# Scattering and Inverse Scattering for First Order Systems

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## Introduction

It is well-known that a number of important nonlinear evolution equations are associated to spectral problems for ordinary differential operators (see [1, 4]). The initial value problem for the evolution equation can, in principle, be solved by solving an inverse scattering problem. Schematically, the unknown function $u(\cdot, t)$ (possibly vector-valued) is identified with or transformed into the coefficients $q(\cdot, t)$ of a differential operator $L_t$. A spectral problem is associated to $L_t$ which carries (at least formally) some asymptotic information called the scattering data $v(\cdot, t)$. The original nonlinear evolution of $u$, or equivalently of $q$, corresponds to a trivially solvable linear evolution of the scattering data $v$.

The analytical theory of scattering and inverse scattering in various cases has been treated, for example, in [1], [6], [10], and other papers of these authors. It should be noted, though, that in much of the literature the expression "solvable by the inverse scattering method" designates evolutions associated to spectral problems for which certain purely formal scattering data would evolve linearly if it existed. The proposed scattering data may exist only for compactly supported or exponentially vanishing $q$, and the support condition or the vanishing condition may be destroyed by the evolution itself. In short, problems may have been termed "solvable" when neither the scattering map $q \mapsto v$ nor the inverse map $v \mapsto q$ has been seriously investigated. (For such problems one has recipes to produce special solutions, such as soliton or multi-soliton solutions, but the general initial value problem may be untouched.)

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A satisfactory analytical treatment of scattering and inverse scattering for a given spectral problem should aim for the following:

i. to formulate a notion of scattering data $v$ which is meaningful for (essentially) all reasonable coefficients $q$, such as $q \in L^1$;

ii. to show that $q \to v$ is injective;

iii. to characterize scattering data by determining all the algebraic or topological constraints such data satisfy;

iv. to show that for (essentially) each set of data satisfying the constraints, there is a corresponding $q$;

v. to discuss the relationship of such analytic properties of $q$ as smoothness or decay at $\infty$ with corresponding properties of $v$.

We summarize here some results of this nature on a class of spectral problems sometimes called generalized AKNS-ZS systems (named after [1] and [11]). This class is directly or indirectly related to most of the interesting nonlinear evolution equations which are said to be solvable by the inverse scattering method. The eigenvalue problem has the form

$$\frac{df}{dx} = zJf(x) + q(x)f(x), \quad z \in \mathbb{C}. \tag{1}$$

Here $f : \mathbb{R} \mapsto \mathbb{C}^n$, $J$ is a constant $(n \times n)$ matrix, and $q$ is a matrix-valued function. The $(2 \times 2)$ case was introduced by Zakharov and Shabat [11] in connection with the cubic nonlinear Schrödinger equation and was studied extensively by Ablowitz, Kaup, Newell, and Segur [1]. The formal theory of the $(n \times n)$ case, including the determination of the appropriate nonlinear evolutions of $q$, has been considered by a number of authors (see [5], [7]). The results described below seem to be new, in some respects, even for the $(2 \times 2)$ case.

Our results on the analytic theory of the scattering and inverse scattering problems for generalized AKNS systems are stated in detail in the first section.

The direct problem is treated in Sections 2–6. The case of compactly supported $q$ is studied in Section 2 and the case of $q$ with small $L^1$ norm in Section 3. The general case is obtained by limiting or patching methods in Sections 4 and 5. The consequences of smoothness of $q$ or decay of $q$ are studied in Section 6.

Sections 7–11 treat the inverse problem. The problem is reformulated as an integral equation in Section 7. The problem is solved for "small" data in Section 8, with refinements for smooth or decaying data in Section 9. In Sections 10 and 11 a rational approximation is used, together with the result for small data, to reduce the general inverse problem to a purely algebraic problem: a system of linear equations with $x$-dependent coefficients.

In Section 12 we consider systems with a symmetry and the relations between symmetry conditions on the potential and on the scattering data. We derive a formula of Hirota type (see [4], [9]) for the soliton and multi-soliton potentials for a system with symmetry.

We have benefitted from discussions with B. Dahlberg, P. Deift, C. Tomei, and E. Trubowitz. Several key observations, in particular the relationship of the winding number constraint to asymptotic solvability of the inverse problem, are due to D. Bar-Yaacov [2] in his work on the case when the matrix $J$ is skew adjoint.

# 1 Summary of Principal Results

We assume throughout that the matrix $J$ in (1) is diagonal, with distinct complex eigenvalues:

$$J = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \quad \lambda_j \neq \lambda_k \text{ if } j \neq k. \tag{1.1}$$

Let $\mathbf{P}$ denote the Banach space of $(n \times n)$ matrix-valued functions on $\mathbb{R}$ which are integrable and off-diagonal: $\mathbf{P} \ni q = (q_{jk})$, where

$$q_{jj} \equiv 0, \quad q_{jk} \in L^1(\mathbb{R}). \tag{1.2}$$

We refer to $q \in \mathbf{P}$ as a *potential.*

The spectral problem (1) leads to the problem of determining a fundamental matrix $\psi(x, z)$:

$$\frac{d}{dx}\psi(x,z) = zJ\psi(x,z) + q(x)\psi(x,z) \quad \text{a.e. } x, \tag{1.3}$$

$$\det \psi(x,z) \neq 0.$$

The desired solution is normalized to be of the form

$$\psi(x,z) = m(x,z)e^{xzJ}, \tag{1.4}$$

$$m(\cdot, z) \text{ bounded and absolutely continuous,} \tag{1.5}$$

$$m(x,z) \to I \text{ as } x \to -\infty.$$

Equation (1.3) is equivalent to

$$\frac{d}{dx}m = z[J,m] + qm \quad \text{a.e. x.} \tag{1.6}$$

Let $\Sigma$ be the following union of lines through the origin in $\mathbb{C}$:

$$\Sigma = \{z : \Re(z\lambda_j) = \Re(z\lambda_k), \text{ some } j \neq k\}. \tag{1.7}$$

**Theorem A.** Suppose $q$ belongs to $\mathbf{P}$.

(a) There is a bounded discrete set $Z \subset \mathbb{C}\backslash\Sigma$ such that for every $z \in \mathbb{C}\backslash(\Sigma \cup Z)$ the problem (1.4)–(1.6) has a unique solution $m(\cdot, z)$ and such that, for every $x \in \mathbb{R}$, $m(x, \cdot)$ is meromorphic in $\mathbb{C}\backslash\Sigma$ with poles precisely at the points of $Z$. Moreover, on $\mathbb{C}\backslash\Sigma$,

$$\lim_{z\to\infty} m(x, z) = I. \tag{1.8}$$

(b) There is a dense open set $\mathbf{P}_0 \subset \mathbf{P}$ such that if $q$ belongs to $\mathbf{P}_0$, then

Z is finite, (1.9)

the poles of $m(x, \cdot)$ are simple, (1.10)

distinct columns of $m(x, \cdot)$ have distinct poles, (1.11)

in each component $\Omega$ of $\mathbb{C}\backslash\Sigma, m(x, \cdot)$ has a continuous extension to $\overline{\Omega}\backslash Z$. (1.12)

The function $m$ is an eigenfunction for the matrix differential equation (1.6); we call it the *eigenfunction associated to q*. The elements of the dense open set $\mathbf{P}_0$ will be called *generic potentials.*

Let $\Omega_1, \Omega_2, \ldots, \Omega_r$ be the sectors which are the components of $\mathbb{C}\backslash\Sigma$, ordered in the positive sense about the origin. Let $\Sigma_\nu$ be the closed ray from the origin which one crosses in passing from $\Omega_\nu$ to $\Omega_{\nu+1}$ in the positive sense. According to (1.12), if $m(x, \cdot)$ is associated to a generic potential, it gives rise to two continuous functions on $\Sigma_\nu$:

$$m_\nu^-(x, \cdot) = \text{limit on } \Sigma_\nu \text{ from } \Omega_\nu, \tag{1.13}$$

$$m_\nu^+(x, \cdot) = \text{limit on } \Sigma_\nu \text{ from } \Omega_{\nu+1}, \quad (\Omega_{r+1} = \Omega_1). \tag{1.14}$$

**Theorem B.** Suppose $q$ is a generic potential with associated eigenfunction $m$.

(a) For $z \in \Sigma_\nu$ there is a unique matrix $v_\nu(z)$ such that, for all x,

$$m_\nu^+(x, z) = m_\nu^-(x, z)e^{xzJ}v_\nu(z)e^{-xzJ}. \tag{1.15}$$

(b) If $m(x, \cdot)$ has poles at $\{z_1, \cdots, z_N\}$, then for each $z_j$ there is a matrix $v(z_j)$ such that the residue satisfies

$$\text{Res } (m(x, \cdot); z_j) = \lim_{z\to z_j} m(x, z) \exp\{xz_jJ\}v(z_j) \exp\{-xz_jJ\}. \tag{1.16}$$

(c) The potential $q$ is uniquely determined by the functions $\{v_\nu\}$, the singularities $\{z_j\}$, and the matrices $\{v(z_j)\}$.

Given $q$ as in Theorem B we denote

$$v = (v_1, \cdots, v_r; z_1, \cdots, z_N; v(z_1), \cdots, v(z_N)) \tag{1.17}$$

and call $v$ the *scattering data* associated to $q$. Note that

$$v_\nu \in C(\Sigma_\nu), \quad v_\nu(z) \to I \text{ as } z \to \infty. \tag{1.18}$$

Part of the scattering data may be recovered from asymptotic information on the singular set $\Sigma$. Let $\Pi_\nu$ be the following projection in the matrix algebra:

$$(\Pi_\nu a)_{jk} = \begin{cases} a_{jk} & \text{if } \Re(z\lambda_j) = \Re(z\lambda_k), \quad z \in \Sigma_\nu, \\ 0 & \text{otherwise.} \end{cases} \tag{1.19}$$

**Theorem C.** Suppose $q$ is a generic potential with associated eigenfunction $m$. If $z$ is in $\Sigma_\nu$, then the limits

$$s_\nu^\pm(z) = \lim_{x \to +\infty} \Pi_\nu(e^{-xzJ} m_\nu^\pm(x, z) e^{+xzJ}) \tag{1.20}$$

exist and uniquely determine $v_\nu(z)$. Moreover, the set of functions $\{s_\nu^\pm\}$ determines the poles $\{z_1, \cdots, z_N\}$ and the columns which have singularities at these points. Conversely, this information determines the $\{s_\nu^\pm\}$.

To describe constraints on the scattering data we introduce additional notation. For any matrix $a$ we let $d_k^+(a)$ and $d_k^-(a)$ denote the upper and lower $(k \times k)$ principal minors:

$$d_k^+(a) = \det((a_{ij})_{i,j \le k}), \tag{1.21}$$

$$d_k^-(a) = \det((a_{ij})_{i,j > n-k}). \tag{1.22}$$

Given $z \in \Omega_\nu$, we introduce an ordering of the eigenvalues $\{\lambda_j\}$ so that $\Re(z\lambda_j)$ is strictly decreasing. Note that the induced ordering of the standard basis gives a new matrix representation of the matrix algebra, denoted

$$a \mapsto a^\nu. \tag{1.23}$$

Thus $a^\nu$ is the matrix $a$ after conjugation by a permutation matrix, and $J^\nu$ has its diagonal entries occurring in the $\nu$-ordering.

**Theorem D.** Suppose $q$ is a generic potential with scattering data $v$. Then

$$\Pi_\nu v_\nu(z) = v_\nu(z), \quad z \in \Sigma_\nu, \tag{1.24}$$

$$v_\nu(0) = a_\nu^{-1} a_{\nu+1}, \tag{1.25}$$

where $(a_\nu)_{jj} = 1$ and $(a_\nu)^\nu$ is upper triangular,

$$d_k^-(v_\nu(z)^\nu) = 1, \quad 1 \le k \le n, z \in \Sigma_\nu, \tag{1.26}$$

$$d_k^+(v_\nu(z)^\nu) \ne 0, \quad 1 \le k \le n, z \in \Sigma_\nu, \tag{1.27}$$

$$\text{if } z_i \text{ is in } \Omega_\nu, \text{ then } v(z_i)^\nu \text{ has a single non-zero entry which is in the } (k, k+1) \text{ position for some } k < n. \tag{1.28}$$

Moreover, let $\alpha_{\nu k}$ be the winding number of the $k$-th upper minor of $(v_\nu)^\nu$:

$$2\pi\alpha_{\nu k} = \int_{\Sigma_\nu} d[\arg d_k^+(v_\nu(z)^\nu)], \tag{1.29}$$

where $\Sigma_\nu$ is oriented from 0 to $\infty$. Let,

$$\beta_{\nu k}\text{=number of } z_i \in \Sigma_\nu \text{ such that } k\text{-th column of } v(z_i) \text{ is } \neq 0. \tag{1.30}$$

Then the $\{\alpha_{\nu k}, \beta_{\nu k}\}$ satisfy $n-1$ independent homogeneous equations

$$\Sigma(\epsilon_{\nu k,j}\alpha_{\nu k} + \eta_{\nu k,j}\beta_{\nu k}) = 0, \quad 1 \le j < n, \tag{1.31}$$

where the coefficients belong to $\{0, \pm 1\}$.

Some analytic properties of the scattering map are summarized in the next theorem.

**Theorem E.** Suppose $q$ is a generic potential with scattering data $v$ and suppose $k$ is a non-negative integer.

(a) If the distribution derivatives of $q$ satisfy

$$D^j q \in L^1, \quad 0 \le j \le k, \tag{1.32}$$

then

$$\lim_{z\to\infty} z^k[v_\nu(z) - I] = 0. \tag{1.33}$$

(b) If

$$x^k q \in L^1, \tag{1.34}$$

then

$$v_\nu \in C^k(\Sigma_\nu) \text{ and } D^j(v_\nu - I) \to 0 \text{ as } z \to \infty, \quad 0 \le j \le k. \tag{1.35}$$

Moreover, let $v_{\nu,k}$ be the Taylor polynomial of degree $k$ for $v_\nu$ at the origin. Then there are matrix-valued polynomials $a_{\nu k}$ as in (1.25), with

$$v_{\nu,k}(z) = a_{\nu,k}(z)^{-1}a_{\nu+1,k}(z) + O(z^k), \quad |z| \le 1. \tag{1.36}$$

(c) If

$$x^j q \in L^1 \cap L^2, \quad 0 \le j \le k, \tag{1.37}$$

then in addition to (1.35) we have

$$D^j(v_\nu - I) \in L^2(\Sigma_\nu), \quad 1 \le j \le k. \tag{1.38}$$

Let $\mathcal{S}(\mathbb{R})$ denote the usual Schwartz space (of matrix-valued functions) and let $\mathcal{S}(\Sigma_\nu)$ denote the space of functions each of whose derivatives is continuous on the closed ray $\Sigma_\nu$ and is rapidly decreasing at $\infty$. Theorem E shows that for a generic potential belonging to $\mathcal{S}(\mathbb{R})$, $v_\nu$ is rapidly decreasing at $\infty$ and all its derivatives are bounded. A rapidly decreasing function with bounded second derivative has rapidly decreasing first derivative; thus we have the following:

**Theorem E′.** *If $q$ is a generic potential belonging to $\mathcal{S}(\mathbb{R})$, then each $v_\nu$ belongs to $\mathcal{S}(\Sigma_\nu)$, and* (1.36) *holds for every integer $k \geq 0$.*

For the inverse problem we introduce the space $\mathbf{S}$ of *formal scattering data*, consisting of elements $v$ of the form (1.17), satisfying the following:

$$I - v_\nu, \quad Dv_\nu \in L^2(\Sigma_\nu); \tag{1.39}$$

this implies

$$v_\nu \to I \text{ as } z \to \infty, \tag{1.40}$$

$$\text{conditions (1.24)–(1.28) and (1.31) hold.} \tag{1.41}$$

In (1.39), $Dv_\nu$ is the distribution derivative along the ray. Condition (1.39) implies (after correction on a set of measure zero) that $v_\nu$ is bounded and belongs to the Hölder space $C^{1/2}(\Sigma_\nu)$, so (1.25) makes sense.

Note that from Theorems D and E it follows that if $q$ is a generic potential and satisfies (1.37) with $k = 2$, then its scattering data belongs to $\mathbf{S}$.

There are two natural ways to topologize $\mathbf{S}$, each of them being, for each fixed $N$, a product topology. The components $v_\nu$ receive the obvious topology, the components $v(z_j)$ the natural matrix topology, and the $z_j$ either the usual $\mathbb{C}$-topology *or* the discrete topology. The map from generic potentials satisfying (1.37) with $k = 2$ to $\mathbf{S}$ is continuous with respect to the first topology on $\mathbf{S}$, while the set $\mathbf{S}_0$ below is open and dense with respect to either topology.

Given $v \in \mathbf{S}$ and $x \in \mathbb{R}$ we look for a matrix-valued function $m(x, \cdot)$ such that

$$m(x,\cdot) \text{ is meromorphic on } \mathbb{C}\backslash\Sigma \text{ with poles at } z_1, \cdots, z_N, \tag{1.42}$$

$$\text{conditions (1.12), (1.15), and (1.16) hold.} \tag{1.43}$$

**Theorem F.** Suppose $v$ belongs to $\mathbf{S}$.

(a) For any real $x$ there is at most one associated eigenfunction; for $|x|$ large, there is exactly one.

(b) There is a dense open set $\mathbf{S}_0 \subset \mathbf{S}$ such that for every $v \in \mathbf{S}_0$ the associated eigenfunction exists for every real $x$. Moreover, $m(\cdot, z)$ is absolutely continuous with respect to $x$ for all $z \notin \Sigma \cup \{z_1, \cdots, z_N\}$ and satisfies the differential equation (1.6), where $q$ is an off-diagonal matrix-valued function with

$$(1 + |x|)q \in L^2 + L^\infty. \tag{1.44}$$

The following is a partial converse of Theorem E.

**Theorem G.** Suppose $v$ belongs to $\mathbf{S}_0$ and suppose $q$ is the corresponding potential.

(a) If $z^k(v_\nu - I) \in L^2(\Sigma_\nu)$, all $\nu$, then the distribution derivatives of $q$ satisfy

$$D^j q \in L^2 + L^\infty, \quad 0 \le j \le k. \tag{1.45}$$

(b) Suppose the distribution derivatives of $v$ satisfy

$$D^j v_\nu \in L^2, \quad 0 < j \le k+1. \tag{1.46}$$

Let $v_{\nu,k}$ be the Taylor polynomial of degree $k$ for $v_\nu$ at the origin, and suppose (1.36) holds. Then

$$x^{k+1} q \in L^2 + L^\infty. \tag{1.47}$$

As above, Theorem G has the following consequence for Schwartz class scattering data.

