Thurston's Hyperbolization of Haken Manifolds

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À Paule Tilouine

in memoriam

INTRODUCTION

In the early 1970's, useful connections between 3-manifolds and Kleinian groups began to emerge and set the scene for Thurston's hyperbolization theorem.

— On the one hand, techniques from 3-dimensional topology improved the understanding of *Kleinian groups*, i.e. discrete torsion-free subgroups of $\mathrm{PSL}_2(\mathbb{C})$, the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^3$. A. Marden recognized in **[Mard]** some important consequences of a theory of Waldhausen for the study of *geometrically finite groups* (cf. §1). A fundamental result of Waldhausen gives a necessary and sufficient condition under which a homotopy equivalence between *Haken manifolds* (see below) can be deformed to a diffeomorphism **[Wa2]**. Using this theorem, Marden obtained a necessary and sufficient condition under which an abstract isomorphism between two geometrically finite groups is induced by a quasi-conformal homeomorphism of $\mathbb{S}^2$, the formal boundary of $\mathbb{H}^3$. This condition was a step to fit the geometrically finite groups into Ahlfors-Bers theory of quasi-conformal deformations. It is also probably Marden who first posed in print the problem of giving conditions on a compact 3-manifold to be hyperbolic **[Mard**, p. 461]. We say that a compact orientable 3-manifold M is *hyperbolic* if its interior is diffeomorphic to the quotient of $\mathbb{H}^3$ by a geometrically finite group. Another equivalent definition is the following. Let M be a compact orientable 3-manifold and let P be the union of the tori contained in ∂M. We say that M is *hyperbolic* if $M - P$ carries *a hyperbolic metric* i.e. a complete metric of constant curvature -1 such that M is locally outwardly convex along $\partial M - P$ (cf. §1). Marden observed that *the irreducibility of* M (see below) and the triviality of the center of $\pi_1(M)$ are necessary conditions.

B. Maskit developed a construction for Kleinian groups from simpler ones. In particular, his *Combination theorems* provide sufficient conditions under which two Kleinian groups can be amalgamated in order to produce a new Kleinian group **[Mas1]**. The topological description of this amalgamation at the level of the quotient 3-manifolds is the gluing of the two corresponding 3-manifolds along a subsurface contained in their boundaries. This is parallel to the key construction used in the study of Haken manifolds, namely as the gluing of two simpler ones along incompressible parts of their boundaries.

R. Riley, exploiting different ideas, wrote a computer program to find discrete and faithful representations into $\mathrm{PSL}_2(\mathbb{C})$ of certain knot groups. Using this program, he gave explicitly the representation of the fundamental group of the figure 8 knot [**Ri**]. A fibering theorem of Stallings [**Stal**] then implied that the quotient of $\mathbb{H}^3$ by this Kleinian group was diffeomorphic to the complement of the figure 8 knot in $\mathbb{S}^3$.

Since the quotient of $\mathbb{H}^3$ by a Kleinian group is a complete hyperbolic 3-manifold which conversely determines the Kleinian group up to conjugacy, 3-manifolds were inevitable side products of Kleinian group theory; however, topologically interesting examples were slow to be discovered. F. Löbell provided in 1931 what was perhaps the first example of a *closed* hyperbolic manifold [**Lö**]. In 1970, E. Andreev succeeded in giving a complete combinatorial classification of 3-dimensional hyperbolic Coxeter groups with compact fundamental domains [**An**]: this result provided a huge family of closed hyperbolic 3-manifolds for it was known by a theorem of Selberg [**Sel**] that each such Coxeter group contains a finite index subgroup which is torsion-free (cf. [**Bo**] for examples of closed quotients of an arbitrary simply connected symmetric space). But the foremost example of a hyperbolic manifold to have been appraised as a topological manifold is probably *the hyperbolic dodecahedral space*, which appeared in 1933 [**WS**]. In this paper, H. Seifert and C. Weber describe this manifold as a 5-fold cover of $\mathbb{S}^3$ ramified over a link with two components and they compute its first homology group showing that it is a torsion group [**WS**, p. 252]. Furthermore, they observe that this manifold is not *a Seifert fibered space* [**Sei**], as a direct consequence of the triviality of the center of a cocompact Kleinian group [**WS**, p. 249].

— On the other hand, the progress made in understanding 3-manifolds, especially Haken manifolds, very gradually led to hyperbolic geometry. *An irreducible manifold* is a 3-manifold in which any embedded 2-sphere bounds a 3-ball. By a theorem of H. Kneser, any compact 3-manifold M without 2-spheres in the boundary, can be written as the connected sum of irreducible manifolds and copies of $\mathbb{S}^2 \times \mathbb{S}^1$ [**Kn**]. J. Milnor [**Mil**] proved that this decomposition is unique up to diffeomorphisms when M is closed and orientable; see [**Hel**] for the generalization to the case with non-empty boundary. This justifies restricting the study of compact orientable 3-manifolds to irreducible orientable manifolds.

The Sphere theorem was proven by C. Papakyriakopoulos in the mid 1970's [**Pa**]: it says that if M is a 3-manifold with $\pi_2(M) \neq 0$, then M contains an embedded 2-sphere which represents a non-zero element of $\pi_2(M)$. Therefore, if M is irreducible, then $\pi_2(M) = 0$. It follows that the universal cover $\widetilde{M}$ of an irreducible manifold M with infinite fundamental group is contractible. It was then natural to ask what the topological type of $\widetilde{M}$ was. For instance, when M is a closed irreducible manifold with infinite (or torsion-free) fundamental group, is $\widetilde{M}$ homeomorphic to $\mathbb{R}^3$? And if the answer is yes, how does $\pi_1(M)$ act? When M is a Seifert fibered space, the answer to the first question is positive. Any Seifert fibered space with infinite fundamental group has a finite cover which is diffeomorphic to the product of a closed surface by the circle [**Sei**]. Therefore, the universal cover of an irreducible

Seifert fibered space M with infinite fundamental group is homeomorphic to $\mathbb{R}^3$. Before Thurston intervened, it seems not to have been realized that the corresponding action of $\pi_1(M)$ is always geometric. Excepting the Seifert fibered spaces which were thoroughly studied in the 1930's [**Sei**], nothing was known about the topological type of a contractible cover of an irreducible manifold. For example, it was plausible that the Whitehead manifold (a contractible open 3-manifold which is not homeomorphic to $\mathbb{R}^3$ [**Wh**]) should cover a compact 3-manifold (as a matter of fact, this was shown only in the late 1980's: the Whitehead manifold covers no manifolds but itself [**My**]). Great progress on this question was made in the late 1960's by F. Waldhausen [**Wa2**]. His methods provided a deep understanding of the vast class of Haken manifolds. *A Haken manifold* is an irreducible manifold M which contains *an incompressible surface*, i.e. a properly embedded connected surface S such that

(i) the fundamental group $\pi_1(S)$ injects into $\pi_1(M)$ and the relative fundamental group $\pi_1(S, \partial S)$ injects into $\pi_1(M, \partial M)$, and

(ii) S cannot be isotoped into a component of ∂M (cf. §7).

Among other results, Waldhausen proved that the universal cover of a Haken manifold M is homeomorphic to $\mathbb{B}^3 - F$, where F is a closed subset of $\mathbb{S}^2 = \partial\mathbb{B}^3$ [**Wa2**, p. 86]. In reality, he offered a picture of $\widetilde{M}$ which is reminiscent of the complement of the limit set of a Kleinian group in the compactified hyperbolic space.

A new light on 3-manifolds came through *the Torus decomposition.* Any compact irreducible orientable 3-manifold M contains a (possibly empty) finite collection $\mathcal{T}$ of disjoint incompressible tori such that any component V of the manifold obtained by splitting M along $\mathcal{T}$ either is a Seifert fibered space or does not contain any incompressible torus. This collection $\mathcal{T}$ is well defined up to isotopy, once we require that it satisfy a minimality condition and the pieces obtained by splitting M along $\mathcal{T}$ form *the Torus decomposition of* M. The existence of the Torus decomposition appears in a cryptical announcement of Waldhausen [**Wa3**] which to some extent guided the development. The complete proof along with important applications was established by W. Jaco and P. Shalen ([**JS1**], [**JS2**]), and independently by K. Johannson ([**Joh1**], [**Joh2**]). Although the Seifert fibered spaces were well understood topologically [**Sei**], there were no general methods to describe the pieces of other type in the Torus decomposition.

Towards the mid 1970's, a number of algebraic properties of Haken manifold fundamental groups were established which may well have helped to convince Thurston that the non-Seifert pieces in the Torus decomposition of a Haken manifold are in fact hyperbolic. The most important is probably *the Torus theorem* ([**Wa3**], [**Fe**]). This theorem asserts that, if M is Haken, any non-Seifert piece V in the Torus decomposition of M is *atoroidal*, i.e. any $\mathbb{Z}+\mathbb{Z}$-subgroup of $\pi_1(V)$ can be conjugated into the fundamental group of a component of ∂V. Any $\mathbb{Z}+\mathbb{Z}$-subgroup of a Kleinian group is *parabolic* (cf. §1); it follows that any hyperbolic manifold is atoroidal. The fundamental group of a Haken manifold M shares other common properties with Kleinian groups.

(i) Each non-zero element of $\pi_1(M)$ is *uniquely divisible*, i.e. it has a unique positive root of maximal order **[Sh]**, and

(ii) if M is atoroidal and *acylindrical* (see below), then the group of outer automorphisms of $\pi_1(M)$ is finite **[Joh2]**.

Property (i) is easily seen to be satisfied when M is a hyperbolic manifold. When M is a closed hyperbolic manifold, (ii) follows from the Mostow rigidity theorem **[Mos]**.

In the spring of 1977, in his lectures on hyperbolic 3-manifolds, Thurston announced his *Hyperbolization theorem for Haken manifolds.* The printed announcement came several years later, along with generalizations that won't be treated in the present article **[Thu3]**. This theorem gives a necessary and sufficient condition for a Haken manifold to be hyperbolic.

Thurston's hyperbolization theorem [Thu3].— *Let M be an irreducible and atoroidal manifold. If M is Haken, then M is hyperbolic.*

Furthermore, at about the same time, Thurston formulated his Geometrization conjecture. This conjecture says that each piece of the Torus decomposition of an irreducible manifold is modeled locally on one of the following *eight geometries*: the three constant curvature ones: $\mathbb{H}^3$, $\mathbb{R}^3$, $\mathbb{S}^3$, and the five fibered ones: $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, Nil and Sol (cf. **[Sco]** for a detailed survey on these geometries). This conjecture is satisfied by Haken manifolds: in view of the Hyperbolization theorem above, the proof amounts to merely observing geometric structures on Seifert fibered spaces. Thus, Thurston's hyperbolization theorem, as a particular case of the Geometrization conjecture was in harmony with the recent Torus decomposition theorem.

Thus, for Haken manifolds, Thurston completely settled the Geometrization conjecture. Recall that every compact irreducible manifold whose boundary is non-empty is necessarily Haken; again, every compact irreducible manifold with infinite first homology group is Haken (cf. **[He1]**, **[Ja]**). However, some of the closed hyperbolic manifolds that had been considered before Thurston were already known to be non-Haken: for instance, Haken himself **[He2]**, and also Waldhausen knew this for the hyperbolic dodecahedral space. Thurston went on to prove that in some sense, most 3-manifolds with finite first homology group are non-Haken. His *Hyperbolic Dehn's surgery theorem* implies that, with only a finite number of exceptions, the manifolds obtained by Dehn surgery on a knot in $\mathbb{S}^3$ whose complement is hyperbolic and does not contain any closed incompressible surface are hyperbolic. At the same time, only finitely many of them are Haken (**[Thu1]**, **[Hat]**). (For the test case of 2-bridge knots which are not torus knots, see **[HT]**.)

According to the Geometrization conjecture, one should be able to replace, in the above Hyperbolization theorem, the phrase "is Haken" by "has torsion-free fundamental group". The later is clearly a necessary hypothesis.

The proof of Thurston's hyperbolization theorem distinguishes two cases according as M is fibered over the circle or not. The difference arises from the fact that, in the fibered case, the quasi-Fuchsian groups appearing in the construction degenerate, while in the non-fibered case, they remain quasi-Fuchsian. The fibered case can be formulated as the beautiful theorem below. Recall that a 3-manifold which fibers over the circle with fiber a surface S, is determined up to diffeomorphism by the isotopy class of its *monodromy*, which is an element of *the mapping class group* $\mathrm{Mod}(S)$ (cf. §1). To any diffeomorphism $\phi \in \mathrm{Mod}(S)$, one can associate a 3-manifold M_ϕ, called *its mapping torus* which is defined as the quotient space of $S \times [0,1]$ by the relation which identifies $(x,1)$ with $(\phi(x),0)$. We say that $\phi \in \mathrm{Mod}(S)$ is *pseudo-Anosov* when its action on the set of conjugacy classes of $\pi_1(S)$ does not act periodically on any non-trivial element [**Thu2**].

