Differential Geometry of Moduli Spaces and its Applications to Soliton Equations and to Topological Conformal Field Theory

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Abstract

We construct flat Riemann metrics on moduli spaces of algebraic curves with marked meromorphic function. This gives a new class of exact algebraic-geometry solutions of some non-linear equations in terms of functions on the A_n moduli spaces. We show that the Riemann metrics on moduli spaces coincide with two-point correlators in topological conformal field theory and calculate the analogue of the partition function for A_n-model for arbirary genus. A universal method for constructing complete families of conservation laws for Whitham-type hierarchies of PDE also is proposed.

Introduction

The recent progress in the study of matrix models [1] of QFT revealed a remarkable connection with hierarchies of integrable equations of the KdV-type. It was shown also [2]–[4] that so-called topological conformal field theories (TCFT) are very important in the study of the low-dimensional string theories and of the matrix models (the general notion of topological field theory was introduced by E. Witten [5]).

The Landau-Ginsburg superpotentials machinery [6], [7] (see below Section 4) in TCFT was analyzed from different points of view. The relation of it with the singularity theory was investigated in refs. [6], [7], [8] (see also ref. [9]). Very recently Krichever [10] has observed the relation of this machinery with the so-called averaged KdV-type hierarchy [11]–[15] (or *Whitham-type hierarchy*). He showed that the target space for this Whitham-type hierarchy coincides with the coupling space of zero genus TCFT and the dependence of the Landau-Ginsburg potential on the coupling constants is determined via solving the equation of this hierarchy (in fact, a very particular solution proved to be involved.)

Our main observation is that the flat metric on the target space of Whitham-type hierarchy being involved in the Hamiltonian description of it (see refs. [11], [12], [13], [15]) coincides with the two-point correlator of the corresponding TCFT. Starting from this point we have found a very general construction of flat Riemann metrics on moduli spaces M of algebraic curves of given genus with marked meromorphic function. This function in TCFT plays the role of Landau-Ginsburg superpotential (we consider only the A_{n-1}-theories) and the relevant moduli space M being the coupling space. It turns out that the equations of flatness of these Riemann metrics coincide with well-known in the soliton theory N-wave interaction system. We obtain therefore a new class of exact solutions of the N-wave system in terms of some special functions on moduli spaces M (the simplest solution of this class has been found in ref. [16]). Some global properties of moduli spaces of the type being described above also follow from our considerations. We construct also the general class of Whitham-type hierarchies of dynamical systems in the loop spaces $\mathcal{L}M$. We describe the bi-Hamiltonian structure and recurrence operator for this hierarchy and construct explicitly the complete family of conservation laws. As a result of these considerations the explicit formula for the non-zero genus TCFT partition function is obtained. In the appendix we discuss the relation of TCFT to the theory of Frobenius algebras.

1 Orthogonal systems of curvilinear coordinates, integrable equations and Hamiltonian formalism

We start with some information on the geometry of curvilinear orthogonal coordinate systems.

Let

$$ds^2 = \sum_{i=1}^{N} g_{ii}(u)\,(du^i)^2 \tag{1.1}$$

be a diagonal metric on some manifold $M = M^N$ (we give all the formulae for positive definite metrics; indefinite metrics can be considered in a similar way). The variables $u^1, \ldots, u^N$ determine a curvilinear coordinate system in Euclidean space iff the curvature of (1.1) vanishes:

$$R_{ijkl}(ds^2) = 0 \tag{1.2}$$

This is a very complicated system of nonlinear PDE. But there is a special subclass [16] of metrics for which the system (1.2) is an integrable one.

Definition 2. The diagonal metric ds is called Egoroff metric (it was proposed by Darboux [18]) iff the rotation coefficients

$$\gamma_{ij} = \frac{\partial_j \sqrt{g_{ii}}}{\sqrt{g_{jj}}}, \quad \partial_j = \partial/\partial u^j \tag{1.3}$$

satisfy the symmetry

$$\gamma_{ji} = \gamma_{ij} \tag{1.4}$$

Equivalently, there exists a potential $V(u)$ for the metric g_{ii}:

$$g_{ii}(u) = \partial_i V(u), \quad i = 1, \ldots, N \tag{1.5}$$

Proposition 1. *(see ref. [16]). The equations of zero curvature for Egoroff metric have the form*

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i, j, k \text{ are distinct,} \tag{1.6}$$

$$\partial \gamma_{ij} = 0, \quad i \neq j, \quad \partial = \sum_{i=1}^{N} \partial_i \tag{1.6$'$}$$

The corresponding linear problem has the form

$$\partial_j \Psi_i = \gamma_{ij} \Psi_j, \quad i \neq j \tag{1.7}$$

$$\partial \Psi_i = \sigma \Psi_i, \quad \sigma \text{ is the spectral parameter.} \tag{1.7$'$}$$

Remark 1. The linear system (1.7), (1.7′) essentially is equivalent to a system of ODE of N-th order. It has N-dimensional space of solutions for given σ. For example, if $\Psi_i^{(\alpha)}, \alpha = 1, \ldots, N$, form the basis of solutions of the system (1.7), (1.7′) for $\sigma = 0$ then the flat coordinates $v^1, \ldots, v^N$ for the metric ds^2 can be found from the system

$$\partial_i v^\alpha = \sqrt{g_{ii}}\, \Psi_i^{(\alpha)}, \quad \alpha = 1, \ldots, N, \tag{1.8}$$

$$\eta^{\alpha\beta} = \sum_{i=1}^{N} g_{ii}^{-1} \partial_i v^\alpha \partial_i v^\beta = \text{ const}, \quad \alpha, \beta = 1, \ldots, N. \tag{1.9}$$

Remark 2. It was shown in ref. [16] that the system (1.6), (1.6′) with the symmetry (1.4) is equivalent to the pure imaginary reduction of the N-wave interaction problem (see e.g. ref. [19]). The system (1.6), (1.6′) is invariant under the scaling transformations

$$u^i \to cu^i, \quad \gamma_{ij} \to c^{-1}\gamma_{ij} \tag{1.10}$$

The corresponding similarity reduction of the system is equivalent to some nonlinear ODE. In the first nontrivial case $N = 3$ this reduction has the form

$$\Gamma_{23}' = \Gamma_{21}\Gamma_{13}, \quad (z\Gamma_{13})' = -\Gamma_{12}\Gamma_{23}, \quad [(z-1)\Gamma_{13}]' = \Gamma_{12}\Gamma_{23} \tag{1.11}$$

Here

$$z = \frac{u^1 - u^3}{u^2 - u^3}, \quad \gamma_{ij}(u) = \frac{1}{u^2 - u^3}\Gamma_{ij}(z). \tag{1.11′}$$

The system (1.11) can be reduced [20] to a system of the second order equivalent to the Painlevé-VI equation using the first integral

$$\Gamma_{23}^2 + (z\Gamma_{13})^2 + [(z-1)\Gamma_{13}]^2 = \text{ const.} \tag{1.11″}$$

Remark 3. If $\vec{r} = \vec{r}(u), \vec{r} = (v^1, \dots, v^N)$ is the realization of the curvilinear orthogonal coordinate system in Euclidean space then the law of transport along the u^j-axis of the corresponding orthonormal frame

$$\vec{\eta_i} = \partial_i \vec{r}/\sqrt{g_{ii}} \tag{1.12}$$

has the form

$$\begin{aligned} \partial_j \vec{\eta_i} &= \gamma_{ij}\vec{\eta_j}, \quad j \neq i \\ \partial_i \vec{\eta_i} &= -\sum_{j\neq i} \gamma_{ji}\vec{\eta_j} \end{aligned} \tag{1.13}$$

This explains the name "rotation coefficients" for γ_{ij}. It follows from (1.13) that this transport of the frame is invariant for Egoroff metric under the diagonal translations of the coordinates:

$$u^i \to u^i + \Delta u, \quad i = 1, \dots, N. \tag{1.14}$$

In general the Egoroff metric is not invariant under these translations. If it is invariant, i.e. $\partial g_{ii} = 0$, then we shall call it ∂-invariant.

