QUASI-MINIMAL SEMI-EUCLIDEAN
LAMINATIONS IN 3-MANIFOLDS

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§0

INTRODUCTION

Thurston's hyperbolization conjecture [Th] asserts that a closed, atoroidal, irreducible 3-manifold with infinite fundamental group has a metric of constant negative curvature. A more modest form of the conjecture asks whether the fundamental group of such a manifold is negatively curved in the sense of Gromov [Gr]. (E.g. see [Bu], [Sch], [Mo1], [MO] or [Ka].) A main result of this paper, the Ubiquity Theorem, provides a technique for addressing the group negative curvature conjecture. We use it in [GK] to show that an atoroidal 3-manifold with a genuine essential lamination has group negative curvature. (These manifolds appear to form a vast subset among the irreducible atoroidal 3-manifolds with infinite π_1. See the survey [G4].)

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In this paper the ubiquity theorem is used to derive new information about quasi least area semi-Euclidean laminations in 3-manifolds.

Gromov stated 6.8.S [Gr] that if M is a compact manifold or finite simplicial complex whose fundamental group is nonnegatively curved, then there is a non constant least area conformal map $f : \mathbb{R}^2 \to M$. Mosher and Oertel [MO] provided the first detailed proof (of a sharper version) in the context of finite 2-complexes. Recently Kleiner [Kl] showed that if M is a closed Riemannian n-manifold such that $\pi_1(M)$ is non negatively curved, then there exists a branched Lipschitz conformal least area immersion $f : \mathbb{R}^2 \to M$. If $n = 3$, then f is an immersion, though the induced metric on $\mathbb{R}^2$ may be incomplete.

In §1 we state basic definitions regarding immersed laminations and branched surfaces in 3-manifolds. Then we outline and discuss the proof of the main technical result of [MO] in the 3-manifold context. Their proof makes essential use of a theorem of Ghys and the Plante argument for immersed surfaces. In §2 we show how to extend [MO] to obtain our first main result.

Theorem 2.1. *If M is a closed non negatively curved 3-manifold, then there exists an immersed strongly least area, Euler characteristic 0, measured semi-Euclidean lamination λ. The induced metric on each leaf is complete.*

Here M has a triangulation τ, and the various measurements of length and area are computed simplicially. Strongly least area [MO] means that if L is a leaf and D is an embedded disc in the universal covering of L, then the induced map $i : D \to M$ is a least area map. *Semi-Euclidean* means that the set of leaves, conformally equivalent to the Euclidean plane, are of full measure and dense in λ.

By replacing immersed by branched immersed in the above statement, we obtain the exact translation of the main result of [MO] to 3-manifolds. The content of §2 is how to eliminate the branched points.

Our second main result states that if k is a smooth simple closed curve in the irreducible group nonnegatively curved 3-manifold M, then either k lies in a 3-cell, or M is toroidal or among least area discs D_α of small isoperimetric ratio, area(D_α)

is approximately the geometric intersection number of D and k. (By definition M is toroidal if and only if $Z \oplus Z \subset \pi_1(M)$. By [CJ], [G2] the topology of closed irreducible toroidal manifolds is completely understood.) More precisely we have the following equivalent results.

Theorem 5.1. *Let k be a smooth simple closed curve in the closed irreducible Riemannian 3-manifold M. Then either M is toroidal or k is contained in a 3-cell or there exists a constant $C > 0$ such that if D is a least area disc with $\partial D \cap k = \emptyset$, then $area(D) \leq C(wr(\partial D, k) + length(\partial D))$.*

Ubiquity Theorem 5.2. *Let $k \not\subset B^3$ be a smooth simple closed curve in the closed, atoroidal, irreducible 3-manifold M. There exists constants K and L such that if D is a least area disc with $\partial D \cap k = \emptyset$ and $length(\partial D)/area(D) < L$, then $wr(\partial D, k)/area(D) > K$.*

If k is a simple closed curve in M, and α is a homotopically trivial curve in M disjoint from k, then we define the *wrapping number* $wr(\alpha, k)$ to be the minimal geometric intersection number between k and all immersed discs D_α spanning α.

Corollary 5.3. *Let λ be a quasi least area semi-Euclidean lamination in the closed atoroidal irreducible 3-manifold M. If $k \subset M - \lambda$ is a smooth simple closed curve, then k lies in a 3-cell.*

Chapter §3 explains how to use cellulations of hyperbolic 3-manifolds by ideal polyhedra to obtain PL versions of various results known in the Riemannian world. In §4 we prove the Ubiquity Theorem in the PL category. Chapter §5 is devoted to proving the Ubiquity Theorem in the smooth category.

In §6 we give a very brief survey of progress on the Thurston geometrization conjecture.

Since the unresolved parts of Thurston's conjecture only concern orientable 3-manifolds, we will assume that **all 3-manifolds in this paper are orientable.**

Readers Advisory. If you are interested in Ubiquity theorem, then skip directly to §4 or §5 and refer back as needed. Except for minor references to earlier stated

definitions or figures, §3-6 are completely independent from the earlier chapters. To get right to the heart of the matter in the PL category, read 4.1-4.4 and then read the proof of the case, $M - k$ is hyperbolic. To read §5 one need only refer back to three self contained arguments of §4.

I would like to thank Igor Rivin for several key conversations about hyperbolic cellulations.

<div align="center">§1</div>

<div align="center">IMMERSED BRANCHED SURFACES AND LAMINATIONS</div>

Definition 1.1. A *generic immersed branched surface* $B = (B^*, \phi)$ in a 3-manifold M is a finite 2-complex B^* which is mapped via ϕ into M as follows. Each $x \in B^*$ has a neighborhood U whose image appears as in Figures 1.1 a-d. Furthermore if $\phi(U)$ appears as in Figures 1.1 a-c, then $\phi|U$ is an embedding. (Figures 1.1 a-c are the standard local models of a generic embedded branched surface [FO].) If $\phi(U)$ appears as in Figure 1.1 d) then U is homeomorphic to the 2-complex shown in Figure 1.1e) and $\phi|U$ fails to be an immersion only at x. Furthermore $\phi(U)$ has an arc of double points which starts at $\phi(x)$. The model shown in Figure 1.1 d) is the *crossed maw* discovered by Joe Christy [Ch]. The singularity is similar to the *undrawable singularity* of Poenaru [Po]. We assume that the various self intersections of ϕ are transverse and generic. Define the *branch locus* $\gamma(B)$ of B to be the set of nonmanifold points of B^*, define a *sector* to be a component of $B^* - \gamma(B)$, and define ∂B to be the union of free edges of B^*. We will often abuse notation by referring to the branch locus and boundary of B as objects in M. We will also refer to generic immersed branched surfaces simply as *branched surfaces*.
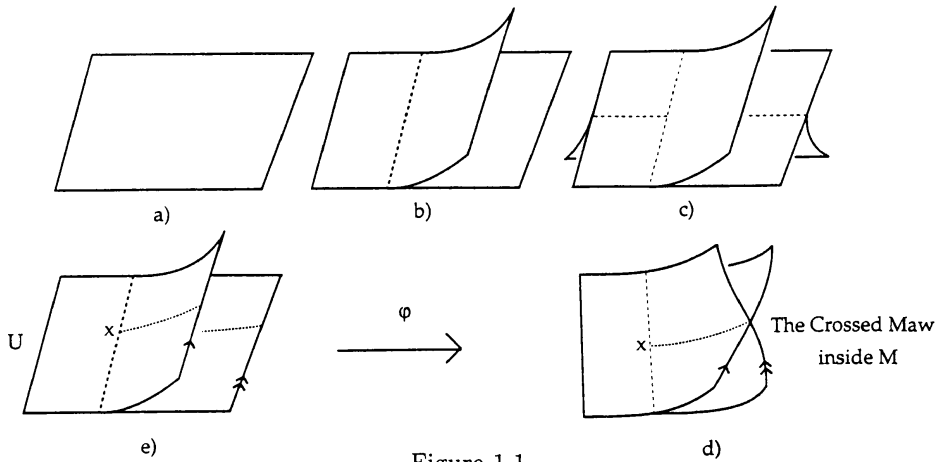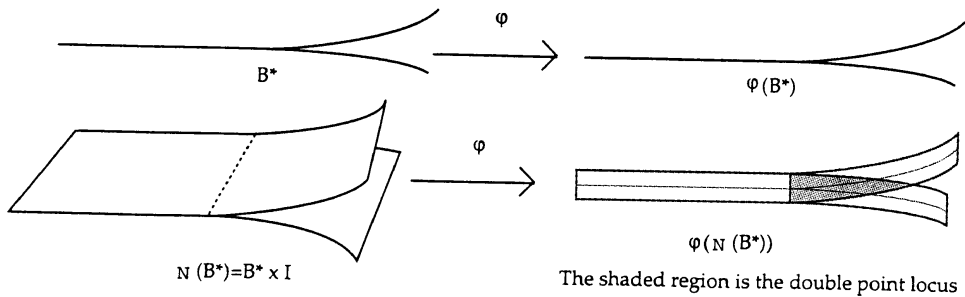
Figure 1.1

Associated to the branched surface B we define the *normal neighborhood* $\mathcal{N}(B) = (\mathcal{N}(B^*), \phi)$ as follows. If B^* is a locally finite union of embedded discs D_i, then define $\mathcal{N}(B^*) = \cup D_i \times [-1, 1]$ modulo the equivalence relation, where $x \times [-1, 1]$ is identified with $y \times [-1,$disjoint from ∂E1], if $x = y \in B^*$. The identification is either the identity or $t \to -t$ depending on whether or not the local orientations on D_i and D_j agree when viewed inside of M. The singular immersion $\phi : B^* \to M$ induces a singular immersion $\phi : \mathcal{N}(B^*) \to M$. The mapping will be non immersive exactly along $\Gamma(B)$ the *normal branched locus* which we define to be $\pi^{-1}(\gamma(B))$ where $\pi : \mathcal{N}(B^*) \to B^*$ denotes the natural projection. See Figure 1.2. We identify B^* with the 0-section of $\mathcal{N}(B^*)$. The notation $\mathcal{N}(B)$ should not be confused with the similar notation $N(B)$ for *fibred neighborhood* of an embedded branched surface.

2-dimensional version of the *normal neighborhood*

Figure 1.2

We say that the branched surface B_1 is obtained from B by *splitting* if there is

a lift of $\phi_1 : B_1 \to M$ to an immersion B_1^* to $\mathcal{N}(B^*)$ such that B_1^* is transverse to the I-fibres, B_1^* intersects each I-factor nontrivally, and finally $\partial B_1^* \subset \pi^{-1}(\partial B^*)$. Conversely we say that B is obtained from B_1 by *squeezing*.

Remark 1.2. Figure 1.3a shows a splitting one dimension lower. Figure 1.3b describes a splitting in a 3-manifolds. Note that one crossed maw is created in Figure 1.3b.
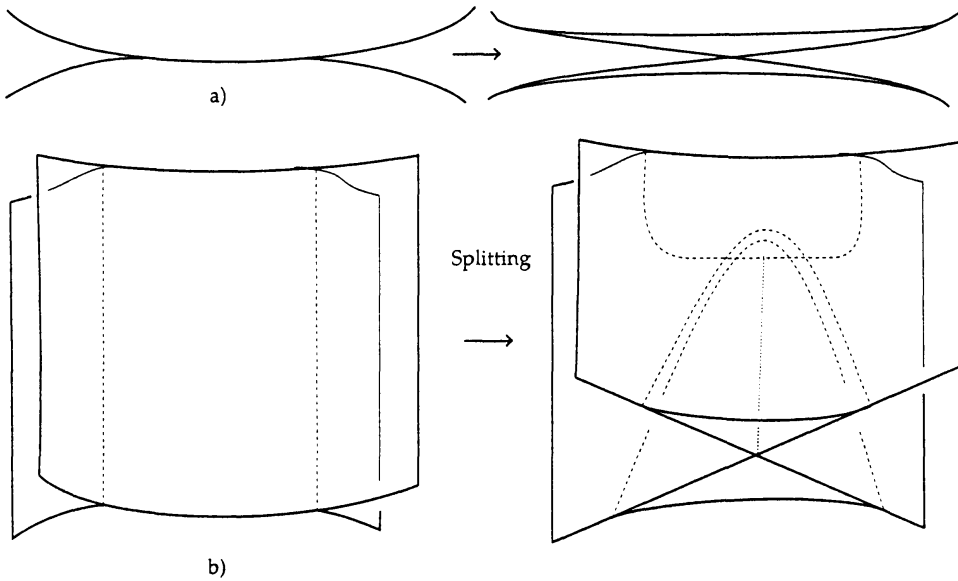

Figure 1.3

Lemma 1.3. *If B_1 is obtained from B by splitting, then B_1 has a normal neighborhood $\mathcal{N}(B_1)$ which lifts to a singular immersion $\mathcal{N}(B_1^*) \subset \mathcal{N}(B^*)$ such that each I-fibre of $\mathcal{N}(B_1^*)$ is properly contained in the associated I-fibre of $\mathcal{N}(B^*)$.* $\square$

Definition 1.4. Suppose that we are given a Riemannian metric on B. If $C > 0$, then we define a *C-splitting* to be a splitting $B \to B_1$ such that if $i : \mathcal{N}(B_1^*) \to \mathcal{N}(B^*)$ is the associated singular immersion then there does not exist paths $\alpha, \beta : [0, t + \epsilon] \to \mathcal{N}(B_1^*)$ such that $\alpha(0), \beta(0)$ lie in the same I-fibre of $\mathcal{N}(B_1^*)$, for all $s \in [0, t], i(\alpha(s)), i(\beta(s))$ lie in the same I-fibre of $\mathcal{N}(B^*)$, and for $s > t; i(\alpha(s)), i(\beta(s))$

lie in distinct I-fibres of $\mathcal{N}(B^*)$. Finally length $\pi(i(\alpha([0,t]))) < C$. Intuitively this means that the branched locus of B has been blasted open at least distance C.

Definition 1.5. By a 2-dimensional *abstract lamination* μ we mean a topological space covered by charts of the form $U_i = T_i \times \mathbb{R}^2$, where T_i is a closed subset of $[0, 1]$. Furthermore if $U_i \cap U_j \neq \emptyset$, then the coordinate transformations are of the form $(x, y) \to (\psi(x), \phi(x, y))$ where $x \in T_i, y \in \mathbb{R}^2$. This is a specialization of a more general definition given by [Ca]. There Alberto Candel initiates a deep investigation of the differential geometry of 2-dimensional abstract laminations.

Throughout this paper we will refer to abstract laminations simply as laminations. We say that the lamination μ is immersed in M^3 if there is a continuous map $J : \mu \to M^3$ whose restriction to each leaf of μ is a smooth or PL immersion. We say that μ is *carried* by the branched surface B if the mapping J factors through an immersion into $\mathcal{N}(B^*)$ such that the leaves of μ are transverse to I-fibres of $\mathcal{N}(B^*)$. If μ intersects each I-fibre of $\mathcal{N}(B^*)$, then μ is *fully carried* by B.

