Integrable systems in Riemannian geometry

Nigel Hitchin*
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB
UK

1 Introduction

What is an integrable system?

There is no precise formulation which will encompass all the mathematical phenomena covered by this term, but we have a host of examples: most mathematicians would agree that any definition of an integrable system should include:

- completely integrable Hamiltonian systems
- systems linearized on an abelian variety
- Painlevé equations

All these have the characteristic property (in Ward's definition of integrability [60]) that "their solutions can in principle be constructed explicitly". Even that vague statement begs the question, for we know that a general solution to a Painlevé equation involves new transcendental functions. What we mean by an integrable system is that there is a systematic method of describing all

*Current address: Mathematical Institute, 24–29 St Giles, Oxford OX1 3LB, UK.

solutions. In an individual problem, we simply have to select the particular one whose constants of integration match the problem.

That sounds like the 19th century approach to differential equations, and it is true that even the modern questions in Riemannian geometry which are amenable to attack using integrable systems frequently involve a journey into the last century to get going. There could not be a stronger contrast between the 20th century use of existence theorems for partial differential equations to discover deep differential geometric results and the method of integrable systems in producing explicit answers, but existence theorems will never give us all the information we want. Nor, as we shall see, does explicitness, when we find it, always answer the interesting questions. What integrability can do is to reveal truths and patterns which are hidden in the existence theory.

There are perhaps three contrasting approaches to mathematics which distinguish the last two centuries:

- local versus global
- explicitness versus existence
- special cases versus unifying structure

I hope to show that, even in those areas of concern to our predecessors, the modern approach as applied to integrable systems has much to commend it.

Riemannian geometry is not *a priori* concerned with abelian varieties or symplectic manifolds—the stuff that integrable systems are made on. The natural questions we ask are those concerning curvature—properties of the extrinsic curvature of manifolds in Euclidean space, for example, or special properties of the Riemann tensor of an abstract manifold. In this paper we shall concentrate on three areas of Riemannian geometry where integrable systems have been used over the past few years:

- Tori of constant mean curvature
- 4-dimensional Einstein manifolds
- Hyperkähler metrics

The first topic is one where the literature is most extensive, and the subject best understood. Whereas ten years ago, one could have argued that the theory of algebraic curves played an important role in the theory of minimal surfaces and harmonic maps, this was essentially based on versions of the twistor construction. In particular, the algebraic curve was the surface itself, with its conformal structure. The integrable system approach to minimal tori is very different (although now integrated into the twistor viewpoint too), and although algebraic curves come in to play, they do so in a more subtle manner: it is a subspace of the Jacobian which models the surface itself, with its linear structure. We give here an account of two approaches to the problem. One is a direct construction of the algebraic curve from holonomy considerations. The other, which is more developed in the literature, concerns the production of a rather special object on the torus—a polynomial Killing field. We show in Section 2 how these two points of view are related by bringing in a correspondence which lies at the heart of many applications of integrable systems. This is the relationship between a line bundle on a curve which is expressed as a covering of the projective line, and a matrix of polynomials. Systems which are linearized on the Jacobian of a curve are usually of this form, and the creation of polynomial Killing fields is one example.

The second topic, treated in Section 3, is broader, but the usefulness of integrable systems in addressing the open problems is less well understood. The specific problem we consider concerns the case of self-dual Einstein manifolds with a three-dimensional group of isometries. Here the aspect of integrability which enters is the isomonodromic deformation problem, as manifested by Painlevé's sixth equation. There are other problems in Riemannian geometry, both classical and modern, where the essential question is to solve a Painlevé equation, but this particular one allows us to produce a complete solution to the original question. In doing so, we can write down some explicit complete non-Kählerian Einstein metrics on the four-dimensional ball. We emphasize here the non-Kähler nature, since existence theorems for Einstein metrics are few and far between in the absence of a complex structure.

In Section 4, we survey the central role of the self-dual Yang-Mills equations as an overarching structure yielding integrable equations in Riemannian geometry as special cases. This is a curious fact, but the role of these equations in explaining integrability is ever-present. We show in particular how the two systems considered above are derived from forms of the Yang-Mills equations, and then study a third system—Nahm's equations—which play a central role in the latter part of this paper.

The first four sections survey material which has already been published by various authors. Section 5 is a new set of results in which the methods of integrable systems are used to provide concrete information about families of metrics which are of interest to both differential geometers and physicists—*hyperkähler metrics*. Many of the examples of such metrics—Kronheimer's metrics on the cotangent bundle, or a coadjoint orbit of, a complex semi-simple Lie group—or monopole moduli spaces—are based on solutions to Nahm's equations. We give a formula for the Kähler potential of those metrics within these families which carry circle actions of a particular type. The formula can be expressed in terms of the data used to construct a solution to Nahm's equations: an algebraic curve and a line bundle on it. In fact it is the logarithmic derivative of the Riemann theta function which provides the essential component of this. In many respects this is the analogue in Riemannian geometry of the formula for the potential in the KdV equation in terms of the τ-function, perhaps the most familiar place in the conventional literature on integrable systems where theta functions arise. Within the general formula we work out some specific examples to compare with alternative methods. These include the Eguchi-Hanson metric and the $\mathcal{L}^2$ metric on the moduli space $\mathcal{M}_k^0$ of centred $SU(2)$-monopoles.

The monopole moduli space and its Riemannian metric has attracted much attention recently as a testing ground for duality theories in physics [53]. Our formula for the Kähler potential ϕ of a given complex structure is the sum of two terms.

$$\phi = \frac{4}{(N+1)(N+2)} \frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} - \frac{1}{3}Q(u,u)$$

The first, which is rotationally invariant, is fundamentally associated to the spectral curve of the monopole: it is defined by the expansion of the Riemann theta function about a distinguished point. The second term, while also expressible in terms of the spectral curve, can also be read off from the monopole itself. It depends on a direction u in $\mathbf{R}^3$, which corresponds to the complex structure in the hyperkähler family. The term $-Q(u,u)/4$ is the coefficient of r^{-3} in the asymptotic expansion of the length of the Higgs field along a radius in the direction u.

2 Tori of constant mean curvature

2.1 Background

The classical problem here is one of determining the shape of a soap bubble—the constant pressure difference between the inside and the outside is translated into the condition on a surface in $\mathbf{R}^3$ that it should have constant mean curvature. Generalizations of this include such surfaces in S^3, hyperbolic space or higher dimensional manifolds.

A.D. Alexandrov [2], in the 1950's, extended the local results of Jellet [32] a hundred years earlier to show that the only constant mean curvature surface *embedded* in $\mathbf{R}^3$ was a round sphere, thus satisfactorily explaining the prevalence of round soap bubbles, but the question remained open as to whether higher genus surfaces could be *immersed* with constant mean curvature.. This problem, in the case of the torus, is attributed to Hopf, and was solved in 1984 by Wente [61]. He found a particular immersed torus which has constant mean curvature. Wente's is a purely 20th century proof—an existence theorem for the sinh-Gordon equation,

$$\omega_{xx} + \omega_{yy} = -\sinh\omega$$

but the subsequent development of the subject, as described by Melko and Sterling [43] is noteworthy.

Wente's existence theorem provoked Abresch to make a computerized numerical analysis of the solution to create pictures of the torus. He noticed, on plotting the curvature lines, that each line of one family appeared to be planar. Taking this as an ansatz, he reduced the partial differential equation to ordinary differential equations solvable by elliptic functions and gave an explicit analytic solution [1]. Ironically, Wente had seen the reformulation of the problem into the sinh-Gordon equation in Eisenhart's classic textbook [20]. Had he read a few pages further, he would have encountered the 19th century version of Abresch's analytical solution [58].

This cautionary tale does not of course reveal the general solution to the problem, which has now developed into a major area of research. The main points of view (see [21] for various articles on the subject) are due to Pinkall and Sterling et al, whose approach can be expressed in terms of Hamiltonian systems and loop groups, Bobenko, who sees it in terms of the much studied finite-gap solutions of the sinh-Gordon equation and the corresponding Baker-Akhiezer function, and the author [25] whose approach is influenced by twistor theory. These viewpoints all embed the specific problem in a bigger structure, and they are all related. In the next sections I want to make the relationship between two of these a little more precise, since it involves a construction which appears in many of the interfaces between integrable systems and differential geometry.

2.2 Harmonic maps

All approaches to constant mean curvature surfaces set the problem in the wider context of harmonic maps of surfaces into Lie groups. It is an old result [50] that the Gauss map of a constant mean curvature surface in $\mathbf{R}^3$ is a harmonic map to S^2. Embedding S^2 totally geodesically as an equator in S^3 we get a harmonic map into $S^3 \cong SU(2)$. Once we are in the 3-sphere we can also think of minimal surfaces in the sphere, which are other examples of harmonic maps. Generalizing, we may consider a Riemann surface M and a harmonic map $f : M \to G$ into a compact semisimple Lie group, with its biinvariant metric.

The equations for a harmonic map have a particularly accessible form. If we trivialize the tangent bundle by left translation, this defines a flat connection ∇_L with trivial holonomy, and by right translation another trivial connection ∇_R. Since the tangent bundle has structure group G, we may equally regard these as connections on a principal G-bundle. The map f relates the two trivializations: $f^{-1}df = \nabla_L - \nabla_R$. However, instead of this formalism we consider

$$\begin{aligned} \nabla &= \frac{1}{2}(\nabla_R + \nabla_L) \\ \phi &= \frac{1}{2}(\nabla_R - \nabla_L) \end{aligned}$$

to define a pair of objects: a connection ∇ and a section $\phi \in \Omega^1(M; \mathrm{ad}\, P)$. Using the conformal structure, we write $\phi = \Phi - \Phi^*$ where $\Phi \in \Omega^{1,0}(M; \mathrm{ad}\, P \otimes \mathbf{C})$ is of type $(1,0)$. Although we write $-\Phi^*$ which makes sense only for the unitary group, for a general Lie group we interpret this as the application of the anti-involution on $\Omega^{1,0}(M; \mathrm{ad}\, P \otimes \mathbf{C})$ defined by the real structure on the complex Lie algebra which gives the compact real form. We then have the following:

Proposition 2.1. *If f is a harmonic map the G^c connection*

$$\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$$

is flat for all $\zeta \in \mathbf{C}^*$. Conversely if ∇, Φ define a flat connection for all ζ which has trivial holonomy for $\zeta = \pm 1$, then it arises from a harmonic map.

The historical origins of this method are unclear: in some respects it is the classical notion of an associated surface, but in the context of integrable systems it seems to begin with Pohlmeyer [49].

2.3 The loop group approach

The approach initiated by Pinkall and Sterling provides a formulation in terms of loop groups and algebras, as studied for example by Burstall et al [8], who we follow in this section. Here one works in the trivial covariant constant gauge at $\zeta = 1$ for the connection $\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$, the connection ∇_R. The connection matrix can be written as

$$A_\zeta = (1-\zeta)\alpha - (1-\zeta^{-1})\alpha^*$$

for some Lie-algebra valued $(1,0)$-form α on the Riemann surface M. The harmonic map f is then defined by the covariant constant trivialization at $\zeta = -1$, the connection ∇_L,

$$f^{-1}df = 2(\alpha - \alpha^*) \tag{2.1}$$

If M is a torus, passing to the universal covering $\mathbf{C}$, we can write $\alpha = adz$, and then comparing coefficients of ζ in the flatness condition gives the equation for f to be harmonic:

$$\frac{\partial a}{\partial \bar{z}} = -[a, a^*] \tag{2.2}$$

The essential point of the loop group approach is to see these equations as the consequence of another equation with more variables, but with a simpler form. One considers a finite Laurent series which vanishes at $\zeta = 1$ (a "polynomial Killing field")

$$\xi = \sum_{|n|\leq d} (1-\zeta^n)\xi_n$$

where the ξ_n are $\mathfrak{g}^c$-valued functions with $\xi_{-n} = -\xi_n^*$ satisfying the equation

$$d\xi + [(1-\zeta)adz - (1-\zeta^{-1})a^*d\bar{z}, \xi] = 0 \tag{2.3}$$

or equivalently

$$\frac{\partial \xi}{\partial z} = [\xi, (1-\zeta)a] \tag{2.4}$$

$$\frac{\partial \xi}{\partial \bar{z}} = -[\xi, (1-\zeta^{-1})a^*] \tag{2.5}$$

and where $a = \xi_d$. Expanding out yields equation (2.2) as one of the many terms.

The principal advantage of using this approach is its integrability in finite-dimensional terms. The highest order coefficient of ξ commutes with a, so a solution of (2.4), (2.5) will still be a Laurent polynomial of the same degree: we can thus rewrite the problem in terms of integrating vector fields on the finite-dimensional vector space Ω_d of Laurent polynomials in the Lie algebra $\mathfrak{g}$ of degree d. These are the two vector fields X_1, X_2 defined by the equations (2.4,2.5)

$$X_1 + iX_2 = Z = [\xi, (1-\zeta)\xi_d] \tag{2.6}$$

They can be simultaneously integrated if they commute, and this happens if and only if $[Z, \bar{Z}] = 0$. But this is true iff

$$\begin{aligned}&[\,[\xi,(1-\zeta^{-1})\xi_{-d}],(1-\zeta)\xi_d] + [\xi,(1-\zeta)[\xi,(1-\zeta^{-1})\xi_{-d}]_d]\\ &= [\,[\xi,(1-\zeta)\xi_d],(1-\zeta^{-1})\xi_{-d}] + [\xi,(1-\zeta^{-1})[\xi,(1-\zeta)\xi_d]_{-d}]\end{aligned}$$

or equivalently

$$\begin{aligned}&(1-\zeta)(1-\zeta^{-1})[\,[\xi,\xi_{-d}],\xi_d] + (1-\zeta)[\xi,[\xi_d,\xi_{-d}]\,]\\ &= (1-\zeta)(1-\zeta^{-1})[\,[\xi,\xi_d],\xi_{-d}] + (1-\zeta^{-1})[\xi,[\xi_{-d},\xi_d]\,]\end{aligned}$$

and verification of this statement follows immediately on writing $(1-\zeta)(1-\zeta^{-1}) = (1-\zeta) + (1-\zeta^{-1})$ and using the Jacobi identity.

The specific form of this approach has certain benefits. Any vector field of the form $X = [\xi, B(\xi)]$ on a Lie algebra (a *Lax pair*) has the property that invariant functions are conserved along the flow. In our case, the algebra is the loop algebra $\Omega\mathfrak{g}$, and we are restricting to the finite-dimensional subspace Ω_d. Any invariant function on $\mathfrak{g}$ then defines a polynomial, all of whose coefficients are constants of integration. In fact, these constants fit into a convenient Hamiltonian formalism for the system, but we shall not go into this here.

A particular consequence of the existence of these conserved quantities is obtained by applying the Killing form. The constant coefficient in the resulting polynomial is the expression

$$\sum_n \langle \xi_n, \xi_{-n}\rangle$$

which, since $\xi_n = -\xi^*_{-n}$, defines an inner product on Ω_d. By compactness the vector fields can be integrated to give a solution to (2.4),(2.5) on the whole of $\mathbf{R}^2$.

A slightly stronger consequence of a Lax form is that the flow is tangential to an orbit of the Lie group, so that (2.6) implies for the complex vector field Z that ξ lies in the orbit of ξ_d under the complex Lie group. In particular, $a = \xi_d : \mathbf{R}^2 \to \mathfrak{g}^c$ itself maps to a fixed orbit.

As it stands, this method is an ansatz designed to produce solutions of the equations on $\mathbf{R}^2$. We shall not go into the Hamiltonian formalism which helps to solve it, involving r-matrices, and the Adler-Kostant-Symes method but (see [9]) it is well-documented.

For the problem at hand, maps of a torus, we need to find doubly periodic solutions. The aim is thus to find the general solution and then choose the right constants of integration to give us maps of a torus. More importantly, what is needed is a proof that the method has general applicability: in particular we need to show that *any* harmonic map from a torus has the property that a lies in a fixed orbit, and then find a polynomial Killing field. Furthermore, if this is true, we need to interpret the equation geometrically in terms of the original problem of harmonic maps to G. The situation here is now quite well understood. In the next section I essentially follow [9].

2.4 Jacobi fields

Suppose we have a harmonic map from a torus to G, then (2.2) shows that evaluating any invariant polynomial on a gives a holomorphic function on the torus, which is constant by compactness. In the simplest case where a is principal semi-simple (has distinct eigenvalues) at some point it follows that it is principal semi-simple everywhere and since all invariant polynomials have the same value, lies in a single complex adjoint orbit. If we make this simplifying assumption for th present, then we see that one condition for the method to apply is satisfied, and this is a consequence of compactness of the torus. It will be compactness again which gives the existence of a polynomial Killing field, but this involves a more complex argument.

The equation (2.3) simply says that ξ is a covariant constant section of the adjoint bundle $\operatorname{ad} P \otimes \mathbf{C}$ with respect to the flat connection $d + A_\zeta$. The flatness of the connection for all ζ implies the existence of a local infinite Laurent series solution, but what we need is a polynomial. We begin by finding a semi-infinite series

$$x = \sum_0^\infty x_n \zeta^{-n}$$

This provides a formal solution if the following recurrence relation holds:

$$dx_{n-1} + [\alpha - \alpha^*, x_{n-1}] - [\alpha, x_n] + [\alpha^*, x_{n-2}] = 0 \tag{2.7}$$

We focus attention on the $(1,0)$ part:

$$\frac{\partial x_{n-1}}{\partial z} + [a, x_{n-1}] - [a, x_n] = 0 \tag{2.8}$$

If we begin with $x_0 = a$, we can solve this recursively so long as $\partial x_{n-1}/\partial z$ lies in the image of $\operatorname{ad}(a)$. Since, as we have seen, a lies in a single orbit, $\partial x_0/\partial z = \partial a/\partial z$ is tangential to the orbit and so is indeed in the image of ad(a). To proceed, one engages in a two-step recursion process modifying x_n by a suitable term in $\ker \mathrm{ad}(a)$ [8]. If a is doubly periodic, so is each x_n, which is thus defined on the torus. We therefore get a formal solution x to

$$d'x + [A_\zeta^{1,0}, x] = 0$$

Attending to the $(0,1)$ part, we note that since $d + A_\zeta$ is flat, $\partial/\partial z + (1-\zeta)a$ and $\partial/\partial\bar{z} - (1-\zeta^{-1})a^*$ commute, so that

$$\partial x/\partial\bar{z} - [(1-\zeta^{-1})a^*, \xi] = \sum_0^\infty \tilde{x}_n\zeta^{-n}$$

is another solution to (2.8). In this case, we have $\tilde{x}_0 = \partial a/\partial\bar{z} - [a^*, a]$ which vanishes by (2.2). Thus, by recurrence, the solution determined above satisfies the full recurrence equation and gives a formal solution to $dx + [A_\zeta, x] = 0$.