**Theorem G′.** *Suppose $v$ belongs to $\mathbf{S}_0$. Suppose also that each $v_\nu$ belongs to $\mathcal{S}(\Sigma_\nu)$ and that* (1.36) *holds for every integer $k \ge 0$. Then the corresponding potential belongs to $\mathbf{S}(\mathbb{R})$.*

The conditions defining $\mathbf{S}$ are preserved under the type of nonlinear evolution of $q$ which is associated to the spectral problem (1.3). Indeed the scattering data $v(\cdot, t)$ for such an evolution evolves with singularities $\{z_j\}$ fixed with

$$\frac{\partial v_\nu}{\partial t}(z,t) = [\mu(z), v_\nu(z,t)], \tag{1.48}$$

$$\frac{\partial v}{\partial t}(z_j,t) = [\mu(z_j), v(z_j,t)], \tag{1.49}$$

where $\mu$ is a diagonal-valued function. Thus

$$v_\nu(z,t) = e^{t\mu(z)} v_\nu(z,0) e^{-t\mu(z)}, \tag{1.50}$$

a similar expression holding for the $v(z_j,t)$. The algebraic conditions on $v$ are clearly invariant under conjugation by a diagonal matrix. Therefore the flow (1.48)–(1.49) maps $\mathbf{S}$ to itself (continuously with respect to either topology) if and only if

$$\mu \text{ is continuous}, \tag{1.51}$$

$$\Re \Pi_\nu \mu(z) = 0, \quad z \in \Sigma_\nu. \tag{1.52}$$

In general, $\mathbf{S}_0$ is not invariant under (1.48), (1.49). For example, if $v_\nu \equiv 0$, all $\nu$, then $q$ is of the form

$$\begin{aligned} q(x,t) &= R(e_1(x,t), \cdots, e_N(x,t)), \\ e_j(x,t) &= \exp\{\tau_j t + \xi_j x\}, \end{aligned} \tag{1.53}$$

where $R$ is a rational function. Depending on the data, $q$ may have algebraic singularities for some real $x$ and $t$.

# 2 Compactly Supported Potentials

With the assumptions and notation of Section 1, we investigate the solution of the eigenvalue problem (1.5), (1.6) when the potential has compact support.

**Proposition 2.1.** *Suppose $q \in \mathbf{P}$ has compact support. For each complex $z$ there is a unique absolutely continuous $m_0(\cdot, z)$ such that*

$$\frac{d}{dx} m_0(x,z) = z[J, m_0(x,z)] + q(x) m_0(x,z), \quad \text{a.e. } x, \tag{2.2}$$

$$m_0(x,z) = I \text{ if } x \ll 0. \tag{2.3}$$

*Moreover, $m_0(x, \cdot)$ is an entire function and*

$$m_0(x,z) = e^{xzJ} s_0(z) e^{-xzJ}, \quad x \gg 0, \tag{2.4}$$

$$\det m_0 \equiv 1. \tag{2.5}$$

Proof: Clearly $m_0$ is the (unique) solution of the Volterra integral equation

$$m_0(x,z) = I + \int_{-\infty}^{x} e^{(x-y)zJ} q(y) m_0(y,z) e^{(y-x)zJ}\, dy. \tag{2.6}$$

The dependence on the parameter $z$ is holomorphic. To prove (2.4) we note that

$$\frac{d}{dx} m_0 = z[J, m_0], \quad x \gg 0. \tag{2.7}$$

To prove (2.5) we let $\psi_0(x,z) = m_0(x,z) e^{xzJ}$. Then

$$\frac{d}{dx} \psi_0 = (zJ + q)\psi_0 \tag{2.8}$$

so that

$$\frac{d}{dx}(\det \psi_0) = z \text{ tr } J \cdot \det \psi_0 \tag{2.9}$$

and thus $\det m_0$ is constant with respect to $x$.

Note that (2.6) gives

$$s_0(z) = I + \int_{\mathbb{R}} e^{-yzJ} q(y) m_0(y,z) e^{yzJ}\, dy. \tag{2.10}$$

**Proposition 2.11.** *Suppose $q \in \mathbf{P}$ has compact support. Then the eigenvalue problem (1.5), (1.6) has a unique solution $m(\cdot, z)$ for every $z \in \mathbb{C}\backslash(\Sigma \cup Z)$, where $Z \subset \mathbb{C}\backslash\Sigma$ is discrete. Moreover, $m(x, \cdot)$ is meromorphic on $\mathbb{C}\backslash\Sigma$.*

Proof: We look for $m$ of the form

$$m(x,z) = m_0(x,z)a_0(x,z). \tag{2.12}$$

Since $m_0$ is invertible we must have

$$\frac{d}{dx}a_0 = z[J, a_0], \tag{2.13}$$

so $a$ must have the form

$$a_0(x,z) = e^{xzJ}a(z)e^{-xzJ}. \tag{2.14}$$

Conversely, (2.12) and (2.14) imply that $m$ is an eigenfunction. Now

$$m(x,z) = e^{xzJ}a(z)e^{-xzJ}, \quad x \ll 0, \tag{2.15}$$

$$m(x,z) = e^{xzJ}s_0(z)a(z)e^{-xzJ}, \quad x \gg 0. \tag{2.16}$$

In order to have $m$ bounded as $x \to -\infty$ it is necessary and sufficient, in view of (2.15) that

$$a(z)_{jk} = 0 \text{ if } \Re(z\lambda_j) < \Re(z\lambda_k). \tag{2.17}$$

In terms of the $\nu$-representation introduced in Section 1, this condition is

$$a(z)^\nu \text{ is upper triangular if } z \in \Omega_\nu. \tag{2.18}$$

Similarly, (2.16) and boundedness as $x \to +\infty$ become

$$s_0(z)^\nu a(z)^\nu \text{ is lower triangular if } z \in \Omega_\nu. \tag{2.19}$$

Convergence to the identity as $x \to -\infty$ requires

$$a(z)^\nu_{jj} = 1, \quad 1 \le j \le n. \tag{2.20}$$

Thus (2.12), (2.14), (2.18)–(2.20) are necessary and sufficient conditions. The algebraic conditions (2.18)–(2.20) determine $a^\nu$, and thus $a$, uniquely, provided that the upper minors do not vanish (see [8], Theorem 1.1):

$$d^+_k(s_0(z)^\nu) \ne 0, \quad 1 \le k \le n, z \in \Omega_\nu. \tag{2.21}$$

Thus $Z$ is precisely the set where (2.21) fails.

*Remark 2.22.* It will be shown in Section 4 that $Z$ is finite. It is clear from this construction that starting from a given sector $\Omega_\nu$, the function $m$ has an extension which is meromorphic in all of $\mathbb{C}$. In particular, except for a discrete set $Z_\nu$, in a ray $\Sigma_\nu$, the limits (1.13) and (1.14)exist. These limits again have determinant 1, and by differentiating the expression $(m^-_\nu)^{-1}m^+_\nu$, we see that it satisfies the equation (2.13). Thus there is $v_\nu(z)$ such that

$$m^+_\nu(x,z) = m^-_\nu(x,z)e^{xzJ}v_\nu(z)e^{-xzJ}, \quad z \in \Sigma_\nu \backslash Z_\nu. \tag{2.23}$$

**Definition 2.24.** Let $q, m, Z$ be as in Proposition 2.11. A singularity $z \in Z$ is *simple* if it is a simple pole for $m$ and only one column of $m$ is singular at $z$.

**Proposition 2.25.** *Suppose $z \in Z \cap \Omega_\nu$ is a simple singularity for $m$. Then there is a matrix $v(z)$ such that*

$$\text{Res } (m(x,\cdot); z) = \lim_{z' \to z} m(x, z') e^{xzJ} v(z) e^{-xzJ}. \tag{2.26}$$

*Moreover, $v(z)^\nu$ is of the form $ce_{k,k+1}$, where $c$ is a constant and $e_{k,k+1}$ a matrix unit.*

Proof: For convenience, replace the original matrix representation with the $\nu$-representation. Thus in $\Omega_\nu$, $m$ is of the form (2.12), (2.14), where

$$a^\nu = I + u, \quad u \text{ strictly upper triangular.} \tag{2.27}$$

It is easily seen that Res $(m(x,\cdot), z)$ satisfies the differential equation (1.6), and it follows that

$$\text{Res } (m(x,\cdot); z) = m_0(x,z) e^{xzJ} v_0(z) e^{-xzJ}. \tag{2.28}$$

Suppose that it is the column $k+1$ of $m$ which is singular at $z$. Taking $x \ll 0$ we see that $v_0$ is the residue of $u$ at $z$; thus $v_0$ is strictly upper triangular, with only column $k+1$ non-null. Let

$$p_k = e_{11} + e_{22} + \cdots + e_{kk}.$$

Then $up_k$ has no singularity at $z$ and we may define

$$v(z) = (1 + u(z)p_k)^{-1} v_0(z). \tag{2.29}$$

In view of (2.12), (2.14), and (2.27)–(2.29), it is clear that (2.26) holds. To see that $v$ has the given form we use the fact that, in $\Omega_\nu$, $m$ is lower triangular for $x \gg 0$. Therefore,

$$\begin{aligned} 0 &= p_k \text{ Res } (s_0(1+u); z) \\ &= p_k s_0(z)(1 + u(z)p_k) v(z) \\ &= p_k b(z) p_k v(z) = p_k b(z) v(z), \end{aligned}$$

where $b$ is lower triangular. By assumption the $(k-1) \times (k-1)$ upper minor of $s_0(z)$ is not zero, so the same is true for $b$. It follows that the first $k-1$ entries of column $k+1$ of $v$ are zero, and the proof is complete.

**Proposition 2.30.** *Suppose $q \in \mathbf{P}$ has compact support. In any neighborhood of $q$ there is a potential whose associated eigenfunction $m$ has properties* (1.10)–(1.12).

Proof: Suppose $r \in \mathbf{P}$ also has compact support. Let $M_n(\mathbb{C})$ be the $(n \times n)$ matrix algebra. Given $\zeta \in M_n(\mathbb{C})$, let $r_\zeta$ be the potential with entries $\zeta_{jk} r_{jk}$. Let $s_0$ be the matrix-valued function associated to $q$ as in the proof of Proposition 2.1, and let $s_{0,\zeta}$ be the corresponding function associated to $r_\zeta$. Note that (2.10) implies

$$s_{0,\zeta}(z) = I + \int_{\mathbb{R}} e^{-yzJ} r_\zeta(y) e^{yzJ}\, dy + O(|\zeta|^2). \tag{2.31}$$

Now suppose that the support of $r$ lies to the right of supp $(q)$. Then it is easily seen that the eigenfunction for $q + r_\zeta$ which satisfies (2.3) is

$$m_{0,\zeta}(x, z) e^{xzJ} s_0(z) e^{-xzJ}, \quad x \gg 0, \tag{2.32}$$

where $m_{0,\zeta}$ is the corresponding eigenfunction for $r_\zeta$. Thus the asymptotic matrix $s_\zeta$ corresponding to $q + r_\zeta$ is

$$s_\zeta(z) = s_{0,\zeta}(z) s_0(z). \tag{2.33}$$

Consider now the map

$$\varphi : \mathbb{C} \times M_n(\mathbb{C}) \mapsto M_n(\mathbb{C}), \tag{2.34}$$

$$\varphi(z, \zeta) = \text{diag } (1 + \zeta_{11}, \cdots, 1 + \zeta_{nn}) s_\zeta(z).$$

This map is holomorphic. For fixed $z$, equation (2.31) shows that the differential of $s_{0,\zeta}$ at $\zeta = 0$ is a matrix whose entries are certain dilations of the Fourier-Laplace transforms of the entries of $r$, evaluated at $z$. In particular, $r$ may be chosen so that none of these entries vanishes at a given point $z$, and it follows that $d\varphi$ is surjective at $(z, 0)$. Let $\Gamma_j \subset M_n(\mathbb{C})$ consist of all matrices for which at least $j$ distinct minors vanish. This is an algebraic variety of complex codimension $j$ in $M_n(\mathbb{C})$. Thus if $d\varphi$ is surjective at $(z, 0)$ it follows that the complex codimension of $\varphi^{-1}(\Gamma_j)$ near $(z, 0)$ is $j$, and the complex codimension of the projection to $M_n(\mathbb{C})$ is at least $j - 1$. In particular, this means that if $z$ is a point where two or more distinct minors of $s_0$ vanish, then there is a neighborhood $U$ of $z$ and a sequence of potentials converging to $q$ such that the minors of the corresponding $s_0$ have distinct zeros in $U$. A similar argument based on the real codimension and the restriction of $\varphi$ to $\mathbb{R} \times M_n(\mathbb{C})$ shows that if a minor of $s_0$ has a real zero $z$, then there is a neighborhood $U$ and a sequence of approximating potentials whose $s_0$ have no minors with real zeros in $U$.

In the proof of Theorem A (a) below we show that there is a constant $C$ such that the eigenfunction $m$ for $q$, or for any sufficiently nearby potential, has no singularities in the region $|z| > C$. A regular point remains regular under small perturbations, so the argument just given implies that we may remove singularities on $\Sigma$ one at a time by arbitrarily small perturbations, and similarly we may split poles of distinct columns which happen to coincide. The last step is to show that any multiple poles can be split into simple poles by small perturbations. If $z \in \Omega_\nu$ is a multiple pole, it corresponds to a multiple zero of an upper minor $d_b^+$ of $s_0^\nu$. We choose $r$ with support to the right of supp $(q)$, such that the dilated Fourier-Laplace transforms of the $r_{jk}$ have a simple zero at $z$. Rewriting matrices in the $\nu$ representation, and multiplying $s_0$ on the right by an upper triangular matrix with ones on the diagonal, we may assume that $(s_0)_{ij} = 0$ for $i \geq j$ if $j \leq k$, near $z$. Since the next minor is not zero at $z$, $(s_0)_{k+1,k}(z) \neq 0$. If we choose $\zeta_{k,k+1} \neq 0$ but small, and the other $\zeta_{ij} = 0$, then $(s_\zeta)_{kk}$ will have simple zeros near $z$ and so will the corresponding minor. This completes the proof.

# 3 Small Potentials

It is convenient here to introduce more notation and structure. We let $\mathbb{C}^n$ have the standard hermitian inner product, and let the matrix algebra $M_n(\mathbb{C})$ operate on $\mathbb{C}^n$ in the standard way. Then $M_n(\mathbb{C})$ is a Hilbert space with respect to the trace form

$$(a, b) = \text{tr } b^*a$$

and we denote the norm by $|\ |$. Then the $L^1$-norm on the space of potentials $\mathbf{P}$ is

$$\|q\|_1 = \int_{\mathbb{R}} |q(x)|dx. \tag{3.1}$$

Define $\mathfrak{J} : M_n(\mathbb{C}) \mapsto M_n(\mathbb{C})$ by

$$\mathfrak{J}a = adJ(a) = [J, a]. \tag{3.2}$$

Then $\mathfrak{J}$ is a normal operator, and for any complex $z$ the operator

$$\Re(z\mathfrak{J}) = ad(\Re(zJ)) = \tfrac{1}{2}(z\mathfrak{J}) + \tfrac{1}{2}(z\mathfrak{J})^*$$

is selfadjoint. Let

$$\Pi_+^z, \Pi_-^z, \Pi_0^z = M_n(\mathbb{C}) \mapsto M_n(\mathbb{C}) \tag{3.3}$$

denote the orthogonal projections of $M_n(\mathbb{C})$ onto the positive, negative, and null subspaces for $\Re(z\mathfrak{J})$, respectively. Let

$$\Pi_0^z = M_n(\mathbb{C}) \mapsto M_n(\mathbb{C}) \tag{3.4}$$

be the orthogonal projection onto the kernel of $\mathfrak{J}$, the set of diagonal matrices. Then the projections (3.3) are constant on each component of $\mathbb{C}\backslash\Sigma$, while

$$\Pi_0^z = \begin{cases} \Pi_0 & \text{if and only if } z \notin \Sigma, \\ \Pi_\nu & \text{if } z \in \Sigma_\nu\backslash(0). \end{cases} \tag{3.5}$$

Furthermore,

$$\Pi_+^z a = a \Leftrightarrow a^\nu \text{ is upper triangular,} \tag{3.6}$$

where $\Omega_\nu \ni z$, a similar statement holding for $\Pi_-^z$ and lower triangularity. Note also that

$$\exp\{t\mathfrak{J}\}(a) = e^{tJ} a e^{-tJ}. \tag{3.7}$$

**Theorem 3.8.** *Suppose $q \in L^1$ has norm $||q||_1 < 1$. Then for each $z \in \mathbb{C}\backslash\Sigma$ there is a unique associated eigenfunction $m(\cdot, z)$ satisfying* (1.5) *and* (1.6). *The function $m$ is holomorphic in $\mathbb{C}\backslash\Sigma$ with values $L^\infty \cap C$. On each component of $\mathbb{C}\backslash\Sigma$, $m$ and its inverse extend continuously to the closure. In addition,*

$$m(x,z) \to I \textit{ as } z \to \infty \textit{ uniformly w.r. to } x, \tag{3.9}$$

$$|m(x,z)| \le (1 - ||q||_1)^{-1} \textit{ for all } x, z, \tag{3.10}$$

$$|m(x,z)^{-1}| \le (1 - ||q||_1)^{-1} \textit{ for all } x, z. \tag{3.11}$$

Proof: Given $q \in \mathbf{P}$ and $z \in \mathbb{C}\backslash\Sigma$, let

$$K_{z,q} : L^\infty \mapsto L^\infty \cap C \qquad \text{(matrix-valued),} \tag{3.12}$$

$$\begin{aligned}[K_{z,q} f](x) = &\int_{-\infty}^{x} e^{(x-y)z\mathfrak{J}} (\Pi_0^z + \Pi_-^z)(q(y)f(y))\,dy \\ &- \int_x^\infty e^{(x-y)z\mathfrak{J}} \Pi_+^z (q(y)f(y))\,dy.\end{aligned}$$

The exponential operators here have norm at most 1 on the subspaces where they act, and these subspaces are orthogonal and invariant for $\mathfrak{J}$, so the operator norm satisfies

$$||K_{z,q}|| \le ||q||_1. \tag{3.13}$$

Clearly,

$$\frac{d}{dx} K_{z,q} f(x) = z\mathfrak{J} K_{z,q} f(x) + q(x) f(x) \quad \text{a.e.} \tag{3.14}$$

It follows that, for $||q||_1 < 1$,

$$m(x,z) = [(Id - K_{z,q})^{-1} I](x) \tag{3.15}$$

is a bounded continuous function of $x$ satisfying the differential equation (1.6) and the estimate (3.10). Moreover, the map $z \to K_{z,q}$ is holomorphic from

$\mathbb{C}\backslash\Sigma$ to the space of bounded operators in $L^\infty \cap C$, so $m$ is holomorphic with values in this space. Similarly, $m$ extends to be continuous on the closure of a component of $\mathbb{C}\backslash\Sigma$, with values in $L^\infty \cap C$.