Using the flat metric (1.4) we introduce the following Poisson structure $\{\,,\} = \{\,,\}_{ds^2}$ on the loop space

$$\mathcal{L}M \equiv M^{S^1}$$

of functions of $x \in S^1$ having their values in M (Poisson brackets of hydrodynamic type [11]–[13], [15]) via the formula

$$\{I_1, I_2\}_{ds^2} = \int \frac{\delta I_1}{\delta u^i(x)} g^{ij}(u) \nabla_k \left(\frac{\delta I_2}{\delta u^j(x)} \right) du^k(x). \tag{1.15}$$

Here

$$g^{ij} = \delta^{ij} \cdot [g_{ii}(u)]^{-1}$$

and ∇_k is the Levi-Cività connection for the metric ds^2. The corresponding Hamiltonian systems for Hamiltonians of the form

$$H = \int h(u)\, dx \tag{1.16}$$

have the form of the first order evolutionary systems of PDE linear in derivatives

$$u^i_t = \{u^i(x), H\}_{ds^2} = \nabla^i \nabla_j h(u) u^j_x. \tag{1.17}$$

In the flat coordinates $v^\alpha = v^\alpha(u)$ (see (1.9)) the P.B. (1.15) has a constant form

$$\{v^\alpha(x), v^\beta(y)\}_{ds^2} = \eta^{\alpha\beta} \delta'(x-y) \tag{1.18}$$

The P.B. $\{\,,\}_{ds^2}$ is degenerate: the functionals

$$\int v^1\, dx, \dots, \int v^N\, dx \tag{1.19}$$

are the Casimirs of it.

Definition 3. (cfr. ref. [21]). The family $\mathcal{H}$ of functionals H on the loop space $\mathcal{L}M$ is called a Lagrangian family if all of them commute pairwise and if it is complete. This means that the skew-gradients of these functionals span the tangent space to their common level surface.

All the Casimirs (1.19) are to belong to $\mathcal{H}$.

It follows from the results of Tsarev [21] that for the P.B. (1.18) Lagrangian families $\mathcal{H}$ of functionals of the form (1.16) are in one-to-one correspondence to systems of curvilinear orthogonal coordinates in the flat space with the metric $\eta^{\alpha\beta}$. The explicit construction of $\mathcal{H}$ is as follows. For P.B. of the form (1.15) for any flat diagonal metric (1.1) the Lagrangian family of functionals of the form (1.16) can be constructed as the family of solutions of the system

$$\partial_i \partial_j h = \Gamma^j_{ji} \partial_j h + \Gamma^i_{ij} \partial_i h, \quad i \neq j. \tag{1.20}$$

The corresponding commuting flows (1.17) have a diagonal form

$$u^i_t = w^i(u) u^i_x, \quad i = 1, \dots, N. \tag{1.21}$$

All of them are completely integrable [21]. The system (1.20) for finding the commuting Hamiltonians of the Lagrangian family $\mathcal{H}$ can be rewritten in the form (1.7) via the substitution

$$\partial_i h = \sqrt{g_{ii}}\, \Psi_i, \quad i = 1, \ldots, N. \tag{1.22}$$

The coefficients $w^i(u)$ of the commuting flows (1.21) also can be found from the same system (1.7) (for Egoroff metric) via the substitution

$$\Psi_i = \sqrt{g_{ii}}\, w^i, \quad i = 1, \ldots, N. \tag{1.23}$$

Therefore we obtain a mapping

$$\text{(commuting Hamiltonians)} \to \text{(commuting flows)} \tag{1.24}$$

of the form

$$\left(H = \int h(u)\, dx\right) \to \left(u_t^i = g_{ii}^{-1} \partial_i h(u) u_x^i, \quad i = 1, \ldots, N\right). \tag{1.25}$$

Warning: this is not the skew-gradient mapping (but in some cases — see below section 3 — it is related to the second Hamiltonian structure of the system (1.21)). For ∂-invariant metric (i.e. $\partial g_{ii} = 0$) the skew-gradient mapping has the form

$$\big(h(u)\big) \to \big(u_t^i = g_{ii}^{-1} \partial_i \partial h(u) u_x^i, \quad i = 1, \ldots, N\big). \tag{1.26}$$

For ∂-invariant metric the operator ∂ plays the role of "recursion operator": if $h(u)$ is one of the Hamiltonians in the Lagrangian family $\mathcal{H}$ then ∂h also belongs to $\mathcal{H}$; also the operator ∂^{-1} can be defined on $\mathcal{H}$ with the same property. It is possible to construct a dense subset [21] in the Lagrangian family $\mathcal{H}$ using the operator ∂^{-1} starting from the Casimirs (1.19). The densities of the functionals of this subset have the form

$$h^{(\alpha,m)}(u) = \partial^{-m} v^\alpha, \quad \alpha = 1, \ldots, N, \quad m = 0, 1, \ldots \tag{1.27}$$

2 Flat metrics on moduli spaces

Let us consider for given integers (g, m, n), $g \geq 0, m > 0, n \geq m$, a moduli space $M = M^N, N = 2g + n + m - 2$ of sets $(C, Q_1, \ldots, Q_m, \lambda)$, where C is a smooth algebraic curve of genus g with m marked points $Q_1, \ldots, Q_m$ and with a meromorphic function λ of degree n such that $\lambda^{-1}(\infty) = Q_1 \cup \ldots \cup Q_m$. To specify a component of M one has to fix also the local degrees $n_1, \ldots, n_m$ of λ in the points $Q_1, \ldots, Q_m$. These are arbitrary positive integers such that $n_1 + \ldots + n_m = n$. We need that the λ-projections $u^1, \ldots, u^N$ of the branch points $P_1, \ldots, P_N$

$$d\lambda|_{P_j} = 0, \quad u^j = \lambda(P_j), \quad j = 1, \ldots, N \tag{2.1}$$

(i.e the critical points of λ) are good local coordinates in an open domain in M. Another assumption is that the one-dimensional affine group acts on M as

$$(C, Q_1, \dots, Q_m, \lambda) \to (C, Q_1, \dots, Q_m, a\lambda + b). \tag{2.2}$$

In the coordinates $u^1, \dots, u^N$ it acts as

$$u^i \to au^i + b, \quad i = 1, \dots, N \tag{2.3}$$

The tautological fiber bundle is defined

$$\underset{M}{\downarrow} C \tag{2.4}$$

such that the fiber over $u \in M$ is the curve $C(u)$. The canonical connection is defined on (2.4): the operators ∂_i are lifted on (2.4) in such a way that

$$\partial_i \lambda \equiv 0 \tag{2.5}$$

Example 1. Here $g = 0, m = 1$. The space M is the set of all polynomials of the form

$$\lambda(p) = p^n + q_{n-2}p^{n-2} + \dots + q_0, \quad q_0, q_1, \dots, q_{n-2} \in \mathbb{C}. \tag{2.6}$$

The branch points $p_1, \dots, p_{n-1}$ can be determined from the equation

$$\lambda'(p) = 0 \tag{2.7}$$

The affine transformations $\lambda \to a\lambda + b$ have the form

$$p \to a^{1/n} p, \quad q_i \to q_i a^{\frac{i}{n}-1}, \quad i > 0, \quad q_0 \to aq_0 + b \tag{2.8}$$

Example 2. Here $g = 0, m = n$ (let us redenote $m = n \to n+1$). The space M consists of all rational functions of the form

$$\lambda(p) = p + \sum_{i=1}^{n} \frac{\eta_i}{p + q_i}, \quad \eta_i, q_i \in \mathbb{C} \tag{2.9}$$

Here $Q_i = \{p = -q_i\}$, $i = 1, \dots, n$, $Q_{n+1} = \{p = \infty\}$.

Example 3. $g > 0, m = 1, n = 2$. Here M is the set of all hyperelliptic curves

$$\mu^2 = \prod_{j=1}^{2g+1} (\lambda - u^j), \tag{2.10}$$

the pairwise distinct parameters $u^1, \dots, u^{g+1}$ are the local coordinates on M.