Definition 1.6. If λ is carried by B, then B_1 is said to be a λ-*splitting* of B if there exists a lift of λ to $\mathcal{N}(B^*)$ which after normal homotopy lifts to $\mathcal{N}(B_1^*)$.

Definition 1.7. Generalizing the similar notion of [GO] we say that an infinite sequence of branched surfaces $B_1, B_2, \cdots$ is a *full splitting* if there exists $C > 0$ such that for each i there exists a $j(i)$ such that the splitting $B_i \to B_{j(i)}$ is a C-splitting. If B is the first term of a full splitting, then we say that B is *fully splittable*.

The following result is the direct analogue of Lemma 4.3 [GO] to immersed laminations. See [MO] for a version for laminations in finite 2-complexes.

Theorem 1.8. *B is fully splittable if and only if B fully carries a lamination.*

Proof. Suppose that B is fully splittable. After making the lengths of the I-fibres of $\mathcal{N}(B_i)$ go to 0 as $i \to \infty$, then the inverse limit of the B_i's is a lamination fully carried by B. Conversely if B fully carries λ, then λ provides the clue to constructing a full splitting. The resulting lamination λ' arising from the inverse

limit construction will generally be distinct from λ, however it will be normally homotopic to a sublamination of λ in $\mathcal{N}(B^*)$.   □

Definition 1.9. Let M be a triangulated 3-manifold, with triangulation τ. Let τ^i denote its i-skeleton. Following Haken [Ha], if J is a 1-manifold, then we say that $f : J \to \tau^2$ is *normal* if f is a PL-immersion transverse to τ^1 and for each edge e of the induced triangulation on J, $f|e$ is an embedding which sends the vertices of e to distinct 1-simplices of τ^1. A normal 2-disc is a properly embedded 2-cell D in a 3-simplex σ such that ∂D is a normal curve crossing a given 1-simplex of σ at most once. If S is a surface, then we say $f : S \to M$ is *normal* or S is a *normal surface* if f is a branched immersion transverse to τ^2 and each 2-cell η in the induced cellulation κ on S gets mapped to a normal disc. Thus, all branch points occur in τ^1. For $v \in \kappa^0$ define $s(v)$, the *spinning* of v, to be the number of 2-cells of κ which meet v. If e is a 1-simplex of τ, define $V(e)$, the *valence* of e, to be the number of 2-cells of τ which meet e. Also for $v \in \kappa^0$, define $V(f(v))$ to be $V(e)$ where $f(v) \in e$. Thus $s(v)/V(f(v))$ measures the local branching of f at v. We say that f has *fake boundary branching* if there exists $v \in \kappa^0 \cap \partial S$, such that f is an immersion at v, but $s(v)/V(f(v)) \geq 1$.

Define $S(D) = \sum_{a \in \kappa^0 \cap \partial D} s(a)$ to be the *total spinning* of D. Define $A(D) = S(D)/(|\kappa^0 \cap \partial D|)$ to be the *average spinning* of D.

Proposition 1.10. *Let τ be a triangulation of the 3-manifold M. There exists a branched surface $\mathcal{B}$, called the canonical normal branched surface, with the following property. If λ is any immersed lamination in M whose leaves are normal with respect to τ, then after normal homotopy λ is carried by $\mathcal{B}$. In particular the conclusion holds if λ is a compact normal surface, possibly with boundary.*

Proof. Figure 1.4 shows how to construct a canonical branched surface B on a 3-simplex σ. It is symmetric under the symmetries of σ and carries the isotopy class of each normal disc in a unique way. Thus by putting this branched surface in each 3-simplex of τ one obtains the desired $\mathcal{B}$. To start with, the three distinct normal quadralaterals and the four distinct normal triangles in σ can be normally

isotoped to appear in σ as follows. The quadralaterals pairwise intersect in arcs whose endpoints lie on edges of σ. Also if η is a triangle then $\eta \cap (\cup$ quads $) = \partial \eta$. Thus the union of the quadralaterals and triangles is isomorphic to the cone on the 1-skeleton of an octahedron together with four faces of the octahedron which pairwise meet in points. (I.e. one color of a two coloring of the octahedron.) See Figure 1.4a. By squeezing the faces in the appropriate manner, we obtain the desired branched surface B. See Figure 1.4b. $\square$
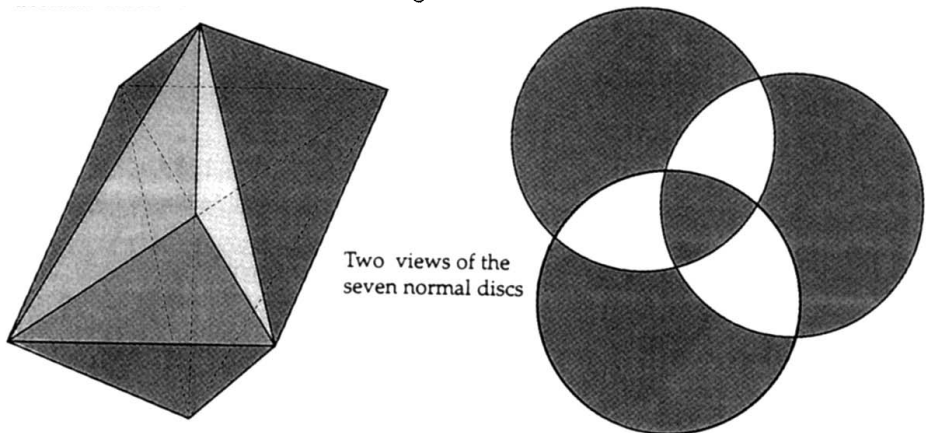
Two views of the
seven normal discs

a)

The shaded regions are the triangles. Attach
hemispheres to each circle to obtain the quadralerals.
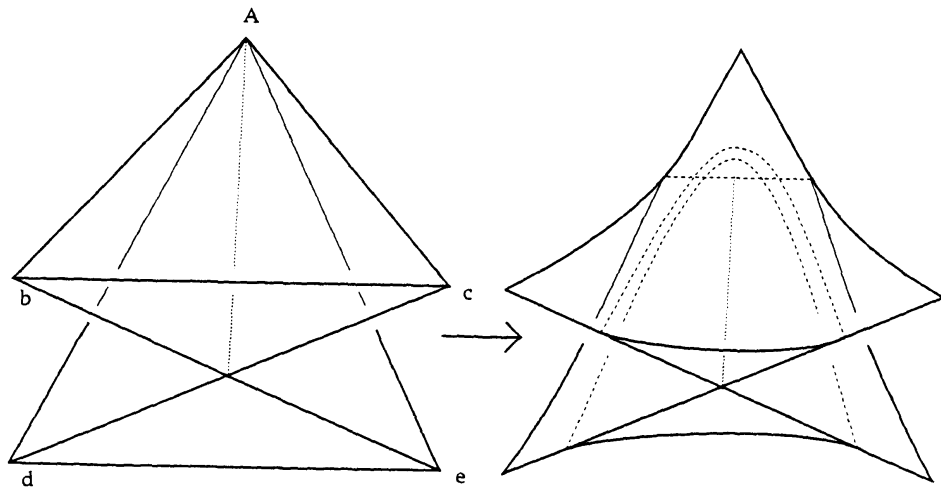
Attach discs to each of Abc, Ade, Abe, Adc.

Smoothing very near corner A of the octahedron creates the branched surface B

b)

Figure 1.4

Definition 1.11. A *measured branched surface* is a branched surface B together with a non negative valued function $m :$ {Sectors of B} $\to \mathbb{R}$ such that if sectors s_1, s_2 merge into sector s_3, then $m(s_1) + m(s_2) = m(s_3)$. A *measured lamination* is a lamination λ together with a transverse measure m. I.e. for each lamination chart $T_i \times \mathbb{R}^2$ there is a Borel measure supported on T_i. Furthermore the transversal overlap functions are measure preserving. If the measured lamination (λ, m) is carried by the branched surface B, then m induces a measure also called m on B. Define the Euler characteristic of a measured branched surface as follows. If (B, m) is a branched surface, then put a cell structure on B so that the branched locus is contained in the 1-skeleton. If t is a vertex or edge, define $m(t)$ to be the maximal m value of a sector which contains t and define $\chi(B, m) = \sum_{vertices} m(v_i) - \sum_{edges} m(e_j) + \sum_{faces} m(f_k)$. This formula is invariant under splitting and choice of cell structure. If (λ, m) is a measured lamination, carried by a branched surface (B, m), then define $\chi(\lambda, m) = \chi(B, m)$.

Definition 1.12. We need the notion of *diagram* [LS] which is a slight generalization of a disc. (The first time reader is strongly advised to pretend that all diagrams are discs.) A diagram D is a simply connected finite 2-complex that is embedded in the plane. Define $\overset{\circ}{D}$ to be the interior of D as a subset of $\mathbb{R}^2$. Define $\partial(D) = D - \overset{\circ}{D}$ and $\mathrm{Bd}(D)$ to be a simplicial map of the circle to the curve which traverses around D. See Figure 1.5. Call a diagram map $f : D \to M$ *admissible* if ∂D has a triangulation ϕ such that f takes each 1-cell of ϕ to a normal arc and f is transverse to τ^2, except at isolated points where f appears as in Figure 1.6. We say that $g : S^1 \to \tau^2$ *extends* to the diagram $G : D \to M$ or G *spans* g, if $G \circ \mathrm{Bd}(D)$ is normally homotopic to g. If $f : D \to M$ is an admissible diagram map, then define length(∂f) to be the number of 1-cells in the induced triangulation on $\mathrm{Bd}(D)$.
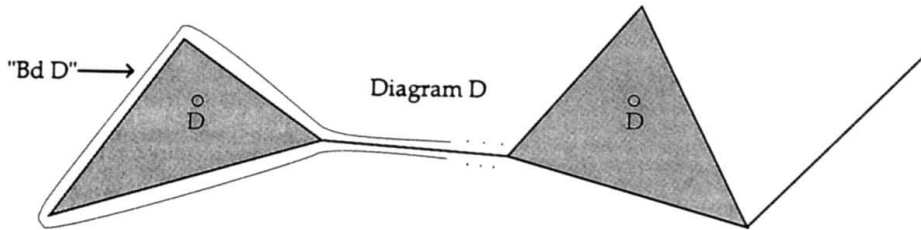
Figure 1.5
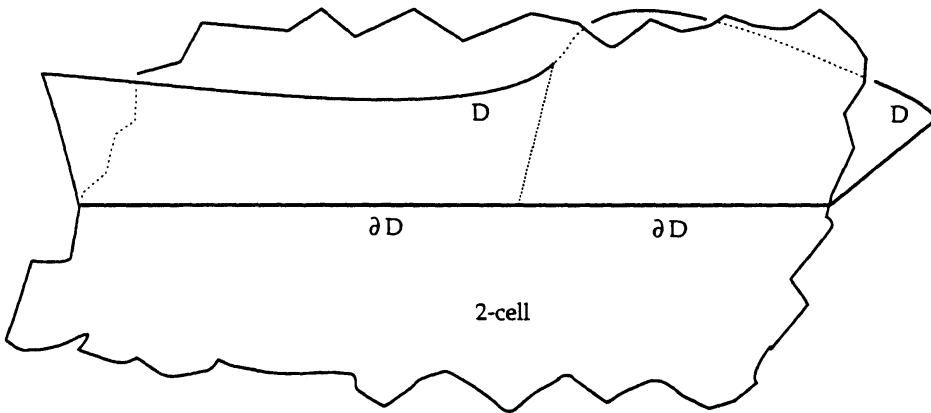

Figure 1.6

Definition 1.13. If $f : S \to M$ is a normal surface, then define "area (f)"
$= \sum_{i=1}^{n} s(v_i)/V(f(v_i))$, where $\{v_1, \cdots, v_n\}$ are the vertices in the induced cel-
lulation κ on S. Thus if $f : S \to M$ is a normal immersion of a closed surface, then
area(f) is the number of vertices in κ, i.e. the Haken weight. This definition of

area takes into account local branching and is additive under union. In a similar manner we can define area(f) where $f : D \to M$ is an admissible diagram map, or if f is a map of a compact surface such that $f|\partial D$ is normal and f is transverse to τ^2 except as in Figure 1.6. In these cases one uses the induced stratification on D to compute $s(v_i)$ and sum over points of $f^{-1}(\tau^1)$.

A diagram $f : D \to M$, and in particular a normal map of a disc, is said to be *least area* (resp. *k-quasi least area*) if area(D) $\leq$ area(E) (resp. area(D) $\leq k$ area(E)) among all diagrams E spanning ∂D.

A normal surface $f : L \to M$ is *least area* if the restriction of f to embedded discs is a least area diagram. L is *strongly least area* [MO] if the induced immersion of the universal covering of L into M is least area. In a similar manner define *strongly k-quasi least area for a normal map* $f : L \to M$. The leaves of a lamination are *quasi strongly least area* if there exists a k such that for each leaf L, the induced immersion of L into M is strongly k-quasi least area.

The following Proposition is a restatement of the main technical result of Mosher and Oertel [MO]. We translate their work about PL maps of discs into finite 2-complexes into a Proposition about maps of discs into 3-manifolds. We generalize slightly to allow for *k-quasi least area* maps.

Proposition 1.14 [MO]. *Let $D_1, D_2, \cdots$ be a sequence of immersed normal discs, in the triangulated compact 3-manifold M, such that $\lim_{i\to\infty} length(\partial D_i)/area(D_i) \to 0$. If no D_i has fake boundary branching, then after passing to a subsequence, D_i converges to a normal Euler characteristic-0 measured lamination λ. If each of the discs is k-quasi least area, then each leaf of λ is strongly k-quasi least area. If each D_i is least area, then λ has no 2-sphere leaves. Finally, the induced metric on each leaf is complete.* $\square$

Remarks. The proof of the existence of λ in [MO] is greatly complicated by the fact that it is happening in the setting of branched immersed discs into 2-complexes. In the setting of immersed normal discs without fake boundary branching in 3-manifolds the argument is considerably simpler. Mosher and Oertel do a competent

job of explaining their proof of strong area minimization. Indeed, their argument shows that if λ had a 2-sphere leaf, then for i sufficiently large, some component of D_i would be a 2-sphere, which gives a contradiction. By construction each leaf of λ is complete. One should be aware that the discussion of diagrams is implicit, though suppressed in [MO].

Idea of the proof of existence of λ. Each D_i is carried by $\mathcal{B}$, the canonical normal branched surface. Fix a small $C > 0$. By the usual diagonal argument one passes to a subsequence of the D_i's and finds a sequence $\mathcal{B} = B_0, B_1, \cdots$ such that B_i is obtained from B_{i-1} via C-splitting where each B_j carries every D_i for $i \geq j$. (This step would fail if there was uncontrolled fake boundary branching.) By counting the number of times a given D_i crosses a sector of B_j, we obtain a function m_{ij} on the sectors of B_j which fails to be a transverse measure only because of the boundary of D_i. Let a_{ij} be the maximal value of m_{ij}. By considering m_{ij}/a_{ij} and using the length area hypothesis, then by passing to a subsequence of the D_i's the measures m_{i1}/a_{i1} converge to a fixed transverse measure m_1 on B_1. By passing to another subsequence we can assume that the D_i's induce a measure m_2 on B_2 so that as measured laminations (B_2, m_2) is obtained by splitting (B_1, m_1). Thus by applying Theorem 1.8 to the sequence $\{(B_i, m_i)\}$ we obtain a measured lamination λ'. Finally take λ to be the sublamination which is the support of the measure m. This type of argument, in tamer form, goes back to Plante [Pl].

