Introduction

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I. The theory of integrable systems has been an active area of mathematics for the past thirty years. Different aspects of the subject have fundamental relations with mechanics and dynamics, applied mathematics, algebraic structures, theoretical physics, analysis and spectral theory and geometry. Most differential geometers have some knowledge and experience with finite dimensional integrable systems as they appear in sympectic geometry (mechanics) or ordinary differential equations, although the reformulation of part of this theory as algebraic geometry is not as commonly known. There are two quite separate methods of extension of these ideas to partial differential equations; one based on algebraic constructions and one based on spectral theory and analysis. These are less familiar still to differential geometers. This volume is a collection of papers intended to introduce and explain some of the more geometric aspects of integrable systems to the community.

This introduction contains a short historical discussion of the early geometers' ideas on "integrable geometric constructions" followed by a brief discussion of the material in the papers. An excellent historical treatment of finite dimensional integrable systems can be found in Moser's article [1980 Mo]. Many expositions on integrable equations (cf. [1991 AC], [1997 Pa]) discuss the history of the Korteveg de Vries (KdV) equation, beginning with the physical observation of Russell [1834 R] and the discovery of the equations by Boussinesq [1871 Bo], Korteveg and de Vries [1895 KdV]. In addition to describing water waves, the KdV equation also arises as a the model for the universal limit of lattice vibrations as the spacing goes to zero. Fermi-Pasta-Ulam's [1955 FPU] surprising numerical experiments on an anharmonic lattice, and the ingenious explanation by Zabuski-Kruskal [1965 ZK] of the results in terms of "solitons" of the KdV equation were quickly followed by a ground-breaking paper of Gardner, Greene, Kruskal and Miura [1967 GGKM] that introduced the method of solving KdV using the inverse scattering transform for the Hill's operator. This represents the approach of applied mathematics to discovering equations, solutions, and their scattering properties. The local differential geometry of the 19th century, on the other hand, foreshadows to some extent modern algebraic constructions. Rather than giving a description of the development of the basic ideas, we describe some of the early geometry and include a bibliography which lists many of the important early papers and a selection from the years between 1975 and 1985 in chronological order. The few recent papers are those we specifically refer to in this introduction.

Many geometric equations are known to have integrable aspects, especially if one takes into account that most experts do not have a good definition of "integrable" as applied to partial differential equations, particularly elliptic examples. In addition to those we mention in our historical discussion, the equations for harmonic maps (sigma models) from surfaces into groups ([1978 ZMi], [1981 Ch], [1989 U]), harmonic tori in symmetric spaces [1993 BFDP], constant mean curvature surfaces in space forms ([1989 PS], [1991 Bo]), isometric immersions of space forms in other space forms ([1919 Ca], [1980 TT]), and the theory of affine spheres ([1910 Tz], [1997 BS]) and affine minimal surfaces ([1980 CT]) are all examples of "elliptic" integrable systems. Likewise, the Yang-Mills equations on C^2 or $E^{2,2}$, their reductions to monopole equations in three variables, and self-dual equations on surfaces are new equations which originated in the physics literature of the late 1970's [1987 Hi]. One atypical development illustrating the great variety in the geometric theory is the reduction of the global $SU(N)$ monopole equation on R^3 to an integrable ordinary differential equation by the physicist Nahm [1984, Do]. The ideas surrounding string theory result in a series of deep and not completely understood connections between representation theory of certain algebras and many of the more classical theories of integrable systems in mathematics. Most recently, supersymmetric quantum field theories produce in a natural way moduli spaces of vacua or ground states which have new geometry generated by the supersymmetry. Since the supersymmetry generalizes the classical symmetries which produce integrals for the Euler-Lagrange equations via Noether's theorem, the connection with integrability is perhaps not surprising. However, this does not explain entirely the use of integrable systems in hyper-Kähler geometry ([1992 Hi]), Seiberg-Witten theory ([1996 DW]), special Kähler geometry ([1996 F], [1997 CRTP], [1997 Fr]) and quantum cohomology ([1991 W], [1994 KM], [1994 RT]).

