic fibration over $\mathbf{F}_n$ to another elliptic fibration over $\mathbf{F}_{n-1}$. In the heterotic string this will correspond to a transition from splitting the $E_8 \times E_8$ bundle into two bundles with c_2 equal to $12 + n$ and $12 - n$, to a splitting of $12 + n - 1$ and $12 - n + 1$ respectively. In this way we may "join up" all the theories considered so far into one connected moduli space. This phenomenon was first observed in [**146**] but we will again follow the argument as presented in [**135, 145**].

As explained in section 6.3, when the $E_8 \times E_8$ heterotic string has a point-like instanton, we expect the dual Calabi–Yau space for the type IIA string to admit a blow-up in the base, Θ, of the elliptic fibration. When all 24 instantons are point-like we saw this as a collision between a curve of II^* fibres and other parts of the discriminant locus as shown in figure 10. Let us concentrate on what happens when one of these points is blown up.

As in section 6.3, we use Δ' to denote the part of discriminant left over after we subtract the contribution from the two curves of II^* fibres. In the first diagram in figure 11 we show locally how Δ' loops around a collision between it and a line, C_∞, of II^* fibres together with the class, f, that passes though this point. Now when we do the blow-up by switching on a scalar in a tensor multiplet we go to the middle diagram. The exceptional divisor is a line of self-intersection -1. The line that was in the class f also becomes a (-1)-line. The middle diagram of figure 11 is obviously symmetric and we may blow-down the latter (-1)-curve to push the loop onto the bottom line of II^* fibres as shown in the last diagram.

The effect of this is to change C_∞ into a line of self-intersection $n - 1$ and C_0 into a line of self-intersection $-n + 1$. Thus we have turned $\mathbf{F}_n$ into $\mathbf{F}_{n-1}$. It is also clear that we have moved the small instanton from one of the E_8's into the other

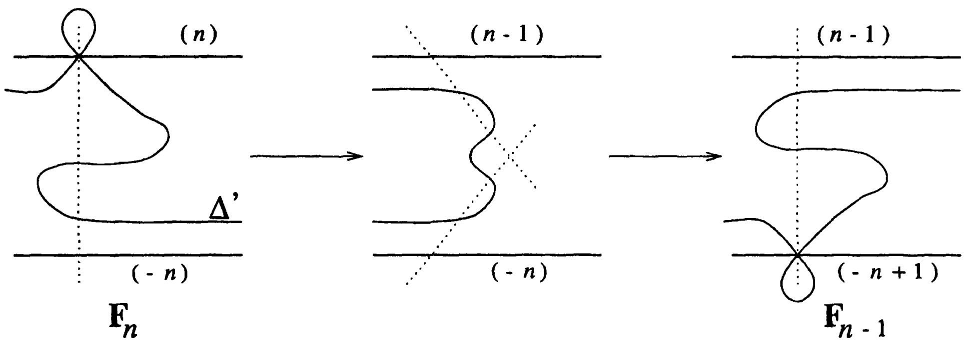

FIGURE 11. The transition $n \to n-1$.

E_8. Thus we may connect up all our theories which are elliptic fibrations over $\mathbf{F}_n$. Note that the use of tensor moduli means that we do not expect a perturbative interpretation of this process in the heterotic string language.

6.5. The Spin(32)/$\mathbb{Z}_2$ heterotic string. Now that we can content ourselves with the knowledge that we know how to build a type IIA dual to the generic $E_8 \times E_8$ heterotic string on a K3 surface we turn our attention to the Spin(32)/$\mathbb{Z}_2$ heterotic string. Whereas the topology of the $E_8 \times E_8$ bundle required specifying how the 24 instantons were divided between the two E_8's, the Spin(32)/$\mathbb{Z}_2$ heterotic string is quite different. In this case the gauge group is not simply-connected and the topology of the bundle is not simply specified by c_2.