Example 4. $g > 0, m = 1, n > g$. Here M is the set of all curves of genus g with marked point Q_1 and with marked meromorphic function $\lambda(P)$ having a pole of n-th order in Q_1 only.

Let $\tilde{M}$ be the covering of M being obtained by fixing a canonical basis $a_1, \dots, a_g$, $b_1, \dots, b_g$ in $H_1(C, \mathbb{Z})$ (for $g = 0, \tilde{M} = M$). We add small cycles $\gamma_1, \dots, \gamma_{m-1}$ around the points $Q_1, \dots, Q_{m-1}$ (for $m > 1$) to obtain a basis in $H_1(C \backslash (Q_1 \cup \ldots \cup Q_m), \mathbb{Z})$. Let us define multivalued Abelian differential on C as Abelian differentials on the universal covering of the punctured curve $C \backslash (Q_1 \cup \ldots \cup Q_m)$ such that

$$\Omega(P + \gamma) = \Omega(P) + \sum_k c_k(\gamma) \lambda^k \, d\lambda \tag{2.11}$$

for any cycle $\gamma \in H_1(C \backslash (Q_1 \cup \ldots \cup Q_m), \mathbb{Z})$. Such a multivalued differential is said to be holomorphic in the point $P \in C$ iff some branch of it is holomorphic in P. It is called normalized iff

$$\oint_{a_\alpha} \Omega = 0, \quad \alpha = 1, \dots, g \tag{2.12}$$

Definition 4. A family $\Omega = \Omega(P, u)$ of multivalued Abelian differentials on the curve $C = C(u)$ smoothly depending on the parameter $u \in \tilde{M}$ is called horizontal if:

1. It is holomorphic for any u on $C \backslash (Q_1 \cup \ldots \cup Q_m)$.
2. Its covariant derivatives $\partial_j \Omega$ are Abelian differentials of the second kind on C (i.e. with zero residues) with double poles only in the branch points $P_1, \dots, P_N$ and with zero a-periods.

Let $\mathcal{D}(\tilde{M})$ be the quotient of the space of all horizontal differentials over the subspace of differentials of the form

$$\sum c_k \lambda^k \, d\lambda, \quad \partial_j c_k \equiv 0. \tag{2.13}$$

Proposition 1. *The basis of the space $\mathcal{D}(\tilde{M})$ can be constructed as follows:*

1. *Normalized Abelian differentials $\Omega_a^{(k)}$ of the second kind with a single pole in the point Q_a and with the principal part*

$$\Omega_a^{(k)}(P) = \frac{dz_a}{z_a^{k+1}} + \textit{regular terms}, \quad P \to Q_a \tag{2.14}$$

$$z_a = \lambda^{-1/n_a}, \tag{2.15}$$

$a = 1, \dots, m, k = 1, 2, \dots$. *The following linear constraints in $\mathcal{D}(\tilde{M})$ hold for these differentials*

$$\sum_{a=1}^{m} n_a \Omega_a^{(k n_a)} = 0, \quad k = 1, 2, \dots. \tag{2.16}$$

2. *Normalized Abelian differentials* $\Psi_a \equiv \Psi_a^{(0)}$ *(for* $m > 1, a \neq m$*) of the third kind with simple poles in* Q_a, Q_m *and with residues* ± 1 *resp.*

3. *Holomorphic differentials* $\omega_\alpha \equiv \omega_\alpha^{(0)}, \alpha = 1, \dots, g$ *normalized as follows:*

$$\oint_{a_\beta} \omega_\alpha = 2\pi i \delta_{\alpha\beta}. \tag{2.17}$$

4. *Multivalued normalized holomorphic on* C *differentials* $\sigma_a^{(k)}, k = 1, 2, \dots, g$ *with increments of the form*

$$\sigma_a^{(k)}(P + b_\alpha) - \sigma_a^{(k)}(P) = -k\lambda^{k-1}\, d\lambda, \tag{2.18}$$

other increments vanish.

5. *Multivalued normalized holomorphic on* C *differentials* $\omega_\alpha^{(k)}, \alpha = 1, \dots, g$, $k = 1, 2, \dots$ *with increments of the form*

$$\omega_\alpha^{(k)}(P + a_\alpha) - \omega_\alpha^{(k)}(P) = -k\lambda^{k-1}\, d\lambda, \tag{2.19}$$

6. *Multivalued normalized differentials* $\Psi_a^{(k)}, a = 1, \dots, m-1, k = 1, 2, \dots,$ *holomorphic on* $C \backslash (Q_a \cup Q_m)$ *with singularities of the form*

$$\begin{aligned} \Psi_\alpha^{(k)} &= -\frac{1}{n_a}\, d\Psi_k(\lambda) + \textit{ regular terms}, \quad P \to Q_a \\ &= -\frac{1}{n_m}\, d\Psi_k(\lambda) + \textit{ regular terms}, \quad P \to Q_m \end{aligned} \tag{2.20}$$

$$\Psi_k(\lambda) = \frac{\lambda^k}{k!}\left[\log\lambda - \left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)\right], \quad k > 0. \tag{2.21}$$

The proof is straightforward.

Lemma 1. *Let* $\Omega^{(i)}, i = 1, 2$ *be any two horizontal differentials such that*

$$\Omega^{(i)} = \sum_k c_{ka}^{(i)} z_a^k\, dz_\alpha + d\sum_{k>0} r_{ka}^{(i)} \frac{\lambda^k \log\lambda}{n_a}, \quad P \to Q_a, \tag{2.22}$$

$$\oint_{a_\alpha} \Omega^{(i)} = 2\pi i A_\alpha^{(i)} \tag{2.23}$$

$$\Omega^{(i)}(P + a_\alpha) - \Omega^{(i)}(P) = dp_\alpha^{(i)}, \quad p_\alpha^{(i)} = \sum_{s>0} p_{s\alpha}^{(i)} \lambda^s, \tag{2.24}$$

$$\Omega^{(i)}(P+b_\alpha)-\Omega^{(i)}(P)=dq_\alpha^{(i)},\quad q_\alpha^{(i)}=\sum_{s>0}q_{s\alpha}^{(i)}\lambda^s, \tag{2.25}$$

Then

$$res_{P_j}\frac{\Omega^{(1)}\Omega^{(2)}}{d\lambda}=\partial_j V_{\Omega^{(1)}\Omega^{(2)}}(u) \tag{2.26}$$

where

$$\begin{aligned} V_{\Omega^{(1)}\Omega^{(2)}}(u) = -\sum_{a=1}^{m}\Bigg[&\sum_{k\geq 0}\frac{C_{-k-2,a}^{(1)}}{k+1}C_{k,a}^{(2)}+\\ &+C_{-1,a}^{(1)}\ v.p.\int_{Q_0}^{Q_a}\Omega^{(2)}-\\ &-\ v.p.\int_{Q_0}^{Q_a}r_{k,a}^{(1)}\lambda^k\Omega^{(2)}+\pi i\sum_{k>0}r_{k,a}^{(1)}C_{n_a k-1}^{(2)}\Bigg]+\\ &+\sum_{\alpha=1}^{g}\Bigg[-\frac{1}{2\pi i}\oint_{a_\alpha}q_\alpha^{(1)}(\lambda)\Omega^{(2)}+\\ &+\frac{1}{2\pi i}\oint_{b_\alpha}p_\alpha^{(1)}(\lambda)\Omega^{(2)}+A_\alpha^{(1)}\oint_{b_\alpha}\Omega^{(2)}\Bigg]. \end{aligned} \tag{2.27}$$

The regularized integrals are defined with respect to the local parameters (2.15). *The sum over a in* (2.27) *does not depend on* $Q_0\in C$. *We recall that all the numbers* $A^{(i)}, p_{s\alpha}^{(i)}, q_{s\alpha}^{(i)}, r_{s\alpha}^{(i)}$ *and* $c_{ka}^{(i)}$ *for negative k are constants.*