Hyperbolization theorem for manifolds which fiber over the circle ([Thu5], [Su]). — *Let S be a closed surface of genus greater than 2 and let $\phi \in Mod(S)$. Then M_ϕ is hyperbolic if and only if ϕ is pseudo-Anosov.*

The proof of this particular case of Thurston's hyperbolization theorem is completely different from the proof in the non-fibered case and it has been already quite well explored (cf. [**McM3**], [**O**]). For this reason, we will restrict our attention in this article to the manifolds which are not fibered.

The proofs of the two halves of the Hyperbolization theorem may nevertheless overlap, as Thurston himself observed. For example, there is the following still unsettled question: does every compact 3-manifold fibered over the circle have a finite cover that contains an incompressible surface which is not a fiber of a fibration over the circle? If this were true, the results of the present article alone would suffice to completely prove the Hyperbolization theorem.

In the proceedings of the Smith Conjecture Symposium [**BM**], J. Morgan gave a survey of a part of Thurston's original proof and the book of M. Kapovitch [**Ka**] is also devoted to the original approach. However, in this article, we will detail the proof from another viewpoint. The crucial part relies more on Teichmüller theory than Thurston's proof. It is due to C. McMullen [**McM2**].

We will here prove the Hyperbolization theorem only under an assumption which is stronger than the assumption "atoroidal". Namely, we will assume that $\pi_1(M)$ *does not contain any* $\mathbb{Z} + \mathbb{Z}$*-subgroups.* The main consequence is that we can completely avoid the study of Kleinian groups containing parabolic elements. (The general case is more complex, partly due to heavier notations, but it is not much more difficult.) It is in a similar spirit that this article excludes all mentions of non-orientable manifolds (see [**To**]).

A crucial property of a Haken manifold is the existence of *hierarchies*, discovered by Haken. By definition, a Haken manifold M contains an incompressible surface S, and by splitting M along S, we obtain a new manifold M_S. The incompressibility of S implies that M_S is irreducible. Moreover, if M_S is not a disjoint union of 3-balls, it is Haken (cf. §7). Therefore, the splitting process can be iterated. It is a fundamental observation of W. Haken that this process ends up, after a finite number of steps, with a disjoint union of 3-balls (cf. [**Ha**]). This sequence of manifolds is called *a hierarchy for M* and the number of terms in this sequence is called *the length* *of the hierarchy.* For technical reasons, we prefer to use hierarchies of another type, called *special hierarchies* (cf. [**Ja**]): this will allow us to prove the Hyperbolization theorem by induction on an integer, which is defined as the greatest length of a special hierarchy, and denoted by $\ell(M)$ (cf. §7).

Like other theorems on Haken manifolds, such as Waldhausen's well-known theorem that homotopy equivalences between Haken manifolds respecting boundary can be deformed respecting boundary to diffeomorphisms, the proof of Thurston's hyperbolization theorem in the non-fibered case involves a finite induction using a hierarchy for M. There is a further parallel between the hyperbolization procedure of Thurston and the proof of this theorem of Waldhausen. Both proofs consist of two distincts parts, one combinatorial and 3-dimensional, and the other 2-dimensional and more geometrical. In each case, the 3-dimensional part is a hierarchical induction. In Waldhausen's theorem, the 2-dimensional part is a known theorem of Nielsen [**Nie**]. The 2-dimensional part of Thurston's theorem however, was an entirely new Fixed point theorem, involving the Teichmüller spaces of the gluing surfaces.

To carry out the proof of Thurston's theorem by induction on the length of M, we will enunciate before long a more general theorem, which applies to *manifolds-with-corners.* First, we explain the basic gluing procedure to which the inductive step will reduce.

Final gluing theorem.— *Let N be a hyperbolic manifold with incompressible boundary. Let τ be an orientation reversing involution of ∂N which exchanges the boundary components by pairs. Suppose that N is not an interval bundle. Then if N/τ is atoroidal, it is hyperbolic.*

This theorem still holds if N is an interval bundle but the proof is entirely different. It corresponds to the case when N/τ is fibered and the hyperbolic structure is obtained as a degeneration of a certain sequence of quasi-Fuchsian structures on N.

Note that the Final gluing theorem never directly provides a hyperbolic structure on a compact manifold with non-empty boundary. However it does so indirectly by a trick of Thurston. This trick makes the boundary of a 3-manifold invisible by covering it with mirrors: in some sense, it converts boundary points to interior points. We present an explication of this trick which is due to F. Bonahon who was first to observe that right-angled corners are sufficient. The following notion of manifold-with-corners is essentially equivalent to the notion of "manifold with (useful) boundary pattern", introduced by Johannson in his work on homotopy equivalences ([**Jo1**], [**Jo2**]). *A manifold-with-corners* is a triple $(M, \mathcal{G}, \partial^0 M)$, where M is a 3-manifold, and $\mathcal{G} \subset \partial M$ is a smooth trivalent graph such that

(i) each component of $\partial M - \mathcal{G}$ equals the interior of its closure, and

(ii) each component of $\partial^0 M$ is the closure of a component of $\partial M - \mathcal{G}$.

The closure of a component of $\partial M - \mathcal{G}$ which is not in $\partial^0 M$ is called *a mirror of* $(M, \mathcal{G}, \partial^0 M)$, and $\partial^0 M$ is called *the boundary of* $(M, \mathcal{G}, \partial^0 M)$. One should think of $(M, \mathcal{G}, \partial^0 M)$ as *a differentiable structure with corners on* M, i.e. an atlas of class C^1 on M with charts modeled on open subsets of $(\mathbb{R}^+)^3$. Then, the graph $\mathcal{G}$ corresponds to points which have a neighborhood diffeomorphic to the neighborhood of a point of $(\mathbb{R}^+)^3$ with 2 or 3 coordinates equal to 0, and the boundary of $(M, \mathcal{G}, \partial^0 M)$ corresponds to distinguishing a set of disjoint mirrors. Such a differentiable structure depends only on the pair $(M, \mathcal{G})$: this follows from [Ce] and [Do].

The notions of irreducibility and atoroidality can be extended to manifolds-with-corners; one can also define the notion of a manifold-with-corners with incompressible boundary (cf §7). Rather than directly prove the Hyperbolization theorem for manifolds possibly with boundary, we will proceed by proving a Hyperbolization theorem for manifolds-with-corners that have empty boundary —which will turn out to be just as strong.

A manifold-with-corners $(M, \mathcal{G}, \partial^0 M)$ is *hyperbolic* when there is a hyperbolic metric on M such that

(i) the mirrors are totally geodesic,

(ii) M is locally outwardly convex along $\partial^0 M$, and

(iii) the components of $\partial M - \mathcal{G}$ meet at right-angles along the edges of $\mathcal{G}$.

Let $(M, \mathcal{G})$ be a hyperbolic manifold-with-corners having empty boundary and let $S' \subset \partial M$ be a surface which is a disjoint union of mirrors. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by erasing the edges whose interior is contained in the interior of S'. Then, after rounding the corners along the erased edges, $(M, \mathcal{G}', S')$ becomes a manifold-with-corners and it follows almost directly by taking δ-neighborhoods in the ambient complete hyperbolic manifold that $(M, \mathcal{G}', S')$ is hyperbolic. In particular, if $(M, \mathcal{G})$ is a hyperbolic manifold-with-corners having empty boundary, then M is hyperbolic.

Hyperbolization theorem for manifolds-with-corners.— *Let $(M, \mathcal{G})$ be a compact irreducible oriented and atoroidal manifold-with-corners having empty boundary. If M is Haken, then $(M, \mathcal{G})$ is hyperbolic.*

Any irreducible and atoroidal manifold M having non-empty boundary can be easily promoted to a manifold-with-corners $(M, \mathcal{G})$ that has empty boundary and is irreducible and atoroidal: this is the mirror trick (absolute version) see §7. Therefore, Thurston's hyperbolization theorem is a consequence of the theorem above. This theorem is proven by induction on the special length of M, viewed as a manifold without corners. The most important advantage of the introduction of manifolds-with-corners is above all that the proof of the inductive step will be in strict parallel with that of the Final gluing theorem and is in the final analysis a consequence of it.

Suppose that $(M, \mathcal{G})$ is an irreducible and atoroidal manifold-with-corners with empty boundary. When M is Haken, we show in §7 the existence of an incompressible surface S which is *a good splitting surface*. This means that the manifold-with-corners $(M_S, \mathcal{G}_S, S')$ obtained by splitting $(M, \mathcal{G})$ along S has incompressible boundary. We show also that $\mathcal{G}_S$ can be extended to a graph $\mathcal{G}'_S$ by adding edges contained in S' so that $(M_S, \mathcal{G}'_S)$ is an irreducible and atoroidal manifold-with-corners with empty boundary. Suppose that $(M_S, \mathcal{G}'_S)$ is hyperbolic. Then, as observed above, $(M_S, \mathcal{G}'_S, S')$ is also hyperbolic, and further the hypotheses of the next theorem are satisfied.

Gluing theorem for manifolds-with-corners.— *Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary. Let S be a good splitting surface for $(M, \mathcal{G})$ and let $(M_S, \mathcal{G}_S, S')$ be the manifold-with-corners obtained from $(M, \mathcal{G})$ by splitting along S. Suppose that $(M_S, \mathcal{G}_S, S')$ is hyperbolic. Then $(M, \mathcal{G})$ is hyperbolic.*

This gluing theorem can be deduced from the statement of the Final gluing theorem (cf. §8). We now apply the Gluing theorem for manifolds-with-corners to prove the Hyperbolization theorem for manifolds-with-corners. The proof is by induction on the special length $\ell(M)$ of M.

The induction starts at $\ell(M) = 0$. Then M is an handlebody and M_S is diffeomorphic to $\mathbb{B}^3$ (cf. §7). Thus $(M_S, \mathcal{G}'_S)$ can be interpreted as a polyhedron. Saying that $(M_S, \mathcal{G}'_S)$ is hyperbolic means that this polyhedron can be realized in $\mathbb{H}^3$ with all dihedral angles equal to $\pi/2$. A characterization of the compact polyhedra which can be embedded in $\mathbb{H}^3$ with prescribed acute dihedral angles is provided by the theorem of Andreev already mentioned [**An**]. In the case of a right-angled polyhedron, the hypothesis of this theorem turn out to be equivalent to the irreducibility and atoroidality of $(M_S, \mathcal{G}'_S)$. Therefore the Andreev theorem asserts that $(M_S, \mathcal{G}'_S)$ is hyperbolic. (Incidently, note that the problem of realizability of a polyhedron in $\mathbb{H}^3$ with various sorts of prescribed data is currently a field of intensive study [**HR**].) Thus, by the Gluing theorem for manifolds-with-corners, $(M, \mathcal{G})$ is hyperbolic when $\ell(M) = 0$. The inductive step reduces similarly to the Gluing theorem. This proves the Hyperbolization theorem for manifolds-with-corners.

Now we sketch the logic of the proof of the Final gluing theorem. Let N be a hyperbolic manifold. Let G be a geometrically finite group such that N is diffeomorphic to the Kleinian manifold $\overline{M}(G)$ associated to G (cf. §1). The space $\mathcal{GF}(G)$ of the geometrically finite groups isomorphic to G can be parametrized by the Teichmüller space $\mathcal{T}(\partial N)$: this is one important application of Ahlfors-Bers theorem on the existence and uniqueness of solutions to the Beltrami equation. Using this parametrization and the Maskit combination theorem (cf. §2), Thurston showed how to reduce the Final gluing theorem to the problem of finding a fixed point for a certain map on Teichmüller space. This map is the composition $\tau^* \circ \sigma$ of two maps: $\tau^* : \mathcal{T}(\overline{\partial N}) \to \mathcal{T}(\partial N)$ is the action induced by the (orientation reversing) diffeomorphism τ and $\sigma : \mathcal{T}(\partial N) \to \mathcal{T}(\overline{\partial N})$ is *the skinning map* (see below). This translates the final gluing theorem into the following:

Thurston's fixed point theorem.— *Let N be a hyperbolic manifold with incompressible boundary which is not an interval bundle. Let τ be an orientation reversing involution of ∂N which permutes the components by pairs. Then, if N/τ is atoroidal, $\tau^* \circ \sigma$ has a fixed point.*

To prove this theorem, we will adopt the approach which was given by McMullen [**McM2**]. This approach originated in an observation that J. Hubbard made shortly after Thurston enunciated the Fixed point theorem. Hubbard noticed that the formula for the coderivative of the skinning map involved a well-known operator in Teichmüller theory, *the Poincaré series operator*. As the proof of Thurston's fixed point theorem presented here relies mostly on complex analysis, we must recall briefly the definition of the Poincaré series.