It will be important to recall some fundamentals of the construction of the Spin(32)/$\mathbb{Z}_2$ heterotic string from [**158**]. The 16 extra right-movers of the heterotic string are compactified on an even self-dual lattice of definite signature. There are two such lattices, which we denote $\Gamma_8 \oplus \Gamma_8$ and Γ_{16}. The former is two copies of the root lattice of $E_8 \times E_8$ with which we have become well-acquainted in these talks. The second lattice is the "Barnes-Wall" lattice [**15**]. This may be constructed by supplementing the root lattice of $\mathfrak{so}(32)$ by the weights of one of its spinors. Such spinor weights are never of length squared 2 are so do not give massless states. Thus, as far as massless states are concerned, the lattice is the root lattice of SO(32) and the string states fill out the adjoint representation. Massive representations may fill out spinor representations for one of the spinors and but we never have representation in the vector representation or the other spinor representation. The gauge group can be viewed as a $\mathbb{Z}_2$ quotient of Spin(32) which does not admit vector representations. Hence π_1 of our gauge group is $\mathbb{Z}_2$. When we try to build Spin(32)/$\mathbb{Z}_2$-bundles the situation is very similar to real vector bundles over M where the fact that $\pi_1(\mathrm{SO}(d)) \cong \mathbb{Z}_2$ leads to the notion of the second "Stiefel-Whitney" class, w_2, of a bundle as an element of $H^2(M, \mathbb{Z}_2)$. Here we have a similar object characterizing the topology of the bundle which we denote $\tilde{w}_2$. If $\tilde{w}_2 \neq 0$ then bundles with fibre in the vector representation are obstructed just as spinor representations are obstructed for non-spin bundles with $w_2 \neq 0$. See [**159**] for a detailed account of this.

Thus, rather than being classified by how 24 is split between second Chern class, which was the case for the $E_8 \times E_8$ string, the topological class of an Spin(32)/$\mathbb{Z}_2$

heterotic string compactified over a K3 surface is characterized by $\tilde{w}_2 \in H^2(S, \mathbb{Z}_2)$. Whereas elements $\tilde{w}_2$ are in one-to-one correspondence with the homotopy classes of $\mathrm{Spin}(32)/\mathbb{Z}_2$-bundles on a fixed (marked) K3 surface, in our moduli space we are also allowed to vary the moduli of the K3 surface. Thus two elements of $H^2(S, \mathbb{Z}_2)$ should be considered to be equivalent if they can be mapped to each other by a diffeomorphism of the K3 surface. That is,

$$\tilde{w}_2 \in \frac{\Gamma_{3,19}/2\Gamma_{3,19}}{\mathrm{O}^+(\Gamma_{3,19})}. \tag{154}$$

It was shown in [**159**] that there are only three possibilities:

1. $\tilde{w}_2 = 0$,
2. $\tilde{w}_2 \neq 0$ and $\tilde{w}_2.\tilde{w}_2 = 0 \pmod 4$,
3. $\tilde{w}_2 \neq 0$ and $\tilde{w}_2.\tilde{w}_2 = 2 \pmod 4$.

We will focus on the case $\tilde{w}_2 = 0$ in this section. This will allow us to use the same arguments as in the last section about shrinking instantons down to retrieve the entire primordial gauge group. If $\tilde{w}_2$ were not zero, the topology of the bundle would obstruct the existence of arbitrary point-like instantons at smooth points in the K3 surface. See [**159**] for an account of this.

We have seen already that if a perturbatively understood heterotic string is dual to a limit of a type IIA string on Calabi–Yau threefold, X, then X must be an elliptic fibration with a section over $\mathbf{F}_n$, where $0 \leq n \leq 12$. Thus, if our duality picture is going to continue working for any $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string then we must have already encountered it in disguise as the $E_8 \times E_8$ heterotic string for a particular n.

