Differential Geometry of the Space of Orbits of a Coxeter Group

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Abstract

Differential-geometric structures on the space of orbits of a finite Coxeter group, determined by Grothéndieck residues, are calculated. This gives a construction of a 2D topological field theory for an arbitrary Coxeter group.

Introduction: Formulation of main results

Let W be a (finite) Coxeter group, i.e. a finite group of linear transformations of a n-dimensional Euclidean space V generated by reflections. The space of orbits

$$M = V/W$$

has a natural structure of affine variety: the coordinate ring of M coincides with the ring $R := S(V)^W$ of W-invariant polynomial functions on V. Due to Chevalley theorem this is a polynomial ring with the generators $x_1, \ldots, x_n$ being invariant homogeneous polynomials. The basic invariant polynomials are not specified uniquely. But their degrees $d_1, \ldots, d_n$ are invariants of the group (see below Sect. 2). The maximal degree h of the polynomials is called Coxeter number of the group W. (More details about Coxeter groups will be given in Sect. 2.)

A clue to understanding of a rich differential-geometric structure of the orbit spaces can be found in the singularity theory. According to this the complexified orbit space of an irreducible Coxeter group of A-D-E type is bi-holomorphic equivalent to the universal unfolding of a simple singularity [1, 10, 31, 41]. Under this identification the Coxeter group coincides with the monodromy group of vanishing cycles of the singularity. The discriminant of the Coxeter group (the set of irregular orbits) is identified with the bifurcation diagram of the singularity. The invariant Euclidean inner product on V coincides with the pairing on the cotangent bundle T_*M defined by the intersection form of vanishing cycles [45]. The bi-holomorphic equivalence is given by the period mapping.

Additional differential-geometric structures on a universal unfolding of an isolated hypersurface singularity are determined by the Grothéndieck residues (see [39]). Let me explain this for the simplest example of the group A_n where the formulae for the residues were rediscovered by R.Dijkgraaf, E. and H.Verlinde [15] (they also found new remarkable properties of these residues, see below). This is obtained from the group of all permutations of the coordinates $\xi_1, \dots, \xi_{n+1}$ of $(n+1)$-dimensional space by restriction onto the subspace

$$V = \{\xi_1 + \ldots + \xi_{n+1} = 0\}.$$

The space of orbits of A_n can be identified with the universal unfolding of the simple singularity $f = z^{n+1}$,

$$M = \left\{ f(z; x_1, \dots, x_n) = z^{n+1} + x_n z^{n-1} + \ldots + x_1 = \prod_{i=1}^{n+1} (z - \xi_i) \right\}.$$

The residue pairing defines a new metric on M: the inner product of two tangent vectors in a point $x = (x_1, \dots, x_n)$ is defined by the formula

$$(\dot{f}(z; x(s_1)), \dot{f}(z; x(s_2)))_x := \operatorname{res}_{z=\infty} \frac{\dot{f}(z; x(s_1)) \dot{f}(z; x(s_2))}{f'(z; x)}. \tag{1}$$

Here the dots mean derivatives w.r.t. the parameters s_1, s_2 resp. on two curves through the point x, the prime means d/dz. This pairing does not degenerate on TM. We can define in a similar way a trilinear form on TM putting

$$c(\dot{f}(z; x(s_1)), \dot{f}(z; x(s_2)), \dot{f}(z; x(s_3)))_x := \operatorname{res}_{z=\infty} \frac{\dot{f}(z; x(s_1)) \dot{f}(z; x(s_2)) \dot{f}(z; x(s_3))}{f'(z; x)}. \tag{2}$$

This gives rise to an operation of multiplication of tangent vectors at any point $x \in M$

$$u,\ v \mapsto u \cdot v, \quad u, v \in T_x M$$

uniquely specified by the equation

$$c(u, v, w)_x = (u \cdot v, w)_x.$$

This is a commutative associative algebra with a unity for any x isomorphic to the algebra of truncated polynomials

$$\mathbf{C}[z]/(f'(z; x)).$$

At the origin $x = 0$ the algebra coincides with the local algebra of the

A_n-singularity $\mathbf{C}[z]/(f'(z))$.

One can define in a similar way polylinear forms

$$c_k(u_1, \ldots, u_k)_x := \mathrm{res}_{z=\infty} \frac{\dot{f}(z; x(s_1)) \ldots \dot{f}(z; x(s_k))}{f'(z; x)}$$

where the tangent vectors in the point x have the form

$$u_i = \dot{f}(z; x(s_i)), i = 1, \ldots, k.$$

For $k > 3$ they can be expressed in a pure algebraic way via the multiplication of vectors and the pairing (,):

$$c_k(u_1, \ldots, u_k) = (u_1 \cdot u_2 \cdot \cdots \cdot u_{k-1}, u_k).$$

Note that this formula coincides with the factorization rule for the primary correlators in two-dimensional TFT [16, 50]. Further details about 2-dimensional topological field theory from the point of view of the theory of singularities can be found in [8].

Let us come back to orbit spaces of arbitrary Coxeter groups. As it was mentioned above the intersection form of the simple singularity corresponding to a Coxeter group (as a metric on the universal unfolding) on the space of orbits can be defined intrinsicaly being induced by the invariant Euclidean structure in V. V.I.Arnol'd in [3] formulated (for $A-D-E$-singularities; for other simple singularities see [28]) the problem of calculation of the local algebra structure in intrinsic terms, i.e. via the metric on the orbit space M (this metric was introduced by Arnol'd in [2]; it is called also *convolution of invariants*). In the same time K.Saito [36, 37] solved the problem of calculation in intrinsic terms the residue pairing metric. The ideas of the papers [3, 36, 37] (and of the paper [39] where the constructions of [36, 37] were developed for extended affine root systems) are very important for constructions of the present paper.

To develop the approaches of these works I am going to contribute to understanding of the problem of giving an intrinsic description of the differential-geometric structures on the space of orbits of a Coxeter group induced by the constructions of the theory of singularities. [This problem was spelled out by K.Saito in [39] (but the structures like (2) were not considered). He considered it as generalised Jacobi's inversion problem: to describe the image of the period mapping. An equivalent problem of axiomatization of the convolution of invariants was formulated by Arnol'd in [5, p.72].] I will give an intrinsic formula for calculation of the Grothéndieck residues for arbitrary Coxeter group without using the construction of the correspondent universal unfolding. My purpose is to obtain a complete differential-geometric characterization of the space of orbits in terms of these structures (see Conjecture at the end of this section).

I came to this problem from another side when I was trying to understand a geometrical foundation of two-dimensional topological field theories (TFT) [14–16, 42, 48–50]. The idea was to extend the Atiyah's axioms [7] of TFT (for the two-dimensional case) by the properties of the canonical moduli space of a TFT model proved in [15] (see also [16]). On this way I found a nice geometrical object that I called Frobenius manifold. Any model of two-dimensional TFT is encoded by a Frobenius manifold and I showed that many constructions of TFT (integrable hierarchies for the partition function, their bi-hamiltonian formalism and τ-functions, string equations, genus zero recursion relations for correlators) can be deduced from geometry of Frobenius manifolds [20, 22]. It looks like Frobenius manifolds play also an important rôle in the theory of singularities. Better understanding of the rôle could elucidate still misterious relations between the theory of singularities, theory of integrable systems, and intersection theories on moduli spaces of algebraic curves [12, 16, 29, 30, 48–52].

In the present paper I show that the orbit spaces of Coxeter groups carry a natural structure of Frobenius manifold. For the groups of $A-D-E$ series this gives an intrinsic description (i.e. only in terms of the Coxeter group) of the residue structures like (1), (2) (this coincides with the primary chiral algebra of the $A-D-E$-topological minimal models [15, 42]).

It's time to proceed to the definition of Frobenius manifold. This is a coordinate-free formulation of a differential equation arised in [15, 49] (I called it WDVV—Witten-Dijkgraaf-E.Verlinde-H.Verlinde equation). This is a system of equations for a function $F(t)$ of n variables $t=(t^1,\dots,t^n)$ resulting from the following conditions:

1. The matrix
$$\eta_{\alpha\beta}:=\frac{\partial^3 F(t)}{\partial t^1\partial t^\alpha\partial t^\beta}$$
should be constant and not degenerate. Let us denote by $(\eta^{\alpha\beta})$ the inverse matrix.

2. The coefficients
$$c^\gamma_{\alpha\beta}(t):=\sum_\epsilon \eta^{\gamma\epsilon}\frac{\partial^3 F(t)}{\partial t^\epsilon\partial t^\alpha\partial t^\beta}$$
for any t should be structure constants of an associative algebra.

3. The function $F(t)$ should be quasihomogeneous of a degree $3-d$ where the degree of the variables are $1-q_\alpha:=\deg t^\alpha$, $q_1=0$. (In physical literature d is called dimension of the TFT-model and q_α are called charges of the primary fields.)