Proof. Let $\tilde{C}$ be the polygon with $4g$ edges obtained by cutting C along the cycles $a_1,\dots,a_g,b_1,\dots,b_g$ passing through a point $Q_0\in C$. Let us choose also some curves in C from Q_0 to $Q_1,\dots,Q_m$ and cut $\tilde{C}$ along these curves to obtain a domain $\tilde{C}_0$. We assume that the λ-images of all of these cuttings do not depend on u — at least in some neighborhood of the point $u\in\tilde{M}$, and that $\lambda(Q_0)\equiv 0$. Then we have an identity

$$\frac{1}{2\pi i}\oint_{\partial C_0}\left[\Omega^{(1)}(P)\int_{Q_0}^{P}\partial_j\Omega^{(2)}\right]=\mathrm{res}_{P_j}\frac{\Omega^{(1)}\Omega^{(2)}}{d\lambda}. \tag{2.28}$$

After calculation of all the residues and of all the contour integrals we obtain (2.26). □

Let $\Omega\in\mathcal{D}(\tilde{M})$ be any non-zero horizontal differential. It defines a metric ds_Ω^2 on $(\tilde{M}_\Omega)$ being diagonal in the coordinates $u^1,\dots,u^N$:

$$ds_\Omega^2=\sum_{i=1}^{N}g_{ii}^{\Omega}(u)\,(du^i)^2 \tag{2.29}$$

via the formula

$$g_{ii}^{\Omega} = \operatorname{res}_{P_i} \frac{\Omega^2}{d\lambda}, \quad i = 1, \dots, N. \tag{2.30}$$

Here the subspace $\tilde{M}_{\Omega}$ consists of all pairs (C, λ) such that Ω does not vanish at $P_1 \dots P_N$ (we recall that $P_1, \dots, P_N$ are the critical points of λ). (In fact we consider the complex analogue of metric. So the coordinates are complex. We need only non-degeneracy of the metric (2.29)).

Theorem 1. The metric (2.29), (2.30) is a flat Egoroff metric on $\tilde{M}_{\Omega}$. Its rotation coefficients $\gamma_{ij} = \gamma_{ij}(u)$ (1.3) do not depend on Ω. They are invariant under scaling transformations (1.10).

Corollary. For any given $g, m, n_1, \dots, n_m (n_1 + \dots + n_m = n)$ the rotation coefficients $\gamma_{ij}(u), u \in (\tilde{M})^N, N = 2g + n + m - 2$, of the metric (2.29), give a self-similar solution of the system (1.6), (1.6′).

Proof. From the Lemma 1 the potential (see (1.5)) of the metric (2.29) has the form

$$g_{ii}^{\Omega} = \partial_i V_{\Omega\Omega}(u), \quad 1 \le i \le N. \tag{2.31}$$

Hence the rotation coefficients $\gamma_{ij}^{\Omega}(u)$ of the metric (2.29),(2.30) are symmetric in i, j. To prove the identity (1.6) for γ_{ij}^{Ω} let us consider the differential

$$\partial_i \partial_j \Omega \int \partial_k \Omega$$

for distinct i, j, k. It has poles only in the branch points P_i, P_j, P_k. The contour integral of the differential along $\partial \tilde{C}_0$ equals zero. Hence the sum of the residues vanishes. This reads as

$$\partial_j \sqrt{g_{ii}^{\Omega}} \partial_k \sqrt{g_{ii}^{\Omega}} + \partial_i \sqrt{g_{jj}^{\Omega}} \partial_k \sqrt{g_{jj}^{\Omega}} = \sqrt{g_{kk}^{\Omega}} \partial_i \partial_j \sqrt{g_{kk}^{\Omega}}.$$

This can be written in the form (1.6) due to the symmetry (1.4).

Let us prove now that the rotation coefficients γ_{ij}^{Ω} do not depend on Ω. Let us consider the differential

$$\partial_i \Omega^{(1)} \int \partial_j \Omega^{(2)}, \quad i \ne j,$$

for any two horizontal differentials $\Omega^{(1)}, \Omega^{(2)}$. From vanishing of the sum of its residues we obtain

$$\sqrt{g_{jj}^{\Omega^{(2)}}} \partial_i \sqrt{g_{jj}^{\Omega^{(1)}}} = \sqrt{g_{ii}^{\Omega^{(1)}}} \partial_j \sqrt{g_{ii}^{\Omega^{(2)}}}.$$

Using the symmetry (1.4) we immediately obtain

$$\gamma_{ij}^{\Omega^{(1)}} = \gamma_{ij}^{\Omega^{(2)}}.$$

Now we are to prove the last identity (1.6′). It is sufficient to prove it for a holomorphic normalized differential Ω. Let us define an operator D on functions $f = f(P, u)$ by the formula

$$Df = \frac{\partial f}{\partial \lambda} + \partial f, \quad \partial = \sum_{j=1}^{N} \partial_j. \tag{2.32}$$

The operator D is extended to differentials Ω in such a way that $Dd = dD$. For any normalized holomorphic differential we have

$$D\Omega = 0$$

(cfr. Lemma 2 below). Hence

$$\partial g_{jj}^{\Omega} = 0 \Rightarrow \partial \gamma_{ij} = 0.$$

The flatness of the metric (2.29) is proved. The scaling invariance can be verified easily again for holomorphic Ω (in this case $g_{ii}^{\Omega}(cu) = c^{-1} g_{ii}^{\Omega}(u)$). The theorem is proved. □

A horizontal differential such that $D\Omega = 0$ we shall call *primary* differential.

Lemma 2. *The subspace in $\mathcal{D}(\tilde{M})$ of all primary differentials is N-dimensional. It is spanned by the differentials (see (2.14)–(2.21) for the notations)*

$$\begin{aligned} &\Omega_a^{(k)}, \quad k = 1, \dots, n_a, \quad a = 1, \dots, m, \quad \sum_{a=1}^{m} n_a \Omega_a^{(n_a)} = 0 \\ &\Psi_a^{(0)}, \quad a = 1, \dots, m-1, \\ &\omega_\alpha, \sigma_\alpha \equiv \sigma_\alpha^{(1)}, \quad \alpha = 1, \dots, g. \end{aligned} \tag{2.33}$$

Proof. It is clear that for any differential Ω of the form (2.33) $D\Omega$ is a holomorphic single-valued differential on C. It is easy to see that is has zero a-periods. Hence $D\Omega = 0$. Conversely, we have

$$\begin{aligned} D\Omega_a^{(k)} &= \frac{n_a - k}{n_a} \Omega_a^{(k-n_a)}, \quad k > n_a \\ D\sigma_\alpha^{(k)} &= k \sigma_\alpha^{(k-1)}, \quad k > 1 \\ D\omega_\alpha^{(k)} &= k \omega_\alpha^{(k-1)}, \quad k > 1 \\ D\Psi_a^{(k)} &= \Psi_a^{(k-1)}, \quad k \geq 1. \end{aligned} \tag{2.34}$$

Hence any primary Ω is a linear combination of the differentials (2.33). Lemma is proved. □

Let us denote by $\mathcal{D}_0(\tilde{M}) \subset \mathcal{D}(\tilde{M})$ the subspace of all primary differentials.

Let us fix any primary differential Ω. It defines the mapping

$$\mathcal{D}(\tilde{M}) \mapsto \text{Functions}(\tilde{M}) \tag{2.35}$$

via the formula

$$\Omega' \mapsto V_{\Omega\Omega'} \tag{2.36}$$

for any horizontal differential Ω'. We call the function $V_{\Omega\Omega'}$ *conjugate* to Ω'. The image of the mapping (2.36) will be denoted as $\mathcal{A}_\Omega(\tilde{M})$. Vice versa, for any function $f \in \mathcal{A}_\Omega(\tilde{M})$ the unique *conjugate* differential $\Omega_f \in \mathcal{D}(\tilde{M})$ is determined such that

$$V_{\Omega\Omega_f} = f. \tag{2.37}$$

The basis in the space $\mathcal{A}_\Omega(\tilde{M})$ is given as follows:

$$\text{res}_{Q_a} z_a^{-k}\Omega, \quad k = 1, 2, \dots ,$$

with linear constraints

$$\sum_{a=1}^{m} \text{res}_{Q_a} z_a^{-kn_a}\Omega = 0, \quad k = 1, 2, \dots$$

$$\oint_{a_\alpha} \lambda^k\Omega, \quad k \geq 1, \qquad \oint_{b_\alpha} \lambda^k\Omega, \quad k \geq 0, \quad \alpha = 1, \dots , g \tag{2.38}$$

$$\text{v.p.} \int_{Q_m}^{Q_a} \lambda^k\Omega, \quad k \geq 1, \quad a \neq m$$

Note that these functions are well-defined globally on $\tilde{M}$.