The statement that the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string compactified on a K3 surface is the same thing as an $E_8 \times E_8$ heterotic string compactified on a K3 surface should not come as a surprise as the same thing has been known to be true for toroidal compactifications for some time [**61, 160**]. The identification of the dilaton as the size of the base of X as a K3 fibration has nothing to do with whether we deal with the $E_8 \times E_8$ or the $\mathrm{Spin}(32)/\mathbb{Z}_2$ string and so the duality between these theories cannot effect the string coupling. Thus there must be some T-duality statement that connects these two theories. This may be highly nontrivial however, as it may mix up the notion of what constitutes the base of the bundle over K3 and what constitutes the fibre. A construction of such a T-duality at a special point in moduli space was given in [**159**].

Now let us return to the issue concerning the extra states in the Γ_{16} lattice not contained in the root lattice of $\mathfrak{so}(32)$. What is the dual analogue of these extra massive states coming from the spinor of $\mathfrak{so}(32)$? Let us think in terms of the type IIA string compactified on a K3 surface versus the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string compactified on a 4-torus as in section 4.3. We may switch off all the Wilson lines of the heterotic string compactification by rotating the space-like 4-plane, Π, so that $\Gamma_{4,20} \cap \Pi^\perp \cong \Gamma_{16}$. Now view this as the "fibre" of the type IIA string compactified on X versus the heterotic string compactified on $\mathrm{K3} \times T^2$. This restores the full $\mathfrak{so}(32)$ gauge symmetry and so can be thought of as shrinking down all 24 instantons. As discussed in section 5.5 we may now vary the T^2 and the Wilson lines by varying the 2-plane, $\mho$, in $\Upsilon \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{2,\rho}$, where Υ is the quantum Picard lattice of the generic fibre of X as a K3 fibration. Thus we see that $\Upsilon \cong \Gamma_{2,2} \oplus \Gamma_{16}$. In other words, *the Picard lattice of the generic K3 fibre of X is $\Gamma_{1,1} \oplus \Gamma_{16} \cong \Gamma_{1,17}$ and is therefore self-dual.*

Let the limit of a type IIA string compactified on X_0 be dual to the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string compactified on a K3 surface (times a 2-torus of large area) with $\tilde{w}_2 = 0$ and all the instantons shrunk down and let X be the blow-up of X_0. This theory will have an $\mathfrak{so}(32)$ gauge symmetry. We know the following:

1. X is a K3 fibration and an elliptic fibration with a section over a Hirzebruch surface.
2. X_0 contains a curve of singularities of type D_{16}.
3. The generic K3 fibre of X has a self-dual Picard lattice (of rank 18).

Let us construct X.

Table 4 tells is that the base, Θ, of X as an elliptic fibration will contain a curve of I_{12}^* fibres. We may put this curve along C_0, the isolated section of $\mathbf{F}_n$. A generic fibre of X as a K3 fibration will be a K3 surface built as an elliptic fibration with a I_{12}^* fibre (and, generically, 6 I_1 fibres). Let us denote this K3 surface as S_t. What is $\mathrm{Pic}(S_t)$?

Let σ denote the section of S_t as an elliptic fibration "at infinity" guaranteed by the Weierstrass form. Let R be the sublattice of $\mathrm{Pic}(S_t)$ generated by the irreducible curves within the fibres not intersecting σ. Let Φ be the set of sections. One may show [**161**]

$$\mathrm{disc}(R) = |\Phi|^2 \,\mathrm{disc}(\mathrm{Pic}(S_t)), \tag{155}$$

where disc denotes the "discriminant" of a lattice, i.e., the determinant of the inner product on the generators.

In our case, R is generated by a set of rational curves forming the Dynkin diagram for D_{16}. Thus, $\mathrm{disc}(R)$ is the determinant of the Cartan matrix of D_{16} which is 4. In order for $\mathrm{Pic}(S_t)$ to be unimodular we see that we require exactly two sections. Writing an elliptic fibration with two sections in Weierstrass form is easy. We are guaranteed one section at infinity. Put the other section along $(y = 0, x = p(s,t))$. Thus the general Weierstrass form with two sections is

$$y^2 = (x - p(s,t))(x^2 + p(s,t)x + q(s,t)), \tag{156}$$

where

$$\begin{aligned} a(s,t) &= q(s,t) - p(s,t)^2 \\ b(s,t) &= -p(s,t)q(s,t). \end{aligned} \tag{157}$$