To give a coordinate-free reformulation of the WDVV equations let me recall the notion of *Frobenius algebra*. This is a commutative* associative algebra with

*Also noncommutative Frobenius algebras are considered by algebraists.

a unity over a commutative ring R supplied with a symmetric nondegenerate R-bilinear inner product $(\ ,\ )$ being invariant† in the following sense

$$(ab, c) = (a, bc)$$

for any a, b, $c \in A$. We will consider the cases where $R = \mathbf{R}$, $\mathbf{C}$ or the ring of functions on a manifold. There is a natural operation of *rescaling* of a Frobenius algebra: for an invertible constant c we change the multiplication law, the unity e and the invariant inner product $(\ ,\ )$ putting

$$a \cdot b \mapsto ca \cdot b, \quad e \mapsto c^{-1}e, \quad (\ ,\ ) \mapsto \varphi(c)(\ ,\ )$$

for an arbitrary $\varphi(c)$.

Definition 1. A manifold M (real or complex) is called *Frobenius manifold* if the tangent planes T_xM are supplied with a structure of Frobenius algebra smoothly depending on the point x and satisfying the following properties.

1. The metric on M specified by the invariant inner product $(\ ,\ )$ is flat (i.e. the curvature of the metric vanishes).

2. The unity vector field e is covariantly constant

   $$\nabla e = 0.$$

   Here ∇ is the Levi-Civita connection for the metric $(\ ,\ )$.

3. Let c be the section of the bundle S^3T_*M (i.e. a symmetric trilinear form on TM) given by the formula

   $$c(u, v, w) := (u \cdot v, w).$$

   Then the tensor

   $$(\nabla_z c)(u, v, w)$$

   should be symmetric in u, v, w, z for any vector fields u, v, w, z.

4. A one-parameter group of diffeomorphisms should be defined on M acting as rescalings on the algebras T_xM.

We will denote by v the generator of the one-parameter group. It can be normalised by the condition on the commutator of the vector fields e, v

$$[e, v] = e.$$

We will call v *Euler vector field* on the Frobenius manifold. The eigenvalues of the linear operator $\nabla_i v^j$ are called *invariant degrees* of the Frobenius manifold.

†Invariant inner product on a Frobenius algebra is not unique: any linear functional $\omega \in A^*$ defines an invariant symmetric inner product on A by the formula $(a, b)_\omega := \omega(ab)$. This does not degenerate for generic ω. We consider a Frobenius algebra with a *marked* invariant inner product.

In the flat coordinates $t^1, \ldots, t^n$ for $(\; , \;)$ the metric is given by a constant matrix $(\eta_{\alpha\beta})$, the unity vector field also has constant coordinates (we can normalize the flat coordinates in such a way that $e = \partial/\partial t^1$) and the Euler vector field has the form

$$v = \sum (1 - q_\alpha) t^\alpha \frac{\partial}{\partial t^\alpha}.$$

The degrees $(1 - q_\alpha)$ of the coordinates t^α coincide with the invariant degrees of the Frobenius manifold. The tensor $c_{\alpha\beta\gamma}$ can be represented (at least localy) as the third derivatives of a function $F(t)$ satisfying WDVV equations.

The nondegenerate form $(\; , \;)$ establishes an isomorphism

$$(\; , \;) : TM \to T_*M.$$

This provides also the cotangent planes with a structure of Frobenius algebra.

Note that the space of vector fields on a Frobenius manifold acquires a natural structure of a Frobenius algebra over the ring R of functions on M. This can be used for algebrization of the notion of Frobenius manifold for a suitable class of rings R as a Frobenius R-algebra structure on the R-module $\operatorname{Der} R$ of derivations of R satisfying the above properties [22]. Particularly, if $R = \mathbf{C}[x_1, \ldots, x_n]$ is a polynomial ring then a Frobenius R-algebra structure on the polynomial vector fields $\operatorname{Der} R$ satisfying the conditions of Definition 1 will be called *polynomial Frobenius manifold* (see the algebraic reformulation of the notion of polynomial Frobenius manifold in Appendix to this paper). In this case $M = \operatorname{Spec} R$ is an affine space and the correspondent solution of WDVV is a quasihomogeneous polynomial. Polynomial solutions of WDVV with integer and rational coefficients are of special interest due to their probable relation to intersection theories on moduli spaces of algebraic curves and their holomorphic maps [6, 52].

Let us come back to the orbit space M of a Coxeter group. We denote by $\langle \; , \; \rangle^*$ the metric on the cotangent bundle T_*M induced by the W-invariant Euclidean structure on V. There are two marked vector fields on M: the Euler vector field

$$E := \sum d_i x_i \frac{\partial}{\partial x_i}$$

and the vector field

$$e := \frac{\partial}{\partial x_1}$$

corresponding to the polynomial of the maximal degree $\deg x_1 = h$. The vector field e is defined uniquely up to a constant factor. The *Saito metric* on M (inner product on T_*M) is defined as

$$(\; , \;)^* := \mathcal{L}_e \langle \; , \; \rangle^*$$

(the Lie derivative along e). This is a flat globaly defined metric on M [36, 37] (for convenience of the reader I reprove this statement in Sect. 2). Our main technical observation inspired by the differential-geometric theory of S.P. Novikov and the author of Poisson brackets of hydrodynamic type [24, 25] and by bi-hamiltonian formalism [33] is that any linear combination $a\langle\ ,\ \rangle^* + b(\ ,\ )^*$ of the flat metrics is again flat.

Theorem 1. *There exists a unique (up to rescaling) polynomial Frobenius structure on the space of orbits of a finite irreducible Coxeter group with the charges and dimension*

$$q_\alpha = 1 - \frac{d_\alpha}{h}, \quad d = 1 - \frac{2}{h}, \tag{3}$$

the unity e, the Euler vector field $\frac{1}{h}E$, and the Saito invariant metric such that for any two invariant polynomials f, g the following formula holds

$$i_v(df \cdot dg) = \langle df, dg\rangle^*. \tag{4}$$

Here i_v is the operator of inner derivative (contraction) along the vector field v.

The formula (4) gives an effective method [23] for calculation of the structure constants of the Frobenius manifold in the flat coordinates for the Saito metric (see formula (2.25) below). If the Saito flat coordinates are chosen to be invariant polynomials with rational coefficients then the polynomial $F(t)$ also has rational coefficients (it follows from (2.25)).

In the origin $t = 0$ the structure constants $c_{ij}^k(0)$ of the Frobenius algebra on T_0M coincide with the structure constants of the local algebra of the correspondent simple singularity $F(z) = 0$

$$\phi_i(z)\phi_j(z) = c_{ij}^k(0)\phi_k(z) \,(\mathrm{mod}\, F'(z)).$$

Here $\phi_i(z) := [\partial F(z; x_1, \dots, x_n)/\partial x_i]_{x=0}$, $F(z; x_1, \dots, x_n)$ is the versal deformation of the singularity $F(z) \equiv F(z; 0) = 0$. In the origin the formula (4) thus coincides with the formula of Arnol'd [3, 28] related the local algebra with the linearized convolution of invariants (i.e., with the linear part of the Euclidean metric) where the identification of T_0M with the cotangent plane $T_{*0}M$ should be given by the Saito metric. But the formula (4) gives more providing a possibility to calculate the local algebra via the convolution of invariants.

Let $R = \mathbf{C}[x_1, \dots, x_n]$ be the coordinate ring of the orbit space M. The Frobenius algebra structure on the tangent planes T_xM for any $x \in M$ provides the R-module $\mathrm{Der}\, R$ of invariant vector fields with a structure of Frobenius algebra over R. To describe this structure let us consider such a basis of invariant polynomials $x_1, \dots, x_n$ of the Coxeter group that $\deg x_1 = h$. Let $D(x_1, \dots, x_n)$ be the discriminant of the group. We introduce a polynomial of degree n in an auxiliary variable u putting

$$P(u; x_1, \dots, x_n) := D(x_1 - u, x_2, \dots, x_n). \tag{5}$$

Let $D_0(x_1, \dots, x_n)$ be the discriminant of this polynomial in u. It does not vanish identicaly on the space of orbits.

Theorem 2. *The map*

$$1 \mapsto e, \quad u \mapsto v \tag{6a}$$

can be extended uniquely to an isomorphism of R-algebras

$$\mathbf{C}[u, x_1, \ldots, x_n]/(P(u; x)) \to \operatorname{Der} R. \tag{6b}$$

Corollary. *The algebra on T_xM has no nilpotents outside the zeroes of the polynomial $D_0(x_1, \ldots, x_n)$.*

A non-nilpotent Frobenius algebra (over $\mathbf{C}$) can be decomposed into a direct sum of one-dimensional Frobenius algebras. Warning: by no means this implies even local decomposability of a massive (see below) Frobenius manifold into a direct sum of one-dimensional Frobenius manifolds.