The coefficients of this series have an interpretation on the torus itself, which helps to prove finiteness of the series. They are (complex) *Jacobi fields*—solutions of the linearization of the harmonic map equation. To see this, recall from (2.1) that the actual harmonic map f is defined by

$$f^{-1}df = 2(adz - a^*d\bar{z})$$

An infinitesimal deformation $\dot{f}$ of the map $f : M \to G$ defines a tangent vector to G along the torus which we represent as the Lie algebra-valued function $\psi = f^{-1}\dot{f}$. Differentiating the above equation along the deformation gives

$$d\psi + 2[\alpha - \alpha^*, \psi] = 2(\dot{\alpha} - \dot{\alpha}^*)$$

We have the harmonic map equation $d''\alpha - [\alpha^*, \alpha] = 0$, and differentiating this along the deformation we get

$$d''\dot{\alpha} - [\dot{\alpha}^*, \alpha] - [\alpha^*, \dot{\alpha}] = 0$$

Writing $\beta = \dot{\alpha}$ and $\gamma = \dot{\alpha}^*$, a complex solution ψ of the Jacobi equation is equivalent to a solution of the three equations

$$\begin{aligned} d'\psi + 2[\alpha, \psi] &= 2\beta \\ d''\psi - 2[\alpha^*, \psi] &= -2\gamma \\ d''\beta - [\gamma, \alpha] - [\alpha^*, \beta] &= 0 \end{aligned}$$

We want to see that $\psi = x_n$ solves these equations. Now from (2.7),

$$\begin{aligned} d'x_{n-1} + [\alpha, x_{n-1}] - [\alpha, x_n] &= 0 \\ d''x_{n-1} - [\alpha^*, x_{n-1}] + [\alpha^*, x_{n-2}] &= 0 \end{aligned}$$

Setting $2\beta = [\alpha, x_n + x_{n-1}]$ and $2\gamma = [\alpha^*, x_{n-2} + x_{n-1}]$ we obtain the first two of the Jacobi equations. As for the third, we have, using $d''\alpha = [\alpha^*, \alpha]$,

$$\begin{aligned} 2d''\beta &= [\,[\alpha^*, \alpha], x_n + x_{n-1}] - [\alpha, d''(x_n + x_{n-1})] \\ &= [\,[\alpha^*, \alpha], x_n + x_{n-1}] - [\alpha, [\alpha^*, x_n]\,] + [\alpha, [\alpha^*, x_{n-1}]\,] \\ &\qquad\qquad\qquad\qquad\qquad\qquad - [\alpha, [\alpha^*, x_{n-1}]\,] + [\alpha, [\alpha^*, x_{n-2}]\,] \\ &= 2[\gamma, \alpha] + 2[\alpha^*, \beta] \end{aligned}$$

using the Jacobi identity.

The coefficients x_n of the formal solution therefore each belong to a fixed vector space of Lie-algebra-valued functions - the space of Jacobi fields. Moreover, since the linearization of the harmonic map equations is elliptic, by compactness of the torus this space is finite-dimensional. This is the crux of the finiteness result from this point of view. To reduce x from a series to a polynomial requires one more step.

We consider the polynomials

$$p_k = \sum_{n=0}^{k} \zeta^{k-n} x_n$$

obtained from the first $(k+1)$ terms of $\zeta^k x$. Since $dx + [A_\zeta, x] = 0$, we have $d(\zeta^k x) + [A_\zeta, \zeta^k x] = 0$, and consideration of the polynomial part gives

$$dp_k + [A_\zeta, p_k] = [\alpha, x_{k+1}] \tag{2.9}$$

We want to show that x_k vanishes for k large enough. Suppose not, then we can find polynomials $p_1, \ldots, p_N$ spanning a space of greater dimension than the vector space J of Jacobi fields. But then, since each x_{k+1} is an element of J, it follows from (2.9) that there is a polynomial $q(\zeta) = \sum_0^N \lambda_k p_k$ such that $dq + [A_\zeta, q] = 0$. Without loss of generality, we can assume that the constant coefficient q_0 is non-zero. But from (2.7) with $n = 0$ $q_0 = ha$ for some function h, and from the same formula with $n = 1$, h must be holomorphic and hence constant on the torus. We can thus scale q to be obtained from the same recurrence relation with the same initial condition as x. Since, moreover, the other coefficients of q are multiples of x_k for $k > 0$, they are in $\mathrm{Im}(\mathrm{ad}(a))$ and so q coincides with x, showing that the x_k do indeed vanish for $k > N$. It is now a simple step using reality and dividing by a power of ζ to obtain the required polynomial Killing field.

This is the loop group approach. We now give the alternative viewpoint which involves the geometry of algebraic curves.

2.5 Hyperelliptic curves

The details of this construction have only been worked out for $G = SU(2)$ [25], so we restrict ourselves to this case and consider, for comparison with the previous method, the situation where a is semisimple. The construction starts with an algebraic curve S defined by the equation

$$\eta^2 = P(\zeta)$$

where $P(\zeta)$ is a polynomial of degree $2p+2$ in ζ such that $\bar{P}(\zeta) = \bar{\zeta}^{2p+2}P(\bar{\zeta}^{-1})$. There must be no roots of P at 0 or ∞ nor on the unit circle. The curve is a branched double covering $\pi : S \to \mathbf{P}^1$—a hyperelliptic curve.

We then take two meromorphic differentials $\theta, \tilde{\theta}$ with double poles over 0 and ∞ but with residue zero, anti-invariant with respect to the hyperelliptic involution $\eta \mapsto -\eta$ and satisfying some reality conditions.

To relate this to the problem of harmonic maps of a torus, we have to find the torus in this mass of algebraic data. The differentials have expansions near $\zeta = 0$

$$\theta = \lambda_{-2}\frac{d\zeta}{\zeta^2} + \lambda_0 + \ldots \quad \tilde{\theta} = \tilde{\lambda}_{-2}\frac{d\zeta}{\zeta^2} + \tilde{\lambda}_0 + \ldots$$

and since their residue is zero, the principal parts are determined by the coefficients $\lambda_{-2}, \tilde{\lambda}_{-2}$.

We write as usual $\mathcal{O}(d)$ for the pull-back to S of the line bundle of degree d on $\mathbf{P}^1$, then the poles of the differentials occur on the divisor $D = \pi^{-1}\{0, \infty\}$ of $\mathcal{O}(2)$. The principal parts $[\theta], [\tilde{\theta}]$ are then, invariantly speaking, elements of $H^0(D, \mathcal{O}(2))$ and the exact cohomology sequence for the natural sequence of sheaves

$$0 \to \mathcal{O} \to \mathcal{O}(2) \to \mathcal{O}(2)_D \to 0$$

gives a coboundary map

$$\delta : H^0(D, \mathcal{O}(2)) \to H^1(S, \mathcal{O}) \tag{2.10}$$

The image of the principal parts of the differentials then spans a (real) 2-dimensional subspace U. This is going to be the *universal covering* of our 2-torus.

The aim is to construct, for each ζ, a flat connection on a certain rank 2 vector bundle over U, and show that this is of the form of Proposition 1. Now each $x \in U \subseteq H^1(S, \mathcal{O})$ defines by exponentiation (and choice of Poincaré bundle) a line bundle L_x of degree zero over S. The *vector bundle* arises by choosing a fixed line bundle E of degree $p+1$ on S (with some reality property we will not go into here) and defining the fibre at $x \in U$ to be

$$V_x = H^0(S, E \otimes L_x).$$

The *connection* on the bundle V is defined by means of parallel translation, and this involves the interpretation of the cohomology classes in the image of the map δ in (2.10). The coboundary construction of the cohomology class means that after exponentiating, the line bundle $L_x = \exp(\delta(a[\theta] + b[\tilde{\theta}]))$ is naturally trivial outside D. The trivialization extends to the whole of S only if we multiply by

$$\exp(\pm\zeta^{-1}(a\lambda_{-2} + b\tilde{\lambda}_{-2})) \tag{2.11}$$

in a neighbourhood of $\pi^{-1}(0)$ and analogous expressions at ∞. For $x, y \in U$ the ratio of these trivializations gives a non-vanishing section P_{xy} of $L_y \otimes L_x^*$ outside D. Now if the line bundle $E \otimes L_x(-1)$ of degree $p-1$ is non-special (i.e. $H^0(S, E \otimes L_x(-1)) = H^1(S, E \otimes L_x(-1)) = 0$), it is easy to see that restricting sections of $E \otimes L_x$ to $D_\zeta = \pi^{-1}(\zeta)$ is an isomorphism, so we have

$$V_x = H^0(S, E \otimes L_x) \cong H^0(D_\zeta, E \otimes L_x) \tag{2.12}$$

Furthermore, if $\zeta \neq 0, \infty$, then multiplication by P_{xy} defines an isomorphism

$$H^0(D_\zeta, E \otimes L_x) \cong H^0(D_\zeta, E \otimes L_y) \tag{2.13}$$

Putting (2.12) and (2.13) together, we have our definition of parallel translation $\Pi_{xy} : V_x \to V_y$. It is clearly independent of the path, and so for each $\zeta \neq 0, \infty$ we have a flat connection on V.

As ζ approaches 0, then (2.11) shows that the connection matrix acquires a simple pole in ζ. A similar consideration at $\zeta = \infty$ shows that the connection is of the form

$$\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$$

as required.

This, in brief, is the construction. We start with the hyperelliptic curve, and for each line bundle E satisfying suitable reality constraints (which actually guarantee it is non-special—see [25]) we obtain a harmonic map from $U = \mathbf{R}^2$ to $SU(2)$. In order to obtain a map of the torus we need the connection to descend to a quotient of the vector space U, and for this we need $U \in H^1(S, \mathcal{O})$ to intersect $H^1(S, \mathbf{Z})$ in a lattice. An equivalent way of saying this is to insist that the periods of the differentials θ and $\tilde{\theta}$ should lie in $2\pi i\mathbf{Z}$, and this of course imposes severe constraints on the curve.

Even if those constraints are satisfied, we only get a harmonic map if the holonomy is trivial at $\zeta = \pm 1$. In fact, given that θ has periods in $2\pi i\mathbf{Z}$, we can write $\theta = dh/h$ for some holomorphic function on $S \backslash \pi^{-1}\{0, \infty\}$, and this 2-valued function on $\mathbf{P}^1 \backslash \{0, \infty\}$ can, by examining the construction above, be seen to be the eigenvalue of the holonomy of the flat connection $\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$ around one generator of the fundamental group of the torus. Similarly $\tilde{\theta}$ gives the holonomy for the other generator. The existence of a harmonic map thus requires this extra constraint, which can also be written in terms of periods of differentials using the reciprocity formula [25].

The algebraic curve method provides another construction in finite-dimensional terms, of harmonic maps from the torus. As with the loop group method, we need to prove its generality—that to every harmonic map from a torus to $SU(2)$ there exists a hyperelliptic curve.

2.6 Spectral curves

Let $f : M \to SU(2)$ be a harmonic map of a 2-torus. Where do we find an algebraic curve? The key idea is to consider the *holonomy* of the flat connection $\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$. Since $\pi_1(M) = \mathbf{Z} \oplus \mathbf{Z}$, we have two holonomy matrices for each $\zeta \neq 0, \infty$ and so holomorphic functions $H, \tilde{H} : \mathbf{C}^* \to SL(2, \mathbf{C})$ which *commute.*

As we saw in the previous section, if the hyperelliptic curve construction applies, then the eigenvalues of the holonomy are single-valued holomorphic functions on $S \backslash \pi^{-1}\{0, \infty\}$, so to construct S we need to consider the eigenvalues

$$h = \frac{\operatorname{tr} H \pm \sqrt{(\operatorname{tr} H)^2 - 4}}{2}$$

This function has branch points where $(\operatorname{tr} H)^2 - 4$ has odd zeros, and the essential point for producing an algebraic curve is to show that there are only finitely many such points. As with the previous method, this finiteness will depend on solutions of an elliptic equation on the compact manifold M. In fact, we have to rule out first the case where the holonomy is trivial for all ζ, but this can rather readily be shown to correspond to the harmonic map defined by a holomorphic or antiholomorphic map to $\mathbf{P}^1$.

The important point to note here is that if $(\operatorname{tr} H)^2 - 4$ has an odd zero at $\zeta = \zeta_0$, so does $(\operatorname{tr} \tilde{H})^2 - 4$. To prove this, we work in the field K of fractions of the convergent power series in $(\zeta - \zeta_0)$. Then H and $\tilde{H}$ are 2×2 matrices with entries in the field. If $(\operatorname{tr} H)^2 - 4$ has an even zero, the eigenvalues of H are in K and are distinct since ζ_0 is an isolated zero. But since $\tilde{H}$ commutes with H, the eigenvectors are also eigenvectors of $\tilde{H}$ which thus has eigenvalues in K, and hence $(\operatorname{tr} \tilde{H})^2 - 4$ also has an even zero.

A consequence of this is that at an odd zero of $(\operatorname{tr} H)^2 - 4$, the eigenvalues of both H and $\tilde{H}$ are ± 1, and since they commute, there is a common eigenvector with eigenvalue 1 for some choice of $\pm H, \pm \tilde{H}$. We now interpret this fact in terms of flat connections. It means that after possibly tensoring with a flat unitary line bundle with holonomy ± 1, we have a global solution s to the equation

$$\nabla s + \zeta_0 \Phi s - \zeta_0^{-1} \Phi^* s = 0$$

and in particular to the elliptic equation

$$\nabla^{1,0} s + \zeta_0 \Phi s = 0$$

For $||\zeta|| \leq 1$, $\nabla^{1,0} + \zeta\Phi$ is a holomorphic family of elliptic operators of index zero which therefore has a *determinant* which is a holomorphic function of ζ.

Being holomorphic, it vanishes at a finite number of points or identically. If the latter, then for $\zeta = e^{i\theta}$, the connection is flat and unitary and we can use a standard Weitzenböck argument (as in the theory of stable bundles) to deduce from $\nabla^{1,0}s + e^{i\theta}\Phi s = 0$ that s is covariant constant and thus the holonomy trivial. Since this is true for all θ, we are back in the trivial holonomy case. So there must be only finitely many odd zeros of $(\mathrm{tr}\, H)^2 - 4$ in the unit disc. Arguing similarly with the (0,1) part we get only finitely many outside the disc.

This is the essential finiteness property, which as the interested reader will find in [25], leads to the fact that any harmonic map from a torus to $SU(2)$ can be constructed from a (possibly singular) hyperelliptic curve—the *spectral curve*—in the manner of the previous section.

We have seen here two rather different methods of integrating the equations for a harmonic map of the torus. One is based on a polynomial with values in the Lie algebra, the other on line bundles over an algebraic curve. The two are in fact closely connected, and the link is provided by the following well-known result, to be found, for example in [5].

2.7 A basic result in integrable systems

Many classical (and not so classical) integrable systems can be linearized on the Jacobian of an algebraic curve. The fundamental idea behind this is the link between a line bundle over a certain curve and a matrix of polynomials. Applications to integrable systems concern the evolution of those matrices as the class of the line bundle follows a straight line on the Jacobian.

As usual, let $\mathcal{O}(d)$ be the line bundle of degree d over $\mathbf{P}^1$. We consider an element $A \in H^0(\mathbf{P}^1, \mathcal{O}(d)) \otimes \mathfrak{gl}(k)$, so that in terms of an affine coordinate ζ on $\mathbf{P}^1$, $A(\zeta)$ is simply a polynomial of degree d with coefficients which are $k \times k$ matrices. Let η denote the tautological section of $\pi^*\mathcal{O}(d)$ over the total space of $\mathcal{O}(d)$ and $S \subset \mathcal{O}(d)$ the curve defined by $\det(\eta - A(\zeta)) = 0$, the *spectral curve* of A. We then have the theorem [5]:

Theorem 1. *Suppose S is smooth, and let X be the space of all $B \in H^0(\mathbf{P}^1, \mathcal{O}(d)) \otimes \mathfrak{gl}(k)$ with spectral curve S. Then $PGL(k, \mathbf{C})$ acts freely on X by conjugation and the quotient can be identified with $J^{g-1}(S)\backslash\Theta$.*

Proof: In algebro-geometric language, a line bundle L on S is equivalent to a vector bundle $V = \pi_*L$ with the structure of a $\pi_*\mathcal{O}$-module. But this is the same thing as a homomorphism

$$A : V \to V(d)$$

satisfying the equation $P(A, \zeta) = 0$ where $\det(\eta - A(\zeta)) = P(\eta, \zeta)$. Since S is smooth and in particular irreducible, then by the Cayley-Hamilton theorem A has characteristic polynomial P. Now L is not contained in the theta divisor if and only if

$$H^0(S, L) = H^1(S, L) = 0$$

and from the functorial properties of the direct image, this is equivalent to

$$H^0(\mathbf{P}^1, V) = H^1(\mathbf{P}^1, V) = 0.$$

But from the Birkhoff-Grothendieck classification of bundles on the projective line, this means that $V \cong \mathcal{O}^k(-1)$, and so $\mathrm{Hom}(V,V) \cong \mathrm{Hom}(\mathcal{O}^k, \mathcal{O}^k)$ and we can thus interpret $A \in H^0(\mathbf{P}^1, \mathcal{O}(d)) \otimes \mathfrak{gl}(k)$

Remarks:

1. Note that the genus g of S is

$$g = \frac{1}{2}(k-1)(dk-2) \tag{2.14}$$

2. As described in [5], the assumption of smoothness is by no means necessary in the proof: if S is irreducible and reduced then we can repeat the argument using torsion-free rank 1 sheaves, and in the general case using invertible sheaves so long as A is regular (i.e. it has a k-dimensional space of commuting matrices) at each point.