Since $\chi(D_i) = 1$, the Euler characteristic of the approximate measure m_{ij}/a_{ij} on B_j is approximately $1/a_{ij}$ and $m_{ij}/a_{ij} \to m_j$, it follows that the Euler characteristic of λ is zero.

Definition 1.15. Let λ be a lamination with a piecewise Riemannian metric on each leaf, which varies continuously in the transverse direction. (E.g. a metric induced from a 3-manifold) We say λ is *conformally Euclidean* if each leaf is conformally equivalent to the Euclidean plane. A measured Riemannian lamination λ is said to be semi-Euclidean if the set of leaves conformally equivalent to the Euclidean plane are of full measure and dense in λ.

The following result provides the crucial link between the topology of the measured laminations and their geometry.

Theorem (Ghys [Gh] 1.16). *Let λ be a measured 2-dimensional abstract Riemannian lamination with out spherical leaves. $\chi(\lambda) = 0$ if and only if almost all leaves are conformally Euclidean.* □

Remark 1.17. This was proven originally for 2-dimensional foliations of manifolds, but Candel [Ca] pointed out that the proof holds for 2-dimensional abstract Riemannian laminations.

§2

IMMERSED LEAST AREA

SEMI-EUCLIDEAN LAMINATIONS

Theorem 2.1. *If M is a closed non negatively curved aspherical 3-manifold, then there exists an immersed strongly least area, Euler characteristic 0, measured semi-Euclidean lamination λ. The induced metric on each leaf is complete.*

Remark 2.2. i) Here M has a triangulation τ, and the various measurements of length and area are computed simplicially.

ii) Theorem 2.1 should be viewed as a refinement of [MO]. Their work translated to closed 3-manifolds yields the above theorem with immersed replace by branched immersed.

Conjecture 2.3. *If K_0 is a non negatively curved 2-complex, then K_0 is simple homotopy equivalent to a 2-complex K_1 such that K_1 supports an immersed least area Euler characteristic zero semi-Euclidean lamination.*

The following is the main result of this section.

Proposition 2.4. *Let M be a closed 3-manifold with triangulation τ. $\pi_1(M)$ is not negatively curved if and only if there exists a sequence of least area normal immersed discs $f_i : D_i \to M$, such that $length(\partial f_i)/area(f_i) \to 0$ and no f_i has fake boundary branching.*

Remark. This result is a serious technical advance beyond the obvious translation of Gromov's theorem, *group negative curvature implies linear isoperimetric inequality,* to normal surfaces in 3-manifolds. The new wrinkle is that "no f_i has fake boundary branching."

Proof of Proposition 2.4.

Step 0. $\pi_1(M)$ is non negatively curved if and only if there exists a sequence of least area normal immersed discs $f_i : D^2 \to M$, such that $\text{length}(\partial f_i)/\text{area}(f_i) \to 0$.

Proof of Step 0. Since $\pi_1(\tau^2) = \pi_1(M)$ is not negatively curved, there exists by Gromov [Gr], [Bo] a sequence of simplicial maps of discs $a_i : A_i \to \tau^2$ such that $\text{length}_\tau(A_i)/\text{area}_\tau(A_i) \to 0$. Where $\text{length}_\tau(\partial A_i)$ (resp. $\text{area}_\tau(A_i)$ is the number of 1-simplices (resp. 2-simplices) in ∂A_i (resp. in A_i.) Furthermore $\text{area}_\tau(A_i)$ is minimal among all such simplicial maps of diagrams that span $a_1|\partial A_i$.

Its routine to translate this statement about 2-complexes to the statement of Step 0. □

From now on we will assume that $\pi_1(M)$ is non negatively curved and that Proposition 2.4 is false. Thus there exists a constant $K_0 \le 1$ such that if $f : D \to M$ is least area normal immersed disc with no fake boundary branching, then $\text{length}(\partial f)/\text{area}(f) > K_0$.

Here is how we derive a contradiction. As a result of Steps 0 and 1 we find a sequence satisfying the conclusion of Proposition 2.4, except that each disc has fake boundary branching at a single vertex. We then delete a small neighborhood of the branching vertex and take (in Step 2) a weak limit of these trimmed discs to obtain an immersed least area measured lamination λ' of Euler characteristic 0, with boundary $S^1 \times K, K$ a closed subset of $[0, 1]$. By capping off the boundary circles with discs, we obtain a measured lamination λ of positive Euler characteristic and derive a contradiction to the least area property of λ.

Step 1. There exists a sequence of least area normal immersed discs $f_i : D^2 \to M$, such that $\text{length}(\partial f_i)/\text{area}(f_i) \to 0$. Furthermore all the fake boundary branching

of f_i occurs at a single vertex w_i of ∂D_i.

Proof of Step 1. Let f_i be as in Step 0 and let Δ_i denote the cellulation induced on D_i by τ. If J is a subcomplex of a cell complex, then let $St(J)$ denote the union of all open cells whose closures intersect J and $\bar{S}t(J)$ to denote the closure of $St(J)$. Define $D_i^{trim} = D_i - St(\partial D_i)$.

Claim 0. We can assume that D_i^{trim} is connected, that Δ_i has no separating 1-cells, and $\bar{S}t(v) \cap D_i^{trim}$ is nonempty and connected for each vertex $v \in \partial D_i$.

Proof of Claim 0. Consider a maximal collection of properly embedded arcs δ_j in D_i such that $\partial \delta_j \subset \Delta^0$, interiors of the δ_j's are pairwise disjoint and the δ_j's satisfy the following additional property. Either δ_j is a separating 1-cell of Δ_i, or $\overset{\circ}{\delta_j} \subset \Delta_i - \Delta_i^2$ and each component of $D_i - \delta_j$ contains at least one open 2-cell of Δ_i. The number of such paths between two given vertices is uniformly bounded by a constant K_1, since M is compact and D_i is least area. Thus for Euler characteristic reasons there are at most $K_1 \text{length}(\partial D_i)$, such δ_j's. If C'_n is a component of $D_i - \cup \delta_j$, then let C_n be the smallest subcomplex of Δ_i which contains C'_n. Since $\sum_n \text{length}(\partial C_n) \leq 4K_1(\text{length}(\partial D_i))$, some C_n will have very small isoperimetric ratio, if D_i does. Finally C_n satisfies the conclusions of Claim 0. Thus by cutting down the D_i's we can assume that they satisfy the conclusions of both Step 0 and Claim 0. □

We will supress cumbersome language by writing as if each D_i^{trim}, as well as other to be defined objects are discs rather than diagrams.

If $v \in \Delta^0 \cap \partial D_i$, then define $D_i^v = D_i^{trim} \cup \bar{S}t(v)$. D_i^v is an immersed disc with possible fake boundary branching only at v_i. Since D_i^{trim} is an immersed least area disc with no fake boundary branching, it follows that for all i, $\text{length}(\partial D_i^{trim})/\text{area}(D_i^{trim}) > K_0$.

Claim 1. The average spinning $A(D_i) \to \infty$.

Proof. Otherwise $A(D_i) \leq A$ all i implies $\text{length}(\partial D_i^{trim})/\text{area}(D_i^{trim}) \leq 2S(D_i)/(\text{area}(D_i) - S(D_i)) \leq 2A\text{length}(\partial D_i)/(\text{area}(D_i) - A\text{length}(\partial D_i)) \to 0$ as

$i \to \infty$. $\square$

We now define $\mathrm{Fan}_j(v)$, which more or less is the smallest simply connected subset of $D_i^v(v)$ which contains all simplices distance j from v. More precisely, if v is a vertex of ∂D_i then define inductively define $St_n^{trim}(v) = \bar{St}(St_{n-1}^{trim}(v))$ where $St_1^{trim}(v) = \bar{St}(v)$, all these complexes being computed in D_i^v. Let $\mathrm{Fan}_j(v)$ be the minimal simply connected subcomplex of D_i^v which contains $St_j^{trim}(v)$.

Claim 2. Given p, there exists $B(p)$ independent of i, such that for any $v_r, v_s \in \partial D_i$, $\mathrm{area}(St_p^{trim}(v_r) \cap St_p^{trim}(v_s)) < B(p)$. (Recall $St_p^{trim}(v_r) \subset D_i^{v_r}$ and $St_p^{trim}(v_s) \subset D_i^{v_s}$.)

Proof. If x is a vertex in $St_p^{trim}(v_r) \cap St_p^{trim}(v_s)$, then there exists a simplicial path α_x of length $\leq 4p$ from v_r to v_s which passes through x. The union of two such paths α_x, α_y gives rise to a least area diagram $D_{xy} \subset D_i$, with $\partial D_{xy} \subset \alpha_x \cup \alpha_y \subset D_{xy}$ and hence length $\partial D_{xy} \leq 8p$. Since M is compact, the number of such diagrams is bounded and hence if $x, y \in St_p^{trim}(v_r) \cap St_p^{trim}(v_s)$, then the distance between x and y in D_i^{trim} is uniformly bounded, independent of i. Since the valence of vertices of D_i^{trim} are uniformly bounded, independent of i, Claim 2 follows. $\square$

Claim 3. Given p, there exists a constant K_3 independent of i such that at most $K_3(\mathrm{length}(\partial D_i)$ distinct pairs $(\mathrm{Fan}_p(v_r), \mathrm{Fan}_p(v_s))$ intersect.

Proof. If $\mathrm{Fan}_p(v_r) \cap \mathrm{Fan}_p(v_s) \neq \emptyset$, then define α_{rs} to be a shortest simplicial path in 1-cells of $D_i^{v_r} \cup D_i^{v_s}$ from v_r to v_s. Thus α_{rs} is embedded of length $\leq 4p$. By rechoosing α_{rs} we can assume that $D_i^{trim} \cap \alpha_{rs}$ is connected, and using Claim 0 that either $\alpha_{rs} \cap D_i^{trim} \neq \emptyset$ or $\alpha_{rs} \subset \partial D_i$. Furthermore, if $\alpha_{rs} \cap D_i^{trim} \neq \emptyset$, then after possibly deleting the first and/or last 1-cells of α_{rs}, the remainder is properly embedded in D_i. Also if $\alpha_{rs} \subset \partial D_i$, then $\mathrm{length}(\alpha_{rs}) \leq 4$. Thus there are at most $4 \, \mathrm{length}(\partial D_i)$ peripheral α_{rs}'s and at most $5N$ non peripheral α_{rs}'s, where N is the number of properly embedded α_{rs}'s. Since at most finitely many paths α_{rs} can pass through a given vertex of D_i^{trim}, it follows that there exists a uniform bound K_2 on the number of α_{rs}'s which can intersect a given α_{uv} inside of D_i^{trim}. The collection

of properly embedded α_{rs}'s can be partitioned into $K_2 + 1$ subsets, with the property that if α_{rs}, α_{uv} are distinct elements of a given subset, then $\overset{\circ}{\alpha}_{rs} \cap \overset{\circ}{\alpha}_{uv} = \emptyset$. For Euler characteristic reasons each subset has at most length(∂D_i) $- 1$ elements, so $N \leq (K_2 + 1)(\text{length}(\partial D_i) - 1)$ and Claim 3 follows with $K_3 = 5(K_2 + 1) + 4$. $\square$

Claim 4. For every $k > 0$, there is a $N > 0$ such that $i > N$ implies that there exists a $v \in \Delta^0 \cap D_i$ and a j such that length($\partial \text{Fan}_j(v)$)/area($\text{Fan}_j(v)$) $< 1/k$.

Proof. Fix k and i. If v is a vertex in ∂D_i, then let $\partial_{ext}\text{Fan}_j(v)$ denote $\partial \text{Fan}_j(v) \cap \overset{\circ}{D}_i^v$. If there exists j such that length($\partial_{ext}\text{Fan}_j(v)$) $\leq s(v)/2k$, then length($\partial\text{Fan}_j(v)$)/area($\text{Fan}_j(v)$) $\leq (s(v)/2k + \text{length}(\text{Fan}_j(v) \cap \partial(D_i^v)))/s(v)$ and hence either length($\partial\text{Fan}_j(v)$)/area($\text{Fan}_j(v)$) $< 1/k$ would hold or length($St_j^{trim}(v) \cap \partial D_i^v$) $= \text{length}(\text{Fan}_j(v) \cap \partial D_i^v) \geq s(v)/2k$. In the latter case, for all $p \geq j$, length($St_p^{trim}(v) \cap \partial D_i^v$) $\geq s(v)/2k$.

Now choose p so that $p/(6k) \geq 8/K_0$. Therefore for all vertices v of ∂D_i

  i) length($\partial\text{Fan}_j(v)$)/area($\text{Fan}_j(v)$) $< 1/k$ for some $j \leq p$ or

  ii) length($St_p^{trim}(v) \cap \partial D_i^v$) $\geq s(v)/2k$ or

  iii) area($St_p^{trim}(v)$) $\geq 8s(v)/K_0$.

Indeed if neither i) nor ii) held for v, then for all $j \leq p$, length($\partial_{ext}\text{Fan}_j(v)$) $\geq s(v)/2k$ and hence area($St_p^{trim}(v)$) $\geq (1/3)\sum_{j=1}^{p}\text{length}(\partial_{ext}\text{Fan}_j(v)) \geq ps(v)/6k \geq 8s(v)/K_0$. This argument was inspired by [Pa].

Let $v_1, \cdots, v_m$ denote the vertices of ∂D_i. It follows that either

  a) length($\partial\text{Fan}_j(v_s)$)/area($\text{Fan}_j(v_s)$) $< 1/k$ for some vertex v_s of ∂D_i and some $j \leq p$ or

  b) $\sum_{n=1}^{m}\text{length}(St_p^{trim}(v_n) \cap \partial D_i^v) \geq S(D_i)/4k$ or

  c) $\sum_{n=1}^{m}\text{area}(St_p^{trim}(v_n)) \geq 4S(D_i)/K_0$.

We show that b) does not hold if i is sufficiently large. Claims 2,3 imply that the sum of the pairwise area overlaps of the St_p^{trim}'s is bounded above by $B(p)K_3\text{length}(\partial D_i)$. Possibility b) implies that the sum of the total pairwise area overlaps of the St_p^{trim}'s is $\geq S(D_i)/12k = A(D_i)\text{length}(\partial D_i)/12k$. Now apply Claim 1.