Definition 2. A Frobenius manifold M is called *massive* if the algebra on the tangent planes T_xM is non-nilpotent for a generic $x \in M$.

In physical language massive Frobenius manifolds correspond to massive TFT models. Examples of massless TFT models where the algebra structure on the tangent planes is identicaly nilpotent are given by topological sigma-models with a Calabi-Yau target space [52].

Conjecture. Any massive polynomial Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group.

In other words, the constructions of Theorem 1 (and their direct products) give all massive polynomial solutions of WDVV with

$$0 \leq q_\alpha \leq d < 1.$$

This could give a simple approach to classification of 2-dimensional topological field theories with $d < 1$. An alternative approach was developed recently by S. Cecotti and C. Vafa [12]. It is based on studying of Hermitean metrics on a Frobenius manifold obeying certain system of differential equations (the so-called equation of topological-antitopological fusion [11], see also [21]). The approach of [12] also gives rise to Coxeter groups (and their generalizations) in classification of topological field theories.

The conjecture can be "improved" a little: instead of polynomiality it is sufficiently to assume that the function $F(t)$ is analytic in the origin. For positive invariant degrees analyticity in the origin implies polynomiality.

The Conjecture can be verified easily for 2- and 3-dimensional manifolds. There are other strong evidences in support of the conjecture. I am going to discuss them in a separate publication.

Historical Remark. I started to think about polynomial solutions of WDVV trying to answer a question of Vafa [43]: what are the 2-dimensional topological field theories (in the approach based on WDVV equations) for which the partition function is a power series in the coupling constants with rational coefficients? The question was motivated by the interpretation, due to E.Witten [42–44], of the logarythm of the partition function as a generating function of intersection numbers of cycles on some orbifolds. On this way I found the solutions (2.46)–(2.48); the sense of the solutions (2.47) and (2.48) from the point of view of known topological field theories was not clear. In December 1992 during my talk at I.Newton institute Arnol'd immediately recognized in the degrees of the polynomials (2.46)–(2.48) the Coxeter numbers (plus one) of the three Coxeter groups in the three-dimensional space. This became the starting point of the present work.

1 Differential-geometric preliminaries

The name *contravariant metric* (or, briefly, *metric*) will mean a symmetric nondegenerate bilinear form $\langle\ ,\ \rangle^*$ on the cotangent bundle T_*M to a manifold M. In a local coordinate system $x^1, \ldots, x^n$ the metric is given by its components

$$g^{ij}(x) := \langle dx^i, dx^j\rangle^* \tag{1.1}$$

where (g^{ij}) is an invertible symmetric matrix. The inverse matrix $(g_{ij}) := (g^{ij})^{-1}$ specifies a *covariant metric* $\langle\ ,\ \rangle$ on the manifold M (usually it is also called metric on the manifold) i.e. a nondegenerate inner product on the tangent bundle TM

$$\langle \partial_i, \partial_j\rangle := g_{ij}(x) \tag{1.2}$$

$$\partial_i := \frac{\partial}{\partial x^i}.$$

The *Levi-Civita connection* ∇_k for the metric is uniquely specified by the conditions

$$\nabla_k g_{ij} := \partial_k g_{ij} - \Gamma^s_{ki} g_{sj} - \Gamma^s_{kj} g_{is} = 0 \tag{1.3a}$$

or, equivalently,

$$\nabla_k g^{ij} := \partial_k g^{ij} + \Gamma^i_{ks} g^{sj} + \Gamma^j_{ks} g^{is} = 0 \tag{1.3b}$$

and

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{1.4}$$

(Summation over twice repeated indices here and below is assumed. We will keep the symbol of summation over more than twice repeated indices.) Here the coefficients Γ^k_{ij} of the connection (the Christoffel symbols) can be expressed via the metric and its derivatives as

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks}\left(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij}\right). \tag{1.5}$$

For us it will be more convenient to work with the *contravariant components* of the connection

$$\Gamma_k^{ij} := \langle dx^i, \nabla_k dx^j \rangle^* = -g^{is}\Gamma_{sk}^j. \tag{1.6}$$

The equations (1.3) and (1.4) for the contravariant components read

$$\partial_k g^{ij} = \Gamma_k^{ij} + \Gamma_k^{ji} \tag{1.7}$$

$$g^{is}\Gamma_s^{jk} = g^{js}\Gamma_s^{ik}. \tag{1.8}$$

It is also convenient to introduce operators

$$\nabla^i = g^{is}\nabla_s \tag{1.9a}$$

$$\nabla^i \xi_k = g^{is}\partial_s \xi_k + \Gamma_k^{is}\xi_s. \tag{1.9b}$$

For brevity we will call the operators ∇^i and the correspondent coefficients Γ_k^{ij} *contravariant connection.*

The *curvature tensor* R_{slt}^k of the metric measures noncommutativity of the operators ∇_i or, equivalently ∇^i

$$(\nabla_s \nabla_l - \nabla_l \nabla_s)\xi_t = -R_{slt}^k \xi_k \tag{1.10a}$$

where

$$R_{slt}^k = \partial_s \Gamma_{lt}^k - \partial_l \Gamma_{st}^k + \Gamma_{sr}^k \Gamma_{lt}^r - \Gamma_{lr}^k \Gamma_{st}^r. \tag{1.10b}$$

We say that the metric is *flat* if the curvature of it vanishes. For a flat metric local *flat coordinates* $p^1, \dots, p^n$ exist such that in these coordinates the metric is constant and the components of the Levi-Civita connection vanish. Conversely, if a system of flat coordinates for a metric exists then the metric is flat. The flat coordinates are determined uniquely up to an affine transformation with constant coefficients. They can be found from the following system

$$\nabla^i \xi_j = g^{is}\partial_s \partial_j p + \Gamma_j^{is}\partial_s p = 0, \; i, j = 1, \dots, n. \tag{1.11}$$

If we choose the flat coordinates orthonormalized

$$\langle dp^a, dp^b \rangle^* = \delta^{ab} \tag{1.12}$$

then for the components of the metric and of the Levi-Civita connection the following formulae hold

$$g^{ij} = \frac{\partial x^i}{\partial p^a}\frac{\partial x^j}{\partial p^a} \tag{1.13a}$$

$$\Gamma_k^{ij} dx^k = \frac{\partial x^i}{\partial p^a}\frac{\partial^2 x^j}{\partial p^a \partial p^b} dp^b. \tag{1.13b}$$

All these facts are standard in geometry (see, e.g., [26]). We need to represent the formula (1.10b) for the curvature tensor in a slightly modified form (cf. [25, formula (2.18)]).

Lemma 1.1. *For the curvature of a metric the following formula holds*

$$R_l^{ijk} := g^{is} g^{jt} R^k_{slt} = g^{is} \left(\partial_s \Gamma_l^{jk} - \partial_l \Gamma_s^{jk} \right) + \Gamma_s^{ij} \Gamma_l^{sk} - \Gamma_s^{ik} \Gamma_l^{sj}. \tag{1.14}$$

Proof. Multiplying the formula (1.10b) by $g^{is} g^{jt}$ and using (1.6) and (1.7) we obtain (1.14). The lemma is proved. □

Let us consider now a manifold supplied with two nonproportional metrics $\langle\ ,\ \rangle_1^*$ and $\langle\ ,\ \rangle_2^*$. In a coordinate system they are given by their components g_1^{ij} and g_2^{ij} resp. I will denote by Γ_{1k}^{ij} and Γ_{2k}^{ij} the correspondent Levi-Civita connections ∇_1^i and ∇_2^i. Note that the difference

$$\Delta^{ijk} = g_2^{is} \Gamma_{1s}^{jk} - g_1^{js} \Gamma_{2s}^{ik} \tag{1.15}$$

is a tensor on the manifold.

Definition 1.1. We say that the two metrics form a *flat pencil* if:

1. The metric
$$g^{ij} = g_1^{ij} + \lambda g_2^{ij} \tag{1.16a}$$
is flat for arbitrary λ and

2. The Levi-Civita connection for the metric (1.16a) has the form
$$\Gamma_k^{ij} = \Gamma_{1k}^{ij} + \lambda \Gamma_{2k}^{ij}. \tag{1.16b}$$

Proposition 1.1. *For a flat pencil of metrics a vector field $f = f^i \partial_i$ exists such that the difference tensor (1.15) and the metric g_1^{ij} have the form*

$$\Delta^{ijk} = \nabla_2^i \nabla_2^j f^k \tag{1.17a}$$

$$g_1^{ij} = \nabla_2^i f^j + \nabla_2^j f^i + c g_2^{ij} \tag{1.17b}$$

for a constant c. The vector field should satisfy the equations

$$\Delta_s^{ij} \Delta_l^{sk} = \Delta_s^{ik} \Delta_l^{sj} \tag{1.18}$$

where

$$\begin{aligned} \Delta_k^{ij} &:= g_{2ks} \Delta^{sij} = \nabla_{2k} \nabla_2^i f^j, \\ (g_1^{is} g_2^{jt} &- g_2^{is} g_1^{jt}) \nabla_{2s} \nabla_{2t} f^k = 0. \end{aligned} \tag{1.19}$$

Conversely, for a flat metric g_2^{ij} and for a solution f of the system (1.18), (1.19) the metrics g_2^{ij} and (1.17b) form a flat pencil.