The slick algebraic proof perhaps disguises the meaning of the correspondence, so let us spell it out. The matrix A has a single-valued eigenvalue η not on $\mathbf{P}^1$, but on the covering S. Over a point $\zeta \in \mathbf{P}^1$, the fibre V_ζ is by definition $H^0(D_\zeta, L)$ where D_ζ is the divisor $\pi^{-1}(\zeta)$. At a generic point the fibre consists of k distinct points $p_1, \ldots, p_k$ and we can find a basis of sections $s_1, \ldots, s_k$ with $s_i(p_j) = 0$ if $i \neq j$. This is a basis of eigenvectors of $A(\zeta)$.

Let us now compare the two methods of solving the harmonic map equations for a 2-torus to $SU(2)$. On the one hand, the loop group method produces a polynomial Killing field

$$\xi = \sum_{-d}^{d} (1-\zeta^n)\xi_n$$

which we may regard as a section of $\mathcal{O}(2d) \otimes \mathfrak{sl}(2, \mathbf{C})$. Its coefficients depend on a point of the torus M. It satisfies the differential equation

$$d\xi + [A_\zeta, \xi] = 0$$

On the other hand, the spectral curve approach produces a hyperelliptic curve S with equation $\eta^2 = P(\zeta)$. Since P is a section of $\mathcal{O}(2p+2)$, the curve S naturally lies inside $\mathcal{O}(p+1)$. Moreover the points x of the torus in the construction correspond to line bundles $E \otimes L_x$ on S. It seems plausible that the two points of view may be linked by taking $d = p+1$ and $k = 2$. This is indeed so:

Proposition 2.2. *Let f be a harmonic map of a torus M to $SU(2)$. Then, if the spectral curve S is smooth, the polynomial Killing field at the point $x \in M$ is obtained by the procedure of Theorem 1 from the line bundle $L = EL_x(-1)$ over S.*

Proof: Recall that the construction of the vector bundle V over the torus gave

$$V_x \cong H^0(D_\zeta, EL_x) = \pi_*(EL_x)_\zeta = (\pi_* L)(1)_\zeta$$

so that $A(\zeta)$ acts naturally as an automorphism of V. We have to prove that it is covariant constant, that is commutes with parallel translation. But as we saw, parallel translation is defined by multiplying $H^0(D_\zeta, EL_x)$ by the section P_{xy} of $L_x^* L_y$. If $s \in H^0(D_\zeta, EL_x)$ vanishes at all but one point in D_ζ, then so does $P_{xy}s$ hence parallel translation preserves the eigenspaces of $A(\zeta)$. Equivalently A commutes with parallel translation and is therefore covariant constant for each ζ. It therefore satisfies the differential equation, and thus agrees, up to a constant, with ξ.

It follows that, given the polynomial Killing field, we obtain the spectral curve from its characteristic polynomial, and given the curve, Theorem 1 produces the Killing field. Seen in this light, the spectral curve construction (at least in the smooth case) clearly works for $G = SU(n)$ for general n.

2.8 Successes and failures

The two methods have different advantages, and it is clear that a synthesis of the two is the best way for further progress. On the one hand, our criteria of integrability are best satisfied by the algebraic curve approach, for here we have a general form of solution, the constants of integration being essentially the coefficients of the curve and the parameters of a point on its Jacobian. Our particularly relevant geometrical problem of harmonic maps to the torus involves choosing those coefficients to satisfy constraints (which are incidentally transcendental in nature and not at all easy to put into effect). On the other hand, it is most effective for the group $SU(2)$, or at best $SU(n)$, since the direct image of a line bundle constructs a vector bundle. For the other classical groups, the Jacobian is replaced by a Prym variety (see [24]), but for a general Lie group one has to consider more general abelian varieties as in [15]. By contrast, the Lie-algebraic formulation of the loop-group approach does not distinguish the linear from non-linear groups.

The generality of the methods is approached in different ways. In the spectral curve approach [25], the case where a is semisimple or nilpotent are treated side-by-side with no essential difference in method. The case of trivial holonomy was simply set aside to be dealt with by different methods. In geometrical terms nilpotency of a corresponds to a conformal harmonic map to $SU(2)$—so its image is a minimal surface—and the trivial holonomy case to a conformal map to S^2 This contrasts somewhat with the loop-group approach where the essential finiteness result seems to require semi-simplicity. Nevertheless, the loop-group approach seems to be more effective for higher rank groups.

One issue which arises in higher rank is the question of how one can incorporate other constructions of harmonic maps, those which factor through holomorphic maps to flag manifolds—the superminimal surfaces, for example—into the integrable system approach. Here the loop-group formalism really comes into its own (see [10], [38]).

What, then, putting both methods together, are the successes?

- There is now a reasonably general method for constructing harmonic tori. Under precisely stated conditions, *any* harmonic torus can be constructed by a particular ansatz.
- The methods allow one to construct explicit new examples, Wente's torus being but the first.
- Some non-existence results ensue due to parameter counting: in particular there are constraints on the conformal structure of a torus to be minimally immersed in S^3 [25], whereas Bryant showed using superminimal surfaces that it can always be minimally immersed in S^4.
- An unexpected feature is the essential presence of deformations. In the loop-group approach we obtain non-trivial Jacobi fields as part of the ansatz, but perhaps more importantly the algebraic curve approach shows that at least some of these integrate to give deformations through harmonic maps. This consists of the choice of the line bundle E on the spectral curve. If the polynomial P is of degree $2p+2$, then curve S is of genus p, so its Jacobian is p-dimensional. The constraints for a harmonic map, severe though they are, are independent of the choice of E, so given one harmonic map, it possesses a p-dimensional family of deformations.

And what are the failures?

- The methods are restricted to the 2-torus (or, as in [8] to pluriharmonic maps of a complex torus of higher dimension). If M is a surface of higher genus, then of course we can still consider the holonomy of the flat connection $\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$ around generators of the fundamental group, but it is not clear how much help that is since the group is now non-abelian. In particular, it was the existence of a commuting holonomy matrix in the torus case which gave finiteness for the branch points. More seriously, it is difficult to see how the surface itself, whose universal covering is now no longer a vector space, can appear in the context of integrable systems linearized on a Jacobian.
- Despite having full control of the parameters, it is very difficult to read off information of a differential-geometric nature from the algebraic-geometric origins. There are formulas for energy in terms of the expansions of the differentials [25], but even estimating the size of the energy from the hyperelliptic curve is currently impossible.
- Some of the longstanding questions in the subject remain unanswered, for example the conjecture of Hsiang and Lawson that the only *embedded*

minimal torus in S^3 is the Clifford torus. All that the method offers in this direction, since the Clifford torus corresponds to the case $p = 0$, is the invitation to prove that an embedded torus has no deformations.

3 Four-dimensional Einstein manifolds

3.1 Background

Since the days of relativity, the search for 4-dimensional solutions to the Einstein equations $R_{ij} = \Lambda g_{ij}$ in both positive definite and Lorentzian signature has been a driving force in differential geometry. As a result of Yau's proof of the Calabi conjecture, and subsequent developments by Yau, Tian and others since, the 20th century approach of global existence theorems has produced many examples, but always restricted to the Kähler case. For a compact non locally-homogeneous example which is not Kähler, we only have the Page metric (see [6]) on the non-trivial 2-sphere bundle over S^2.

The non-compact case is less amenable to analysis, but offers more opportunities for constructions. Here again, both analysis and explicit construction are easier in the Kähler case. I want to indicate now how solutions of Painlevé's sixth equation can enter into the construction of some complete *non-Kählerian* Einstein metrics of negative scalar curvature on the unit ball in $\mathbf{R}^4$, providing deformations of both the hyperbolic metric and the Bergmann metric. The details are in [26].

The Painlevé equations are well-known not to be solvable in terms of "known functions", and require the introduction of the so-called Painlevé transcendants in general. Fortuitously, the problem discussed here does not need these and can be solved explicitly in terms of theta functions. The reason, as we shall see, is fundamentally associated with the sense in which these equations can be thought of as being integrable.

The starting point for this construction is the work of Tod [57], who directly approached the question of finding Einstein metrics in four dimensions which have self-dual Weyl tensor, and are invariant under the action of $SU(2)$. He showed that the conformal structure could be written, using the standard basis $\sigma_1, \sigma_2, \sigma_3$ of left-invariant 1-forms on $SU(2)$, in the form:

$$g_0 = \frac{ds^2}{s(1-s)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-s)\sigma_2^2}{\Omega_2^2} + \frac{s\sigma_3^2}{\Omega_3^2}$$

where the functions Ω_i satisfy the differential equations:

$$\begin{aligned}\Omega_1' &= -\frac{\Omega_2\Omega_3}{s(1-s)}\\ \Omega_2' &= -\frac{\Omega_3\Omega_1}{s}\\ \Omega_3' &= -\frac{\Omega_1\Omega_2}{1-s}.\end{aligned} \tag{3.1}$$

and the value of $\Omega_1^2 - \Omega_2^2 - \Omega_3^2$, which is a constant as a consequence of (3.1), is $-1/4$. This is not the metric itself. The Einstein metric is $g = e^{2u}g_0$ and has scalar curvature 4Λ where

$$-4\Lambda e^{2u} = \frac{8s\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3(s(\Omega_1^2+\Omega_2^2) - (1-4\Omega_3^2)(\Omega_2^2-(1-s)\Omega_1^2))}{(s\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2-(1-s)\Omega_1^2))^2} \tag{3.2}$$

This equation is one whose integrability we shall consider. Again, as is characteristic of the subject, its history goes back beyond Einstein and his equations to the 19th century study of *orthogonal coordinates.*

3.2 Orthogonal coordinates

Consider a metric given locally by

$$g = \frac{\partial\phi}{\partial x_1}dx_1^2 + \cdots + \frac{\partial\phi}{\partial x_n}dx_n^2$$

Clearly $x_1, \ldots, x_n$ are orthogonal coordinates, so if the metric is *flat* we have an orthogonal coordinate system on $\mathbf{R}^n$. Flatness leads to a nonlinear equation for ϕ, which can be put in various forms. Metrics of this type were studied by Darboux and Egorov (see [14, Chapter VIII], and [18]) at the turn of the century. In more recent times, such metrics arise in Dubrovin's theory [17] of Frobenius manifolds with its connections to conformal field theories and quantum cohomology. Such metrics have extra properties:

- $d\phi$ is covariant constant
- ϕ is homogeneous as a function of $x_1, \ldots, x_n$

The first condition is that the 1-form

$$d\phi = \frac{\partial\phi}{\partial x_1}dx_1 + \cdots + \frac{\partial\phi}{\partial x_n}dx_n$$

is covariant constant, which means that, using the metric, its dual vector field

$$X = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$$

is covariant constant. In particular X is an infinitesimal isometry. It generates the $\mathbf{R}$-action $x_i \mapsto x_i + t$. The homogeneity condition clearly gives an action of $\mathbf{R}^*$ by conformal transformations, so we are considering a problem of orthogonal coordinates invariant under the two symmetries:

$$x_i \mapsto x_i + t$$
$$x_i \mapsto \lambda x_i$$

In positive definite signature, the degree of homogeneity of the coefficients of the metric must be zero, because X being covariant constant implies that $g(X, X)$ is a non-zero constant. Thus the coefficients are functions of $\mathbf{R}^n$ invariant under this two-parameter group. In three dimensions, this means that they are functions of the single variable

$$s = \frac{x_1 - x_2}{x_3 - x_2}$$

and this is where the differential equation (3.1) enters:

Proposition 3.1. *Let $\Omega_1, \Omega_2, \Omega_3$ be functions of $s = (x_1 - x_2)/(x_3 - x_2)$, then the metric*

$$g = -\Omega_1^{-2} dx_1^2 + \Omega_2^{-2} dx_2^2 + \Omega_3^{-2} dx_3^2$$

is flat iff $\Omega_1, \Omega_2, \Omega_3$ satisfy equation (3.1).

Darboux's 1910 book [14] on orthogonal coordinates has more pages than Besse's comprehensive 1987 book [6] on Einstein manifolds, which is perhaps an indicator of the importance of the subject at the time, but in some sense we are seeing here another manifestation of the same piece of mathematics in a different context. We face the same problem, though: how to solve the differential equation (3.1).

3.3 Isomonodromic deformations

The equation (3.1) is integrable because it can be reduced to the geometrical problem of isomonodromic deformations, which we describe briefly next.

Consider a meromorphic connection on a trivial bundle over $\mathbf{P}^1$ with connection form (in an affine coordinate z)

$$A = \sum_{i=1}^{n} \frac{A_i dz}{z - z_i}$$

An isomonodromic deformation $A_i(z_1, \ldots, z_n)$ for $(z_1, \ldots, z_n) \in U \subset \mathbf{C}^n$ is a family of such connections with constant holonomy (up to conjugation). It necessarily satisfies the Schlesinger equation [40]:

$$dA_i + \sum_{j \neq i} [A_i, A_j] \frac{dz_i - dz_j}{z_i - z_j} = 0. \tag{3.3}$$

Suppose we have four points $z_1, \ldots, z_4$. By a projective transformation we can make these points $0, 1, s, \infty$. Then

$$A(z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-s}$$

and Schlesinger's equation becomes:

$$\begin{aligned} \frac{dA_1}{ds} &= \frac{[A_3, A_1]}{s} \\ \frac{dA_2}{ds} &= \frac{[A_3, A_2]}{s-1} \\ \frac{dA_3}{ds} &= \frac{[A_1, A_3]}{s} + \frac{[A_2, A_3]}{s-1} \end{aligned} \tag{3.4}$$

where the last equation is equivalent to

$$A_1 + A_2 + A_3 = -A_4 = \text{const.}$$

Assume that the A_i are 2×2 matrices of trace 0. Note that from Schlesinger's equation (3.3),

$$d(\operatorname{tr} A_i^2) = -2 \sum_{j \neq i} \operatorname{tr}(A_i[A_i, A_j]) \frac{dz_i - dz_j}{z_i - z_j} = 0,$$

so that each $\operatorname{tr} A_i^2$ is independent of z_1, z_2, z_3, z_4, and is a constant of integration. By its very origin, the full constants of integration of this equation consist of the prescription of the holonomy: a representation of the fundamental group of the 4-punctured sphere in $SL(2, \mathbf{C})$. Since this group is a free group on three generators (taken to be loops passing once around three of the punctures), these constants are essentially the choice of a triple of elements M_1, M_2, M_3 in $SL(2, \mathbf{C})$ modulo conjugation. When the eigenvalues of A_j do not differ by an integer, the holonomy around a small loop surrounding the pole z_j is conjugate to

$$M_j = \exp(2\pi i A_j)$$

so that if $\operatorname{tr} A_j^2 = k$,

$$\operatorname{tr} M_j = 2\cos(\pi\sqrt{2k}) \tag{3.5}$$

The constancy of k is thus just part of the full constants of motion—invariants of the holonomy.

From the point of view of integrable systems, we have here the general solution. A specific one is determined by the choice of a holonomy representation. That is precisely what we shall do when we relate the equations for the Ω_i to isomonodromic deformations.

To make that relationship, we consider solutions to the isomonodromic deformation problem for which the holonomies for small curves surrounding the poles are all conjugate. This means that the residues A_i satisfy

$$\operatorname{tr} A_1^2 = \operatorname{tr} A_2^2 = \operatorname{tr} A_3^2 = \operatorname{tr}(A_1 + A_2 + A_3)^2 = k \tag{3.6}$$

Now if we set

$$\Omega_1^2 = -(k + \operatorname{tr} A_1A_2) \quad \Omega_2^2 = (k + \operatorname{tr} A_2A_3) \quad \Omega_3^2 = (k + \operatorname{tr} A_3A_1) \tag{3.7}$$

The Schlesinger equation for isomonodromic deformations (3.4) shows that

$$\begin{aligned} 2\Omega_1 \frac{d\Omega_1}{ds} &= \frac{\operatorname{tr}([A_1, A_2]A_3)}{s(s-1)} \\ 2\Omega_2 \frac{d\Omega_2}{ds} &= -\frac{\operatorname{tr}([A_1, A_2]A_3)}{s} \\ 2\Omega_3 \frac{d\Omega_3}{ds} &= \frac{\operatorname{tr}([A_1, A_2]A_3)}{s-1} \end{aligned} \tag{3.8}$$

On the other hand, in the Lie algebra of $SL(2, \mathbf{C})$, we have the identity

$$(\operatorname{tr}([A_1, A_2]A_3))^2 = -2 \det \begin{pmatrix} \operatorname{tr} A_1^2 & \operatorname{tr} A_1A_2 & \operatorname{tr} A_3A_1 \\ \operatorname{tr} A_1A_2 & \operatorname{tr} A_2^2 & \operatorname{tr} A_2A_3 \\ \operatorname{tr} A_3A_1 & \operatorname{tr} A_2A_3 & \operatorname{tr} A_3^2 \end{pmatrix}$$

Expanding the determinant and using (3.6), we see from this that $\operatorname{tr}([A_1, A_2]A_3)$ is a distinguished square root of $4\Omega_1^2\Omega_2^2\Omega_3^2$, which gives the required equations (3.1):

$$\begin{aligned} \Omega_1' &= -\frac{\Omega_2\Omega_3}{s(1-s)} \\ \Omega_2' &= -\frac{\Omega_3\Omega_1}{s} \\ \Omega_3' &= -\frac{\Omega_1\Omega_2}{1-s}. \end{aligned}$$

Does this approach give explicit solutions? In some ways, the answer is no, for a simple substitution

$$\frac{dy}{dx} = \frac{y(y-1)(y-x)}{x(x-1)} \left(2z - \frac{1}{2y} - \frac{1}{2(y-1)} + \frac{1}{2(y-x)}\right)$$

(which defines the auxiliary variable z), and

$$\begin{aligned} \Omega_1^2 &= \frac{(y-x)^2y(y-1)}{x(1-x)} \left(z - \frac{1}{2(y-1)}\right) \left(z - \frac{1}{2y}\right) \\ \Omega_2^2 &= \frac{y^2(y-1)(y-x)}{x} \left(z - \frac{1}{2(y-x)}\right) \left(z - \frac{1}{2(y-1)}\right) \\ \Omega_3^2 &= \frac{(y-1)^2y(y-x)}{(1-x)} \left(z - \frac{1}{2y}\right) \left(z - \frac{1}{2(y-x)}\right). \end{aligned}$$

reduces the general isomonodromic deformation problem for four points to a Painlevé equation:

$$\frac{d^2y}{dx^2} = 1/2\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right)$$

and the general solutions of such equations are known to involve new transcendental functions. However, the specific problem on Einstein manifolds we began with concerned finding solutions to this equation with $\Omega_1^2-\Omega_2^2-\Omega_3^2=-1/4$. This has a natural interpretation in terms of the holonomy, for

$$\Omega_1^2-\Omega_2^2-\Omega_3^2 = -3k - \operatorname{tr} A_1A_2 - \operatorname{tr} A_2A_3 - \operatorname{tr} A_3A_1 = -2k$$

and so we must take $k = 1/8$. From (3.5) this means that the holonomy around each pole has eigenvalues $\pm i$ and so is of order 4. We can use this fact to give explicit solutions.