From (146) the divisors P and Q, given by the zeros of p and q, are in the class $4C_0 + (4+2n)f$ and $8C_0 + (8+4n)f$ respectively. The discriminant is

$$\begin{aligned} \delta &= 4a^3 + 27b^2 \\ &= (q + 2p^2)^2(4q - p^2). \end{aligned} \tag{158}$$

The fact that the discriminant factorizes will have some profound consequences. In terms of divisor classes let us write $\Delta = 2M_1 + M_2$, where M_1 is the divisor given by $q + 2p^2$ and M_2 corresponds to $4q - p^2$.

We know that $2M_1 + M_2$ contains $18C_0$ from the I_{12}^* fibres. Using the fact that a and b vanish to an order no greater than 2 and 3 respectively along C_0 and the fact that M_1 and M_2 must contain a nonnegative number of C_0 and f, one may show that M_1 contains $8C_0$ and M_2 contains $2C_0$ as their contributions towards the I_{12}^* fibres. What remains is

$$\begin{aligned} M_1' &= (8+4n)f \\ M_2' &= 6C_0 + (8+4n)f. \end{aligned} \tag{159}$$

Putting $M_2'.C_0 \geq 0$ fixes $n \leq 4$. There are no other constraints.

We have not yet achieved our goal in determining what n is for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string. All we know is that $n \leq 4$. Note however that along the $8+4n$ zeros of M_1 we have a double zero of Δ. Thus, generically we have $8+4n$ parallel lines along the f direction of I_2 fibres. Thus means the gauge algebra acquires an extra $\mathfrak{su}(2)^{8+4n}$ factor. The necessary appearance of an extra gauge symmetry can ultimately be traced to the way the determinant factorized. We will denote this as $\mathfrak{sp}(1)^{8+4n}$ to fit in with later analysis. It is very striking how different this behaviour is from the $E_8 \times E_8$ analysis of the last section. In the latter case we acquired 24 massless tensors when the instantons were shrunk down to zero size, whereas for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string we acquire new massless gauge fields.

The natural thing to do would be to identify each $\mathfrak{sp}(1)$ with a small instanton. To do this fixes $n = 4$ as one can have 24 small instantons, at least in the case that the underlying K3 surface is smooth and $\tilde{w}_2 = 0$. Thus we arrive at the following

PROPOSITION 10. *The* $\mathrm{Spin}(32)/\mathbb{Z}_2$ *heterotic string compactified smoothly on a K3 surface with* $\tilde{w}_2 = 0$ *is dual to (a limit of) the type IIA string compactified on an elliptic fibration over* $\mathbf{F}_4$.

This proposition depends on the assumptions governing proposition 6 and the somewhat schematic way we associated each of the $\mathfrak{sp}(1)$ factors to a small instanton to show $n = 4$.

Another way [**135**] to argue $n = 4$ is that the generic point in moduli space of $n = 4$ theories has gauge symmetry $\mathfrak{so}(8)$ and then compare to the analysis in section 6.4. We can try to break as much of $\mathfrak{so}(32)$ as possible by finding the largest subalgebra of $\mathfrak{so}(32)$ consistent with (153). Since $\tilde{w}_2 = 0$ we may consider vector representations in which case $\mathfrak{so}(24)$ is the largest such algebra with $c_2 = 24$ and this breaks $\mathfrak{so}(32)$ down to $\mathfrak{so}(8)$. A glance at table 5 then confirms that $n = 4$.

Actually, the analysis of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string was first done by Witten [**152**], where the appearance of the $\mathfrak{sp}(1)$ factors from small instantons was argued directly. Unfortunately this involves methods beyond the scope of these talks. It is worth contrasting Witten's method with the above. It is a remarkable achievement of string duality that the same answer results.

We may consider a specialization of the moduli to bring together some of the components of M_1' along f. Suppose we bring k of the 24 lines together. This will produce an I_{2k} line of fibres. As usual we have to worry about monodromy to get the full gauge group. In this case the transverse collision between the I_{2k} line and the I_{12}^* line produces monodromy acting on the I_{2k} fibres to give an $\mathfrak{sp}(k)$ gauge symmetry.[32] This corresponds to k of the 24 point-like instantons *coalescing* [**152**].