Let $Y \to X$ be a cover of Riemann surfaces. A holomorphic L^1-integrable quadratic differential ϕ on Y can be summed over the sheets of the cover to define a holomorphic integrable quadratic differential $\Theta_{Y/X}\phi$ on X. If we denote by $\mathcal{Q}(X)$, $\mathcal{Q}(Y)$ the space of integrable holomorphic quadratic differentials on X and Y respectively, this defines a map $\Theta_{Y/X} : \mathcal{Q}(Y) \to \mathcal{Q}(X)$, called *the Poincaré series operator* or *Theta operator*. When $\mathcal{Q}(X)$ and $\mathcal{Q}(Y)$ are endowed with their respective L^1-norms, the norm of $\Theta_{Y/X}$ is less than or equal to 1.

In view of the formula for the coderivative of σ, Hubbard suggested that the existence of a fixed point for $\tau^* \circ \sigma$ would be easier to establish if one could prove a conjecture of Kra [**Kr**]. This conjecture asserts that the norm of the Theta operator associated to the universal cover of a finite volume hyperbolic Riemann surface X is *strictly* less than 1. It was McMullen who, in 1989, succeeded in carrying out the program of Hubbard. In [**McM1**] he proves a generalized version of Kra's conjecture, giving a necessary and sufficient condition on a cover $Y \to X$ for the norm of $\Theta_{Y/X}$ to be strictly less than 1. In [**McM2**] he shows how this result applies to give a new proof of Thurston's fixed point theorem.

§2 begins with a proof of the particular case of the Maskit combination theorem we need in order to show the equivalence between the Final gluing theorem and Thurston's fixed point theorem. Next, we study the skinning map which is defined as follows. By the Ahlfors-Bers theorem, there corresponds to any point $s = (s_1, \cdots, s_k) \in \mathcal{T}(\partial N)$ a Kleinian manifold N^s diffeomorphic to N, such that $\partial N^s = s$ (cf. §1). By taking the cover of N^s associated to the component S_i of ∂N, we obtain a quasi-Fuchsian structure on $S_i \times [0,1]$: the Ahlfors-Bers parameters of this structure are (s_i, s_i'), where s_i' is a complex structure on $\overline{S_i}$, the surface S_i with the reversed orientation. Then, the skinning map assigns to ∂N^s the point $\sigma(\partial N^s) \in \mathcal{T}(\overline{\partial N})$, whose i-th coordinate is s_i'. We see *spots* on the Riemann surface $\sigma(\partial N^s)$: in the cover of N^s associated to S_i they are the components of the preimage of ∂N^s others than the canonical lift of S_i. In particular, to each spot U is associated a cover $U \to X_U$ of a component X_U of ∂N^s. The topological configuration of the spots on ∂N^s and the topological type of the covers associated to them are independent of s. The cover $U \to X_U$ associated to a spot is *geometric*: it arises from a compact incompressible surface contained in X_U. Also the shape of the spots reflects some important topological properties of N. The basic one is that each curve on U which is not homotopically trivial projects under the covering $U \to X_U$ to a curve on ∂N which is one boundary component of an essential annulus in N. In particular, the spots are all simply connected if (and only if) N is *acylindrical*, i.e. if N does not contain essential annuli. In §2, we compute also the coderivative of σ: it is a convex linear combination of the Theta operators associated to the spots.

§4 and 5 are devoted to prove the McMullen theorem that the Theta operator $\Theta_{Y/X}$ associated to a geometric cover $Y \to X$ has norm bounded away from 1 by a constant depending only on $\chi(X)$ and on *the systole of* X, i.e. the length of the shortest closed geodesic of X. We will follow the approach given by Barrett and Diller **[BD]**.

§3 contains auxiliary results on Riemann surfaces which are used during the proof. The most important is Theorem 3.1 which is due to McMullen **[McM1]**. It establishes a property of convergence for a sequence of triples (X_i, x_i, ϕ_i) where ϕ_i is an integrable holomorphic quadratic differentials on the Riemann surface X_i and $x_i \in X_i$. Recall first what it means for a sequence of pointed hyperbolic Riemann surfaces (X_i, x_i) having a fixed topological type to converge to a Riemann surface (X, x). There are two cases to consider, according the behaviour of the injectivity radius $\mathrm{inj}(x_i)$ at x_i. If $\mathrm{inj}(x_i)$ remains bounded away from 0, it means that (X_i, x_i) converges to (X, x) for the Hausdorff-Gromov topology on pointed metric spaces: in this case, X is a hyperbolic Riemann surface with finite volume. If $\mathrm{inj}(x_i)$ tends to 0, it means that (X_i, x_i) with the hyperbolic metric rescaled by $1/\mathrm{inj}(x_i)$ converges to (X, x): in that case, $X = \mathbb{C}^*$ with a complete flat metric. Then, if $\phi_i \in \mathcal{Q}(X_i)$ and if ϕ is a holomorphic quadratic differential on X, we say that (ϕ_i) *converges uniformly to* ϕ, when the local expression of ϕ_i (in a chart for X_i which converges to a chart for X) converges uniformly to the local expression of ϕ. Theorem 3.1 asserts that, if (X_i, x_i) converges to (X, x) and if (ϕ_i) is a sequence in $\mathcal{Q}(X_i)$ with $||\phi_i|| = 1$, (ϕ_i) *converges uniformly* to a non-zero holomorphic quadratic differential ϕ on X, up to multiplying ϕ_i by a constant and up to extracting a subsequence. This theorem is proved in two steps. First, we produce a sequence of non-zero $\theta_i \in \mathcal{Q}(X_i)$ which converges uniformly to a holomorphic quadratic differential ϕ in the above sense. This reduces the uniform convergence of $\phi_i = (\phi_i/\theta_i)\theta_i$ to the uniform convergence of (the functions) (ϕ_i/θ_i). This follows from the classical theorems of Montel and Picard. The proof we give here of the first step is slightly different from the original one and relies on less advanced machinery than **[McM1]**.

§4 concerns the solution of a certain $\bar{\partial}$-problem on *an open Riemann surface* Y; an open Riemann surface is a Riemann surface with finite topological type but with infinite volume. On any open Riemann surface Y, the hyperbolic volume form dv is exact. Moreover, since any open Riemann surface is a Stein manifold, dv is $\bar{\partial}$-exact. Theorem 4.1, which is due to Diller **[Di]**, provides a *well-behaved* solution to the equation $\bar{\partial}\eta = dv$. This solution η is well-behaved in the sense that it is a 1-form of type $(1,0)$ whose hyperbolic norm is finite, bounded by a function of $\chi(Y)$ and of the systole of Y. We consider only the case when Y has no cusps. Then Y is the union along the boundary of a compact surface Y_0 with geodesic boundary and a finite collection of half-infinite annuli. Using the formula for the hyperbolic metric on an annulus, one can define an explicit 1-form η_0 of type $(1,0)$ which is supported on $Y - Y_0$ and which solves $\bar{\partial}\eta_0 = dv$ on a neighborhood of the ends of Y. By construction, the hyperbolic norm of η_0 is independent of Y, and therefore, we are led to find a well-behaved solution η' of $\bar{\partial}\eta' = dv - \bar{\partial}\eta_0$, for then $\eta = \eta_0 + \eta'$ will be the required solution. Since the 2-form $dv - \bar{\partial}\eta_0$ has compact support, this equation can be solved using the Green's function on Y. Estimates on the circular averages of this Green's function provide then the required bound on the hyperbolic norm of η'.

Also in §4, we prove a refinement of Theorem 4.1, in which one we no longer have control on the systole of Y. Let ε a constant smaller than the Margulis constant. Rather than finding a well-behaved solution to the equation $\bar{\partial}\eta = dv$ which is defined on the entire surface Y, we find such a solution on the *unbounded* (i.e. non-compact) components of the ε-thick part $Y^{[\varepsilon,\infty[}$ of Y, and whose norm is bounded by a function of ε and $\chi(Y)$.

In **[BD]**, D. Barrett and J. Diller show how to deduce from Theorem 4.1 the McMullen theorem about the norm of the Theta operator $\Theta_{Y/X}$ associated to a geometric cover $Y \to X$. This short proof is explained in §5. For the applications, one further needs to control how the norm of $\Theta_{Y/X}$ can approach 1, when the topological type of the cover $Y \to X$ is fixed but when there is no information on the systole of X. This control can be formulated in terms of *the ε-amenable part of the cover* $Y \to X$. Recall that the cover $Y \to X$ arises from a proper incompressible surface $S \subset X$. Thus, any component of $X^{[0,\varepsilon]}$ or of $X^{[\varepsilon,\infty[}$ which can be isotoped into S can be lifted homeomorphically to a surface contained in Y. The ε-amenable part of the cover $Y \to X$ is then defined as the union of the preimage of $X^{[0,\varepsilon]}$ and the lifts of the components of $X^{[\varepsilon,\infty[}$ which can be isotoped into S. Theorem 5.1 is made more precise by the next statement (Theorem 5.3): if $||\Theta_{Y/X}\phi||$ is more than $1-\delta$ for a unit norm $\phi \in \mathfrak{Q}(Y)$, then the ϕ-mass of *the ε-amenable part* of the cover $Y \to X$ is more than $1 - c(\delta)$, where $c(\delta)$ depends on ε and tends to 0 with δ. This result is also due to McMullen, who deals not only with geometric covers, but also with *non-amenable covers* **[McM1]**.

In §6, we prove, following McMullen **[McM2]**, Thurston's fixed point theorem. The existence of a fixed point for $\tau^* \circ \sigma$ is related to a contraction property for $\tau^* \circ \sigma$ with respect to the Teichmüller distance. Since τ^* is an isometry, the contraction properties of $\tau^* \circ \sigma$ follows from the contraction properties of σ. The results of §5 have direct consequences for the norm of $d^*\sigma$ at $s \in \mathcal{T}(\partial N)$. From Theorem 5.1, it follows that $||d_s^*\sigma||$ is bounded away from 1 by a constant which depends only on $\chi(\partial N)$ and on the systole of s (Proposition 6.1). Proposition 6.2 is also a consequence of Theorem 5.3: it says that if $||d_s^*\phi|| \geq 1-\delta$ for a unit norm $\phi \in \mathfrak{Q}(\sigma(\partial N^s))$, then the ϕ-mass of the ε-amenable part of $\sigma(\partial N^s)$ is more than $1 - c'(\delta)$ where $c'(\delta)$ tends to 0 with δ (the ε-amenable part of $\sigma(\partial N^s)$ is the union of the ε-amenable parts of the covers $U \to X_U$ associated to the spots $U \subset \sigma(\partial N^s)$). However, these two results of 2-dimensional nature, don't suffice to solve the Fixed point problem and another argument is needed. For this, we observe that the ε-amenable part of $\sigma(\partial N^s)$ can be decomposed into the union of the simply connected components of the preimage of $X_U^{]0,\varepsilon]}$ and the lifts of the components of $X_U^{]0,\varepsilon]}$ or $X_U^{[\varepsilon,\infty[}$: the (possibly empty) union of these lifts forms a compact surface, called *the ε-liftable part of* $\sigma(\partial N^s)$. With this terminology, Proposition 6.3 asserts that, for sufficiently small ε, if $||d_s^*\sigma\phi|| \geq 1-\delta$ for a unit norm $\phi \in \mathcal{Q}(\sigma(\partial N^s))$, then the ϕ-mass of the ε-liftable part of $\sigma(\partial N^s)$ is more than $1-c''(\delta)$ where $c''(\delta)$ only depends on ε and tends to 0 with δ. One deduces Proposition 6.3 from Proposition 6.2 and from Proposition 6.4, whose proof rests on a more global argument: after a normalization of the limit set of G^s in $\overline{\mathbb{C}}$ (which uses the geometry of the 3-dimensional Margulis tubes) it is a consequence of the compacity theorem for holomorphic quadratic differentials (Theorem 3.1).

To prove the Fixed point theorem, we exploit the geometric consequences of Proposition 6.3. There are two cases to consider according as N is acylindrical or not.

When N is acylindrical, all the spots are simply connected, and in particular, the ε-liftable part is empty. This implies that σ contracts uniformly the Teichmüller distance and therefore $\tau^* \circ \sigma$ also (for any gluing data τ). Since Teichmüller space is complete, $\tau^* \circ \sigma$ has a fixed point.