The basic idea is that by taking a double covering of $\mathbf{P}^1$ branched over the four poles of the meromorphic connection, we pull back the connection to one on an elliptic curve E. This time the eigenvalues of the holonomy around each pole are ± 1 and since the connection has holonomy in $SL(2,\mathbf{C})$, this is a multiple of the identity. In $PSL(2,\mathbf{C})$ this is trivial, and so the holonomy extends to a representation of the fundamental group of the elliptic curve, which is abelian. We can now rewrite the connection in terms of meromorphic connections on line bundles and the holonomy is then defined by the periods of meromorphic differentials on the elliptic curve E. This allows us to find explicit solutions of the isomonodromic deformation problem, and consequently (see [26] for details) explicit forms for Einstein metrics. For information, the general solution to this particular Painlevé equation, where the coefficients in the equation are $\alpha = 1/8, \beta = -1/8, \gamma = 1/8, \delta = 3/8$, is

$$y(x) = \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3}\left(1+\frac{\vartheta_3^4(0)}{\vartheta_4^4(0)}\right)$$
$$+\frac{\vartheta_1'''(\nu)\vartheta_1(\nu) - 2\vartheta_1''(\nu)\vartheta_1'(\nu) + 4\pi i c_1(\vartheta_1''(\nu)\vartheta(\nu) - {\vartheta_1'}^2(\nu))}{2\pi^2\vartheta_4^4(0)\vartheta_1(\nu)(\vartheta_1'(\nu) + 2\pi i c_1\vartheta_1(\nu))}$$

where $\nu = c_1\tau + c_2$ and $x = \vartheta_3^4(0)/\vartheta_4^4(0)$. It is then a matter of using estimates for theta functions to describe global properties of the resulting Einstein metrics in terms of the two constants of integration c_1 and c_2.

This is a classical approach to the problem, but just as with the case of harmonic tori, it gives some new examples and illuminates the area in a number of ways.

3.4 Metrics on the ball

The *complete* metrics produced this way are of different forms. There are two obvious ones—the well-known (and unique) *compact* self-dual Einstein manifolds S^4 and CP^2 with positive scalar curvature. The isomonodromic problem they correspond to has finite holonomy group: the binary dihedral group for a triangle and a square respectively (see [27]). The three matrices M_1, M_2, M_3 cover three reflections in the dihedral group.

In the case of zero scalar curvature the metric on the moduli space of 2-monopoles [3] is the unique complete metric and corresponds to the holonomy representation

$$M_1 = \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}, \quad M_2 = \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}, \quad M_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

The rest are metrics of negative scalar curvature, and correspond to representations of the following form:

$$M_1 = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \lambda e^{i\theta} \\ -\lambda^{-1} e^{-i\theta} & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix} \tag{3.9}$$

They live naturally on the unit ball in $\mathbf{R}^4$ and fall into two types. The first consists of taking a general value for λ and θ. It has the following properties:

- The conformal structure extends over the boundary 3-sphere, and induces there a left-invariant conformal structure on $SU(2)$. If we add in the hyperbolic metric and Pedersen's metrics [48], then we have a two-parameter family of self-dual Einstein metrics which induce *any* left-invariant conformal structure on the 3-sphere. Thus these conformal structures are, in the language of LeBrun [36] of "positive frequency". There are obstructions in general to the positive frequency condition, involving the eta-invariant of the boundary (see [28]).

- The metrics approach the hyperbolic metric near the boundary of the ball, but not fast enough to be covered by the rigidity theorem of Min-Oo [44] which would force it to be the hyperbolic metric itself. It is interesting to note that Min-Oo's approach is based on Witten's proof of the positive mass theorem, concerning zero scalar curvature. The Taub-NUT metric (see e.g. [6]) is a self-dual Einstein metric on $\mathbf{R}^4$ with zero scalar curvature which does not fall within the scope of Witten's theorem. Our metrics are in some sense hyperbolic analogues of this metric. Indeed, the Pedersen metrics (which have an extra degree of symmetry) can be obtained by a quaternionic-Kähler quotient construction which generalizes the hyperkähler quotient which yields the Taub-NUT metric.

- There are no quotients of these metrics by discrete groups: the central fixed point of the $SU(2)$ is distinguished by its local geometry.

- The full family is parametrized by the left-invariant conformal structures on the 3-sphere.

The second family corresponds to taking a special value $\theta = 0$ in the holonomy (3.9). It has the following features:

- One of the coefficients of σ_i^2 in the metric decays much faster than the others as the boundary 3-sphere is approached and so the conformal structure of the metric does not extend over the sphere. In the limit it induces a degenerate conformal structure, which we can think of as a CR-structure on the boundary. If we add in the Bergmann metric on the unit disc in $\mathbf{C}^2$, then we get a family which realizes every left-invariant CR-structure on S^3.
- Although the Bergmann metric is a Kähler metric, this is not so for any other member of the family.
- There are no quotients by discrete groups.
- The full family is parametrized by the left-invariant CR-structures on the 3-sphere. In fact, if $\mu\sigma_1^2 + \sigma_2^2$ represents this structure, with $\mu < 1$, we take the holonomy with
$$\lambda = \pm\sqrt{\frac{1+\mu}{1-\mu}}$$

4 Integrability and self-duality

4.1 Background

We remarked initially that among the distinguishing characteristics of 20th century mathematics is the goal of setting up unifying structures linking together disparate areas. The most noteworthy case for us concerns integrable systems and their relationship with the self-dual Yang-Mills equations. For a thorough account of this, we refer to the book of Mason and Woodhouse [42]. The essential idea is to see integrable systems as *dimensional reductions* of solutions to the equations
$$F_A = *F_A$$
for the curvature F_A of a connection A on a principal G-bundle P over $\mathbf{R}^4$. In fact, it is unwise to rely on the Euclidean signature alone. We may consider complex connections, or more commonly real ones which are self-dual in signature $(2,2)$ (recall that there are no real self-dual 2-forms in Lorentzian signature since in that case $*^2 = -1$).

By "dimensional reduction" we mean that we take solutions which are invariant under a group H of diffeomorphisms of $\mathbf{R}^4$ which preserve the equations. Since the Hodge star operator is conformally invariant in the middle degree, this must be a group of conformal transformations, i.e. a subgroup of $SO(5,1)$ for

Euclidean signature, or $SO(3,3)$ in the other case. If $\dim H = d < 4$, then the self-duality equations can be reinterpreted as equations in a $(4-d)$-dimensional quotient space, which in the case of translations consists of another vector space, but where the induced metric may be degenerate.

The equations on the quotient are not for just a connection, but involve extra data—the "Higgs fields". These arise, invariantly speaking, from the fact that H-invariance of the connection only has meaning if we have a lifting of the action of H from $\mathbf{R}^4$ to the principal bundle P. If X is a vector field on $\mathbf{R}^4$ generated by the action of H, then for any section s of a vector bundle V associated to P there is another section, the Lie derivative $\mathcal{L}_X s$. If we have a connection, we can also define the covariant derivative $\nabla_X s$ and the difference $(\nabla_X - \mathcal{L}_X)s$ is a zero-order operator: an endomorphism of V. If the connection is H-invariant, then this endomorphism is defined on the quotient space, and is called a Higgs field. Thus for a d-dimensional group H, we have d Higgs fields.

To relate the corresponding coupled equations for connections and Higgs fields to known integrable systems, two processes are involved. In the first place, since systems are not necessarily written in gauge-theoretical terms, some gauge choices have to be made. Secondly, a curious phenomenon takes place with a reduction to two dimensions: the equations become, after a suitable reinterpretation, conformally invariant. This has nothing to do with the conformal invariance in four dimensions, and is much stronger since there is an infinite dimensional pseudogroup of such transformations in two dimensions. In the case that the quotient metric is degenerate, this is the pseudogroup of Galilean transformations. What it means is that a reduction to a relatively few canonical models can be achieved.

In this manner, choosing G to be an appropriate real form of $SL(2,\mathbf{C})$, the KdV equation and non-linear Schrödinger equations appear as dimensional reductions, as pointed out by Mason and Sparling [41].

We have considered so far two examples of integrable systems arising in Riemannian geometry: harmonic maps of a surface and the isomonodromic deformation problem corresponding to Painlevé's sixth equation. We can see these now as dimensional reductions of the Yang-Mills equations.

4.2 Two examples

We begin with $\mathbf{R}^4$ with metric $g = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$ and and volume form $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. The Hodge star operator is then:

$$\begin{aligned} *dx_1 \wedge dx_2 &= dx_3 \wedge dx_4 \\ *dx_2 \wedge dx_3 &= -dx_4 \wedge dx_2 \\ *dx_1 \wedge dx_4 &= -dx_2 \wedge dx_3 \end{aligned}$$

We take for the group H the 2-dimensional group of translations

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3 + a_1, x_4 + a_2)$$

for $(a_1, a_2) \in \mathbf{R}^2$. The quotient space is $\mathbf{R}^2$ with Euclidean metric $dx_1^2 + dx_2^2$ and we have two Higgs fields $\phi_1(x_1, x_2)$ and $\phi_2(x_1, x_2)$. In these terms, the self-dual connection has connection form

$$A_1 dx_1 + A_2 dx_2 + \phi_1 dx_3 + \phi_2 dx_4$$

Equating the three anti-self-dual coefficients of the curvature to zero gives the three equations:

$$\begin{aligned} F_{12} &= [\phi_1, \phi_2] \\ \nabla_1 \phi_1 &= \nabla_2 \phi_2 \\ \nabla_1 \phi_2 &= -\nabla_2 \phi_1 \end{aligned}$$

Putting $\Phi = (\phi_1 + i\phi_2)d(x_1 + ix_2)$, we obtain the equations

$$\begin{aligned} \nabla^{0,1}\Phi &= 0 \\ F &= [\Phi, \Phi^*] \end{aligned}$$

for a harmonic map in the form of Proposition 1.

For the next example (following [42]) we pass to complex coordinates, setting $z = x_1 + ix_2, \tilde{z} = x_1 - ix_2, w = x_3 + ix_4, \tilde{w} = x_3 - ix_4$, so that the metric is given by

$$g = dz\, d\tilde{z} - dw\, d\tilde{w}.$$

In this case, the anti-self-dual 2-forms have as basis $dz \wedge dw, d\tilde{z} \wedge d\tilde{w}, dz \wedge d\tilde{z} - dw \wedge d\tilde{w}$.

Consider the 3-dimensional group $H = \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^*$ acting by conformal transformations as follows:

$$(z, \tilde{z}, w, \tilde{w}) \mapsto (\lambda\nu^{-1}z, \mu\tilde{z}, \lambda w, \mu\nu^{-1}\tilde{w})$$

The quotient space is 1-dimensional. Since $s = z\tilde{z}/w\tilde{w}$ is invariant under the group action, we can take it to be a parameter on the quotient. Now introduce coordinates

$$\begin{aligned} p &= -\log w \\ q &= -\log \tilde{z} \\ r &= \log(\tilde{w}/\tilde{z}) \end{aligned}$$

Under the group action these transform as

$$(p, q, r) \mapsto (p - \log\lambda, q - \log\mu, r - \log\nu)$$

and so an H-invariant connection defines Higgs fields P, Q, R which are Lie-algebra valued functions of s. In one dimension, a gauge transformation locally trivializes any connection, so there is a gauge for which the connection form is

$$Pdp + Qdq + Rdr.$$

In these coordinates the anti-self-dual 2-forms are spanned by

$$ds \wedge dp + sdr \wedge dp, \quad dq \wedge dr, \quad (s-1)dp \wedge dq + dp \wedge dr - ds \wedge dq$$

and the curvature is

$$P'ds \wedge dp + Q'ds \wedge dq + R'ds \wedge dr + [P,Q]dp \wedge dq + [Q,R]dq \wedge dr + [R,P]dr \wedge dp$$

For self-duality of the connection, the product with the anti-self-dual 2-forms must vanish and this leads to the three equations

$$\begin{aligned} P' &= 0 \\ Q' &= \frac{1}{s}[R,Q] \\ R' &= \frac{1}{(s-1)}[R,P] + \frac{1}{s(s-1)}[R,Q] \end{aligned}$$

and taking

$$\begin{aligned} P &= -A_1 - A_2 - A_3 \\ Q &= A_1 \\ R &= A_3 \end{aligned}$$

we obtain Schlesinger's equation in the form (3.4). So Schlesinger's equation, and the isomonodromic deformation problem for four singular points, is a dimensional reduction of the self-dual Yang-Mills equations.

The two concrete applications of integrable systems to problems in Riemannian geometry which we have considered thus arise in a natural way by choosing a group H of conformal transformations and studying the self-dual Yang-Mills equations invariant under H, for differing gauge groups G. Much more can be said, in particular with regard to the twistor methods of solving the equations, but that will take us too far afield. Suffice it to note that the indeterminate ζ in the flat connection $\nabla + \zeta\Phi - \zeta^{-1}\Phi^*$ for a harmonic map is essentially a complex parameter on a twistor line. Many of the standard, inverse scattering methods of solving integrable systems have a reinterpretation in twistor terms (see [42]).

The one feature which does emerge from this general point of view, is that we can't expect all of the interesting problems in Riemannian geometry to succumb to the method of integrable systems. As Ward has pointed out in [59], the self-dual Yang-Mills equations have the "Painlevé property" whereas the full Yang-Mills equations do not. By analogy, it would be surprising if the full Einstein equations could be solved by any integrable system method. In four dimensions, as we have seen, self-duality may lead to integrability, and we shall see later a very direct relationship between certain integrable systems and the construction of hyperkähler metrics in higher dimensions. For this, though, we need to study another dimensional reduction, that of Nahm's equations.

4.3 Nahm's equations

Take $\mathbf{R}^4$ with positive definite Euclidean metric $dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$ and consider the 3-dimensional group H of translations of the form

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1 + a_1, x_2 + a_2, x_3 + a_3)$$

An invariant connection now gives three Higgs fields T_1, T_2, T_3, functions of x_0, and, as in the previous example, trivializing the connection in one dimension, the connection form can be written $T_1 dx_1 + T_2 dx_2 + T_3 dx_3$. The curvature of the connection is

$$\begin{aligned} &T_1' dx_0 \wedge dx_1 + T_2' dx_0 \wedge dx_2 + T_3' dx_0 \wedge dx_3 \\ &\qquad + [T_1, T_2] dx_1 \wedge dx_2 + [T_2, T_3] dx_2 \wedge dx_3 + [T_3, T_1] dx_3 \wedge dx_1 \end{aligned}$$

The self-dual Yang-Mills equations now become *Nahm's equations*

$$\begin{aligned} T_1' &= [T_2, T_3] \\ T_2' &= [T_3, T_1] \\ T_3' &= [T_1, T_2] \end{aligned}$$

Since these are obtained by the action of a three-dimensional group of translations, it is not surprising that there is a close relationship to the equations for a harmonic map, where H was a two-dimensional translation group. The only difference is in the signature on the metric on $\mathbf{R}^4$. In fact, harmonic maps of a torus which are S^1-invariant reduce to the very similar equations

$$\begin{aligned} T_1' &= [T_2, T_3] \\ T_2' &= -[T_3, T_1] \\ T_3' &= -[T_1, T_2] \end{aligned}$$

which also arise in the theory of variations of Hodge structure [52]. Given that we can linearize the equations for harmonic maps of a torus on the Jacobian of a curve, it is not surprising that Nahm's equations can be too. We write, with an indeterminate ζ,

$$A(\zeta) = (T_2 + iT_3) - 2iT_1\zeta + (T_2 - iT_3)\zeta^2 \tag{4.1}$$

so that $A \in H^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes \mathfrak{g}$. When $G = U(n)$, Proposition 1 tells us that, modulo overall conjugation, this corresponds to a line bundle L on the spectral curve $S \subset \mathcal{O}(2)$ defined by $\det(\eta - A(\zeta)) = 0$.