[32] Actually, since the f curves are topologically spheres, there had better be more than one point within them around which there is nontrivial monodromy. There are three other points within each f curve where there is monodromy provided by non-transverse collisions of f with what is

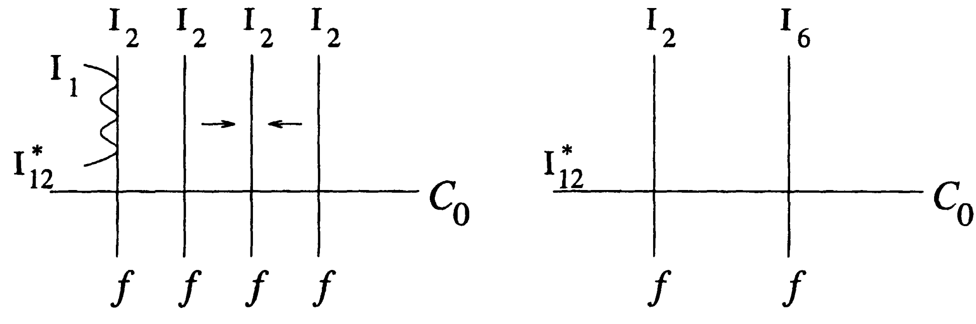

FIGURE 12. Three $\mathfrak{so}(32)$ point-like instantons coalescing to form $\mathfrak{sp}(3)$.

We show an example of this in figure 12. As an extreme case, all 24 instantons may be pushed to the same point. This results in a gauge algebra of $\mathfrak{so}(32) \oplus \mathfrak{sp}(24)$.

As we said above, it is interesting to contrast the behaviour of the point-like E_8 instantons with the point-like $\mathfrak{so}(32)$ instantons. The former produced massless tensor multiplets whereas the latter produce extra massless vector multiplets to enhance the gauge group. Of course, since both theories are meant to live in the same moduli space for $n = 4$, it should be possible to continuously deform one type of point-like instanton into another. This may well require a deformation of the base K3 surface as well as the bundle. The T-duality analysis of **[159]** should provide a good starting point for such a description.

6.6. Discussion of the heterotic string. We have analyzed both the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string compactified on a K3 surface in terms of a dual type IIA string compactified on an elliptically fibred Calabi–Yau threefold. In both cases we discovered curious nonperturbative behaviour associated to point-like instantons.

It is worth noting that in the case of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ string we appear to have acquired a free lunch concerning the analysis of the gauge group. To see this note the following. In old perturbative string theory a type IIA string compactified on a Calabi–Yau manifold never has a nonabelian gauge group. Similarly a heterotic string may have a subgroup of the original $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ as its gauge group together with perhaps a little more from massless strings. This latter can only have rank up to a certain value limited by the central charge of the corresponding conformal field theory (see, for example, **[108]** for a discussion of this). In the case of point-like instantons we have claimed above to have found gauge algebras as large as $\mathfrak{so}(32) \oplus \mathfrak{sp}(24)$. Such a gauge group must be nonperturbative from both the type IIA *and* heterotic point of view.

When discussing duality one would normally claim that its power lies in its ability to relate nonperturbative aspects of one theory to perturbative aspects of another. This then allows the nonperturbative quantities to be calculated. In the above we appear to have discovered things about a subject that was nonperturbative in both theories at the same time! How did we do this? The answer is that we first

left of the discriminant. We show this on one of the f curves in figure 12. We omit this on the other f curves to make the figure more readable.

related perturbative nonabelian gauge groups in the heterotic string to curves of orbifold singularities in the type IIA picture. To do this the curves of singularities were always formed by each K3 fibre of X, as a K3 fibration, acquiring a singular point. There is no reason why a curve of singularities need be in this form — it may be completely contained in a bad fibre and not seen by the generic fibre at all. Why should the type IIA string intrinsically care about the fibration structure? Assuming it does not, this latter kind of curve should have just as much right to produce an enhanced gauge group as those we could understand perturbatively from the heterotic string. This is exactly the type of curve that gives the $\mathfrak{sp}(24)$ gauge symmetry which is now nonperturbative in terms of the heterotic string.