We show that c) does not hold if i is sufficiently large. If c) holds then we obtain area$(D_i^{trim}) \geq$ area$((\cup_{n=1}^m St_p^{trim}(v_n)) -$ St$(\partial D_i)) \geq 4S(D_i)/K_0 - B(p)K_3$ length$(\partial D_i) - S(D_i) \geq 2S(D_i)/K_0 + S(D_i) - B(p)K_3$length$(\partial D_i) \geq$ length$(\partial D_i^{trim})/K_0 + (A(D_i) - B(p)K_3)$length$(\partial D_i) >$ area(D_i^{trim}). By Claim 1, the latter inequality holds if i is sufficiently large.

Thus for fixed k, a) holds for i sufficienlty large. $\square$

Thus we can assume that each D_i has fake branching except at the vertex w_i. Furthermore by the compactness of M we can assume that for all i, $f(w_i) = w \in \tau^1$. Let $E_i = D_i - \overset{\circ}{St}(v_i)$ and $h_i = f_i|(D_i - \overset{\circ}{St}(w_i))$. Let γ be the 1-simplex of τ which contains w and let $\alpha_i = \partial$ St$(w_i) - \partial D_i$. Since h_i has no branching or fake boundary branching we can assume that for all i, length$(\partial h_i)/$area$(h_i) > K_0$. Thus there exists $K_7 > 0$ so that for i sufficiently large length $(\alpha_i)/$area$(E_i) > K_7$ which in particular implies that length$(\partial E_i - \alpha_i)/$length$(\alpha_i) \to 0$. Since the restriction of f_i to $St(w_i)$ is an immersion, $h_i(\alpha_i)$ is an embedded arc which spirals around $\partial \bar{S}t(\gamma)$.

Step 2. Either Proposition 2.4 is true or there exists an immersed normal measured lamination (λ, m) by strongly least area leaves such that $\chi(\lambda) = 0$. Furthermore $\partial \lambda$ is a lamination of the form $S^1 \times K \subset \partial \bar{S}t(\gamma)$ where K is compact and $m(K) = 1$.

Proof of Step 2. Let n_i be the number of times $h(\alpha_i)$ spirals around St(γ). Here we abuse notation by letting St(γ) denote the union of all 3-simplices of τ which nontrivially intersect $\overset{\circ}{\gamma}$. As in the proof of Proposition 1.14 normalize the various weights associated to the discs E_i by dividing by n_i. Since area$(D_i)/n_i$ is uniformly bounded, thus we can apply the proof of Proposition 1.14 to let the normalized discs E_i/n_i weakly converges to the measured lamination (λ, m). Since $\partial \lambda$ is the weak limit of the sequence $\partial E_i/n_i$ and length$(\partial E_i - \alpha_i)/n_i = 0$, any weak limit of $\partial E_i/n_i$ is a weak limit of $h_i(\alpha_i)/n_i$. Because each D_i is immersed in M, α_i is an embedded spiral in ∂St(γ), so it follows that any weak limit is of the form $S^1 \times K$ where K is compact and $m(K) = 1$. $\square$

Step 3. Obtain the contradiction.

Proof of Step 3. Let (λ', m') be the immersed measured lamination obtained from (λ, m) by capping off each component of $S^1 \times K$ by a disc. $\chi(\lambda') = 1$ and so by the Connes sphere theorem [Co] (or [MO1] for an elementary proof) there exists a measured sublamination $\mathcal{S}$ of λ' consisting only of 2-spheres and $\chi(\mathcal{S}) \geq 1$.

Since λ is strongly area minimizing it has no 2-sphere leaves. The rest of the argument follows implicitly from the proof of "strong area minimization of leaves theorem" [MO]. See the next paragraph for a hint. If some leaf S of $\mathcal{S}$ intersects $D^2 \times K$ in a single disc F, then for i sufficiently large, there exists a subdisc H_i of D_i which is normally parallel to $S - \overset{\circ}{F}$, and $\partial H_i \subset \alpha_i$. This implies that D_i is a sphere. Now suppose that some leaf S of $\mathcal{S}$ intersects $D^2 \times K$ in exactly 2-discs F_1, F_2. Given $n \in \mathbb{N}$, and an essential arc β in the annulus $A - \overset{\circ}{F}_1 \cup \overset{\circ}{F}_2$, then for i sufficiently large, there exists a subdics H_i of D_i of the form $[0,1] \times [0,n]$ where $(\partial[0,1]) \times [0,n] \subset \alpha$, and H_i naturally projects to A as part of an n-fold cyclic cover. Finally each component of $[0,1] \times \{0,n\}$ naturally projects isomorphically onto β. Thus by coning off $H_i \cap \alpha_i$ one obtains a subdisc of D_i whose boundary length is constant, independent of i, but whose area $\to \infty$ as $i \to \infty$. This thereby contradicts the least area property of D_i. A similar argument works when $S \cap D^2 \times K$ has $r > 2$ components. However the cases $r \leq 1$, together with $m(K) = 1$, implies $\chi(\mathcal{S}) = 1$ and that the generic leaf of $\mathcal{S} - D^2 \times K$ is an annulus thereby contradicting the $r = 2$ argument.

Here is the idea behind what was used in the previous paragraph. Recall that we had, by Proposition 1.14, a sequence of measured branched surfaces (B_i, m_i) which converged to a measured lamination with measure m, and that λ was the minimal sublamination which supported m. Also $j > i$ implies that (B_j, m_j) is obtained by splitting the measured branched surface (B_i, m_i). Therefore if T is a compact leaf of λ, then for i sufficiently large T embeds in B_i. If $r = 1$, then T is a disc so for C sufficiently large, any C-splitting either isolates or destroys T. Since the latter does not occur, T must appear as an isolated sector in B_j, for j sufficiently large. If

$r = 2$, then T is an annulus and a holonomy argument shows that for i sufficiently large near $T \subset B_i$, part of each $D_j, j \geq i$, spirals around T. The spiraling must get arbitrarily large to obtain the annulus T in the limit. Our $[0,1] \times [0,n]$ is a piece of that spiral. This completes the proof of Proposition 2.4 $\square$

Proof of Theorem 2.1. Combine Propositions 2.4 and 1.14 with Theorem 1.16. $\square$

§3

HYPERBOLIC CELLULATIONS

Definition 3.1. Let ψ be a piecewise linear cellulation of the 3-manifold M. This means that M is obtained by taking a disjoint union of 3-dimensional polyhedra $\sigma_1, \cdots, \sigma_n$ and pairwise identifying 2-dimensional faces in a PL fashion. Thus to each cell C_i of ψ, there is a piecewise linear mapping $f_i : C_i \to M$ which restricts to an embedding on $\overset{\circ}{C}_i$. We will usually suppress mentioning the f_i's and will denote the i-skeleton of ψ by ψ^i.

As in the setting of triangulations define a *normal curve* (resp. *local ψ-embedding*) to be an immersion $f : \alpha \to \psi^2$ transverse to ψ^1, where α is a compact 1-manifold and for each 1-simplex η in the induced triangulation on $\alpha, f|\alpha = f_i \circ t$, where $t : \eta \to C_i$ is an embedding into the 2-cell C_i with $\partial\eta$ going to distinct edges (resp. η not necessarily going to distinct edges.).

Similary define a *normal disc* to be a map $f : D^2 \to M$, which factors through an embedding into a 3-cell σ of ψ, so that $f|\partial D \to M$ is normal and (the lift into σ) crosses a given edge of σ at most once. If S is a surface, define a map $f : (S, \partial S) \to (M, \psi^2)$ to be *normal* if f is transverse to ψ^2, $f|\partial S$ is a normal curve, and for each 2-cell η in the induced cellulation on S, $f|S$ is a normal disc. If S is a surface, define a map $f : (S, \partial S) \to (M, \psi^2)$ to be *a-transverse* to ψ^2 if f is transverse to ψ^2 except at isolated points of ∂S where f appears as in Figure 1.6 and furthermore $f|\partial S$ is a local ψ-embedding. We say that $f : S \to N$ is *normal*, or S is a *normal surface*, if f is a branched immersion a-transverse to ψ^2, inducing a cellulation κ on S such that each 2-cell of κ gets mapped to a normal disc in ψ.

In particular $f|\partial S$ is a normal curve.

In order to suppress notation, we will often view S as lying in M. E.g. if Y is a submanifold of M we may refer to $S \cap Y$ rather than $f^{-1}(Y)$.

If α is a curve in ψ^2 transverse to ψ^1, then define $length(\alpha)$ to be the number of 1-cells crossed by α. If $f : S \to N$ is a-transverse to ψ^2, then f induces a stratification κ on S, where $f^{-1}(\psi^i) = \kappa^i$ the i-th strata. We'll abuse notation by calling a component of $\kappa^i - \kappa^{i-1}$ an i-cell. Finally define $area(S) = \sum_{i=1}^n s(v_i)/\text{valence}(f(v_i))$. Here $v_1, \cdots, v_n$ are the vertices of κ, $s(v_i)$ is the local number of 2-cells of κ touching v_i, and $valence(f(v_i))$ is the local number of 2-cells of ψ which come into $f(v_i)$. For example if $f : S \to M$ is an immersion of a closed surface, then $area(S)$ is simply the intersection number of S with ψ^1. This definition of area keeps track of the local branching at vertices and is additive under unions, e.g. if $S = S_1 \cup S_2, S_1 \cap S_2 \subset \partial S_1 \cap \partial S_2$, and $f|S$ is normal, then $area(S_1)+ area(S_2) = area(S)$.

View the mass of M as being concentrated near ψ^1. Think of a 1-cell ϕ as a $D^2 \times I$. If k 3-cells of ψ touch ϕ, then pretend the $D^2 \times I$ is subdivided into k wedges, one for each 3-cell. Finally imagine that each $D^2 \times t, t \in I$ is subdivided into k pie slices, the area of each slice being $1/k$.

We may use the notation $length_\psi(\alpha)$ or $area_\psi(D)$ to make clear that we are measuring with respect the cellulation ψ.

Lemma 3.2. *If $f : S \to M$ is a mapping of a closed surface into the aspherical manifold M with cellulation ψ, then f can be homotoped to a normal surface, provided that for each essential simple closed curve in S, $1 \neq [f_*(\alpha)] \in \pi_1(M)$ (i.e. f injects on simple loops).* $\square$

Proof. First homotope f to a generic least area immersion transverse to ψ^2. After a further homotopy we can assume that the induced statification κ is a cellulation. Among all such maps choose one that minimizes double points of $f|f^{-1}(\psi^2)$. If η is a 2-cell of κ which maps into the 3-cell σ, then $f|\partial\eta$ is an embedding (when lifted) into σ. Otherwise a homotopy of f supported near η reduces the number of double

points. Thus we can assume that the restriction of f to each 2-cell is an embedding (when lifted) into its associated 3-cell. Such an f is normal. □

Lemma 3.3. *(Epstein - Penner [EP]). If N is a complete noncompact hyperbolic 3-manifold of finite volume, then N has a cellulation by ideal polyhedra.* □

Question 3.4. (Igor Rivin) Does every complete noncompact hyperbolic 3-manifold of finite volume have a cellulation by ideal triangles?

Definition 3.5. If the complete noncompact finite volume orientable hyperbolic 3-manifold N has a cellulation by ideal polyhedra, then removing a neighborhood of the ends of N one obtains a compact manifold M whose boundary is a non empty union of tori. The induced cellulation on M is a cellulation by truncated ideal polyhedra. We call such a cellulation a *relative hyperbolic cellulation*. Call the newly created 2-cells (on ∂M) the *facets*.

Definition 3.6. Define a *combinatorial geometry* (κ, angle) on the compact surface S to be a cellulation κ together with a function angle:{vertices of 2-cells of ψ} $\to [0, \pi]$ such that if $v \in \kappa^0$, $w_1, \cdots, w_n$ are the vertices of 2-cells of κ which are identified with the vertex v of κ and Angle(v) $\overset{def}{=} \sum_{i=1}^n$ angle(w_i), then Angle(v) $= 2\pi$ for $v \in \overset{\circ}{S}$.

If C is a 2-cell with vertices $v_1, \cdots, v_n$, then define $\int_C K = \sum_{i=1}^n$ angle(v_i) $- (n-2)\pi = 2\pi - \sum_{i=1}^n (\pi -$ angle(v_i)) and $\int_S K = \sum_{i=1}^m \int_{C_i} K$, where $C_1, \cdots, C_m$ are the 2-cells of κ. If v is a vertex of $\kappa, v \in \partial S$, then define $\gamma_g(v) = \pi -$Angle(v) and $\int_{\partial S} \gamma_g = \sum_{v_i \in \kappa^0 \cap \partial S} \gamma_g(v_i)$. Call a 2-cell C respectively *negatively curved, flat* or *positively curved* if $\int_C K$ is respectively $< 0, 0,$ or > 0.

PL Gauss Bonnet theorem [We] 3.7. *If S has a combinatorial geometry (ψ,angle), then $\int_S K + \int_{\partial S} \gamma_g = 2\pi\chi(S)$.* □

Remark 3.8. The proof is an elementary combinatorial exercise. See [We] for the case of a closed surface.

Remark 3.9. The following result is the obvious generalization of an observation of Andrew Casson which was stated for normal closed surfaces with respect to ideal

triangulations on hyperbolic 3-manifolds.

Lemma 3.10. *Let ψ be a relative hyperbolic cellulation on Y. If S is an immersed surface in Y which is normal with respect to ψ. Then f induces on S a combinatorial geometry (κ, angle) where κ is the induced cellulation on S and if $v \in \overset{\circ}{Y}$, then angle(v) is the dihedral angle at the edge of the associated 3-cell of ψ. Otherwise angle(v) = $\pi/2$.* $\square$

Corollary 3.11. *(Casson [Ca]) Let ψ be a relative hyperbolic cellulation on Y. If S is an immersed closed surface in Y which is is normal with respect to ψ, then $\chi(S) \leq 0$. Equality holds if and only if S is a normally parallel to a peripheral torus.*

Proof. Combine the the following result with Lemma 3.10 and Theorem 3.7. $\square$

Rivin's Lemma 3.12 [Ri]. *If α is a normal closed curve on the boundary of an ideal hyperbolic polygon which crosses (not necessarily distinct) edges $e_1, \cdots, e_n$, then $2\pi \leq \sum_{i=1}^{n}(\pi - d(e_i))$, where $d(e_i)$ is the dihedral angle at e_i. Equality holds if and only if α is a simple closed curve which separates off a single facet from the rest.*

Proof. Use the fact that any ideal polygon is the union of ideal tetrahedra and use the fact that for an ideal tetrahedron, dihedral angles of opposite edges are equal. $\square$

Remark 3.13. Its a famous theorem of Igor Rivin [Ri], that the converse is also true. That is any combinatorial geometry on a polygon satisfying the conclusion of Lemma 3.12 arises from a unique ideal hyperbolic polygon.

Remark 3.14. A closed surface of genus g which is (Riemannian) least area in a hyperbolic 3-manifold has area bounded above by $(2g - 2)\pi$. The rest of §3 is devoted to establishing a similar result for least area surfaces with respect to relative hyperbolic cellulations. Unlike the Riemannian setting, it may happen that $\int_C K > 0$, where C is a normal disc. However, we show that if S is a normal surface in the relative hyperbolic cellulation, then after a small homotopy and redefinition

of the combinatorial geometry, all such "positively curved" discs can be eliminated. It will then follow that the total integral over the negatively curved normal discs is bounded below by $2\pi\chi(S) - \pi$ length(∂S). Therefore by the finiteness of normal disc types the negative curvature is concentrated in a finite area subsurface. After shoving, the flat part of S out of Y, we are left with a surface S_Y whose area is bounded above by $C_0($ length$(\partial S) - \chi(S))$ where C_0 is a constant depending only on ψ.

