The Geometry of the Seiberg-Witten Invariants

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The Seiberg-Witten invariants were defined by Witten [W1] for any compact, oriented 4-manifold; and they are determined by the underlying differentiable structure when the Betti number b^{2+} is larger than 1. After the choice of orientation for the real line $\det^+ = H^0 \otimes \det(H^1) \otimes \det(H^{2+})$, the Seiberg-Witten invariants constitute a map from the set, $\mathcal{S}$, of $\mathrm{Spin}^{\mathbb{C}}$ structures on the 4-manifold to the integers. There is also an extension of SW in the case where the Betti number b^1 is positive to a map SW: $\mathcal{S} \to \Lambda^* H^1(X;\mathbb{Z})$. (Here, $\Lambda^* H^1(X;\mathbb{Z}) = \mathbb{Z} \oplus H^1 \oplus \Lambda^2 H^1 \oplus \cdots \oplus \Lambda^{b_1} H^1$. Note that the projection of the image of SW on the summand $\mathbb{Z}$ reproduces the original map as defined from $\mathcal{S}$ to $\mathbb{Z}$ in [W1].) In either guise, this map, SW, is computed by an algebraic count of solutions to a certain non-linear system of differential equations on the manifold. The invariant SW and the Seiberg-Witten equations were introduced to the mathematical community by Witten [W1] after his ground breaking work with Seiberg in [SW1], [SW2]. See also [KM], [Mor], [KKM] and [T1].

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It is hard to deny that Seiberg and Witten's equation has revolutionized the study of smooth, 4-dimensional manifolds. This advance is due primarily to the fact that the Seiberg-Witten invariants are eminently computable, much more so than the Donaldson's invariants. (Witten has conjectured that the information coded in the two invariants is equivalent.) This is to say that the basic strategies for computing Donaldson invariants can be applied with comparitively little effort (not no effort, mind you) to computing the Seiberg-Witten invariants. Thus, our knowledge has expanded greatly due to the relative ease of the calculations involved. (See, e.g. [KM], [MST], [FS1].)

However, a second impetus for the Seiberg-Witten revolution comes directly from the observation of an intimate relationship between the Seiberg-Witten invariants and symplectic geometry. This last observation has brought symplectic geometry into the 4-manifold picture, with surprising gains both for 4-manifold topology and for symplectic geometry.

As remarked in [T1], a symplectic manifold has a natural orientation as does the line $\det^+$. Furthermore, there is a canonical identification of the set $\mathcal{S}$ with $H^2(X;\mathbb{Z})$. Thus, on a symplectic 4-manifold, SW can be viewed as a map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, or, more generally, from $H^2(X;\mathbb{Z})$ to $\Lambda^* H^1(X;\mathbb{Z})$. Meanwhile, a compact symplectic 4-manifold has a second natural map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, its Gromov invariant, Gr. The map Gr also extends on a $b^1 > 0$ symplectic 4-manifold to a map from $H^2(X;\mathbb{Z})$ into $\Lambda^* H^1(X;\mathbb{Z})$; the extension is sometimes called the Gromov-Witten invariant, but it will be denoted here by Gr as well. In either guise, Gr, assigns to a class e a certain weighted count of compact, symplectic submanifolds whose fundamental class is Poincare dual to e.

The Gromov invariant was introduced initially by Gromov in [Gr] and then generalized by Witten [W2] and Ruan [Ru]. See also [T2] and [T5]. (Note that Gr here does not count maps from a fixed complex curve. It differs in this fundamental sense from the Gromov-Witten invariant introduced in [W2].) The precise definitions of SW and Gr are provided in the first section of this paper.

Here is the main theorem which relates SW to Gr:

Theorem 1. *Let X be a compact, symplectic manifold with $b^{2+} > 1$. Use the symplectic structure to orient X and the line* $\det^+$; *and use the symplectic structure to define* SW *as a map from $H^2(X;\mathbb{Z})$ to $\Lambda^* H^1(X;\mathbb{Z})$. Also, use the symplectic structure to define* Gr: $H^2(X;\mathbb{Z}) \to \Lambda^* H^1(X;\mathbb{Z})$. *Then* SW=Gr.

Theorem 1 is proved in [T5].

The equivalence between the Gromov invariant and the original SW map into Z was announced by the author in [T1]. The proof of Theorem 1 can be divided into three main parts. The first part explains how a non-zero Seiberg-Witten invariant implies the existence of symplectic submanifolds. The second part explains how a symplectic submanifold can be used to construct a solution to a version of the Seiberg-Witten equations. The final part compares the counting procedures for the two invariants. The first and second parts of the proof can be found in [T3] and [T4], respectively and the final part (together with an overview of the whole strategy) is in [T5]. Some of the early applications of Theorem 1 are also described in [Kol].

A restricted version of Theorem 1 holds in the case when $b^{2+} = 1$. Here, a fundamental complication is that the Seiberg-Witten invariant depends on more than the differentiable structure. This is to say that there is a dependence on a "so called" choice of chamber. However, the symplectic form selects out a unique chamber, and with this understood, one has:

Theorem 2. *Let X be a compact, oriented 4-manifold with $b^{2+} = 1$ and with a symplectic form. Then the symplectic form canonically defines a chamber in which the equivalence* SW = Gr *holds for classes $e \in H^2(X;\mathbb{Z})$ which obey $\langle e, s\rangle \geq -1$ when ever $s \in H^2(X;\mathbb{Z})$ is represented by an embedded, symplectic sphere with self-intersection number -1.*

(Here, $\langle , \rangle$ denotes the pairing between cohomology and homology.) Theorem 2 is also proved in [T5].

By the way, when X is a $b^{2+} = 1$ symplectic manifold and $e \in H^2(X;\mathbb{Z})$ is a class for which the conditions of Theorem 2 do not hold, it is still the case that the Seiberg-Witten invariant SW(e) still counts pseudo-holomorphic subvarieties of some sort; Li and Liu [LL] have very recently sorted out the details. See also Theorem 1.6 below. (The Gromov invariant as defined below is not the correct symplectic invariant for such e since the subspaces involved can have singularities.)

The remainder of this essentially expository article is devoted to three related topics, one for each numbered section. The first topic is Theorems 1 and 2 and related results about the Seiberg-Witten invariants on a symplectic manifold. To be more precise, the first section below starts with a summary of the definitions of both the Seiberg-Witten invariant and the Gromov-Witten invariant. This first section also lists other special properties of the Seiberg-Witten invariants on symplectic manifolds. (See Theorems 1.6-1.8, below.) And, the section ends with an instructive example.

The second topic concerns the role of symplectic manifolds in the broader arena of 4-dimensional differential topology. And, the final topic concerns a possible geometric interpretation of the Seiberg-Witten invariants in the non-symplectic case. In particular, the final section contains, for the most part, speculations about future avenues of research.

1 The Seiberg-Witten and the Gromov-Witten invariants.

The purpose of this first section is to give a precise definition of the Seiberg-Witten invariants and also the Gromov invariants for a symplectic 4-manifold. The former is considered in Subsections 1a-c, and the latter in Subsections 1d,e. Then, Subsection 1f describes the basic geometric underpinning of the SW= Gr theorems and Subsection 1g summarizes some of other properties of SW on a symplectic manifold. The final subsections here discuss interpretations, applications and examples of the SW=Gr theorems.

a) The Seiberg-Witten equations.

The Seiberg-Witten equations were first introduce by Seiberg and Witten in [SW1], and [SW2] and see also [W1]. A purely mathematical approach to these equations was first taken in [KM]. The book by Morgan [Mor] is a more complete reference, as is the exposition by Kotschick, Kronheimer and Mrowka [KKM].

In this subsection, X is a compact, connected, oriented, 4-dimensional manifold. Let $b^1 = \dim(H^1(X))$ denote the first Betti number of X and let b^{2+} denote the dimension of a maximal subspace, $H^{2+}(X;\mathbb{R}) \subset H^2(X)$ where the cup product form is positive.

Fix a smooth Riemannian metric on X. The metric defines the principle SO(4) bundle of orthornormal frames, Fr$\to X$. Of the various associated bundles to this frame bundle, two in particular play central roles. These are the bundles Λ_+ of self-dual 2-forms and Λ_- of anti-self dual 2-forms. Note that $\Lambda^2TX \approx \Lambda_+ \oplus \Lambda_-$.

By definition, a Spin$^{\mathbb{C}}$ structure on X is an equivalence class of lifts of Fr to a principal Spin$^{\mathbb{C}}(4)$ bundle $F \to X$. In this regard, recall that the group Spin$^{\mathbb{C}}(4)$ is the group $(SU(2) \times SU(2) \times U(1))/\{\pm 1\}$, this being a central extension of $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$ by the circle $U(1)$. The homomorphism Spin$^{\mathbb{C}} \to (SU(2) \times SU(2))/\{\pm 1\}$ simply forgets the factor of $U(1)$.

A Spin$^{\mathbb{C}}$ lift F of Fr has two canoncial associated $\mathbb{C}^2$ bundles, $S_\pm \to$

X which are defined using the two evident homomorphisms of $\text{Spin}^{\mathbb{C}}$ to $U(2) = (SU(2) \times U(1))/\{\pm 1\}$. Note that S_+ is distinguished by the fact that the projective bundle is the unit 2-sphere bundle in Λ_+. There is, of course, an analagous relationship between S_- and Λ_-.

With the preceding understood, the original version of Seiberg and Witten's equations can now be defined. These are equations for a pair (A, ψ), where A is a connection on $\det(S_+)$ and where ψ is a section of S_+. The equations read:

$$\begin{aligned} D_A \psi &= 0, \\ P_+ F_A &= \frac{1}{4}\tau(\psi \oplus \psi^*) + \mu. \end{aligned} \tag{1.1}$$

In the first line above, D_A is the Dirac operator, a first order differential operator which maps sections of S_+ to sections of S_-. This D_A is defined as the composition of Clifford multiplication (a homomorphism from $S_+ \otimes T^*X$ to S_-) with covariant differentiation using the connection on S_+ which comes from the Levi-Civita connection on Fr and the connection A on $\det(S_+)$. In the second line of (1.1), P_+ denotes the orthogonal projection from $\Lambda^2 T^*X$ to Λ_+ and F_A denotes the curvature 2-form of A. Meanwhile, τ is the adjoint of the Clifford multiplication endomorphism from $\Lambda_+ \otimes \mathbb{C}$ into $\text{End}(S_+)$ and μ is a fixed, imaginary valued, anti-self dual 2-form on X. Any choice for μ will do.

There is a natural action of the group of smooth maps from X to $U(1)$ on the set of solutions to (1.1). The action sends a map g and a pair (A, ψ) to $(A + 2g dg^{-1}, g\psi)$. Use $\mathcal{M}$ to denote the set of orbits under this group action. Typically, notational distinctions will not be made between a pair (A, ψ) and its orbit in $\mathcal{M}$.

Topologize $\mathcal{M}$ as follows: First, introduce the manifold $\text{Conn}(\det(S_+))$ of Hermitian connections on $\det(S_+)$. This is an affine Frechet manifold modelled on $i \cdot \Omega^1$. Here, Ω^1 denotes the vector space of smooth 1-forms on X. With $\text{Conn}(\det(S_+))$ understood, introduce the space $\text{Conn}(\det(S_+)) \times C^\infty(S_+)$. The group $C^\infty(X; S^1)$ acts smoothly on the latter (as indicated above), and the space of orbits of this group action, $(\text{Conn}(\det(S_+) \times C^\infty(S_+))/C^\infty(X; S^1)$, is given the quotient topology. The space $\mathcal{M}$ sits in this quotient, and the implicit topology on $\mathcal{M}$ is the subspace topology inherited from the orbit space $(\text{Conn}(\det(S_+) \times C^\infty(S_+))/C^\infty(X; S^1)$.