The dominated convergence theorem implies that, for fixed $z$,

$$K_{z,q}f(x) \to 0 \text{ as } x \to -\infty, \tag{3.16}$$

uniformly on bounded sets in $L^\infty \cap C$. In particular, since

$$m(x,z) = I + [K_{z,q}m(\cdot,z)](x), \tag{3.17}$$

it follows that $m$ satisfies (1.5).

As in the proof of (2.5) we see that

$$\det m \equiv 1; \tag{3.18}$$

thus $m$ is invertible and $m(\cdot,z)^{-1}$ is bounded. If $m_1$ were a second solution of (1.5), (1.6), then as in the proof of Proposition 2.11 we would have

$$m_1(x,z) = m(x,z)e^{xz\mathfrak{J}}(a(z)), \quad z \notin \Sigma. \tag{3.19}$$

But $\exp\{xz\mathfrak{J}\}(a(z))$ would have to be bounded with respect to $x$, which implies that $a(z)$ is diagonal, and then the asymptotic condition (1.5) for $m$ and $m_1$ implies $a(z) = I$.

To prove the estimate (3.11) for $m^{-1}$, let $q_2(x) = -q(x)^*$ and let $m_2$ be the corresponding eigenfunction. Then

$$\left(\frac{d}{dx} - z\mathfrak{J}\right)[m(x,z)m_2(x,-\bar{z})^*] = 0 \tag{3.20}$$

and as above we can conclude that

$$m(x,z)^{-1} = m_2(x,-\bar{z})^* \tag{3.21}$$

which implies (3.11).

Finally, consider the asymptotic behavior in $z$. Suppose first that $dq/dx$ is also in $L^1$. Let

$$n(x,z) = I - z^{-1}\mathfrak{J}^{-1}q(x), \tag{3.22}$$

where $\mathfrak{J}^{-1} : \text{ran } \mathfrak{J} \mapsto \text{ran } \mathfrak{J}$. (Note that (1.2) implies $q(x) \in \text{ran } \mathfrak{J}$ for all $x$; this is our first essential use of this fact.) Then

$$\left(\frac{d}{dx} - z\mathfrak{J}\right)n = qn + z^{-1}f, \tag{3.23}$$

where $f \in L^1$. Then using the asymptotic information and

$$\left(\frac{d}{dx} - z\mathfrak{J}\right)(m^{-1}n) = z^{-1}m^{-1}f = g(x,z), \tag{3.24}$$

we obtain

$$(m^{-1}n)(x,z) = I + \int_{-\infty}^{x} e^{(x-y)z\partial}(\Pi_0^z + \Pi_-^z)g(y,z)\,dy - \int_{x}^{\infty} e^{(x-y)z\partial}\Pi_+^z g(y,z)dy. \tag{3.25}$$

Thus

$$|m(x,z) - n(x,z)| \le C|z|^{-1} \text{ for all } x, z. \tag{3.26}$$

This proves (3.9) when the derivative of $q$ is in $L^1$. In general we approximate $q$ by such potentials and note that the corresponding eigenfunctions converge uniformly with respect to $z$ and $x$. In fact, suppose $q$ and $q_1$ are in $L^1$ with norm less than 1 and let $m, m_1$ be the corresponding eigenfunctions. Then

$$\left(\frac{d}{dx} - z\partial\right)(m^{-1}m_1) = m^{-1}(q_1 - q)m_1, \tag{3.27}$$

and we obtain an integral expression analogous to (3.25) which implies

$$|m_1(x,z) - m(x,z)| \le \|q - q_1\|_1 \|m(\cdot,z)^{-1}\|_\infty \|m(\cdot,z)\|_\infty^2. \tag{3.28}$$

# 4 Proof of Theorem A and Theorem B

Suppose $q$ belongs to $\mathbf{P}$. When $\|q\|_1 < 1$, part (a) of Theorem A is just Theorem 3.8, and the set $Z$ of singularities is empty. To complete the proof of part (a) we induce on the least integer $N \ge 0$ such that $\|q\|_1 < 2^N$. Note that the eigenfunction corresponding to a translate of $q$ is the translate (with respect to $x$) of the eigenfunction. Thus after translation we may assume that

$$\int_{-\infty}^{0} |q(x)|\,dx = \int_{0}^{\infty} |q(x)|\,dx. \tag{4.1}$$

Let $q_1(x) = q(x)$ for $x < 0$, $q_1(x) = 0$ for $x \ge 0$, and $q_2 = q - q_1$. The induction assumption implies that $q_j$ has an eigenfunction $m_j$ for which Theorem A (a) holds. Any eigenfunction $m$ for $q$ must be of the form

$$\begin{aligned} m(x,z) &= m_1(x,z)e^{xz\partial}a_1(z), \quad x \le 0, \\ &= m_2(x,z)e^{xz\partial}a_2(z), \quad x \ge 0. \end{aligned} \tag{4.2}$$

For boundedness, continuity, and the asymptotic condition as $x \to -\infty$ it is necessary and sufficient to have

$$m_1(0,z)a_1(z) = m_2(0,z)a_2(z), \tag{4.3}$$

$$\Pi_-^z a_1(z) = 0 = \Pi_+^z a_2(z), \tag{4.4}$$

$$\Pi_0 a_1(z) = I, \quad z \in \mathbb{C}\backslash\Sigma. \tag{4.5}$$

In the matrix representation corresponding to the sector $\Omega_\nu \ni z$, this is a factorization problem:

$$[m_2(0,z)^{-1}m_1(0,z)]^\nu = a_2(z)^\nu[a_1(z)^{-1}]^\nu,$$

$$a_2(z)^\nu \text{ lower triangular}, \quad a_1(z)^\nu \text{ upper triangular}, \tag{4.6}$$

$$a_1(z)_{jj} = 1, \quad 1 \le j \le n.$$

As before this problem has a unique solution as long as the upper minors of $(m_2^{-1}m_1)^\nu$ are non-zero. The latter matrix approaches $I$ as $z \to \infty$, so the factorization problem (4.6) introduces at most a bounded, discrete set of new singularities in the construction of $m$. Moreover, $a_j(z) \to I$ as $z \to \infty$, so $m(x,z) \to I$ as $z \to \infty$. This completes the proof of Theorem A (a).

To prove that the set $\mathbf{P}_0$ of generic potentials is dense, we note first that the set of compactly supported potentials is dense in $\mathbf{P}$. Second, since the set of singularities is now shown to be bounded, the construction in the proof of Proposition 2.11 shows that any compactly supported potential has only finitely many singularities, including the singularities of the extensions to $\Sigma$. Therefore, the potentials obtained in Proposition 2.30 are generic and dense in $\mathbf{P}$.

Finally, we need to show that $\mathbf{P}_0$ is open.

**Lemma 4.7.** *Suppose $q \in \mathbf{P}$ has associated eigenfunction $m$. Suppose $K$ is a compact subset of the one-point compactification of the closure of a sector $\Omega_\nu$ and suppose that $m$ extends to be continuous on $\mathbb{R} \times K$. Given $\epsilon > 0$, there is $\delta > 0$ such that, if $q_1 \in \mathbf{P}$ and $\|q - q_1\|_1 < \delta$, then the associated eigenfunction $m_1$ extends to $\mathbb{R} \times K$ and $|m - m_1| < \epsilon$ on $\mathbb{R} \times K$.*

Proof: The argument leading to the inequality (3.28) proves this when $\|q\|_1 < 1$, and the inductive construction of this section gives the general result.

Suppose now that $q$ is a generic potential. Lemma 4.7 implies that the eigenfunction for a nearby potential will extend to $\Sigma$ and will have singularities only near those of the eigenfunction $m$ of $q$. Moreover, if the potential is sufficiently close, the singularities will be simple poles occurring in one column at a time, by a contour integration argument. Thus $\mathbf{P}_0$ is open, and the proof of Theorem A is complete.

If $q$ is a generic potential with associated eigenfunction $m$, then by assumption the limits $m_\nu^\pm$ exist on $\Sigma_\nu$. If $z$ belongs to $\Sigma_\nu$, then

$$\left(\frac{d}{dx} - z\mathcal{J}\right)[m_\nu^-(x,z)^{-1}m_\nu^+(x,z)] = 0 \tag{4.8}$$

so (1.15) holds. If $z_j \in \mathbb{C}\backslash\Sigma$ is a singularity for $m$, to prove (1.16) we approximate $q$ by compactly supported generic potentials. It follows from Lemma 4.7 and the construction in (2.29) that we may pass to the limit in (1.16) for the approximating potentials to obtain (1.16) for $m$. Note also that the limit $v(z_j)$ is of the form $ce_{k,k+1}$ in the $\nu$-matrix representation.

Finally, to see that a generic potential is uniquely determined by its scattering data, suppose $q_1$ and $q_2$ are generic potentials with eigenfunctions $m_1$, $m_2$ and having the same scattering data. For any fixed $x \in \mathbb{R}$ the function

$$f(z) = m_1(x,z)m_2(x,z)^{-1} \tag{4.9}$$

is meromorphic on $\mathbb{C}\backslash\Sigma$ and converges to $I$ as $z \to \infty$. It is enough to show that the apparent singularities are removable. On $\Sigma_\nu$, $f$ has limits $f_\nu^\pm$. A trivial calculation shows that $f_\nu^+ = f_\nu^-$, so $f$ is holomorphic except possibly at the sigularities $\{z_j\}$ of $m_1$ and $m_2$. Now

$$m_1(x,z) = (z - z_j)^{-1}a + b + O(|z - z_j|), \tag{4.10}$$

and (1.16) implies

$$av_j = 0, \quad bv_j = a, \tag{4.11}$$

where $v_j = \exp\{xz_j\mathfrak{J}\}[v(z_j)]$. Thus

$$m_1(x,z) = b(I + (z - z_j)^{-1}v_j) + O(|z - z_j|). \tag{4.12}$$

Since $v_j^2 = 0$, (4.12) implies that

$$m_1(x,z)(I - (z - z_j)^{-1}v_j) = m_1(x,z)w_j(z) \tag{4.13}$$

has a removable singularity at $z_j$. The same is true for $m_2$, so $f = m_1m_2^{-1} = (m_1w_j)(m_2w_j)^{-1}$ has a removable singularity at $z_j$. Then $f \equiv I$ and the proof of Theorem B is complete.

# 5 Proof of Theorem C and Theorem D

Assume first that the generic potential $q$ has compact support. Let $m_0$ be the eigenfunction of Proposition 2.1 and $s_0$ the function in (2.4) which gives the asymptotic behavior of $m_0$. We know that

$$m(x,z) = m_0(x,z)e^{xz\mathfrak{J}}a(z). \tag{5.1}$$

The function $a$ has limits $a_\nu^\pm$ on $\Sigma_\nu$. Now

$$m(x,z) = e^{xz\mathfrak{J}}s_1(z), \quad x \gg 0, \tag{5.2}$$

where $s_1 = s_0a$. Again, $s_1$ has limits on $\Sigma_\nu$ and we set

$$s_\nu^\pm = \Pi_\nu(s_0a_\nu^\pm) = \Pi_0^z(s_0a_\nu^\pm), \quad z \in \Sigma_\nu. \tag{5.3}$$

Clearly, (1.20) is satisfied. We may now pass to the limit from compactly supported potentials to obtain the existence of (1.20) in the general case.

To see that the limits $s_\nu^\pm$ determine $v_\nu$, we return to the compactly supported case. Recall that (5.2) and boundedness imply that $(s_1)^\nu$ is lower triangular when $z$ is in $\Omega_\nu$. Thus

$$(\Pi_0^z + \Pi_-^z)(s_0 a_\nu^\pm) = s_0 a_\nu^\pm \text{ on } \Omega_\nu. \tag{5.4}$$

For $x \gg 0$ and $z \in \Omega_\nu$,

$$\begin{aligned} v_\nu(z) &= e^{-xz\partial}[m_\nu^-(x,z)^{-1} m_\nu^+(x,z)] \\ &= [s_0(z)a_\nu^-(z)]^{-1} s_0(z) a_\nu^+(z). \end{aligned} \tag{5.5}$$

Boundedness implies

$$v_\nu(z) = \Pi_0^z v_\nu(z), \quad z \in \Sigma_\nu. \tag{5.6}$$

From (5.4)–(5.6) we obtain

$$v_\nu = \Pi_0^z (s_0 a_\nu^-)^{-1}(s_0 a_\nu^+) = (s_\nu^-)^{-1} s_\nu^+ \tag{5.7}$$

since $\Pi_0^z$ is multiplicative on the range of $\Pi_0^z + \Pi_+^z$, which is an algebra. Again passage to the limit gives (5.6) and (5.7) for general generic potentials.

To see that the limits $s_\nu^\pm$ determine the location of the singularities, we return again to the compactly supported case. In the matrix representation corresponding to $\Omega_\nu$, a singularity in column $k+1$ occurs at a zero of the $k$-th upper minor of $s_0^\nu$. Since $a^\nu$ is upper triangular with 1's on the diagonal,

$$d_k^+(s_0^\nu) = d_k^+(s_0^\nu a^\nu). \tag{5.8}$$

Now $s_0^\nu a^\nu$ is lower triangular, so its upper minors are the same as those of the corresponding diagonal matrix $\delta^\nu$, where

$$\begin{aligned} \delta(z)_{jj} &= (s_0 a)(z)_{jj} \\ &= \lim_{x\to\infty} m(x,z)_{jj}. \end{aligned} \tag{5.9}$$

The elements of $\delta$, being ratios of the minors which are holomorphic in $\mathbb{C}\backslash\Sigma$, are themselves meromorphic in $\mathbb{C}\backslash\Sigma$ with continuous extensions to $\Sigma$. They are therefore determined by their restrictions to $\Sigma$, which are just the diagonal elements of the $s_\nu^\pm$. Once again we may pass to the limit from compactly supported $q$.

To complete the proof of Theorem C, we want to show that $\{v_\nu\}$ and the location of the singularities of $m$ determine $s_\nu^\pm$. Again we start with the compactly supported case and pass to the limit. In the $\nu$ representation the range of $\Pi_0^z$ consists of matrices whose non-zero elements occur only in certain diagonal blocks, when $z \in \Omega_\nu$. Moreover, in these blocks $(s_\nu^-)^\nu$ is lower triangular, while

$(s_\nu^+)^\nu$ is upper triangular. Thus, within the range of $\Pi_0^z$, (5.7) gives a triangular factorization of $v_\nu^\nu$. This factorization is unique up to left multiplication of $s_\nu^\pm$ by a diagonal matrix; hence it is determined once we know the diagonal parts of $s_\nu^+$ or $s_\nu^-$. The factorization (5.7) implies that the upper minors of $v_\nu^\nu$ are quotients of the upper minors of $(s_\nu^+)^\nu$ and $(s_\nu^-)^\nu$, and this means that $v_\nu$ determines the ratios of corresponding diagonal elements of $s_\nu^-$ and $s_\nu^+$. Returning to the diagonal matrix $\delta$, we conclude that the ratios of its elements on $\Sigma$ are determined by the $\{v_\nu\}$, while the zeros and poles are determined by the singularities of $m$. This information determines $\delta$ and thus it determines $s_\nu^\pm$. This completes the proof of Theorem C.

We have already proved (1.24) of Theorem D, as (5.6) above, and also (1.28). When $q$ has compact support and $a$ is as in (5.1), we take $x \ll 0$ to obtain

$$\begin{aligned} v_\nu &= (a_\nu^-)^{-1} a_\nu^+ \\ &= (b_\nu^-)^{-1} b_\nu^+, \end{aligned} \tag{5.10}$$

where

$$b_\nu^\pm = \Pi_0^z a_\nu^\pm, \quad z \in \Omega_\nu. \tag{5.11}$$

In the diagonal blocks where $v_\nu^\nu$ lives, $(a_\nu^-)^\nu$ is upper triangular and $(a_\nu^+)^\nu$ is lower triangular. Thus $(b_\nu^-)^\nu$ is upper, $(b_\nu^+)^\nu$ is lower, and each has 1's on the diagonal. It follows that (1.25) is true and also that the lower minors of $v_\nu^\nu$ are $\equiv 1$. Similarly, the factorization (5.7) implies that the upper minors of $v_\nu^\nu$ are non-zero. Once again, passage to the limit gives (1.25), (1.26), and (1.27) in general.

Finally we come to the constraints (1.31). Let $\delta$ be the diagonal matrix (5.9). As we have already observed, the ratios of the limits $(\delta_\nu^\pm)_{jj}$ on $\Sigma_\nu$ are determined by the upper minors of $v_\nu^\nu$; in fact, the latter ratios are certain products of the former. On the other hand, the singularities of $m$ and the columns in which they occur determine the zeros and poles of the $\delta_{jj}$. For each $j$, there is a compatibility condition between the ratios $(\delta_{\nu,jj}^-)^{-1}\delta_{\nu,jj}^+$ and the zeros and poles on $\delta_{jj}$. This condition takes the form (1.31); the $n$ conditions are not independent because of the single constraint

$$\Pi \delta_{jj} = \det \delta \equiv 1, \tag{5.12}$$

which follows from the fact that

$$\delta(z) = \lim_{z \to +\infty} m(x, z). \tag{5.13}$$

In fact, (5.13) is clear from (2.16) when $q$ has compact support and then follows from a limit argument for general generic $q$. For details on the compatibility condition, see part 1 of the appendix.