If we now set A_+ to be the polynomial part of $A\zeta^{-1}$, Nahm's equations become equivalent to the Lax pair (putting $s = x_0$)

$$\frac{dA}{ds} = [A, A_+]$$

As a consequence of the above Lax form, the spectral curve remains the same, and the line bundle evolves along a curve in the Jacobian, which in fact is a straight line. This is very similar to the case of harmonic maps from a torus, but there the points $\zeta = 0, \infty$ were distinguished. In the present case, because A is only quadratic in ζ, these points play no particular role. The direction in the Jacobian in which the straight line evolution takes place is determined by a canonical element in $H^1(S, \mathcal{O})$, (the tangent space to the Jacobian at any point). We take the canonical generator x of $H^1(\mathbf{P}^1, K) \cong H^1(\mathbf{P}^1, \mathcal{O}(-2))$ and the tautological element $\eta \in H^0(\mathcal{O}(2), \pi^*\mathcal{O}(2))$ and define $\eta x \in H^1(\mathcal{O}(2), \mathcal{O})$. Restricting ηx to $S \subset \mathcal{O}(2)$ gives a canonical element in $H^1(S, \mathcal{O})$.

The principal result (see e.g.[22]) is that (T_1, T_2, T_3) satisfy Nahm's equations if and only if the line bundle L_s evolves in a straight line on the Jacobian in the distinguished direction ηx.

Nahm's equations originated in the study of magnetic monopoles, but they have become a means of constructing concrete Einstein metrics in higher dimensions than four. We shall use the integrable systems approach to study these in some detail, and find explicit formulae.

5 Hyperkähler metrics

5.1 Background

It is now 20 years since Yau's proof of the Calabi conjecture. This theorem provided a great many compact manifolds satisfying the Einstein equation $R_{ij} = 0$. Given a compact Kähler manifold with first Chern class zero, the theorem asserts the existence of an essentially unique Kähler metric cohomologous to the initial one, but with zero Ricci tensor. The first examples, of K3 surfaces, are also the first examples of *hyperkähler manifolds*—Riemannian manifolds whose holonomy is contained in $Sp(n) \subseteq SU(2n)$, and, as pointed out in [4], a minor extension of Yau's theorem can be used to prove the existence of a hyperkähler metric on any compact Kähler manifold with a non-degenerate holomorphic 2-form. In particular, the Hilbert scheme $X^{[n]}$ of 0-cycles of length n on a K3 surface or an abelian surface X has a hyperkähler metric.

These existence proofs are impressive, especially given the state of affairs 25 years ago, when no complete non-trivial Ricci-flat metric was known to exist. They have provided a great source of information for algebraic geometers, in particular for studying moduli problems, but there are simple questions which they cannot hope to answer. For example, is the hyperkähler metric on the Hilbert scheme in any way locally defined by that on the K3 surface itself?

The existence theorems work best in the compact situation, and despite more than 15 years of study, the situation in the non-compact case—trying to put a Ricci-flat Kähler metric on the complement of an anticanonical divisor—is still not fully understood (see [56]). On the other hand, some construction methods have arisen over the last few years to give plenty of explicit hyperkähler metrics on noncompact manifolds. These are of three types:

- twistor space methods
- finite-dimensional hyperkähler quotients
- infinite-dimensional hyperkähler quotients

Sometimes all three methods can be used to give a single metric, for example the ubiquitous Taub-NUT metric on $\mathbf{R}^4$, whose twistor construction is given in [6] (Chapter 13) is expressed as a quotient of $\mathbf{C}^2 \times \mathbf{C}^2$ in [6] (Addendum E), and has recently appeared in the context of duality as the natural metric obtained by an infinite-dimensional quotient construction on a certain family of $SU(3)$ monopoles [37].

Among the infinite-dimensional quotient constructions is a series of metrics defined on coadjoint orbits of complex semisimple Lie groups. These ideas were initiated by Kronheimer [35] and developed more fully by Biquard [7] and Kovalev [33]. The construction makes use of Nahm's equations, as do other metrics of interest. In [34] Nahm's equations are used to put complete hyperkähler metrics on the total space of the cotangent bundle to a complex Lie group. Finally, Nakajima's result [45] show that the natural metrics on the moduli spaces of $SU(2)$ monopoles on $\mathbf{R}^3$ as described in [3] may also be obtained from the Nahm matrices which are used to construct the monopole. A more recent result of Nakajima and Takahasi [46], [55] shows that this applies also to the $SU(3)$ monopole metrics studied by Dancer [11].

As we have seen, Nahm's equations are solvable in terms of a linear flow on the Jacobian of a curve, and we might hope to be able to write down the hyperkähler metric explicitly using the data of the curve. In some sense, as we shall see, this is the case.

5.2 Hyperkähler quotients and Nahm's equations

Recall that a hyperkähler manifold is a Kähler manifold M with a nondegenerate covariant constant holomorphic 2-form ω^c. The real and imaginary parts of ω^c together with the Kähler form constitute a triple of closed 2-forms $\omega_1, \omega_2, \omega_3$, each one symplectic and satisfying some algebraic constraints. These constraints can be summed up as saying that the stabilizer of all three at each point is conjugate to the quaternionic unitary group $Sp(n) \subset GL(4n, \mathbf{R})$. Each form $\omega_1, \omega_2, \omega_3$ is the Kähler form of a complex structure I, J, K and these generate an action of the quaternions on the tangent bundle.

If G is a Lie group acting on M, preserving all three 2-forms, we have three moment maps, which can be collected into a single function:

$$\mu : M \to \mathfrak{g} \otimes \mathbf{R}^3$$

If $\mu^{-1}(0)$ is smooth, then the induced metric is G-invariant and descends to the quotient. The hyperkähler quotient construction [23] consists of the observation that this quotient metric is again hyperkähler. For most purposes, the initial space M is taken to be a flat quaternionic vector space, so that in this case the hyperkähler metric on the quotient is simply induced from the restriction of a Euclidean metric to a submanifold. This is quite explicit, except for the fact that the non-linear algebraic equation $\mu(m) = 0$ may not be easy to solve.

A standard class of examples can be obtained by taking a compact semisimple Lie group G, setting

$$M = \mathfrak{g} \otimes \mathbf{H}$$

and taking the adjoint action of G. The moment map equations are then

$$[A_0, A_1] = [A_2, A_3], \quad [A_0, A_2] = [A_3, A_1], \quad [A_0, A_3] = [A_1, A_2]$$

where $A \in M$ is $A = A_0 + iA_1 + jA_2 + kA_3$. An infinite-dimensional version of this is to consider the interval $[0,1]$ and a connection $A_0 = d/ds + B_0(s)$ on a trivial G-bundle, with A_1, A_2, A_3 replaced by Higgs fields $B_i : [0,1] \to \mathfrak{g}$. We can then consider the infinite-dimensional quaternionic affine space $\mathcal{A}$ of differential operators of the form

$$\frac{d}{ds} + B_0 + iB_1 + jB_2 + kB_3$$

as a hyperkähler manifold, using the $\mathcal{L}^2$ inner product. The appropriate group is now the infinite-dimensional gauge group $\mathcal{G}_0^0$ of smooth functions $g : [0,1] \to G$ such that $g(0) = g(1) = 1$. The moment map equations for this action read

$$\begin{aligned} B_1' + [B_0, B_1] &= [B_2, B_3] \\ B_2' + [B_0, B_2] &= [B_3, B_1] \\ B_3' + [B_0, B_3] &= [B_1, B_2] \end{aligned}$$

Formally speaking, we expect a hyperkähler metric to be induced on the quotient space. The appropriate analysis was carried out in [34] and gives a complete hyperkähler metric on the cotangent bundle of the complex group G^c. A related paper which describes properties of these metrics is [13].

The identification of the hyperkähler quotient as a cotangent bundle proceeds as follows: define

$$\alpha = B_0 - iB_1 \qquad \beta = B_2 + iB_3$$

and let $f : [0,1] \to G^c$ be the solution to the equation

$$\frac{df}{ds} = f\alpha \tag{5.1}$$

satisfying the initial condition $f(1) = 1$. Consider the map defined by

$$\psi(\alpha, \beta) = (f(0)^{-1}, \beta(1)) \tag{5.2}$$

Then because the group $\mathcal{G}_0^0$ consists of functions vanishing at the end-points, the map ψ is easily seen to be defined on the quotient, and as shown in Kronheimer's paper [34] (where the arguments are modelled on those of Donaldson [16]), this gives a diffeomorphism to $G^c \times \mathfrak{g}^c \cong T^*G^c$.

Now there is a unique gauge transformation $g : [0,1] \to G$ with $g(0) = 1$ such that

$$\frac{dg}{ds} = -B_0 g$$

and after applying this and putting $T_i = \mathrm{Ad}(g)B_i$, the moment map equations for B_i become Nahm's equations, so that in principle we only have to solve Nahm's equations to determine the zero set of the moment map and hence the metric.

The other uses of Nahm's equations to give hyperkähler metrics depend on different boundary conditions for the Nahm matrices, and we shall deal with these separately later.

5.3 Kähler potentials

It is easy to ask for an explicit form of a metric, but less easy to decide in what form one would really like it. When we ask for explicitness and receive it, it may not be what we really wanted, since the questions we pose initially may not be readily answered by using some complicated expression in transcendental functions. The examples of metrics on the ball in Section 3 are borderline in this respect: it is just possible to determine global behaviour of the metric, but it involves a mixture of expansions of theta functions—well-known because of their long lineage—and consequences of the differential equations they satisfy. So what would be a good answer for a Kähler metric on a manifold? Perhaps the simplest is to find a *Kähler potential*—a locally-defined function ϕ such that the Kähler form is expressible as

$$i\partial\bar{\partial}\phi = \omega$$

This is just a single function on the manifold, and as a consequence has an interpretation independent of coordinates. It disguises the fact that one needs to know the holomorphic coordinates as well in order to write down the metric, but it may well be that some properties of the manifold can be deduced from the potential itself. This is what we shall do for the metrics constructed from solutions of Nahm's equations. It turns out that there is a natural global Kähler potential for one of the complex structures of the hyperkähler family. This is a consequence of the following result [23]

Proposition 5.1. *Let M be a hyperkähler manifold with Kähler forms $\omega_1, \omega_2, \omega_3$ and suppose X is a vector field on M such that*

$$\mathcal{L}_X\omega_1 = 0, \quad \mathcal{L}_X\omega_2 = \omega_3, \quad \mathcal{L}_X\omega_3 = -\omega_2$$

Let μ be the moment map for X with respect to ω_1, then 2μ is a Kähler potential for the complex structure J.

Proof: By definition, μ satisfies $d\mu = \iota(X)\omega_1$. Using the complex structure J, we have

$$d\mu(JY) = (\partial_J\mu + \bar{\partial}_J\mu)(JY) = i(\partial_J\mu - \bar{\partial}_J\mu)(Y)$$

But we also have

$$d\mu(JY) = \iota(X)\omega_1(JY) = g(IX, JY) = g(KX, Y) = \omega_3(X, Y)$$

Hence

$$\iota(X)\omega_3 = i(\partial_J\mu - \bar{\partial}_J\mu)$$

and so

$$-2i\partial_J\bar{\partial}_J\mu = d(\iota(X)\omega_3) = \mathcal{L}_X\omega_3 = -\omega_2$$

giving, as required,

$$\omega_2 = 2i\partial_J\bar{\partial}_J\mu$$

Remarks:
1. Note that so long as the moment map μ is globally defined (and this will certainly be true if $b_1(M) = 0$), so is the Kähler potential. This has serious implications for the complex structure J. In particular, since the Kähler form is cohomologically trivial, there can be no compact complex subvarieties.
2. A circle action which acts non-trivially on the 3-dimensional space of covariant constant 2-forms spanned by $\omega_1, \omega_2, \omega_3$ will, after some orthogonal change of basis, always be of the above form. The vector field X generating it is normalized by the conditions $\mathcal{L}_X\omega_2 = \omega_3, \quad \mathcal{L}_X\omega_3 = -\omega_2$.
3. It is clear from the symmetry of the problem that the complex structure J is not determined by the circle action, and any complex structure $\cos\theta J + \sin\theta K$ orthogonal to I in the 2-sphere of all complex structures will share the same Kähler potential 2μ.

In the case of Nahm's equations, there is an obvious action of $SO(3)$ on the space of (B_0, B_1, B_2, B_3) given by

$$B_0 \mapsto B_0, \quad B_i \mapsto \sum_i^3 P_{ij}B_j$$

where $P \in SO(3)$. This rotates the Kähler forms, as does the action which descends to the quotient. Choosing the $SO(2)$ subgroup which leaves fixed the Kähler form ω_1 then gives an action which differentiates to a vector field X of precisely the nature of Proposition 5.1. Thus Kronheimer's metric on T^*G^c has a globally defined Kähler potential for one of the complex structures. In fact, since the $SO(3)$ action acts transitively on the hyperkähler complex structures, these are all equivalent to the standard one.

The chosen circle action acts trivially on $\alpha = B_0 - iB_1$ and takes $\beta = B_2 + iB_3$ to $e^{i\theta}\beta$. In terms of the parametrization (5.2) above, this is just scalar multiplication by $e^{i\theta}$ in the fibres of the cotangent bundle.

Now on the affine space $\mathcal{A}$, the vector field X generated by the circle action is given by

$$X = jB_3 - kB_2$$

On a Kähler manifold, the moment map μ for the Hamiltonian vector field X satisfies $\operatorname{grad}\mu = IX$, but since

$$IX = jB_2 + kB_3$$

we see that for $A \in \mathcal{A}$,

$$\mu(A) = \frac{1}{2}\int_0^1 \langle B_2, B_2\rangle + \langle B_3, B_3\rangle ds$$

This descends to the quotient, and so defines a potential for the metric there.

The inner product on the Lie algebra is Ad-invariant, so we may equally write μ as a function on the space of solutions (T_1, T_2, T_3) of Nahm's equations:

$$\mu(A) = \frac{1}{2}\int_0^1 \langle T_2, T_2\rangle + \langle T_3, T_3\rangle ds \tag{5.3}$$

The challenge now is to express this in terms of the data which yields the solution to Nahm's equations for the group $G = SU(n)$: a family of line bundles over an algebraic curve.

5.4 Theta functions

We consider the integrand in the formula (5.3) in the case that $G = SU(n)$. With the invariant inner product $\langle U, U\rangle = \operatorname{tr} UU^* = -\operatorname{tr} U^2$, this can be expressed in terms of

$$\operatorname{tr}(T_2^2 + T_3^2).$$

Since this does not involve the adjoint, it makes sense for arbitrary $T_i \in \mathfrak{sl}(k, \mathbf{C})$.

Consider then what $\operatorname{tr}(T_2^2 + T_3^2)$ expresses: a conjugation-invariant function on the space of triples (T_1, T_2, T_3) of matrices of trace zero. If we put these together, as in (4.1)

$$\begin{aligned} A(\zeta) &= (T_2 + iT_3) - 2iT_1\zeta + (T_2 - iT_3)\zeta^2 \\ &= A_0 + A_1\zeta + A_2\zeta^2 \end{aligned}$$

we obtain an element of $H^0(\mathbf{P}^1, \mathcal{O}(2)) \otimes \mathfrak{gl}(k)$. According to Theorem 1, a dense open set in the space of triples modulo conjugation corresponds to the choice of a non-singular curve S

$$\det(\eta - A(\zeta)) = \eta^k + a_1(\zeta)\eta^{k-1} + a_2(\zeta)\eta^{k-2} \ldots + a_k(\zeta) = 0$$

in $\mathcal{O}(2)$ and a point in the Jacobian of line bundles of degree $g-1$, not in the theta divisor Θ. Since $A(\zeta)$ has trace zero, in fact the coefficient $a_1(\zeta)$ vanishes. From the algebraic nature of the correspondence, $\operatorname{tr}(T_2^2+T_3^2)$ is a meromorphic function on $J^{g-1}(S)$ having a pole on Θ, and our task is to determine what it is.

There are quadratic functions of $(T_1.T_2.T_3)$ which are *constant* on the torus: the coefficients of the polynomial $a_2(\zeta)$ in the formula for the spectral curve. In fact,

$$\begin{aligned} a_2(\zeta) &= -\frac{1}{2}\operatorname{tr} A(\zeta)^2 \\ &= c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 \end{aligned}$$

so that

$$\begin{aligned} c_0 + c_4 &= \operatorname{tr}(T_3^2) - \operatorname{tr}(T_2^2) \\ c_0 - c_4 &= -2i\operatorname{tr} T_2T_3 \\ c_1 + c_3 &= 2i\operatorname{tr} T_1T_2 \\ c_1 - c_3 &= -2\operatorname{tr} T_3T_1 \\ c_2 &= 2\operatorname{tr}(T_1^2) - \operatorname{tr}(T_2^2) - \operatorname{tr}(T_3^2) \end{aligned} \tag{5.4}$$

It is clear that $\operatorname{tr}(T_2^2+T_3^2)$ is not in the space spanned by these coefficients, and indeed it is not constant on the Jacobian. Nevertheless, it is more natural to ask for the more symmetrical invariant

$$\Delta = \operatorname{tr}(T_1^2+T_2^2+T_3^2)$$

and we can then obtain

$$\operatorname{tr}(T_2^2+T_3^2) = \frac{1}{3}(2\Delta - c_2) \tag{5.5}$$

Before we give the answer, recall some basic features of $J^{g-1}(S)$. It has a distinguished linear vector field $\mathcal{X} = \eta x \in H^1(S,\mathcal{O})$. ¿From (2.14), the genus of the spectral curve is $g=(k-1)^2$, so $\pi^*\mathcal{O}(k-2)$, which has degree $k(k-2) = (k-1)^2-1 = g-1$ is a distinguished point, which we can use to identify with $J(S)$. Furthermore, it is easy to see [22] that $K_S \cong \pi^*\mathcal{O}(2k-4)$, so the distinguished origin is a theta-characteristic: a line bundle L such that $L^2 \cong K$. Finally, since $\mathcal{O}(k-2)$ has $k-1$ sections on $\mathbf{P}^1$ (and usually even more on S itself), the distinguished point always lies on the theta divisor, and if $k>2$ it is a singular point. With these preliminaries, we can state:

Theorem 2. *Let (T_1, T_2, T_2) be a triple of trace-free $k\times k$ matrices for which the spectral curve S is smooth. Then*

$$\Delta = \operatorname{tr}(T_1^2+T_2^2+T_3^2) = \frac{3}{2}\frac{d^2}{ds^2}\log\vartheta + c$$

where ϑ is the Riemann theta function translated by a half-period, d/ds denotes the derivative along the vector field $\mathcal{X}$, and the constant c is given by

$$c = -\frac{3}{(N+2)(N+1)}\frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)}$$

with $N = k(k^2-1)/6$.