When N is cylindrical, some gluing data τ may produce non-atoroidal manifolds, thereby forbidding the existence of a fixed point for $\tau^* \circ \sigma$. This occurs for instance when τ maps one boundary component of an essential annulus to the other. Therefore, we must take into account τ in proving the Fixed point theorem. In the cylindrical case, we won't prove that $\tau^* \circ \sigma$ is uniformly contracting, but only that some iterate $(\tau^* \circ \sigma)^K$ is uniformly contracting on a certain $\tau^* \circ \sigma$-invariant closed subset of $\mathcal{T}(\partial N)$ (since $(\tau^* \circ \sigma)^K$ and $\tau^* \circ \sigma$ commute, this suffices to prove the Fixed point theorem). The principal geometric consequence of Proposition 6.3 and indeed, the only one necessary to prove the Fixed point theorem is: if $||d_s^*\sigma||$ is sufficiently near to 1, there is an essential annulus in N which joins two curves α and γ such that α is shorter than ε for the metric s and such that γ is shorter than ε for the metric $\sigma(s)$. If we suppose that $||d_s^*(\tau^* \circ \sigma)^K||$ is near 1, then since σ is contracting and since τ^* is an isometry, the norm of $d^*\sigma$ at $\tau^*(\tau^* \circ \sigma)^k(s)$ is also near one, for all $0 \leq k \leq K-1$. Therefore, by successive applications of Fact 6.14, we produce a sequence of K essential annuli A_i in N with boundary the union of two simple closed curves α_i and γ_i, such that $\alpha_{i+1} = \tau(\gamma_i)$ and such that the curves α_i are shorter for the hyperbolic metric s than the Margulis constant. Then, by the well known Margulis lemma, each of the curves α_i is homotopic to one of finitely many disjoint simple closed curves on ∂N of length less than the Margulis constant. Therefore, if we choose K bigger than the maximal number of disjoint pairwise non-homotopic simple closed curves on ∂N, two of the curves α_i are homotopic on ∂N. It is then easy to produce an essential singular torus in N/τ, contradicting the hypothesis of atoroidality on N/τ.

§3, 4, 5, 6 are completely self-contained. They form the main part of this paper and give a complete and detailed proof of Thurston's fixed point theorem.

§7 develops the theory of manifolds-with-corners which was sketched above. It is the "3-dimensional core" of the proof. Proofs are given with details, but to shorten the exposition, we use the equivariant versions of the Dehn lemma, the Sphere theorem and the Torus theorem.

In §8, we explain the equivariant machinery which allows one to deduce the Final gluing theorem for manifolds-with-corners from the Final gluing theorem. Then, we deduce the Hyperbolization theorem.

Thus, granting the by now standard material on Kleinian groups laid out with appropriate references in §1, the above three equivariant theorems, and the case of Andreev's theorem for right-angled compact polyhedra in $\mathbb{H}^3$, this survey comprises a complete proof of Thurston's hyperbolization theorem for non-fibered Haken manifolds whose fundamental group does not contain $\mathbb{Z}+\mathbb{Z}$. The extension to the general case, i.e. to atoroidal manifolds, or to "pared manifolds" (cf. [**Mor**]) does not encounter any difficulties that are unfamiliar or deep. The reader is invited to find the modifications necessary to establish this general case. For this, he or she needs to extend the results concerning the norm of Theta operators to geometric covers of finite volume Riemann surfaces. The topological results of §7 also need to be extended to deal with the case of "pared manifolds-with-corners". The Hyperbolization theorem for pared manifolds is covered by the (still informal) notes [**OP**].

I wish to thank Peter Shalen and John Stallings who have told me their memories of the Hyperbolization theorem, some of which I have tried to evoke in the first part of this introduction. Larry Siebenmann clarified their points of view and induced me to investigate the earlier literature. I thank him for being so exacting. I felt obliged to reorganize the combinatorial part of this article after he convinced me that the most direct way from Thurston's fixed point theorem to his Hyperbolization theorem goes via the natural generalization to compact manifolds-with-corners of the case for right-angled polyhedra of Andreev's hyperbolization theorem. This approach nicely complements Francis Bonahon's observation [**OP**] that right-angles are sufficient. The pictures which illustrate this article were drawn by Greg McShane who also patiently commented on early drafts. With a lot of criticism, Saar Hersonsky helped to bring the all analytical part of this article to its present form. During the writing, I benefitted also from several fruitful discussions with Frederic Paulin. In particular, §7 and §8 emerged from chapters he contributed to the notes [**OP**].

CHAPTER 1

Kleinian groups and Teichmüller theory

We refer the reader to the books [**Bea**], [**BP**] and [**Ra**] for more details on the first section.

1.1 Kleinian groups

The *hyperbolic space of dimension* n is the complete and simply connected n-dimensional Riemannian manifold of constant curvature -1 (we think of n as being equal to 2 or 3). This manifold has two well-known isometric models, *the Riemann model*

$$\mathbb{D}^n = \{(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \quad \sum \xi_i^2 < 1\},$$

endowed with the Riemannian metric

$$ds^2 = \frac{4(d\xi_1^2 + \cdots + d\xi_n^2)}{(1 - (\sum \xi_i^2))^2},$$

and *the upper-half space model*

$$\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n, \quad x_n > 0\},$$

endowed with the Riemannian metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

Any isometry of $\mathbb{H}^n$ extends continuously to the boundary $\overline{\mathbb{R}^{n-1}}$ where it induces a conformal or anticonformal map according to whether it preserves the orientation or not. Let $\mathrm{Isom}(\mathbb{H}^n)$ be the group of orientation preserving isometries of $\mathbb{H}^n$. Therefore, $\mathrm{Isom}(\mathbb{H}^2)$ and $\mathrm{Isom}(\mathbb{H}^3)$ can be identified with $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$ respectively.

Definition.— *A Kleinian group* is a discrete, finitely generated subgroup of $\mathrm{Isom}(\mathbb{H}^n)$.

Let γ be an isometry of $\mathbb{H}^n$ which is different from the identity. It is well known that γ is either *hyperbolic* (it has exactly two fixed points in $\partial\mathbb{H}^n$) or *parabolic* (it has a unique fixed point in $\partial\mathbb{H}^n$), or *elliptic* (it has a fixed point in $\mathbb{H}^n$). When γ is hyperbolic, it leaves invariant the geodesic of $\mathbb{H}^n$ joining its two fixed points. This geodesic $A(\gamma)$ is called *the axis of* γ. The isometry γ acts on $A(\gamma)$ as a translation of a certain distance $\ell(\gamma)$ called the *translation distance of* γ.

From now on, all the Kleinian groups will be supposed to be torsion-free.

Let G be a Kleinian group. Then G does not contain elliptic elements and its action on $\mathbb{H}^n$ is properly discontinuous. The quotient space $M(G) = \mathbb{H}^n/G$ is a complete Riemannian manifold of constant curvature -1.

The limit set and the domain of discontinuity of a Kleinian group.

Definition.— A group is *elementary* if it contains an Abelian subgroup of finite index.

One can show that elementary Kleinian groups are characterized among all Kleinian groups as those which act on $\partial\mathbb{H}^n$ by fixing one or two points.

Let G be a non-elementary Kleinian group.

Definition.— *The limit set of* G is the smallest non-empty closed subset of $\partial\mathbb{H}^n$ which is invariant under G. It is denoted by $L(G)$.

One can also define $L(G)$ as the closure in $\partial\mathbb{H}^n$ of the set of fixed points of the non-zero elements of G. It is a perfect subset of $\partial\mathbb{H}^n$.

Let $C(G)$ be the smallest closed convex subset of $\mathbb{H}^n$ whose closure in $\overline{\mathbb{H}^n}$ contains $L(G)$. It is a convex subset invariant by G. The quotient space $N(G) = C(G)/G$ is contained in $M(G)$ and is called *the Nielsen core of* $M(G)$. It is the smallest closed convex subset of $M(G)$ such that the inclusion into $M(G)$ is a homotopy equivalence [**Th1**].

For $n = 2$, $\partial N(G)$ is totally geodesic, but for $n = 3$, $N(G)$ is not a differentiable submanifold of $M(G)$ in general: its boundary is "bent" along certain geodesics. One way to avoid this problem is to replace $N(G)$ by its neighborhood of radius δ in $M(G)$. Denote this neighborhood by $N_\delta(G)$. It is not difficult to see that, for any $\delta > 0$, $N_\delta(G)$ is a submanifold of $M(G)$ of class C^1 which is strictly convex (i.e. any geodesic arc of $M(G)$ joining two points of $N_\delta(G)$ is contained in the interior of $N_\delta(G)$, except possibly its endpoints). Furthermore, $N_\delta(G)$ does not depend on $\delta > 0$ up to diffeomorphism. It is called *the thickened Nielsen core.*

Although $\partial N(G)$ is not a differentiable submanifold of $M(G)$, the convexity of $N(G)$ allows to consider the path metric that the metric of $M(G)$ induces. When $n = 3$, a basic property of this distance is that it is "hyperbolic": with this induced metric, $\partial N(G)$ is locally isometric to $\mathbb{H}^2$ (cf. [**Th1**], [**Ro**]). This important property will be used at the end of §6.

Definition.— *The domain of discontinuity of* G is the complement of $L(G)$ in $\partial\mathbb{H}^n$. It is denoted by $\Omega(G)$.

When $\Omega(G) \neq \emptyset$, the action of G on $\mathbb{H}^n \cup \Omega(G)$ is properly discontinuous. This can be seen using *the nearest point retraction.* The map which assigns to $x \in \mathbb{H}^n$ the point of (the closed subset) $C(G)$ which is nearest to x extends continuously to a map $\widetilde{r} : \mathbb{H}^n \cup \Omega(G) \to C(G)$ called the nearest point retraction. The map $\widetilde{r}$ commutes with the isometries of $\mathbb{H}^n$ which leave $C(G)$ invariant. In particular $\widetilde{r}$ commutes with the elements of G. Therefore, since G acts properly and discontinuously on $C(G) \subset \mathbb{H}^n$, G acts properly and discontinuously on $\mathbb{H}^n \cup \Omega(G)$ also and one can form the quotient space of $\mathbb{H}^n \cup \Omega(G)$ by G. It is a smooth (analytical) manifold with boundary denoted by $\overline{M}(G)$ whose interior equals $M(G)$. For $n = 3$, its boundary is a Riemann surface.

Since $\widetilde{r}$ commutes with the elements of G, it induces a retraction $r : \overline{M}(G) \to N(G)$. We can define in a similar way a retraction $r_\delta : \overline{M}(G) \to N_\delta(G)$. For $\delta > 0$, it follows from the strict convexity of $N_\delta(G)$ that $r_\delta^{-1}(\partial N_\delta(G))$ is diffeomorphic to $\partial N_\delta(G) \times [0, 1]$. Thus, $\overline{M}(G)$ is diffeomorphic to $N_\delta(G)$.

Thus one can associate to G three manifolds:

(i) the manifold with boundary $\overline{M}(G)$, whose interior is

(ii) the complete Riemannian manifold $M(G)$ with constant sectional curvature -1, and

(iii) the Nielsen core $N(G)$ or its δ-neighborhood $N_\delta(G)$.

The Margulis decomposition.

Let G be a non-elementary Kleinian group.

Definition.— Let $\varepsilon > 0$. *The ε-thin part of* $M(G)$ is the set of points $x \in M(G)$ through which goes a geodesic loop of length less than or equal to ε. We denote the ε-thin part by $M(G)^{]0,\varepsilon]}$. Equivalently $M(G)^{]0,\varepsilon]}$ is the set of points where the injectivity radius is less than $\varepsilon/2$. The closure of the complement of $M(G)^{]0,\varepsilon]}$ in $M(G)$, denoted by $M(G)^{[\varepsilon,\infty[}$, is called *the ε-thick part of* $M(G)$.

Margulis lemma [Marg].— *There exists a constant $\varepsilon(n) > 0$ such that, if $G \subset \mathrm{Isom}(\mathbb{H}^n)$ is a Kleinian group and if $x \in \mathbb{H}^n$, then the subgroup of G generated by the elements which move x a distance smaller than $\varepsilon(n)$ is elementary.*

The constant $\varepsilon(n)$ depends on n but not on G. It is called *the Margulis constant.*

As a consequence of Margulis lemma, we describe now the geometry of $M(G)^{]0,\varepsilon]}$ for $n = 3$ when G has no parabolic elements and for $n = 2$ (cf. **[Thl]**).