# 6 Proof of Theorem E

Suppose $q$ is a generic potential with eigenfunction $m$ and scattering data $v$. Part (a) of Theorem E is an immediate consequence of the following.

**Theorem 6.1.** *Suppose $q$ belongs to* **P** *and suppose*

$$D^j q \in L^1, \quad 0 \le j \le k. \tag{6.2}$$

*Then there are unique functions*

$$m_j : \mathbb{R} \mapsto M_n(\mathbb{C}), \quad 0 \le j \le k, \tag{6.3}$$

*such that $m_0 \equiv I$ and*

$$\left| m(x,z) - \sum_{j=0}^{k} z^{-j} m_j(x) \right| = O(|z|^{-k}) \tag{6.4}$$

*as $z \to \infty, z \in \mathbb{C}\backslash\Sigma$, uniformly with respect to $x \in \mathbb{R}$.*

Proof: Uniqueness is clear. Since $m(x,z) \to I$ as $x \to -\infty$ we need

$$m_j(x) \to 0 \text{ as } x \to -\infty \text{ if } j > 0. \tag{6.5}$$

We determine $m_0, \cdots, m_k$ from (6.5) and

$$\frac{d}{dx} m_j - q m_j = \mathfrak{J} m_{j+1} \qquad \text{a.e.} \tag{6.6}$$

In fact, given $a \in M_n(\mathbb{C})$, write

$$a = a' + a'', \tag{6.7}$$

where $a'$ is diagonal and $a''$ is off-diagonal. Then (6.6) determines $m''_{j+1}$, given $m_j$. Note also that (6.6) requires

$$\frac{d}{dx} m'_j = (q m_j)' \quad \text{a.e.} \tag{6.8}$$

Now $q = q''$ so that $(q m_j)' = (q m''_j)'$. Thus (6.5) and (6.8) give

$$m_j(x)' = \int_{-\infty}^{x} (q(y) m_j(y)'')' \, dy. \tag{6.9}$$

Hence (6.6) and (6.9) allows us at least formally to determine $m''_1, m'_1, m''_2, m'_2, \cdots, m'_k$. Moreover, $D^j q \in L^\infty$ if $j < k$, and $\mathfrak{J} m_1 = -q$. Inductively we obtain

$$D^r m''_j \in L^1 \text{ if } 0 \le r \le k + 1 - j, \tag{6.10}$$

$$D^r m'_j \in L^1 \text{ if } 0 < r < k + 2 - j. \tag{6.11}$$

Let

$$m^k(x,z) = \sum_{j=0}^{k} z^{-j} m_j(x). \tag{6.12}$$

Then from (6.6) we obtain

$$(D - z\mathfrak{J})m^{-1}m^k = z^{-k}m(x,z)^{-1}f(x) = g(x,z), \tag{6.13}$$

where $f \in L^1$. Therefore for large $z$ we have

$$\begin{aligned}(m^{-1}m^k)(x,z) = I + \int_{-\infty}^{x} e^{(x-y)z\mathfrak{J}}(\Pi_0^z + \Pi_-^z)g(y,z)\,dy \\ - \int_x^\infty e^{(x-y)z\mathfrak{J}}\Pi_+^z g(y,z)\,dy.\end{aligned} \tag{6.14}$$

This implies that, for all $x$ and all large $z$,

$$|m(x,z) - m^k(x,z)| \le C|z|^{-k}. \tag{6.15}$$

The map $q \mapsto m_j$ is continuous to $L^\infty$ with respect to the $L^1$ norms of the derivatives (6.2), and for large $z$ the map $q \mapsto m(\cdot, z)$ is continuous from $L^1$ to $L^\infty$ uniformly with respect to $z$. Considering (6.13) for the functions corresponding to two nearby potentials and taking the difference, we conclude that, for large $z$,

$$|m_1(x,z) - m_2(x,z)| \le C(q_1,q_2)|z|^{-k}, \tag{6.16}$$

where, for fixed $q_1$, $C$ is small when the derivatives of $q_2 - q_1$ have small $L^1$ norm. Because of this it is enough to prove (6.4) for a dense set of $q$. But when, in addition to (6.2), we have $D^{k+1}q \in L^1$, then (6.15) with $k$ replaced by $k+1$ implies (6.4).

We turn now to part (b) of Theorem E.

**Lemma 6.17.** *Suppose $k$ is a positive integer, and suppose*

$$\|q\|_1 < 1, \quad q_k \in L^1, \tag{6.18}$$

*where $q_k(x) = (-x\mathfrak{J})^k q(x)$. Then for $x$ real and $z \in \mathbb{C}\setminus\Sigma$ we have*

$$\left|\left(\frac{\partial}{\partial z} - x\mathfrak{J}\right)^k m(x,z)\right| \le C_k(1 - \|q\|_1)^{-k-1}(1 + \|q\|_1 + \|q_k\|_1)^k. \tag{6.19}$$

Proof: Let $K_{z,q}$ be the operator (3.12), and let $K_q$ be the operator on functions of two variables:

$$[K_q f](x,z) = [K_{q,z} f(\cdot, z)](x). \tag{6.20}$$

Let $A = \partial/\partial z - x\eth$. The commutator is

$$[A, K_q] = K_{q_1}, \quad q_1 = -x - \eth q. \tag{6.21}$$

Also,

$$[A, K^N] = \sum_{j=1}^{N} K^{N-j}[A, K]K^{j-1}. \tag{6.22}$$

If $I$ denotes the identity constant function,

$$AK_q^N(I) = [A, K_q^N](I). \tag{6.23}$$

Using (6.21)–(6.23) we obtain

$$|AK_q^N(I)| \leq N\|q\|_1^{N-1}\|q_1\|_1 \tag{6.24}$$

on $\mathbb{R} \times (\mathbb{C}\backslash\Sigma)$. Thus we may differentiate the Neumann series term by term to obtain

$$\begin{aligned}\left(\frac{\partial}{\partial z} - x\eth\right)m &= \sum_{N=1}^{\infty}\sum_{j=1}^{N} K_q^{N-j}K_{q_1}K_q^{j-1}(I) \\ &= (1 - K_q)^{-1}K_{q_1}(1 - K_q)^{-1}(I) \\ &= (1 - K_q)^{-1}K_{q_1}m.\end{aligned} \tag{6.25}$$

This gives the estimate

$$\left|\left(\frac{\partial}{\partial z} - x\eth\right)m\right| \leq (1 - \|q\|_1)^{-2}\|q_1\|_1. \tag{6.26}$$

The estimate (6.19) for $k > 1$ is obtained by an elaboration of this procedure. The argument just given shows more generally that

$$[A, (1 - K_q)^{-1}] = (1 - K_q)^{-1}A(1 - K_q)^{-1} \tag{6.27}$$

applied to functions $f$ with $f$ and $Af$ bounded. In particular,

$$\begin{aligned}A^2m &= A(1 - K_q)^{-1}K_{q_1}m \\ &= (1 - K_q)^{-1}AK_{q_1}m + (1 - K_q)^{-1}K_{q_1}(1 - K_q)^{-1}K_{q_1}m \\ &= (1 - K_q)^{-1}K_{q_2}m + 2(1 - K_q)^{-1}K_{q_1}(1 - K_q)^{-1}K_{q_1}m.\end{aligned} \tag{6.28}$$

Thus

$$\begin{aligned}\left|\left(\frac{\partial}{\partial z} - x\eth\right)^2 m\right| &\leq (1 - \|q\|_1)^{-2}\|q_2\|_1 + 2(1 - \|q\|_1)^{-3}\|q_1\|_1^2 \\ &\leq 3(1 - \|q\|_1)^{-3}(1 + \|q\|_1 + \|q_2\|_1)^2.\end{aligned} \tag{6.29}$$

The general case is proved by the obvious elaboration of this argument.

**Lemma 6.30.** *Suppose $q \in \mathbf{P}$ and*

$$\int_{\mathbb{R}} (1+|x|)^k |q(x)|\, dx < \infty. \tag{6.31}$$

*Then, for large $z \in \mathbb{C}\backslash\Sigma$ and all $x \in \mathbb{R}$,*

$$\left| \left(\frac{\partial}{\partial z}\right)^k m(x,z) \right| \le C_q(1+|x|)^k. \tag{6.32}$$

Proof: As in the proof of Theorem A (a), we induce on the least integer $r \ge 0$ such that $\|q\|_1 < 2^r$; the case $r = 0$ is part of Lemma 6.17. If $\|q\|_1 < 2^r$ we choose $y$ so that

$$\int_{\infty}^{y} |q(x)|\, dx = \int_{y}^{\infty} |q(x)|\, dx,$$

and let $q = q_1 + q_2$ with

$$q_1(x) = q(x) \text{ if } x \le y, \quad q_1(x) = 0 \text{ if } x > y.$$

Again the eigenfunction associated to $q$ is of the form

$$m(x,z) = \begin{cases} m_1(x,z)e^{(x-y)zJ}a_1(z), & x \le y, \\ m_2(x,z)e^{(x-y)zJ}a_2(z), & x > y, \end{cases}$$

where $m_j$ is associated to $q_j$. The $a_j$ solve a factorization problem for

$$m_2(y,z)^{-1}m_1(y,z). \tag{6.33}$$

We know that this problem is solvable as $z \to \infty$ since the matrix (6.33) tends to $I$. The solution has entries which are rational functions of the entries of (6.33), and the denominators are bounded away from 0 for large $z$. Therefore the induction assumption applied to $m_1$, $m_2$ gives the desired estimates for $m$.

We may now prove Theorem E (b). Given a generic $q \in \mathbf{P}$ satisfying (6.30), we may write $q = q_1 + q_2 + q_3$, where $q_1$ is supported on $(-\infty, y_1]$, $q_2$ is supported in $[y_1, y_2]$, and $q_3$ in $[y_2, \infty)$. Moreover, we may assume

$$\|q_1\|_1 < 1, \quad \|q_3\|_1 < 1, \quad q_2 \in \mathbf{P}_0. \tag{6.34}$$

Let $m_j$ be the associated eigenfunction. Again

$$m(x,z) = m_j(x,z)e^{xzJ}a_j(z) \tag{6.35}$$

on the support of $q_j$ and the $a_j$ solve a certain algebraic factorization problem. Now since $q_2$ has compact support, $m_2$ is meromorphic with respect to $z$. From Lemma 6.17, $m_1$ and $m_3$ are $C^k$ with respect to $z$, on the closure of any sector

$\Omega_\nu$. Therefore, the $a_j$ in (6.35) are $C^k$ with respect to $z$, and so then is $m$. Thus

$$v_\nu(z) = m_\nu^-(0,z)^{-1}m_\nu^+(0,z), \quad z \in \Sigma_\nu,$$

is $C^k$ in $z$ on $\Sigma_\nu$. Lemma 6.30 gives boundedness of the derivative. To prove that $D^j(v_\nu - I)$ converges to zero at $\infty$, it is enough to show this for a dense set of generic potentials. Part (a) of Theorem E implies that if the generic potential $q$ belongs to the Schwartz class, then $v_\nu - I$ is rapidly decreasing, while we have just shown that the derivatives are bounded. As noted in Section 1, these facts imply that the derivatives also are of rapid decrease. This completes the proof of (1.35).

Finally, let $v_{\nu,k}$ be the Taylor expansion of $v_\nu$ at the origin. When $q$ has compact support, let $a$ be the function in (2.18) and let $a_{\nu,k}$ be the Taylor polynomial of $a$ in the sector $\Omega_\nu$, at the origin. Then we have (1.25) and (1.36). Note that if we assume that our polynomials are of degree $k$, then they are uniquely determined by (1.25) and (1.36). In fact, $s = 2p$ is even and the $\nu$ and $\nu + p$ orderings are opposite. Thus (letting $a_{\nu+s,k} = a_{\nu,k}$ and $v_{\nu+s,k} = v_{\nu,k}$) we have

$$(a_{\nu+p,k})^\nu = (a_{\nu,k})^\nu\{v_{\nu,k}v_{\nu+1,k}\cdots v_{\nu+p-1,k}\}^\nu + O(z^k), \tag{6.36}$$

an upper- and lower-triangular factorization problem with a unique solution; the condition that it have a solution is that the term in braces has appropriate minors which are $1 + O(z^k)$. Therefore we may pass to the limit from potentials with compact support.

To prove Theorem E (c), we assume first

$$||q||_1 < 1, \quad ||q||_2^2 = \int |q(x)|^2\,dx < \infty. \tag{6.37}$$

Consider the operator $K_q$ of (6.20); we extend if to $z \in \Sigma_\nu$ from either side of $\Sigma_\nu$ and consider it as mapping the space of matrix-valued functions

$$L^\infty(\mathbb{R}; L^2(\Sigma)) \cap C(\mathbb{R}; L^2(\Sigma)) \tag{6.38}$$

to itself. Then, as a mapping in this space, $K$ is easily seen to have operator norm

$$||K||_q \leq ||q||_1. \tag{6.39}$$

The function

$$g = K_q(I) \tag{6.40}$$

has entries which are Fourier or Fourier-Laplace transforms of products of translates of $q$ with the characteristic function of $\mathbb{R}_\pm$, so $g$ belongs to the space (6.38). Under assumption (6.37),

$$\begin{aligned} m - I &= (Id - K)^{-1}(I) - I \\ &= (Id - K)^{-1}g; \end{aligned}$$

thus $m - I$ belongs to the space (6.38). Then, on $\Sigma_\nu$,

$$v_\nu(z) - I = m_\nu^-(0,z)^{-1}\{m_\nu^+(0,z) - m_\nu^-(0,z)\}$$

belongs to $L^2$. We may now induce again on the smallest integer $N \geq 0$ such that $\|q\|_1 < 2^N$ to establish the following:

**Lemma 6.41.** *Suppose $q$ is in $\mathbf{P} \cap L^2$. Then there is a bounded set $A_\nu \subset \Sigma_\nu$ such that $m_\nu^\pm$ exists on $\Sigma_\nu \backslash A_\nu$ and*

$$m_\nu^\pm - I \in L^2(\Sigma_\nu \backslash A_\nu). \tag{6.42}$$

Proof: By induction, using the method of proof of Theorem A (a), and noting that since $m_2(0,z)^{-1}m_1(0,z) - I$ is in $L^2$ near $\infty$ on any ray, the same is true for $a_1 - I$ and $a_2 - I$.

This lemma and Theorem E (b) give Theorem E (c) when $k = 0$. The extension to positive $k$ is analogous to the argument for Theorem E (b) when $k > 0$, operating again in the space (6.37). We omit the details.

# 7 A Reformulation of the Inverse Problem

Suppose $q$ is a generic potential with eigenfunction $m$, and suppose that $m$ has no singularities in $\mathbb{C} \backslash \Sigma$. On $\Sigma_\nu$, $m$ has an (additive) jump

$$\begin{aligned} g_\nu(x,z) &= m_\nu^+(x,z) - m_\nu^-(x,z) \\ &= m_\nu^-(x,z)[e^{xz\partial}v_\nu(z) - I]. \end{aligned} \tag{7.1}$$

We may expect $m$ to be given by the corresponding Cauchy integral,

$$m(x,z) = I + \Sigma \frac{1}{2\pi i} \int_{\Sigma_\nu} (\zeta - z)^{-1} g_\nu(x,\zeta)\, d\zeta, \tag{7.2}$$

where $\Sigma_\nu$ is oriented from 0 to $\infty$. In fact, suppose

$$I - v_\nu \in L^2(\Sigma_\nu), \quad Dv_\nu \in L^2(\Sigma_\nu), \tag{7.3}$$

so that $g_\nu$ belongs to $L^2(\Sigma_\nu)$ for each $x$. Then well-known results for $\mathbb{R}$, carried over to the rays $\Sigma_\nu$, imply that the function defined by (7.2) has the additive jump $g_\nu$ on $\Sigma_\nu$, from which we can deduce that (7.2) is valid.

We want to formulate (7.2) as an integral equation for $m(x,\cdot)$ on $\Sigma$, and it is convenient to make the following choice for $m$ on $\Sigma$.

**Proposition 7.4.** *If $q \in \mathbf{P}$ is generic, then for each $z \in \Sigma_\nu \backslash (0)$ there is a unique function $m(\cdot, z)$ satisfying (1.5) and (1.6). This function has a continuous extension to the closed ray $\Sigma_\nu$.*

Proof: Suppose first that $q$ has compact support. As in (5.1),

$$m(x,z) = e^{xz\partial}a(z), \quad x \ll 0, z \in \mathbb{C}\backslash\Sigma. \tag{7.5}$$

The function $a$ has limits $a_\nu^\pm$ on $\Sigma_\nu$. As in (5.12), take

$$b_\nu^\pm = \Pi_\nu a_\nu^\pm \tag{7.6}$$

and on $\Sigma_\nu$ set

$$\begin{aligned} m(x,z) &= m_\nu^+(x,z)e^{xz\partial}[b_\nu^+(z)^{-1}] \\ &= m_\nu^-(x,z)e^{xz\partial}[b_\nu^-(z)^{-1}]; \end{aligned} \tag{7.7}$$

the second equality comes from (5.11). Now

$$\begin{aligned} \Pi_\nu m(x,z) = I \quad &\text{if } x \ll 0, \\ (Id - \Pi_\nu)m \to 0 \quad &\text{as } x \to -\infty. \end{aligned} \tag{7.8}$$

Thus $m \to I$ as $x \to -\infty$.

Recall that $b_\nu^\pm$ are the unique solutions of the factorization problem

$$\begin{aligned} &b_\nu^- v_\nu = b_\nu^+, \quad \Pi_\nu b_\nu^\pm = b_\nu^\pm, \quad (b_\nu^\pm)_{jj} = 1, \\ &(b_\nu^+)^\nu \text{ is lower triangular}, \\ &(b_\nu^-)^\nu \text{ is upper triangular}. \end{aligned} \tag{7.9}$$

In the general case we approximate by compactly supported potentials. The corresponding scattering data converge; thus the solutions to (7.9) converge, and so the eigenfunctions on $\Sigma_\nu$ converge and give the desired eigenfunction. Finally, any other solution would have the form

$$m(x,z)e^{xz\partial}c(z). \tag{7.10}$$

But boundedness in $x$ implies $\Pi_\nu c = c$ and the normalization at $-\infty$ implies that $\Pi_\nu c = I$.