Definition 3.15. A compact immersed surface S in the 3-manifold Y with relative hyperbolic cellulation ψ is *h-normal* if S is a normal surface with respect to ψ and satisfies the following additional properties.

i) $\partial S \cap \partial Y = \emptyset$,

ii) S has no fake boundary branching, and

iii) for each 1-cell β in κ (the induced cellulation on S) which lies in the 2-cell η of ψ, if β separates in η, a single vertex from the rest, then either $\beta \subset \partial Y$, or $\beta \cap \partial S, \neq \emptyset$. Call an arc of the latter type *bad*. See Figure 3.1.
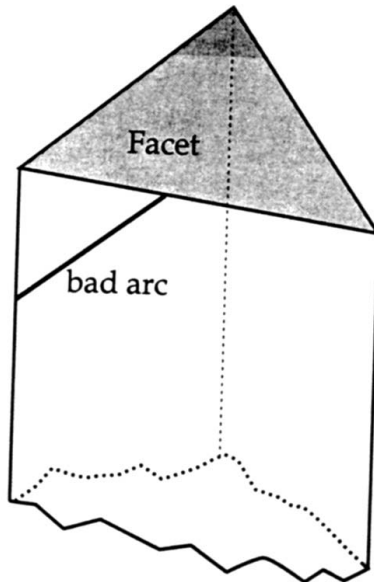

Figure 3.1

The next lemma is a version of Lemma 3.2 for surfaces in relative hyperbolic cellulations. It says that at the cost of pushing part of S out of Y, S can be homotoped to be h-normal, without decreasing $\chi(S)$.

Lemma 3.16. *Let Y be a codimension-0 submanifold of the compact aspherical 3-manifold M with $Y \cap \partial M = \emptyset$. Let ψ be a relative hyperbolic cellulation on Y and $f : S \to M$ satisfy $\partial S \subset \overset{\circ}{Y}, f|\partial S$ is normal, f injects on simple loops, f is transverse to ψ^2 near ∂Y (and in particular transverse to ∂Y) and f is boundary incompressible in the following sense. There is no properly embedded arc $\alpha \subset S \cap Y$ with endpoints in distinct 1-cells of the induced triangulation on ∂S, such that $f|\alpha$ can be homotoped rel $\partial\alpha$ into a 2-cell of ψ. Assume also that no component of $f^{-1}(Y)$ is a 2-disc disjoint from ∂S. Then S can be homotoped rel ∂S to S_1 such that $S_1 \cap Y$ is h-normal respect to ψ and $\chi(S_1 \cap Y) \geq \chi(S \cap Y)$. Furthermore no component of $S_1 \cap Y$ is a disc disjoint from ∂S_1.*

Proof. Using the boundary incompressibility of f, first homotope f rel $\partial S \cup (S \cap \partial Y)$ to an immersion a-transverse to ψ^2. Homotope f near ∂S to eliminate fake boundary branching at the expense of introducing intersections as in Figure 1.6. If some component of $f^{-1}(Y) - f^{-1}(\psi^2)$ is not a disc, then f can be homotoped rel ∂S to reduce $|S \cap \psi^1|$. If for some 2-cell C of $\kappa, f(\partial C)$ crossed a 1-cell e of ψ more than once, then a homotopy of f reduces $|S \cap \psi^1|$. (But $\chi(S \cap Y)$ may rise if $e \subset \partial Y$.) If S had a bad arc β, then a small homotopy eliminates this arc thereby reducing $|S \cap \overset{\circ}{Y} \cap \psi^1|$ and possibly increasing $|S \cap \psi^1|$. Argue as in the proof of Lemma 3.2 to homotope f so that the restriction of f to each 2-cell of κ is an embedding into its associated 3-cell. All of these homotopies can be accomplished without decreasing $\chi(S \cap Y)$ or introducing disc components of $S \cap Y$. The result follows by induction on $(|S \cap \overset{\circ}{Y} \cap \psi^1|, |S \cap \psi^1|)$ ordered lexicographically. $\square$

Proposition 3.17. *Let ψ be a relative hyperbolic triangulation on the compact 3-manifold Y which is contained in the interior of the compact 3-manifold M. Let $f : S \to M$ be a map of a compact surface transverse to ∂Y such that $f|\partial S \subset \overset{\circ}{Y}$. There exists a constant C_0 such that if $S \cap Y$ is an h-normal surface, then S is homotopic*

rel ∂S to a surface S_1, such that area$(S_1 \cap Y) \leq C_0(length(\partial S) - \chi(S \cap Y))$.

Corollary 3.18. *Under the hypothesis of Proposition 3.17, there exists a C_1, such that length$(\partial(S_1 \cap Y)) \leq C_1(length(\partial S) - \chi(S \cap Y))$.*

Proof. Apply the following Lemma 3.19 taking $C_1 = vC_0$. □

Remark. If S is a Riemannian least area surface in Y which goes far out the cusps, then the region out in the cusp accounts for little Riemannian area, but large PL area. The passage from S to S_1 amounts to shoving the cusp stuff out of Y.

Lemma 3.19. *Let v be the maximal valence of an edge of ψ. If T is a-transverse to ψ, then area(T) $\geq$ length(∂T)/v. □*

Proof of Proposition 3.17. Let κ be the induced cellulation on $S \cap Y$. Define a combinatorial geometry (κ, angle) on $S \cap Y$ as follows. Let v be a vertex of the 2-cell η of κ.

If $f(v) \in \partial Y$, define angle$(v) = \pi/2$.

If $v \in \partial S \cap \psi^1$, then define angle$(v) = 0$.

If $v \in \partial S - \psi^1$, then define angle$(v) = \pi/2$ (Such vertices arise as in Figure 1.6.)

Otherwise angle(v) is the associated dihedral angle of the 3-cell σ of ψ which contains η.

Claim. If C is a 2-cell of κ, then C is nonpositively curved. If C is flat, then either

a) $f(C)$ is parallel to a facet and C is disjoint from ∂S, or

b) $f(\partial C)$ encircles a vertex of ψ, or

c) $f(C)$ encircles a single edge e on the boundary of the 3-cell which contains $f(C)$, $e \not\subset \partial Y$ and $C \cap \partial S = \emptyset$. See Figure 3.2.

Type a)   Type b)   Type c)
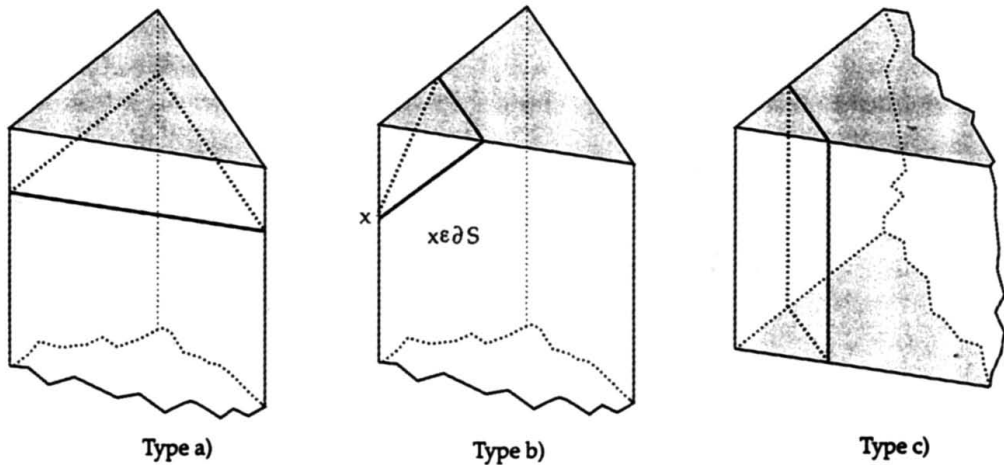
Flat 2-cells
Figure 3.2

Proof of Claim. If σ is the 3-cell which contains $f(C)$, then $f|C$ lifts to an embedding (also called f) of C into σ. First suppose that $f(\partial C)$ is disjoint from the facets. Let $\hat{C}$ denote C with the following combinatorial geometry. The 1-skeleton of $\hat{C}$ equals $\partial \hat{C}$, $f^{-1}(\psi^1) \cap C$ are the vertices of $\partial \hat{C}$, and each point x of $f^{-1}(\psi^1)$ is assigned the dihedral angle of σ at $f(x)$. Thus by Rivin's lemma, $\hat{C}$ is either negatively curved or is flat and parallel to a facet. If $C \cap \partial S = \emptyset$, then C and $\hat{C}$ will have the same combinatorial geometry. If $C \cap \partial S \neq \emptyset$, then by comparing the combinatorial geometry of $\hat{C}$ with that of C it follows that $\int_C K < \int_{\hat{C}} K \leq 0$.

If ∂C crosses exactly one facet and has no bad subarcs, then by pushing ∂C off of the facet, in at least one of the two possible ways, one obtains a new normal curve bounding a disc C' which is not more curved than C. Any two facets of a truncated polyhedron are connected by at most one edge. Thus if C' is flat, then C is negatively curved. If ∂C has a bad subarc β, then by decree angle$(v) = 0$ for the non facet vertex v of $\partial \beta$. Another application of Rivin's Lemma implies that such a C is flat if and only if ∂C crosses exactly 3 edges of ψ.

If ∂C crosses exactly two facets, and $C \subset \sigma$ is a 3-cell of ψ, then a similar argument to the one above shows that either C is negatively curved or ∂C separates off a single edge of σ disjoint from ∂S.

If ∂C crosses more than two facets, then C is negatively curved.   □

Even accounting for the possibility that part of a 2-cell lies in ∂S, there are only a bounded number of combinatorial possibilities for negatively curved h-normal 2-cells in ψ. Thus there exists $c_1 > 0$ which depends only on ψ such that $-c_1$ is the maximal possible value of $\int_C K$ where C is a negatively curved h-normal 2-cell.

By definition for $v \in \kappa^0 \cap \partial S$ either $f(v) \in \psi^1$ in which case $\gamma_g(v) = \pi$ or $f(v) \notin \psi^1$ in which case $\gamma_g(v) = 0$. Also if α is a component of $f^{-1}(\partial Y)$, then $\int_\alpha \gamma_g = 0$. Therefore $\int_S K = 2\pi\chi(S) - \pi\mathrm{length}(\partial S)$ and hence there are at most $\frac{1}{c_1}(\pi\mathrm{length}(\partial S) - 2\pi\chi(S))$ negatively curved 2-cells of κ.

Let c_2 be the maximal valence of an edge of ψ. The Claim and the fact that there is no fake boundary branching implies that there are at most $c_2\,\mathrm{length}(\partial S)$ flat 2-cells of type b). Since a flat 2-cell of type c) can only share an edge with either another type c) flat 2-cell or a negatively curved 2-cell, and that a string of c_2 type c) flat 2-cells leads to an area reducing homotopy of S, it follows that after homotopy there can be at most $\frac{c_3 c_2}{c_1}(\pi\mathrm{length}(\partial S) - 2\pi\chi(S))$ type c) flat 2-cells. Here c_3 is the maximal length of the boundary of an h-normal 2-cell. Thus for some constant c_4, there are at most $c_4(\mathrm{length}(\partial S) - \chi(S))$ 2-cells of κ which are not flat of type a). Now let $W \subset S$ be the union of type a) flat 2-cells. Homotope S to S_1 by a homotopy supported in $\overset{\circ}{W}$ to push most of W out of Y. So if T denotes W with a small collar of ∂W removed, then $S_1 \cap Y = ((S \cap Y) - \overset{\circ}{T}) \cup A$, where A is a union of annuli connecting ∂T straight to ∂Y. Therefore $\mathrm{area}(S_1 \cap Y) < \mathrm{area}((S - \overset{\circ}{W}) \cap Y) + 2c_2\mathrm{length}(\partial W) < c_5\mathrm{area}((S - \overset{\circ}{W}) \cap Y) < c_5 c_3 c_4(\mathrm{length}(\partial S) - \chi(S)) = C_0(\mathrm{length}(\partial S) - \chi(S))$.   □

Proposition 3.20. *Let ψ be a relative hyperbolic triangulation on the compact 3-manifold $Y \subset \overset{\circ}{M}$. Suppose that $f : S \to M$ is a map of a compact surface such that for each component δ of ∂S, either $\delta \cap Y = \emptyset$ or $f|\delta$ is an immersed curve in $\psi^2 \cap \overset{\circ}{Y}$ which is transverse to ψ^1, f injects on simple loops, f is transverse to ∂Y, f is a-transverse to ψ^2 and no component of $f^{-1}(Y)$ is a disc disjoint from ∂S. Then S can be homotoped rel ∂S to S_1 such that $\mathrm{area}(S_1 \cap Y) \leq Max(0, C_0(\mathrm{length}((\partial S) \cap Y) - \chi(S \cap Y)))$.*

Proof of Proposition 3.20. The main improvement of this Proposition over the last is that $f : S \to M$ is not h-normal. We argue by induction on $c(S) = \text{length}((\partial S) \cap Y) - \chi(S \cap Y)$ and can assume that S is connected. If $c(S) = -2$, then S is a 2-sphere in Y and can be homotoped out of Y. Now assume that the Proposition is true for $c(S) < k$. Using the induction hypothesis, its routine to verify the Proposition if $(\partial S) \cap Y$ is not a union of normal curves.

Suppose that there is an arc $\alpha \subset f^{-1}(Y)$ with endpoints in distinct 1-cells of κ such that $f|\alpha$ is homotopic into a 2-cell of ψ, via a homotopy fixing $\partial \alpha$ and supported in Y. Boundary compress along α to get a new map $f_1 : T \to M$ which satisfies the hypothesis of the Proposition, and $c(f_1) < c(f)$. By induction, f_1 is homotopic rel ∂T to g_1 which satisfies the conclusions of the Proposition. If T_1 is the resulting surface, then attach a little 1-handle to T_1 (missing ψ^1) to obtain a surface S_1 which is homotopic to S rel ∂S and satisfies the conclusion.