Lemma 3. *For any two functions $f(u)$, $h(u) \in \mathcal{A}_\Omega(\tilde{M})$ the following identities hold:*

$$\langle df, dh\rangle_\Omega = \sum_{j=1}^{N} res_{P_j} \frac{\Omega_f\Omega_h}{d\lambda} = \partial V_{\Omega_f\Omega_h}. \tag{2.39}$$

Here $\langle\, ,\rangle_\Omega$ is the scalar product of gradients of the functions f, h with respect to the metric ds^2_Ω.

Proof. This immediately follows from the definition of the conjugate differentials (2.37) and from the Lemma 1. □

This Lemma gives us a bridge between Riemannian geometry of the moduli spaces and TCFT (see Section 4 below).

We want the explicit formulae for acting of the translation generator ∂ on the basis (2.38).

Lemma 4. *For any primary differential $\Omega \in \mathcal{D}_0(\tilde{M})$ the following identities hold:*

$$\partial\ res_{Q_a}(z_a^{-k}\Omega) = \frac{k}{n_a} res_{Q_a}(z_a^{-k+n_a}\Omega)$$

$$\partial \oint_{a_\alpha} \lambda^k \Omega = k \oint_{a_\alpha} \lambda^{k-1}\Omega, \quad \partial \oint_{b_\alpha} \lambda^k \Omega = k \oint_{b_\alpha} \lambda^{k-1}\Omega \tag{2.40}$$

$$\partial\ v.p. \int_{Q_0}^{Q_a} \Omega = \left.\frac{\Omega}{d\lambda}\right|_{Q_0}, \qquad \partial\ v.p. \int_{Q_0}^{Q_a} \lambda^k \Omega = 0, \quad k > 0.$$

The proof is straightforward.

Theorem 2. For any primary differential Ω the flat coordinates for the metric ds^2_Ω have the form

$$t_{k,a} = -\frac{1}{k}\mathrm{res}_{Q_a} z_a^{-k}\Omega, \quad 1, \dots, m, \quad k = 1, \dots, n_a,$$

$$t_{0,a} = \ \mathrm{v.p.} \int_{Q_0}^{Q_a} \Omega + t_0, \tag{2.41}$$

$$t'_\alpha = -\frac{1}{2\pi i}\oint_{a_\alpha} \lambda\Omega, \quad t''_\alpha = \oint_{b_\alpha} \Omega, \quad \alpha = 1, \dots, g$$

with two constraints

$$\sum_{a=1}^m t_{0,a} = \sum_{a=1}^m t_{n_a,a} = 0. \tag{2.42}$$

The matrix η of the metric ds^2_Ω in the coordinates (2.41) can be obtained from the matrix

$$\begin{aligned} \eta^{(a,k),(b,l)} &= \delta^{a,b}\delta^{k+1,n_a} \\ \eta^{\alpha',\beta''} &= \delta^{\alpha,\beta} \end{aligned} \tag{2.43}$$

other components vanish, via the restriction on the subspace (2.42). The conjugate differentials have the form

$$\Omega_{t_{(k,a)}} = \Omega_a^{(k)}, \quad 1 \le k \le n_a$$

$$\Omega_{[t_{(0,a)} - t_{(0,m)}]} = \Psi_a^{(0)}, \quad a \neq m \tag{2.44}$$

$$\Omega_{t'_\alpha} = \omega_\alpha, \quad \Omega_{t''_\alpha} = \sigma_\alpha$$

The proof immediately follows from the formula (2.27) and from the Lemmas 3, 4.

Corollary 6. *For any primary differential* Ω *the metric* ds^2_Ω *is well-defined and non-degenerate globally on* $\tilde{M}$.

Corollary 7. *For any primary differential* Ω *the mapping* $\tilde{M}_\Omega \mapsto \mathbb{C}^N$ *being given by the flat coordinates* (2.41) *is regular everywhere and therefore is a covering.*

It is interesting to find the degree of this covering. For $g = 0$ it equals one.

Remark. For any horizontal differential Ω it is possible to construct another flat metric on $\tilde{M}'_\Omega \equiv \{(C, \lambda) \in \tilde{M}_\Omega \,|\, \lambda(P_i) \neq 0, i = 1, \dots, N\}$,

$$d\tilde{s}^2_\Omega = \sum_{i=1}^N \tilde{g}^\Omega_{ii}(du^i)^2 \tag{2.45}$$

where

$$\tilde{g}^\Omega_{ii} = \text{res}_{P_i} \frac{\Omega^2}{\lambda d\lambda} = \frac{g^\Omega_{ii}}{u^i}. \tag{2.46}$$

It is an Egoroff metric in the coordinates

$$z^i = \log u^i, \quad i = 1, \dots, N \tag{2.47}$$

with the rotation coefficients

$$\tilde{\gamma}_{ij}(z^1, \dots, z^N) = \exp\left(\frac{z^i + z^j}{2}\right)\gamma_{ij}(e^{z^1}, \dots, e^{z^N}) \tag{2.48}$$

Hence the functions $\tilde{\gamma}_{ij}(z)$ also enjoy the system (1.6), (1.6′) (but they are not scaling invariant!) The flat coordinates for (2.45) also can be calculated explicitly for scaling invariant Ω.

3 Poisson structures on the loop space $\mathcal{L}\tilde{M}$.

We recall that the flat metric ds^2_Ω on $\tilde{M}$ determines a Poisson structure of the form (1.15) on the loop space $\mathcal{L}\tilde{M}$. Let us denote it by $\{\,,\}_\Omega$. Let Ω be a primary differential.

Theorem 3. 1. For any horizontal differential Ω' the t-flow on the loop space $\mathcal{L}\tilde{M}$ of the form

$$\partial_t \Omega = \partial_x \Omega' \tag{3.1}$$

is a Hamiltonian flow with respect to P.B. $\{\,,\}_\Omega$ with the Hamiltonian $H = \int h(u)\, dx$ such that

$$\partial h = V_{\Omega\Omega'} \tag{3.2}$$

2. The functionals $H = \int h(u)\,dx$, $h(u) \in \mathcal{A}_\Omega(\tilde{M})$, form a Lagrangian family for the P.B. $\{\,,\}_\Omega$.

3. For any horizontal Ω' the flow (3.1) is completely integrable. The (locally) general solution of (3.1) can be written in the form

$$\{x\Omega + t\Omega' + \Omega_0\}_{P_j} = 0, \quad j = 1, \ldots, N \tag{3.3}$$

for any horizontal differential Ω_0.

Proof. The equation (3.2) can be obtained from the definition (1.26) of the P.B. $\{\,,\}_\Omega$ (note that the metric ds^2_Ω is ∂-invariant). The completeness of the functionals with densities in $\mathcal{A}_\Omega(\tilde{M})$ follows from the Lemma 4. Indeed, these functionals can be constructed starting from the Casimirs (2.41) using the recursion procedure (1.27). The formula (3.3) for general solution can be proved as in ref. [14]. □

Remark 1. The flow (3.1) can be considered also as x-flow on the space of functions on t (cf. ref. [22]). Its Hamiltonian structure is defined by the bracket P.B. $\{\,,\}_{\Omega'}$.