The 19th century geometers were mainly interested in the local theory of surfaces in R^3, which we might regard as the prehistory of these modern constructions. The sine-Gordon equation arose first through the theory of surfaces of constant Gauss curvature -1, and the reduced 3-wave equation can be found in Darboux's work on triply orthogonal systems of R^3 ([1866 Da]). In 1906, a student of Levi Civitas, da Rios, wrote a master's thesis, in which he modeled the movement of a thin vortex filament in a viscous liquid using the equations of a curve propagating in R^3 along its binormal ([1906 dR]). It was much later that Hasimoto ([1971 Ha]) showed the equivalence of this system with the nonlinear Schrödinger equation. Since the equations were rediscovered somewhat independently of their geometric history, the main contribution of the classical geometers lies in the methods they developed for constructing explicit solutions of these equations. Ribaucour, Bianchi, Lie, and Bäcklund worked out the theory of what are now called Backlund transformations ([1883 Ba]). Lie developed a general theory of transformations known as "Lie transformations" which includes as examples scaling transformations and Lorentz transformations ([1979 L]). Darboux transformations for the Hills operator (the isospectral operator for KdV) were constructed by Darboux ([1882 Da]), but had been found even earlier by Moutard ([1875 M]). Most of the integrable partial differential equations are known to have Bäcklund, Lie and Darboux transformations.

II. The three-wave equation and the sine-Gordon equation arise out of the study of the classical differential geometry of R^3. We take a short trip back to this subject. A triply orthogonal system on R^3 is a local coordinate system $X(x_1, x_2, x_3)$ on R^3 such that the Euclidean metric ds^2 on R^3 has the form

$$ds^2 = a_1(x)^2 dx_1^2 + a_2(x)^2 dx_2^2 + a_3(x)^2 dx_3^2$$

for some smooth functions a_1, a_2, a_3, i.e., $X_{x_i} \cdot X_{x_j} = a_i^2 \delta_{ij}$ for $1 \leq i, j \leq 3$. Then the three one-parameter families of surfaces in R^3, defined by setting the different x_i to be constant, are mutually orthogonal. Already in 1813, Dupin ([1813 Du]) published a proof that the x_i-curves are lines of curvature for these surfaces. The flatness of ds^2 implies that

$$\begin{cases} (\beta_{ij})_{x_j} + (\beta_{ji})_{x_i} + \beta_{ik}\beta_{kj} = 0, \\ (\beta_{ij})_{x_k} = \beta_{ik}\beta_{kj}, \ i, \ j, \ k \text{ distinct}, \end{cases}$$

where $\beta_{ij} = (a_i)_{x_j}/a_j$ for $i \neq j$. These are the *Equations of Lamé* or *The Lamé System*. Given a solution (β_{ij}) of the Lamé system, finding a triply orthogonal system with these (β_{ij}) means finding the (a_1, a_2, a_3) that satisfy

$$(a_i)_{x_j} = \beta_{ij} a_j, \quad \text{for all } i \neq j.$$

Solutions of this latter system depend on three functions of one variable. Thus, given one triply orthogonal system, we can derive from it infinitely many others, parametrized by three functions of one variable. These transformations of a triply orthogonal system are referred to as Combescure transformations ([1867 Co]).

Triply orthogonal systems are easy to construct, and consequently many solutions to the equations of Lamé are known. For example, let $f(x_1, x_2)$ be a surface in R^3 parametrized by line of curvatures coordinates, and let N be the unit normal field. Then $X(x_1, x_2, x_3) = f(x_1, x_2) + x_3 N(x_1, x_2)$ is an orthogonal coordinate system for a domain in R^3. Moreover, if the two fundamental forms of the surface f are given by

$$I = b_1^2 dx_1^2 + b_2^2 dx_2^2, \quad II = \lambda_1 b_1^2 dx_1^2 + \lambda_2 b_2^2 dx_2^2,$$

then the Euclidean metric of R^3 written in this coordinate is $ds^2 = \sum_{i=1}^3 a_i^2 dx_i^2$, where

$$a_i(x_1, x_2, x_3) = (1 - \lambda_i(x_1, x_2)x_3)b_i(x_1, x_2), \quad \text{for } i = 1, 2, \quad a_3 = 1.$$

In this case, the equations of Lamé for $\beta_{ij} = (a_i)_{x_j}/a_j$ are precisely the Gauss and Codazzi equations of the surface f.