Here are some basic properties of $\mathcal{M}$ (see, [W1] or [KM], [Mor], [KKM]):

- $\mathcal{M}$ is always compact.

- If $b^{2+} > 0$, then there is a Baire set of $\mathcal{U} \subset C^\infty(X; i\Lambda_+))$ of choices for μ in (1.1) whose corresponding $\mathcal{M}$ has the structure of a smooth manifold of dimension

$$2d = -\frac{1}{4}(2\chi + 3\tau) + \frac{1}{4}c_1 \bullet c_1.$$

Here, χ is the Euler characteristic of X and τ is the signature of X. Also, "$\bullet$" signifies the pairing on $H^2(X;\mathbb{Z})$ which is cup product composed with evaluation in the fundamental class. Furthermore, when $\mu \in \mathcal{U}$, then

a) There are no points in $\mathcal{M}$ where the corresponding ψ is zero.

b) $\mathcal{M}$ is orientable, and an orientation of $\det^+$ canonically orients $\mathcal{M}$.

c) The subspace of orbits $(p, (A, \psi)) \in (\det(S_+) \times \text{Conn}(\det(S_+)) \times C^\infty(S_+))/C^\infty(X; S^1)$ where $(A, \psi) \in \mathcal{M}$ naturally defines a smooth, principal S^1 bundle $\mathcal{E} \to X \times \mathcal{M}$.

(1.2)

A Baire set is a countable intersection of open and dense sets and so is dense. The Baire set in question is characterized by the condition that a certain family of first order, elliptic differential operators that is parameterized by the points in $\mathcal{M}$ has, at each point in $\mathcal{M}$, trivial cokernel.

Here are some additional comments about (1.2):

- The number $2d$ in (1.2) can be even or odd. Its parity is the same as that of $\frac{1}{2}(\chi + \tau) = 1 - b^1 + b^{2+}$.

- Equation (1.2) implies the following: When $d < 0$ and $\mu \in \mathcal{U}$, then $\mathcal{M} = \varnothing$. This is because there are no negative dimensional manifolds.

- In the case $d = 0$ and $\mu \in \mathcal{U}$, then $\mathcal{M}$ consists of a finite set of points. In this case, an orientation on $\mathcal{M}$ simply assigns either $+1$ or -1 to each point. (This is because $H_0(\text{point}; \mathbb{Z})$ already has a canonical generator, which is the point itself. The orientation assigns a fundamental class which is either the point, or - the point.)

- Let $c_1(\mathcal{E})$ denote the first Chern class of the principal S^1 bundle $\mathcal{E} \to X \times \mathcal{M}$. Then slant product with $c_1(\mathcal{E})$ defines a map, $\phi : H_*(X;\mathbb{Z}) \to H^{2-*}(\mathcal{M};\mathbb{Z})$.

(1.3)

b) The Seiberg-Witten invariant.

Let $\mathcal{S}$ denote the set of SpinC structures on X. Although $\mathcal{S}$ requires a choice of Riemannian metric for its definition, there is a natural identification between such sets defined by any two metrics. (Remember that the space of metrics on X is convex.) Thus, one can speak unambiguously about $\mathcal{S}$ with out reference to a particular metric. Note that $\mathcal{S}$ is an affine space modelled on $H^2(X;\mathbb{Z})$.

Likewise, the definition of SW requires a choice of Riemannian metric; and it also requires a choice of perturbing form μ in the set $\mathcal{U}$ of (1.2). Here is the definition of SW:

Definition 1.1. Fix an orientation for the line $\det^+$. Fix a Riemannian metric on X and the fix a $\mathrm{Spin}^{\mathbb{C}}$ structure in $\mathcal{S}$. Also, fix $\mu \in \mathcal{U}$ so that the conclusions of (1.2) and (1.3) are valid. Let d be as defined in (1.2). Then, the value of $\mathrm{SW} \in \Lambda^* H^1(X;\mathbb{Z})$ on the given $\mathrm{Spin}^{\mathbb{C}}$ structure is defined as follows:

- If $d < 0$, then SW$= 0$.

- If $d = 0$, then $\mathcal{M}$ is a finite set of points and the chosen orientation for $\det^+$ defines a map $\varepsilon : \mathcal{M} \to \{\pm 1\}$. With this understood, then
$$\mathrm{SW} \equiv \Sigma_{\Xi \in \mathcal{M}}\, \varepsilon(\Xi) \tag{1.4}$$
which is an element in the $\mathbb{Z}$ summand of $\Lambda^* H^1$.

- In general, $\mathrm{SW} \in \mathbb{Z} \oplus H^1 \oplus \cdots \oplus \Lambda^{2d} H^1$ with non-zero projection in $\Lambda^p H^1$ only if p has the same parity $1 - b^1 + b^{2+}$. In this case, SW is defined by its values on the set of decomposable elements in $\Lambda^p(H_1(X;\mathbb{Z})/\,\mathrm{Torsion})$; and
$$\mathrm{SW}(\gamma_1 \wedge \cdots \wedge \gamma_p) = \int_{\mathcal{M}} \phi(\gamma_1) \wedge \cdots \wedge \phi(\gamma_p) \wedge \phi(*)^{d-p/2}, \tag{1.5}$$
where $*$ is the class of a point generating $H_0(X)$.

The next proposition asserts that the apparent dependence of SW on the choice of metric and μ is spurious when $b^{2+} > 1$.

Proposition 1.2. *Let X be a compact, connected, oriented 4-manifold with $b^{2+} > 1$. Then the value of* SW *is independent of the choice* *of Riemannian metric and form* μ. *In fact,* SW *depends only on the diffeomorphism type of* X. *Furthermore,* SW *pulls back naturally under orientation preserving diffeomorphisms. This is to say that if* $\varphi : X \to X'$ *is a smooth, orientation preserving diffeomorphism, then* $SW_X(\varphi^*\eta) = \varphi^* SW_{X'}(\eta)$. *Finally,* SW *changes sign when the orientation of the line* $\det^+$ *is switched.*

(Note that $\mathrm{Spin}^{\mathbb{C}}$ structures pull-back under diffeomorphisms because metrics do.)

See, e.g. [Mor] or [KKM] for a proof of this proposition.

The preceding proposition does not hold in general in the case where the 4-manifold X has $b^{2+} = 1$. However, the failure of this proposition can be readily analyzed, and the results are summarized in Proposition 1.3, below. To state the proposition precisely, it is convenient to make a short digression to consider some special features of $b^{2+} = 1$ manifolds.

To begin the digression, introduce $\mathrm{Met}(X)$ to denote the Frechet space of smooth, Riemannian metrics on X. Given a metric g on X, let ω_g denote the unique (up to multiplication by $\mathbb{R}^*$), non-trivial, self-dual, harmonic 2-form on X. With ω_g understood, then each $c \in H^2(X; \mathbb{Z})$ defines a "wall" in $\mathrm{Met}(X) \times i \cdot \Omega^{2+}$ whose elements consist of pairs (g, μ) where $2\pi \cdot [\omega_g] \bullet c = i \cdot \int_X \omega_g \wedge \mu$. The wall divides $\mathrm{Met}(X) \times i \cdot \Omega^{2+}$ into two open sets, each of which is called a "c-chamber".

With the preceding, then Proposition 1.2 has the following $b^{2+} = 1$ version:

Proposition 1.3. *Let* X *be a compact, connected, oriented 4-manifold with* $b^{2+} = 1$. *Let* s *be a* $\mathrm{Spin}^{\mathbb{C}}$ *structure on* X. *Then the value of* $\mathrm{SW}(s) \in \Lambda^* H^1(X; \mathbb{Z})$ *is constant on any* $c = c_1(\det(S_+))$ *chamber in* $\mathrm{Met}(X) \times i \cdot \Omega^{2+}$.

See, e.g. [Mor] or [KKM] for a proof. However, the point is that the arguments for Proposition 1.2 work on an open set in $\mathrm{Met}(X) \times i \cdot \Omega^{2+}$ where the corresponding $\mathcal{M}$ contains no elements where the corresponding ψ vanishes identically. Indeed, the count for SW can change along a path in $\mathrm{Met}(X) \in i \cdot \Omega^{2+}$ only when the path intersects elements in $\mathcal{M}$ where the corresponding ψ vanishes identically. And, such elements occur if and only if (g, μ) lies in the wall. The change in SW as the wall is crossed can be computed. See [KM], [LL2], [OO].

c) The Seiberg-Witten invariants on symplectic manifolds.

As remarked in the introduction, a symplectic 4-manifold has a natural orientation, a natural orientation for the line $\det^+$ and a natural identification between $\mathcal{S}$ and $H^2(X;\mathbb{Z})$. The introduction also asserted that a symplectic manifold with $b^{2+} = 1$ also has a natural chamber. The purpose of this subsection is to explain these assertions.

The orientation of X. A symplectic 4-manifold is, by definition, a pair (X, ω), where X is a smooth 4-manifold and where ω is a closed 2-form on X with $\omega \wedge \omega$ nowhere zero. (The characteristic number $\frac{1}{2}(\chi + \tau) = 1 - b^1 + b^{2+}$ must be even for X to admit a symplectic form.) Because $\omega \wedge \omega$ is nowhere zero, this form orients X. This is the orientation that the introduction referred to. It will be assumed throughout.

The orientation of $\det^+$. The description of the orientation for the line $\det^+$ is conveniently divided into five steps.

Step 1. A choice of orientation for $\det_+$ is equivalent to a choice of orientation for the virtual vector space $H^1(X;\mathbb{R}) - H^0(X;\mathbb{R}) - H^{2+}(X;\mathbb{R})$. After a metric on X is chosen, the latter can be viewed using Hodge theory as the formal difference between the kernel and the cokernel of the operator $\delta_0 = (P_+d, d^*) : \Omega^1 \to \Omega^0 \oplus \Omega^{2+}$. Here, Ω^1 is the space of smooth 1-forms, Ω^0 is the space of smooth functions and Ω^{2+} is the space of smooth, self dual 2-forms.

Step 2. Every symplectic manifold admits almost complex structures, endomorphisms J of TX with square -1. As noted by Gromov [Gr], one can find almost complex structures with the property that the bilinear form

$$g = \omega(\cdot, J(\cdot)) \tag{1.6}$$

defines a Riemannian metric on TX. Such a J will be called *ω-compatible.*

The almost complex structure J decomposes $TX \otimes \mathbb{C} \approx T_{1,0} \oplus T_{0,1}$ into a sum of complex 2-plane bundles such that J has eigenvalue i on the former and $-i$ on the latter. The complexified cotangent bundle decomposes analogously as $T^{1,0} \oplus T^{0,1}$.

Thus, the endomorphism J acts, by definition on the domain of the operator δ_0.

Step 3. If the metric g is chosen as in (1.6), then there is also a natural almost complex structure (call it J_R) which acts on the range of δ_0. The latter is induced from a square -1 endomorphism (also called J_R) on the vector bundle $\varepsilon_{\mathbb{R}} \oplus \Lambda_+$ whose sections define δ_0's range. Here,

$\varepsilon_{\mathbb{R}} \to X$ denotes the product bundle $X \times \mathbb{R}$. Likewise, $\varepsilon_{\mathbb{C}}$, below, will denote the product complex line bundle.