Set

$$w_\nu(z) = b_\nu^+(z) - b_\nu^-(z), \quad z \in \Sigma_\nu, \tag{7.11}$$

so that

$$(w_\nu)_{jj} = 1, \quad \Pi_\nu w_\nu = w_\nu. \tag{7.12}$$

Now $w_\nu$ is determined from $v_\nu$ by (7.9) and (7.11). Conversely, given $w_\nu$ satisfying (7.12), let $w_\nu^\pm$ be defined so that $(w_\nu^+)^\nu$ is the lower triangular part of $(w_\nu)^\nu$ and $(w_\nu^-)^\nu$ is the upper triangular part. Then set

$$b_\nu^\pm = I \pm w_\nu^\pm, \quad v_\nu = (b_\nu^-)^{-1}b_\nu^+. \tag{7.13}$$

Then $v_\nu$ satisfies the constraints (1.24), (1.26), (1.27), and $w_\nu$ is determined from $v_\nu$ by (7.9) and (7.11). There is a complete equivalence between scattering data (or formal scattering data)

$$v = (v_1, \cdots, v_r; z_1, \cdots, z_N; v(z_1), \cdots, v(z_N)) \tag{7.14}$$

and *transformed scattering data*

$$w = (w_1, \cdots, w_r; z_1, \cdots, z_N; v(z_1), \cdots, v(z_N)). \tag{7.15}$$

We shall show eventually that when $m$ has no singularities and is extended to $\Sigma$ as in Proposition 7.4, then it satisfies on $\Sigma$ an integral equation

$$m(x, \cdot) = I + C_{w,x} m(x, \cdot), \tag{7.16}$$

where $w$ is the transformed scattering data. Here

$$C_{w,x} f = C^+(f w^-(x, \cdot)) + C^-(f w^+(x, \cdot)), \tag{7.17}$$

where

$$w^\pm(x, z) = e^{xz\partial} w_\nu^\pm(z), \quad z \in \Sigma_\nu, \tag{7.18}$$

and where $C^\pm$ are suitably normalized Cauchy integrals which we proceed to describe. When $\mu \neq \nu$, we let $C_{\mu,\nu}$ map functions on $\Sigma_\nu$ to functions on $\Sigma_\mu$ (oriented from 0 to $\infty$) by

$$C_{\mu,\nu} f(z) = \frac{1}{2\pi i} \int_{\Sigma_\nu} (\zeta - z)^{-1} f(\zeta)\, d\zeta, \quad z \in \Sigma_\mu. \tag{7.19}$$

Let $C_\nu^\pm$ map functions on $\Sigma_\nu$ to functions on $\Sigma_\nu$:

$$C_\nu^\pm f(z) = \lim_{z' \to z} \frac{1}{2\pi i} \int_{\Sigma_\nu} (\zeta - z)^{-1} f(\zeta)\, d\zeta, \tag{7.20}$$

where the limit is taken from $\Omega_{\nu+1}$ for $C_\nu^+$ and from $\Omega_\nu$ for $C_\nu^-$. These maps will be discussed more thoroughly in later sections. It is classical that

$$C_{\mu,\nu} : L^2(\Sigma_\nu) \mapsto L^2(\Sigma_\mu), \quad \nu \neq \mu, \tag{7.21}$$

$$C_\nu^\pm : L^2(\Sigma_\nu) \mapsto L^2(\Sigma_\nu), \tag{7.22}$$

$$\pm C_\nu^\pm \text{ are complementary orthogonal projections.} \tag{7.23}$$

For a function $f \in L^2(\Sigma)$ write $f = (f_\nu), f_\nu \in L^2(\Sigma_\nu)$, and define $C^\pm f$ by

$$(C^\pm f)_\mu = \sum_{\nu \neq \mu} C_{\mu,\nu} f + C_\mu^\pm f. \tag{7.24}$$

Suppose that $m(x,\cdot)$ is a function on $\Sigma$ which solves the integral equation (7.16), where $C_{w,x}$ is defined in (7.17), (7.22). We extend $m$ to $\mathbb{C}\backslash\Sigma$ by taking the natural extension of (7.16), which is

$$m(x,z) = I + \Sigma \frac{1}{2\pi i} \int_{\Sigma_\nu} (\zeta - z)^{-1} m(x,z) e^{xz\partial} w(\zeta)\, d\zeta. \tag{7.25}$$

Then (7.23) implies for the limits of $m$ on $\Sigma_\nu$ that

$$\begin{aligned} m_\nu^+ &= I + C^+(m,w) = I + C_{w,x} m + m w_\nu^+(x,\cdot) \\ &= m + m w_\nu^+(x,\cdot) \\ &= m(x,z)(I + e^{xz\partial} w_\nu^+(z)) = m(x,z) e^{xz\partial} b_\nu^+(z). \end{aligned} \tag{7.26}$$

Similarly,

$$m_\nu^-(x,z) = m(x,z) e^{xz\partial} b_\nu^-(z). \tag{7.27}$$

Thus

$$m_\nu^+(x,z) = m_\nu^-(x,z) e^{xz\partial} v_\nu(\zeta) \tag{7.28}$$

as desired.

# 8 The Inverse Problem with Small Data, I

We begin our study of the integral equation (7.16) with a lemma which is classical; it is convenient for later use to record the proof.

**Lemma 8.1.** *The operators $C^\pm$ of* (7.24) *are bounded in $L^2(\Sigma)$. Moreover, if $\alpha_1, \cdots, \alpha_r$ are constants such that*

$$\alpha_\nu z > 0 \text{ if } z \in \Sigma_\nu\backslash(0) \tag{8.2}$$

*and if $e_\lambda$ denotes the function on $\Sigma$ with*

$$e_\lambda(z) = \exp\{i\lambda\alpha_\nu z\}, \quad z \in \Sigma_\nu, \lambda \in \mathbb{R}, \tag{8.3}$$

*then for any $f \in L^2(\Sigma)$ we have*

$$||C^\pm(e_\lambda f)||_2 \to 0 \text{ as } \pm\lambda \to -\infty. \tag{8.4}$$

Proof: It is enough to consider the operators $C_{\mu,\nu}$ and $C_\nu^+$, and we may rotate and assume $\Sigma_\nu = R_+$. Parametrize $\Sigma_\mu$ by $R_+$ also. For $\mu \neq \nu$, we see that $C_{\mu,\nu}$ is an integral operator with kernel

$$k(s,t) = (2\pi i)^{-1}(\alpha s + t)^{-1}, \quad \alpha \in \mathbb{C}\backslash\mathbb{R}, s,t \geq 0. \tag{8.5}$$

Then $|k(s,t)| \leq C(s+t)^{-1}$; $L^2$-boundedness is classical. To see that

$$||C_{\mu,\nu}(e_\lambda f)|| \to 0 \text{ as } |\lambda| \to \infty, \tag{8.6}$$

where $e_\lambda(t) = e^{i\lambda t}$, $\lambda \in \mathbb{R}$, note that it is enough to prove (8.6) when $f$ is smooth and has compact support in $(0,\infty)$; then (8.6) follows from an integration by parts.

When $\Sigma_\nu = R_+$ the operator $C_\nu^+$ can be computed on test functions:

$$\begin{aligned} C_\nu^+ f(s) &= \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int (s-t-i\epsilon) f(t)\, dt \\ &= \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \iint (s-t-i\epsilon)^{-1} e^{it\xi} \hat{f}(\xi)\, d\xi\, dt \\ &= \int_{-\infty}^{0} e^{it\xi} \hat{f}(\xi)\, d\xi = (h_0 \hat{f})\check{}(s), \end{aligned} \tag{8.7}$$

where $h_a(\xi) = 1$ if $\xi \leq a$, $h_a(\xi) = 0$ if $\xi > a$, and $\hat{}, \check{}$ denote the Fourier transform and its inverse. This gives $L^2$-boundedness. With $e_\lambda(t) = e^{i\lambda t}$, $\lambda \in \mathbb{R}$, the same calculation gives

$$C_\nu^+(e_\lambda f)(s) = (h_\lambda \hat{f})\check{}(s) \tag{8.8}$$

which yields $L^2$ convergence to 0 as $\lambda \to -\infty$ and to $f$ as $\lambda \to +\infty$.

**Theorem 8.9.** *Suppose $w = (w_\nu)$ satisfies the conditions (7.12), belongs to $L^2(\Sigma) \cap L^\infty(\Sigma)$, and $w(z) \to 0$ as $z \to \infty$. Let $C_{w,x}$ be the operator defined by (7.17), (7.24), let $C^\pm$ be defined by (7.24), and let*

$$||C^+|| = ||C^-||$$

*be the operator norm in $L^2(\Sigma)$. Suppose*

$$2||w||_\infty ||C^\pm|| < 1. \tag{8.10}$$

*Then for every real $x$ there is a unique function $m(x,\cdot) \in L^2(\Sigma) + L^\infty(\Sigma)$ which satisfies the integral equation (7.16). If $m$ is extended to $\mathbb{C}\backslash\Sigma$ by (7.25), then for each $z \in \mathbb{C}\backslash\Sigma$, $m(\cdot, z)$ is bounded and absolutely continuous with respect to $x$, and*

$$m(x,z) \to I \text{ as } x \to -\infty. \tag{8.11}$$

*Let*

$$q(x) = \frac{1}{2\pi i} \partial \int_\Sigma m(x,z) e^{xz\partial} w(z)\, dz. \tag{8.12}$$

*Then*

$$q \in L^\infty + L^2 \tag{8.13}$$

*and, for $z \in \mathbb{C}\backslash\Sigma$,*

$$\frac{\partial}{\partial x} m(x,z) = z\partial m(x,z) + q(x) m(x,z) \quad \text{a.e. } x. \tag{8.14}$$

Proof: The operator $C_{w,x}$ maps $L^\infty(\Sigma)$ to $L^2(\Sigma)$, since $w$ is assumed to belong to $L^2(\Sigma)$. As an operator in $L^2(\Sigma)$, $C_{w,x}$ has norm dominated by the expression (8.10). Therefore, $Id - C_{w,x}$ is invertible as an operator in $L^2 + L^\infty$, and (7.16) has the unique solution

$$m(x,\cdot) = (Id - C_{w,x})^{-1}(I). \tag{8.15}$$

Then

$$m(x,\cdot) - I = (Id - C_{w,x})^{-1}(g(x,\cdot)), \tag{8.16}$$

where

$$g(x,\cdot) = C_{w,x}(I). \tag{8.17}$$

Now it follows from Lemma 8.1 that the $L^2$-norm of $g(x,\cdot)$ approaches zero as $x \to -\infty$. From this and from (8.16) we obtain

$$\sup_x \|m(x,\cdot) - I\|_2 < \infty, \tag{8.18}$$

$$\|m(x,\cdot) - I\|_2 \to 0 \text{ as } x \to -\infty. \tag{8.19}$$

An easy consequence of (8.18) is that $m(\cdot, z)$ is bounded as a function of $x$ for every $z \in \mathbb{C}\backslash\Sigma$. Lemma 8.1 and (8.19) imply (8.11) for $z \in \mathbb{C}\backslash\Sigma$.

Let $q$ be defined by (8.12) and write it as a sum of two integrals, involving $m(x,z) - I$ and $I$, respectively. The integrand in the first integral is a product of $L^2$ functions with norms bounded as $x$ varies, so the first term is in $L^\infty$. (In fact $x \mapsto C_{w,x}$ is continuous to the strong operator topology, by the dominated convergence theorem, which implies that $x \mapsto m(x,\cdot) - I$ is continuous to $L^2$, so this term is even continuous.) The entries of the second term are easily seen to be (dilates of) Fourier transforms of the entries of $w$; thus the second term is in $L^2(\mathbb{R})$.

In this region where (8.10) holds, this construction shows that

$$w \mapsto m \text{ is continuous from } L^2(\Sigma) \cap L^\infty(\Sigma) \text{ to } C(\mathbb{R}; L^\infty(\Sigma) + L^2(\Sigma)), \tag{8.20}$$

$$w \mapsto q \text{ is continuous from } L^2(\Sigma) \cap L^\infty(\Sigma) \text{ to } L^\infty(\Sigma) + L^2(\Sigma), \tag{8.21}$$

Because of (8.20) and (8.21), it is enough to prove (8.14) when $w$ belongs to a dense set. We shall assume, in fact, that $w$ has compact support. In that case it is clear that $x \mapsto C_{w,x}$ is analytic from $\mathbb{R}$ to the bounded operators in $L^\infty + L^2$, and so $x \mapsto m(x,\cdot)$ is analytic to $L^\infty + L^2$. Consider $A$ and $C_w$ as mapping the space

$$C^\infty(\mathbb{R}; L^\infty + L^2) \tag{8.22}$$

to itself,

$$[Af](x,z) = \frac{d}{dx} f(x,z) - z\mathfrak{d} f(x,z), \tag{8.23}$$

$$[C_w, f](x,\cdot) = C_{w,x} f(x,\cdot). \tag{8.24}$$

Then the commutator

$$[A, C_w]f(x,\cdot) = \frac{1}{2\pi i}\mathfrak{d}\int_\Sigma f(x,\zeta)e^{x\zeta\mathfrak{d}}w(\zeta)\,d\zeta \tag{8.25}$$

maps to functions which are constant with respect to $z$. As in (6.22)–(6.25), since $A(I) = 0$ we have

$$\begin{aligned} Am(x,\cdot) &= (I - C_w)^{-1}[A, C_w](I - C_w)^{-1}(I) \\ &= (I - C_w)^{-1}q. \end{aligned} \tag{8.26}$$

Since $q = qI$ and $C_w$ commutes with left multiplication by functions independent of $z$, we have

$$\left(\frac{d}{dx} - z\mathfrak{d}\right)(m,\cdot) = q(x)m(x,\cdot) \tag{8.27}$$

as functions in $C^\infty(\mathbb{R}; L^\infty + L^2)$. Now for $z \in \mathbb{C}\backslash\Sigma$ we differentiate (7.25) to obtain

$$\begin{aligned} 2\pi i\left(\frac{\partial}{\partial x} - z\mathfrak{d}\right)m(x,z) &= \int_\Sigma (\zeta - z)^{-1}\left[\frac{\partial}{\partial x} - \zeta\mathfrak{d} + (\zeta - z)\mathfrak{d}\right]m(x,\zeta)e^{x\zeta\mathfrak{d}}w(\zeta)\,d\zeta \\ &= \int_\Sigma (\zeta - z)^{-1}q(x)m(x,\zeta)e^{x\zeta\mathfrak{d}}w(\zeta)\,d\zeta + 2\pi i q(x) \\ &= 2\pi i q(x)\{m(x,z) - I + I\} = 2\pi i q(x)m(x,z). \end{aligned} \tag{8.28}$$

# 9 The Inverse Problem with Small Data, II

In this section we strengthen the hypotheses on the function $w$ of (8.1), with respect to decay at $\infty$ and with respect to smoothness, to obtain results corresponding to Theorem G. We consider first the condition

$$(1 + |z|)^k w_\nu \in L^2(\Sigma_\nu). \tag{9.1}$$

**Theorem 9.2.** *Let $w$, $m$, and $q$ be as in Theorem 8.9. If $w$ satisfies* (9.1), *then*

$$D^j q \in L^\infty + L^2, \quad j \le k. \tag{9.3}$$

Proof: Assume first that $w$ has compact support. As noted above, this implies that $m$ is analytic in $x$ with $L^\infty + L^2$ values, and $q$ is analytic. It is enough to establish bounds on $D^j q$ in $L^\infty + L^2, j \le k$, which (under the assumption (8.10)) depend only on the pair

$$N_k = \{||w||_\infty, ||(1 + |z|)^k w||_2\}. \tag{9.4}$$

We have this result for $k = 0$. Note also that in (8.12) we have

$$2\pi i q(z) = \int (m(x,z) - I)e^{xz\mathfrak{J}}w(z)\,dz + \int e^{xz\mathfrak{J}}w(z)\,dz. \tag{9.5}$$

As pointed out above, the first term is in $L^\infty$. The second term has $L^1$ norm dominated by

$$\begin{aligned}\|w\|_1 &= \|(1+|z|)^{-1}(1+|z|)w\|_1 \\ &\le \|(1+|z|)^{-1}\|_2\|(1+|z|)w\|_2,\end{aligned}$$

again because it is essentially a Fourier transform. This shows that $\|q\|_\infty$ is dominated by $N_1$.

We now induce: suppose we know that $D^j q$ in $L^\infty + L^2$, $j \le k-1$ is controlled by $N_{k-1}$, and also that $\|D^{k-1}q\|_\infty$ is controlled by $N_k$. Repeated differentiation of (8.12) gives an expression for $D^k q$ as a linear combination of integrals with integrands which (apart from occurrences of the operator $\mathfrak{J}$) are of the forms

$$z^j p(x)[m(x,z) - I]e^{xz\mathfrak{J}}w(z), \tag{9.6}$$

$$z^j p(z)e^{xz\mathfrak{J}}w(z), \quad j \le k, \tag{9.7}$$

where $p$ is a product of derivatives of order less than $k$ of $q$. By the induction assumption, $\|p\|_\infty$ is controlled by $N_k$ and it can be ignored. The term (9.6) gives a function with $L^\infty$ norm controlled by $\|m(x,\cdot) - I\|_2$ and $N_k$, hence by $N_k$. The term (9.7), as before, has $L^2$ norm controlled by $N_k$ and $L^\infty$ norm controlled by $N_{k+1}$. This completes the induction, and the proof.

In order to consider the effect of smoothness of $w$, we introduce two spaces of functions on $\Sigma$ and an extension of Lemma 8.1. Recall that

$$D^j f_\nu \in L^2(\Sigma_\nu), \quad 0 \le j \le k+1, \tag{9.8}$$

implies, after correction on a set of measure zero,

$$f_\nu \in C^k(\Sigma_\nu), D^j f_\nu \to 0 \text{ as } z \to \infty, \quad j \le k. \tag{9.9}$$

**Definition 9.10.** For $k$ an integer greater than or equal to 0, we denote by $H^{k+1}(\Sigma)$ the space of matrix-valued functions $f = \{f_\nu\}$ satisfying (9.8) and such that

$$D^j f_\mu(0) = D^j f_\nu(0) \text{ for all } \mu,\nu, \quad j \le k. \tag{9.11}$$

We denote by $H_0^{k+1}(\Sigma)$ the subspace consisting of $f$ such that

$$D^j f_\nu(0) = 0 \text{ for all } \nu, \quad j \le k. \tag{9.12}$$

The Sobolev norm

$$\|f\|_{2,k+1}^2 = \sum_{j\leq k+1} \|D^j f\|_2^2 \tag{9.13}$$

makes $H^{k+1}(\Sigma)$ a Hilbert space. $H^{k+1}(\Sigma)$ is also an algebra under pointwise multiplication, and $H_0^{k+1}(\Sigma)$ is a closed ideal.