Darboux considered the Lamé system restricted to the case when $\beta_{ij} = \beta_{ji}$. This condition is easily seen to be equivalent to the condition that $ds^2 = \frac{\partial \phi}{\partial x_1} dx_1^2 + \frac{\partial \phi}{\partial x_2} dx_2^2 + \frac{\partial \phi}{\partial x_3} dx_3^2$ for some function ϕ. Darboux called a metric with this property an *Egorof* metric. In this case, The Lamé system is equivalent to the reduced 3-wave equation studied more recently by [1973 ZM]. Local analytic solutions of this system were classified by Cartan (cf. [1945 Ca]) using the methods which he and Kähler developed, now usually refered to as the "Cartan-Kähler theory". In fact, Cartan proved that (i) if c_1, c_2, c_3 are distinct, then the line $t(c_1, c_2, c_3)$ is non-characteristic, and (ii) that if L is a fixed, non-characteristic line, then given any three analytic functions f_1, f_2, f_3 defined on L, there exists a local analytic solution $\beta_{12}, \beta_{13}, \beta_{23}$ of the reduced 3-wave system such that $\beta_{12}, \beta_{13}, \beta_{23}$ are equal to f_1, f_2, f_3 on L. However, the ability to handle the global problem came only with the insights which come from scattering theory, and the solution of the problem with arbitrary smooth and rapidly decaying data is a recent development.

In the first construction we can find of a "Bäcklund transformation", Ribaucour ([1870 Ri]) produced in 1870 an interesting triply orthogonal system. We include a description of this result to give a flavor of the geometry of that time. Note that classical geometers referred to a surface in R^3 of constant Gauss curvature -1 as *pseudospherical*, and we will keep this terminology. Start with a surface M in R^3, and let C_p denote the unit circle in the physical tangent plane T_pM in R^3 centered at p. Ribaucour proved that if there exist a one-parameter family of surfaces that meet these tangent circles orthogonally, then, first of all both M and all the surfaces in the one-parameter family are pseudoshperical. In addition these pseudospherical surfaces are one of the families of surfaces in a triply orthogonal system. To describe this sytem, we let $f(u,v)$ denote the embedding of M in R^3 parametrized by line of curvature coordinates (u,v), with e_1 and e_2 the unit vectors along the lines of curvature in the directions of f_u and f_v respectively. Fix a base point p_0 in M, and find for each angle τ the surface P_τ which is orthogonal to C_{p_0} at the point

$$f(u_0, v_0) + \cos\tau \, e_1(u_0, v_0) + \sin\tau \, e_2(u_0, v_0).$$

Define the function $\theta(u, v, \tau)$ to be the angle between $f(u,v) + e_1(u,v)$ and $P_\tau(u,v) \cap C_{f(u,v)}$. Then

$$X(u, v, \tau) = f(u,v) + \cos\theta(u,v,\tau)e_1(u,v) + \sin\theta(u,v,\tau)e_2(u,v)$$

is an orthogonal coordinate system in R^3, and the surface defined by $\tau = \tau_0$ is P_{τ_0}.

The sine-Gordon equation first arose in the study of pseudospherical surfaces. If M is a surface of negative curvature, then there exist asymptotic coordinates (s, t). The Codazzi equations of a pseudospherical surface implies that the (s, t) can be chosen to be arc length. There is a reduction of the Gauss equation in which the fact that the Gauss curvature is -1 translates into the sine-Gordon equation, $q_{st} = \sin q$, where q is the angle between the s and t-curves. Thus, by The Fundamental Theorem of surfaces, there is a local bijective correspondence between solutions of the sine-Gordon equation and pseudospherical surfaces, up to rigid motion. The surfaces typically have singlarities and self-intersections. Modern computer graphics provide beautiful pictures of such surfaces which one believes the classical geometers would have appreciated very much.