Remark 2. Let the primary differential Ω be scaling invariant:

$$\lambda \to c\lambda, \quad u^i \to cu^i, \quad \Omega \to c^q\Omega. \tag{3.4}$$

Then all the flows (3.1) are Hamiltonian flows also for the P.B. $\{\,,\}\tilde{}_\Omega$ being determined by the flat metric (2.45). The corresponding recursion operator coincides with ∂ (up to some constant). If $f \in \mathcal{A}_\Omega(\tilde{M})$ is a homogeneous function then the corresponding flow

$$u^i_t = \left\{u^i(x), \int f\,dx\right\}\tilde{}_\Omega \tag{3.5}$$

can be written in the form

$$\partial_t\Omega = \partial_x\Omega_f. \tag{3.6}$$

Here the differential Ω_f is defined by the formula (2.37). The definition of the conjugate differential Ω_f can be written therefore in the form

$$\Omega_f = \partial_x^{-1}\left\{\Omega(P, u(x)), \int f\,dx\right\}\tilde{}_\Omega \tag{3.7}$$

The system of equations of the form (3.1) where Ω' is any of the basic differentials (2.14)–(2.20) we shall call *Whitham-type hierarchy* (or W-hierarchy) for given primary differential Ω. It is put in order by action of the recursion operator D^{-1} on the differentials Ω'.

4 Main examples. Application to TCFT.

Example 1. For the family of polynomials $M = \{\lambda = p^n + q_{n-2}p^{n-1} + \ldots + q_0\}$ the equations of the W-hierarchy for $\Omega = dp$ have the form

$$\partial_{t_i} dp(\lambda) = \partial_x dr^{(i)}(\lambda), \quad i \neq kn \tag{4.1}$$

where

$$r^{(i)} = \frac{[\lambda^{i/n}(p)]_+}{i} \tag{4.2}$$

(the polynomial part in p). Note that $t_1 = x$. They can be rewritten in the form [10]

$$\partial_{t_i}\lambda = \partial_p r^{(i)} \partial_x \lambda - \partial_x r^{(i)} \partial_p \lambda. \tag{4.3}$$

Here $\lambda = \lambda(p)$, $r^{(i)} = r^{(i)}(p)$ are polynomials. Equation (4.3) can be obtained by averaging of the Kdv-type hierarchy (or the Gel'fand-Dikii hierarchy)

$$\partial_i L = \frac{1}{i}[L, [L^{i/n}]_+], \quad i \neq kn, \tag{4.4}$$

$$L = \partial^n + q_{n-2}\partial^{n-2} + \ldots + q_0 \tag{4.4'}$$

over the family of the constant solutions $q_j = \text{const}$, $j = 0, \ldots, n-2$. The metric ds^2_Ω has the form

$$ds^2_\Omega = \sum_{i=1}^{n-1} \frac{(du^i)^2}{\lambda''(p_i)} \tag{4.5}$$

Here $p_1, \ldots, p_{n-1}$ are the critical points of $\lambda(p)$,

$$\lambda'(p_i) = 0$$

and

$$u^i = \lambda(p_i).$$

For $n = 4$ the rotation coefficients of the metric (4.5) give a nontrivial algebraic solution of the system (1.6), (1.6') (and, therefore, of the system (1.11)). The flat coordinates $t_1, \ldots, t_{n-1}$ have the form

$$t_{n-i} = \text{res}_{\lambda=\infty} \frac{\lambda^{i/n}}{i} dp, \quad i = 1, \ldots, n-1, \tag{4.6}$$

$$\langle dt^i, dt^j \rangle_{dp} = \frac{\delta_{i+j,n}}{n}. \tag{4.6'}$$

Remark. We also can take any other differential $\Omega = dr^{(i)}$, $i = 2, \ldots, n-1$ to determine a flat metric on M. All these flat geometries are inequivalent one to another.

To explain the relation [10] of (4.2), (4.3) to TCFT we recall here the Landau-Ginsburg superpotentials approach [6], [7]. In TCFT all the correlation functions do not depend on coordinates (but do depend on coupling parameters $t_1, \ldots, t_N$) and can be expressed in terms of the two-point and the three-point correlation functions

$$\eta_{\alpha\beta} = \langle \phi_\alpha \phi_\beta \rangle = \text{const}, \quad \det(\eta_{\alpha\beta}) \neq 0 \tag{4.7}$$

$$c_{\alpha\beta\gamma}(t) = \langle \phi_\alpha \phi_\beta \phi_\gamma \rangle \tag{4.8}$$

by the factorization formulae

$$\begin{gathered} \langle \phi_\alpha \phi_\beta \phi_\gamma \phi_\delta \rangle = c^\epsilon_{\alpha\beta} c_{\epsilon\gamma\delta} \\ \langle \phi_\alpha \phi_\beta \phi_\gamma \int \phi_\delta \rangle = \partial_\delta c_{\alpha\beta\gamma}, \quad \partial_\delta = \frac{\partial}{\partial t_\delta} \end{gathered} \tag{4.9}$$

etc. (the raising of indices using the metric $\eta_{\alpha\beta}$). For the primary free energy $F = F(t_1, \ldots, t_N)$ of the model the following identities hold:

$$\partial_1 \partial_\alpha \partial_\beta F = \eta_{\alpha\beta}, \tag{4.10}$$

$$\partial_\alpha \partial_\beta \partial_\gamma F = c_{\alpha\beta\gamma}. \tag{4.11}$$

The function $F(t)$ is quasihomogeneous in $t_1, \ldots, t_N$. To find these correlation functions for genus zero let us consider the set of polynomials $\lambda(p)$ of the form given above (Landau-Ginsburg superpotentials, A_{n-1}-model) depending on the coupling parameters $t_1, \ldots, t_N$ $(N = n-1)$ in such a way that

$$\partial_\alpha \lambda = -\phi_\alpha, \quad \alpha = 1, \ldots, N. \tag{4.12}$$

Here $\phi_1, \ldots, \phi_N$ is the basis of polynomials of degrees $0, 1, \ldots, N-1$ orthogonal with respect to the scalar product

$$\langle \phi, \varphi \rangle = \operatorname{res}_{\lambda=\infty} \frac{\phi\varphi}{d\lambda/dp}, \tag{4.13}$$

$$\langle \phi_\alpha, \varphi_\beta \rangle = \delta_{\alpha+\beta,n} \tag{4.14}$$

Proposition 1. *The family* $\lambda = \lambda(p, t_1, \ldots, t_{n-1})$ *is a particular solution of the system* (4.1) *where* $i = 1, \ldots, n-1$ *with the initial data*

$$\lambda|_{t_1=\ldots=t_{n-1}=0} = p^n. \tag{4.15}$$

The crucial point in the proof is in the observation [4] that the orthogonal polynomials (4.13), (4.14) have the form

$$\phi_\alpha = dr^{(\alpha)}/dp, \quad \alpha = 1, \dots, n-1. \tag{4.16}$$

The dependence of the coefficients $q_0, q_1, \dots, q_{n-2}$ of the polynomial $\lambda(p)$ on $t_1, \dots, t_{n-1}$ can be found from the equations (4.6). It can be represented in the form [10]

$$\begin{aligned} &f(p_k) = 0, \quad k = 1, \dots, n-1, \\ &f(p) = \frac{d}{dp}\left[r^{(n+1)}(p) + \sum_{i=1}^{n-1} t_i r^{(i)}(p)\right] \end{aligned} \tag{4.16'}$$

The triple correlators have the form [6], [7]

$$c_{\alpha\beta\gamma}(t) = \operatorname{res}_{p=\infty} \frac{\phi_\alpha \phi_\beta \phi_\gamma}{d\lambda/dp}$$

where the polynomials $\phi_\alpha = \phi_\alpha(p;t)$ are determined by (4.16). The coefficients $c^\gamma_{\alpha\beta}(t) = c_{\alpha\beta\epsilon}(t)\eta^{\epsilon\gamma}$ for any $t = t_1, \dots, t_{n-1}$ are the structure constants of an associative commutative algebra A with a unit ϕ_1 and with a constant (in t) invariant scalar product $\eta_{\alpha\beta}$ (chiral algebra of primary fields). We recall that those algebras are called Frobenius algebras [25]. In this example it is isomorphic to the truncated polynomials

$$A \cong C[p]/(\lambda'(p)).$$

The free energy $F = F(t)$ was found by Krichever. It has the form

$$F(t) = \frac{1}{2}\operatorname{res}_{p=\infty}\left\{\left[\int f(p)\,dp\right]_+ f(p)\,dp\right\}.$$

The function $f(p)$ is determined by (4.16′); for any function $g(p)$ the symbol $[g(p)]_+$ means the polynomial part of $g(p)$ with respect to the parameter $z = \lambda^{1/n}$. Krichever also argued that the function $F(t)$ should be considered as a logarithm of τ-function of the hierarchy (4.1) for the particular solution (4.16′).