Let $G \subset \mathrm{PSL}_2(\mathbb{C})$ be a Kleinian group without parabolic elements. Any elementary subgroup of G is then a cyclic group generated by a hyperbolic isometry. Let $x \in M(G)^{]0,\varepsilon]}$. For any $\widetilde{x}$ in the preimage of x in $\mathbb{H}^3$, there is an isometry in G other than the identity, which moves $\widetilde{x}$ a distance less than or equal to ε. Let $g \in G$ be a hyperbolic isometry. The set of points in $\mathbb{H}^3$ which are moved by g a distance less than or equal to ε is non-empty only when the translation distance of g is less than or equal to ε. Then, by reasons of symmetry, it is a neighborhood of constant radius of the invariant axis $A(g)$ of g. We denote this neighborhood by $n^\varepsilon(g)$. Let $\langle\langle g \rangle\rangle$ be the maximal Abelian subgroup of G containing g. Since

g is an hyperbolic isometry $\langle\langle g\rangle\rangle$ is a cyclic group generated by a root of g. Let $\mathcal{N}^\varepsilon(g)$ be the union of the neighborhoods $n^\varepsilon(h)$ over all non-zero elements h in $\langle\langle g\rangle\rangle$. Suppose $\varepsilon \leq \varepsilon(3)$. By the Margulis lemma, the restriction of the covering map $p : \mathbb{H}^3 \to M(G)$ to $\mathcal{N}^\varepsilon(g)$ identifies points only when they are in the same orbit of $\langle\langle g\rangle\rangle$. Hence $p(\mathcal{N}^\varepsilon(g)) \subset M(G)$ is a solid tube diffeomorphic to $\mathcal{N}^\varepsilon(G)/\langle\langle g\rangle\rangle$. This image is a neighborhood of constant radius of the (embedded) closed geodesic $p(A(g)) = A(g)/\langle\langle g\rangle\rangle$. It is called *the Margulis tube around* $p(A(g))$. By Margulis lemma again, $M(G)^{]0,\varepsilon]}$ is a disjoint union of Margulis tubes. It is a nice exercice to show that the radius of the Margulis tube around a (very short) geodesic of length $\ell(g)$ is equivalent to $\log(\varepsilon/\ell(g))$, independently of G ([**Th5**], [**OP**]). Qualitatively, the shorter the geodesic, the larger the Margulis tube around it.

When G is a Kleinian group contained in $\mathrm{PSL}_2(\mathbb{R})$, $M(G)^{]0,\varepsilon]}$ can be similarly described. The set of points which are moved a distance less than or equal to ε by a hyperbolic isometry $g \in G$ is a regular neighborhood of the axis of g. For $\varepsilon \leq \varepsilon(2)$ the quotient of this neighborhood by $\langle\langle g\rangle\rangle$ embeds in $M(G)$. Its image, called a Margulis tube, is diffeomorphic to an annulus. The set of points which are moved a distance less than or equal to ε by a parabolic isometry $g \in \mathrm{PSL}_2(\mathbb{R})$ is an horoball. For $\varepsilon \leq \varepsilon(2)$, the quotient of this horoball by $\langle\langle g\rangle\rangle$ embeds in $M(G)$. Its image is called *a cusp*. A cusp is conformally equivalent to the punctured closed unit disc.

When $n = 2$, $M(G)^{]0,\varepsilon]}$ is a disjoint union of Margulis tubes and cusps and by Margulis lemma, no two of them are homotopic. Therefore $M(G)^{]0,\varepsilon]}$ has finitely many components. When $n = 3$, this is not true in general, but will be satisfied by the groups we will consider.

1.2 Quasi-conformal homeomorphisms

We refer to the books [**Ahl**], [**Ga**] and [**LV**] for more details on this section.

Definition. — Let $\varphi : U \to V$ be an homeomorphism between two open sets in the complex plane. The map φ is called *quasi-conformal* when the following three conditions hold:

(i) φ is orientation preserving,

(ii) the derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ in the sense of distributions exist and are locally square-integrable, and

(iii) there exists $\mu \in L^\infty(U, \mathbb{C})$ with $||\mu||_\infty < 1$, such that, for almost all $z \in U$, $\bar\partial\varphi(z) = \mu(z)\partial\varphi(z)$, where

$$\partial\varphi = \frac{1}{2}(\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}) \qquad \text{and} \qquad \bar\partial\varphi = \frac{1}{2}(\frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y}).$$

The quasi-conformal homeomorphism φ is said to be K*-quasi-conformal* for

$$K = K(\varphi) = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}.$$

The function μ is called *the Beltrami coefficient of* φ and $K(\varphi)$ *the eccentricity of* φ.

A 1-quasi-conformal homeomorphism is conformal. Therefore $\log K$ measures the deviation of φ from being conformal.

When the frontier of U in $\mathbb{C}$ is locally connected, any quasi-conformal homeomorphism φ of U extends continuously to ∂U. In particular, any quasi-conformal homeomorphism of the upper half-space extends continuously to the boundary.

Note that the right or left composition of a K-quasi-conformal homeomorphism with a conformal homeomorphism is K-quasi-conformal. Therefore one can define the notion of being K-quasi-conformal for a homeomorphism between two Riemann surfaces.

The fundamental result about quasi-conformal homeomorphisms is due to L. Ahlfors and L. Bers.

Ahlfors-Bers theorem [AB].— *Let* $\mu \in L^\infty(\overline{\mathbb{C}})$ *with* $||\mu||_\infty < 1$. *Then, there exists a unique quasi-conformal homeomorphism* φ_μ *of* $\overline{\mathbb{C}}$ *such that*

(i) $\dfrac{\bar\partial \varphi_\mu}{\partial \varphi_\mu}(z) = \mu(z)$ *for almost all points* z,

(ii) φ_μ *fixes* 0, 1 *and* ∞.

The function $\mu \to \varphi_\mu$ *is continuous for the topology of the uniform convergence over compact sets. Furthermore, for* $t \le 1$ *we have*

$$\varphi_{t\mu}(z) = z + tF_\mu(z) + O(t^2),$$

where

$$F_\mu(z) = -\frac{z(z-1)}{\pi} \iint_{\mathbb{H}^2} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta.$$

Equation (i) is called *the Beltrami equation.* A quasi-conformal homeomorphism of $\mathbb{C}$ (resp. of $\overline{\mathbb{H}^2}$) which satisfies (ii) is said to be *normalized.*

The following is a corollary of Ahlfors-Bers theorem (existence and unicity).

Theorem [AB].— *Let* $\mu \in L^\infty(\mathbb{H}^2)$ *with* $||\mu||_\infty < 1$. *Then there exists a unique normalized quasi-conformal homeomorphism* φ_μ *of* $\mathbb{H}^2$ *which satisfies*

$$\frac{\bar\partial \varphi_\mu}{\partial \varphi_\mu}(z) = \mu(z).$$

1.3 Teichmüller space

Definition.— *A Fuchsian group* is a Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$. We say that Γ is *cocompact* (resp. *has finite covolume*) if $\mathbb{H}^2/\Gamma$ is compact (has finite volume). If $\mathbb{H}^2/\Gamma$ has finite volume, it is conformally equivalent to the complement of a finite number of points in a compact Riemann surface. A point in this finite set is called a *puncture*. A connected Riemann surface X is *hyperbolic* when its universal cover is conformally equivalent to $\mathbb{H}^2$. A Riemann surface is hyperbolic if each of its components is hyperbolic.

Let X be a connected Riemann surface of negative Euler characteristic. By the Poincaré-Koebe uniformization theorem, the universal cover of X is conformally equivalent to $\mathbb{H}^2$. Hence X is conformally equivalent to the quotient of $\mathbb{H}^2$ by a group of conformal automorphisms. The group of conformal automorphisms of $\mathbb{H}^2$ equals $\mathrm{PSL}_2(\mathbb{R})$ acting by homographies. Therefore X is hyperbolic. Since $\mathrm{PSL}_2(\mathbb{R})$ is the group of isometries of $\mathbb{H}^2$, the hyperbolic metric on $\mathbb{H}^2$ projects to a complete Riemannian metric of constant curvature -1 on X which is in the given conformal class. This Riemannian metric is called *the hyperbolic metric on* X. This construction gives us our first examples of Kleinian groups.

The goal of Teichmüller theory is to describe all hyperbolic metrics on a fixed surface or equivalently, all the "deformations" of a given Fuchsian group Γ. We will restrict our attention to cocompact Fuchsian groups.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a cocompact Fuchsian group. Let $X = \mathbb{H}^2/\Gamma$.

Definition. — *A Fuchsian deformation of* Γ is a couple $(\rho, \widetilde{\varphi})$, where ρ is a representation of Γ in $\mathrm{PSL}_2(\mathbb{R})$ and where $\widetilde{\varphi}$ is a normalized quasi-conformal homeomorphism of $\mathbb{H}^2$ which conjugates Γ and $\rho(\Gamma)$, i.e. such that

(i) for all $\gamma \in \Gamma$,

$$\rho(\gamma) = \widetilde{\varphi} \circ \gamma \circ \widetilde{\varphi}^{-1}, \quad \text{and}$$

(ii) the continuous extension of $\widetilde{\varphi}$ to $\overline{\mathbb{R}}$ fixes 0, 1 and ∞.

Define an equivalence relation on the set of Fuchsian deformations of Γ by $(\rho, \widetilde{\varphi}) \simeq (\rho', \widetilde{\varphi}')$ if and only if $\rho = \rho'$.

Observe that the extension of $\widetilde{\varphi}$ (resp. $\widetilde{\varphi}'$) to the real axis conjugates the action of Γ to the one of $\rho(\Gamma)$ (resp. $\rho'(\Gamma)$). Thus, the density of the set of fixed points of elements of Γ in $\mathbb{R}$ implies that: $\rho = \rho'$ if and only if $\widetilde{\varphi}|\mathbb{R} = \widetilde{\varphi}'|\mathbb{R}$. The quotient of the set of Fuchsian deformations of Γ by this equivalence relation is called *the Teichmüller space of* Γ. We denote it by $\mathcal{T}(\Gamma)$ or by $\mathcal{T}(X)$.

If $(\rho, \widetilde{\varphi})$ is a Fuchsian deformation of Γ, $\rho(\Gamma)$ is discrete because its action on $\mathbb{H}^2$ is conjugate to that of Γ. So $\widetilde{\varphi}$ projects to a homeomorphism φ between $\mathbb{H}^2/\Gamma$ and $\mathbb{H}^2/\rho(\Gamma)$. One checks that the equivalence relation which defines Teichmüller space identifies two Fuchsian deformations $(\rho, \widetilde{\varphi})$ and $(\rho', \widetilde{\varphi}')$ when the composed homeomorphism $\varphi^{-1} \circ \varphi'$ of $\mathbb{H}^2/\Gamma$ is homotopic to the identity. Therefore a point in $\mathcal{T}(X)$ can be interpreted as a hyperbolic surface with a homeomorphism from X, which is well defined up to homotopy.

The Teichmüller distance.

For two points σ_1, σ_2 in $\mathcal{T}(\Gamma)$ we define

$$d(\sigma_1, \sigma_2) = \frac{1}{2} \inf \log K(\widetilde{\varphi}_2 \circ \widetilde{\varphi}_1^{-1}),$$

where the infimum is taken over all representatives $(\rho_1, \widetilde{\varphi}_1)$ (resp. $(\rho_2, \widetilde{\varphi}_2)$) of σ_1 (resp. σ_2).

Then d is a distance, called *the Teichmüller distance*, which turns $\mathcal{T}(\Gamma)$ into a complete metric space.

In §6, we will need the following result (cf. **[Ga]**).

Distortion lemma. — *Let* $\sigma_1 = (\rho_1, \widetilde{\varphi}_1)$, $\sigma_2 = (\rho_2, \widetilde{\varphi}_2)$ *be two points in* $\mathcal{T}(\Gamma)$. *Let* $\gamma \in \Gamma$ *be a hyperbolic element. Then, we have*

$$e^{-2d(\sigma_1,\sigma_2)}\ell(\rho_1(\gamma)) \leq \ell(\rho_2(\gamma)) \leq e^{2d(\sigma_1,\sigma_2)}\ell(\rho_1(\gamma)).$$

The modular group.

Definition. — *The modular group of* X is the group of homotopy equivalences of orientation preserving diffeomorphisms of X. It is denoted by $\mathrm{Mod}(X)$.

We will denote by $\overline{X}$ the Riemann surface X with the opposite orientation, i.e. the quotient of the lower half-plane $\overline{\mathbb{H}^2}$ by Γ. The space of equivalences classes of Fuchsian deformations of the action of Γ on $\overline{\mathbb{H}^2}$ is denoted by $\mathcal{T}(\overline{\Gamma})$ (or by $\mathcal{T}(\overline{X})$). There is a natural map from from $\mathcal{T}(\Gamma)$ to $\mathcal{T}(\overline{\Gamma})$ induced by the map which assigns to the Fuchsian deformation $(\rho, \widetilde{\varphi})$ of Γ the deformation $(\rho, \bar{\varphi})$ of Γ acting on $\overline{\mathbb{H}^2}$, where $\bar{\varphi}$ is the conjugate of $\widetilde{\varphi}$ by the complex conjugation. It is denoted by $s \to \bar{s}$ and called *the complex conjugation.* Its inverse is a map from $\mathcal{T}(\overline{\Gamma})$ to $\mathcal{T}(\Gamma)$ which is defined similarly.