**Lemma 9.14.** *The operators $C^\pm$ of (7.24) are bounded from $H_0^k(\Sigma)$ to $H^k(\Sigma)$. Moreover, if $\{\alpha_\nu\}$ and $e_\lambda$ are as in (8.1), then*

$$\|C^\pm(e_\lambda f)\|_2 \leq C_k|\lambda|^{-k}\|f\|_{2,k}, \quad f \in H_0^k(\Sigma), \pm\lambda < 0. \tag{9.15}$$

Proof: For a smooth function $f$ with support not containing the origin, it is clear that, along any line through the origin,

$$\frac{d}{dz}C^\pm(f) = C^\pm\left(\frac{df}{dz}\right). \tag{9.16}$$

Such functions are dense in $H_0^k(\Sigma)$, so the first statement follows from $L^2$-boundedness of $C^\pm$. To prove (9.15) we argue as in the proof of Lemma 8.1. For $C_\nu^+$, note in (8.8) that $f \in H^k(\Sigma)$ implies $(1+|\xi|)^k\hat{f}(\xi) \in L^2(R)$; thus

$$\|h_\lambda \hat{f}\|_2 \leq C_k(1+|\lambda|)^{-k}\|f\|_{2,k}, \quad \lambda < 0.$$

Consider now $C_{\mu,\nu}$ when $\mu \neq \nu$. For ease of notation we suppose $\Sigma_\nu = R_+$, $\Sigma_\mu = R_-$, and then change signs on $\Sigma_\mu$ to consider the map in $L^2(R_+)$ with kernel $(t+s)^{-1}$. We want to estimate

$$h(\lambda, t) = \int_0^\infty (t+s)^{-1}e^{i\lambda s}f(s)\,ds, \quad t \geq 0. \tag{9.17}$$

We have

$$\begin{aligned} i\lambda h(\lambda,t) &= \int_0^\infty \frac{d}{ds}(e^{i\lambda s})(t+s)^{-1}f(s)\,ds \\ &= -\int_0^\infty e^{i\lambda s}(t+s)^{-1}Df(s)\,ds + \int_0^\infty e^{i\lambda s}(t+s)^{-2}f(s)\,ds. \end{aligned} \tag{9.18}$$

The first term on the right has $L^2$ norm in $t$ dominated by $\|Df\|_2$. The second term is dominated by

$$\int_0^\infty (t+s)^{-2}\int_0^s |Df(u)|\,du = \int_0^\infty (t+u)^{-1}|Df(u)|\,du \tag{9.19}$$

and again the $L^2$ norm in $t$ is dominated by $\|Df\|_2$. This proves (9.15) for $C_{\mu,\nu}$ when $k = 1$, and the argument extends in the obvious way to larger $k$.

**Theorem 9.20.** *Let $w$, $m$, and $q$ be as in Theorem 8.9. Suppose*

$$w \in H_0^{k+1}(\Sigma), \quad \|w\|_{2,k+1} < \delta_k, \tag{9.21}$$

*where $\delta_k > 0$ is sufficiently small. Then, for all $x$ in $R$,*

$$m(x,\cdot) - I \in H^{k+1}(\Sigma), \tag{9.22}$$

$$\|m(x,\cdot) - I\|_2 = O(|x|^{-k-1}), \quad x < 0. \tag{9.23}$$

*Moreover, there is a function $s$ on $\Sigma$ with properties*

$$s - I \in H^{k+1}(\Sigma), \tag{9.24}$$

$$\|m(x,\cdot) - s(x,\cdot)\|_2 = O(x^{-k-1}), \quad x > 0, \tag{9.25}$$

*where $s(x,z) = e^{xz\partial}s(z)$. Finally,*

$$(1+|x|)^{k+1} q \in L^\infty(R) + L^2(R). \tag{9.26}$$

Proof: Fix $x$ and consider the operator

$$B_x f(z) = \frac{d}{dx} f(z) - x\partial f(z). \tag{9.27}$$

Let

$$\|f\|^2_{2,k+1,x} = \sum_{j \le k+1} \|B_x f\|^2. \tag{9.28}$$

This is equivalent to the $H^{k+1}(\Sigma)$ norm. Set

$$w_x(z) = e^{xz\partial} w(z). \tag{9.29}$$

Clearly,

$$B_x(w_x f) = w_x B_x f + (Dw)_x f. \tag{9.30}$$

The $L^1$ norm of the Fourier transform of $f$ can be estimated by the Schwarz inequality to obtain $\|f\|_\infty \le c\|f\|_{2,1,x}$ with $c$ independent of $x$. Thus, iterating (9.30) and estimating $L^2$ norms, we get

$$\|w_x f\|_{2,k+1,x} \le c_k \|w_x\|_{2,k+1} \|f\|_{2,k+1,x}. \tag{9.31}$$

Here and below $c_k$ will denote various constants depending only on $\Sigma$ and $k$. Recall that $H_0^{k+1}(\Sigma)$ is an ideal in the algebra $H^{k+1}(\Sigma)$. Integration by parts shows that the operators $C_\pm$ of (7.24) map

$$C_\pm : H_0^{k+1}(\Sigma) \mapsto H^{k+1}(\Sigma). \tag{9.32}$$

Thus $C_{w,x}$ maps $H^{k+1}(\Sigma)$ to itself with norm

$$||C_{w,x}f||_{2,k+1,x} \le c_k ||w||_{2,k+1} ||f||_{2,k+1,x}. \tag{9.33}$$

Also, clearly

$$||C_{w,x}(I)||_{2,k+1,x} \le c_k ||w||_{2,k+1}. \tag{9.34}$$

It follows from (9.33) and (9.34) that (9.21) with $\delta_k$ small enough gives

$$m(x,\cdot) - I = \sum_{n=1}^{\infty} C_{w,x}^n(I) \in H^{k+1}(\Sigma). \tag{9.35}$$

To get the $L^2$ estimate (9.23) we note that Lemma 9.14 implies

$$||C_{w,x}(I)||_2 = O(|x|^{-k-1}), \quad x < 0. \tag{9.36}$$

Thus (9.23) follows from the identity (9.35).

To obtain the function $s$ we set

$$\begin{aligned} C_{w,x,0}f(z) &= f(z)e^{xzJ}[w^-(z) - w^+(z)] \\ &= f(z)e^{xzJ}\tilde{w}(z), \end{aligned} \tag{9.37}$$

$$\begin{aligned} C_{w,x,1}f &= C_{w,x}f - C_{w,x,0}f \\ &= C^+(fw^+(x,\cdot)) - C^-(fw^-(x,\cdot)), \end{aligned} \tag{9.38}$$

in the notation of (7.17), (7.18). Dropping the subscripts $w$ and $x$, we write

$$(C_0 + C_1)^N = C_0^N + \sum_{M=1}^{N} (C_0 + C_1)^{N-M} C_1 C_0^{M-1}. \tag{9.39}$$

It is clear from Lemma 9.14 that the off-diagonal part of $C_1 C_0^{M-1}(I)$ has $L^2$ norm less than or equal to

$$x^{-k-1} c_k^M ||w||_{2,k+1}^M, \quad x > 0. \tag{9.40}$$

The diagonal part of $C_1 C_0^{M-1}(I)$ is independent of $x$. We apply $(C_0 + C_1)^{N-M}$ to this diagonal part and use the identity (9.39) with $N$ replaced by $N - M$. At the next occurrence of $C_1$ we again dominate the $L^2$ norm of the off-diagonal part by an expression like (9.40), and iterate for the diagonal part. This procedure yields

$$(C_0 + C_1)^N(I) = \sum_{M=0}^{N} C_0^{N-M} \delta_{N,M} + r_N, \tag{9.41}$$

where $\delta_{N,M}$ is diagonal and independent of $x$, while, for $x > 0$,

$$x^{k+1} ||r_N||_2 \le N c_k^N ||w||_{2,k+1}^N, \tag{9.42}$$

$$||\delta_{N,M}||_{2,k+1} \le c_k^M ||w||_{2,k+1}^M. \tag{9.43}$$

Now we set

$$\begin{aligned} s(z) &= e^{-z\mathfrak{J}}\Big(I + \sum_{N=1}^{\infty}\sum_{M=0}^{N} C_0^{N-M}\delta_{N,M}(z)\Big) \\ &= I + \sum_{N=1}^{\infty} \tilde{w}(z)^{N-M}\delta_{N,M}(z). \end{aligned} \tag{9.44}$$

The estimates (9.43) give (9.24) if $\delta_k$ in (9.21) is small enough, and the estimates (9.42) yield (9.25).

Finally, we want to obtain information on the potential $q$. We use (8.12) again and assume $x \geq 0$; the argument for $x < 0$ is the same but uses $I$ in place of $s(x,z)$. We have

$$\begin{aligned} q(x) &= \mathfrak{J}\int m(x,z)e^{xz\mathfrak{J}}w(x)\,dz \\ &= \mathfrak{J}\int [m(x,z) - s(x,z)]w(x,z)\,dz + \mathfrak{J}\int e^{xz\mathfrak{J}}[s(z)w(z)]\,dz. \end{aligned} \tag{9.45}$$

The $L^\infty$ norm of the first term on the right above is dominated by the $L^2$ norm of $m(x,\cdot) - s(x,\cdot)$, since $w(x,\cdot)$ is in $L^2$ uniformly with respect to $x$. Thus (9.25) gives the desired estimate, $O(x^{-k-1})$. From (9.24) we have $sw \in H_0^{k+1}$. Because of the operator $\mathfrak{J}$, only off-diagonal entries appear, and these are dilates of Fourier transforms of the entries of $sw$, hence have $L^2$ norm which is $O(x^{-k-1})$.

# 10 The Inverse Problem Near $-\infty$

Suppose $v$ belongs to the space $\mathbf{S}$ of formal scattering data,

$$v = (v_1, \cdots, v_r; z_1, \cdots, z_N; v(z_1), \cdots, v(z_N)).$$

We shall see that a rational approximation and the results of Sections 8 and 9 will allow us to reduce the inverse problem for $v$ to a finite set of linear equations, with $x$ a parameter.

**Definition 10.1.** A matrix-valued function $u$ defined on $\mathbb{C}\backslash\Sigma$ is *piecewise rational* if on each component $\Omega_\nu$ of $\mathbb{C}\backslash\Sigma$ it coincides with a rational function which has no singularities on the boundary $\Sigma_\nu \cup \Sigma_{\nu+1}$.

As before we denote by $u_\nu^-$ and $u_\nu^+$ the limits on $\Sigma_\nu$ from $\Omega_\nu$ and $\Omega_{\nu+1}$.

**Lemma 10.2.** *Given $v \in \mathbf{S}$ and $\epsilon > 0$, there is a piecewise rational function $u$ with the properties*

$$u_{jj} \equiv 1, \quad u_j(z)^\nu \text{ is upper triangular in } \Omega_\nu, \tag{10.3}$$

$$u \to I \text{ as } z \to \infty, \tag{10.4}$$

$$\|u_\nu^- v_\nu (u_\nu^+)^{-1} - I\|_\infty < \epsilon, \tag{10.5}$$

$$u_\nu^-(0)v_\nu(0)u_\nu^+(0)^{-1} = I. \tag{10.6}$$

Proof: Let $\{a_\nu\}$ be the (unique) matrices satisfying (1.25). Choose a piecewise rational function $a$ having no singularities, such that $a$ satisfies (10.3) and (10.4), and such that

$$a(z) \to a_\nu \text{ as } z \to 0, \quad z \in \Omega_\nu. \tag{10.7}$$

The matrices

$$[a_\nu^-(z)v_\nu(z)a_\nu^+(z)^{-1}]^\nu, \quad z \in \Sigma_\nu \tag{10.8}$$

have lower minors $\equiv 1$, because of (1.26) for $v$ and (10.3) for $a$. It follows that there is a unique factorization of (10.8) as

$$[b_\nu^-(z)^{-1}b_\nu^+(z)]^\nu, \tag{10.9}$$

where

$$(b_\nu^\pm)_{jj} = 1, \quad (b_\nu^-)^\nu \text{ is upper triangular;} \tag{10.10}$$

$$\Pi_\nu b_\nu^+ = b_\nu^+, \quad (b_\nu^+)^\nu \text{ is lower triangular.} \tag{10.11}$$

In fact, $(\Pi_\nu b_\nu^\pm)^\nu$ are the triangular factors of the $\Pi_\nu$ projection of (10.9), and $b_\nu^-$ is then determined from (10.11) and the equality of (10.8) and (10.9). The uniqueness implies

$$b_\nu^\pm(0) = I. \tag{10.12}$$

From condition (10.11) it follows that $(b_\nu^+)^{\nu+1}$ is upper triangular on $\Sigma_\nu$. Since $(b_{\nu+1}^-)^{\nu+1}$ is upper triangular on $\Sigma_{\nu+1}$ and both are the identity at the origin, continuous, and approach $I$ at $\infty$, we may approximate both on the boundary of $\Omega_{\nu+1}$ by a rational function; see part 2 of the Appendix. Thus, given $\delta > 0$, there is a piecewise rational function $c = c_\delta$ which satisfies (10.3) and also

$$c_\nu^\pm(0) = I, \quad \|c_\nu^\pm - b_\nu^\pm\|_\infty < \delta. \tag{10.13}$$

With $\delta$ to be chosen later, set

$$u(z) = c(z)a(z). \tag{10.14}$$

Then

$$\begin{aligned} u_\nu^- v_\nu (u_\nu^+)^{-1} &= c_\nu^-[a_\nu^- v_\nu (a_\nu^+)^{-1}](c_\nu^+)^{-1} \\ &= c_\nu^-(b_\nu^-)^{-1}b_\nu^+(c_\nu^+)^{-1}, \end{aligned} \tag{10.15}$$

and (10.13) with $\delta$ sufficiently small gives (10.5) and (10.6).

Define

$$v_\nu^\# = u_\nu^- v_\nu (u_\nu^+)^{-1}. \tag{10.16}$$

Because of (10.3), $v_\nu^\#$ satisfies the defining conditions (1.25) and (1.26) for elements of **S**. Because of (10.4), it is also clear that $v_\nu^\# - I$ and its derivative belong to $L^2(\Sigma_\nu)$. It follows that if $\epsilon$ in (10.5) is small enough, we may apply Theorem 8.9 and obtain an associated eigenfunction $m^\#$, piecewise holomorphic, with

$$(m^\#)_\nu^+(x,z) = (m_\nu^\#)^-(x,z)e^{xz\partial}v_\nu^\#(z), \quad z \in \Sigma_\nu. \tag{10.17}$$

**Lemma 10.18.** *Suppose $v \in$ **S**, and suppose $v_\nu^\#$ is given by* (10.16), *where $u$ is as in Lemma 10.2 and $\epsilon$ is small enough so that $\{v_\nu^\#\}$ has associated eigenfunction* (10.17) *for all $x$. Then, for any $x \le 0$, if $v$ has an associated eigenfunction $m(x,\cdot)$, it is of the form*

$$m(x,z) = r(x,z)m^\#(x,z)e^{xz\partial}u(z), \tag{10.19}$$

*where $r(x,\cdot)$ is rational.*

Proof: First, set

$$m_0(x,z) = m^\#(x,z)e^{xz\partial}u(z) = m^\#(x,z)u^x(z) \tag{10.20}$$

and note that

$$\begin{aligned}(m_{0,\nu})^+ &= (m_\nu^\#)^+(u_\nu^x)^+ \\ &= (m_{0,\nu})^- v_\nu^x\end{aligned} \tag{10.21}$$

by (10.17) and (10.16). The differential equation (8.14) and asymptotic condition (8.11) imply once again that

$$\det m^\# \equiv 1 \tag{10.22}$$

and the same is true of $m_0$. Therefore, if $m(x,\cdot)$ is an eigenfunction associated to $v$ and if we define

$$r(x,z) = m(x,z)m_0(x,z)^{-1}, \tag{10.23}$$

we find that

$$r_\nu^+ = r_\nu^-. \tag{10.24}$$

Clearly, $r(x,\cdot)$ is meromorphic in $\mathbb{C}\backslash\Sigma$ since $m^\#$, $m$, and $u$ are; hence $r(x,\cdot)$ is rational.

*Remark 10.25.* The piecewise rational function $u$ has the same singularities as the function $c$ in the proof of Lemma 10.2. The latter function can be chosen to have only simple poles, and the locations can be chosen to be distinct from the $\{z_j\}$ of $v$ and to be distinct for distinct entries. It follows from this that at any singularity in $\Omega_\nu$ the residue of $u$ is strictly upper triangular in the $\nu$-representation and has only one non-zero row; thus its square is zero. We say that such a function $u$ is *regular*, and we assume that $u$ is chosen to be regular.

We now fix $x \in \mathbb{R}$ and look for a rational function $r(x,\cdot)$ so that when $m(x,\cdot)$ is *defined* by (10.19), it is the associated eigenfunction for $v$. (We remark at this point that the uniqueness proof in the case $q \mapsto v$ shows that formal scattering data has at most one associated eigenfunction, given $x$). The function $r$ should have only simple poles and should be $I$ at $\infty$; thus

$$r(x,z) = I + \sum_{k=1}^{P} (z - z_k)^{-1} a_k, \tag{10.26}$$

where $z_1, \cdots, z_N$ are the singularities of $m$ and $z_{N+1}, \cdots, z_P$ are the singularities of $u$. Then

$$r(x,z) = (z - z_j)^{-1} a_j + b_j + O(|z - z_j|). \tag{10.27}$$

If $j \le N$, then $m_0 = m^{\#} u^x$ is regular at $z_j$,

$$m_0(x,z) = c_j + (z - z_j) d_j + O(|z - z_j|^2), \tag{10.28}$$

where $c_j = m_0(x, z_j)$ is invertible. Let

$$v_j = \exp\{x z_j \mathcal{J}\} v(z_j). \tag{10.29}$$

We would like to have

$$\text{Res}\,(m(x,\cdot), z_j) = \lim_{z \to z_j} m(x,z) v_j, \tag{10.30}$$

which is equivalent to

$$a_j c_j v_j = 0, \quad j \le N, \tag{10.31}$$

$$(a_j d_j + b_j c_j) v_j = a_j c_j, \quad j \le N. \tag{10.32}$$

Note that the condition (1.28) in the definition of formal scattering data implies

$$v_j^2 = 0. \tag{10.33}$$

Therefore, (10.31) is a consequence of (10.32).