To define J_R, remark first that the metric in (1.6) splits $\Lambda^2 T^*X$ as $\Lambda_+ \oplus \Lambda_-$. The form ω is self dual with respect to this splitting and has norm $\sqrt{2}$ everywhere. Conversely, if g is any metric for which ω is self-dual and has norm $\sqrt{2}$, then $J \equiv g^{-1}\omega$ defines an almost complex structure J on TX such that (1.6) holds.

Note that J induces an endomorphism of $\Lambda^2 T^*X$ with square 1 which preserves Λ_+. The $+1$ eigenspace of this endomorphism on Λ_+ is the span of ω. The orthogonal compliment is the -1 eigenspace. The latter is an oriented, 2-plane bundle over X which is the underlying real bundle of the complex line bundle $K^{-1} = \Lambda^2 T^{0,1}$.

With the preceding understood, view $\varepsilon_{\mathbb{R}} \oplus \Lambda_+$ as a complex 2-plane bundle by writing the latter as $\varepsilon_{\mathbb{C}} \oplus K^{-1}$, where $x + y \cdot \omega \in \varepsilon_{\mathbb{R}} \oplus \Lambda_+$ is identified with $x + \sqrt{-1} \cdot y \in \varepsilon_{\mathbb{C}}$. Multiplication by $\sqrt{-1}$ on $\varepsilon_{\mathbb{C}} \oplus K^{-1}$ defines the endomophism J_R on $\varepsilon_{\mathbb{R}} \oplus \Lambda_+$.

Step 4. In general, $\delta_0 J - J_R \delta_0 \neq 0$. However, this difference is always a zero'th order operator. (The symbol of δ_0 intertwines J with J_R; and δ_0 itself intertwines J with J_R when J is an integrable almost complex structure.)

The fact that $\delta_0 J - J_R \delta_0$ is zero'th order implies that there is a relatively compact perturbation of δ_0 which does intertwine J and J_R. For example, $\delta_1 = 2^{-1} \cdot (\delta_0 - J_R \cdot \delta_0 \cdot J)$ has this property.

Since δ_1 differs from δ_0 by a zero'th order operator, both its kernel and cokernel are finite dimensional. Furthermore, because δ_1 intertwines J with J_R, its kernel and cokernel have natural structures as complex vector spaces. And, since complex vector spaces have canonical orientations, the virtual vector space kernel(δ_1) $-$ cokernel(δ_1) has a canonical orientation.

Step 5. The complex orientation for kernel(δ_1) $-$ cokernel(δ_1) canonically orients the line $\det^+ =$ kernel(δ_0) $-$ cokernel(δ_0). The argument here is standard K-theory since the family of operator $\{\delta_t = t \cdot \delta_0 + (1-t) \cdot \delta_1\}_{t \in [0,1]}$ defines a continuous map of Fredholm operators with respect to appropriate Hilbert space completions of Ω^1 and $\Omega^0 \oplus \Omega^{2+}$. (The Sobolev spaces L^2_1 for the range and L^2 for the domain will suffice.) The point is that the association of the virtual vector space kernel$(\delta_t)-$ cokernel(δ_t) to $t \in [0,1]$ defines an element in the real K-theory of the interval (see the Appendix in [At].) Since the interval is contractible, this element is trivial. In particular, it has vanishing first Stieffel-Whitney class, so it is orientable and an orientation at $t = 1$ induces one at $t = 0$.

Note that the purpose of orienting the line det+ is to obtain a reasonably canonical orientation for the moduli space $\mathcal{M}$. In this regard, the symplectic orientation of $\det^+$ induces an orientation on $\mathcal{M}$ which is described directly in Section 4.

The identification of S with $H^2(X;\mathbb{Z})$. As remarked, the set S has the natural structure of an affine space modelled on $H^2(X;\mathbb{Z})$. This implies that the identification in question arises immediately with the specification of a "canonical" Spin$^{\mathbb{C}}$ structure. And, as observed in [T1], there is a canonical Spin$^{\mathbb{C}}$ structure on a symplectic manifold.

With the metric chosen from an ω-compatible J, the canonical Spin$^{\mathbb{C}}$ structure is characterized by the identifications

$$S_+ = \mathbb{I} \oplus K^{-1} \quad \text{and} \quad S_- = T^{0,1}, \tag{1.7}$$

where $K^{-1} = \Lambda^2 T^{0,1}$. This splitting of S_+ is defined as follows: Clifford multiplication defines an endomorphism from Λ_+ into the bundle of skew hermitian endomorphisms of S_+. With the preceding understood, the splitting of S_+ in (1.7) is the decomposition of S_+ into eigenbundles for the action of ω; here ω acts with eigenvalue $-2i$ on the trivial summand $\mathbb{I}$, and it acts with eigenvalue $+2i$ on the K^{-1} summand.

As just remarked, the identification in (1.5) of a canonical element in s identifies

$$\mathcal{S} \approx H^2(X;\mathbb{Z}). \tag{1.8}$$

Under this identification, a class $e \in H^2(X;\mathbb{Z})$ is sent to the Spin$^{\mathbb{C}}$ structure whose $S_\pm$ bundles are given by

$$S_+ = E \oplus (K^{-1} \otimes E) \quad \text{and} \quad S_- = T^{0,1} \otimes E, \tag{1.9}$$

where E is a complex line bundle whose first Chern class is isomorphic to e. Once again, this splitting of S_+ is into eigenbundles for the action of ω on S_+; and the convention is that the bundle where ω acts as $-2i$ is written first.

By the way, after the identification in (1.8), the dimension $2d$ of the Seiberg-Witten moduli space (as given in (1.2)) can be rewritten as follows: If $e \in H^2(X;\mathbb{Z})$ and if e is used to determine the Spin$^{\mathbb{C}}$ structure as in (1.9), then the formal dimension of the moduli space $\mathcal{M}$ is

$$2 \cdot d = e \bullet e - c \bullet e \tag{1.10}$$

where $c = c_1(K)$ with $K = \Lambda^2 T^{1,0}$. (The number $e \bullet e - c \bullet e$ is even because the class c is characteristic: Its mod 2 reduction is the second Stieffel-Whitney class of X.)

The symplectic chamber when $b^{2+} = 1$. Suppose now that X is a compact, oriented 4-manifold with $b^{2+} = 1$ and with a symplectic form ω. The latter defines a canonical c-chamber for each $c \in H^2(X;\mathbb{Z})$ by requiring μ in (1.1) to obey $i \cdot \int_X \omega \wedge \mu > 2\pi \cdot [\omega] \bullet c$. This last chamber will be called the "symplectic chamber".

Note, by the way, that two symplectic forms ω and ω' on X define the same chamber when $[\omega] \bullet [\omega'] > 0$. Thus, the symplectic chamber depends only on the form ω up to continuous deformations through closed forms ν with $[\nu] \bullet [\nu] > 0$.

In the subsequent discussions, the Seiberg-Witten invariant for such a pair (X, ω) will always denote the map SW from Proposition 1.3 as defined in the symplectic chamber. This $b^{2+} = 1$ definition of SW will be implicit in the subsequent discussions.

d) Pseudo-holomorphic submanifolds.

As noted in the introduction, the Gromov-Witten invariant is defined by counting (in a suitable sense) pseudo-holomorphic submanifolds on the symplectic manifold X. Thus, a more complete description of this invariant must start with a digression to discuss pseudo-holomorphic submanifolds. There are four parts to this discussion.

Part 1. The complex line bundle $K = \Lambda^2 T^{1,0}$ is called the canonical bundle. Note that the isomorphism class of K, and thus its first Chern class $c \in H^2(X;\mathbb{Z})$, are independent of the choice of ω-compatible almost complex structure J. Furthermore, this isomorphism class and also c are both unchanged if ω is changed through a continuous family of symplectic forms. (Note the sign convention here: $c \bullet [\omega] < 0$ when $X = \mathbb{CP}^2$.)

Part 2. A submanifold Σ in X is called pseudo-holomorphic when J preserves $T\Sigma$. It follows from the non-degeneracy of (1.6) that ω is non-degenerate on $T\Sigma$ and so orients Σ. Infact, J induces the structure of a complex curve on Σ. Then, the inclusion map of Σ into X is pseudo-holomorphic in the sense of Gromov [Gr].

If Σ is a connected and compact pseudo-holomorphic submanifold, then the genus of Σ is constrained by the adjunction formula to equal

$$\text{genus} = 1 + \frac{1}{2}(e \bullet e + c \bullet e) \tag{1.11}$$

where e is the Poincare' dual to the fundamental class $[\Sigma]$ of Σ.

Henceforth, all pseudo-holomorphic submanifolds in this Section 1 should be assumed to be compact unless stated to the contrary.

Part 3. Fix a pseudo-holomorphic submanifold Σ. Since J preserves $T\Sigma$, it must also preserve the orthogonal compliment in TX of $T\Sigma$.

The latter is the normal bundle, N, of Σ. Thus, N has a natural structure as a complex line bundle over Σ. The metric from TX defines a connection on $N \to \Sigma$, and thus endows N with a holomorphic structure as a bundle over the complex curve Σ. With this understood, one can introduce the associated d-bar operator, $\bar{\partial}$, to map sections of N to sections of $N \otimes T^{0,1}C$. (Here, $T^{0,1}C$ is the usual anti-holomorphic summand of $T^*C \otimes_{\mathbb{R}} \mathbb{C}$.)

One's first guess is that the kernel of $\bar{\partial}$ corresponds to the vector space of deformations of Σ in X which are pseudo-holomorphic to first order. However, this guess is wrong, in general. Rather, this vector space corresponds to the kernel of certain canonical, zero'th order deformation of $\bar{\partial}$. This deformation is an $\mathbb{R}$ linear operator, D, which also maps sections of N to sections of $N \otimes T^{0,1}C$ and which is defined as follows: The 1-jet off of Σ of the almost complex structure defines a pair (ν, μ) of section of $T^{0,1}C$ and $N^{\otimes 2} \otimes T^{0,1}C$. (See (2.3) in [T4].) Then

$$Dh = \bar{\partial}h + \nu \cdot h + \mu \cdot \bar{h}. \tag{1.12}$$

Part 4. Note that the index of D is given by the Riemann-Roch formula, which is to say that it equals $2d$ in (1.10) in the case where $e \in H^2(X;\mathbb{Z})$ is Poincare' dual to $[\Sigma]$. As the index is, by definition, the difference between the dimensions (over $\mathbb{R}$) of the kernel and the cokernel of D, a necessary condition for the triviality of cokernel(D) is that $2 \cdot d$ be non-negative. In general, this condition is not sufficient. However, if $2d \geq 0$, all pseudo-holomorphic submanifolds have trivial cokernel if the almost complex structure is chosen from a certain Baire subset of ω compatible almost complex structures. (This fact is proved in, e.g. [MS].)

e) The Gromov-Witten type invariants.

Fix $e \in H^2(X;\mathbb{Z})$. This subsection defines $\mathrm{Gr}(e) \in \Lambda^* H^1(X;\mathbb{Z})$. (The reader is referred to [T2] for the proofs of the assertions below.) The discussion here is broken into seven parts.

Part 1. Introduce the integer $d = d(e)$ as defined by (1.10). Then $\mathrm{Gr}(e)$ lies in the summand $\mathbb{Z} \oplus \Lambda^2 H^1 \oplus \cdots \oplus \Lambda^{2d} H^1$. Its projection into $\Lambda^{2p} H^1$ (for $0 \leq p \leq d$) can be determined by evaluating $\mathrm{Gr}(e)$ on a decomposable element in $\Lambda^{2p}(H^1(X;\mathbb{Z})/\,\mathrm{Torsion})$. Of course, when $p = 0$, the corresponding component of $\mathrm{Gr}(e)$ is simply an integer.