Proof. Let us assume that $x^1, \dots, x^n$ is the flat coordinate system for the metric g_2^{ij}. In these coordinates we have

$$\Gamma_{2k}^{ij} = 0, \ \Delta_k^{ij} := g_{2ks}\Delta^{sij} = \Gamma_{1k}^{ij}. \tag{1.20}$$

The equation $R_l^{ijk} = 0$ in these coordinates reads

$$\left(g_1^{is} + \lambda g_2^{is}\right)\left(\partial_s \Delta_l^{jk} - \partial_l \Delta_s^{jk}\right) + \Delta_s^{ij}\Delta_l^{sk} - \Delta_s^{ik}\Delta_l^{sj} = 0. \tag{1.21}$$

Vanishing of the linear in λ term provides existence of a tensor f^{ij} such that

$$\Delta_k^{ij} = \partial_k f^{ij}.$$

The rest part of (1.21) gives (1.18). Let us use now the condition of symmetry (1.8) of the connection (1.16b). In the coordinate system this reads

$$\left(g_1^{is} + \lambda g_2^{is}\right)\partial_s f^{jk} = \left(g_1^{js} + \lambda g_2^{js}\right)\partial_s f^{ik}. \tag{1.22}$$

Vanishing of the terms in (1.22) linear in λ provides existence of a vector field f such that

$$f^{ij} = g_2^{is}\partial_s f^j.$$

This implies (1.17a). The rest part of the equation (1.22) gives (1.19). The last equation (1.7) gives (1.17b). The first part of the proposition is proved. The converse statement follows from the same equations. □

Remark. The theory of S.P. Novikov and the author establishes a one-to-one correspondence between flat contravariant metrics on a manifold M and Poisson brackets of hydrodynamic type on the loop space

$$L(M) := \{\text{smooth maps } S^1 \to M\}$$

with certain nondegeneracy conditions [24, 25]. For a flat metric $g^{ij}(x)$ and the correspondent contravariant connection ∇^i the Poisson bracket of two functionals of the form

$$I = I[x] = \frac{1}{2\pi}\int_0^{2\pi} P(s, x(s))\, ds, \quad J = J[x] = \frac{1}{2\pi}\int_0^{2\pi} Q(s, x(s))\, ds,$$

$x = (x^i(s))$, $x(s + 2\pi) = x(s)$ is defined by the formula

$$\{I, J\} := \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I}{\delta x^i(s)}\nabla^i \frac{\delta J}{\delta x^j(s)}\, dx^j(s) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I}{\delta x^i(s)} g^{ij}(x) d_s \frac{\delta J}{\delta x^j(s)}.$$

Here the variational derivative $\delta I/\delta x^i(s) \in T_*M|_{x=x(s)}$ is defined by the equality

$$I[x + \delta x] - I[x] = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta I}{\delta x^i(s)}\delta x^i(s)\, ds + o(|\delta x|);$$

$\delta J/\delta x^j(s)$ is defined by the same formula, $d_s := ds\frac{\partial}{\partial s}$. The Poisson bracket can be uniquely extended to all "good" functionals on the loop space by Leibnitz rule [24, 25]. Flat pencils of metrics correspond to compatible pairs of Poisson brackets of hydrodynamic type. By the definition, Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are called compatible if an arbitrary linear combination

$$a\{\ ,\ \}_1 + b\{\ ,\ \}_2$$

again is a Poisson bracket. Compatible pairs of Poisson brackets are important in the theory of integrable systems [33].

The main source of flat pencils is provided by the following statement.

Lemma 1.2. *If for a flat metric in some coordinate system $x^1, \dots, x^n$ both the components $g^{ij}(x)$ of the metric and $\Gamma^{ij}_k(x)$ of the correspondent Levi-Civita connection depend linearly on the coordinate x^1 then the metrics*

$$g_1^{ij} := g^{ij} \text{ and } g_2^{ij} := \partial_1 g^{ij} \tag{1.23}$$

form a flat pencil if $\det(g_2^{ij}) \neq 0$. *The correspondent Levi-Civita connections have the form*

$$\Gamma^{ij}_{1k} := \Gamma^{ij}_k,\ \Gamma^{ij}_{2k} := \partial_1 \Gamma^{ij}_k. \tag{1.24}$$

Proof. The equations (1.7), (1.8) and the equation of vanishing of the curvature have constant coefficients. Hence the transformation

$$g^{ij}(x^1, \dots, x^n) \mapsto g^{ij}(x^1 + \lambda, \dots, x^n),\ \Gamma^{ij}_k(x^1, \dots, x^n) \mapsto \Gamma^{ij}_k(x^1 + \lambda, \dots, x^n)$$

for an arbitrary λ maps the solutions of these equations to the solutions. By the assumption we have

$$g^{ij}(x^1 + \lambda, \dots, x^n) = g_1^{ij}(x) + \lambda g_2^{ij}(x),\ \Gamma^{ij}_k(x^1 + \lambda, \dots, x^n) = \Gamma^{ij}_{1k}(x) + \lambda \Gamma^{ij}_{2k}(x).$$

The lemma is proved. □

All the above considerations can be applied also to complex (analytic) manifolds where the metrics are quadratic forms analyticaly depending on the point of M.

2 Frobenius structure on the space of orbits of a Coxeter group

Let W be a *Coxeter group*, i.e. a finite group of linear transformations of real n-dimensional space V generated by reflections. We always can assume the transformations of the group to be orthogonal w.r.t. a Euclidean structure on V. The complete classification of irreducible Coxeter groups was obtained in [13]; see also [9]. The complete list consists of the groups (dimension of the space V equals the subscript in the name of the group) A_n, B_n, D_n, E_6, E_7, E_8, F_4, G_2 (the Weyl groups of the correspondent simple Lie algebras), the groups H_3 and H_4 of symmetries of the regular icosahedron and of the regular 600-cell in the 4-dimensional space resp. and the groups $I_2(k)$ of symmetries of the regular k-gon on the plane. The group W also acts on the symmetric algebra $S(V)$ (polynomials of the coordinates of $V = V^*$) and on the $S(V)$-module $\Omega(V)$ of differential forms on V with polynomial coefficients. The subring $R = S(V)^W$ of W-invariant polynomials is generated by n algebraicaly independent homogeneous polynomials $x^1, \ldots, x^n$ [9]. The submodule $\Omega(V)^W$ of the W-invariant differential forms with polynomial coefficients is a free R-module with the basis $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ [9]. Degrees of the basic invariant polynomials are uniquely defined by the Coxeter group. They can be expressed via the *exponents* $m_1, \ldots, m_n$ of the group, i.e. via the eigenvalues of a Coxeter element C in W [9]

$$d_i := \deg x^i = m_{n-i+1} + 1, \tag{2.1a}$$

$$\{\text{eigen}\, C\} = \left\{ \exp \frac{2\pi i(d_1 - 1)}{h}, \ldots, \exp \frac{2\pi i(d_n - 1)}{h} \right\}. \tag{2.1b}$$

The maximal degree h is called *Coxeter number* of W. I will use the reversed ordering of the invariant polynomials

$$d_1 = h > d_2 \geq \ldots \geq d_{n-1} > d_2 = 2. \tag{2.2}$$

The degrees satisfy the *duality condition*

$$d_i + d_{n-i+1} = h + 2, \; i = 1, \ldots, n. \tag{2.3}$$

The list of the degrees for all the Coxeter groups is given in Table 1.

W	$d_1, \ldots, d_n$
A_n	$d_i = n + 2 - i$
B_n	$d_i = 2(n - i + 1)$
$D_n, \; n = 2k$	$d_i = 2(n - i), \; i \leq k,$ $d_i = 2(n - i + 1), \; k + 1 \leq i$
$D_n, \; n = 2k + 1$	$d_i = 2(n - i), \; i \leq k,$ $d_{k+1} = 2k + 1,$ $d_i = 2(n - i + 1), \; k + 2 \leq i$
E_6	12, 9, 8, 6, 5, 2
E_7	18, 14, 12, 10, 8, 6, 2
E_8	30, 24, 20, 18, 14, 12, 8, 2
F_4	12, 8, 6, 2
G_2	6, 2
H_3	10, 6, 2
H_4	30, 20, 12, 2
$I_2(k)$	k, 2

Table 1.