Therefore we can assume that f satisfies the hypothesis of Lemma 3.16. Now apply Lemma 3.16 and Proposition 3.17. $\square$

Corollary 3.21. *If S_1 is as in Proposition 3.20, then $\text{length}(\partial(S_1 \cap Y))$ $\leq Max(0, C_1(\text{length}((\partial S) \cap Y) - \chi(S \cap Y)))$.* $\square$

Remark 3.22. The main results of §1-2 can all be stated in terms of cellulations rather than triangulations. In particular Proposition 1.14, Theorem 2.1 and Proposition 2.4 hold by substituting triangulation with cellulation. The proofs parallel those of §1-2. Of course a version of Proposition 1.10 also holds for cellulations, but there is not an obvious canonical branched surface which carries all normal laminations. For the sake of reference we single out

Proposition 3.23. *Let M be a closed 3-manifold with cellulation ψ. $\pi_1(M)$ is not negatively curved if and only if there exists a sequence of least area normal immersed discs $f_i : D_i \to M$, such that $\text{length}(\partial f_i)/\text{area}(f_i) \to 0$ and no f_i has fake boundary branching.* $\square$

<center>§4</center>

<center>THE UBIQUITY THEOREM</center>

Definition 4.1. If α is a homotopically trivial curve in the 3-manifold M and $k \subset M - \alpha$ is a smooth simple closed curve, then define the *wrapping number* $\mathrm{wr}(\alpha, k) = \min\{|E \cap k|$ where $E \subset M$ is an immersed 2-disc transverse to k such that $\partial E = \alpha\}$.

Theorem 4.2. *Let k be a smooth simple closed curve in the closed, irreducible 3-manifold M with cellulation ψ such that k is transverse to ψ^2, disjoint from ψ^1 and for each 2-cell $\eta \subset \psi$, $|k \cap \eta| \leq 1$. Then either M is toroidal, or k is contained in a 3-cell or there exists a constant C such that for any least area disc D a-transverse to ψ^2 with $\partial D \cap k = \emptyset$ we have $area(D) \leq C(wr(\partial D, k) + length(\partial D))$.*

Remark. Here length and area are measured as in Definition 3.1. Recall that normal immersed discs are examples of a-transverse discs.

The following result is an immediate consequence of Theorem 4.2.

Ubiquity Theorem 4.3. *Let $k \not\subset B^3$ be a smooth simple closed curve in the closed, atoroidal, irreducible 3-manifold M with cellulation ψ such that k is transverse to ψ^2, disjoint from ψ^1 and for each 2-cell $\eta \subset \psi$, $|k \cap \eta| \leq 1$. There exists constants K and L such that if D is a least area disc a-transverse to ψ^2 with $\partial D \cap k = \emptyset$ and $length(\partial D)/area(D) < L$, then $wr(\partial D, k)/area(D) > K$.* $\square$

Corollary 4.4. *Let λ be a quasi-least area semi-Euclidean lamination in the closed atoroidal irreducible 3-manifold M. If $k \subset M - \lambda$ is a smooth simple closed curve, then k lies in a 3-cell.*

Proof of Corollary 4.4. Suppose that λ is normal with respect to the cellulation ψ and all measurements are taken with respect to ψ. After passing to a subdivision of ψ and isotopy of k we can assume that for each 2-cell η of ψ that $|k \cap \eta| \leq 1$. Note that the property of a leaf being conformally Euclidean or quasi-least area is preserved after passing to subdivision. If L is a conformally Euclidean leaf of λ, then by the Ahlfors Lemma, [Ca] p. 499, there exists a sequence of embedded discs $\tilde{E}_1 \subset \tilde{E}_2 \subset \cdots \subset \tilde{L}$ the universal covering of L, such that $length(\partial \tilde{E}_i)/area(\tilde{E}_i) <$

$1/i$. Since λ is quasi least area there exists a constant K_1 such that for each i, area($\tilde{E}_i$) $\leq K_1$area(D_i) where D_i is a least area disc in $\tilde{M}$ with $\partial \tilde{E}_i = \partial D_i$. If E_i is a slightly perturbed projection of $\tilde{E}_i$ to M such that ∂E is embedded, then $\mathrm{wr}(\partial E_i, k) = 0$ all i. Also if $k \not\subset B^3$, and i is sufficiently large then we obtain a contradiction to Theorem 4.3. $\square$

Remarks 4.5. i) In words the Ubiquity Theorem says that the area of a least area disc D of small isoperimetric ratio is proportional to the wrapping number of ∂D and k.

ii) The proof we give relies on Thurston's hyperbolization theorem for Haken 3-manifolds. One can give a much less elegant proof which uses only [Gr] and standard 3-manifold topology.

Proof of Theorem 4.2. To obtain a clue, read the proof of case 1. For a complete and more detailed argument read the proof of case 2.

Case 1. $M - k$ is hyperbolic.

Proof of Case 1. It suffices to consider the case that ψ is obtained by attaching a 2-cell and a 3-cell to a relative hyperbolic cellulation on $M - \overset{\circ}{N}(k)$. And it suffices to prove Theorem 4.2 for ψ-least area discs D such that $\partial D \subset M - N(k)$. Given such a disc D, let $E \subset M$ be an immersed disc such that $\partial E = \partial D$ and $\mathrm{wr}(\partial D, k) = |E \cap k| = |E \cap \partial N(k)|$. By Proposition 3.20 and Corollary 3.21 E can be homotoped to a disc E_1 rel ∂E, such that area(D) $\leq$ area(E_1) $=$ area($E_1 \cap (M - \overset{\circ}{N}(k))$)$+$ area($E_1 \cap N(k)$) $\leq$ area($E_1 \cap (M - \overset{\circ}{N}(k))$)$+$ length($\partial E_1 \cap N(k)$) $\leq (C_0 + C_1)$(length(∂E)$- \chi(E \cap (M - \overset{\circ}{N}(k)))$) $= (C_0 + C_1)$(length(∂D)$+$ $\mathrm{wr}(\partial D, k) - 1$) $< (C_0 + C_1)$(length(∂D)$+$ $\mathrm{wr}(\partial D, k)$). $\square$

Case 2. General case

Proof of Case 2.

If k does not lie in a 3-cell, then $M - \overset{\circ}{N}(k)$ is irreducible and hence there exists by the characteristic manifold theory of [JS], [J] a collection of π_1-injective pair wise disjoint embedded tori $T_0, T_1, \cdots, T_n \subset M$ such that each component of

$(M - \overset{\circ}{N}(k)) - \cup_{i=0}^{n} T_i$ is either a finite volume hyperbolic 3-manifold or the interior of a graph manifold. (Recall that all 3-manifolds in this paper are orientable and that a graph manifold is the union of finitely many Seifert fibred spaces glued together along some of their boundary components.) Here $T_0 = \partial N(k)$. By Thurston [Th] $M - k$ is hyperbolic if and only if $\{T_0, T_1, \cdots, T_n\} = \{T_0\} = \{\partial N(k)\}$ and $M - \overset{\circ}{N}(k)$ is a not a graph manifold.

If T is an embedded π_1-injective non boundary parallel torus in $M - \overset{\circ}{N}(k)$, then either T is π_1-injective in M and hence Theorem 4.2 is proved or T is compressible. By standard 3-manifold topology, if T is compressible, then either T bounds a unique solid torus W in M or T bounds a *cube with knotted hole* W. In the former case $k \subset W$. In the latter case there exists an embedded $B^3 \subset M$ and a knotted properly embedded arc $\beta \subset B^3$ such that $W = B^3 - \overset{\circ}{N}(\beta)$ and thus $k \cap W = \emptyset$. Note that $k \not\subset B^3$ implies that T cannot bound both a solid torus and a cube with knotted hole. For each i, let W_i be the associated solid torus or cube with knotted hole bounded by T_i. Observe that if $T_i \subset \overset{\circ}{W}_j$, then $W_i \subset \overset{\circ}{W}_j$.

The W_i's are partially ordered by inclusion, so reorder the T_i's so that W_0 is the maximal solid torus region (which is necessarily nonempty since for some $i, W_i = N(k)$) and $W_1, \cdots W_r$ are the maximal cube with knotted hole regions. Let $Y = M - \overset{\circ}{W}_0 \cup \cdots \cup \overset{\circ}{W}_r$.

Claim 1. Either Theorem 4.2 is true or Y is atoroidal and hence by Thurston [Th] has a complete hyperbolic metric of finite volume.

Proof of Claim 1. By construction Y is either atoroidal or is a graph manifold. We will assume the latter and derive a contradiction. If $r > 0$, then Figure 4.1 shows how to reembed $Y \cup W_0 \cup \cdots \cup W_{r-1}$ into M in such a way that $\partial W_r'$ bounds a solid torus V_r in $M - Y' \cup W_0' \cup \cdots \cup W_{r-1}'$, where W_i' (resp. $\partial W_r', Y'$) denotes the reembedded W_i (resp. $\partial W_r, Y$). There are an infinite number of such reembeddings, by "twisting the neck". Now fix a graph structure on Y. This structure induces a Seifert fibering on ∂W_r. At most one reembedding corresponds to spanning Seifert fibres of $\partial W_r'$ by meridinal discs of V_r. After any other reembedding, $M - W_0' \cup$

$\cdots \cup W'_{r-1}$ is a graph manifold. By repeating this operation $r - 1$ more times, we conclude that M is obtained by attaching a solid torus to the boundary of a graph manifold. Since such a manifold is either reducible, has finite π_1 or has a π_1-injective immersed torus, Claim 1 is established. Note that if $\pi_1(M)$ is finite, then M has a linear isoperimetric inequality. $\square$
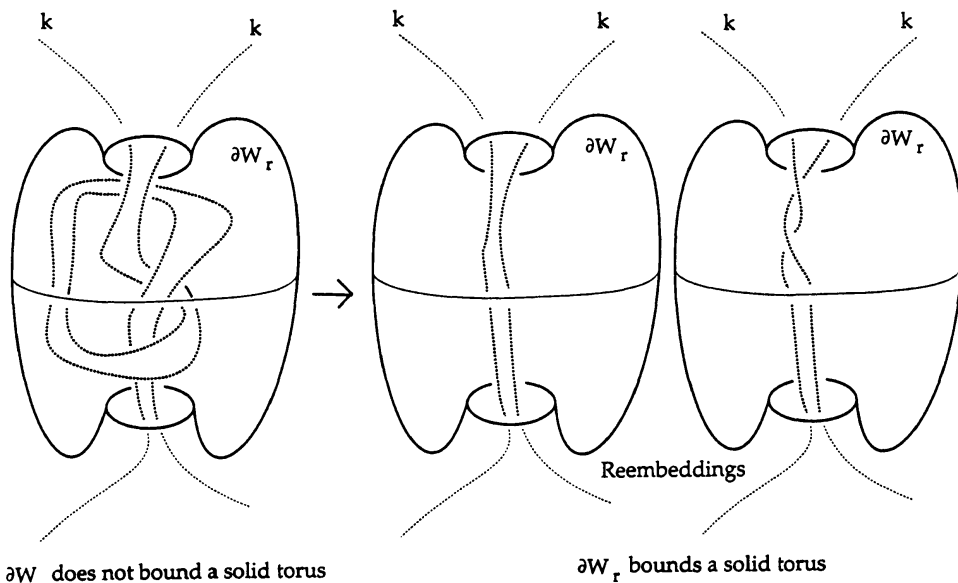

∂W does not bound a solid torus   ∂W_r bounds a solid torus

Figure 4.1

Given a ψ-least area disc D, define

$$W_\psi(D, k) = \frac{\mathrm{wr}(\partial D, k) + \mathrm{length}_\psi(\partial D)}{\mathrm{area}_\psi(D)}.$$

We may suppress the subscript ψ when the cellulation is understood.

Claim 2. We can assume that ψ satisfies

a) $\psi|Y$ is a relative hyperbolic cellulation and in particular for each $i \leq r, \psi$ restricts to a cellulation on W_i.

b) There exists a $D^2 \times S^1 \subset \overset{\circ}{W}_0(k)$ called $N_1(k)$ such that $k \subset \overset{\circ}{N}_1(k)$ is a core of $N_1(k)$, $\psi^2 \cap N_1(k)$ consists of $\partial N_1(k)$ and a finite number of pairwise disjoint meridinal discs. Furthermore if σ is a 3-cell of $\psi|N_1(k)$, then $k \cap \sigma$ is a properly embedded unknotted arc. Note that $\psi^1 \cap \overset{\circ}{N}_1(k) = \emptyset$. See Figure 4.2
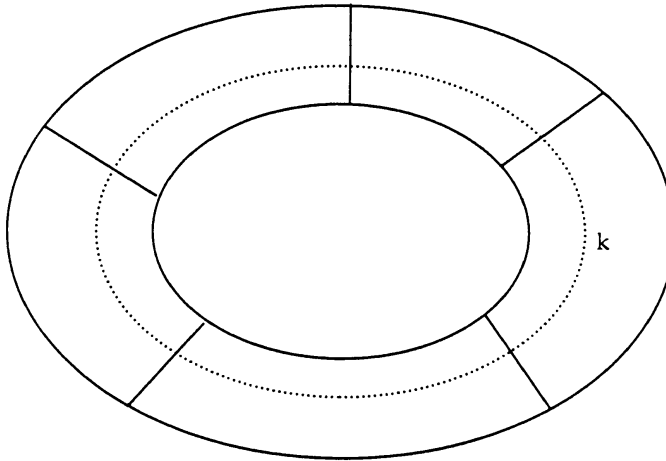
A schematic view of $N_1(k)$

Figure 4.2

Proof of Claim 2. If ϕ is a subdivision of ψ and Theorem 4.2 holds for ϕ, then Theorem 4.2 holds for ψ. Use the fact that there exists a $K_1 > 0$ such that if D is a ψ-least area disc, then there is a ψ least area disc, also called D, with $W_\psi(D, k)$ unchanged such that D is a K_1-quasi ϕ-least area disc. Furthermore $\text{length}_\psi(\partial D) \leq \text{length}_\phi(\partial D) \leq K_1 \text{length}_\psi(\partial D)$. The last inequality uses the fact that an embedded arc α in a 2-cell of ψ is isotopic rel $\partial \alpha$ to an arc crosing ϕ^1 a uniformly bounded number of times, via an isotopy disjoint from k. (This is where we use the hypothesis that for each 2-cell η of ψ, $|k \cap \eta| \leq 1$.) Thus a sequence $\{D_i\}$ of ψ-least area discs with $W_\psi(D_i) \to 0$ gives rise to a sequence $\{E_i\}$ of ϕ-least area discs with $W_\phi(E_i) \to 0$.

Conversely suppose that ϕ is a subdivision of ψ such that if σ is a 3-cell of ψ and $\sigma \cap k \neq \emptyset$, then $\phi|\sigma = \psi|\sigma$. Then standard arguments show that there exists a $K_2 \geq 1$ such that a ϕ-least area disc D give rise to a ψ-least area disc E where $\frac{\text{length}_\psi(\partial E)}{\text{area}_\psi(E)} \leq \frac{K_2 \text{length}_\phi(\partial D)}{\text{area}_\phi(D)}$ and ∂E is homotopic to ∂D via a homotopy disjoint from k. Thus if Theorem 4.2 holds for ψ, then it holds for ϕ. Similarly this result follows in the following case. Here if σ is a 3-cell of ψ, then either $k \cap \sigma = \emptyset$ or $k \cap \sigma$

is an unknotted arc and ϕ is a subdivision of ψ such that $\phi^2 \cap k = \psi^2 \cap k$.