Proof: First recall that the Riemann theta function is a holomorphic function on the universal covering $\mathbf{C}^g$ of $J(S)$. We may write

$$\vartheta(z) = \sum_{m\in\mathbf{Z}^g} \exp[\pi i(\langle Bm, m\rangle + 2\langle z, m\rangle)]$$

where A_i, B_i are the integrals of the holomorphic differentials on the curve over a canonical basis, and B_i are the columns of the matrix B. The Riemann theta function depends on the choice of canonical basis, which in turn determines a theta characteristic, providing an isomorphism $J(S) \cong J^{g-1}(S)$. This isomorphism carries the theta divisor $\Theta \subset J^{g-1}$ to the zero set of the theta function ϑ. In our geometrical problem, we have another theta characteristic, independent of the choice of canonical basis, and the difference is a half-period. We shall continue to denote this translate by ϑ.

The theta function satisfies the basic properties:

$$\vartheta(z + A_i) = \vartheta(z) \tag{5.6}$$

$$\vartheta(z + n_1B_1 + \cdots + n_gB_g) = \exp[-\pi i(\langle Bn, n\rangle + 2\langle z, n\rangle)]\vartheta(z) \tag{5.7}$$

It is a consequence of these relations that

$$\frac{\partial^2}{\partial z_i \partial z_j}\log\vartheta$$

is invariant under translation by the lattice generated by A_i, B_i and hence is a meromorphic function on $J(S)$. In particular, so is

$$\frac{d^2}{ds^2}\log\vartheta = \sum_{i,j}\mathcal{X}_i\mathcal{X}_j\frac{\partial^2}{\partial z_i\partial z_j}\log\vartheta$$

Both Δ and $(\log\vartheta)''$ are meromorphic with poles along the theta-divisor. The strategy of proof will be to show that, taking an appropriate multiple of Δ, the principal parts coincide, so the difference is holomorphic and constant. A calculation at the distinguished origin will then evaluate the constant. To analyze the pole, we shall use Nahm's equations.

We begin by considering a generic smooth point of Θ where the vector field $\mathcal{X}$ is transversal. From [5], the holomorphic structure on the vector bundle $V(1)$

jumps from $\mathcal{O}^k$ to $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{k-2}$. We need to investigate the behaviour of $A(\zeta)$ near this point. It acquires a singularity clearly from Theorem 1, but its precise nature requires further investigation.

In fact, in [22] (§5), it is shown that at the origin in $J(S)$, a solution to Nahm's equations acquires a simple pole and thus does so at a general point of Θ. Expanding, we have

$$T_i(s) = \frac{\rho_i}{s} + \sigma_i + \tau_i s \ldots$$

and applying Nahm's equations about any simple pole gives the three sets of relations:

$$-\rho_1 = [\rho_2, \rho_3] \quad (5.8)$$
$$0 = [\rho_2, \sigma_3] + [\sigma_2, \rho_3] \quad (5.9)$$
$$\tau_1 = [\tau_2, \rho_3] + [\sigma_2, \sigma_3] + [\rho_2, \tau_3] \quad (5.10)$$

and similar expressions obtained by cyclic permutation of the indices. From (5.8), the residues ρ_i define a k-dimensional representation of $SL(2, \mathbf{C})$. At the origin, the purpose of the long argument in [22] is to show that this is the unique irreducible k-dimensional representation, a fact we shall need at a later stage.

Consider then a solution of Nahm's equations which acquires a pole as $s \to 0$, for which the line bundle L_s approaches a *smooth* point of Θ. The construction of Theorem 1 always yields a vector bundle V_s on $\mathbf{P}^1$ and a homomorphism $A_s : V_s \to V_s(2)$. The bundle $V_s(1)$ is trivial for $s \neq 0$. At $s = 0$, $V_0(1) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{k-2}$, and then $V_0(1)$ still has a k-dimensional space of sections, just as in the trivial case. It follows that we can choose a basis $v_1(\zeta, s), \ldots, v_k(\zeta, s)$ of sections of V_s, holomorphic for s in a neighbourhood of 0. For $s \neq 0$ these span the fibre at each point $\zeta \in \mathbf{P}^1$, but for $s = 0$, they all lie in the subbundle $\mathcal{O}(1) \oplus \mathcal{O}^{k-2}$. Now since the Nahm matrices acquire a simple pole,

$$A_s(\zeta) v_i(\zeta, s) = \sum_j \frac{R(\zeta)_{ji} v_j(\zeta, s)}{s} + \ldots$$

and so since this is finite at $s = 0$,

$$\sum_j R(\zeta)_{ji} v_j(\zeta, 0) = 0$$

But $v_1(\zeta, 0), \ldots, v_k(\zeta, 0)$ span a $(k-1)$-dimensional subspace, so the rank of $R(\zeta)$ is 1. The residues of the Nahm matrices thus define a representation for which

$$R(\zeta) = (\rho_2 + i\rho_3) - 2i\rho_1\zeta + (\rho_2 - i\rho_3)\zeta^2$$

has rank 1. But on the irreducible representation of dimension n this has rank $n - 1$, so the representation is the sum of a $(k-2)$-dimensional trivial representation and the irreducible 2-dimensional one. The residues thus only have a pole on the 2-dimensional component, and this, up to conjugacy, is

$$\rho_1 = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho_2 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rho_3 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

It follows that

$$\Delta(s) = \mathrm{tr}(T_1^2 + T_2^2 + T_3^2)(s) = -\frac{3}{2s^2} + \frac{c_{-1}}{s} + \dots \tag{5.11}$$

We now need to determine the constant

$$c_{-1} = 2\sum_i \mathrm{tr}(\rho_i \sigma_i)$$

Note that from Nahm's equations (5.9), we have

$$[\rho_2, \sigma_3] + [\sigma_2, \rho_3] = 0 \quad \text{etc.}$$

and so

$$[\rho_1, [\rho_2, \sigma_3]] = [\rho_1, [\rho_3, \sigma_2]]$$

But from the Jacobi identity,

$$\begin{aligned}[\rho_1, [\rho_2, \sigma_3]] &= -[\sigma_3, [\rho_1, \rho_2]] - [\rho_2, [\sigma_3, \rho_1]] \\ &= [\sigma_3, \rho_3] + [\rho_2, [\rho_1, \sigma_3]]\end{aligned}$$

Doing a similar calculation on the right hand side, we obtain

$$\begin{aligned}[\sigma_3, \rho_3] + [\sigma_2, \rho_2] &= -[\rho_2, [\rho_1, \sigma_3]] + [\rho_3, [\rho_2, \sigma_1]] \\ &= -[\rho_2, [\rho_3, \sigma_1]] + [\rho_3, [\rho_2, \sigma_1]] \\ &= -[\sigma_1, \rho_1]\end{aligned}$$

As a consequence of this relation

$$\begin{aligned}0 &= [\rho_1, [\rho_1, \sigma_1]] + [\rho_1, [\rho_2, \sigma_2]] + [\rho_1, [\rho_3, \sigma_3]] \\ &= [\rho_1, [\rho_1, \sigma_1]] - [\rho_2, [\sigma_2, \rho_1]] - [\sigma_2, [\rho_1, \rho_2]] - [\rho_3, [\sigma_3, \rho_1]] - [\sigma_3, [\rho_1, \rho_3]] \\ &= [\rho_1, [\rho_1, \sigma_1]] - [\rho_2, [\sigma_1, \rho_2]] + [\sigma_2, \rho_3] - [\rho_3, [\sigma_1, \rho_3]] - [\sigma_3, \rho_2] \\ &= ((\mathrm{ad}\,\rho_1)^2 + (\mathrm{ad}\,\rho_2)^2 + (\mathrm{ad}\,\rho_3)^2)\sigma_1\end{aligned}$$

Consider for a moment the case of any simple pole for Nahm's equations. The residues ρ_1, ρ_2, ρ_3 give $\mathbf{C}^k$ the structure of a representation space for $SL(2, \mathbf{C})$. We decompose $\mathbf{C}^k = \bigoplus_n E_n$ where E_n is an irreducible representation space. Since from the calculation above, $((\mathrm{ad}\,\rho_1)^2 + (\mathrm{ad}\,\rho_2)^2 + (\mathrm{ad}\,\rho_3)^2)\sigma_j = 0$, σ_j commutes with each ρ_i. Its component in each $\mathrm{Hom}(E_n, E_n)$ is thus a scalar by irreducibility. It follows that $\mathrm{tr}(\rho_i\sigma_i) = 0$ since $\mathrm{tr}(\rho_i) = 0$. Hence the coefficient of s^{-1} vanishes and

$$\Delta(s) = -\frac{3}{2s^2} + c_0 + c_1 s + \ldots \tag{5.12}$$

Now compare this with the Riemann theta function ϑ. We know that this has a simple zero at a smooth point on Θ, so in terms of the parameter s, we have an expansion

$$\vartheta(s) = s(a_0 + a_1 s + a_2 s^2 + \ldots)$$

so

$$\frac{d^2}{ds^2}\log\vartheta = -\frac{1}{s^2} + \frac{2a_0a_2 - a_1^2}{a_0^2} + \ldots$$

Hence from (5.12),

$$\Delta - \frac{3}{2}\frac{d^2}{ds^2}\log\vartheta$$

is holomorphic on $J(S)$ and so is equal to a constant, depending on the curve S.

To evaluate the constant, we focus attention on the origin. Here, as proved in [22], the residues ρ_i define the k-dimensional irreducible representation of $SL(2, \mathbf{C})$. Thus $\mathbf{C}^k = E_1$, and by the argument above each σ_i is a scalar. However, $\mathrm{tr}\,\sigma_i = 0$ so

$$\sigma_i = 0$$

Now consider the constant term

$$\sum_i (2\,\mathrm{tr}(\rho_i\tau_i) + \mathrm{tr}(\sigma_i)^2) = 2\sum_i \mathrm{tr}(\rho_i\tau_i)$$

in the expansion of Δ at the origin. From (5.10), with $\sigma_i = 0$, we have

$$\tau_1 = [\tau_2, \rho_3] + [\rho_2, \tau_3]$$

and so

$$\begin{aligned}\mathrm{tr}(\rho_1\tau_1) &= \mathrm{tr}(\rho_1[\tau_2, \rho_3]) + \mathrm{tr}(\rho_1[\rho_2, \tau_3]) \\ &= \mathrm{tr}([\rho_3, \rho_1]\tau_2) + \mathrm{tr}([\rho_1, \rho_2]\tau_3) \\ &= -\,\mathrm{tr}(\rho_2\tau_2) - \mathrm{tr}(\rho_3\tau_3)\end{aligned}$$

from the definition of representation (5.8). Consequently

$$\sum_i \mathrm{tr}(\rho_i\tau_i) = 0$$

and Δ has no constant term at the origin.

Compare this with the expansion of ϑ at the origin. Since this is a singular point of the theta-divisor, ϑ vanishes to high order N. Since the origin is a theta-characteristic, our translate ϑ of the theta function is either odd or even and so

$$\vartheta(s) = s^N(a_0 + a_2 s^2 + \dots)$$

where

$$a_0 = \frac{1}{N!}\vartheta^{(N)}(0), \quad a_2 = \frac{1}{(N+2)!}\vartheta^{(N+2)}(0)$$

hence

$$\frac{d^2}{ds^2}\log\vartheta(s) = -\frac{N}{s} + \frac{2\vartheta^{(N+2)}(0)}{(N+1)(N+2)\vartheta^{(N)}(0)} + \dots$$

Now for the k-dimensional irreducible representation,

$$\sum_i \operatorname{tr}(\rho_i^2) = -\frac{k(k^2-1)}{4}$$

so near the origin,

$$\Delta(s) = -\frac{k(k^2-1)}{4s^2} + \dots$$

and

$$\frac{3}{2}\frac{d^2}{ds^2}\log\vartheta = -\frac{3N}{2s^2} + \dots$$

Hence since

$$\Delta - \frac{3}{2}\frac{d^2}{ds^2}\log\vartheta$$

is constant, we must have $N = k(k^2-1)/6$ and finally

$$\Delta = \frac{3}{2}\frac{d^2}{ds^2}\log\vartheta - \frac{3}{(N+1)(N+2)}\frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)}$$

as required.

Example:
When $k = 2$, the curve S is elliptic, and we can identify $J(S)$ with S itself. The theta divisor is a single point, which we take as the origin, and the translate of the theta function which vanishes at the origin is traditionally called ϑ_1, and regarded as a function of $\nu = u/2\omega_1$. The classical formula for the Weierstrass zeta function

$$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \frac{1}{2\omega_1}\frac{\vartheta_1'(\nu)}{\vartheta_1(\nu)} \tag{5.13}$$

shows that

$$-\wp(u) = \zeta'(u) = \frac{\eta_1}{\omega_1} + \frac{d^2}{du^2}\log\vartheta_1$$

and as is well known, the Weierstrass $\wp$-function has an expansion at the origin

$$\wp(u) = \frac{1}{u^2} + c_2 u^2 + \dots$$

Thus there is no constant term for $\wp$. Now $u = \kappa s$ for some constant κ. This constant is a function of the coefficients of the spectral curve: it expresses the canonical vector field $\mathcal{X}$ in terms of the Weierstrass vector field d/du. As a consequence of the theorem, we can say that

$$\mathrm{tr}(T_1^2 + T_2^2 + T_3^2)(s) = -\frac{3}{2}\kappa^2 \wp(\kappa s) \tag{5.14}$$

The constant κ can be evaluated by referring to Hurtubise's calculation [29]. The spectral curve can be reduced by a rotation to the canonical form

$$\eta^2 = r_1\zeta^3 - r_2\zeta^2 - r_1\zeta$$

with r_1, r_2 real and $r_1 \geq 0$. In this form $\kappa = r_1^{1/2}$.

If we return to the question of Kähler potentials, then Theorem 2 provides an answer for Kronheimer's metrics on T^*G^c. From Proposition 5.1 and (5.3) and (5.5) we have

$$\begin{aligned}\phi = 2\mu(A) &= -\int_0^1 \mathrm{tr}(T_2^2 + T_3^2)ds \\ &= -\int_0^1 \frac{1}{3}(2\Delta - c_2)\end{aligned}$$

so from Theorem 2, we have

$$\phi = \int_0^1 -\frac{d^2}{ds^2}(\log\theta)ds + \frac{2\vartheta^{N+2}(0)}{(N+1)(N+2)\vartheta^N(0)} + \frac{c_2}{3}$$

Before we state this as a theorem, we need to understand better the role of the coefficient c_2 in the formula. As it stands, the formula for the Kähler potential is the sum of two terms. The first, involving the theta function, depends only on the modulus of the curve, and so is invariant under the $SO(3)$ action

$$B_0 \mapsto B_0, \quad B_i \mapsto \sum_i^3 P_{ij}B_j$$

The second term consists essentially of the coefficient of ζ^2 in the quartic polynomial $a_2(\zeta)$ appearing in the formula for the spectral curve. This is picked out by the action of the circle subgroup in $SO(3)$ which preserves the complex structure I. Invariantly speaking, $a_2(\zeta)$ lies in the vector space $H^0(\mathbf{P}^1, \mathcal{O}(4))$. The circle action generates the vector field $X = \zeta d/d\zeta$, a holomorphic section of the tangent bundle $\mathcal{O}(2)$, and so $X^2 \in H^0(\mathbf{P}^1, \mathcal{O}(4))$. But the space $H^0(\mathbf{P}^1, \mathcal{O}(4))$ is an irreducible representation space for $SL(2, \mathbf{C})$ and has an invariant inner product. If $a \in H^0(\mathbf{P}^1, \mathcal{O}(4))$ is written as

$$a = c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4$$

then the inner product is

$$\langle a, a\rangle = 12c_0c_4 - 3c_1c_3 + c_2^2$$

From this point of view, the coefficient c_2 has the invariant meaning

$$c_2 = \langle a_2, X^2\rangle$$

We are aiming to give a formula for the Kähler potential of the complex structure J, but our formula seems to involve the holomorphic vector field X which vanishes at I, and I is not canonically associated to J. There are circle actions in $SO(3)$ fixing *any* complex structure orthogonal to J. We cannot expect a formula for the Kähler potential to be independent of the complex structure I, because as remarked above (following Proposition 5.1), it is already independent of the complex structure orthogonal to I and putting both facts together, we would have an $SO(3)$-invariant potential which is not the case.

The explanation, of course, is that Kähler potentials for a given complex structure are not uniquely defined: we may add on any pluriharmonic function—a function f satisfying

$$\partial\bar{\partial} f = 0.$$

With this in mind, consider the effect of rotating the complex structure I by a circle action preserving J. A rotation of θ replaces c_2 by

$$c_2 + \frac{3}{2}\sin^2\theta(c_0 + c_4 - c_2) - \frac{3}{2}\sin\theta\cos\theta(c_1 - c_3) \tag{5.15}$$

Note from (5.4) that

$$2\,\mathrm{tr}(T_3 + iT_1)^2 = c_0 + c_4 - c_2 - 2i(c_1 - c_3)$$

and $\mathrm{tr}(T_3 + iT_1)^2 = \mathrm{tr}(B_3 + iB_1)^2$ is holomorphic relative to the complex structure J. Its real and imaginary parts $c_0 + c_4 - c_2$ and $c_1 - c_3$ are thus pluriharmonic. Hence from (5.15) any one of these choices gives the same Kähler metric.