With the preceding understood, make the following choices when $d > 0$: First, choose $p \in \{0, \ldots, 2d\}$ and if $p > 0$, choose an element $\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p} H^1$. Then, for each $j \in \{1, \ldots, 2p\}$, choose an oriented, embedded circle in X to represent the class γ_j. To simplify notation, the chosen circle will be denoted by γ_j also. Make these choices of $\Gamma = \{\gamma_j\}_{1 \le j \le 2p}$ so that the distinct circles are disjoint. With Γ chosen, choose a set $\Omega \subset X$ of $d - p$ distinct points which miss each circle in Γ.

Part 2. Let $\mathcal{H} \equiv \mathcal{H}(e, J, \Gamma, \Omega)$ denote the set whose typical element is an unordered set, h, of pairs $\{(C_k, m_k)\}$ where each C_k is a compact, oriented, pseudo-holomorphic submanifold in X, and where the corresponding m_k is a positive integer. The elements in h should be constrained as follows:

1. For each k, introduce e_k to denote the Poincare' dual to $[C_k]$, and $d_k = e_k \bullet e_k - c \bullet e_k$. Require $d_k \ge 0$.
2. Require that $m_k = 1$ unless $d_k = 0$ and the genus of C_k is also 0. (Thus, C_k is a torus with trivial normal bundle.)
3. Require that $\Sigma_k m_k e_k = e$.
4. There is a partition $\Gamma = \cup_k \Gamma_k$, where each Γ_k contains some even number $2 \cdot p_k$ elements with $0 \le p_k \le d_k$. Furthermore, C_k intersects precisely once each $\gamma \in \Gamma_k$; and no $\gamma \in \Gamma_k$ is tangent to C_k at their intersection point. Moreover, C_k has empty intersection with the elements of $\Gamma - \Gamma_k$.
5. Each C_k contains precisely $d_k - p_k$ points of Ω.
6. Require that the $C_k \cap C_{k'} = \varnothing$ when $k \ne k'$.

(1.13)

(The final condition implies that $e_k \bullet e_{k'} = 0$ when $k \ne k'$. And this implies that $\mathcal{H}$ is empty whenever d (from (1.10)) is negative.)

Part 3. Suppose that $h \in \mathcal{H}$, and that $(C_k, m_k) \in h$ is such that $d_k > 0$. This data can be used to define a real vector space V_k of dimension $2 \cdot d_k$ as follows: First of all, each $z \in C_k \cap \Omega$ contributes a summand $N|_z$ to V_k, where $N \to C_k$ is the normal bundle to C_k in X. Meanwhile, each $\gamma \in \Gamma_k$ contributes a real line summand to V_k; the latter being the line $N|_z / p(T\gamma|_z)$, where $z = \gamma \cap C_k$ and where $p : TX|_z \to N|_z$ is the tautological projection.

Note that N is naturally oriented, as is each $\gamma \in \Gamma_k$. This means that each of the summands of V_k has a natural orientation. Thus, V_k inherits an orientation with the choice of an ordering for the set Γ_k. For, this ordering gives the order of the oriented real line summands in

V_k. (The summands which are indexed by the points in $C_k \cap \Omega$ are each naturally complex, and so their order in V_k is immaterial.)

With V_k understood, note that any section α of the normal bundle N defines a tautological element in V_k by restricting α to the points in $C_k \cap \Omega$ and to the points where the elements of Γ_k intersect Ω. The preceding defines a tautological map from $C^\infty(N)$ to V_k whose restriction to the kernel of the operator D in (1.12) will be denoted by G_k.

Part 4. Here are some salient properties of the set $\mathcal{H}$:

- Each $h \in \mathcal{H}$ is a finite set.

- There is a Baire set $\mathcal{W}$ of triples (J, Γ, Ω) for which J is ω compatible and for which the corresponding set $\mathcal{H}$ is finite. Furthermore when $h \in \mathcal{H}$ and when $(C_k, m_k) \in h$, then

  a) The operator D in (1.12) has trivial cokernel.

  b) If $d_k > 0$, the homomorphism G_k : kernel$(D) \to V_k$ is an isomorphism.

  c) If $d_k = 0$ and $m_k > 1$, then the pull-back of D to any finite cover of the torus C_k also has trivial cokernel (and kernel).

(1.14)

These facts are proved in [T2], see also [Mc1].

Part 5. Assume now that the data (J, Γ, Ω) is chosen from the set $\mathcal{W}$ in (1.14). Let $h \in \mathcal{H}$ and let $(C, m) = (C_k, m_k) \in h$. The purpose of this part of the discussion is to associate to such a pair an integer, $r(C, m)$. There are three cases to consider.

If $m = 1$ and $d_k = 0$. Here, $r(C, 1) \in \{\pm 1\}$ and it counts (mod 2) the spectral flow for a path of zero'th order deformations of D which starts with D and ends with an operator $D_1 = \bar{\partial} + \nu' : C^\infty(C; N) \to C^\infty(C; N \otimes T^{0,1}C)$ whose kernel and cokernel are also trivial. The path $t \to D_t$ can be chosen so that

- The set of t where cokernel$(D_t) \neq \{0\}$ is a finite number, N.

- At such t where cokernel$(D_t) \neq \{0\}$, the dimension of this cokernel is 1.

- At such t where cokernel$(D_t) \neq \{0\}$, the restriction of the t-derivative of D_t to kernel(D_t) composes with projection onto cokernel(D_t) as an isomorphism.

(1.15)

(Because D' is $\mathbb{C}$-linear, the set of ν' where kernel$(D') \neq \{0\}$ is a codimension 2 variety in $C^\infty(C; T^{0,1}C)$. This insures that $r(C,1)$ depends only C and, in particular, not on the details of D''s deformation.)

If $m = 1$ and $d_k > 0$. The integer $r(C,1) \in \{\pm 1\}$ again. However, the definition in this case requires the choice of an ordering of the elements of Γ_k. As remarked above, the latter serves to orient the vector space V_k. Next, choose a continuous path $t \to D_t$ so that:

- For each $t \in [0,1]$, D_t is a zero order deformation of D.
- $D_0 = D$.
- D_t has trivial cokernel for all t.
- $D_1 = \bar{\partial} + \nu'$ is $\mathbb{C}$-linear.

(1.16)

With the preceding understood, the association of the kernel(D_t) to $t \in [0,1]$ defines a $2 \cdot d_k$ dimensional, real vector bundle over $[0,1]$. The fiber of this vector bundle over $t = 1$ is complex, so naturally oriented; and the latter induces an orientation of the kernel(D), the fiber over $t = 0$. With this orientation for kernel(D), the linear map G_k is then an isomorphism between two oriented vector spaces. Now define $r(C,1) = +1$ if G_k preserves orientation, and otherwise define $r(C,1) = -1$.

Note that $r(C,1)$ is independent of the choice of the path $\{D_t\}$, but it will change sign if the ordering of Γ_k is changed by a permutation with odd parity.

If $m > 1$. As noted above, this requires C to be a torus (and N to be topologically trivial). There are three distinct isomorphism classes of non-trivial real line bundles over C, and by tensoring (over $\mathbb{R}$) the range and domain of D with any one of these, one obtains a suite of 3 new operators. Agree to call any one of these a "twisted version" of D. Note that the index of D and any of its twisted versions is zero. This is because d_k is zero when C is a torus with trivial normal bundle.

With the preceding understood, the value of $r(C,m)$ depends only on the various possibilities for the mod(2) spectral flow for the operator

D and for its twisted versions. (Once again, the genericity assumptions on J are such as to insure that these spectral flows are well defined.) This is to say, that $r(C,m)$ depends only on the mod 2 spectral flow for D and on the number of D's twisted version which have non-trivial spectral flow. (And, of course, it depends on m.) In this regard, once m is fixed, there are eight possibilities for $r(C,m)$; and it is convenient to label the possibilities with a tag, $\pm k$, where the $\pm$ indicates whether the spectral flow for D is $+1$ or -1, and where $k \in \{0,1,2,3\}$ indicates the number of the twisted versions of D which have non-trivial spectral flow.

For a fixed tag, $\pm k$, it proves convenient to present the data $\{r(C,m)\}_{m=1,2,\ldots}$ with the help of a "generating function", $f_{\pm k}(t)$. This is to say that $f_{\pm k}$ is, by definition, that formal power series for which the coefficient of t^m is $r(C,m)$. This sort of presentation is convenient here only because $f_{\pm k}(t)$ is, in all cases, a fairly simple analytic function of t. Here are the eight generating functions:

- $f_{+0}(t) = \frac{1}{1-t}$.
- $f_{+1}(t) = 1 + t$.
- $f_{+2}(t) = \frac{1+t}{1+t^2}$.
- $f_{+3}(t) = \frac{(1+t)(1-t^2)}{1+t^2}$.
- $f_{-0}(t) = 1 - t$.
- $f_{-1}(t) = \frac{1}{1+t}$.
- $f_{-2}(t) = \frac{1+t^2}{1+t}$.
- $f_{-3}(t) = \frac{1+t^2}{(1+t)(1-t^2)}$.

(1.17)

End the digression.

Part 6. Suppose $e \in H^2(X;\mathbb{Z})$ has been chosen and suppose that $d = e \bullet e - c \bullet e \geq 0$. Let $p \in \{0,\ldots,d\}$ and let $\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p}H^1(X;\mathbb{Z})$. Fix $(J,\Gamma,\Omega) \in \mathcal{W}$ so that the conclusions of (1.14) hold. Then, let $h = \{(C_k, m_k)\} \in \mathcal{H}$. The preceding step defined an integer weight $r(C_k, m_k)$ for each $(C_k, m_k) \in h$ from the given data and from the choice of an ordering on the corresponding Γ_k. The purpose of this step is to use $\{r(C_k, m_k)\}$ to define an integer weight, $q(h)$, to the set h. The definition of $w(h)$ is simplest when $p = 0$, whence

$$q(h) = \Pi_k r(C_k, m_k). \tag{1.18}$$

In the case where $p > 0$, each $r(C_k, m_k)$ depends on the choice of an ordering for the corresponding Γ_k. This dependence is compensated for in the definition of $q(h)$ as follows: The chosen orderings of the Γ_k also induce an ordering of Γ which differs from the given labeling by a permutation, σ, of the set $\{1, \dots, 2p\}$. The latter has a parity, which will be denoted by $\varepsilon(\sigma) \in \{\pm 1\}$. Note that $\varepsilon(\sigma)$ is insensitive to the choice of ordering for $\{(C_k, m_k)\}$ as each Γ_k has an even number of elements.

With the preceding understood, associate the weight

$$q(h) = \varepsilon(\sigma) \cdot \Pi_k r(C_k, m_k) \tag{1.19}$$

to each $h \in \mathcal{H}$ in the case when $p > 0$. (Note that $q(h)$ in (1.19) is insensitive to the chosen orderings of each of the Γ_k's.)