Example 2. For the family of rational functions (2.9) the W-hierarchy has the form

$$\partial_{t_{i,a}} dp = \partial_x d[\lambda^i]_a, \qquad a = 1, \dots, n, \quad i = 0, 1, \dots \tag{4.17}$$

$$\partial_{s_{i,a}} dp = \partial_x d[\varphi_i(\lambda)]_a, \qquad a = 1, \dots, n, \quad i = 0, 1, \dots \tag{4.18}$$

Here the operation $[\,]_a$ means that one should kill singularities in all the points $Q_1, \dots, Q_n$ but Q_a; the functions $\varphi_i(\lambda)$ are defined in (2.21). The flat coordinates for the primary differential $\Omega = dp$ are η_i, q_i. The hierarchy (4.17), (4.18) is the hierarchy of the "highest Benney equations" — it can be deduced from Zakharov's paper [23].

Example 3. For the moduli space of hyperelliptic curves (2.10) the differential $\Omega = \Omega^{(1)}$ is a primary one. The part of the corresponding W-hierarchy of the form

$$\partial_{t_i}\Omega = \partial_x \Omega^{(2i+1)}, \quad i = 0, 1, \dots \tag{4.19}$$

coincides with the KdV hierarchy averaged over the family of g-gap solutions. This was proved by Flaschka, Forest and McLaughlin [24]. One should add the equations

$$\begin{aligned} \partial_{t'_{\alpha,k}}\Omega &= \partial_x \omega_\alpha^{(k)}, \quad k = 0, 1, \dots \\ \partial_{t''_{\alpha,k}}\Omega &= \partial_x \sigma_\alpha^{(k)}, \quad k = 1, 2, \dots \end{aligned} \tag{4.20}$$

to obtain the complete hierarchy ($\alpha = 1, \dots, g$). The functionals $H = \int h(u)\,dx$ with the densities of the form

$$\operatorname{res}_{\lambda=\infty} \lambda^{(2i+1)/2}\Omega, i = 0, 1, \dots, \qquad \oint_{a_k} \lambda^k \Omega, k > 0, \qquad \oint_{b_k} \lambda^k \Omega, k \geq 0, \tag{4.21}$$

give the complete family of conservation laws for the hierarchy (4.19), (4.20) (the Lagrangian family $\mathcal{A}_\Omega(\tilde{M})$). The P.B. $\{\,,\}_\Omega$ coincides [16] with the averaged [11]–[13], [15] Gardner-Zakharov-Faddeev (GZF) P.B. of the KdV-hierarchy. The flat coordinates (2.41) for the metric ds^2_Ω are:

$t_1 =$ annihilator of the GZF P.B.,
$t'_1, \dots, t'_g$ action variables for the GZF P.B.
$t''_1, \dots, t''_g$ components of the wave-number vector

(see ref. [13] for details). It can be proved also that the second P.B. $\{\,,\tilde{\}}_\Omega$ coincides with the averaged Magri bracket [13].

Example 4. For the moduli space M of all curves of genus g with a marked point Q_1 and a meromorphic function λ with pole in Q_1 of order n the primary differentials are

$$\Omega^{(i)} = \frac{dz}{z^{i+1}} + \text{regular terms}, \quad z = \lambda^{-1/n}, \quad i = 1, \dots, n-1$$

$$\oint_{a_\alpha} \Omega^{(i)} = 0, \quad \alpha = 1, \dots, g, \tag{4.22}$$

$$\omega_\alpha, \sigma_\alpha, \quad \alpha = 1, \dots, g,$$

(for the definitions of ω_α, $\sigma_\alpha \equiv \sigma_\alpha^{(1)}$ see (2.17), (2.18)). Let us use these differentials for calculations of correlation functions of genus g minimal model of TCFT. Here the space $\tilde{M}_\Omega$ plays the role of the coupling space (we recall that $\dim M = N = 2g + n - 1$). Let us redenote the primary differentials (4.22) as $\Phi_1, \ldots, \Phi_N$ in such a way that

$$\begin{aligned} \Phi_\alpha &= -n\Omega^{(\alpha)}, \quad \alpha = 1, \ldots, n-1, \\ \left.\begin{aligned} \Phi_{n-1+\alpha} &= \omega_\alpha \\ \Phi_{g+n-1+\alpha} &= \sigma_\alpha \end{aligned}\right\} & \quad \alpha = 1, \ldots, g. \end{aligned} \tag{4.23}$$

Let $\wedge = (\wedge_{\alpha\beta}(u))$ be the symmetric $N \times N$-matrix of the form

$$\wedge_{\alpha\beta}(u) = V_{\Phi_\alpha \Phi_\beta}. \tag{4.24}$$

Under scaling transformations

$$\lambda \to c^n \lambda \tag{4.25}$$

the matrix transforms as follows:

$$\wedge \to S \wedge S \tag{4.26}$$

where $S = \mathrm{diag}(c^{-1}, \ldots, c^{-n+1}, 1, \ldots, 1, c^n, \ldots, c^n)$. For any fixed β the variables $t_1 = \wedge_{\beta\beta}$, $t_2 = \wedge_{\beta\beta+1}, \ldots$ provide the flat coordinates for the metric $ds^2_{\Phi_\beta}$.

Let us choose the differential $\Omega = \Omega^{(1)}$ for defining the flat metric ds^2_Ω (this choice seems to be natural since it does not depend on n and g). Let us denote it as $\Omega = dp$. The flat coordinates on $\tilde{M}$ are

$$\begin{aligned} t_{n-\alpha} &= \mathrm{res}_{\lambda=\infty} \frac{\lambda^{\alpha/n}}{\alpha} dp, \quad \alpha = 1, \ldots, n-1, \\ \left.\begin{aligned} t_{n-1+\alpha} &= \tfrac{1}{2\pi i} \oint_{a_\alpha} \lambda\, dp \\ t_{g+n-1+\alpha} &= \oint_{b_\alpha} dp \end{aligned}\right\} & \quad \alpha = 1, \ldots, g \end{aligned} \tag{4.27}$$

The point of $\tilde{M}$ with coordinates $t_1, \ldots, t_N$ is determined by the following system (cf. (3.3))

$$(n\Omega^{(n+1)} + \hat{\Omega})|_{P_j} = 0, \; j = 1, \ldots, N \tag{4.28}$$

where

$$\hat{\Omega} = \sum_{i=1}^{N} t_i \Phi_i \tag{4.29}$$

Indeed, if $u \in \tilde{M}$ is the point with the coordinates $t_1, \ldots, t_N$ then the following identity holds on the corresponding curve $C = C(u)$:

$$p\, d\lambda = n\Omega^{(n+1)} + \hat{\Omega} \tag{4.30}$$

The l.h.s. of (4.30) vanishes in all the branch points $P_1, \dots, P_N$. This proves (4.28).

The correlation functions have the form

$$\langle \Phi_\alpha \Phi_\beta \rangle = \sum \operatorname{res}_{d\lambda=0} \frac{\Phi_\alpha \Phi_\beta}{d\lambda} = \eta_{\alpha\beta} \tag{4.31}$$

$$\langle \Phi_\alpha \Phi_\beta \Phi_\gamma \rangle = \sum \operatorname{res}_{d\lambda=0} \frac{\Phi_\alpha \Phi_\beta \Phi_\gamma}{d\lambda dp} = c_{\alpha\beta\gamma} \tag{4.32}$$

etc. The correlation functions (4.31) do not depend on t. This follows from Lemma 3 and from Theorem 2. They have the form

$$\begin{aligned} \langle \Phi_\alpha \Phi_\beta \rangle &= n\delta_{\alpha+\beta,n}, \quad 1 \le \alpha, \beta \le n-1 \\ \langle \Phi_{n-1+\alpha} \Phi_{g+n-1+\beta} \rangle &= \delta_{\alpha\beta}, \quad 1 \le \alpha, \beta \le g \end{aligned} \tag{4.33}$$

otherwise zero.