Bäcklund transformations arose from a geometrical construction on pseudospherical surfaces, but to understand them we need to review yet more classical geometry. A *congruence of lines* is a two-parameter family of lines $L(u, v)$ in R^3. We describe the line $L(u, v)$ as the line through $x(u, v)$ and in the direction of $\xi(u, v)$ parameterized by t,

$$L(u,v): \quad x(u,v) + t\xi(u,v).$$

A surface M given by $Y(u,v) = x(u,v) + t(u,v)\xi(u,v)$ for some smooth function t is called a *focal surface* of the congruence of lines if the line $L(u, v)$ is tangent to M at $Y(u, v)$ for all (u, v). Hence $\xi(u, v)$ lies in the tangent plane of M at $Y(u, v)$, which is spanned by $x_u + t_u\xi + t\xi_u$ and $x_v + t_v\xi + t\xi_v$. This implies that t satisfies the following quadratic equation:

$$\det(\xi, x_u + t\xi_u, x_v + t\xi_v) = 0.$$

In general, this quadratic equation has two distinct solutions for t. Hence generically each congruence of lines has two focal surfaces, M and M^*. This results in a diffeomorphism $\ell : M \to M^*$ such that the line joining p and $p^* = \ell(p)$ is tangent to both M and M^*. We will call ℓ a *line congruence*. If you have difficulty understanding this construction, it is helpful to think through the construction of a congruence of lines (a one-parameter family) in R^2 and the single focal curve for this family.

A line congruence $\ell : M \to M^*$ is called *pseudospherical* with constant θ if the distance between p and $p^* = \ell(p)$ is $\sin\theta$ and the angle between the normal of M at p and the normal of M^* at p^* is θ for all $p \in M$. Bäcklund ([1883 Ba]) showed that if ℓ is a pseudospherical line congruence, then both M and M^* are pseudospherical, and ℓ maps asymptotic lines to asymptotic lines. However, the transformations come about from showing that this construction can always be realized. Given a pseudospherical surface, a constant θ, and a unit vector $v_0 \in TM_{p_0}$, not a principal direction, then there exists a unique surface M^* and a pseudospherical congruence $\ell : M \to M^*$ with constant θ such that $\overline{p_0p_0^*} = \sin\theta\, v_0$. Analytically this is equivalent to the statement that if q is a solution of the sine-Gordon equation, then the following overdetermined system of ordinary differential equations is solvable for q^*:

$$\begin{cases} q^*_x &= q_x + 4s\sin\left(\frac{q^*+q}{2}\right), \\ q^*_t &= -q_t + \frac{1}{s}\sin\left(\frac{q^*-q}{2}\right), \end{cases} \qquad (BT_{q,s})$$

where $s = \csc\theta - \cot\theta$. Moreover, a solution, q^* is again a solution of the sine-Gordon equation.

Bäcklund's theorem, but for $\theta = \frac{\pi}{2}$, was already known to Bianchi [1879 Bi]. In fact this case had been discovered even earlier (1870) in a different form by Ribaucour ([1870 Ri]). Note that if $X(u, v, \tau)$ is the Ribaucour triply orthogonal system defined above and τ is a fixed constant, then the map $\ell : M \to P_\tau$ defined by $\ell(f(u, v)) = X(u, v, \tau)$ is a pseudospherical congruence with $\theta = \pi/2$. However, Ribaucour did not realize that his theorem gave a method for constructing new pseudospherical surfaces from a given one.

The classic permutability Theorem for Bäcklund transformations is due to Bianchi (cf. [1909 Ei]). Given two pseudospherical congruences $\ell_i : M_0 \to M_i$ with angles θ_i respectively and $\sin\theta_1^2 \neq \sin\theta_2^2$, then there exist an algebraic construction of a unique surface M_3, and two pseudospherical congruences $\tilde{\ell}_i : M_i \to M_3$, such that $\tilde{\ell}_2\ell_1 = \tilde{\ell}_1\ell_2$. The analytic reformulation of this permutabiliy formula is the following. Suppose q is a solution of the sine-Gordon equaton and q_1, q_2 are two solutions of the above system of ordinary differential equations $(BT)_{q,s}$ with constants $s = s_1, s_2$ respectively, where $s_i = \csc\theta_i - \cot\theta_i$. The Bianchi permutability theorem gives a third solution to the sine-Gordon equation

$$q_3 = q + 4\tan^{-1}\left(\frac{s_1 + s_2}{s_1 - s_2}\tan\left(\frac{q_1 - q_2}{4}\right)\right).$$