Let f be an orientation preserving diffeomorphism of $X = \mathbb{H}^2/\Gamma$. Choose a lift $\widetilde{f}$ of f to the universal cover $\mathbb{H}^2$. Since X is compact, f is K-quasi-conformal for some constant K and $\widetilde{f}$ is K-quasi-conformal also. Let $(\rho, \widetilde{\varphi})$ be a Fuchsian deformation of Γ. Then $\widetilde{\varphi} \circ \widetilde{f}^{-1}$ is a quasi-conformal homeomorphism which conjugates Γ to a certain Fuchsian group. Let $a^{-1} \in \mathrm{PSL}_2(\mathbb{R})$ be an element which takes the same values as $\widetilde{\varphi} \circ \widetilde{f}^{-1}$ on the points 0, 1 and ∞. Then $a \circ \widetilde{\varphi} \circ \widetilde{f}^{-1}$ is a normalized quasi-conformal homeomorphism which conjugates the trivial representation of Γ to a representation ρ_f. Therefore, the couple $(\rho_f, a \circ \widetilde{\varphi} \circ \widetilde{f}^{-1})$ is a Fuchsian deformation of Γ. One checks easily that the equivalence class of this deformation depends only of the equivalence class of $(\rho, \widetilde{\varphi})$, and defines therefore a map of $\mathcal{T}(\Gamma)$ to itself, which depends only on the homotopy class of f. We denote this map by f^*: it is an isometry for the Teichmüller distance. Thus $f \to f^*$ defines an isometric action of Mod (X) on $\mathcal{T}(\Gamma)$.

An orientation reversing diffeomorphism f of X induces also a map f^* from $\mathcal{T}(\Gamma)$ to $\mathcal{T}(\overline{\Gamma})$ (and also from $\mathcal{T}(\overline{\Gamma})$ to $\mathcal{T}(\Gamma)$). This map is an isometry which commutes with the complex conjugation.

The differentiable structure of Teichmüller space.

Teichmüller space has been defined so far as a metric space. In fact it is a smooth manifold, a property which is crucial in McMullen's proof of Thurston's fixed point theorem. We sketch below how Teichmüller space can be viewed as a complex manifold (cf. **[Ga]**). In order to do this, we first introduce two objects which are fundamental for understanding Teichmüller space from an infinitesimal viewpoint, *the Beltrami forms* and *the holomorphic quadratic differentials.*

Beltrami forms and holomorphic quadratic differentials.

Let Γ be a (not necessarily cocompact) Fuchsian group. Let $X = \mathbb{H}^2/\Gamma$.

Definition. — *A Beltrami form* μ on $\mathbb{H}^2/\Gamma$ is a measurable function $\widetilde{\mu} : \mathbb{H}^2 \to \mathbb{C}$, with finite L^∞-norm and such that

$$\forall \gamma \in \Gamma, \qquad \widetilde{\mu}(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \widetilde{\mu}(z)$$

for almost all $z \in \mathbb{H}^2$. We denote by $\mathcal{B}(\Gamma)$ (or $\mathcal{B}(X)$) the Banach space of Beltrami forms on $\mathbb{H}^2/\Gamma$ endowed with the L^∞-norm.

Let $\mathcal{B}^1(\Gamma)$ be the open unit ball in $\mathcal{B}(\Gamma)$. Let $\mu \in \mathcal{B}^1(\Gamma)$. By Ahlfors-Bers theorem, there is a quasi-conformal homeomorphism φ_μ of $\mathbb{H}^2$ with Beltrami coefficient $\widetilde{\mu}$. A short computation shows that the Beltrami coefficient of the quasi-conformal homeomorphism $\rho_\mu(\gamma) = \varphi_\mu^{-1} \circ \gamma \circ \varphi_\mu$ vanishes so that $\rho_\mu(\gamma)$ belongs to $\mathrm{PSL}_2(\mathbb{R})$. Clearly, $\gamma \to \rho_\mu(\gamma)$ is a homomorphism of Γ into $\mathrm{PSL}_2(\mathbb{R})$. On this way, we define a map $\Pi : \mathcal{B}^1(\Gamma) \to \mathcal{T}(\Gamma)$ by assigning to μ the equivalence class of the couple (ρ_μ, φ_μ). By definition of $\mathcal{T}(\Gamma)$, Π is onto.

Definition. — *A holomorphic quadratic differential* ϕ on $\mathbb{H}^2/\Gamma$ is a holomorphic function $\widetilde{\phi}$ on $\mathbb{H}^2$ which satisfies

$$\forall \gamma \in \Gamma, \quad \forall z \in \mathbb{H}^2, \quad \widetilde{\phi}(\gamma(z))(\gamma'(z))^2 = \widetilde{\phi}(z).$$

This transformation rule means that the tensor $\widetilde{\phi}(z)dz^2$ is invariant under the action of Γ and projects therefore to a tensor on X. In particular the expression $|\widetilde{\phi}(z)||dz^2|$ is invariant under Γ. It defines a measure on X. For $\phi \in \mathcal{Q}(X)$ and for $E \subset X$ a measurable set, the measure of E, $\int_E |\phi|$ is called *the* ϕ*-mass of* E. When the ϕ-mass of X is finite, ϕ is said to be *integrable.* We denote by $\mathcal{Q}(\Gamma)$ (or $\mathcal{Q}(X)$) the Banach space of integrable holomorphic quadratic differentials endowed with the L^1-norm.

When Γ is cocompact, any holomorphic quadratic differential is of course integrable. When Γ is not cocompact but has finite covolume, the integrability condition is not necessarily satisfied. It means precisely that when ϕ is expressed in a conformal chart around each puncture, it has at worst a simple pole. When Γ has finite covolume, it follows then (for instance from Riemann-Roch theorem) that $\mathcal{Q}(\Gamma)$ is a finite dimensional complex vector space. Its dimension is $3g - 3 + p$, where g is the genus of X and p is the number of punctures of X. When Γ has infinite covolume, it is not hard to see that $\mathcal{Q}(\Gamma)$ is infinite dimensional.

There is a natural pairing between $\mathcal{B}(\Gamma)$ and $\mathcal{Q}(\Gamma)$. If $\phi \in \mathcal{Q}(\Gamma)$ and $\mu \in \mathcal{B}(\Gamma)$, the local expression $\widetilde{\phi}(z)\widetilde{\mu}(z)|dz^2|$ is invariant under Γ. It projects to a complex measure on X which has finite total mass. One defines:

$$\langle \phi, \mu \rangle = \Re\left(\int_X \phi(z)\mu(z)|dz^2|\right).$$

We denote by $\mathcal{N}(\Gamma)$ the kernel of this pairing:

$$\mathcal{N}(\Gamma) = \{\mu \in \mathcal{B}(\Gamma) \mid \forall \phi \in \mathcal{Q}(\Gamma), \quad \langle \phi, \mu \rangle = 0\}.$$

It follows from the Hahn-Banach theorem and from the Riesz representation theorem that $\langle .,. \rangle$ induces a duality between $\mathcal{Q}(\Gamma)$ and the quotient space $\mathcal{B}(\Gamma)/\mathcal{N}(\Gamma)$.

Definition. — Let Y be a hyperbolic Riemann surface with finitely many components. *The space of Beltrami forms* $\mathcal{B}(Y)$ is defined as the product of the spaces $\mathcal{B}(X)$ where X varies over the components of Y. *The space of integrable holomorphic quadratic differentials* $\mathcal{Q}(Y)$ is defined as the product of the spaces $\mathcal{Q}(X)$, where X varies over the components of Y. The norm on $\mathcal{B}(Y)$ (resp. on $\mathcal{Q}(Y)$) is the supremum norm of the norms on the spaces $\mathcal{B}(X)$ (resp. the sum of the norms on the spaces $\mathcal{Q}(X)$). The pairing between $\mathcal{Q}(Y)$ and $\mathcal{B}(Y)$ is defined as the sum of the pairings on the components of Y.

Poincaré series.

Consider a covering of Riemann surfaces $\pi : Y \to X$. Let $\phi \in \mathcal{Q}(Y)$. If $U \subset X$ is an embedded disc, the cover $\pi^{-1}(U) \to U$ is trivial so that for each component V_i of $\pi^{-1}(U)$, the map π admits a section $s_i : U \to V_i$. In any holomorphic chart, the restriction of ϕ to V_i can be expressed in the form $\phi_i(z)dz^2$. By Cauchy's formula and since ϕ is integrable, the series

$$\sum_i \phi_i \circ s_i(u)(s_i'(u))^2$$

is absolutely convergent on compact subsets of U. It defines therefore a holomorphic function on X. This function transforms under changes of charts like a quadratic differential and defines an element in $\mathcal{Q}(X)$ denoted by $\Theta_{Y/X}\phi$. One has: $||\Theta_{Y/X}\phi|| \leq ||\phi||$. The operator $\Theta_{Y/X} : \mathcal{Q}(Y) \to \mathcal{Q}(X)$ is called *the Theta operator associated to the cover* $Y \to X$. Its norm is less than or equal to 1. The study of the contraction properties of the operator $\Theta_{Y/X}$ will be the main theme of §5. For the moment, we note that $\Theta_{Y/X}$ is dual to *the pull-back operator on Beltrami forms* for the pairing between Beltrami forms and quadratic differentials. For $\mu \in \mathcal{B}(X)$, one can define a Beltrami form $\pi^*(\mu) \in \mathcal{B}(Y)$ by setting $\widetilde{\pi^*(\mu)} = \widetilde{\mu}$. This form is called *the pull-back of* μ. For all $\phi \in \mathcal{Q}(Y)$ and for all $\mu \in \mathcal{B}(X)$, we have

$$\langle \Theta_{Y/X}\phi, \mu \rangle = \langle \phi, \pi^*(\mu) \rangle.$$

The Bers embedding.

The key ingredient in the proof that the Teichmüller space carries a complex structure is a result of L. Bers which allows us to embed $\mathcal{T}(\Gamma)$ in the complex vector space $\mathcal{Q}(\overline{\Gamma})$. We outline below this construction.

Let Γ be a cocompact Fuchsian group. Let $\mu \in \mathcal{B}^1(\Gamma)$. Consider the measurable function on $\overline{\mathbb{C}}$ which agrees with $\widetilde{\mu}$ on $\mathbb{H}^2$ and which vanishes identically on the lower half-space $\overline{\mathbb{H}^2}$. This function has a L^∞-norm strictly less than 1, and by the Ahlfors-Bers theorem it is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism of $\overline{\mathbb{C}}$. We denote this homeomorphism by φ^μ.

Lemma. — *For any two elements μ, μ' in $\mathcal{B}^1(\Gamma)$, we have:*

$$\varphi_\mu = \varphi_{\mu'} \Leftrightarrow \varphi^\mu|\overline{\mathbb{H}^2} = \varphi^{\mu'}|\overline{\mathbb{H}^2}.$$

By construction, the restriction $\varphi^\mu|\overline{\mathbb{H}^2}$ is a univalent map. Recall that *the Schwarzian derivative* of a univalent map f is the holomorphic function defined by

$$S(f) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2.$$

The transformation rule of μ under Γ implies that $S(\varphi^\mu)$ transforms under the action of Γ on $\overline{\mathbb{H}^2}$ like a holomorphic quadratic differential. Hence $S(\varphi^\mu) \in \mathcal{Q}(\overline{\Gamma})$, the space of integrable holomorphic quadratic differentials on $\overline{\mathbb{H}^2}/\Gamma$. Consider the map $\Phi : \mathcal{B}^1(\Gamma) \to \mathcal{Q}(\overline{\Gamma})$ which assigns to $\mu \in \mathcal{B}^1(\Gamma)$ the holomorphic quadratic differential $S(\varphi^\mu) \in \mathcal{Q}(\overline{\Gamma})$. By the lemma above, Φ can be factorized through a map $\mathcal{B} : \mathcal{T}(\Gamma) \to \mathcal{Q}(\overline{\Gamma})$: $\Phi = \mathcal{B} \circ \Pi$. By this lemma also, $\mathcal{B}$ is injective since any univalent map is characterized by its Schwarzian derivative, up to post-composition with a Möbius map. The map $\mathcal{B}$ is called *the Bers embedding.* We identify $\mathcal{T}(\Gamma)$ with its image under $\mathcal{B}$ and we continue to denote this image by $\mathcal{T}(\Gamma)$. By the Ahlfors-Bers theorem, Φ is holomorphic and its derivative $D\Phi_0$ at the origin is the map

$$D\Phi_0(\nu)(z) = \frac{-6}{\pi} \iint_{\mathbb{H}^2} \frac{\nu(\zeta)}{(\zeta - z)^4} d\xi d\eta, \quad \text{for} \quad z \in \overline{\mathbb{H}^2}.$$

Since Γ is cocompact, $\mathcal{Q}(\overline{\Gamma})$ is a finite dimensional vector space, and the following result is easy to prove (when Γ is an arbitrary Fuchsian group, it is still true but is a deep theorem due to Bers [**Ber**]).