Part 7. Here is the definition of Gr:

Definition 1.4. Define Gr: $H^2(X;\mathbb{Z}) \to \Lambda^* H^1(X;\mathbb{Z})$ as follows: Set $\mathrm{Gr}(0) = 1$. For $e \in H^2(X;\mathbb{Z}) - \{0\}$, set $d = e \bullet e - c \bullet e$. Then

- $\mathrm{Gr}(e) = 0$ if $d < 0$.
- If $d \geq 0$,

  a) Fix $J \in \mathcal{W}$, and use J to define $\mathcal{H} = \mathcal{H}(e, J, \varnothing, \varnothing)$. With $\mathcal{H}$ understood, define the projection of $\mathrm{Gr}(e)$ in the $\mathbb{Z}$ summand of $\Lambda^* H^1(X;\mathbb{Z})$ to equal $\Sigma_{h \in \mathcal{H}} q(h)$ where $w(h)$ is given by (1.18).

  b) Fix $p \in \{1, \dots, d\}$ and then fix $\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p}(H^1(X;\mathbb{Z})/$ Torsion). Then, choose $(J, \Gamma, \Omega) \in \mathcal{W}$ and use this data to define $\mathcal{H} = \mathcal{H}(e, J, \Gamma, \Omega)$. With $\mathcal{H}$ understood, define $\mathrm{Gr}(e)(\gamma_1 \wedge \cdots \gamma_{2p}) \equiv \Sigma_{h \in \mathcal{H}} q(h)$ where $q(h)$ is given by (1.19).

The following proposition describes the salient properties of the preceding definition:

Proposition 1.5. *Let (X, ω) be a pair consisting of a smooth, compact, connected 4-manifold X with a symplectic form ω. If $e \in H^2(X;\mathbb{Z})$, then the value of $Gr(e)$ as given in Definition 1.4 is independent of the precise choice for the data (J, Γ, Ω) and thus depends only on the symplectic form ω. Furthermore, $Gr(\cdot)$ is constant if ω is changed through a continuous path of symplectic forms. Finally, Gr behaves naturally with respect to diffeomorphisms of X in the following sense: Let $\varphi : X \to X$ be a diffeomorphism and let Gr_ω and $Gr_{\varphi^*\omega}$ denote the respective Gromov invariants as defined by ω and $\varphi^*\omega$. Then $Gr_{\varphi^*\omega}(\varphi^* e) = \varphi^*(Gr_\omega(e))$.*

A proof of this proposition can be found in [T2].

f) A geometric relationship between SW and Gr.

The relationship between SW and Gr in Theorems 1 and 2 is ultimately a consequence of a certain geometric property of the zero set of a component of the spinor ψ when a pair (A, ψ) solves a certain version of (1.1). To make this more precise, it is necessary to first rewrite (1.1) on a symplectic manifold X which exploits the decomposition in (1.9). This rewriting of (1.1) requires a preliminary, two part digression. Part 1 of the digression observes that the bundle K^{-1} comes equipped with a canonical connection (up to the action of $C^\infty(X; S^1)$ (see, e.g. [T6]). To define this canonical connection, remember first that for any fixed $\mathrm{Spin}^{\mathbb{C}}$ structure, the choice of a connection on $\det(S_+)$ and the Levi-Civita connection on the bundle Fr defines a connection on the $\mathrm{Spin}^{\mathbb{C}}$ lift F. Thus, the choice of a connection (say A) on $\det(S_+)$ gives a covariant derivative, ∇_A, on sections of S_+. Now consider the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure in (1.7). Restriction of ∇_A to a section of the trivial summand $\mathbb{I}$ and projection of the resulting covariant derivative onto $\mathbb{I} \otimes T^*X$ defines a covariant derivative ∇_A on the trivial complex line bundle. With the preceding understood, remark that there is a unique choice of connection A_0 (up to the afore-mentioned gauge equivalence) on $\det(S_+) = K^{-1}$ for which the corresponding covariant derivative on the trivial line bundle admits a non-trivial, covariantly constant section.

For Part 2 of the digression, consider the general $\mathrm{Spin}^{\mathbb{C}}$ structure in (1.7). Since $\det(S_+) = E^2 \otimes K^{-1}$, the choice of the connection A_0 on K^{-1} allows any connection A on $\det(S_+)$ to be written uniquely as

$$A = A_0 + 2a, \tag{1.20}$$

where a is a connection on the complex line bundle E. Thus, with A_0 chosen, the Seiberg-Witten equations in (1.1) can be thought of as equations for a pair (a, ψ) where a is a connection on E and where ψ is a section of S_+ in (1.9).

End the digression. With this reinterpretation of (1.1) understood, remark now that it proves useful to "renormalize" the form μ in (1.1) by writing

$$\mu = \frac{ir}{4}\omega + P_+F_{A_0} + i\mu_0. \tag{1.21}$$

Here, r can be any non-negative number and μ_0 can be any section of Λ_+. (In practice, think of μ_0 as being close to 0.) Furthermore, in the case where $r > 0$, it also proves useful to write

$$\psi = r^{1/2}(\alpha, \beta) \tag{1.22}$$

to correspond with the splitting in (1.9). Then, with the preceding understood, the Seiberg-Witten equations in (1.1) read

$$D_A(\alpha, \beta) = 0.$$
$$P_+F_a + \frac{ir}{8}(1 - |\alpha|^2 + |\beta|^2)\omega - \frac{r}{4}(\alpha\beta^* - \alpha^*\beta) - i\mu_0 = 0. \qquad (1.23)$$

Here, $\alpha\beta^*$ and $\alpha^*\beta$ are sections of K and K^{-1} where the latter are naturally identified as the orthogonal compliment of the span of ω in $\Lambda_+ \otimes \mathbb{C}$. (Note that this last equation differs from the analogous equations in [T1], [T3] and [T6] in that the β used here is $-i$ times that used in the previous papers. The insertion of this factor of $-i$ here avoids numerous factors of i later on.)

Rewriting (1.1) as in (1.23) realizes the Seiberg-Witten equations as equations for data $(a, (\alpha, \beta))$ where a is a connection on the line bundle E, α is a section of E, and β is a section of $K^{-1} \otimes E$.

End the digression. The following theorem provides the geometric underpinnings of the theorems in the introduction:

Theorem 1.6. *Let X be a compact, oriented, symplectic manifold. Fix an ω-compatible almost complex structure on X, and use the resulting metric to define the Seiberg-Witten equations. Fix $e \in H^2(X;\mathbb{Z})$ and use e to define the* Spin$^{\mathbb{C}}$ *structure in (1.9). Also, fix a finite (maybe empty) collection $\{\varsigma_k\}$ of closed subsets of X. Given $\varepsilon > 0$, and then given r sufficiently large, the following is true: If $(a, (\alpha, \beta))$ solves the r and $\mu_0 = 0$ version of (1.23) and has $\alpha^{-1}(0)$ intersect each ς_k, then there is a compact (not necessarily connected), complex curve C with a pseudo-holomorphic map $\varphi : C \to X$ with*

- *$\varphi_*[C]$ equal to the Poincare dual of e;*
- *$\varphi(C) \cap \varsigma_k \neq 0$ for each k.*
- $\sup_{x:\alpha(x)=0} \operatorname{dist}(x, \varphi(C)) + \sup_{x\in\varphi(C)} \operatorname{dist}(x, \alpha^{-1}(0)) < \varepsilon.$

(This theorem follows from Theorem 1.3 in [T3].)

g) Symplectic manifold constraints on SW.

Apart from Theorems 1 and 2 of the introduction and Theorem 1.6, above, the following summarizes the main results about the Seiberg-Witten invariants for a symplectic manifold X with $b^{2+} > 1$.

Theorem 1.7. *Let X be a compact 4-manifold with $b^{2+} > 1$ and with symplectic form ω. Then SW, as a map from $H^2(X;\mathbb{Z})$ to $\Lambda^*H^1(X;\mathbb{Z})$, has the following properties:*

- *If $SW(e) \neq 0$, then $e \bullet e = c \bullet e$ so d in (1.5) is zero.*
- *SW maps into the $\Lambda^0 = \mathbb{Z}$ summand of $\Lambda^* H^1(X;\mathbb{Z})$.*
- *$SW(0) = 1$ and $SW(c) = \pm 1$, where $c = c_1(K)$.*
- *$SW(e) = \pm\,\mathrm{SW}(c - e)$.*
- *If $SW(e) \neq 0$, then $0 \leq e \bullet [\omega] \leq c \bullet [\omega]$. Furthermore, the lower bound is strict except for $e = 0$, and the upper bound is strict except for $e = c$.*

(1.24)

Remark that third assertion is the main theorem in [T6]. (Note that an easier proof is had by using the analysis in [T3].) The fourth assertion follows from Witten's observation that the set $\mathcal{S}$ admits an involution which changes the Seiberg-Witten invariant by a sign, at most. The fifth assertion is proved in [T7], and the first assertion in [T3]. (The second assertion says that X has "simple type".) The second assertion follows from the first one.

In the case when $b^{2+} = 1$, similar arguments yield:

Theorem 1.8. *Let X be a compact, symplectic 4-manifold with $b^{2+} = 1$. Then the invariant SW obeys:*

- *$SW(0) = 1$.*
- *If $SW(e) \neq 0$, then $0 \leq e \bullet \omega$ with equality if and only if $e = 0$.*

h) An interpretation of the classes e with $\mathbf{SW}(e) \neq 0$.

There is a sense in which SW for symplectic X measures solely properties of the class c. Indeed, if e is any class with $\mathrm{Gr}(e) \neq 0$, then any connected, pseudo-holomorphic submanifold which contributes to the formula for $\mathrm{Gr}(e)$ also contributes to that for $\mathrm{Gr}(c)$. This comment is justified by the following observation: If e has $\mathrm{Gr}(e) \neq 0$, then the same is true for $c - e$ using the fourth point in (1.24). Thus, both $\mathcal{H}(e)$ and $\mathcal{H}(c - e)$, as defined in (1.14), are non-empty. Pick arbitrary elements $h \in \mathcal{H}(e)$, and let $h' \in \mathcal{H}(c - e)$. If there are no tori of square zero which appear both in h and h', then the union of these two sets is in $\mathcal{H}(c)$. More generally, let $\{C_a\}$ denote the list of tori which are shared between the sets h and h'. Let $h^0 \subset h$ denote the subset which is obtained by deleting any (C, m) where $C \in \{C_a\}$ is a shared torus. Define h'^0 analagously. Then $h^0 \cup h'^0 \cup \{(C_a, m_a + m'_a)\}$ is in $\mathcal{H}(c)$. This last point follows from the condition $e \bullet (c - e) = 0$ (due to the first point in (1.24)), since the latter implies that except for the sharing of tori, any representation of e by a union of disjoint, pseudo-holomorphic submanifolds is disjoint from any representation of $c - e$ by such a set. (Here, remember that distinct pseudo-holomorphic submanifolds intersect with locally positive intersection number. Thus, the only way e and $c - e$ can be have intersecting representatives is if e and $c - e$ have pseudo-holomorphic representatives which share some number of tori with self intersection zero and/or spheres of self-intersection -1. The case where e and $c - e$ share a sphere of self intersection -1 can be ruled out using the adjunction formula and the $d = 0$ condition.)

Note that the preceding picture was more or less conjectured by Witten in the case where X is Kahler.

i) Applications.

The main application to date of Theorems 1 and 2 and 1.7-1.8 has been to find certain kinds of pseudo-holomorphic or symplectic submanifolds in X. In particular, in the $b^{2+} > 1$ case, Theorems 1 and 1.7 imply that every class e with SW$\neq 0$ is Poincare' dual to the fundamental class of a symplectic submanifold of X. Furthermore, for generic choice of metric (for which ω is still self dual with norm $\sqrt{2}$), every class e with SW$\neq 0$ will be Poincare' dual to the fundamental class of a pseudo-holomorphic submanifold. For example, one corollary states that if X has $b^{2+} > 1$ and $c \bullet c < 0$, then $X = Y\#(-\mathbb{CP}^2)$, where Y is a symplectic manifold. (Note that in every case, this existence theorem holds for the class c.)