If $j > N$, then $u$ is singular at $z_j$,

$$e^{xz\mathcal{J}} u(z) = (z - z_j)^{-1} u_j + n_j + O(|z - z_j|). \tag{10.34}$$

Note that $n_j^\nu$ is upper triangular if $z_j \in \Omega_\nu$, and the diagonal part is $I$; thus $n_j$ is invertible. Then, as in the remark above,

$$(u_j n_j^{-1})^2 = 0. \tag{10.35}$$

The function $m^{\#}(x,\cdot)$ is regular at $z_j$; therefore,

$$m_0(x,z) = (z - z_j)^{-1} a_j u_j + (\beta_j u_j + \alpha_j n_j), \tag{10.36}$$

where $\alpha_j = m^{\#}(x, z_j)$ is invertible. We want $m(x, \cdot)$ to have a removable singularity at $z_j, j > N$. From (10.27) and (10.36) this is equivalent to

$$a_j \alpha_j u_j = 0, \quad j > N, \tag{10.37}$$

$$b_j \alpha_j u_j + a_j(\beta_j u_j + \alpha_j n_j) = 0, \quad j > N. \tag{10.38}$$

These in turn are equivalent to

$$a_j \alpha_j u_j n_j^{-1} = 0, \quad j > N, \tag{10.39}$$

$$a_j \alpha_j = (b_j a_j - a_j \beta_j) u_j n_j^{-1}, \quad j > N. \tag{10.40}$$

Because of (10.35), equations (10.39) are a consequence of (10.40), or (10.38).

Consider now the necessary and sufficient conditions (10.32), (10.38). The $c_j$, $d_j$, $\alpha_j$, $\beta_j$, $u_j$, and $n_j$ are determined by $m^{\#}$ and $u$. We have also

$$b_j = I + \sum_{k \neq j} (z_j - z_k)^{-1} a_k. \tag{10.41}$$

Thus (10.32), (10.38) are $Pn^2$ equations for the $Pn^2$ unknown coefficients of the $a_k$. Since $c_j, \alpha_j$, and $n_j$ are invertible, these equations would have only the trivial solution $a_k = 0$, all $k$, if we had

$$v_j = 0, j \leq N, \quad u_j = 0, j > N. \tag{10.42}$$

Thus (10.32), (10.38) have a unique solution for almost all choices of the matrices $\alpha_j$, $\beta_j$, $c_j$, $d_j$, $u_j$, $n_j$, and the entries are rational functions $\rho_i$ of the entries of these matrices. The functions $\rho_i$ are independent of $x$.

As $x \to -\infty$, we have, near points $z_j, j \leq N$,

$$m^{\#}(x, z) \to I, \quad \exp\{x z_j \mathfrak{J}\} u(z) \to I, \quad \exp\{x z_j \mathfrak{J}\} v(z_j) \to 0. \tag{10.43}$$

Thus

$$c_j(x) \to I, \quad d_j(x) \to 0, \quad v_j(x) \to 0. \tag{10.44}$$

Similarly, for $j > N$ as $x \to -\infty$ we have

$$\alpha_j \to I, \quad \beta_j \to 0, \quad u_j \to 0, \quad n_j \to I. \tag{10.45}$$

*Remark 10.46.* We have proved half of Theorem F (a), namely the fact that there is an associated eigenfunction as $x \to -\infty$. Note that the convergence of $v_j$ and $u_j$ in (10.43) and (10.44) is exponential; examination of (10.32), (10.38) shows that we may conclude that $a_j(x) \to 0$ exponentially at $-\infty$. From this we obtain, for some $\delta > 0$,

$$\|m_\nu^{\pm}(x, \cdot) - (m^{\#})_\nu^{\pm}(x, \cdot)\|_\infty = O(e^{-\delta |x|}) \text{ as } x \to -\infty. \tag{10.47}$$

We have not used the winding number conditions (1.31). In the next section we show that these conditions allow us to transform the scattering data in a way which corresponds to normalizing the eigenfunctions at $+\infty$ instead of $-\infty$. The renormalized problem may then be handled in an analogous fashion, leading to a linear system with coefficients having limits at $+\infty$. It follows, indeed, that (1.31) implies solvability of (10.32), (10.38) for $x \to +\infty$ as well; however, the coefficients grow exponentially in this direction; hence the renormalized system is easier to study theoretically and to solve in practice.

## 11 Solvability Near $+\infty$; Theorems F and G

To investigate solvability of the inverse problem at $+\infty$, let us suppose first that $m$ is the eigenfunction for a generic potential $q$. Let

$$\delta(z) = \lim_{x\to+\infty} m(x,z) \tag{11.1}$$

be the diagonal matrix of (5.9). Then $\tilde{m} = m\delta^{-1}$ is an eigenfunction normalized at $+\infty$. We have, clearly,

$$\tilde{m}_\nu^+(x,z) = \tilde{m}_\nu^-(x,z)e^{xzJ}\tilde{v}_\nu(z), \tag{11.2}$$

$$\tilde{v}_\nu = \delta_\nu^- v_\nu (\delta_\nu^+)^{-1}. \tag{11.3}$$

Thus $\{\tilde{v}_\nu\}$ is the scattering data for the renormalized problem on $\Sigma$. The singularities of $\tilde{m}$ are the same as those of $m$, since $\delta$ and $\delta^{-1}$ are regular where $m$ is. Consider a singularity $z_j \in \Omega_\nu$. For convenience, we suppose that the $\nu$-ordering of the basis vectors coincides with the original ordering. Suppose $m$ is singular at $z_j$ in column $k+1$. According to the discussion in Section 2, this means that the $k$-th diagonal entry of $\delta$, $\delta_k$, has a simple zero at $z_j$; since the $k+1$ upper minor is not zero at $z_j$, $\delta_{k+1}$ must have a simple pole at $z_j$. Thus

$$\delta_k(z)^{-1} = \alpha(z-z_j)^{-1} + O(1), \tag{11.4}$$

$$\delta_{k+1}(z)^{-1} = \alpha(z-z_j) + O(|z-z_j|^2). \tag{11.5}$$

Also,

$$m(x,z) = (z-z_j)^{-1}a + b + O(|z-z_j|), \tag{11.6}$$

where

$$0 \neq a = bv(z_j) = \beta b e_{k,k+1}, \quad \beta \in \mathbb{C}, \tag{11.7}$$

where $e_{ij}$ is the usual matrix unit. There is a similar expression for $\tilde{m}(x,z)$, with

$$0 \neq \tilde{a} = \tilde{b}\tilde{v}(z_j) = \tilde{\beta}\tilde{b}e_{k,k+1}, \quad \tilde{\beta} \in \mathbb{C}; \tag{11.8}$$

in fact the argument giving the form $\beta e_{k,k+1}$ for $v(z_j)$ gives this corresponding form for $\tilde{v}(z_j)$. Since $\tilde{m} = m\delta^{-1}$, we may infer from (11.4) and (11.5) and inspection of columns $k$ and $k+1$ of $\tilde{m}$ that (11.7), (11.8) imply

$$abe_{kk} = \tilde{a} = \tilde{b}\tilde{v}_j = a\gamma\tilde{v}_j = \gamma b v_j \tilde{v}_j; \tag{11.9}$$

therefore

$$\alpha e_{kk} = \gamma v(z_j)\tilde{v}(z_j) = \gamma\beta\tilde{\beta}e_{kk}. \tag{11.10}$$

This may be written in invariant form, using the trace

$$\text{tr } (v(z_j)\tilde{v}(z_j)) = \text{tr } (\text{Res } (\delta, z_j)) \text{ tr } (\text{Res } (\delta^{-1}, z_j)). \tag{11.11}$$

We have shown that the scattering data for the renormalized eigenfunction $\tilde{m}$ is computable from that for $m$, once we know the diagonal matrix $\delta$. To determine $\delta$ from the scattering data for $m$ we note that the condition corresponding to (1.26) is

$$d_k^+(\tilde{v}_\nu(z)^\nu) = 1, \quad 1 \le k \le n, z \in \Sigma_\nu. \tag{11.12}$$

From (11.3) we see that these conditions determine the ratios

$$d_k^+((\delta_\nu^-)^\nu)/d_k^+((\delta_\nu^+)^\nu) = d_k^+(v_\nu^\nu) \tag{11.13}$$

and therefore the ratios of the $(\delta_k)_\nu^\pm$ on $\Sigma_\nu$. The zeros and poles of the $\delta_k$ are also determined by the scattering data; this information, together with the ratios of the limits of the rays $\Sigma_\nu$, uniquely determines the $\delta_k$. In fact, the winding number constraints (1.31) are exactly the conditions that all this data be compatible; see part 1 of the Appendix. Thus starting with $v \in \mathbf{S}$ we may determine uniquely the data $\tilde{v}$ which would correspond to a normalization at $+\infty$. Repeating the procedure of Section 10, we reduce to an algebraic problem which is uniquely solvable as $x \to +\infty$. We obtain eigenfunctions $\tilde{m}(x, \cdot)$ associated to $\tilde{v}$; then $m(x, \cdot)$ defined by $m(x, \cdot) = \tilde{m}(x, \cdot)\delta(\cdot)$ is the eigenfunction associated to $v$.

We have now proved part (a) of Theorem F. To prove part (b) we suppose first that each $v_\nu$ has compact support. The same is true of the transformed data $v_\nu^\#$ of Section 10, and it follows that $m^\#$ is analytic with respect to $x$. Therefore the system of equations (10.32), (10.39) has coefficients depending analytically on $x$. We know now that the system is solvable for $|x|$ large, and hence for all by finitely many values of $x$. Consider the map

$$\varphi : \mathbb{R} \times \mathbb{C}^P \mapsto \mathbb{C} \tag{11.14}$$

obtained by taking the determinant of the system (10.32), (10.38) at $x$, when $v_j$ has been replaced by $\zeta_j v_j, j \le N$, and $u_j$ has been replaced by $\zeta_j u_j, j > N$. For $\zeta_j \cong 1$ this is the system corresponding to a slight perturbation of the original scattering data $v$. Now $\varphi^{-1}(0)$ has real codimension 2, so its projection on

$\mathbb{C}^P$ has real codimension at most 1, and we conclude that there are arbitrarily small perturbations of $v$ for which the associated eigenfunction exists for every $x$. Data with compact support are dense, so $\mathbf{S}_0$ is dense in $\mathbf{S}$.

To see that $\mathbf{S}_0$ is open, we note that in the construction in Section 10, the piecewise rational function $u$ can be chosen to vary continuously with $v$, so $m^\#$ will also vary continuously with $v$. Thus the coefficients of (10.32), (10.38) vary continuously with $v$ and $x$; the system is solvable for large $|x|$ for all $v'$ near a given $v$, and it follows that if $v$ is in $\mathbf{S}_0$ and $v'$ is sufficiently close, then $v'$ is in $\mathbf{S}_0$.

Finally, we need to establish the differential equation (1.6) and prove (1.44). The additive jump of $m(x,\cdot)$ across $\Sigma_\nu$ is

$$\begin{aligned} m_\nu^+(x,z) - m_\nu^-(x,z) &= m_\nu^-(x,z)[e^{xz\mathfrak{J}}v_\nu(z) - I] \\ &= m_\nu^-(x,z)w_\nu(x,z), \end{aligned} \tag{11.15}$$

while if we define

$$m(x,z_j) = \frac{1}{2\pi i}\int_{C_j}(\zeta - z_j)^{-1}m(x,\zeta)\,d\zeta, \tag{11.16}$$

where $C_j$ is a small circle with center $z_j$, then (1.16) is

$$\begin{aligned} \text{Res }(m(x,\cdot);z_j) &= m(x,z_j)\exp\{xz_j\mathfrak{J}\}v(z_j) \\ &= m(x,z_j)v_j(x). \end{aligned} \tag{11.17}$$

From (11.15), (11.16), and the asymptotic behavior as $z \to I$ we see that $m(x,\cdot)$ is a solution of

$$\begin{aligned} m(x,z) = I &+ \frac{1}{2\pi i}\int_\Sigma (z-\zeta)^{-1}m^-(x,\zeta)w(x,\zeta)\,d\zeta \\ &+ \Sigma(z-z_j)^{-1}m(x,z_j)v_j(x). \end{aligned} \tag{11.18}$$

Suppose now that $w$ has compact support. It is then obvious that any solution of (11.18) is asymptotically $I$ as $z \to \infty$ and satisfies (11.15), (11.17). The eigenfunction $m(x,\cdot)$ constructed in Section 10 is invertible where it is regular, so we repeat the proof of uniqueness in Theorem B to conclude that $m_1 m^{-1} \equiv I$. Now still assuming that $w$ has compact support, once again $m$ is analytic in $x$ and we may differentiate (11.18) to see that $m_2 = (\partial/\partial x - z\mathfrak{J})m$ satisfies

$$\begin{aligned} m_2(x,z) = q(x) &+ \frac{1}{2\pi i}\int_\Sigma (\zeta - z)^{-1}m_2(x,\zeta)w(x,\zeta)\,d\zeta \\ &+ \Sigma(z-z_j)^{-1}m_2(x,z_j)v_j(x), \end{aligned} \tag{11.19}$$

where

$$q(x) = \frac{1}{2\pi i}\mathfrak{J}\int_\Sigma m(x,\zeta)w(x,\zeta)\,d\zeta - \Sigma\mathfrak{J}[m(x,z_j)v_j(x)]. \tag{11.20}$$

Again, this equation implies that $m_2$ satisfies (11.15) and (11.17), while $m_2 \sim q$ as $z \to \infty$. Consequently, $m_2 m^{-1} \equiv q$, and this is our differential equation.

To complete the proof of Theorem F (b) it is only necessary to estimate the norm of $(1+|x|)q$ in $L^\infty + L^2$ in terms of the norms of $v$ and $Dv$ in $L^2(\Sigma)$, locally, since we may then pass to the limit from compactly supported $v$. (Observe, in this passage to the limit, that the piecewise rational function $u$ of Lemma 10.2 can be held fixed.) As noted at the end of Section 10, $m_\nu^+(x,\cdot)$ is exponentially close to $(m^\#)_\nu^+(x,\cdot)$ as $x \to -\infty$; the same is true of derivatives with respect to $z$. Since $v_j(x)$ in (11.20) is also exponentially small at $-\infty$, we may estimate $q$ in the same way here as in Section 8, for $x \le 0$. For $x \ge 0$ we repeat the renormalization at $+\infty$ and have formulas of the same type with exponential convergence at $+\infty$.

*Remark 11.21.* The arguments here show that if $v$ belongs to $\mathbf{S}$ but not to $\mathbf{S}_0$, the associated eigenfunction $m(x,\cdot)$ exists on an open set and satisfies the differential equation on that open set, again with $q$ given by (11.20).

When the scattering data evolve according to (1.50), we may let the piecewise rational function $u$ of Lemma 10.2 evolve in the same way. It continues to satisfy the algebraic constraints, and in the stable case (1.52) it also satisfies (10.5). In short, the rational approximation only needs to be computed twice (at $-\infty$ and at $+\infty$) for an equation of evolution.

Proof of Theorem G: For part (a) we may argue exactly as in the proof of Theorem 9.2, except for considering separately the cases $x \le 0$ and $x \ge 0$ in order to have exponential decrease in the discrete terms in (11.20).

For part (b) we examine Lemma 10.2. Using the assumption (1.36) we may suppose that the piecewise rational function $a$ is chosen so as to have the correct Taylor expansion to order $k$ at 0 from $\Omega_\nu$, so that (1.36) holds also for $a$. For the factors $b_\nu^\pm$ this will imply that they are $I + O(z^k)$ at the origin. We approximate the $b$ on the boundary of $\Omega_\nu$ in $C^k(\Omega_\nu)$, and the result is that the new data $\{v_\nu^\#\}$ will have transformed data $\{w_\nu^\#\}$ which satisfies the conditions of Theorem 9.20. Thus $m^\#$ is as in Theorem 9.20. Now once again $m$ is exponentially close to $m^\#$ on $\Sigma$ or on the circles $C_j$ as $x \to -\infty$, so we may argue as in the proof of Theorem 9.20 to obtain (1.47) for $x \le 0$; again the renormalization at $+\infty$ completes the argument.

# 12 Systems with Symmetry; Multisolitons

Suppose $a \to a^\sigma$ is an automorphism of the matrix algebra $M_n(\mathbb{C})$, and suppose $J$ is an eigenvector:

$$J^\sigma = \alpha^{-1} J. \tag{12.1}$$

Let $\mathbf{P}_0$ denote the space of generic $\sigma$-symmetric potentials:

$$\mathbf{P}_0^\sigma = \{q \in \mathbf{P}_0 : q(x)^\sigma \equiv q(x)\}. \tag{12.2}$$

**Theorem 12.3.** *Under assumption* (12.1), $\alpha$ *is a root of unity and* $\Sigma$ *is invariant under multiplication by* $\alpha$. *If* $q$ *belongs to* $\mathbf{P}_0^\sigma$ *and* $v = \{v_\nu, z_j, v(z_j)\}$ *is the associated scattering data, then*

$$v(\alpha z)^\sigma = v(z), \quad z \in \Sigma, \tag{12.4}$$

$$\{z_j\} \textit{ is invariant under multiplication by } \alpha, \tag{12.5}$$

$$\alpha^{-1} v(\alpha z_j)^\sigma = v(z_j). \tag{12.6}$$

*Conversely, if* $q$ *belongs to* $\mathbf{P}_0$ *and the associated scattering data satisfies* (12.4)–(12.6), *then* $q$ *is in* $\mathbf{P}_0^\sigma$.