The vector field Y conjugate to X which fixes J is

$$Y = \frac{1}{2i}(\zeta^2 + 1)\frac{d}{d\zeta}$$

and this gives

$$\langle a_2, Y^2\rangle = -\frac{3}{2}(c_0 + c_4 - c_2) - 2c_2$$

so that

$$\partial_J \bar{\partial}_J c_2 = -\frac{1}{2}\partial_J \bar{\partial}_J \langle a_2, Y^2 \rangle \tag{5.16}$$

Thus $-\langle a_2, Y^2\rangle/2$ provides an alternative to the c_2 term in the Kähler potential, which is now invariantly defined by the complex structure J under consideration.

Perhaps a better way to express this is to use the identification of the 5-dimensional irreducible representation space of $SU(2)$ which here appears as $H^0(\mathbf{P}^1, \mathcal{O}(4))$ as the space of 3×3 symmetric matrices of trace zero, in other words to define from $q \in H^0(\mathbf{P}^1, \mathcal{O}(4))$ a quadratic form Q on the 3-dimensional space $H^0(\mathbf{P}^1, \mathcal{O}(2))$. This is easily achieved by defining for $x \in H^0(\mathbf{P}^1, \mathcal{O}(2))$

$$Q(x, x) = \langle q, x^2 \rangle$$

From this point of view, we think of a complex structure in the hyperkähler family as being a point $u \in S^2 \subset \mathbf{R}^3$ and then (5.16) becomes

$$\partial\bar{\partial} c_2 = -\frac{1}{2}\partial\bar{\partial}\langle a_2, Y^2\rangle = -\frac{1}{2}\partial\bar{\partial} Q(u, u)$$

We can now formulate the theorem in a more natural form:

Proposition 5.2. *Let $u \in S^2$ be a complex structure of the hyperkähler family on $T^*SL(k, \mathbf{C})$, then the Kähler potential ϕ is given by the formula*

$$\phi = \frac{\vartheta'(a)}{\vartheta(a)} - \frac{\vartheta'(b)}{\vartheta(b)} + \frac{2\vartheta^{(N+2)}(0)}{(N+1)(N+2)\vartheta^{(N)}(0)} - \frac{1}{6}Q(u, u)$$

where the spectral curve S given by $\eta^k + a_2(\zeta)\eta^{k-2} + \ldots + a_k(\zeta) = 0$, Q is the quadratic form defined by the coefficient a_2, the points $a, b \in J(S)$ are the line bundles corresponding to the triple (T_1, T_2, T_3) at $s = 0$ and $s = 1$ respectively, and ϑ is a translate by a half-period of the Riemann theta function on $J(S)$.

Remark: Although this is an explicit formula, it is only useful in the parametrization of the space $T^*SL(k, \mathbf{C})$ by the integrable system approach. The data involves firstly $(k+1)^2 - 4$ parameters for the coefficients of the spectral curve (these are real because the curve satisfies reality conditions), and secondly $(k-1)^2$ real parameters for the points a on the Jacobian, the initial point for the flow of the vector field $\mathcal{X}$. These provide

$$(k+1)^2 - 4 + (k-1)^2 = 2(k^2 - 1)$$

parameters. The full space has real dimension $4(k^2 - 1)$ since $\dim SU(k) = k^2 - 1$, but there is a free isometric action of $SU(k) \times SU(k)$ given by the two quotient groups $\mathcal{G}_0/\mathcal{G}_0^0$ and $\mathcal{G}^0/\mathcal{G}_0^0$ where $\mathcal{G}_0$, $\mathcal{G}^0$ are the groups of gauge transformations $g : [0,1] \to G$ for which $g(0) = 1$ or $g(1) = 1$ respectively (see [13]), and this provides the extra degrees of freedom. Since $SU(k) \times SU(k)$ preserves the metric and all three complex structures, the Kähler potential naturally exists on the quotient.

A more satisfying answer would result if we could relate these coordinates to the holomorphic parametrization given by (5.2). This, however, involves solving a supplementary linear differential equation $df/ds = f\alpha$ to determine f, and even in the case of $SL(2,\mathbf{C})$, where solutions to Nahm's equations can be explicitly written down with elliptic functions, this is not a practical prospect. Instead we can consider another class of metrics obtained by solutions to Nahm's equations: those on *coadjoint orbits.*

5.5 Hyperkähler metrics on coadjoint orbits

Kronheimer's use of Nahm's equations to construct hyperkähler metrics on coadjoint orbits of a complex semi-simple Lie group [35] has been extended from the semi-simple or nilpotent orbit case which he originally considered by Biquard and Kovalev [7],[33], but we shall restrict ourselves here to the semi-simple case. It is the same basic set-up as above, except that one studies solutions of Nahm's equations on a semi-infinite instead of finite interval. We also make a special choice of boundary condition to ensure the existence of a circle action, for only this will give us our Kähler potential by the above method.

Choose a point ξ in the Lie algebra of the maximal torus of a compact semi-simple group G, and consider solutions to Nahm's equations on the interval $(-\infty, 0]$ with boundary condition

$$(T_1, T_2, T_3) \to (\mathrm{Ad}(g)\xi, 0, 0)$$

for some $g \in G$. Because of this choice, we have the same circle action as before: $(T_1, T_2 + iT_3) \mapsto (T_1, e^{i\theta}(T_2 + iT_3))$. The Lax form of Nahm's equations implies

$$(T_1 + iT_2)' = [iT_3, T_1 + iT_2]$$

so that for all $s \in (-\infty, 0]$, $(T_1 + iT_2)(s)$ lies in the G^c-orbit of ξ. Kronheimer showed in [35] that

$$(T_1, T_2, T_3) \mapsto (T_1 + iT_2)(0)$$

identifies the moduli space of solutions diffeomorphically with the adjoint ($\cong$ coadjoint) orbit of ξ.

Now for $G = SU(k)$, the spectral curve $\det(\eta - A(\zeta, s)) = 0$ is independent of s. Letting $s \to -\infty$, $(T_1, T_2, T_3) \to (\mathrm{Ad}(g)\xi, 0, 0)$ so we have

$$\begin{aligned}\det(\eta - A(\zeta, s)) &= \det(\eta - ((T_2 + iT_3) - 2iT_1\zeta + (T_2 - iT_3)\zeta^2)) \\ &= \det(\eta + 2i\xi\zeta) \\ &= (\eta - \lambda_1\zeta)(\eta - \lambda_2\zeta)\ldots(\eta - \lambda_k\zeta)\end{aligned}$$

and the spectral curve is reducible to a union of projective lines, all meeting at $\zeta = 0$ and $\zeta = \infty$. Biquard's analysis [7] shows that any solution of Nahm's equations arising from this spectral curve also satisfies the boundary conditions at $s = -\infty$, and also shows that the Kähler potential defined by

$$\phi = -\int_{-\infty}^{0} \mathrm{tr}(T_2^2 + T_3^2)ds \tag{5.17}$$

is finite.

What is required then is to study such solutions from the point of view of integrable systems, involving line bundles and the construction of triples from curves of the above type. Such a study was made in the paper of Santa Cruz [51]. We would like to apply Theorem 2 to obtain a description in terms of theta functions, but that was only stated for a non-singular spectral curve S. One might hope in general that the statement of the theorem would hold for these curves too. The line bundles on the degenerate spectral curve S can be described as in [51] in terms of transition functions and from that, a means of determining those triples (T_1, T_2, T_3) for which $A(\zeta)$ is regular (in the Lie group sense) for all ζ is provided. As Santa Cruz points out, this leads to explicit determinantal formulas for the theta divisor, and rationality in terms of $e^{\lambda_i s}$ for the integrand $\mathrm{tr}(T_2^2 + T_3^2)$ in the Kähler potential. With such concrete forms for each side it seems likely that the theorem would still hold.

The case $k = 2$ is somewhat easier, for then the spectral curve consists of two rational curves meeting transversely at two points, and this is a situation where degeneration methods tell us what the theta divisor should be [39]. The metric is the well known Eguchi-Hanson metric [19]. This is more commonly described as being defined on the cotangent bundle of the projective line, but this is an exceptional complex structure among the hyperkähler family. The general one is that of an affine quadric in $\mathbf{C}^3$—a semisimple orbit of $SL(2, \mathbf{C})$.

Consider then the spectral curve

$$\eta^2 - \lambda^2\zeta^2 = (\eta - \lambda\zeta)(\eta + \lambda\zeta) = 0$$

since ξ lies in the Lie algebra of $SU(2)$ and $\pm\lambda$ are the eigenvalues of $2i\xi$, λ is real. The curve has genus $(k-1)^2 = 1$ and any line bundle of degree zero is obtained as $\exp(u\eta x)$ for the canonical element $\eta x \in H^1(S, \mathcal{O})$. Setting U_0 to be the complement of $\zeta = \infty$ and U_∞ the complement of U_0, this is the line bundle with transition function

$$\exp(u\eta/\zeta).$$

On the component $\eta = \lambda\zeta$, this can be trivialized by the constant functions $e^{\lambda u}$ on U_0 and 1 on U_∞, and on the other component $\eta = -\lambda\zeta$ by the functions $e^{-\lambda u}$ and 1. For these ordinary singular points the bundle is trivial if and only if the two trivializations agree at $\zeta = 0$ and $\zeta = \infty$, and this is true if and only if

$$\exp(\lambda u) = \exp(-\lambda u)$$

The Jacobian of S is thus $\mathbf{C}/(\pi i/\lambda)\mathbf{Z}$, and following [39], the theta function is

$$\vartheta(u) = \sinh(\lambda u)$$

According to Theorem 2, we have

$$\begin{aligned}\Delta &= \frac{3}{2}\frac{d^2}{du^2}\log\vartheta - \frac{\vartheta'''(0)}{2\vartheta'(0)}\\ &= -\frac{3}{2}\lambda^2 \operatorname{cosech}^2 \lambda u - \frac{1}{2}\lambda^2\end{aligned}$$

Now the spectral curve has the equation $\eta^2 - \lambda^2\zeta^2 = 0$, so the coefficient $a_2(\zeta)$ is given by

$$a_2(\zeta) = -\lambda^2\zeta^2$$

and the constant c_2 is $-\lambda^2$. Consequently, we have from (5.5)

$$\begin{aligned}\operatorname{tr}(T_2^2 + T_3^2) &= \frac{1}{3}(2\Delta - c_2)\\ &= -\lambda^2 \operatorname{cosech}^2 \lambda u - \frac{1}{3}\lambda^2 + \frac{1}{3}\lambda^2\\ &= -\lambda^2 \operatorname{cosech}^2 \lambda u\end{aligned}$$

The solution to Nahm's equations for $s \in (-\infty, 0]$ is derived from a linear flow along the Jacobian using the vector field $\mathcal{X}$ and this corresponds to setting $u = s - a$. Thus at $s = 0$, $u = -a$. Since the solution must be non-singular for $s \in (-\infty, 0]$, the theta divisor $u = 0$ must not be in this interval, so $a > 0$. From (5.17) we can now evaluate the Kähler potential as

$$\begin{aligned}\phi &= -\int_{-\infty}^{0} \operatorname{tr}(T_2^2 + T_3^2)ds\\ &= \int_{-\infty}^{-a} \lambda^2 \operatorname{cosech}^2(\lambda u)du\\ &= \lambda(1 - \coth \lambda a)\end{aligned}$$

Now we must relate the parameter a to the complex coordinates of the coadjoint orbit. This is the more difficult part in higher rank groups, but it is purely algebraic: there is no extra differential equation to solve. Recall that a solution to Nahm's equations defines the matrix

$$X = (T_1 + iT_2)(0)$$

on the orbit of ξ. Note that

$$\operatorname{tr} XX^* = -\operatorname{tr}(T_1^2 + T_2^2) = -\operatorname{tr}(T_2^2 + T_3^2) + \operatorname{tr}(T_3^2 - T_1^2)$$

but from the coefficients of $a_2(\zeta) = -\lambda^2\zeta^2$ (5.4) we see that $\operatorname{tr} T_3^2 = \operatorname{tr} T_2^2$ and $2 \operatorname{tr} T_1^2 - \operatorname{tr} T_2^2 - \operatorname{tr} T_3^2 = -\lambda^2$ and hence

$$\operatorname{tr}(T_3^2 - T_1^2) = \frac{1}{2}\lambda^2$$

so that now

$$\operatorname{tr} XX^* = \lambda^2 \operatorname{cosech}^2 \lambda a + \frac{1}{2}\lambda^2$$

This now provides the final formula for the potential in terms of the coadjoint orbit:

$$\phi(X) = \lambda + \sqrt{(\operatorname{tr} XX^* + \lambda^2/2)}$$

Remark: This formula has been encountered before. In [51], Santa Cruz uses conjugates of the Nahm matrices explicitly derived from the line bundle approach to give the integrand $\operatorname{tr}(T_2^2 + T_3^2)$. Stenzel [54] obtains the same expression by using the $SU(2)$ symmetry to reduce the problem to an ordinary differential equation.

5.6 Monopole moduli spaces

Nahm's equations were originally produced in order to solve another set of differential equations: the Bogomolny equations. These are dimensional reductions of the self-dual Yang-Mills equations by the group H of translations

$$(x_0, x_1, x_2, x_3) \mapsto (x_0 + a, x_1, x_2, x_3)$$

The quotient space is $\mathbf{R}^3$ with the Euclidean metric and we have a G-connection A and a single Higgs field $-\phi$. The equations are then

$$\nabla\phi = *F_A$$

A solution to these equations with the boundary conditions that the curvature F_A is $\mathcal{L}^2$ is called a magnetic monopole. The boundary conditions and equations imply that as $r \to \infty$, ϕ tends to a particular orbit in $\mathfrak{g}$. Let $G = SU(n)$, then up to conjugation the Higgs field has an asymptotic expansion in a radial direction

$$\phi = i \operatorname{diag}(\lambda_1, \ldots, \lambda_n) - \frac{i}{2r} \operatorname{diag}(k_1, \ldots, k_n) + \ldots$$

where for topological reasons $k_1, \ldots, k_n$ are integers.

In 1981, W. Nahm proposed a construction of $SU(2)$ monopoles by performing a non-linear Fourier transform to reduce the Bogomolny equations to the ordinary differential equations which have now become known as Nahm's equations. We assume then that $n = 2$, and

$$\phi = i \operatorname{diag}(\lambda, -\lambda) - \frac{i}{2r} \operatorname{diag}(k, -k) + \ldots$$

The interpretation of the integer k is as the first Chern class of the $i\lambda$ eigenspace bundle of ϕ at large distances.

The formalism for the Nahm transform is rather like that of the hyperkähler quotient construction for

$$A = A_0 + iA_1 + jA_2 + kA_3$$

in Section 5.2. In $\mathbf{R}^4$ the formally written operator

$$D = \nabla_0 + i\nabla_1 + j\nabla_2 + k\nabla_3$$

can be thought of as the Dirac operator coupled to the vector bundle with connection. It has a dimensional reduction to three dimensions

$$D = -\phi + i\nabla_1 + j\nabla_2 + k\nabla_3$$

which is also a Dirac operator, now with the zero-order Higgs term ϕ. One considers an eigenvalue problem for this operator, showing that for $s \in (-\lambda, \lambda)$ the space of $\mathcal{L}^2$ solutions ψ of

$$(D - is)\psi = 0$$

is of dimension k. For varying s this space is a vector bundle of rank k inside the space of $\mathcal{L}^2$ functions. Orthogonal projection then defines a connection $d/ds + B_0$ on the vector bundle over the interval $(-\lambda, \lambda)$ and orthogonal projection of the operations of multiplication $\psi \mapsto x_i\psi$ defines three Higgs fields $B_1(s)$. One then shows (cf. [22]) that, gauging B_0 to zero, the matrices $B_i = T_i$ satisfy Nahm's equations. They acquire poles at the end points of the interval whose residues are irreducible k-dimensional representations of $SL(2, \mathbf{C})$.

It is for this reason that so much attention in [22] was expended on this singular behaviour, but which was also of some considerable use in our proof of Theorem 2.

The return journey, from a solution to Nahm's equations to an $SU(2)$ monopole, involves the same procedure, considering the $\mathcal{L}^2$ solutions to the equation $(\hat{D} - ix)\psi = 0$ on the interval, where

$$\hat{D} = \frac{d}{ds} + iT_1 + jT_2 + kT_3$$

and $x = x_1 i + x_2 j + x_3 k$ is a "quaternionic eigenvalue".

The Fourier transform analogy enables one to prove a "Plancherel formula" (see [45]) that the natural $\mathcal{L}^2$ metrics from both points of view coincide. For physical reasons (see [3]), the metric from the Bogomolny equation point of view is the most fundamental, but for calculation it is the Nahm equation metric which is most accessible. Nakajima and Takahasi [46], [55] have studied not only the case of $SU(2)$, but also $SU(n)$ where the $\lambda_2 = \lambda_3 = \cdots = \lambda_n$ and

$k_2 = k_3 = \cdots = k_n$. This means that the Higgs field breaks the symmetry from $SU(n)$ to $U(n-1)$ at infinity. The Chern class of the line bundle at infinity for Nakajima must be $n-1$ which we call the charge of the $SU(n)$ monopole. Dancer [11] made a close study of this case where $n = 3$. At present these are the cases where the Plancherel formula is known and where the monopole metric is the natural metric on the moduli space of solutions to Nahm's equations. Let us consider this in more detail, first in the more studied $SU(2)$ case.