What are now called Darboux transformations were discovered by Darboux during his investigation of Liouville metrics. A metric $ds^2 = A(x, y)(dx^2 + dy^2)$ is Liouville if there is a coordinate system (u, v) such that ds^2 is of the form

$$ds^2 = (f(u) - g(v))(du^2 + dv^2)$$

for some f and g of one varible. The classical geometers were interested in such metrics at least in part because Liouville had shown that all geodesics on such surfaces can be obtained by quadratures. The question of deciding whether a metric ds^2 is Liouville led to the study of the following special second order linear partial differential equation

$$w_{xy} = (f(x + y) - g(x - y))w.$$

Darboux was led to look for transformations of Hill's operators in the process of separating variables in this equation. The original analytic version of Darboux transformation ([1882 Da] and [1889 Da] Chap. 9) is the following: Let q be a smooth function of one variable, λ_0 a constant, and suppose that f satisfies $f'' = (q + \lambda_0)f$. Set

$$q^\sharp = f(f^{-1})'' - \lambda_0.$$

If $y(x, \lambda)$ is the general solution of the Hills operator with potential q:

$$y'' = (q + \lambda)y,$$

then $z = y' - (f/f')y$ is the general solution of the Hills operator with potential $q^\sharp$:

$$z'' = (q^\sharp + \lambda)z.$$

Next, suppose that we factor

$$D^2 - q - \lambda_0 = (D + v)(D - v),$$

In other words, suppose that v satisfies $v_x + v^2 = q + \lambda_0$. (Here $D = d/dx$.) Choose f so that $f'/f = v$. Then

$$(D - v)(D + v) = D^2 - q^\sharp - \lambda_0.$$

This Darboux theorem gives an algebraic algorithm (without quadrature) to transform general solutions of $D^2 - q - \lambda$ to those of $D^2 - q^\sharp - \lambda$.

Examples where the classical geometry appears in the modern theory of integrable systems are common and easily found. For example, orthogonal systems and their Combescure transformations arise in the local Hamiltonian theory of hydrodynamics of weakly deformed soliton lattices, and are known as the Whitham equations. The latter is the work of Dubrovin and Novikov ([1983 DN]) and Tsarëv ([1991 Ts]). An Egorof metric that is homogeneous of degree m, i.e., $\phi_{x_i}(rx) = r^m\phi_{x_i}(x)$ for all i, gives rise to a Frobenius manifold in the sense of Dubrovin, and such metrics give solutions to the WDVV equation in the study of Gromov-Witten invariants ([1991 W], [1995 RT], [1996 Du]). One of the best known examples is perhaps the Darboux transformations. If $q(x, t)$ is a solution of KdV then the Hills operators with potential $q(\cdot, t)$ are isospectral. It follows that the Darboux transformations of the Hills operators induce transformations on the space of solutions of KdV. This is a critical observation due to Adler and Moser ([1978 AM]) and Deift ([[1978 De]).

III. Many of the topics mentioned in the previous paragraphs will reappear in the papers in this volume, including the classical differential geometry. It is easy to understand the importance of the subject of integrable systems in modern mathematics once one notes that all of the papers except the last three mention either in passing or as fundamental motivation applications to constructions which are relevant in string theory and supersymmetric quantum field theory. Techniques from integrable systems, despite their limitations in classical geometry and applications, have provided one among the successful tools for rigorous mathematical interpretation of ideas in quantum field theory. It should be emphasized that it is mainly geometric ideas about classical geometric moduli spaces which can be treated. These spaces arise only as vacua for the field theories, but the special properties of the various field theories induce geometry on the moduli spaces which is not the usual Riemannian or symplectic geometry familiar to differential geometers. Because of the nature of the questions, the smooth part of the moduli space is not necessarily compact, so local algebraic results reminiscent of the classical 19th century have relevance if a more complete global analysis is not feasible.