As noted in Theorem 2, when $b^{2+} = 1$, the Seiberg-Witten equations typically yield invariants of X with some additional structure chosen. In any event, Theorem 2 again produces (under the correct hypothesis) pseudo-holomorphic submanifolds in $b^{2+} = 1$ manifolds. Of particular interest are pseudo-holomorphic spheres with non-negative square, because the existence of such spheres can be used to determine X. The point is that spheres with non-negative square come in positive dimensional families, and thus sweep out "coordinates" on X. This surprising observation was first made in the symplectic context by Gromov [Gr]; and then McDuff [Mc2] considerably extended Gromov's original application. In any event, the game to date in the $b^2_+ = 1$ case has been to find pseudo-holomorphic spheres. This game has been played with much cleverness by Li and Liu [LL3], [Liu], and also by [OO], [Bi], [Mc3] to the extent that a great deal is now known about the classification of symplectic manifolds with $b^2_+ = 1$. McDuff and Salamon [MS2] have a nice review of the $b^2_+ = 1$ story.

By the way, Donaldson [D1] has a remarkable existence theorem for symplectic submanifolds of very high degree ($[\omega] \bullet e \gg 0$) on any symplectic manifold. Donaldson's existence theorem is proved by a very different sort of argument.

j) Examples from 3-manifolds.

Let M be a compact, oriented 3-manifold which admits a fibering $\phi : M \to S^1$. Then, one can choose a metric on M so that ϕ is a harmonic map. This implies that the ϕ-pull back, ν, of the standard volume form on the circle is a harmonic 1-form. That is, $d\nu = 0$ and also $d * \nu = 0$. Here, $*$ is the Hodge star operator on M. With the preceding understood, let $X = S^1 \times M$. Then X has a symplectic form, namely $\theta \wedge \nu + *\nu$, where θ is the pull-back to X of the standard volume form on S^1 via the projection on the first factor of S^1. Note that $b^2_+(X)$ is equal to $b^1(M)$.

Let $F \subset M$ denote a typical fiber of ϕ. The manifold M can be realized as the quotient of $[0,1] \times F$ by the equivalence relation which identifies $(0, x)$ with $(1, \varphi(x))$, where $\varphi : F \to F$ is a diffeomorphism. In this guise, the map φ is induced by the tautological projection from $[0,1] \times F$ to $[0,1]$. Let $A : H^1(F) \to H^1(F)$ denote the action of φ^*. The characteristic polynomial of A is defined to be $\det(1 - t \cdot A)$.

It turns out that the characteristic polynomial of A can be computed from the Seiberg-Witten invariants of X by as described by:

Proposition 1.9. *Let X be as described above. For each integer $n \geq 1$, let $sw_n \in \mathbb{Z}$ denote the sum of the Seiberg-Witten invariants from the set of* Spin$^{\mathbb{C}}$ *structures on X which have the property that $\langle c_1(\det(S_+)), F \rangle = n$. Let $sw_0 = 1$. Then*

$$\Sigma_n sw_n t^n = \frac{\det(t^{-1} - tA)}{(t^{-1} - t)^2}. \tag{1.25}$$

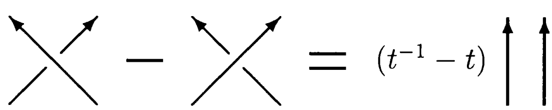
Remark that the right side of (1.25) is equal to $t^{-2g+2} \cdot \det(1 - t^2 \cdot A)/(1 - t^2)^2$ where $g = \text{genus}(F)$, and so defines a formal power series in t and t^{-1}.

(Meng-Taubes [MT] and also Salamon [Sa] have arrived at (1.25) independently, and by different techniques.)

Here are some explanatory remarks: In the case where $b^2_+(X) > 1$, there are only finitely many Spin$^{\mathbb{C}}$ structures with non-zero Seiberg-Witten invariants. This implies that the left hand sum is finite, and so $(t^{-1} - t)^2$ should divide $\det(t^{-1} - tA)$. That such is the case can be argued as follows: As remarked, $b^2_+(X) = b^1(M)$, and the latter is 1 more than the number of zero eigenvalues of the matrix $A - 1$. Furthermore, the number of such zero eigenvalues is always even because A preserves the symplectic cup product pairing on $H^1(F)$. Thus, if $b^2_+(X) > 1$, then $A - 1$ has at least two zero eigenvalues. And, if $A - 1$ has n zero eigenvalues, then $\det(t^{-1} - tA)$ is divisible by $(t^{-1} - t)^n$. (Note that the determinant in question is equal to t^{-g} times the characteristic polynomial of A.)

In the case when $b^2_+(X) = b^1(M) = 1$, the formula in (1.25) should be interpreted in the symplectic chamber.

By the way, examples of such fibering 3-manifolds with $b^1 = 1$ can be obtained from a so-called fibered knot $K \subset S^3$. Indeed, if K is any knot in S^3, take out a solid torus neighborhood of the knot, and then glue the latter back in so that the meridian and the zero framed longitude are switched. The result will be a compact, oriented 3-manifold $M = M_K$ with $b^1 = 1$. For fibered knots, $S^3 - K$ fibers over S^1, and for these knots, the manifold M_K will also fiber over S^1. In this case, the polynomial $\det(t^{-1} - tA)$ can be identified (up to sign) with a certain normalization of the Alexander polynomial of the knot, $\Delta_K(t)$. With this understood, then (1.25) becomes a special case of a general formula (valid for any knot in S^3) of the theorem of Meng and the author [MT] which is discussed in Section 3b, below. (The Alexander polynomial Δ_K which appears here is normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$. This version of Δ_K is determined [Co] by the skein relations



(1.26)

with the normalization that its value on the unknot is 1.)

Here is an outline of the proof of Proposition 1.9: The equivalence between SW and Gr turns the problem into one of counting pseudo-holomorphic submanifolds in the appropriate homology class. This process is simplified when the metric for X is taken to be the obvious product metric. In this case, the connected, pseudo-holomorphic submanifolds that are relevant to the computation are tori of square zero which have the form $S^1 \times \gamma$, where $\gamma \subset M$ is a closed, integral curve of the vector field on M which is metrically dual to the 1-form ν. Furthermore, for the calculation of sw_n, the curve γ should intersect the fiber F exactly n times. If the metric on M is chosen appropriately, then such a curve γ is determined by a fixed point of the diffeomorphism $\varphi^n : F \to F$; thus the problem comes down to counting fixed points of φ^n with appropriate weights. To see the results of this count, it proves convenient to introduce the set $S(n) \subset F$ of points which are fixed by φ^n, but not by φ^m for any positive $m < n$. From the SW=Gr equivalence, the left hand side of (1.14) is equal to

$$Q(t) = t^{-2g+2}\Pi_n\Pi_{q\in S(n)}(1 - \varepsilon'(q) \cdot t^{2n})^{-\varepsilon(q)/n} \tag{1.27}$$

Here, $\varepsilon(q)$ is the sign of the determinant of the identity minus the differential of φ^n at q and $\varepsilon'(q) = 1$ if $\varepsilon = -1$, but ε' can be either 1 or -1 when $\varepsilon = 1$. (This ε' is 1 if ε also gives the sign of the determinant of the identity minus the differential of φ^{2n}, otherwise ε' is -1.) Note that the factor of $1/n$ in the exponent arises for the following reason: A point $x \in S(n)$ determines the same γ as does $\varphi(x)$, and so counting fixed points in $S(n)$ will over count by a factor of n the number of embedded, pseudo-holomorphic tori in X which have intersection number n with the fiber F. The factor of t^{-2g+2} arises because the canonical class for the symplectic structure evaluates as $2g - 2$ on the fiber F. Here, one must remember that when the S_+ bundle of the $\mathrm{Spin}^{\mathbb{C}}$ structure is given by (1.9), then $c_1(\det(S_+)) = 2e - c$, where $e = c_1(E)$ and where c is the first Chern class of the canonical bundle.

With (1.27) understood, the proof of Proposition 1.9 is completed by identifying (1.27) with $\det(t^{-1} - tA)/(t^{-1} - t)^2$. The argument here (as shown to the author by R. Bott) uses the Lefschetz fixed point formula on F. Indeed, consider that the natural log of Q has the expansion:

$$\begin{aligned}\ln(Q) + (2g-2)\cdot\ln(t) &= \Sigma_{n\geq 0}\Sigma_{q\in S(n)}(-\varepsilon n^{-1}\ln(1-\varepsilon' t^{2n})) \\ &= -\Sigma_{n\geq 0}\Sigma_{q\in S(n)}\Sigma_{k\geq 0}\ \varepsilon\cdot(\varepsilon')^k\frac{t^{2nk}}{nk}.\end{aligned} \tag{1.28}$$

Now, rewrite this last expression by reordering the summation and introducing $m = nk$ to obtain

$$\ln(Q) + (2g-2)\cdot\ln(t) = -\Sigma_{m\geq 0}\ a_m \cdot \frac{t^{2m}}{m}, \tag{1.29}$$

where $a_m = \Sigma_{k:k \text{ is a root of } m}(\Sigma_{q\in S(k)}\ \varepsilon(q)\cdot\varepsilon'(q)^k)$. This sum for a_m is simply the sum over all fixed points of φ^m weighted by the sign of the determinant of the differential of φ^m. And, according the the Lefschetz fixed point theorem, the latter is nothing but $2 - \mathrm{trace}(A^m)$, the Lefschetz number of φ^m. Thus,

$$\begin{aligned}\ln(Q) + (2g-2)\cdot\ln(t) &= -2\cdot\Sigma_{m\geq 0}\frac{t^{2m}}{m} + \Sigma_{m\geq 0}\mathrm{trace}(A^m)\frac{t^{2m}}{m} \\ &= -\ln((1-t^2)^2) + \ln(\det(1-t^2A)).\end{aligned} \tag{1.30}$$

(The author is indebted to E. Ionel and T. Parker for pointing out an omission from (1.27) and (1.28) that was present in an earlier version of this article. Note as well that Ionel and Parker can compute Gromov invariant of $S^1 \times M$ directly, see [IP2].)

2 The smooth classification of 4-manifolds.

Up until recently, one could conjecture that a compact, orientable, simply connected 4-manifold was a connect sum of parts which were symplectic with one or the other orientation. This conjecture was proved false in the summer of 1996 by Zoltan Szabo [Sz] who exhibited indecomposable manifolds whose Seiberg-Witten invariants (for either orientation) violate the constraints in Theorem 1.7 or 1.8. (They are proved indecomposable by virtue of the fact that their Seiberg-Witten invariants for one orientation are non-zero.) Subsequently, Fintushel and Stern [FS2] have found a vast array of such examples. For example, Fintushel and Stern find infinitely many manifolds with the homotopy type of the K3 surface which are not symplectic and not mutually diffeomorphic.

Prior to Szabo's constructions, Shugang Wang [Sw] (see also [Ko2]) constructed (with the help of the Sieberg-Witten invariants) non-symplectic but irreducible 4-manifolds with $\pi_1 = \mathbb{Z}/2$. Wang's manifolds are quotients of algebraic surfaces by anti-holomorphic involutions. (On the other hand, Gompf [Go1] has shown that every finitely presentable group is the fundamental group of a symplectic manifold.)