I will extend the action of the group W to the complexified space $V \otimes \mathbf{C}$. The space of orbits

$$M = V \otimes \mathbf{C}/W$$

has a natural structure of an affine algebraic variety: the coordinate ring of M is the (complexified) algebra R of invariant polynomials of the group W. The coordinates $x^1, \dots, x^n$ on M are defined up to an invertible transformation

$$x^i \mapsto x^{i'}(x^1, \dots, x^n), \tag{2.4}$$

where $x^{i'}(x^1, \dots, x^n)$ is a graded homogeneous polynomial of the same degree d_i in the variables $x^1, \dots, x^n$, $\deg x^k = d_k$. Note that the Jacobian $\det(\partial x^{i'}/\partial x^j)$ is a constant (it should not be zero). The transformations (2.4) leave invariant the vector field $\partial_1 := \partial/\partial x^1$ (up to a constant factor) due to the strict inequality $d_1 > d_2$. The coordinate x^n is determined uniquely within a factor. Also the vector field

$$E = d_1 x^1 \partial_1 + \dots + d_n x^n \partial_n = p^a \frac{\partial}{\partial p^a} \tag{2.5}$$

(the generator of scaling transformations) is well-defined on M.

Let $\langle \ , \ \rangle$ denotes the W-invariant Euclidean metric in the space V. I will use the orthonormal coordinates $p^1, \dots, p^n$ in V with respect to this metric. The invariant x^n can be chosen as

$$x^n = \frac{1}{2}((p^1)^2 + \dots + (p^n)^2). \tag{2.6}$$

We extend $\langle \ , \ \rangle$ onto $V \otimes \mathbf{C}$ as a complex quadratic form.

The factorization map $V \otimes \mathbf{C} \to M$ is a local diffeomorphism on an open subset of $V \otimes \mathbf{C}$. The image of this subset in M consists of *regular orbits* (i.e. the number of points of the orbit equals $\# W$). The complement is the *discriminant* $\operatorname{Discr} W$. By the definition it consists of all irregular orbits. Note that the linear coordinates in V can serve also as local coordinates in small domains in $M \setminus \operatorname{Discr} W$. This defines a metric $\langle \ , \ \rangle$ (and $\langle \ , \ \rangle^*$) on $M \setminus \operatorname{Discr} W$. The contravariant metric can be extended onto M according to the following statement (cf. [39, Sections 5 and 6]).

Lemma 2.1. *The Euclidean metric of V induces polynomial contravariant metric $\langle \ , \ \rangle^*$ on the space of orbits*

$$g^{ij}(x) = \langle dx^i, dx^j \rangle^* := \frac{\partial x^i}{\partial p^a} \frac{\partial x^j}{\partial p^a} \tag{2.7}$$

and the correspondent contravariant Levi-Civita connection

$$\Gamma^{ij}_k(x) dx^k = \frac{\partial x^i}{\partial p^a} \frac{\partial^2 x^j}{\partial p^a \partial p^b} dp^b \tag{2.8}$$

also is a polynomial one.

Proof. The right-hand sides in (2.7)/(2.8) are W-invariant polynomials/ differential forms with polynomial coefficients. Hence $g^{ij}(x)/\Gamma_k^{ij}(x)$ are polynomials in $x^1, \dots, x^n$. Lemma is proved. □

Remark. The matrix $g^{ij}(x)$ does not degenerate on $M \setminus \operatorname{Discr} W$ where the factorization $V \otimes \mathbf{C} \to M$ is a local diffeomorphism. So the polynomial (also called *discriminant* of W)

$$D(x) := \det(g^{ij}(x)) \tag{2.9}$$

vanishes precisely on the discriminant $\operatorname{Discr} W$ where the variables $p^1, \dots, p^n$ fail to be local coordinates. Due to this fact the matrix $g^{ij}(x)$ often is called *discriminant matrix* of W. The operation $x^i, x^j \mapsto g^{ij}(x)$ is also called *convolution of invariants* (see [2]). Note that the image of V in the real part of M is specified by the condition of positive semidefiniteness of the matrix $(g^{ij}(x))$ (cf. [34]). The Euclidean connection (2.8) on the space of orbits is called *Gauss-Manin connection.*

Corollary 2.1. *The functions $g^{ij}(x)$ and $\Gamma_k^{ij}(x)$ depend linearly on x^1.*

Proof. From the definition one has that $g^{ij}(x)$ and $\Gamma_k^{ij}(x)$ are graded homogeneous polynomials of the degrees

$$\deg g^{ij}(x) = d_i + d_j - 2 \tag{2.10}$$

$$\deg \Gamma_k^{ij}(x) = d_i + d_j - d_k - 2. \tag{2.11}$$

Since $d_i + d_j \leq 2h = 2d_1$ these polynomials can be at most linear in x^1. Corollary is proved. □

Corollary 2.2 (K. Saito). *The matrix*

$$\eta^{ij}(x) := \partial_1 g^{ij}(x) \tag{2.12}$$

has a triangular form

$$\eta^{ij}(x) = 0 \text{ for } i + j > n + 1, \tag{2.13}$$

and the antidiagonal elements

$$\eta^{i(n-i+1)} =: c_i \tag{2.14}$$

are nonzero constants. Particularly,

$$c := \det(\eta^{ij}) = (-1)^{\frac{n(n-1)}{2}} c_1 \dots c_n \neq 0. \tag{2.15}$$

Proof. One has

$$\deg \eta^{ij}(x) = d_i + d_j - 2 - h.$$

Hence $\deg \eta^{i(n-i+1)} = 0$ (see (2.3)) and $\deg \eta^{ij} < 0$ for $i + j > n + 1$. This proves triangularity of the matrix and constancy of the antidiagonal entries c_i. To prove nondegenerateness of $(\eta^{ij}(x))$ we consider, following Saito, the discriminant (2.9) as a polynomial in x^1

$$D(x) = c(x^1)^n + a_1(x^1)^{n-1} + \ldots + a_n$$

where the coefficients $a_1, \ldots, a_n$ are quasihomogeneous polynomials in $x^2, \ldots, x^n$ of the degrees $h, \ldots, nh$ resp. and the leading coefficient c is given in (2.15). Let γ be the eigenvector of a Coxeter transformation C with the eigenvalue $\exp(2\pi i/h)$. Then

$$x^k(\gamma) = x^k(C\gamma) = x^k(\exp(2\pi i/h)\gamma) = \exp(2\pi i d_k/h)x^k(\gamma).$$

For $k > 1$ we obtain

$$x^k(\gamma) = 0,\ k = 2, \ldots, n.$$

But $D(\gamma) \neq 0$ [9]. Hence the leading coefficient $c \neq 0$. Corollary is proved. □

Corollary 2.3. *The space M of orbits of a finite Coxeter group carries a flat pencil of metrics $g^{ij}(x)$ (2.7) and $\eta^{ij}(x)$ (2.12) where the matrix $\eta^{ij}(x)$ is polynomialy invertible globaly on M.*

We will call (2.12) *Saito metric* on the space of orbits. This metric will be denoted by $(\ ,\ )^*$ (and by $(\ ,\ )$ if considered on the tangent bundle TM). Let us denote by

$$\gamma_k^{ij}(x) := \partial_1 \Gamma_k^{ij}(x) \tag{2.16}$$

the components of the Levi-Civita connection for the metric $\eta^{ij}(x)$. These are quasihomogeneous polynomials of the degrees

$$\deg \gamma_k^{ij}(x) = d_i + d_j - d_k - h - 2. \tag{2.17}$$

Corollary 2.4 (K. Saito). *There exist homogeneous polynomials $t^1(p), \ldots, t^n(p)$ of degrees $d_1, \ldots, d_n$ resp. such that the matrix*

$$\eta^{\alpha\beta} := \partial_1 \langle dt^\alpha, dt^\beta \rangle^* \tag{2.18}$$

is constant.