The example of heterotic-heterotic duality studied in section 5.6 gives a clear picture of how nonperturbative effects are viewed in this way. Begin with a perturbative gauge group in the heterotic string. This corresponds to a curve of singularities in the type IIA's K3 fibration passing through each generic fibre. Heterotic-heterotic duality corresponds to exchanging the two $\mathbb{P}^1$'s in the base of the Calabi–Yau viewed as an elliptic fibration. This gives another K3 fibration but now the curve of singularities lies totally within a bad fibre. Thus it has become a nonperturbative gauge group for the dual heterotic string.

An issue which is very interesting but we do not have time to discuss here concerns the appearance of extra massless hypermultiplets at special points in the moduli space. This question appears to be rather straight forward in the case of hypermultiplets in the adjoint representation, as shown in [**162**]. The more difficult question of other representations has been analyzed in [**96, 145, 149, 155, 163**]. It is essential to do this analysis to complete the picture of possible phase transitions in terms of Higgs transitions in the heterotic string. As usual, in discussions about duality, it also hints at previously unsuspected relationships in algebraic geometry.

Another very important issue we have not mentioned so far concerns anomalies. The heterotic string compactified on a K3 surface produces a chiral theory with potential anomalies. This puts constraints on the numbers of allowed massless supermultiplets (see [**164**] for a discussion of this). One constraint may be reduced to the condition [**165, 166, 167**]

$$273 - 29n_T - n_H + n_V = 0, \tag{160}$$

where n_T, n_H and n_V count the number of massless tensors, hypermultiplets, and vectors respectively. The reason we have been able to ignore this seemingly important constraint is that, assuming one does the geometry of elliptic fibrations correctly, it always appears to be obeyed. At present this appears to be another string miracle!

As an example consider the following. Compactify the $E_8 \times E_8$ heterotic string, with c_2 split as $12+n$ and $12-n$ between the two E_8 factors, on a K3 surface so that the unbroken gauge symmetry is precisely E_8. This means that the corresponding elliptic threefold, X, has a curve of II* fibres which we may assume lies along C_0 in the Hirzebruch surface $\mathbf{F}_n$. We will calculate n_T, n_H, and n_V. What is left of the divisors A, B, and Δ after subtracting the contribution from the curve of II* fibres is given by

$$\begin{aligned} A' &= 4C_0 + (8+4n)f \\ B' &= 7C_0 + (12+6n)f \\ \Delta' &= 14C_0 + (24+12n)f. \end{aligned} \tag{161}$$

B' collides with C_0 generically $12-n$ times. Each of these points corresponds to a point-like instanton required to produce the E_8 gauge group. Each such point must be blown up within the base to produce a Calabi–Yau threefold. Thus we have $12-n$ massless tensor multiplets in addition to the one from the six-dimensional dilaton. Since the gauge group is E_8, we have 248 massless vectors furnishing the adjoint representation.

To count the massless hypermultiplets we require $h^{2,1}(X)$. It is relatively simple to compute $h^{1,1}(X)$ (assuming there is a finite number of sections) as this is given by 2, from the Hirzebruch surface, plus $12-n$ from the blow-ups within the base, plus 1 from the generic fibre, plus 8 from II* fibres. That is, $h^{1,1}(X) = 23-n$. Now $h^{2,1}$ may be determined from the Euler characteristic of X, $\chi(X) = 2(h^{1,1} - h^{2,1})$. To find this, recall that the Euler characteristic of a smooth bundle is given by the product of the Euler characteristic of the base multiplied by the Euler characteristic of the fibre. Thanks to the nice way Euler characteristics behave under surgery, we may thus apply this rule to each part of fibration separately. The Euler characteristic of any fibre is given by N in table 4.