The corresponding Landau-Ginsburg superpotential is $\lambda = \lambda(p)$, $p = \int dp$. In other words

$$\partial_{t_\alpha} (\lambda\, dp)_{p=const} = -\Phi_\alpha, \quad \alpha = 1, \dots, N. \tag{4.34}$$

This follows from identities

$$\partial_{t_\alpha} (p\, d\lambda) = \Phi_\alpha, \quad \alpha = 1, \dots, N \tag{4.35}$$

and

$$\partial_{t_\alpha} (p\, d\lambda)_{\lambda=const} = -\partial_{t_\alpha} (\lambda\, dp)_{p=const}. \tag{4.36}$$

Theorem 4. The free energy $F = F(t_1, \dots, t_N)$ satisfying (4.10), (4.11) for (4.31), (4.32) has the form

$$-2F = V_{p\,d\lambda, p\,d\lambda} \equiv n^2 V_{\Omega^{(n+1)}, \Omega^{(n+1)}} + 2n \sum_{i=1}^{N} t_i V_{\Omega^{(n+1)}, \Phi_i} + \sum_{i,j=1}^{N} t_i t_j V_{\Phi_i, \Phi_j}. \tag{4.37}$$

Proof. From (4.28), (4.29) we obtain

$$\sum_{i=1}^{N} t_i \partial_{t_\alpha} V_{\Phi_i, \Phi_j} + n \partial_{t_\alpha} V_{\Omega^{(n+1)}, \Phi_j} = 0, \quad j = 1, \dots, N,$$

$$\sum_{i=1}^{N} t_i \partial_{t_\alpha} V_{\Omega^{(n+1)}, \Phi_j} + n \partial_{t_\alpha} V_{\Omega^{(n+1)}, \Omega^{(n+1)}} = 0.$$

Hence

$$\partial_{t_\alpha}\partial_{t_\beta}F = -V_{\Phi_\alpha\Phi_\beta} \tag{4.38}$$

Let us prove now that

$$\partial_{t_\gamma}V_{\Phi_\alpha\Phi_\beta} = -c_{\alpha\beta\gamma}. \tag{4.39}$$

Indeed,

$$\partial_{t_\gamma}V_{\Phi_\alpha,\Phi_\beta} = \sum_{i=1}^{N}\frac{\partial}{\partial u^\delta}V_{\Phi_\alpha,\Phi_\beta}\frac{\partial u^\delta}{\partial t_\gamma} = -\sum_{\delta=1}^{N}\operatorname{res}_{P_\delta}\frac{\Phi_\alpha\Phi_\beta}{d\lambda}\frac{\Phi_\gamma}{dp}\bigg|_{P_\delta} = -c_{\alpha\beta\gamma}.$$

Here we use (2.26) and (4.34). This completes the proof. □

The corresponding Frobenius algebra of primary fields (see above) has the form

$$\Phi_i\Phi_j = c_{ij}^k\Phi_k\,dp\,(\operatorname{mod}\,\mathcal{D}_0(\tilde{M})\cdot d\lambda) \tag{4.40}$$

where

$$c_{ij}^k = \eta^{kl}c_{ijl}. \tag{4.40'}$$

Remark 1. It follows from (4.38) that the Hessian

$$\tau_{\alpha\beta} = -(\partial_{t_{n-1+\alpha}}\partial_{t_{n-1+\beta}}F), \quad 1\le\alpha,\beta\le g \tag{4.41}$$

coincides with the period matrix of the curve C. We shall consider the linear Virasoro-type constraints for F in the next publication.

Remark 2. Probably the exactness of the differential $d\lambda$ is not necessary. Almost all the constructions of this section seem to be realizable also for any normalized Abelian differential $d\lambda$ with poles in $Q_1,\dots,Q_m$. This possibility also is to be investigated.

The quasihomogeneous property of the partition function (4.37) has the form

$$F(c^n t_1,\dots,c^2 t_{n-1},\dots,c^{n+1}t_{n+g-1},ct_{n+g},\dots,ct_N) = c^{2(n+1)}F(t). \tag{4.42}$$

We shall also analyze the problem of glueing all the Riemann manifolds $\tilde{M}$ with different genera g in the next publication.

Appendix. Deformation of Frobenius algebras and partition functions of TCFT.

The logarithm of partition function $F = F(t_1,\dots,t_N)$ of TCFT satisfies the following system of nonlinear equations: its third derivatives (after raising of an index) for any t form a set of structure constants of a commutative associative N-dimensional algebra with a unity with invariant nondegenerate scalar product (in fact, only the equation of associativity is nontrivial). Such algebras are well known as Frobenius algebras. We see that the free energy $F(t)$ determines some deformation of Frobenius algebra

$$c^i_{jk}(t) = \eta^{is} c_{sjk}(t), \quad c_{ijk}(t) = \partial_{t_i}\partial_{t_j}\partial_{t_k} F(t) \tag{A.1}$$

such that the corresponding invariant scalar product η^{ij} does not depend on t (let us call (A.1) F-deformations). Here we shall construct some class of F-deformations of any Frobenius algebra using the results of ref. [25].

Let A be any N-dimensional Frobenius algebra and $M = A^*$ (the dual space). A multiplication is defined on T^*M: if $u^1, \ldots, u^N$ is a basis in A (providing the coordinate system in M) then

$$du^i \cdot du^k = c^{ij}_k du^k, \tag{A.2}$$

c^{ij}_k being the structure constants of A. The non-degenerate scalar product on T^*M (and, therefore, a metric on M) is defined by the formula

$$\langle df, dg \rangle = 2 i_\partial (df \cdot dg), \tag{A.3}$$

$\partial = u^i \frac{\partial}{\partial u^i}$ is the dilation generator. It was observed [25] that the metric (A.3) is flat and the corresponding Levi-Civita connection has the form

$$\nabla^i T^j = \partial^i T^j - c^{ij}_s T^s \tag{A.4}$$

(raising of indices using the metric (A.3)). The flat coordinates $t_1, \ldots, t_N$ can be introduced via appropriate quadratic substitution of the form [25]

$$u^i = \frac{1}{2} a^i_{\alpha\beta} t^\alpha t^\beta \tag{A.5}$$

$$\langle dt^\alpha, dt^\beta \rangle = \eta^{\alpha\beta} = \text{const}. \tag{A.5$'$}$$

Let us introduce the coefficients

$$c^{i_1, \ldots, i_n} = 2 c^{i_1 i_2}_{s_1} c^{s_1 i_3}_{s_2} \ldots c^{s_{n-2} i_n}_{s_{n-1}} u^{s_{n-1}} \tag{A.6}$$

and the functions

$$\tilde{c}^{\alpha\beta}_\gamma(t) = \frac{\partial t^\alpha}{\partial u^i} \frac{\partial t^\beta}{\partial u^j} \frac{\partial u^k}{\partial t^\alpha} c^{ij}_k. \tag{A.7}$$

Proposition. The function (A.7) defines a F-deformation of the Frobenius algebra A with constant scalar product (A.5$'$) and with the "free energy"

$$F(t) = \sum_{n=3}^{\infty} \frac{2^{n-1}}{n} c^{i_1 \ldots i_n} u_{i_1} \ldots u_{i_n}. \tag{A.8}$$

Proof. It is sufficient to prove that in the curvilinear coordinates $u^1, \dots, u^N$ the function (A.8) satisfies the equation

$$\nabla^i \nabla^j \nabla^k F = c^{ijk}. \tag{A.9}$$

The proof of (A.9) is straightforward using the identities

$$\nabla^l c_k^{ij} = -c_s^{ij} c_k^{sl}, \quad \nabla^k u_i = \frac{1}{2}\delta_i^k. \tag{A.10}$$

□

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