Given ψ, we first construct a subdivision ψ_1 with the property that for each 3-cell σ of $\psi_1, k \cap \sigma = \emptyset$ or $k \cap \sigma$ is an unknotted arc. The subdivision is created on each 3-cell B of ψ as follows. Let $t = B \cap k$. Think of B as $D^2 \times I$ with $\partial t \subset D^2 \times 0$ and t in generic position with respect to the height function coming from the I-factor. Now attach a finite number of pairwise disjoint properly embedded 2-cells $C_1, \cdots, C_n \subset B$, one for each local minimum of $t \subset \overset{\circ}{B}$, with the property that for each i, C_i separates off a 3-cell B_i with $B_i \cap t$ is an unknotted arc and $t \cap (B - \cup_{i=1}^n \overset{\circ}{B}_i)$ is a trivial B^3-link on s components. Now subdivide each C_i along a properly embedded arc such that each "half" of of C_i hits t exactly once. See Figure 4.3. After subdividing along another $s - 1$ 2-cells we obtain a cell division on B such that each 3-cell intersects t in an unknotted arc. ψ_1 is the result of these $n + s - 1$ subdivisions of ψ. One readily checks in $n + s - 1$ steps that Theorem 4.2 holds for ψ if and only if it holds for ψ_1.

a)   Figure 4.3   b)
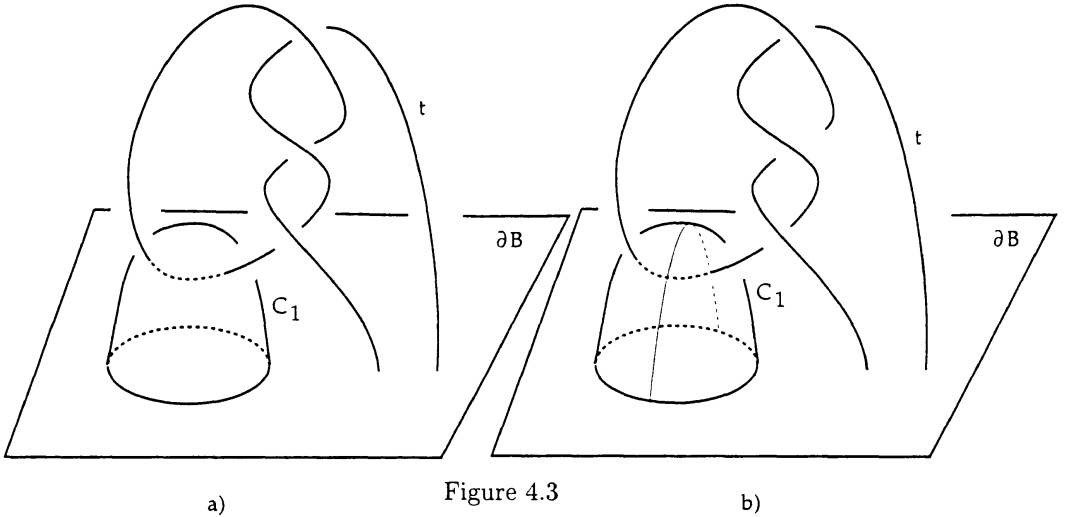
Let $N_1(k)$ be a $D^2 \times S^1$ in $\overset{\circ}{W}_0$ whose core is k such that $N_1(k) \cap \psi_1^1 = \emptyset$ and $N_1(k) \cap \psi_1^2$ is a disjoint union of meridinal discs. Subdivide ψ_1 to ψ_2 so that $N_1(k) \cap \psi_2^2 = N_1(k) \cap \psi_1^2 \cup \partial N_1(k)$. By the second paragraph of this proof, Theorem 4.2 holds for ψ_2 if and only if it holds for ψ_1.

By [EP] there exists a hyperbolic cellulation on Y and hence there exists a

cellulation ψ_3 on M satisfying a) and b) and which agrees with ψ_2 near $N_1(k)$. By the uniqueness of PL structures for 3-manifolds [Mo], there exists a cellulation ψ_4 on M isomorphic to ψ_3 via an isomorphism which restricts to the identity on $N_1(k)$ such that ψ_4 and ψ_2 have a common subdivision ψ_5 and $\psi_5|N_1(k) = \psi_4|N_1(k) = \psi_2|N_1(k)$. By the first two paragraphs Theorem 4.2 holds for ψ_4 if and only if it holds for ψ_5 if and only if it holds for ψ_2. Therefore Theorem 4.2 holds for ψ_4 if and only if it holds for ψ. Since ψ_4 satisfies the conclusions of Claim 2, that result follows. $\square$

Claim 3. It suffices to prove Theorem 4.2 for discs D such that $\partial D \subset \overset{\circ}{Y}$.

Proof of Claim 3. It suffices to prove Theorem 4.2 for discs D such that $\partial D \cap N_1(k) = \emptyset$. Indeed by construction of $N_1(k)$, a small homotopy disjoint from k, pushes ∂D off of $N_1(k)$ and so D gives rise to a least area disc E a-transverse to ψ with $E \cap N_1(k) = \emptyset, \mathrm{wr}(\partial E, k) = \mathrm{wr}(\partial D, k)$ and D, E have approximate isoperimetric inequalities.

Since balls and solid tori have linear isoperimetric inequalities it follows that there exists a K_3 such that if γ is a locally ψ-embedded homotopically trivial curve in $M - N_1(k)$ then there exists a homotopy $H : S^1 \times I \to M$ of γ to γ_1 such that γ_1 is locally ψ-embedded, $\gamma_1 \subset \overset{\circ}{Y}$ and $\mathrm{area}(H) \leq K_3\mathrm{length}(\gamma)$. This implies that if D (resp. E) is a least area disc spanning γ, (resp. γ_1), then

$$\mathrm{area}(E) \geq \mathrm{area}(D) - \mathrm{area}(H)$$

$$\mathrm{length}(\gamma_1) \leq \mathrm{length}(\gamma) + v(\mathrm{area}(H)) \leq (1 + vK_3)\mathrm{length}(\gamma)$$

$$\mathrm{wr}(\gamma_1, k) \leq \mathrm{wr}(\gamma, k) + 3\mathrm{area}(H) \leq \mathrm{wr}(\gamma, k) + 3K_3\mathrm{length}(\gamma)$$

The first and third inequalities are immediate. The second follows from Lemma 3.19. To obtain the fourth, note that H can be chosen so that if η is a 2-cell in the induced cellulation on $S^1 \times I$, and $H(\eta) \cap \overset{\circ}{N}_1(k) \neq \emptyset$, then $H|\eta$ is an embedding and $H(\eta) \cap \psi^1 \neq \emptyset$. (This uses the fact $\gamma \cap N_1(k) = \emptyset$.) Since H can be chosen so that $|\eta \cap k| \leq 1$ and $\mathrm{area}(H|\eta) \geq \frac{1}{3}$ it follows that $|H^{-1}(k)| \leq 3\mathrm{area}(H)$.

Thus a sequence of discs $\{D_i\}$ such that $W(D_i, k) \to 0$ gives rise to a sequence $\{E_i\}$ with $W(E_i, k) \to 0$ and $\partial E_i \subset \overset{\circ}{Y}$. $\square$

Let E be an immersed disc in M a-transverse to ψ^2 such that $\partial E = \partial D$ and $|E \cap k| = wr(\partial D, k)$. Among all such discs with these properties choose one that minimizes $|\partial Y \cap E|$. Let $E_Y = E \cap Y$. Assume that length$(\partial E) > 0$.

Claim 4. $-\chi(E_Y) < \text{wr}(\partial E, k)$

Proof of Claim 4. Since Y has no essential annuli, and $|\partial Y \cap E|$ is minimal, each component of E_Y disjoint from ∂E is a disc with at least two open discs removed. Define a partial ordering on the components $F_1, \cdots, F_m$ of E_Y, by $F_i < F_j$ if F_i is separated from ∂E by F_j. Let $\tilde{F}_i = \cup_{F_j \leq F_i} F_j$ and $\hat{F}_i$ be the subdisc of E bounded by the outermost component of ∂F_i. Since for some $i, E_Y = \tilde{F}_i$ and $E = \hat{F}_i$ it suffices to show that for each i, $-\chi(\tilde{F}_i) \leq |\hat{F}_i \cap k| - 1$. To prove this let F_i be a minimal component of E_Y. All but one component of ∂F_i bounds a disc in E whose interior is disjoint from F_i. Let G be one such disc. The incompressibility of ∂W_i in $W_i, i > 0$ implies that $G \subset W_0$. The incompressibility of ∂W_0 in $W_0 - k$ implies that $G \cap k \neq \emptyset$. Hence $-\chi(F_i) < |\hat{F}_i \cap k|$. Now inductively assume that for all $F_i < F_j, -\chi(F_i) \leq |\hat{F}_i \cap k| - 1$. Again let $G \subset E$ be the disc spanning an innermost component of ∂F_j. By either the induction hypothesis, if $\overset{\circ}{G} \cap E_Y \neq \emptyset$ or the previous argument, if $\overset{\circ}{G} \cap E_Y = \emptyset$ it disjoint from ∂E follows that $-\chi(G \cap E_Y) \leq |G \cap k| - 1$ and therefore $-\chi(\tilde{F}_j) \leq |\hat{F}_j \cap k| - 1$. $\square$

Claim 5. E can be homotoped to E' rel ∂E such that if F is the component of E'_Y which contains ∂E, then

   1) area$(F) < C_0(\text{wr}(\partial D, k) + \text{length}(\partial D))$ and

   2) length$(\partial F) < C_1(\text{wr}(\partial D, k) + \text{length}(\partial D))$. $\square$

Proof. Let S be the compact codimension-0 submanifold of E such that $S \cap Y$ is exactly the component of $E \cap Y$ which contains ∂E. By Proposition 3.20 and Claim 4, S can be homotoped rel ∂S to S_1 such that area$(S_1 \cap Y) \leq C_0(\text{length}((\partial S) \cap Y) - \chi(S \cap Y)) < C_0(\text{length}(\partial E) + \text{wr}(\partial E, k))$. Since the homotopy of S to S_1

extends trivially to a homotopy of E and $\partial E = \partial D$ conclusion 1) follows. (Here E' is the homotoped E and F is the component of $S_1 \cap Y$ containing ∂E.) Conclusion 2) follows from Proposition 3.21. $\square$

Claim 6. There exists a constant k_2 such that if α is a closed curve in W_i, which is homotopically trivial in M, then α bounds a disc A such that area(A) $< k_2$(length(α)). $\square$

Proof. Since balls and solid tori have linear isoperimetric inequalities and each $W_i, i \geq 1$, lies in a 3-cell, Claim 6 holds (using the same constant k_3) for all $\alpha \subset W_i, i \geq 1$, and for all $\alpha \subset W_0$ which are homotopically trivial in W_0.

Let $\beta \subset W_0$ be a normal curve which generates $\pi_1(W_0)$. There exists a k_4 such that if $\alpha \subset W_0$ is not null homotopic in W_0, then α is homotopic in W_0 to $\beta^n, n \neq 0$ via a homotopy $L : S^1 \times I \to M$ such that area(L) $+ |n| \leq k_4$length(α). Since α is homotopically trivial in M, β^n and hence β (since $\pi_1(M)$ is torsion free) are homotopically trivial in M. If B (resp. A) is a least area disc bounded by β (resp. α), then

$$\text{area}(A) \leq |n|\text{area}(B) + \text{area}(L) \leq (1 + \text{area}(B))(|n| + \text{area}(L))$$

$$\leq 2(1 + \text{area}(B))k_4\text{length}(\alpha) = k_5\text{length}(\alpha),$$

where $k_5 = (1 + \text{area}(B))k_4$. Finally, take $k_2 = \max(k_3, k_5)$. $\square$

To complete the proof of Theorem 4.2 observe that ∂D bounds a disc J which is the union of F (which was defined in Claim 5) and least area discs bounded by $\partial F - \partial D$. Thus by Claims 5 and 6 we obtain area(D) $\leq$ area(J) $\leq$ area(F) $+ k_2$length(∂F) $\leq (C_0 + C_1 k_2)(\text{wr}(\partial D, k) + \text{length}(\partial D))$. $\square$

§5

UBIQUITY IN THE SMOOTH CATEGORY

Theorem 5.1. *Let k be a smooth simple closed curve in the closed irreducible Riemannian 3-manifold M. Then either M is toroidal or k is contained in a 3-cell*

or there exists a constant $C > 0$ such that if D is a least area disc with $\partial D \cap k = \emptyset$,

then $area(D) \leq C(wr(\partial D, k) + length(\partial D))$.

Proof: Step 1. Either $k \subset B^3$ or M is toroidal or $M = Y \cup W_0 \cup \cdots \cup W_r$ where

W_0 is a $D^2 \times S^1, k \subset \overset{\circ}{W}_0, \partial W_0$ is π_1-injective in $W_0 - k, W_i \cap W_j = \emptyset$ if $i \neq j$,

and if $i \geq 1$, then W_i lies in a 3-cell and ∂W_i is a torus which is π_1-injective in

W_i. Furthermore $Y \neq \emptyset$ and $\overset{\circ}{Y}$ has a hyperbolic structure of finite volume. Finally

$Y \cap (W_0 \cup \cdots \cup W_r) = \partial Y$.

Proof of Step 1. For the proof of step 1, read (in §4) from the beginning of the

proof of case 2, through the end of the proof of Claim 1.   □

Step 2. It suffices to establish Theorem 5.1 for any Riemannian metric on M.

Proof of Step 2. Use the fact that changes in the Riemannian metric changes arc

length as well as area by uniformly bounded multiplicative amounts.   □

Let X be a space diffeomorphic to $\overset{\circ}{Y}$ and give X a complete hyperbolic metric.

Let $T \times [0, \infty)$ cut off the ends of X and be parametrized so that each $T \times i$ is a

union of $r + 1$ horospherical tori and for $z \in T, z \times [0, \infty)$ is a geodesic parametrized

by arc length.

Let $N(\partial Y) \subset (W_0 \cup \cdots \cup W_r) - k$ be a closed regular neighborhood of ∂Y. Let g be

a Riemannian metric on M such that $g|Y$ is isometric to $X - T \times (0, \infty)$ and $g|N(\partial Y)$

is isometric to $T \times [0, 1]$. I.e. $g|Y \cup N(Y)$ is the pullback metric induced from a

diffeomorphism $f : Y \cup N(Y) \to X - T \times (1, \infty)$ such that $f(Y) = X - T \times (0, \infty)$

and $f(N(\partial Y)) = T \times [0, 1]$.

Step 3. a) There exists $\gamma > 0$ with the following property. It suffices to prove

Theorem 5.1 for discs D where the geodesic curvature at each point of $\partial D \cap Y$ is

bounded above by γ.

b) We can assume that $\partial D \cap N(\partial Y)$ is a union of curves of the form $z \times [0, 1]$.

c) We can assume the ∂D is embedded in M.   □

Let D be a least area disc in M such that $\partial D \cap k = \emptyset$. Let E be a smooth disc

which spans ∂D, is transverse to $k \cup N(\partial Y)$, $|E \cap k| = wr(\partial D, k)$ and satisfies the

following two conditions.

i) No component of $E \cap Y$ disjoint from ∂E is either a disc or an annulus

ii) If $\alpha \subset E \cap Y$ is a simple closed curve, then either α bounds a disc in $E \cap Y$ or α is homotopically nontrivial in Y.