Over most of the base, the fibre is a smooth elliptic curve which has Euler characteristic zero. Thus only the degenerate fibres contribute to our calculation. We have a curve of II* fibres over C_0 which contributes 10×2. The rest of the contribution comes from Δ'. Let us determine the geometry of Δ'. Firstly we know that, prior to blowing up the base, Δ' has $12-n$ double points as seen in the upper part of figure 10. Secondly, whenever A' and B' collide, which happens at $A'.B' = 24n+104$ points, Δ' will have a cusp (assuming everything is generic). Naïvely, the Euler characteristic of Δ' can be given by the adjun

$$-\Delta'.(\Delta' + K) = -596 - 130n. \tag{162}$$

However, each cusp will increase this by value by 2 and each double point by 1. In addition, when we do the blow-up of the base, the double points will be resolved and an additional 1 must be added for each double point. Thus

$$\begin{aligned}\chi(\Delta') &= -596 - 130n + 2(24n + 104) + 2(12-n)\\ &= -364 - 84n.\end{aligned} \tag{163}$$

Over most of the points in Δ' the fibre will be type I_1 but there will be type II fibres over each cusp. Thus the Euler characteristic of X is given by

$$\begin{aligned}\chi(X) &= 10.2 + 1.(-364 - 84n - (24n+104)) + 2.(24n+104)\\ &= -240 - 60n,\end{aligned} \tag{164}$$

and so $h^{2,1}(X) = 143 + 29n$. To obtain the number of hypermultiplets we add one to this figure from the four-dimensional dilaton in the type IIA string. In summary, we have

$$\begin{aligned}n_T &= 13 - n\\ n_H &= 144 + 29n\\ n_V &= 248.\end{aligned} \tag{165}$$

It can be seen that this satisfies (160). See [**164, 145**] for more examples.

In section 6.5 we analyzed the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string compactified on a K3 surface with $\tilde{w}_2 = 0$. What about the case $\tilde{w}_2 \neq 0$? An example of this was studied in [**159**] based on the construction of Gimon and Polchinski [**168**]. This uses open string models which we have not discussed here. In this case, $\tilde{w}_2$ is dual to $\frac{1}{2}\sum_{i=1}^{16} C_i$, where $\{C_i\}$ are sixteen disjoint (-2)-curves. The fact that such a class lies in $H^2(S,\mathbb{Z})$ can be seen in [**16**]. Such a $\tilde{w}_2$ is conjectured to be dual to the case of the $E_8 \times E_8$ heterotic string with $n=0$. It would be interesting to analyze this from the elliptic fibration point if view presented here. One indication that things will work is that the algebra $\mathfrak{sp}(8)$ appears naturally in the nonperturbative group in the $n=0$ case from the analysis following (159). This agrees with the model of Gimon and Polchinski. The problem is that we have the full $\mathfrak{so}(32)$ present as a gauge symmetry which does not appear in Gimon and Polchinski's model. One can also show that massless tensor multiplets appear from the type IIA approach. These issues are resolved in [**156**].

An interesting issue we had no time to pursue is that concerning what happens as we take the coupling of the heterotic string to be strong. All of the above analysis was done for a weakly-coupled heterotic string. It has been noted that one can expect some kind of phase transition to occur as the coupling reaches a particular value [**133**]. This can be analyzed in the context of elliptic fibrations [**145**].

As concluding remarks to the discussion concerning the heterotic string on a K3 surface we should note that this analysis is far from complete. In the case of the type IIA and type IIB string we were able to give a fairly complete picture of the entire moduli space. For the heterotic string we have been able to study a few components of the moduli space and a few points at extremal transitions. A first requirement to study the full moduli space will be a classification of elliptic fibrations. A more immediate shortcoming in our discussion is that we do not yet have an explicit map between the moduli of the elliptically fibered Calabi–Yau manifold and the K3 surface and bundle on which the heterotic string is compactified. This must be understood before we can really answer questions such as what happens when point-like instantons collide with singularities of the K3 surface.

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