The coordinates $t^1, \ldots, t^n$ on the orbit space will be called *Saito flat coordinates.* They can be chosen in such a way that the matrix (2.18) is antidiagonal

$$\eta^{\alpha\beta} = \delta^{\alpha+\beta, n+1}.$$

Then the Saito flat coordinates are defined uniquely up to an η-orthogonal transformation

$$t^\alpha \mapsto a_\beta^\alpha t^\beta,$$

$$\sum_{\lambda+\mu=n+1} a_\lambda^\alpha a_\mu^\beta = \delta^{\alpha+\beta, n+1}.$$

Proof. From flatness of the metric $\eta^{ij}(x)$ it follows that the flat coordinates $t^\alpha(x)$, $\alpha = 1,\dots,n$ exist at least localy. They are to be determined from the following system

$$\eta^{is}\partial_s\partial_j t + \gamma_j^{is}\partial_s t = 0 \tag{2.19}$$

(see (1.11)). The inverse matrix $(\eta_{ij}(x)) = (\eta^{ij}(x))^{-1}$ also is polynomial in $x^1,\dots,x^n$. So rewriting the system (2.19) in the form

$$\partial_k\partial_l t + \eta_{il}\gamma_k^{is}\partial_s t = 0 \tag{2.20}$$

we again obtain a system with polynomial coefficients. It can be written as a first-order system for the entries $\xi_l = \partial_l t$,

$$\partial_k\xi_l + \eta_{il}\gamma_k^{is}\xi_s = 0,\ k,l = 1,\dots,n \tag{2.21}$$

(the integrability condition $\partial_k\xi_l = \partial_l\xi_k$ follows from (1.4)). This is an overdetermined completely integrable system. So the space of solutions has dimension n. We can choose a fundamental system of solutions $\xi_l^\alpha(x)$ such that $\xi_l^\alpha(0) = \delta_l^\alpha$. These functions are analytic in x for sufficiently small x. We put $\xi_l^\alpha(x) =: \partial_l t^\alpha(x)$, $t^\alpha(0) = 0$. The system of solutions is invariant w.r.t. the scaling transformations

$$x^i \mapsto c^{d_i}x^i,\ i = 1,\dots,n.$$

So the functions $t^\alpha(x)$ are quasihomogeneous in x of the same degrees $d_1,\dots,d_n$. Since all the degrees are positive the power series $t^\alpha(x)$ should be polynomials in $x^1,\dots,x^n$. Because of the invertibility of the transformation $x^i \mapsto t^\alpha$ we conclude that $t^\alpha(x(p))$ are polynomials in $p^1,\dots,p^n$. Corollary is proved. □

We need to calculate particular components of the metric $g^{\alpha\beta}$ and of the correspondent Levi-Civita connection in the coordinates $t^1,\dots,t^n$ (in fact, in arbitrary homogeneous coordinates $x^1,\dots,x^n$).

Lemma 2.2. *Let the coordinate t^n be normalized as in (2.6). Then the following formulae hold:*

$$g^{n\alpha} = d_\alpha t^\alpha \tag{2.22}$$

$$\Gamma_\beta^{n\alpha} = (d_\alpha - 1)\delta_\beta^\alpha. \tag{2.23}$$

(In the formulae there is no summation over the repeated Greek indices!)

Proof. We have

$$g^{n\alpha} = \frac{\partial t^n}{\partial p^a}\frac{\partial t^\alpha}{\partial p^a} = p^a\frac{\partial t^\alpha}{\partial p^a} = d_\alpha t^\alpha$$

due to the Euler identity for the homogeneous functions $t^\alpha(p)$. Furthermore,

$$\Gamma_\beta^{n\alpha}dt^\beta = \frac{\partial t^n}{\partial p^a}\frac{\partial^2 t^\alpha}{\partial p^a\partial p^b}dp^b = p^a\frac{\partial^2 t^\alpha}{\partial p^a\partial p^b}dp^b = p^a d\left(\frac{\partial t^\alpha}{\partial p^a}\right) =$$

$$d\left(p^a\frac{\partial t^\alpha}{p^a}\right) - \frac{\partial t^\alpha}{\partial p^a}dp^a = (d_\alpha - 1)dt^\alpha.$$

Lemma is proved. □

We can formulate now the main result of this section.

Main lemma. *Let $t^1, \dots, t^n$ be the Saito flat coordinates on the space of orbits of a finite Coxeter group and*

$$\eta^{\alpha\beta} = \partial_1 \langle dt^\alpha, dt^\beta \rangle^* \tag{2.24}$$

be the correspondent constant Saito metric. Then there exists a quasihomogeneous polynomial $F(t)$ of the degree $2h+2$ such that

$$\langle dt^\alpha, dt^\beta \rangle^* = \frac{(d_\alpha + d_\beta - 2)}{h} \eta^{\alpha\lambda} \eta^{\beta\mu} \partial_\lambda \partial \mu F(t). \tag{2.25}$$

The polynomial $F(t)$ determines on the space of orbits a polynomial Frobenius structure with the structure constants

$$c_{\alpha\beta}^\gamma(t) = \eta^{\gamma\epsilon} \partial_\alpha \partial_\beta \partial_\epsilon F(t) \tag{2.26a}$$

the unity

$$e = \partial_1 \tag{2.26b}$$

and the invariant inner product η.

Proof. Because of Corollary 2.3 in the flat coordinates the tensor $\Delta_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta}$ should satisfy the equations (1.17)–(1.19) where $g_1^{\alpha\beta} = g^{\alpha\beta}(t)$, $g_2^{\alpha\beta} = \eta^{\alpha\beta}$. First of all according to (1.17a) we can represent the tensor $\Gamma_\gamma^{\alpha\beta}(t)$ in the form

$$\Gamma_\gamma^{\alpha\beta}(t) = \eta^{\alpha\epsilon} \partial_\epsilon \partial_\gamma f^\beta(t) \tag{2.27}$$

for a vector field $f^\beta(t)$. The equation (1.8) (or, equivalently, (1.19)) for the metric $g^{\alpha\beta}(t)$ and the connection (2.27) reads

$$g^{\alpha\sigma} \Gamma_\sigma^{\beta\gamma} = g^{\beta\sigma} \Gamma_\sigma^{\alpha\gamma}.$$

For $\alpha = n$ because of Lemma 2.2 this gives

$$\sum_\sigma d_\sigma t^\sigma \eta^{\beta\epsilon} \partial_\sigma \partial_\epsilon f^\gamma = (d_\gamma - 1) g^{\beta\gamma}.$$

Applying to the l.h.s. the Euler identity (here $\deg \partial_\epsilon f^\gamma = d_\gamma - d_\epsilon + h$) we obtain

$$(d_\gamma - 1) g^{\beta\gamma} = \sum_\epsilon \eta^{\beta\epsilon} (d_\gamma - d_\epsilon + h) \partial_\epsilon f^\gamma = (d_\gamma + d_\beta - 2) \eta^{\beta\epsilon} \partial_\epsilon f^\gamma. \tag{2.28a}$$

From this one obtains the symmetry

$$\frac{\eta^{\beta\epsilon} \partial_\epsilon f^\gamma}{d_\gamma - 1} = \frac{\eta^{\gamma\epsilon} \partial_\epsilon f^\beta}{d_\beta - 1}.$$

Let us denote

$$\frac{f^\gamma}{d_\gamma - 1} =: \frac{F^\gamma}{h}. \tag{2.28b}$$

We obtain

$$\eta^{\beta\epsilon}\partial_\epsilon F^\gamma = \eta^{\gamma\epsilon}\partial_\epsilon F^\beta.$$

Hence a function $F(t)$ exists such that

$$F^\alpha = \eta^{\alpha\epsilon}\partial_\epsilon F. \tag{2.28c}$$

It is clear that $F(t)$ is a quasihomogeneous polynomial of the degree $2h+2$. From the formula (2.28) one immediately obtains (2.25).

Let us prove now that the coefficients (2.26a) satisfy the associativity condition. It is more convenient to work with the dual structure constants

$$c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F.$$

Because of (2.27), (2.28) one has

$$\Gamma_\gamma^{\alpha\beta} = \frac{d_\beta - 1}{h} c_\gamma^{\alpha\beta}.$$

Substituting this in (1.18) we obtain associativity. Finaly, for $\alpha = n$ the formulae (2.22), (2.23) imply

$$c_\beta^{n\alpha} = h\delta_\beta^\alpha.$$

Since $\eta^{1n} = h$, the vector (2.26b) is the unity of the algebra. Lemma is proved. □

Proof of Theorem 1. Existence of a Frobenius structure on the space of orbits satisfying the conditions of Theorem 1 follows from Main lemma. We are now to prove uniqueness. Let us consider a polynomial Frobenius structure on M with the charges and dimension (3) and with the Saito invariant metric. In the Saito flat coordinates we have

$$dt^\alpha \cdot dt^\beta = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F(t)dt^\gamma.$$

The l.h.s. of (4) reads

$$i_v(dt^\alpha \cdot dt^\beta) = \sum_\gamma d_\gamma t^\gamma \eta_{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F(t) = (d_\alpha + d_\beta - 2)\eta_{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t).$$

This should be equal to $h\langle dt^\alpha, dt^\beta\rangle^*$. So the function $F(t)$ should satisfy (2.25). It is determined uniquely by this equation up to terms quadratic in t^α. Such an ambiguity does not affect the Frobenius structure. Theorem is proved. □

An algebraic remark: let T be a n-dimensional space and $U : T \to T$ an endomorphism (linear operator). Let

$$P_U(u) := \det(U - u \cdot 1)$$

be the characteristic polynomial of U. We say that the endomorphism U is semisimple if all the n roots of the characteristic polynomial are simple. For a semisimple endomorphism there exists a cyclic vector $e \in T$ such that

$$T = \mathrm{span}(e, Ue, \ldots, U^{n-1}e).$$

The map

$$\mathbf{C}[u]/(P_U(u)) \to T, \quad u^k \mapsto U^k e, \; k = 0, 1, \ldots, n-1 \tag{2.29}$$

is an isomorphism of linear spaces.