Fact. — *The map $D\Phi_0$ is surjective and induces an isomorphism from $\mathcal{B}(\Gamma)/\mathcal{N}(\Gamma)$ to $\mathcal{Q}(\overline{\Gamma})$.*

Thus, by the implicit function theorem, $\mathcal{T}(\Gamma)$ contains a neighborhood of 0 in $\mathcal{Q}(\overline{\Gamma})$. It says also that in any sufficiently small neighborhood of $0 \in \mathcal{T}(\Gamma)$, Φ admits local sections.

Let $\mu \in \mathcal{B}^1(\Gamma)$ and denote by Γ_μ the group conjugated to Γ by φ_μ. Define a holomorphic map $\alpha_\mu : \mathcal{B}^1(\Gamma) \to \mathcal{B}^1(\Gamma_\mu)$ by

$$\alpha_\mu(\nu) = (\frac{\nu - \mu}{1 - \nu\bar{\mu}})(\frac{\partial\varphi_\mu}{|\partial\varphi_\mu|}) \circ \varphi_\mu^{-1}.$$

Then α_μ is biholomorphic. Furthermore it conjugates Φ and $\Phi_\mu : \mathcal{B}^1(\Gamma_\mu) \to \mathcal{T}(\Gamma_\mu)$. In this way, a neighborhood of Γ_μ in $\mathcal{T}(\Gamma)$ becomes biholomorphically equivalent to a neighborhood of $0 \in \mathcal{Q}(\overline{\Gamma_\mu})$. In particular, $\mathcal{T}(\Gamma)$ is an open subset of $\mathcal{Q}(\overline{\Gamma})$ and so it inherits the complex structure from $\mathcal{Q}(\overline{\Gamma})$. The derivative $D\Phi_0$ induces an isomorphism from $\mathcal{B}(\Gamma)/\mathcal{N}(\Gamma)$ to the tangent space of $\mathcal{T}(\Gamma)$ at point Γ (identified with $0 \in \mathcal{Q}(\overline{\Gamma})$). The map α_μ induces an identification between the tangent space of $\mathcal{T}(\Gamma)$ at Γ_μ with the tangent space of $\mathcal{T}(\Gamma_\mu)$ at the origin. Hence the tangent space to $\mathcal{T}(\Gamma)$ at Γ_μ is isomorphic to $\mathcal{B}(\Gamma_\mu)/\mathcal{N}(\Gamma_\mu)$.

The infinitesimal form of Teichmüller distance.

For the smooth structure constructed above, the Teichmüller distance has an infinitesimal form, that of a Finsler metric.

On $\mathcal{B}^1(\Gamma)$, consider the distance

$$\bar{d}(\mu_1,\mu_2) = -\frac{1}{2}\log K(\phi_{\mu_2}\circ\phi_{\mu_1}^{-1}).$$

A short computation gives

$$K(\phi_{\mu_2}\circ\phi_{\mu_1}^{-1}) = (\frac{\mu_2-\mu_1}{1-\mu_2\overline{\mu_1}})(\frac{\partial\phi_{\mu_1}}{|\partial\phi_{\mu_1}|})\circ\phi_{\mu_1},$$

from which we deduce the nice formula $\bar{d}(\mu_1,\mu_2) = \sup_{z\in\mathbb{H}^2} d_{\mathbb{D}^2}(\mu_1(z),\mu_2(z))$, where $d_{\mathbb{D}^2}$ is the hyperbolic distance on $\mathbb{D}^2$.

By definition, the Teichmüller distance d satisfies: $d(\sigma_1,\sigma_2) = \inf\bar{d}(\mu_1,\mu_2)$, where the infimum is taken over all pairs (μ_1,μ_2) such that $\Pi(\mu_i)=\sigma_i$. This means that d is the quotient distance of $\bar{d}$ by Π. Now we construct *a Finsler metric* on $\mathcal{T}(\Gamma)$ with associated distance d. It will appear as the quotient of a Finsler metric on $\mathcal{B}^1(\Gamma)$ with associated distance $\bar{d}$.

Definition.— *A Finsler metric* on a Banach manifold $\mathcal{B}$ (like for instance an open set in a Banach space) is a continuous function on the tangent space $T\mathcal{B}$ which induces a Banach norm on the tangent space at each point and which is locally equivalent to the (ambient) Banach norm.

When $\mathcal{B}$ is connected, one can associate a distance to a Finsler metric on $\mathcal{B}$: the distance between two given points is equal to the infimum of the length of a smooth arc joining these two points.

On the tangent space of $\mathcal{B}^1(\Gamma)$ ($\subset\mathcal{B}(\Gamma)$) consider the function

$$\bar{\beta}_\mu(\nu) = 2\|\frac{\nu}{1-|\mu|^2}\|,$$

where $\nu\in\mathcal{B}(\Gamma)$ is a tangent vector at $\mu\in\mathcal{B}^1(\Gamma)$. It is a Finsler metric on $\mathcal{B}^1(\Gamma)$ (which is formally reminiscent of the hyperbolic metric on $\mathbb{D}^2$). From the formula

$$\bar{d}(\mu_1,\mu_2) = \sup_{z\in\mathbb{H}^2} d_{\mathbb{D}^2}(\mu_1(z),\mu_2(z)),$$

one deduces that the distance on $\mathcal{B}^1(\Gamma)$ associated to $\bar{\beta}$ is $\bar{d}$.

The Finsler metric $\bar{\beta}$ projects to a Finsler metric on $\mathcal{T}(\Gamma)$. Let $s\in\mathcal{T}(\Gamma)$. Let $\mu\in\mathcal{B}^1(\Gamma)$ be such that $s=\Pi(\mu)$. Let us define

$$\beta_s(v) = \inf\bar{\beta}_\mu(\nu),$$

where the infimum is taken over all vectors ν such that $D\Pi_\mu(\nu)=v$. One shows that $\beta_s(v)$ does not depend on μ and that β is a Finsler metric on $\mathcal{T}(\Gamma)$. Since d is the quotient distance of $\bar{d}$ by Π, it follows that d is the distance associated to β. This is a general result on Finsler metrics due to O'Byrne **[O'B]**.

On the tangent space at the origin of Teichmüller space $\mathcal{T}(\Gamma)$ (i.e. at point Γ), β equals the quotient norm of the L^∞-norm on $\mathcal{B}(\Gamma)/\mathcal{N}(\Gamma)$. The identification of the tangent space to $\mathcal{T}(\Gamma)$ at an arbitrary point Γ_μ with $\mathcal{B}(\Gamma_\mu)/\mathcal{N}(\Gamma_\mu)$ implies that β equals on this tangent space the quotient of the L^∞-norm on $\mathcal{B}(\Gamma_\mu)/\mathcal{N}(\Gamma_\mu)$.

Since the cotangent space of $\mathcal{T}(\Gamma)$ at Γ_μ is identified with $\mathcal{Q}(\Gamma_\mu)$ via the pairing $\langle .,. \rangle$, the norm dual to β equal the L^1-norm on $\mathcal{Q}(\Gamma_\mu)$.

1.4 Geometrically finite groups

An important feature of the proof of the Hyperbolization theorem for non-fibered manifolds is that we need only to consider Kleinian groups which are *geometrically finite.* Since we will restrict to Kleinian groups without parabolic elements, we adopt the following definition (for a detailed discussion of geometrically finite groups in $PSL_2(\mathbb{C})$, see [**Mor**]).

Definition. — *A geometrically finite group* is a non-elementary Kleinian group $G \subset PSL_2(\mathbb{C})$ without parabolic elements, such that $N(G)$ is compact.

A cocompact Fuchsian group Γ is geometrically finite and $N_\delta(G)$ is diffeomorphic to the product $\mathbb{H}^2/\Gamma \times [0,1]$. If G is a geometrically finite group which is not Fuchsian, $N(G)$ is a 3-manifold with compact boundary and, for all $\delta > 0$, $N_\delta(G)$ is a compact 3-manifold (of class C^1).

Hyperbolic manifolds.

Definition. — Suppose that M is a compact 3-manifold which admits an atlas $\mathcal{A}$ of class C^1 with charts modelled on convex subsets of $\mathbb{H}^3$ and with coordinate changes in $PSL_2(\mathbb{C})$. Then the constant curvature -1 Riemannian metric of $\mathbb{H}^3$ can be pulled back to a Riemannian metric $m_\mathcal{A}$ on M (of class C^1).

We say that M is *hyperbolic* if M admits an atlas $\mathcal{A}$ such that

(i) the distance associated to $m_\mathcal{A}$ is complete, and

(ii) the volume of $m_\mathcal{A}$ is finite.

A Riemannian metric $m_\mathcal{A}$ with these properties is called *a hyperbolic metric on* M.

To a hyperbolic metric $m_\mathcal{A}$ on M, one can associate *its developping map* ([**Th1**], [**CEG**]): it is a local isometry Dev from the universal cover $\widetilde{M}$ of M (endowed with the lift of the metric $m_\mathcal{A}$) to $\mathbb{H}^3$ which conjugates the canonical action of $\pi_1(M)$ to the action of a subgroup of $PSL_2(\mathbb{C})$ denoted by $G_\mathcal{A}$, and called *the holonomy group of* $m_\mathcal{A}$.

Let $c(p,q)$ be a length-minimizing geodesic connecting the points p and q of $\widetilde{M}$. It follows from the fact that the charts in $\mathcal{A}$ are modelled on convex subsets of $\mathbb{H}^3$ that $\mathrm{Dev}(c(p,q))$ is a geodesic of $\mathbb{H}^3$. The uniqueness of the geodesic connecting two points of $\mathbb{H}^3$ implies then that Dev is a diffeomorphism onto its image. Therefore $G_\mathcal{A}$ is a Kleinian group. Furthermore, Dev induces an isometric embedding from

M into $M(G_A)$, which identifies isometrically M with a closed convex subset of $M(G_A)$.

Examples.— 1) The manifold $(\mathbb{S}^1 \times \mathbb{B}^2)$ is hyperbolic: $\mathbb{S}^1 \times \mathbb{B}^2$ is diffeomorphic to the quotient of a constant radius neighborhood of a geodesic in $\mathbb{H}^3$ by a hyperbolic element fixing this geodesic.

2) Let G be a geometrically finite group. Then, for $\delta > 0$, $N_\delta(G)$ is a hyperbolic manifold.

Fact.— *Every hyperbolic manifold is diffeomorphic to one of the above examples.*

Proof.— Let m_A be a hyperbolic metric on M. Let G_A $(\simeq \pi_1(M))$ be its holonomy group. Since M is compact, each element of $\pi_1(M)$ is represented by a closed geodesic. Therefore, if G_A is elementary, it is a cyclic group generated by a hyperbolic isometry and M is diffeomorphic to the first example.

Suppose that G_A is non-elementary. Since $C(G_A)$ is the smallest closed convex subset of $\mathbb{H}^3$ invariant under G_A, it is contained in the image of Dev. Therefore $N(G_A)$ is naturally contained in M. Since M is compact, G_A has no parabolic elements, and $N(G_A)$ is compact. Therefore, G_A is geometrically finite. Since M is convex in $M(G_A)$, M is diffeomorphic to $N_\delta(G_A)$. □

Thurston's hyperbolization theorem gives a sufficient condition on the topology of M that guarantees that M is hyperbolic.

Thurston's hyperbolization theorem.— *Let M be a Haken manifold whose fundamental group does not contain $\mathbb{Z}+\mathbb{Z}$-subgroups. Then M is hyperbolic.*

The definition of a Haken manifold will be given in §7.

Except the arithmetic constructions which give little control on the topology of the resulting quotient manifold [**Bo**], there are essentially three ways to construct Kleinian groups:

(i) *Andreev's theorem* yields a lot of examples of hyperbolic polyhedra in $\mathbb{H}^3$ [**An**] (cf. introduction);

(ii) *the deformation theory*, using quasi-conformal maps, gives us a way to deform a given geometrically finite group;

(iii) *Maskit's combination theorem* give conditions under which two geometrically finite groups can be amalgamated to form a new geometrically finite group [**Mas2**].

All of these techniques are used in the proof of Thurston's hyperbolization theorem. Andreev's theorem will be discussed in §8. Maskit's combination theorem will be explained in §2. Now, we describe how quasi-conformal homeomorphisms can be used to deform a given geometrically finite group.