Note that the issue in all of the above is the C^∞ classification of X. Mike Freedman [Fr], [FQ], has completed the classification up to homeomorphism for simply connected, topological 4-manifolds. The topological classification data consists simply of a triple $(V, Q, \pm 1)$ up to equivalence, where V is a finite $\mathbb{Z}$-module (identified with $H_2(X;\mathbb{Z})$) and $Q : V \otimes V \to \mathbb{Z}$ is a unimodular, symmetric pairing (identified with the intersection pairing on $H_2(X;\mathbb{Z})$). The final $\mathbb{Z}/2$ data is the Kirby-Seibenman invariant (which is automatically 1 when Q is even). (The equivalence here is that of the pair (V, Q) under the action of $Gl(V;\mathbb{Z})$.)

Donaldson's work shows that the smooth and topological classification problems have radically different solutions. In particular, in [D2], topological manifolds with no stable smoothing obstruction are shown not to be smoothable, and in [D3], a simply connected, topological manifold is exhibited with at least two inequivalent smoothings. (With the help of the Seiberg-Witten invariants, a vast number of examples of non-smoothable manifold have now been found; see e.g. [Fu].)

By the way, the C^∞ classification of compact symplectic 4-manifolds is completely up in the air. For example, as of January 1997, the following questions have no answer:

- Fix a finitely presentable group π and list all compact, oriented, smooth 4-manifolds (up to diffeomorphism) which are symplectic and have $\pi_1 = \pi$.

- Let X be a compact, oriented 4-manifold with symplectic forms ω and ω'. Does there exist a diffeomorphism φ of X, and a continuous family $\{\omega_t\}_{t\in[0,1]}$ of symplectic forms such that $\omega_0 = \omega$ and $\omega_1 = \varphi^*\omega'$?

(2.1)

Note that there is a classical obstruction to a manifold having a symplectic form, this being the parity of $b^{2+} - b^1$. Odd parity is required. (The parity here is the obstruction to finding a nowhere vanishing section of the bundle of self-dual 2-forms. This is the same obstruction as that of reducing the structure group from the SO(4) to U(2).)

The Seiberg-Witten invariants and in particular, Theorem 1.6 give additional constraints. For example, the Seiberg-Witten invariants prove that the multiple connect sum $\#_{2n+1}\mathbb{CP}^2$ has no symplectic structure when $n > 0$. (The even connect sums violated the parity condition for b^{2+}.)

This last non-existence result follows from Theorem 1.7 and:

Proposition 2.1. *Let Z be a compact, oriented, $b^{2+} > 1$ four-manifold which smoothly decomposes as a connect sum $X\#Y$ with both X and Y having $b^{2+} \geq 1$. Then all Seiberg-Witten invariants of Z are zero.*

The proof of this last statement is very much like the proof of a similar statement about the Donaldson invariants (see, e.g [DK] for the Donaldson invariant assertion, and [KKM] for the Seiberg-Witten statement).

Note that Proposition 2.1 suggests the following very small subcase of (2.1):

Let Z be a $b^{2+} > 1$, symplectic 4-manifold with a smooth decomposition $Z = X\#Y$. As one summand, say Y, must have $b^{2+} = 0$, must X be symplectic and $Y = \#_n(-\mathbb{CP}^2)$?

(2.2)

(It follows from the main theorem in [D4] that the intersection form of Y is diagonalizable over $\mathbb{Z}$. It is also known [KMT] that Y can have no non-trivial finite covering spaces. Furthermore, using Theorem 1, Kotschick proved [Ko2] that Z must also decompose smoothly as $X'\#_n(-\mathbb{CP}^2)$ for some n with X' being symplectic.)

Vis a vis (2.1), potentially interesting cases also come from algebraic surfaces of general type. Perhaps the most well known examples are the so called Horikawa surfaces [Ho]. These are minimal algebraic surfaces whose geometric genus p and first Chern class c satisfy the equality $c \bullet c = 2p - 4$. There are two deformation types when $c \bullet c$ is divisable by 16, but it is not known whether the two types are diffeomorphic. Or, if diffeomorphic, whether the two symplectic structures are equivalent in the manner noted in the second point of (2.1).

Also note that Morgan and Szabo [MSz] have a construction which yields infinite families of pairs of manifolds which

1. are homeomorphic,
2. may not be diffeomorphic,
3. if they are diffeomorphic, may have inequivalent symplectic structures

(2.3)

As long as C^∞ classifications are under discussion, remark that as of January 1997, the smooth dimension four Poincare' conjecture has yet to be settled. (The conjecture posits a unique smooth structure on the topological 4-sphere.) Of course, as of January 1997, the three dimensional Poincare' conjecture is also up in the air, but unlike the case of dimension three, there are a slew of potentially fake 4-spheres. A family of such examples was constructed by Cappell and Shaneson [CS] along the following lines: Let A be an integer values, 3×3 matrix with determinent 1 such that $\det(A - \mathbb{I}) = 1$ also. Because A has determinant 1, A defines a self diffeomorphism of the 3-torus, $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. The mapping torus of A is the 4-manifold Y which is obtained as the quotient space $(T^3 \times [0,1])/\sim$, where the equivalence is $(x,0) \sim (A \cdot x, 1)$. Because $\det(A - \mathbb{I}) = 1$, the space Y has the homology of $S^3 \times S^1$. Let $B \subset T^3$ be a small ball about the origin. Surger out $B \times S^1$ from Y and glue in $S^2 \times D^2$ in its stead. (Here, D^2 is the standard 2-disk.) There are two inequivalent ways to make this surgery; they differ by a "Gluck twist", which is to say that the gluing map for the second differs from that for the first by the fact that as one moves around the S^1, the S^2 is rotated once around its axis relative to a fixed identification of $S^2 \times S^1$ with the boundary of $S^2 \times D^2$. In any event, let $X_\pm$ denote the two manifolds so constructed. Both have the homotopy type of S^4. Some of these are known to be standard [AK], [Go2], but the identity of a large class has not been determined. (Note that when A is a square, then these manifolds have involutions whose quotients give fake $\mathbb{RP}^4$'s. See [CS].)

3 Can a symplectic form be made?

As remarked previously, a symplectic form is characterized as being both closed and non-degenerate. Lacking an existence theorem for symplectic forms, one can try to study 4-manifolds with a non-degenerate, but not closed form, or with a closed form which is degenerate in places. This section discusses some preliminary steps along the second path.

The second of the approaches above is based on the observation that a compact, oriented, Riemannian manifold X has, thanks to Hodge theory, a b^2_+ dimensional space of closed, self dual forms. Such a form is symplectic where it is not zero. Indeed, when ω is a self dual form, then $\omega \wedge \omega = |\omega|^2 \cdot d\mathrm{vol}$. Furthermore, if the metric is suitably generic, then there will be closed, self dual forms ω which vanish transversely as sections of the $\mathbb{R}^3$-bundle L_+. In this case, $Z = \omega^{-1}(0)$ is a disjoint union of embedded circles. Thus, any compact, oriented 4-manifold has a symplectic form on the compliment of a union of embedded circles.

Meanwhile, on the compliment of Z,

$$J = \frac{\sqrt{2}}{|\omega|} g^{-1} \cdot \omega \tag{3.1}$$

defines an ω-compatible, almost complex structure. This allows one to talk unambiguously about pseudo-holomorphic submanifolds on $X - Z$.

Needless to say, the circles in Z embody (in some mysterious sense) the obstructions to making a symplectic form on X.

At this juncture, there are two self-evident courses of action:

1. Pseudo-holomorphic curves in $X - Z$ are studied with the object of using them (as done in the symplectic case) to understand the differential topology of X.

2. The form ω is manipulated (keeping $d\omega = 0$) so as to remove (or otherwise simplify) components of Z.

(3.2)

The examples below indicated a possible close relation between these two courses of action.

a) A non-compact example.

Here is a (non-compact) example: The manifold in question is $\mathbb{R}^4$, with its standard flat metric. To write down the form, introduceEuclidean coordinates (x, y, z, t) and then the complex coordinates $\eta = x + i \cdot y$ and $\xi = z + i \cdot t$. Now consider the following 2-form:

$$\omega = \frac{i}{2}\Big\{(1 - |\eta|^2 + |\xi|^2) \cdot (d\eta \wedge d\bar{\eta} + d\xi \wedge d\bar{\xi}) \\ - \eta \cdot \xi \cdot d\bar{\eta} \wedge d\bar{\xi} + \bar{\eta} \cdot \bar{\xi} \cdot d\eta \wedge d\xi\Big\} \quad (3.3)$$

It is left as an exercise to verify that ω is both closed and self-dual.

The set Z for ω is the circle $|\eta| = 1$ in the plane where $\eta = 0$. Notice that the disk where $\eta = 0$ and $|\eta| < 1$ is a pseudo-holomorphic submanifold in $X - Z$.

b) Examples from 3-manifolds.

Let M be a compact, oriented, Riemannian 3-manifold with $b^1 \geq 1$. Via Hodge theory, every class in $H^1(M; \mathbb{Z})$ is represented by a harmonic, $\mathbb{R}/\mathbb{Z}$-valued function on M. This is an $\mathbb{R}/\mathbb{Z}$ valued function f which obeys $d * df = 0$, where $*$ is the Hodge star of the Riemannian metric.

Now, let $X = S^1 \times M$, and use the product metric to give X a Riemannian structure. Then

$$\omega = d\theta \wedge df + *df. \quad (3.4)$$

In particular, note that ω is symplectic when f has no critical points. (In the notation of Section 1i, the fibration ϕ is given by $e^{2\pi i f}$ and the 1-form df is denoted by ν. As noted in Section 1i, if $\phi : M \to S^1$ is a fibration, then M has a metric for which $f = (2\pi i)^{-1} \cdot \ln(\phi)$ is harmonic.) In general,

$$Z = S^1 \times \ \mathrm{crit}(f) \quad (3.5)$$

where crit(f) is the set of critical points of the $\mathbb{R}/\mathbb{Z}$ valued function f. By the way, it is not hard to prove that f has only non-degenerate critical points when the metric on M is suitably generic.

Any submanifold of the form $\{\text{Point in } S^1\} \times \{f^{-1}(\text{constant})\}$ is a pseudo-holomorphic submanifold of X. A second family of pseudo-holomorphic submanifolds is given by $S^1 \times \{\text{Gradient flow line of } f\}$.

This last example is instructive for a number of reasons. First, as indicated in the previous section, there is something about the fundamental group of X that may give obstructions to making X symplectic.

When such is the case, a first guess is that the elements in Z define generators of this group. However, (3.5) suggests that the relationship between Z and the fundamental group of X is much more subtle than this.

Furthermore, this example indicates that the problem of manipulating ω to cancel elements in Z is, in the S^1-invariant context, the same problem as that of cancelling critical points of an $\mathbb{R}/\mathbb{Z}$-valued function on a 3-manifold M. (This, of course, "explains" how Z sees the fundamental group of M.) This correspondence between a critical points of f and a component of Z is a manifestation of the afore mentioned fact that when $\gamma : \mathbb{R} \to M$ is a gradient flow line of f, then $S^1 \times \gamma(\mathbb{R})$ is a pseudo-holomorphic submanifold of X (and any S^1-invariant pseudo-holomorphic submanifold of X has this form.)

Thus, in the S^1-invariant context, there is a dictionary which translates Morse theoretic constructions into constructions with self-dual forms and pseudo-holomorphic submanifolds:

1. Critical points of $f \leftrightarrow$ zeros circles of ω.
2. Gradient flow lines of $f \leftrightarrow$ pseudo-holomorphic cylinders.
3. Cancelling critical points of $f \leftrightarrow$ removing components of Z.