Let us fix a point $x \in M$. We define a linear operator

$$U = (U^i_j(x)) : T_x M \to T_x M \tag{2.30}$$

(being also an operator on the cotangent bundle) taking the ratio of the quadratic forms g^{ij} and η^{ij}

$$(U\omega_1, \omega_2)^* = \langle \omega_1, \omega_2 \rangle^* \tag{2.31}$$

or, equivalently,

$$U^i_j(x) := \eta_{js}(x) g^{si}(x). \tag{2.32}$$

Lemma 2.3. *The characteristic polynomial of the operator $U(x)$ is given up to a nonzero factor c^{-1} (2.15) by the formula (5).*

Proof. We have

$$\begin{aligned} P(u; x^1, \ldots, x^n) &:= \det(U - u \cdot 1) = \det(\eta_{js}) \det(g^{si} - u\eta^{si}) = \\ & c^{-1} \det(g^{si}(x^1 - u, x^2, \ldots, x^n) = c^{-1} D(x^1 - u, x^2, \ldots, x^n). \end{aligned}$$

Lemma is proved. □

Corollary 2.5. *The operator $U(x)$ is semisimple at a generic point $x \in M$.*

Proof. Let us prove that the discriminant $D_0(x^1, \ldots, x^n)$ of the characteristic polynomial $P(u; x^1, \ldots, x^n)$ does not vanish identicaly on M. Let us fix a Weyl chamber $V_0 \subset V$ of the group W. On the inner part of V_0 the factorization map

$$V_0 \to M_{Re}$$

is a diffeomorphism. On the image of V_0 the discriminant $D(x)$ is positive. It vanishes on the images of the n walls of the Weyl chamber:

$$D(x)_{i\text{-th wall}} = 0, \quad i = 1, \ldots, n. \tag{2.33}$$

On the inner part of the i-th wall (where the surface (2.33) is regular) the equation (2.33) can be solved for x^1:

$$x^1 = x^1_i(x^2, \ldots, x^n). \tag{2.34}$$

Indeed, on the inner part

$$(\partial_1 D(x))_{i\text{-th wall}} \neq 0.$$

This holds since the polynomial $D(x)$ has simple zeroes at the generic point of the discriminant of W (see, e.g., [2]) .

Note that the functions (2.34) are the roots of the equation $D(x) = 0$ as the equation in the unknown x^1. It follows from above that this equation has simple roots for generic $x^2, \ldots, x^n$. The roots of the characteristic equation

$$D(x^1 - u, x^2, \ldots, x^n) = 0$$

are therefore

$$u_i = x^1 - x_i^1(x^2, \ldots, x^n), \quad i = 1, \ldots, n. \tag{2.35}$$

Generically these are distinct. Lemma is proved. □

Lemma 2.4. *The operator U on the tangent planes T_xM coincides with the operator of multiplication by the Euler vector field $v = \frac{1}{h}E$.*

Proof. We check the statement of the lemma in the Saito flat coordinates:

$$\sum_\sigma \frac{d_\sigma}{h} t^\sigma c_{\sigma\beta}^\alpha = \frac{h - d_\beta + d_\alpha}{h} \eta^{\alpha\epsilon} \partial_\epsilon \partial_\beta F =$$

$$\sum_\lambda \frac{d_\lambda + d_\alpha - 2}{h} \eta_{\beta\lambda} \eta^{\alpha\epsilon} \eta^{\lambda\mu} \partial_\epsilon \partial_\mu F = \eta_{\beta\lambda} g^{\alpha\lambda} = U_\beta^\alpha.$$

Lemma is proved. □

Proof of Theorem 2. Because of Lemmas 2.3, 2.4 the vector fields

$$e,\ v,\ v^2, \ldots, v^{n-1} \tag{2.36}$$

genericaly are linear independent on M. It is easy to see that these are polynomial vector fields on M. Hence e is a cyclic vector for the endomorphism U acting on $\operatorname{Der} R$. So in generic point $x \in M$ the map (6a) is an isomorphism of Frobenius algebras

$$\mathbf{C}[u]/(P(u;x)) \to T_xM.$$

This proves Theorem 2. □

Remark 1. The Euclidean metric (2.7) also defines an invariant inner product for the Frobenius algebras (on the cotangent planes T_*M). It can be shown also that the trilinear form

$$\langle \omega_1 \cdot \omega_2, \omega_3 \rangle^*$$

can be represented (localy, outside the discriminant $\operatorname{Discr} W$) in the form

$$(\hat\nabla^i \hat\nabla^j \hat\nabla^k \hat F(x)) \partial_i \otimes \partial_j \otimes \partial_k$$

for some function $\hat F(x)$. Here $\hat\nabla$ is the Gauss-Manin connection (i.e. the Levi-Civita connection for the metric (2.7)). The unity dt^n/h of the Frobenius algebra on T_*M is not covariantly constant w.r.t. the Gauss-Manin connection.

Remark 2. The vector fields

$$l^i := g^{is}(x)\partial_s, \quad i = 1, \dots, n \tag{2.37}$$

form a basis of the R-module $\mathrm{Der}_R(-\log(D(x))$ of the vector fields on M tangent to the discriminant [2]. By the definition, a vector field $u \in \mathrm{Der}_R(-\log(D(x))$ iff

$$uD(x) = p(x)D(x)$$

for a polynomial $p(x) \in R$. The basis (2.37) of $\mathrm{Der}_R(-\log(D(x))$ depends on the choice of coordinates on M. In the Saito flat coordinates commutators of the basic vector fields can be calculated via the structure constants of the Frobenius algebra on T_*M. The following formula holds:

$$[l^\alpha, l^\beta] = \frac{d_\beta - d_\alpha}{h} c_\epsilon^{\alpha\beta} l^\epsilon. \tag{2.38}$$

This can be proved using (2.25).

Remark 3. The eigenvalues $u_1(x), \dots, u_n(x)$ of the endomorphism $U(x)$ can be chosen as new local coordinates near a generic point $x \in M$ (such that $D_0(x) \neq 0$). As it follows from [20, 22] these are *canonical coordinates* on the Frobenius manifold M: by the definition, this means that the law of multiplication of the coordinate vector fields has the form

$$\partial_i \cdot \partial_j = \delta_{ij}\partial_i \tag{2.39}$$

$$\partial_i = \frac{\partial}{\partial u_i}.$$

In these coordinates the Saito metric (,) is given by a diagonal Egoroff metric (see [20] for the definition)

$$\eta_{ij}(u) = \eta_{\alpha\beta}\frac{\partial t^\alpha}{\partial u_i}\frac{\partial t^\beta}{\partial u_i}\delta_{ij}. \tag{2.40}$$

The Euclidean metric $\langle\ ,\ \rangle$ outside of the discriminant $u_1 \dots u_n = 0$ in these coordinates is written as another diagonal Egoroff metric with the diagonal entries $\eta_{ii}(u)/u_i$. The unity vector field has the form

$$e = \sum_{i=1}^n \partial_i \tag{2.41}$$

and the Euler vector field

$$\frac{1}{h}E = v = \sum_{i=1}^n u_i\partial_i. \tag{2.42}$$

I recall that, according to the theory of [20] the metric (2.40) satisfies the Darboux-Egoroff equations

$$\partial_k\gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i, j, k \text{ are distinct,} \tag{2.43a}$$

$$\sum_{k=1}^{n} \partial_k \gamma_{ij} = 0 \tag{2.43b}$$

$$\sum_{k=1}^{n} u_k \partial_k \gamma_{ij} = -\gamma_{ij} \tag{2.43c}$$

where the rotation coefficients $\gamma_{ij}(u) = \gamma_{ji}(u)$ are defined by the formula

$$\gamma_{ij}(u) := \frac{\partial \sqrt{\eta_{jj}(u)}}{\sqrt{\eta_{ii}(u)}}, \quad i \neq j. \tag{2.44}$$

The system (2.43) is empty for $n = 1$; it is linear for $n = 2$. For the first nontrivial case $n = 3$ it can be reduced to a particular case of the Painlevé-VI equation [27] using the first integral

$$(u_1 - u_2)^2 \gamma_{12}^2 + (u_1 - u_3)^2 \gamma_{13}^2 + (u_2 - u_3)^2 \gamma_{23}^2 = R^2. \tag{2.45}$$

For any $n \geq 3$ the system (2.43) can be reduced to a system of ordinary differential equations. It coincides with the equations of isomonodromy deformations of a certain linear differential operator with rational coefficients [20, 22]. Thus the eqs. (2.43) can be called a high-order analogue of the Painlevé-VI. The constructions of the present paper for the groups A_3, B_3, H_3 specify three distinguished solutions of the correspondent Painlevé-VI eqs.. The function $F(t)$ for these groups has the form

$$F_{A_3} = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^2 t_3^2}{4} + \frac{t_3^5}{60} \tag{2.46}$$

$$F_{B_3} = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3}{6} + \frac{t_2^2 t_3^3}{6} + \frac{t_3^7}{210} \tag{2.47}$$

$$F_{H_3} = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3^2}{6} + \frac{t_2^2 t_3^5}{20} + \frac{t_3^{11}}{3960}. \tag{2.48}$$

The correspondent constants R in (2.45) equal 1/4, 1/3 and 2/5 resp.