Remark. The reader may have noticed that we have abused notation by identifying the disc E with its image in M thereby suppressing the mapping $f : E \to M$ which defines that image. Thus it would have been more precise to have first defined $f : E \to M$ and then rephrased i) by "no component of $f^{-1}(Y)$ is a disc or annulus disjoint from ∂E" and ii) by "If α is a simple closed curve in $f^{-1}(Y)$, then either α bounds a disc in $f^{-1}(Y)$ or $f|\alpha$ is homotopically nontrivial as a map into Y. In the following statement $E \cap Y$ denotes a subset of the disc E.

Step 4. $-\chi(E \cap Y) < \mathrm{wr}(\partial E, k)$

Proof of Step 4. By hypothesis $\mathrm{wr}(\partial E, k) = |E \cap k|$. Define a partial ordering on the components $F_1, \cdots, F_m$ of $E \cap Y$, by $F_i < F_j$ if F_i is separated from ∂E by F_j. Let $\tilde{F}_i = \cup_{F_j \leq F_i} F_j$ and $\hat{F}_i$ be the subdisc of E bounded by the outermost component of ∂F_i. Since $E \cap Y$ is the union of $\tilde{F}_{i_1}, \cdots, \tilde{F}_{i_n}$ where $F_{i_1}, \cdots, F_{i_n}$ are the maximal components, it suffices to show that for each j, $-\chi(\tilde{F}_{i_j}) \leq |\hat{F}_{i_j} \cap k| - 1$. The proof of this fact is given after Claim 4 §4. Ignore the assertion, true only in §4, that $\partial E = \partial F_i$ for some i. In that proof E_Y denotes $E \cap Y$. [Note that for each i, no component of $\hat{F}_i \cap Y$ is a disc or annulus disjoint from ∂E, each disc component of $\hat{F}_i \cap W_j$ disjoint from ∂E must lie in W_0 and nontrivially intersect k, and that $\mathrm{wr}(\partial \hat{F}_i, k) = |\hat{F}_i \cap k|$.)] $\square$

Step 5. There exists constants C_4, C_5 such that after homotopy of E rel ∂E, there exists a codimension-0 submanifold H of E such that $E \cap Y \subset H \subset Y \cup N(\partial Y)$ and

a) $\mathrm{area}(H) \leq C_4(\mathrm{length}(\partial E) + \mathrm{wr}(\partial E, k))$

b) $\mathrm{length}(\partial H) \leq C_5(\mathrm{length}(\partial E) + \mathrm{wr}(\partial E, k))$

Proof of Step 5. Let $H_1 = E \cap Y$ and view H_1 as lying in $X - T \times (0, \infty)$. After a

homotopy of E rel ∂E, supported near ∂Y we can assume that $H_1 \cap T \times 0$ is straight in the Euclidean tori $T \times 0$. Let $H_2 \subset X$ be $H_1 \cup ((H_1 \cap T \times 0) \times [0, s])$. [More precisely, if $f : H_1 \to Y$, then define $H_2 = H_1 \cup (f^{-1}(\partial Y) \times [0, s]) / \sim$ where $x \in f^{-1}(\partial Y)$ is identified with $x \times 0$. Finally extend f to H_2 by $f(x, t) = (f(x), t) \in T \times t$.] Choose s sufficiently large so that, $\text{length}(H_2 \cap (T \times s)) < \frac{1}{2}$. After an extremely tiny perturbation of ∂H_2 near $T \times s$ we can assume that ∂H_2 is embedded in X. By Theorem 3 of Meeks - Yau [MY] (generalizing [SY]) H_2 is homotopic rel ∂H_2 (in X) to a least area surface H_3. It follows that for s sufficiently large we obtain

$$
\begin{aligned}
\text{area}(H_3) &= \int_{H_3} dA \\
&\leq \int_{H_3} -K \, dA \\
&= -2\pi\chi(H_3) + \int_{\partial H} \gamma_g \\
&\leq -2\pi\chi(H_3) + \gamma\text{length}(\partial H_3 \cap Y) + \pi|\partial E \cap (M - \overset{\circ}{Y})| + \frac{1}{2} \\
&< 2\pi\text{wr}(\partial E, k) + \gamma\text{length}((\partial E \cap Y)) + \frac{\pi}{2}\text{length}(\partial E \cap N(\partial Y)) \\
&\leq C_4(\text{length}(\partial E) + \text{wr}(\partial E, k))
\end{aligned}
$$

where $C_4 = \max(2\pi, \gamma)$. The second equation follows from the fact that H_3 is a minimal surface and hence its sectional curvature is ≤ -1 everywhere. The third equation is Gauss - Bonnet. To establish the fourth equation use the fact that $\partial H_3 \cap Y$ has geodesic curvature bounded above by γ, $\partial H_3 \cap T \times [0, s)$ is geodesic and hence has zero geodesic curvature and ∂H_3 near $T \times s$ is essentially horospherical, hence has geodesic curvature equal extremely close to 1. Finally note that there are two right angled corners of ∂H_3 for each arc component of $(\partial H_1) \cap \partial Y$. The number of such components is equal to $|\partial E \cap (M - \overset{\circ}{Y})|$. To obtain the fifth equation observe that $-\chi(H_3) = -\chi(H_2) = -\chi(H_1) = -\chi(E \cap Y) < \text{wr}(\partial E, k) - \frac{1}{2}$, the last inequality following by Step 4. Also each component of $\partial E \cap (M - \overset{\circ}{Y})$ corresponds to two components of $\partial E \cap N(\partial Y)$ each of which has length 1.

After a small perturbation of H_3 we can assume that the conclusion of the above series of equations still hold and except for finitely many levels, H_3 is

transverse to each $T \times i$. Since $\int_0^1 \text{length}(H_3 \cap (T \times t))dt \leq \text{area}(H_3 \cap (T \times [0,1])) \leq \text{area}(H_3)$, it follows that for some $t \in [0,1), H_3$ is transverse to $T \times t$ and $\text{length}(H_3 \cap (T \times t)) \leq \text{area}(H_3) \leq C_4(\text{length}(\partial E) + \text{wr}(\partial E, k))$. Let $H = H_3 \cap X - T \times (t, \infty)$. Thus $\text{area}(H) \leq \text{area}(H_3) \leq C_4(\text{length}(\partial E) + \text{wr}(\partial E, k))$ and $\text{length}(\partial H) \leq \text{length}(\partial E) + \text{length}(H_3 \cap T \times t) \leq 2C_4(\text{length}(\partial E) + \text{wr}(\partial E, k))$. Let $C_5 = 2C_4 + 1$. To complete the proof of Step 5, view H as lying in $Y \cup N(\partial Y)$ and observe that there is a homotopy of E rel ∂E supported in $Y \cup N(\partial Y)$ such that after the homotopy $E \cap (Y \cup T \times [0,t]) = H$. $\square$

Proof of Theorem 5.1. Let G be the closure of $E - H$. By the proof of Step 5 $\text{length}(\partial G) \leq \text{length}(\partial E) + \text{length}(H_3 \cap T \times t) \leq C_5(\text{length}(\partial E) + \text{wr}(\partial E, k))$. Each component of $\partial G \subset W_i$ for some i and is homotopically trivial in M. The argument of Claim 6 (§4) shows that there exists a constant C_6 such that if α is a closed curve in some W_i and α is homotopically trivial in M, then α bounds a disc $D_\alpha \subset M$ such that $\text{area}(D_\alpha) \leq C_6\text{length}(\alpha)$. Therefore we can span ∂G by a collection of discs $\mathcal{D}$ such that $\text{area}(\mathcal{D}) \leq C_6\text{length}(\partial G)$.

One can construct a disc F spanning ∂D by piecing together a subset of the components of H together with a subset of the least area discs bounded by components of ∂G. In fact, use those components of H which, as subsets of E, non-trivially intersect ∂E and those components of ∂G which are outermost in E. It follows that $\text{area}(D) \leq \text{area}(F) \leq \text{area}(\mathcal{D} \cup H) \leq \text{area}(H) + C_6\text{length}(\partial G) \leq (C_4 + C_6C_5)(\text{length}(\partial E) + \text{wr}(\partial E, k))$. $\square$

Ubiquity Theorem 5.2. *Let $k \not\subset B^3$ be a smooth simple closed curve in the closed, atoroidal, irreducible 3-manifold M. There exists constants K and L such that if D is a least area disc with $\partial D \cap k = \emptyset$ and $\text{length}(\partial D)/\text{area}(D) < L$, then $\text{wr}(\partial D, k)/\text{area}(D) > K$.* $\square$

Corollary 5.3. *Let λ be a quasi-least area semi-Euclidean lamination in the closed atoroidal irreducible 3-manifold M. If $k \subset M - \lambda$ is a smooth simple closed curve, then k lies in a 3-cell.* $\square$

Remark 5.4. By defining $\text{wr}(\alpha, k)$ for all reasonable homotopically trivial curves α

(rather than just α with $\alpha \cap k = \emptyset$) in the appropriate manner we can then obtain 5.1, 5.2, 5.3 with the hypothesis $\partial D \cap k = \emptyset$ eliminated.

§6 A Very Brief Survey On The Geometrization Of 3-Manifolds

Thurston Geometization Conjecture 6.1 [Th]. *Let M be a closed connected irreducible 3-manifold, then either*

a) $M = S^3/\Gamma$ where $\Gamma \subset Isom(S^3)$ or

b) $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ or

c) $M = \mathbb{H}^3/\Gamma$ where $\Gamma \subset Isom(\mathbb{H}^3)$

Remarks 6.2. i) These conclusions exactly parallel the conclusions of the corresponding geometrization theorem for closed surfaces.

ii) The three conclusions are mutually exclusive. Conclusion a) is the geometrization conjecture for 3-manifolds with finite fundamental group. That conjecture is equivalent to positive solutions to each of the following longstanding open problems.

Poincare Conjecture 6.3. *If M is a closed connected simply connected 3-manifold, then M is homeomorphic to S^3.*

Generalized Space Form Conjecture 6.4. *If the finite group Γ acts freely on S^3, then Γ is conjugate to a linear action.*

Remark 6.5. See [Liv], [My], [Ru], and [T] for partial results on this conjecture. In particular by [T], Conjecture 6.4 for cyclic groups implies a positive solution to the following classical question.

Space Form Conjecture 6.6. *If M is covered by S^3, then $\pi_1(M)$ is isomorphic to a subgroup of $SO(4)$*

As of this writing there is one class of groups that still needs to be eliminated [Mi], [Lee].

Remark 6.7. Conclusions b) and c) of the Geometrization conjecture treat the case that $\pi_1(M)$ is infinite. The topology of closed irreducible 3-manifolds with $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ is now completely understood, thanks to the work of [CJ], [G2], [M], and

[Sc1-2]. These works in turn built on ideas of many others e.g. Tukia, Ghering, Martin, Jaco, Shalen, Johannson, Waldhausen, Burde, Zieschang, Neuwirth, and Murasugi. (See [G2] for a more detailed history.) Such manifolds contain either embedded π_1-injective surfaces or are Seifert fibred spaces. The [JS][J]-theory, discussed in §4, details the structure of 3-manifolds with π_1-injective embedded tori. The nature of 3-manifolds M with $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ was not known when the Geometrization conjecture was originally formulated and conclusion b) was spelled out with all its possibilities.

Conclusion c) is also known as the

Thurston Hyperbolization Conjecture 6.8. *Let M be a closed, connected, irreducible 3-manifold with $\pi_1(M)$ infinite and $\mathbb{Z} \oplus \mathbb{Z} \not\subset \pi_1(M)$, then $M = \mathbb{H}^3/\Gamma$ where $\Gamma \subset Isom(\mathbb{H}^3)$.*

Remark 6.9. In 1978 Thurston proved this conjecture for Haken 3-manifolds [Th]. A result known in those days as the Monster Theorem.

At the JDG 96 conference, the author announced the following result [GMT] which was joint with Robert Meyerhoff and Nathaniel Thurston.

Theorem 6.10. *Let N be a closed hyperbolic 3-manifold. Then*

a) If $f : M \to N$ is a homotopy equivalence where M is an irreducible 3-manifold, then f is homotopic to a homeomorphism.

b) If $f, g : M \to N$ are homotopic homeomorphisms, then f is isotopic to g.

c) The space of hyperbolic metrics on N is path connected.

Remark 6.11. i) The analogue of a) for spherical manifolds is false. Its a classical theorem of Reidemeister and Whitehead that the lens space $L(7,1)$ is homotopy equivalent but not homeomorphic to $L(7,2)$. Its a much more recent observation of the author that there exists an orientation preserving self homotopy equivalence of a lens space which is not homotopic to a homeomorphism [G3]. For example, consider the homotopy equivalence of $L(8,1)$ to itself whose action on π_1 is multiplication by 3.

ii) Theorem 6.10 reduces the Hyperbolization Conjecture to

Conjecture 6.12. *If M is a closed, connected, aspherical 3-manifold with $\mathbb{Z} \oplus \mathbb{Z} \not\subset \pi_1(M)$, then M is homotopy equivalent to a hyperbolic 3-manifold. Equivalently [since M is a $K(\pi, 1)$], $\pi_1(M)$ is isomorphic to $\pi_1(N)$ where N is a hyperbolic 3-manifold.*

Conjecture 6.12 is equivalent to the following two well known open problems.

Group Negative Curvature Conjecture 6.13. *If M is a closed, connected, aspherical 3-manifold and $\mathbb{Z} \oplus \mathbb{Z} \not\subset \pi_1(M)$, then $\pi_1(M)$ is negatively curved.*

See [Gr], [Bu], [Sch], [Mo1], [MO], and [Ka] for some contributions towards Conjecture 6.13.

Cannon Conjecture 6.14. *If M^3 is closed, connected, aspherical and $\pi_1(M)$ is negatively curved, then $\pi_1(M)$ is isomorphic to the fundamental group of a closed hyperbolic 3-manifold.*

It was first proved by Casson [Ca] and Poenaru [Po], that if M is closed, aspherical with a negatively curved fundamental group, then the universal covering of M is $\mathbb{R}^3$. Also by [HRS] and [M] a closed irreducible 3-manifold M with $Z \oplus Z \subset \pi_1(M)$ is covered by $\mathbb{R}^3$. Thus a special case of Conjecture 6.13 is the longstanding

Conjecture 6.15. *If M is a closed irreducible connected 3-manifold with $\pi_1(M)$ infinite, then M is covered by $\mathbb{R}^3$.*

Remarks 6.16. i) See p. 149-151 [Ki] for a list of contributions to this problem.

ii) Bestvina - Mess [BM] show that the sphere at infinity of an aspherical 3-manifold with negatively curved fundamental group is a 2-sphere.

Remark 6.17. See [Can], and [CFP] for some very remarkable and beautiful developments towards Jim Cannon's program to establish Conjecture 6.14. In particular the introduction to [CFP] gives a lucid overview of both the general program and the state of their work.

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