(3.6)

Furthermore, recent work of the author and G. Meng [MT] plus work of Hutchings and Lee [HL] have extended the dictionary in (3.6) to include the Seiberg-Witten invariants. This is to say that the Seiberg-Witten invariants of $S^1 \times M$ can be interpreted in Morse theoretic terms on M. The simplest case for this interpretation has M given by zero-framed surgery on a knot, K, in the 3-sphere and so has the homology of $S^1 \times S^2$. Here, $X = S^1 \times M$ has $b^{2+} = 1$, but there are only two chambers, and a choice of generator $o \in H^1(M;\mathbb{Z})$ determines a chamber as follows: Let θ be any closed one form on M whose cohomology class gives o and consider (1.1) in the case where $\mu = -i \cdot r \cdot P_+(dt \wedge \theta)$ and r is positive and very large.

With the preceding understood, introduce the formal Laurent series

$$\underline{SW} = \Sigma_n sw_n \cdot t^n, \tag{3.7}$$

where sw_n is the sum of the Seiberg-Witten invariants for $\mathrm{Spin}^{\mathbb{C}}$ structures for which the cup product of $c_1(\det(S_+))$ and o evaluates as n on the fundamental class. Then, [MT] prove that the formal series $\underline{SW} = \Delta_K(t)/(t - t^{-1})^2$, where Δ_K is the symmetrized, Alexander-Conway polynomial of K (see (1.26)). Meanwhile, Hutchings and Lee

[HL] show how to compute the invariant $\Delta_K(t)/(t-t^{-1})^2$ as a suitable, weighted count of gradient flow lines of a circle valued Morse function $\phi : M \to \mathbb{R}/\mathbb{Z}$ which generates H^1.

In the general case, [MT] relates Seiberg-Witten invariants of $X = S^1 \times M$ in the case where $b^1(M) \geq 1$ to a purely topological 3-manifold invariant known as the Milnor torsion (see, e.g. [Mi1], [Tu]). And, [HL] show how to compute the latter using Morse theory. The reader is referred to [MT] and [HL] for the details. See also the very recent [Tu2].

By the way, Theorems 1.7 and 1.8 and the results in [MT] give new obstructions to the existence of a symplectic form on $X = S^1 \times M$. For example, when M is obtained by zero surgery on a knot K in S^3, then it follows from Theorem 1.8 that X has a symplectic form only if Δ_K is a monic polynomial. (This is to say that the highest order term in t begins with 1.) For a fibered knot, the fact that Δ_k is monic follows from (1.25). On the other hand, there are knots K with monic Δ_K which are not fibered. This leads to the following question:

- Can $S^1 \times M$ have a symplectic structure when K is not fibered?

- If K is fibered, is the S^1 invariant symplectic structure (as described in Section 1j, above) unique up to deformation and diffeomorphism?

(3.8)

Kronheimer [K] has the most recent result on these questions.

c) Does Morse theory generalize?

It is not known at present how many of these Morse theoretic notions survive the transition to the non-S^1-invariant world. For example, there are well known techniques (i.e. Morse theory) for manipulating the critical points of a function (see, e.g [Mi2]), and so a relevant question is: Which Morse theory techniques have analogs in the world of self-dual 2-forms on an arbitrary, compact and oriented 4-manifold?

In particular, here is the famous cancellation lemma from Morse theory: Supppose that f is an $\mathbb{R}/\mathbb{Z}$-valued function with a pair of non-degenerate critical points. Suppose further that this pair of critical points is joined by a single, minimal gradient flow line along which the Morse index drops by one. Then there is a new $\mathbb{R}/\mathbb{Z}$ valued function on M with two less critical points. (A gradient flow line between two critical points is minimal when the drop in the value of f along the flow line is no greater than the drop along any other gradient flow line between the two critical points.)

This Morse theoretic cancellation lemma suggests the following question: Is it possible to use pseudo-holomorphic cylinders (or higher genus surfaces) in X to modify the given closed form so as to "simplify" its zero set? In this regard, it is not clear precisely what one means be "simplify". However, one can imagine the following (perhaps naive) version of the simplification question: Suppose that $C \subset X - Z$ is a pseudo-holomorphic submanifold. Define the energy of C to be the integral over C of the form ω. Say that C is minimal if C has least energy amongst all closed, pseudo-holomorphic submanifolds in $X - Z$ which have these components of Z as boundary. Suppose further that C is stable under perturbations of the given metric. Now, assume that C is a cylinder and is the only minimal, stable pseudo-holomorphic submanifold in $X - Z$ with its boundary. Does it then follow that ω can be changed so as to remove the two components of Z which are boundary circles of C, but without modifying either the remaining components of Z or their bounding pseudo-holomorphic curves?

By the way, an analogy with Morse theory is supported by the following observations: Let X be a compact, oriented 4-manifold with vanishing H^1. Suppose that ω is a closed, self dual form on X with transversal zero set. Let $\mathcal{T}$ denote the Frechet manifold of compact, oriented, dimension 1 submanifolds of X whose fundamental class is zero in $H^1(X;\mathbb{Z})$. Define an $\mathbb{R}/\mathbb{Z}$-valued function f on $\mathcal{T}$ as follows: Given a submanifold γ, choose an oriented surface $\Sigma \subset X$ with boundary γ. Now define

$$f(\gamma) = \int_{\Sigma} \omega. \tag{3.8}$$

Here are some simple observations about this function f: Let Z' denote the set of pairs consisting of a component of $\omega^{-1}(0)$ with an orientation. Thus, Z' sits naturally in $\mathcal{F}$. With this understood, consider:

- Z' coincides with the critical points of f.
- A gradient flow line of f defines a map of a cylinder into X which intersects $X - Z$ as a pseudo-holomorphic subvariety.

(3.9)

Thus (3.9) provides a second indication that the pseudo-holomorphic cylinders in $X - Z$ carry some of the obstruction to making X symplectic.

d) Pseudo-holomorphic submanifolds in $X - Z$.

The preceding indicates that pseudo-holomorphic submanifolds in $X - Z$ play some sort of role in the obstruction problem to making

X symplectic. However, even without this as motivation, one might study these objects for the interesting analytical issues that arise. In this regard, the author offers two results (with proofs to appear in a forthcoming article) on this subject. One result is an existence assertion, and the other is a regularity assertion. The statement of both results requires the following definition:

Definition 3.1. A subset $C \subset X - Z$ will be called a *pseudo-holomorphic subvariety* when C is the image of a complex curve C' under a proper, pseudo-holomorphic map

$$\varphi : C' \to X$$

with the following two properties:

- φ is 1-1 except for at most a countable set of points.
- $\int_{C'} \varphi^* \omega < \infty$.

The statement of the first result also requires the notion of a homological boundary. Here is the definition: Say that Z is the homological boundary of a pseudo-holomorphic subvariety $C \subset X$ if C has intersection number ± 1 with every linking 2-sphere of Z. (As an embedded circle in X, each component of Z has a tubular neighborhood whose boundary is diffeomorphic to $S^1 \times S^2$. Any 2-sphere of the form (point) $\times S^2$ in this boundary is a linking 2-sphere.)

Proposition 3.2 ([T8]). *Let X be a compact, oriented, Riemannian 4-manifold with $b^2_+ \geq 2$ and with non-trivial Seiberg-Witten invariants. Let ω be a closed, self-dual form on X which vanishes transveraly. Then the set Z is the homological boundary of a pseudo-holomorphic subvariety in $X - Z$.*

(The set of circles Z with appropriate orientations is always null-homologous in $H^1(X; \mathbb{Z})$. This can be seen as follows: Choose a Spin$^{\mathbb{C}}$ structure for X, and choose a section ψ of the corresponding S_+ bundle which vanishes at isolated points. Then $\tau(\psi)$ is a section of Λ_+ which vanishes only where ψ does. A generic one parameter family of sections which interpolates between ω and $\tau(\psi)$ will define, by 1-parameter family of zero sets, a null-homology for Z.)

The second assertion is a regularity theorem for pseudo-holomorphic subvarieties. In this regard, remark that the "classical" regularity theory (see, for example [MS], [Ye], [PW]) holds away from Z. (Basically, the singularities are no worse than those which appear in the complex holomorphic case.) The issue here is the behavior near to Z. Here is the fundamental conjecture:

Conjecture 3.3. *Let $C \subset X$ be a pseudo-holomorphic subvariety. If some subset $\{Z_a\} \subset Z$ form the homological boundary of C, then there are arbitrarily small perturbations of C which result in a smooth, submanifold with boundary, $C' \subset X$, whose interior is symplectic and whose boundary is $\cup_a Z_a$.*

At the time of this writing, a good part of this last conjecture can be proved. In particular, one has

Proposition 3.4 ([T9]). *Suppose the metric and form ω obey certain special, but generally realizable (unobstructed) constraints near Z. Then, let C be as in Conjecture 3.3. There is at most a finite set of points in $\cup_a Z_a$, and given a neighborhood of this set of points, there are arbitrarily small perturbations of C which obey the conclusions of Conjecture 3.3, except in the given neighborhood.*

Note that the behavior of C near the finite set in Proposition 3.4 can be characterized precisely and it is likely that the singularities at these points can be shown to be removable after a further perturbation. The constraints on the metric and form near Z are described in detail in [T9] to which the reader is referred.

With regard to the behavior of C' where the latter is smooth near a point in Z, one can be quite a bit more precise. For this purpose, let ε be a small positive number and let $B^1 \subset \mathbb{R}$ and $B^3 \subset \mathbb{R}^3$ denote the balls of radius ε. Now, let $p \in Z$ be the point in question. Then, there exists $\varepsilon > 0$ and a neighborhood $U \subset X$ of p in X with coordinates $(t, x) \in B^1 \times B^3$ which has the following properties:

1. $(0, 0)$ corresponds to p.
2. $\{x = 0\}$ corresponds to $Z \cap U$ and the 1-form dt canonically orients $Z \cap U$.
3. $\omega = (dt \wedge x^\dagger A dx + x^\dagger A * dx) + \mathcal{O}(|x|^2)$ where $A = A(t)$ is a symmetric, trace zero, invertible 3×3 matrix. Here, $*$ is the Hodge star on $\mathbb{R}^3$.
4. $ds^2 = dt^2 + dx^t dx + \mathcal{O}(|x|)$

$$(3.10)$$

Some comments are in order: First, as pointed out to me by M. Hutchings, the submanifold Z has a canonical orientation since both TX and Λ_+ are oriented. Second, the matrix A is everywhere invertible because of the assumption that ω vanishes transversely along Z. Third A is symmetric and trace zero to insure that $d\omega = 0$ along Z.

Because A has trace zero everywhere, there is, at each $t \in B^1$, either two negative eigenvalues of A or else two positive eigenvalues. This is to say that $A(t)$ defines an orthogonal decomposition of $\mathbb{R}^3$ as $\mathbb{R}_t \oplus W_t$, where $\mathbb{R}_t$ is an eigenspace of $A(t)$, and where $A(t)$ is definite on the 2-plane W_t. Of course, this decomposition varies smoothly with $t \in B^1$.

With the preceding understood, remark that when U has small radius, then it should be possible to choose C' so that

$$C' \cap U = \{t, s \cdot v(t) : (t, s) \in B^1 \times [0, \varepsilon)\}, \tag{3.11}$$

where $v(t)$ is a unit length vector which depends smoothly on t and which either lies in W_t or in $\mathbb{R}_t$. Moreover, if the metric on X is generic, the latter case will not arise.

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