Concluding remarks

1. The results of this paper can be generalised for the case where W is the Weyl group of an extended affine root system of codimension 1 (see the definition in [39]). In this case the Frobenius structure will be polynomial in all the coordinates but one and it will be a modular form in this exceptional coordinate. The solutions of WDVV of [32, 46] are just of this type. We consider the orbit spaces of these groups in a subsequent publication.

2. The two metrics on the space of orbits of the group A_n are closely related to the two hamiltonian structures of the nKdV hierarchy (see [18–20, 22]).

The Saito metric is obtained by the semiclassical limit of [24, 25] from the first Gelfand-Dickey Poisson bracket of nKdV, and the Euclidean metric is obtained by the same semiclassical limit from the second Gelfand-Dickey Poisson bracket. The Saito and the Euclidean coordinates on the orbit space are the Casimirs for the corresponding Poisson brackets. The factorization map $V \to M = V/W$ is the semiclassical limit of the Miura transformation. Probably, the semiclassical limit of the bi-hamiltonian structure of the $D - E$ Drinfeld-Sokolov hierarchies [17] give the two flat metrics on the orbit spaces of the groups D_n and E_6, E_7, E_8 resp. But this should be checked.

It is still an open question if it is possible to relate integrable hierarchies to the Coxeter groups not of $A - D - E$ series. A partial answer to this question is given in [20, 22]: the unknown integrable hierarchies for any Frobenius manifold are constructed in a semiclassical (i.e., in the dispersionless) approximation.

3. A closely related question: what is the algebraic-geometrical description of the TFT models related to the polynomial solutions of WDVV constructed in this paper? For $A - D - E$ groups the correspondent TFT models are the topological minimal models of [15]. For other Coxeter groups the TFT can be constructed as equivariant topological Landau-Ginsburg models using the results of [44, 47] for $W \neq H_4$ (the singularity theory related to H_4 was partialy developed in [35, 40]). For the group A_n a nice algebraic-geometrical reformulation of the correspondent TFT as the intersection theory on a certain covering over the moduli space of stable algebraic curves, was proposed in [50, 51] (for the topological gravity $W = A_1$ this conjecture was proved by M.Kontsevich [29, 30]). What are the moduli spaces whose intersection theories are encoded by the orbit spaces of other Coxeter groups? Note that a part of these intersection numbers should coincide with the coefficients of the polynomials $F(t)$ (these are rational but not integer numbers since the moduli spaces are not manifolds but orbifolds).

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Appendix: Algebraic version of the definition of polynomial Frobenius manifold

Let k be a field of the characteristic $\neq 2$ and

$$R := k[x^1, \dots, x^n] \tag{A.1}$$

be the ring of polynomials with the coefficients in k. By $\operatorname{Der} R$ we denote the R-module of k-derivations of R. This is a free R-module with the basis

$$\partial_i := \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n.$$

A map

$$\operatorname{Der} R \times R \to R$$

is defined by the formula

$$(u = u^i \partial_i, \ p) \mapsto up := u^i \partial_i p. \tag{A.2}$$

A R-bilinear symmetric inner product

$$\begin{aligned} &\operatorname{Der} R \times \operatorname{Der} R \to R \\ &u, \ v \mapsto (u, v) \in R \end{aligned} \tag{A.3}$$

is called nondegenerate if from the equations

$$(u, v) = 0 \ \text{ for any } v \in \operatorname{Der} R$$

it follows that $u = 0$.

As it was mentioned in the introduction, a polynomial Frobenius manifold is a structure of Frobenius R-algebra on $\operatorname{Der} R$ satisfying certain conditions. We obtain here these conditions by reformulating the Definition 1 in a pure algebraic way.

The first standard step is to reformulate the notion of the Levi-Civita connection. By the definition, this is a map

$$\begin{aligned} &\operatorname{Der} R \times \operatorname{Der} R \to \operatorname{Der} R \\ &u, \ v \mapsto \nabla_u v \end{aligned} \tag{A.4}$$

R-linear in the first argument and satisfying the Leibnitz rule in the second one

$$\nabla_u (pv) = p \nabla_u v + (up) v \tag{A.5}$$

uniquely specified by the equations

$$u(v, w) = (\nabla_u v, w) + (v, \nabla_u w) \tag{A.6a}$$

$$\nabla_u v - \nabla_v u = [u, v] \tag{A.6b}$$

(the commutator of the derivations). Equivalently, it can be determined from the equation

$$\begin{aligned} \langle \nabla_u v, w \rangle = &\frac{1}{2} [u \langle v, w \rangle + v \langle w, u \rangle - w \langle u, v \rangle \\ &+ \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle] \end{aligned} \tag{A.6c}$$

for arbitrary u, v, $w \in \operatorname{Der} R$.

Now the assumptions 1–3 of Definition 1 for the Frobenius R-algebra $\operatorname{Der} R$ can be reformulated as follows:

1. For any u, v, w the following identity holds
$$(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})w = 0.$$
2. For the unity $e \in \operatorname{Der} R$ and for arbitrary $u \in \operatorname{Der} R$
$$\nabla_u e = 0.$$
3. The identity
$$\nabla_u(v \cdot w) - \nabla_v(u \cdot w) + u \cdot \nabla_v w - v \cdot \nabla_u w = [u, v] \cdot w \tag{A.7}$$
holds for any three derivations fields u, v, w.

To reformulate the assumption 4 of Definition 1 let us assume that $\operatorname{Der} R$ is a graded algebra over a graded ring R with a graded invariant inner product $(\ ,\ )$. That means that two gradings deg and deg$'$ are defined on R and on $\operatorname{Der} R$ resp., i.e. real numbers
$$P_i := \deg x^i, \quad Q_i := \deg' \partial_i \tag{A.8}$$
are assigned to the generators $x^1, \ldots, x^n$ and to the basic derivations $\partial_1, \ldots, \partial_n$ resp. By the definition, the degree of a monomial
$$p = (x^1)^{m_1} \ldots (x^n)^{m_n}$$
equals
$$\deg p := m_1 P_1 + \cdots + m_n P_n.$$
Homogeneous elements of $\operatorname{Der} R$ are defined by the assumption that the operators $p \mapsto up$ shifts the grading in R to $\deg' u - Q_0$ for a constant Q_0, i.e.
$$\deg(up) = \deg' u + \deg p - Q_0. \tag{A.9}$$
The R-algebra structure on $\operatorname{Der} R$ should be consistent with the grading, i.e. for any homogeneous elements p, q of R and u, v of $\operatorname{Der} R$ the following formulae hold:
$$\deg'(pu) = \deg' u + \deg p \tag{A.10}$$
$$\deg(pq) = \deg p + \deg q \tag{A.11}$$
$$\deg'(u \cdot v) = \deg' u + \deg' v. \tag{A.12}$$
The invariant inner product $(\ ,\ )$ should be graded of a degree D, i.e.
$$(u, v) = 0 \ \text{ if } \ \deg' u + \deg' v \neq D \tag{A.13}$$
for arbitrary homogeneous u, $v \in \operatorname{Der} R$. Note that the Euler vector field is homogeneous of the degree Q_0. We consider only the case $Q_0 \neq 0$.

The numbers P_i, Q_i, Q_0, D are defined up to rescaling. One can normalise these in such a way that $Q_0 = 1$. Then we have
$$Q_i := q_i, \ P_i = 1 - q_i, \ D = d$$
in the notations of Introduction.

The constructions of this paper give such an algebraic structure for $k = \mathbf{Q}$.

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