Proof: The automorphism is inner:

$$a^\sigma = \pi^{-1} a \pi, \quad \text{some } \pi \in M^n(\mathbb{C}). \tag{12.7}$$

From (12.1) it follows that $\pi$ maps the eigenspace for $J$ with eigenvalue $\lambda$ to the eigenspace for eigenvalue $\alpha^{-1}\lambda$, and it follows that $\alpha$ is a root of unity and that $\Sigma$ is invariant under multiplication by $\alpha$. For a matrix-valued function defined on a subset of $\mathbb{C}$ invariant under multiplication by $\alpha$, set

$$f^\#(z) = f(\alpha z)^\sigma. \tag{12.8}$$

In particular note that if $f(z) = zJ$, then $f = f^\#$. It follows for $q \in \mathbf{P}_0^\sigma$ with associated eigenfunction $m$ that $m(x,\cdot)^\#$ satisfies the differential equation also. Therefore, $m = m^\#$, and (12.4), (12.5) follow immediately. The residue at a singularity satisfies

$$\text{Res } (m(x,\cdot), \zeta_j) = \alpha^{-1} \text{ Res } (m(x,\cdot), \alpha\zeta_j)^\sigma \tag{12.9}$$

and (12.6) is a consequence.

Conversely, if the scattering data satisfy (12.4)–(12.6), then it is easy to see that $m(x,\cdot)^\#$ has the same relationship to the scattering data as $m(x,\cdot)$; since $m(x,\cdot)^\#$ also is $I$ at $\infty$, we have $m \equiv m^\#$ and the differential equation implies that $q \equiv q^\sigma$.

We suppose now that $\alpha$ is a primitive $n$-th root of unity, which is equivalent to assuming that $\pi$ is a cyclic permutation of the eigenspaces of $J$. Then $\pi^n$ is scalar, and we may replace $\pi$ by a scalar multiple so that $\pi^n = I$. After a change of basis and rescaling of the eigenvalue problem we may assume

$$J = \text{diag } (\alpha, \alpha^2, \cdots, \alpha^{n-1}, 1), \tag{12.10}$$

$$\pi = e_{12} + e_{23} + \cdots + e_{n1}, \tag{12.11}$$

where the $e_{jk}$ are the matrix units in $M^n(\mathbb{C})$.

The key fact is then that the subalgebra fixed by $\sigma$,

$$M_n(\mathbb{C})^\sigma = \{a \in M_n(\mathbb{C}) : a^\sigma = a\}, \tag{12.12}$$

is *commutative*: it is the commutator of $\pi$ and consists of polynomials in $\pi$.

Under these assumptions we consider the construction of an eigenfunction for scattering data which vanish on $\Sigma$. As above, the problem becomes an algebraic one. In this case the symmetries and the commutativity allow an explicit computation. Let the singularities be

$$\{\alpha^{-k} z_j : 1 \le j \le N, 0 \le k < n\}, \tag{12.13}$$

and let these points be distinct. The symmetry condition implies that if one column of $m$ has a singularity at point $z_0$, then the last column has a singularity at $\alpha^k z_0$, some $k$. Therefore we may assume for convenience that it is the last column which is to be singular at $z_1, \cdots, z_N$. The matrix $v(z_j)$ is of the form $c_j e_{d_j n}$ for some constant $c_j$ and some index $d_j < n$. Then

$$\exp\{x z_j \mathfrak{J}\} v(z_j) = \exp\{x(\zeta_j - z_j)\} v(z_j), \tag{12.14}$$

$$\zeta_j = \alpha^{d_j} z_j \neq z_j. \tag{12.15}$$

Given a rational matrix-valued function $f$, we define as before

$$f(z_j) = \frac{1}{2\pi i} \int_{C_j} (z - z_j)^{-1} f(z)\, dz, \tag{12.16}$$

where $C_j$ is a small circle around $z_j$. We set

$$C_{v,x} f(x) = \sum_{j=1}^{N} \exp\{x(\zeta_j - z_j)\} [f(z_j) v_j]^{\text{sym}} (zJ - z_j)^{-1}, \tag{12.17}$$

where $b^{\text{sym}}$ is the symmetrized version of the matrix $b$:

$$b^{\text{sym}} = \sum_{k=0}^{n-1} \pi^{-k} b \pi^k. \tag{12.18}$$

Then

$$(C_{v,x} f)^{\#} = C_{v,x} f \tag{12.19}$$

and

$$\text{Res}\,(C_{v,x} f; z_j) = f(z_j) \exp\{x z_j \mathfrak{J}\} v(z_j). \tag{12.20}$$

From the symmetry condition (12.19) we see that (12.20) also holds with $z_j$ replaced by $\alpha^{-k} z_j$. Therefore the desired eigenfunction $m(x, \cdot)$ is precisely the solution of

$$m(x, \cdot) = I + C_{v,x} m(x, \cdot). \tag{12.21}$$

Consider the formal Neumann series solution of (12.21). We have

$$C_{v,x}(I) = \Sigma \exp\{x(\zeta_j - z_j)\} v_j (zJ - z_j)^{-1}, \tag{12.22}$$

where

$$v_j = v(z_j)^{\text{sym}}. \tag{12.23}$$

In general, if $f$ is of the form

$$f(z) = \Sigma a_j (zJ - z_j)^{-1}, \quad a_j = a_j^\sigma, \tag{12.24}$$

then

$$C_{v,x} f(z) = \Sigma b_j (zJ - z_j)^{-1}, \tag{12.25}$$

where

$$b_k = \Sigma a_j A(x)_{jk}, \tag{12.26}$$

$$A(x)_{jk} = \exp\{x(\zeta_k - z_k)\}(\zeta_k - z_j)^{-1} v_k. \tag{12.27}$$

We consider $A(x)$ as an $(N \times N)$ matrix with entries in the commutative algebra $M_n(C)^\sigma$ and write it as a product of such matrices:

$$A(x) = \Delta(x) B(x) V \Delta(x)^{-1}, \tag{12.28}$$

where $\Delta(x)$ and $V$ are diagonal:

$$\Delta(x)^{jj} = \exp\{x z_j\} I, \quad V_{jj} = v_j, \tag{12.29}$$

and

$$B(x)_{jk} = \exp\{x(\zeta_k - z_j)\}(\zeta_k - z_j)^{-1} I. \tag{12.30}$$

Let $\Delta_2(x)$ be the diagonal $M_n(C)^\sigma$-valued matrix with

$$\Delta_2(x)_{jj} = \exp\{x\zeta_j\} I. \tag{12.31}$$

Let $\mathbf{1}$ denote the $M_n(C)^\sigma$-valued row vector with $N$ entries, each of them the identity matrix. Then from the above considerations we see that the formal Neumann series solution of (12.21) is given by

$$m(x, z) = I + \Sigma a_j(x)(zJ - z_j)^{-1}, \tag{12.32}$$

where

$$\begin{aligned} a(x) &= (a_1(x), a_2(x), \cdots, a_N(x)) \\ &= \sum_{s=0}^{\infty} \mathbf{1} V \Delta_2(x)(B(x)V)^s \Delta_1(x)^{-1}. \end{aligned} \tag{12.33}$$

The corresponding potential, as in Section 11, is

$$q(x) = -\mathcal{J} \Sigma \text{ Res } (m(x, \cdot)) \tag{12.34}$$

and from (12.32) we calculate that the sum of the residues is

$$\Sigma \text{ Res } (m(x,\cdot)) = \Sigma a_j J^{-1}. \tag{12.35}$$

Thus

$$q(x) = \Sigma a_j(x) - J(\Sigma a_j(x))J^{-1}. \tag{12.36}$$

Now we can represent $\Sigma a_j(x)$ as the matrix-valued trace of the matrix-valued matrix

$$\mathbf{1}' \cdot a(x) = \sum_{s=0}^{\infty} \mathbf{1}' \cdot \mathbf{1} V \Delta_2(x)(B(x)V)^s \Delta_1(x)^{-1}. \tag{12.37}$$

Relation (12.30) shows that

$$\frac{d}{dx} B(x) = \Delta_1(x)^{-1} \mathbf{1}' \cdot \mathbf{1} \Delta_2(x). \tag{12.38}$$

Note also that $V$ and $\Delta_2(x)$ commute. Since the trace of (12.37) is unchanged under conjugation by $\Delta_1(x)$, it is the same as the trace of

$$\left(\frac{d}{dx} B(x)V\right)(I - B(x)V)^{-1}. \tag{12.39}$$

The trace of (12.37) is the derivative of the trace of $-\log(I - B(x)V)$, which is the negative of the logarithm of $\det(I - B(x)V)$. Therefore we have the (formal) calculation

$$q = JF^{-1}\frac{dF}{dx}J^{-1} - F^{-1}\frac{dF}{dx}, \tag{12.40}$$

where $F$ is the matrix-valued determinant,

$$F(x) = \det(I - B(x)V). \tag{12.41}$$

When the formal scattering data belongs to $\mathbf{S}$, the exponentials are rapidly vanishing at $-\infty$ and the series (12.33) converges for $x \ll 0$. It follows that (12.40) defines the corresponding potential wherever $m(x,\cdot)$ exists.

# Appendix

We sketch the derivation of two facts used above which are extensions of well-known results.

### A.1. The Scalar Factorization Problem

As before, let $\Sigma$ be a union of lines through the origin. Write $\Sigma \backslash (0)$ as a union of open rays $\Sigma_1, \Sigma_2, \cdots, \Sigma_r$, where $\Sigma_\nu$ and $\Sigma_{\nu+1}$ form (with the origin) the boundary of a component $\Omega_\nu$ of $\mathbb{C}\backslash\Sigma$, and $\Sigma_{r+1} = \Sigma_1$. The $\Sigma_\nu$ are indexed in order of increasing argument. The problem to be considered is the following.

A1. Suppose for $1 \le \nu \le r$ that $\varphi_\nu$ is a continuous nonvanishing complex function on the closure of $\Sigma_\nu$ with $\varphi_\nu - 1$ and $D\varphi_\nu$ in $L^2$. Find functions $\delta_\nu$ meromorphic on $\Omega_\nu$ with simple zeros and simple poles at prescribed points of $\Omega_\nu$ and no other zeros in $\Omega_\nu$ such that $\delta_\nu$ extends continuously to the boundary of $\Omega_\nu$, has no zeros on the boundary, and has limit 1 at $\infty$; moreover $\delta_\nu = \delta_{\nu-1}\varphi_\nu$ on $\Sigma_\nu$, where $\delta_0 = \delta_r$.

**Theorem A2.** *Problem* A1 *has at most one solution. A solution exists if and only if*

$$\varphi_1(0)\varphi_2(0)\cdots\varphi_r(0) = 1, \tag{A3}$$

$$\sum_{\nu=1}^{r} \int_{\Sigma_\nu} d(\arg\varphi_\nu) = 2\pi(N - P), \tag{A4}$$

*where $N$ is the number of zeros, $P$ the number of poles, and the $\Sigma_\nu$ are oriented from 0 to $\infty$.*

Proof: *Uniqueness.* In a simpler version of the argument at the end of Section 4, the quotient of two solutions has removable singularities at the prescribed zeros and poles and on $\Sigma$ and is 1 at $\infty$.

*Necessity.* Since $\varphi_\nu(0) = \delta_{\nu+1}(0)\delta_\nu(0)^{-1}$, with $\delta_{r+1} = \delta_1$, condition (A3) is immediate. If $N_\nu$ and $P_\nu$ are the numbers of zeros and poles at $\Omega_\nu$, then the argument principle gives

$$-\int_{\Sigma_{\nu+1}} d(\arg\delta_\nu) + \int_{\Sigma_\nu} d(\arg\delta_\nu) = 2\pi(N_\nu - P_\nu). \tag{A5}$$

On $\Sigma_\nu$, $\arg\delta_\nu = \arg\delta_{\nu-1} + \arg\varphi_\nu$. Inserting this identity into the second term on the left in (A5) and summing, we get (A4).

*Sufficiency.* It is convenient to consider transformations of the problem. Suppose $f_1, \cdots, f_r$ are rational functions having only simple zeros and poles, having no zeros or poles on $\Sigma$, and equal to 1 at $\infty$. Look for the $\delta_\nu$ in the form

$$\delta_\nu = f_\nu\delta_\nu^*. \tag{A6}$$

Then the $\delta_\nu^*$ must solve a similar problem with data $\varphi_\nu^*$, where

$$\varphi_\nu^* = \varphi_\nu f_\nu^{-1} f_{\nu-1}, \quad f_0 = f_r, \tag{A7}$$

and where the prescribed zeros and poles are altered to take into account those created or destroyed by the $f_\nu$. Condition (A4) will be satisfied for one problem if and only if it is for the other, by (A5) for $f_\nu$. In particular we may choose the

$f_\nu$ to have the prescribed zeros and poles, so that the $\delta_\nu^*$ are to have no zeros and poles. Also, by choosing $f_\nu$ with

$$f_\nu(0) = \varphi_1(0)\varphi_2(0)\cdots\varphi_\nu(0), \tag{A8}$$

we may ensure that $\varphi_\nu^*(0) = 1$ for all $\nu$.

Now induce on $s = \frac{1}{2}r$. When $s = 1$ we have a single line which we may assume is the real axis with $\Sigma_1 = \mathbb{R}_+$. Set $\varphi(s) = \varphi_1(s)$, $s \geq 0$, and $\varphi(s) = \varphi(s)^{-1}$, $s < 0$. The problem is a trivial Wiener-Hopf factorization problem: with the zeros and poles removed, we want to find $\delta_+$ and $\delta_-$, holomorphic and non-zero in the upper and lower half-planes, respectively, with $\delta_+ = \varphi\delta_-$ on $\mathbb{R}$. The winding number of $\varphi$ is zero; thus $\varphi = \exp\psi$ and the solution is obtained by expressing $\psi$ as $\psi = \psi_+ - \psi_-$, where $\psi_+$ and $\psi_-$ are boundary values of functions holomorphic in the upper and lower half-planes, respectively.

For $s > 1$ we first reduce to the case $\varphi_\nu(0) = 1$, all $\nu$. Having done so we note that

$$\frac{1}{2\pi}\int_{\Sigma_1} d(\arg\varphi_1) + \frac{1}{2\pi}\int_{\Sigma_{s+1}} d(\arg\varphi_{s+1}) \tag{A9}$$

is an integer, since each summand is a winding number. A suitable transformation as above by rational functions will then give us a problem for which the integer (A9) is zero. This means that (A4) will be satisfied for the problem for the configuration $\Sigma'$ in which the (collinear) rays $\Sigma_1$ and $\Sigma_{s+1}$ have been removed. By the induction assumption this problem has a piecewise meromorphic solution $\varphi'$ with the prescribed zeros and poles. We look for $\varphi = \varphi' f$ and the problem reduces to the Wiener-Hopf factorization problem for a function on the line $\Sigma_1 \cup (0) \cup \Sigma_{s+1}$.

*Remark.* If the $\varphi_\nu$ satisfy conditions like

$$(1+|z|)^k[\varphi_\nu(z) - 1] \in L^2(\Sigma_\nu) \text{ or } D^j(\varphi_\nu - 1) \in L^2, \quad j \leq k+1,$$

then the same will be true of $\delta_\nu$ on $\Sigma_\nu$ and $\Sigma_{\nu+1}$. This follows readily from the construction when $s = 1$, and then inductively. Similarly, if the $\varphi_\nu$ satisfy conditions like

$$D^j(\varphi_\nu - 1) \in L^2(\Sigma_\nu) \text{ or } L^\infty(\Sigma_\nu), \quad 0 \leq j \leq k$$

and (A3) holds to order $k-1$, then the same will be true of $\delta_\nu$ on $\Sigma_\nu$ and $\Sigma_{\nu+1}$. It follows that the renormalization at $+\infty$ in Section 10 does not destroy these conditions.

### A.2. Rational Approximation

Here we consider a single sector $\Omega$ bounded by the origin and two rays $\Sigma_1$ and $\Sigma_2$.

**Theorem A10.** *Suppose $f$ is a continuous complex function on the boundary of $\Omega$, with limit 0 at $\infty$. Then $f$ may be approximated uniformly by (restrictions* *to $\partial\Omega$ of) rational functions $f_n$ which vanish at $\infty$. Moreover, suppose*

$$D^j f \in L^\infty \cap L^2 \text{ on } \Sigma_i, \quad 0 \le j \le k, i = 1,2, \tag{A11}$$

$$\lim_{z\to 0} D^j f(z) = 0 \text{ on } \Sigma_i, \quad 0 \le j < k, i = 1,2. \tag{A12}$$

*Then the $f_n$ may be chosen so that*

$$D^j f_n \mapsto D^j f \text{ in } L^\infty \cap L^2 \text{ for } j < k,$$

$$D^k f_n \mapsto D^k f \text{ in } L^2, \tag{A13}$$

$$\{D^k f_n\} \text{ is bounded in } L^\infty.$$

Proof: Recall one version of the argument when $\partial\Omega = \mathbb{R}$. Given $\epsilon > 0$, let

$$f_\epsilon(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} [(s-t-i\epsilon)^{-1} - (s-t+i\epsilon)^{-1}] f(s)\, ds. \tag{A14}$$

This is just the convolution of $f$ with the Poisson kernel $P_\epsilon$, which is an approximate identity, so the $f_\epsilon$ converge uniformly to $f$ and one has the requisite convergence by derivatives as well. For a fixed $\epsilon$, $f_\epsilon$ itself (and derivatives) may be approximated by Riemann sums

$$f_{\epsilon,N}(t) = \sum_{j=-N^2}^{N^2} f(j/N) P_\epsilon(s - j/N), \tag{A15}$$

which are rational functions.

With a little more effort, the same construction works for a general sector. We may assume that the positive imaginary axis bisects $\Omega$ and define $f_\epsilon$ by (A14) with $\mathbb{R}$ replaced by $\partial\Omega$. We no longer have a convolution kernel, but

$$f_\epsilon(t) = \int_{\partial\Omega} P_\epsilon(t,s) f(s)\, ds, \tag{A16}$$

where $P_\epsilon$ has the essential features of an approximate identity:

$$\int |P_\epsilon(t,s)|\, |ds| \le C,$$

$$\int P_\epsilon(t,s)\, ds \to 1 \text{ as } \epsilon \searrow 0, \tag{A17}$$

$$\int_{|t-s|>\delta} |P_\epsilon(t,s)|\, |ds| \to 0 \text{ as } \epsilon \searrow 0 \text{ for all } \delta > 0.$$

Thus $f_\epsilon \to f$ uniformly. Under assumptions (A11) and (A12) we also have appropriate convergence of the derivatives, since (complex) differentiation of $f_\epsilon$ can be passed onto $f$ in (A16). Finally, the Riemann sums approximating (A16) are again rational functions which vanish at $\infty$.

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