Nigel Hitchin's paper "Integrable systems in Riemannian Geometry" starts with a discussion of what integrability means, and of the contrast between this century and the previous one. He goes on to discuss three problems familiar to differential geometers: constant mean curvature surfaces (tori) in R^3, Einstein metrics with certain symmetries in four dimensions, and hyper-Kähler manifolds. We meet loop groups in the section on tori, the flat Egorov metrics studied by Darboux in the section on Einstein manfiolds, and a moduli space (for a sypersymmetric quantum field theory) which has one of the special geometries, the hyper-Kähler one. The paper gives three different, beautiful illustrations of reduction of partial differential equations to finite dimensional flows and a nice explanation of "algebraically integrable".

The paper "Seiberg-Witten Integrable Systems" by Ron Donagi deals entirely with finite dimensional algebraically integrable systems which arise directly from a supersymmetric quantum field theory. The integrable system comes from the study of Higg's bundles on curves, and the paper is the only one to attempt a useful discussion of how an integrable system can relate to high energy physics. A host of references to the physics literature are given for the ambitious geometer, and a certain amount of algebraic geometry is needed to follow the details.

The paper "Five Lectures on Soliton Equations" by Edward Frenkel is motivated by the study of deformations of conformal field theories, and affine Lie algebras play a major role. The basic theory for the KdV equation, which is usally presented in terms of isospectral flows for a second order Hill's operator, is rederived for extensions of modified KdV, a system which is contained in the Non-linear Schrodinger hierarchy. This is particularly helpful in understanding the generality with which the algebraic constructions for KdV can be carried over to other classes of integrable systems and their Poisson brackets.

We are fortunate to include two papers by Boris Dubrovin which are most closely related to the classical geometry, but which in addition have important applications in two dimensional topological quantum field theory. In the paper "Differential Geometry of Moduli Spaces and Applications to Soliton Equations and Topological Conformal Field Theory", the relationship among Egorov metrics, n-wave equations, flat metrics on moduli spaces of curves with marked points and topological conformal field theories is discussed. The moduli space in the second paper "Differential Geometry of the Space of Orbits of a Coxeter Group" is a space of orbits, and carries the natural structure of a Frobenius manifold, a coordinate free description of the WDVV (Witten-Dijkgraaf, Verlinde, Verlinde) equations.

The emphasis in the paper "Symplectic Forms in the Theory of Solitons" by Igor Krichever and Duong Phong lies on the construction of a universal symplectic form, which can be specialized to many specific cases. We meet, for the first time, global questions in the identification of scattering data for the initial data given in two dimensions for the KP equation. A description of the WKB or Whitham flows on the moduli spaces of exact solutions leads into a discussion of leafs of the various moduli spaces of curves and applications to the topological Landau-Ginsburg models treated by Dubrovin. The paper includes applications to the Seiberg-Witten moduli spaces discussed by Donagi. The careful discussion of the analytic versus algebraic constructions is particularly helpful to geometers in the habit of thinking in terms of actual solutions to partial differential equations.

The final recent paper in the collection, "Poisson Actions and Scattering Theory" is our own. While partially motivated by questions from modern physics, it represents an attempt to reconcile the local algebraic methods with an understanding of global inverse scattering theory on the line. Reality conditions consistent with a compact group permit this, but restrict the number of integrable systems to whcih the global theory is applicable. We also give an explanation of Bäcklund (or Darboux) transformations in terms of the action of a rational subgroup of a loop group. The implication is that not all constructions are best described in terms of Lie groups and infinitesimal actions, especially if one is interested in global questions.

We have reprinted two influential papers which where originally published in 1984 and 1985. The paper "Loop groups and Equations of KdV type" by Graeme Segal and George Wilson is an important introduction to both subjects. A description of the KP hierarchy as commuting flows on an infinite dimensional Grassman manifold was originally due to Sato ([1981 S], [1983 DJKM]), but the work of Segal and Wilson is written up with a geometeric emphasis very much appreciated by some of us. We argued for the inclusion of the seminal paper "Scattering and Inverse Scattering for First Order Systems" by Richard Beals and Ronald Coifman. The algebraic geometry needed to treat the periodic theory is referred to by many authors in this volume, but it seems important for geometers to also have some understanding of analytic issues involved in constructing solutions to equations on non-compact domains.

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