5.7 $SU(2)$ monopoles

The hyperkähler quotient setting of Section 5.2 needs to be modified for the case of $SU(2)$ monopoles since the Nahm matrices are singular at the end-points of the interval. The standard normalization here, achievable by rescaling the metric on $\mathbf{R}^3$, is to take the eigenvalues of the Higgs field at infinity to be $\pm i$. The interval for Nahm's equations then has length 2 and it is convenient to take it, as in [22], to be $[0,2]$. We consider solutions T_1, T_2, T_3 to Nahm's equations on this interval, for which $\operatorname{tr} T_i = 0$, (this being equivalent to centering the monopole [3]) and which have simple poles at the endpoints whose residue defines an irreducible representation of $SL(2,\mathbf{C})$. The circle action

$$(T_2 + iT_3) \mapsto e^{i\theta}(T_2 + iT_3)$$

is well-defined on this space, but the $\mathcal{L}^2$ metric is not, since the residues may vary within a conjugacy class. We therefore have to adopt the point of view of fixing the residues at the poles. This necessitates the reintroduction of the connection matrix B_0. We thus consider the space $\mathcal{A}$ of operators $d/ds + B_0 + iB_1 + jB_2 + kB_3$ with $B_i : (0,2) \to \mathfrak{su}(k)$ on a rank k complex vector bundle over the interval $[0,2]$ where at $s = 0$, B_0 is smooth and for $i > 0$,

$$B_i = \frac{\rho_i}{s} + \dots$$

for a *fixed* irreducible representation defined by ρ_i. At $s = 2$ we have the same behaviour:

$$B_i = \frac{\rho_i}{s-2} + \dots$$

Tangent vectors (A_0, A_1, A_2, A_3) to this space are then smooth at the end-points, and using the group $\mathcal{G}_0^0$ of smooth maps $g : [0,2] \to SU(k)$ for which $g(0) = g(1) = 1$, and a little analysis, we obtain a hyperkähler metric on the space $\mathcal{B}$ of solutions to the equations

$$\begin{aligned} B_1' + [B_0, B_1] &= [B_2, B_3] \\ B_2' + [B_0, B_2] &= [B_3, B_1] \\ B_3' + [B_0, B_3] &= [B_1, B_2] \end{aligned} \tag{5.18}$$

modulo the action of the gauge group $\mathcal{G}_0^0$.

This is our metric, but to find the Kähler potential we need the circle action. Having fixed the residues, this is less easy to describe, because it involves a compensating gauge transformation outside $\mathcal{G}_0^0$. The potential only depends on the infinitesimal version of the action, and this is represented by a vector field

$$(\psi' + [B_0, \psi], [B_1, \psi], B_3 + [B_2, \psi], -B_2 + [B_3, \psi])$$

This is a vector field on the space $\mathcal{B}$ in the infinite-dimensional flat space $\mathcal{A}$: we are using the linear structure of the ambient space to write down tangent vectors. It must be smooth at the end-points, so

$$\psi(0) = \psi(2) = -\rho_1.$$

The Kähler potential is defined in terms of the moment map μ for this vector field, which uses the symplectic form on the quotient. But the quotient construction tells us that its pull-back to $\mathcal{B}$ is the restriction of the constant symplectic form on $\mathcal{A}$:

$$\int_0^2 [-\operatorname{tr}(A_0\tilde{A}_1) + \operatorname{tr}(A_1\tilde{A}_0) + \operatorname{tr}(A_2\tilde{A}_3) - \operatorname{tr}(A_3\tilde{A}_2)]ds$$

The moment map thus satisfies

$$\begin{aligned} d\mu(A) = \int_0^2 & -\operatorname{tr}(A_0[B_1,\psi]) + \operatorname{tr}(A_1(\psi' + [B_0,\psi])) + \operatorname{tr}(A_2(-B_2 + [B_3,\psi])) \\ & \qquad - \operatorname{tr}(A_3(B_3 + [B_2,\psi]))ds \\ = \int_0^2 & -\operatorname{tr}([A_0,B_1]\psi) + \operatorname{tr}(A_1\psi') - \operatorname{tr}([B_0,A_1]\psi) - \operatorname{tr}(A_2B_2) \\ & \qquad + \operatorname{tr}([A_2,B_3]\psi) - \operatorname{tr}(A_3B_3) + \operatorname{tr}([B_2,A_3]\psi)ds \end{aligned}$$

On the other hand A is tangent to $\mathcal{B}$, so A satisfies the linearization of the equations (5.18), so in particular

$$A_1' + [A_0, B_1] + [B_0, A_1] = [A_2, B_3] + [B_2, A_3]$$

Substituting in the formula for $d\mu$ then gives

$$\begin{aligned} d\mu(A) &= \int_0^2 \operatorname{tr}(A_1'\psi) + \operatorname{tr}(A_1'\psi)ds - \int_0^2 \operatorname{tr}(A_2B_2 + A_3B_3)ds \\ &= [\operatorname{tr}(A_1\psi)]_0^2 - \int_0^2 \operatorname{tr}(A_2B_2 + A_3B_3)ds \end{aligned}$$

Now as we saw in the proof of Theorem 2, at an irreducible representation, the Nahm matrix has an expansion

$$T_i = \frac{\rho_i}{s} + \tau_i s + \dots$$

so the conjugate B_i by a smooth gauge transformation behaves like

$$B_i = \frac{\rho_i}{s} + [\varphi, \rho_i] + \dots$$

hence

$$\mathrm{tr}(B_i\rho_i) = -\frac{N}{2s} + b_1 s + \dots$$

Thus, differentiating with respect to a parameter, a tangent vector A to $\mathcal{B}$ has the property

$$A_i(0) = 0$$

and similarly at $s = 2$, which tells us that the boundary term vanishes and

$$d\mu(A) = -\int_0^2 \mathrm{tr}(A_2B_2 + A_3B_3)ds$$

The integrand is conjugation-invariant, so we can write this in terms of Nahm matrices

$$d\mu(\dot{T}) = -\int_0^2 \mathrm{tr}(T_2\dot{T}_2 + T_3\dot{T}_3)ds \tag{5.19}$$

To find the potential, we would therefore like to make sense of the divergent integral

$$-\frac{1}{2}\int_0^2 \mathrm{tr}(T_2^2 + T_3^2)ds$$

Since again there is no constant term in the expansion of T_i at an irreducible pole, we have

$$\mathrm{tr}(T_i^2) = -\frac{N}{2s^2} + 2\,\mathrm{tr}(\rho_i\tau_i) + \dots$$

the most obvious function μ that will satisfy (5.19) is thus

$$\mu(T) = -\frac{1}{2}\int_0^2 \left(\mathrm{tr}(T_2^2 + T_3^2) + \frac{N}{s^2} + \frac{N}{(s-2)^2}\right) ds \tag{5.20}$$

We can cast this in another form if we recall how Nahm's equations are to be solved with a linear flow in the direction of the canonical vector field $\mathcal{X}$ on the Jacobian of the spectral curve. As we saw, a pole which defines an irreducible representation corresponds to the flow passing through the origin of $J(S)$, so an irreducible pole at both ends of the interval means that the flow is periodic. (This is the $L^2 \cong \mathcal{O}$ condition on the spectral curve—see [3]). Thus the function

$$\mathrm{tr}(T_2^2 + T_3^2)$$

is periodic in s of period 2.

Now let $f(s)$ be a meromorphic function with an expansion about $s = 0$ of

$$f(s) = \frac{a_{-2}}{s^2} + a_0 + a_1 s + \dots$$

and Γ_ϵ a contour consisting of an interval $[a, b]$ indented around the origin with a semicircular contour of radius ϵ. The contour integral is the limit as $\epsilon \to 0$ of

$$\int_a^{-\epsilon} f(s)\, ds + \int_\epsilon^b f(s)\, ds - \frac{2a_{-2}}{\epsilon}$$

Comparing with (5.20) and using the periodicity, we can take μ to be the integral of $-\operatorname{tr}(T_2^2 + T_3^2)/2$ around a closed contour in the Jacobian which is an orbit of the flow missing the origin.

Now return to Theorem 2. We saw that

$$\operatorname{tr}(T_2^2 + T_3^2) = \frac{1}{3}(2\Delta - c_2)$$

and

$$\Delta = \frac{3}{2}\frac{d^2}{ds^2}\log\vartheta + c$$

so apart from the constants c and c_2, the Kähler potential is obtained by integrating $(\log\vartheta)''$ around a closed cycle. This would be zero if $(\log\vartheta)'$ were single valued, but (5.7) shows that this is not so. However (5.6) gives

$$\vartheta(z + A_i) = \vartheta(z)$$

so if we choose a canonical basis so that our cycle is in the linear span of the A-cycles, then the integral is indeed zero. We thus obtain a formula for the Kähler potential:

$$\phi = \frac{4}{(N+1)(N+2)}\frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} - \frac{1}{3}Q(u,u) \tag{5.21}$$

In the case of monopoles the term $Q(u, u)$ has a direct interpretation in terms of the Higgs field ϕ. For this, we have to use Hurtubise's approach to the asymptotic Higgs field.

Recall that the boundary conditions of a monopole imply an expansion

$$|\phi| = 1 - \frac{k}{2r} + O(r^{-2})$$

Hurtubise shows in [30] that this extends to a complete asymptotic expansion which defines a harmonic function, corresponding to some distribution of charge. Moreover, this can be calculated from the spectral curve.

If we consider the expansion of $|\phi|$ along the ray $(r,0,0)$, then we set $\eta = -2r\zeta = R\zeta$ in the equation of the spectral curve

$$\eta^k + a_2(\zeta)\eta^{k-2} + \cdots + a_k(\zeta)$$

to obtain a polynomial of degree $2k$ in ζ whose coefficents are functions of R. As $R \to \infty$, the $2k$ roots of this equation separate unambiguously into two groups: k which tend to 0 and k which tend to ∞. If $\zeta_1, \ldots, \zeta_k$ are the first group, then Hurtubise's formula is

$$|\phi| \sim 1 - \frac{\partial}{\partial R}\log(\zeta_1\zeta_2\ldots\zeta_k)$$

Making the substitution in the equation of the spectral curve, we obtain

$$\zeta^k + a_2(\zeta)\frac{\zeta^{k-2}}{R^2} + \cdots + \frac{a_k(\zeta)}{R^k} = 0 \tag{5.22}$$

Now each root ζ_i has an asymptotic expansion

$$\zeta_i = \frac{\alpha_i}{R} + \ldots$$

and it follows that each term of the form $c\zeta^{k+m}$ for $m \geq 1$ in (5.22) decays at least as fast as $R^{-(k+3)}$. Collecting together the terms of the form $c\zeta^n$, for $n \leq k$ we have a polynomial of degree k such that for each $\zeta = \zeta_i$,

$$(1 + c_2R^{-2})\zeta^k + \cdots + z_0R^{-k} = O(R^{-(k+3)})$$

where $a_2(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \cdots + c_4\zeta^4$ and $a_k(\zeta) = z_0 + z_1\zeta + \cdots + z_{2k}\zeta^{2k}$. Thus the product of the roots satisfies

$$\zeta_1\zeta_2\ldots\zeta_k = (-1)^k\frac{z_0R^{-k}}{(1 + c_2R^{-2})} + O(R^{-(k+3)})$$

and so

$$\frac{\partial}{\partial R}\log(\zeta_1\zeta_2\ldots\zeta_k) = -\frac{k}{R} - \frac{2c_2}{R^3} + \ldots$$

Replacing R by $-2r$, we have

$$|\phi| \sim 1 - \frac{\partial}{\partial R}\log(\zeta_1\zeta_2\ldots\zeta_k) = 1 - \frac{k}{2r} - \frac{c_2}{4r^3} + \ldots$$

In our parametrization, the unit direction $u = (1,0,0)$ corresponds to $\zeta = 0$, and so as in 5.4, $c_2 = \langle a_2, X^2\rangle = Q(u,u)$ The asymptotic expansion in the direction u is thus

$$|\phi| = 1 - \frac{k}{2r} - \frac{Q(u,u)}{4r^3} + \ldots$$

We thus have the following theorem about the moduli space metric:

Theorem 3. *Let $\mathcal{M}_k^0$ denote the moduli space of centred $SU(2)$ monopoles on $\mathbf{R}^3$ and let $u \in S^2$ be a complex structure of the hyperkähler family on $\mathcal{M}_k^0$. Then the Kähler potential for the natural $\mathcal{L}^2$ metric in this complex structure is*

$$\phi = \frac{4}{(N+1)(N+2)} \frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} - \frac{1}{3} Q(u,u)$$

where ϑ is a translate of the Riemann theta function of the spectral curve and $-Q(u,u)/4$ is the coefficient of r^{-3} in the asymptotic expansion of the length of the Higgs field in the radial direction u.

As with the metric on the cotangent bundle of the group, to express this potential in terms of the natural complex coordinates of $\mathcal{M}_k^0$, considered as a space of rational functions [15] requires the solution of a linear differential equation whose coefficients are Nahm matrices. This clearly places constraints on explicitness. Even the coordinates determined by the integrable system approach are complicated to determine because of the constraints. The periodicity condition imposes $g = (k-1)^2$ conditions on the coefficients of the spectral curve which are $5 + 7 + (2k+1) = (k+1)^2 - 4$ in number, giving the moduli space $\mathcal{M}_k^0$ dimension $4k - 4$.

Example:
Consider the 2-monopole moduli space, where the metric is fully described in [3]. Here, $k = 2$, the curve S is elliptic, and the periodicity of the Nahm flow for $s \mapsto s + 2$ requires us in standard Weierstrassian coordinates to put

$$s = u/\omega_1 = 2\nu$$

In this case $N = 1$ and

$$\frac{4}{(N+1)(N+2)} \frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} = \frac{2}{3} \frac{\vartheta^{(3)}(0)}{\vartheta^{(1)}(0)} = \frac{1}{6} \frac{\vartheta'''(0)}{\vartheta'(0)}$$

using differentiation with respect to ν. From the the standard formula in elliptic functions (cf. (5.13)), we have

$$2\eta_1\omega_1 = -\frac{1}{6} \frac{\vartheta'''(0)}{\vartheta'(0)} \tag{5.23}$$

so that the $SO(3)$-invariant term in the potential is

$$-2\eta_1\omega_1$$

In [47], D.Olivier gives a derivation of this potential, directly from the formula in [3]

$$g = \beta^2\gamma^2\delta^2 \frac{(dk^2)^2}{(4k^2k'^2K^2)^2} + \beta^2\sigma_1^2 + \gamma^2\sigma_2^2 + \delta^2\sigma_3^2$$

where

$$\beta\gamma = -K^2(k'^2 + u)$$
$$\gamma\delta = K^2(k^2 - u)$$
$$\beta\delta = -K^2 u$$

and

$$u = \frac{E}{K} - k'^2$$

All these expressions use the Legendre notation for complete elliptic integrals. For Olivier, the Kähler potential is

$$\phi = \frac{\beta\gamma + \gamma\delta + \delta\beta}{4} - J$$

where J is direction-dependent. After changing from Legendrian formulae to Weierstrassian ones, this corresponds (up to a constant multiple) to the formula here.

5.8 $SU(n)$ monopoles

The study of $SU(n)$ monopoles is more complicated because of the choice of boundary conditions

$$\phi = i\,\mathrm{diag}(\lambda_1, \dots, \lambda_n) - \frac{i}{2r}\mathrm{diag}(k_1, \dots, k_n) + \dots$$

for the Higgs field. The most studied case is that of maximal symmetry breaking: where the eigenvalues at infinity are distinct. Thus the structure group $SU(n)$ is reduced to its maximal torus at infinity. Here there is a construction involving Nahm's equations [31]. In this case one solves the equations on a sequence of intervals corresponding to vertices of the Dynkin diagram. It seems reasonable to believe that the same methods advanced here may work in that situation, but the full correspondence, including the Plancherel formula, has not been worked out. We restrict ourselves instead to a special case of minimal symmetry breaking:

$$\phi = i\,\mathrm{diag}(n-1, -1 \dots, -1) - \frac{i}{2r}\mathrm{diag}(n-1, -1, \dots, -1) + \dots$$

where the group is reduced from $SU(n)$ to $U(n-1)$ at infinity. Here the work of Nakajima, Takahasi [46] and Dancer [11], [12] provide a fuller picture.

The arguments involving the Dirac operator can be extended in this case so that there is an $\mathcal{L}^2$ nullspace of dimension $k = n - 1$ for s between the two distinct eigenvalues k and -1 of ϕ at infinity, or normalizing so that the origin is one end-point, for $s \in (0, k+1)$. Nahm matrices can then be defined just as in the $SU(2)$ case, but with the boundary condition that there is a simple pole at $s = 0$ whose residue defines the irreducible k-dimensional representation of $SL(2, \mathbf{C})$. At the other end-point $s = k + 1$, the matrices are finite.

Using matrices (B_0, B_1, B_2, B_3) as above, we can define a hyperkähler metric on the moduli space of solutions to Nahm's equations, and by the Plancherel theorem this is the natural metric on the moduli space of monopoles with this structure. There is, as with all monopole metrics, an action of $SO(3)$ rotating the complex structures and in this case an extra action of the group $\mathcal{G}_0/\mathcal{G}_0^0 \cong SU(k)$ where $\mathcal{G}_0$ is the group of smooth maps $g : [0, k+1] \to SU(k)$ with $g(0) = 1$. Using the integrable system approach we can count dimensions, since solving Nahm's equations with these boundary conditions consist of starting at the origin and following the flow for a time $t = k + 1$. We therefore have $(k+1)^2 - 4$ parameters for the spectral curve and $k^2 - 1 = \dim SU(k)$ for the gauge action giving a moduli space M_k of dimension $(k+1)^2 - 4 + k^2 - 1 = 2(k^2 + k - 2)$.

As above, we remove the singularity at $s = 0$ by defining

$$\mu(T) = -\frac{1}{2}\int_0^{k+1}\left(\mathrm{tr}(T_2^2 + T_3^2) + \frac{N}{s^2}\right) ds$$

and using

$$\mathrm{tr}(T_2^2 + T_3^2) = \frac{1}{3}(2\Delta - c_2)$$

with

$$\Delta = \frac{3}{2}\frac{d^2}{ds^2}\log\vartheta + c$$

This gives the potential

$$\phi = -\frac{\vartheta'(k+1)}{\vartheta(k+1)} + \frac{k(k-1)}{6} + \frac{2(k+1)}{(N+2)(N+1)}\frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} + \frac{k+1}{3}c_2$$

Example:
For $k = 2$ (the case considered by Dancer) we can again express this using Weierstrass functions. We obtain from (5.14)

$$\Delta(s) = -\frac{3}{2}\kappa^2\wp(\kappa s)$$

and so using the Weierstrass zeta-function and ignoring the constant term,

$$\phi = -\kappa\zeta(3\kappa) + c_2 \qquad (5.24)$$

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