The deformation space of a geometrically finite group.

The deformation theory of a geometrically finite group in $\mathrm{PSL}_2(\mathbb{C})$ can be formulated in exactly the same terms as the deformation theory of a Fuchsian group in $\mathrm{PSL}_2(\mathbb{R})$.

Let G be a geometrically finite group such that $\Omega(G) \neq \emptyset$ and such that each component of $\Omega(G)$ is conformally equivalent to $\mathbb{D}^2$. This is equivalent to say that $M = \overline{M}(G)$ is a compact manifold with non-empty and *incompressible boundary* (i.e. the fundamental group of any component of ∂M injects into $\pi_1(M)$, see §7). We will suppose also that $L(G) \subset \overline{\mathbb{C}}$ contains 0, 1 and ∞. This situation can always be achieved by conjugating G in $\mathrm{PSL}_2(\mathbb{C})$.

Definition. — *A quasi-conformal deformation of* G is a couple $(\rho, \widetilde{\varphi})$, where ρ is a representation of G into $\mathrm{PSL}_2(\mathbb{C})$ and where $\widetilde{\varphi}$ is a normalized quasi-conformal homeomorphism of $\overline{\mathbb{C}}$ which conjugates the actions of G and $\rho(G)$ on $\overline{\mathbb{C}}$, i.e. which satisfies, for all $g \in G$

$$\rho(g) = \widetilde{\varphi} \circ g \circ \widetilde{\varphi}^{-1}.$$

Consider the equivalence relation on the set of quasi-conformal deformations of G defined by $(\rho, \varphi) \simeq (\rho', \varphi')$ if and only if $\rho = \rho'$. The set of equivalence classes of quasi-conformal deformations of G is denoted by $\mathcal{GF}(G)$ (or $\mathcal{GF}(M)$).

(i) in the next Proposition is one reason for this notation.

Proposition. — *Let* $(\rho, \widetilde{\varphi})$ *be a quasi-conformal deformation of* G. *Then*

(i) $\rho(G)$ *is geometrically finite, and*

(ii) $\widetilde{\varphi}$ *induces a quasi-conformal homeomorphism* φ *between* $\Omega(G)/G$ *and* $\Omega(\rho(G))/\rho(G)$ *which extends to a homeomorphism* Φ *between* $\overline{M}(G)$ *and* $\overline{M}(\rho(G))$.

One way for proving this Proposition is to use the following result of Thurston which provides a natural extension of any quasi-conformal homeomorphism of $\overline{\mathbb{C}}$ to a homeomorphism of $\mathbb{H}^3 \cup \overline{\mathbb{C}}$ ([**Th1**], [**Re**]). Let $\widetilde{\varphi}$ be a quasi-conformal homeomorphism of $\overline{\mathbb{C}}$. Then there is a homeomorphism $\widetilde{\Phi} = \widetilde{\Phi}(\widetilde{\varphi})$ such that

(i) $\widetilde{\Phi}(\widetilde{\varphi})$ extends continuously to a homeomorphism of $\mathbb{H}^3 \cup \overline{\mathbb{C}}$ and restricts to $\widetilde{\varphi}$ on $\overline{\mathbb{C}}$, and

(ii) for any isometries g, h of $\mathbb{H}^3$, $\widetilde{\Phi}(g \circ \widetilde{\varphi} \circ h) = g \circ \widetilde{\Phi}(\widetilde{\varphi}) \circ h$.

Definition. — The homeomorphism $\widetilde{\Phi}(\widetilde{\varphi})$ is called *the natural extension of* $\widetilde{\varphi}$.

Let us go back to the Proposition. The natural extension $\widetilde{\Phi}(\widetilde{\varphi})$ induces a homeomorphism $\Phi = \Phi(\varphi)$ from $\overline{M}(G)$ to $\overline{M}(\rho(G))$, called also the natural extension of φ. This proves (ii). Under our hypothesis, $\overline{M}(G)$ is compact. Therefore (ii) implies that $\overline{M}(\rho(G))$ is compact also. Thus, $\rho(G)$ is geometrically finite.

The Ahlfors-Bers map.

The Ahlfors-Bers theorem gives us a way to parametrize $\mathcal{GF}(M)$ by a product of Teichmüller spaces.

Notations. — Let G be a geometrically finite group such that the manifold $M = \overline{M}(G)$ has a non-empty and incompressible boundary. Denote by $(S_i)_{i=1,..,k}$ the components of the Riemann surface $\partial M = \Omega(G)/G$. For $1 \leq i \leq k$, choose a component Ω_i of the preimage of S_i in $\Omega(G)$. Denote by Γ_i' the stabilizer of Ω_i

in G. By Koebe's uniformization theorem, there is a conformal homeomorphism $f_i : \Omega_i \to \mathbf{H}^2$ which conjugates Γ_i' to a Fuchsian group $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$.

Let $(\rho, \widetilde{\varphi})$ be a quasi-conformal deformation of G. For $1 \leq i \leq k$, let $f_i' : \widetilde{\varphi}(\Omega_i) \to \mathbf{H}^2$ be a conformal homeomorphism which conjugates the action of $\rho(\Gamma_i')$ on Ω_i with the action of some Fuchsian group $\rho_i'(\Gamma_i)$ on $\mathbf{H}^2$. Then $f_i' \circ \widetilde{\varphi} \circ f_i^{-1}$ is a quasi-conformal homeomorphism. Up to post-composing f_i' with a Möbius map, we can suppose that $f_i' \circ \widetilde{\varphi} \circ f_i^{-1}$ is normalized. Then $(\rho_i', f_i' \circ \widetilde{\varphi} \circ f_i^{-1})$ is a Fuchsian deformation of Γ_i. One checks easily that the class of $(\rho_i', f_i' \circ \widetilde{\varphi} \circ f_i^{-1})$ in $\mathcal{T}(\Gamma_i)$ depends only on the class of $(\rho, \widetilde{\varphi})$ in $\mathcal{GF}(M)$.

Set $\mathcal{T}(\partial M) = \times_i \mathcal{T}(\Gamma_i)$.

Definition. — *The Ahlfors-Bers map* is the map from $\mathcal{GF}(M)$ to $\mathcal{T}(\partial M)$ which assigns to $(\rho, \widetilde{\varphi})$ the k-tuple whose i-th coordinate is the class of $(\rho_i', f_i' \circ \widetilde{\varphi} \circ f_i^{-1})$. It is denoted by ∂.

Theorem. — *∂ is a bijection.*

Proof. — Let $s = (s_1, \cdots, s_k) \in \mathcal{T}(\partial M)$. Choose $\mu_i \in \mathcal{B}^1(S_i)$ such that $s_i = (\rho_i', \varphi_{\mu_i})$. By taking the pull-back of each Beltrami form μ_i to the preimage of S_i in $\Omega(G)$, we define a function $\widetilde{\mu} \in L^\infty(\Omega(G))$ which satisfies

$$\widetilde{\mu}(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \widetilde{\mu}(z),$$

for all $\gamma \in G$. Extend $\widetilde{\mu}$ to a function $\bar{\mu} \in L^\infty(\mathbb{C})$ with norm strictly smaller than 1 by setting it equal to 0 on $L(G)$. Then the quasi-conformal homeomorphism $\phi_{\bar{\mu}}$ provided by the Ahlfors-Bers theorem conjugates G to a group $\rho(G)$. The uniqueness of the solution of the Beltrami equation implies

$$\partial(\rho, \phi_{\bar{\mu}}) = (s_1, \cdots, s_k).$$

This proves that ∂ is onto. The injectivity is more delicate. The proof uses the Ahlfors measure 0 theorem, which states that the limit set of any geometrically finite group has Lebesgue measure 0 or equals the whole sphere [**Ahl**]. □

Notations. — Let $s \in \mathcal{T}(\partial M)$. Let $(\rho, \widetilde{\varphi})$ such that $\partial(\rho, \widetilde{\varphi}) = s$. We will denote by G^s the group $\rho(G)$ and by M^s the manifold $\overline{M}(G^s)$. The point s will be then identified with ∂M^s, thought of as a hyperbolic metric on ∂M.

Ahlfors' lemma.

The following (particular case of a) lemma of Ahlfors compares the lengths of closed geodesics in ∂M^s and in $M(G^s)$. Its proof is an application of Koebe's $1/4$-theorem. We keep the same hypothesis and notations as for the definition of the Ahlfors-Bers map.

Lemma [Ahl]. — *Let g be a closed curve on ∂M^s which is homotopic to a geodesic of length $\ell^s(g)$ for the hyperbolic metric ∂M^s. Then g is homotopic in M^s to a geodesic contained in $M(G^s)$ of length smaller than $2\ell^s(g)$.*

Quasi-Fuchsian groups.

Let Γ be a cocompact Fuchsian group. Then $L(\Gamma) = \overline{\mathbb{R}}$ and $\Omega(\Gamma)$ has two invariant components, the upper-half plane $\mathbb{H}^2$ and the lower half-plane $\overline{\mathbb{H}^2}$. Clearly, $\overline{M}(\Gamma)$ is homeomorphic to $(\mathbb{H}^2/\Gamma) \times [0,1]$; also, one component of $\partial\overline{M}(\Gamma)$ is conformally equivalent to $\mathbb{H}^2/\Gamma$ and the other to $\overline{\mathbb{H}^2}/\Gamma$. Thus the Ahlfors-Bers map leads to a new class of Kleinian groups, the quasi-Fuchsian groups.

Definition. — *A quasi-Fuchsian group* is a geometrically finite G such that for some cocompact Fuchsian group Γ, there is a quasi-conformal deformation $(\rho, \widetilde{\varphi})$ of Γ with $G = \rho(\Gamma)$.

If G is quasi-Fuchsian, $\overline{M}(G)$ is diffeomorphic to the product of a closed surface by an interval.

Maskit's theorem.

Same hypothesis and notations as for the definition of the Ahlfors-Bers map.

Let S be a component of ∂M. Then the inclusion $\pi_1(S) \subset \pi_1(M) \simeq G$ gives a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, which is faithful and has discrete image. With these notations, we have:

Theorem [Mas2]. — *The group $\rho(\pi_1(S))$ is quasi-Fuchsian.*

Remark. — This last result is a particular case of a theorem of Maskit which asserts that any finitely generated subgroup of a geometrically finite group G (eventually with parabolic elements) with $\Omega(G) \neq \emptyset$, is geometrically finite ([**Mor**], [**OP**]). However, this weaker statement will be sufficient for us: its most important application will be to define *the skinning map* (cf.§2).

The Kleinian groups that will appear during the proof of the Hyperbolization theorem are constructed by induction. It follows from this construction that the Maskit theorem can be checked for these groups by the same induction.

Remark. — An essential hypothesis of Maskit's theorem is $\Omega(G) \neq \emptyset$. A finitely generated subgroup of an arbitrary geometrically finite group is not necessarily geometrically finite. The basic example to keep in mind is the fundamental group of the fiber of a hyperbolic manifold which fibers over the circle. This is precisely the reason why the proof of Thurston's hyperbolization theorem needs to consider separately the cases of fibered manifolds and non-fibered manifolds.

One corollary of Maskit's theorem is the next result which will be used in §2.

Proposition. — *Let G be a geometrically finite group. Then, for all $\eta > 0$, $\Omega(G)$ has only a finite number of components with a diameter bigger than η.*

Hyperbolic annuli.

We give some formulas for the hyperbolic metric on annuli, which will be used in §3 and §4.

For $r < r'$, we denote by $A_{e^r,e^{r'}}$ the annulus:

$$A_{e^r,e^{r'}} = \{z \in \mathbb{C}, \quad e^r < |z| < e^{r'}\}.$$

Any hyperbolic annulus $\mathbb{H}^2/\langle\gamma\rangle$, where γ is an hyperbolic isometry, is conformally equivalent to an annulus A_{e^{-R},e^R}. The hyperbolic metric on A_{e^{-R},e^R} is given by:

$$ds = \frac{\pi|dz|}{2R|z|\cos(\pi\frac{\log|z|}{2R})}.$$

The circle of radius 1 is the only embedded closed geodesic of this annulus. Its hyperbolic length is $\ell(\gamma) = \pi^2/R$.

For $r < R$, the two circles of radius e^r and e^{-r} are equidistant curves to the circle of radius 1 at distance D such that

$$\tanh D = \sin(\pi\frac{r}{2R}).$$

The hyperbolic length of these two circles is

$$\frac{\pi^2}{R\cos(\pi\frac{r}{2R})}.$$

The injectivity radius of the hyperbolic metric of A_{e^{-R},e^R} is constant on the circle of radius e^r: if it is bounded above, say by $\varepsilon(2)$, the injectivity radius on the circle of radius e^r is equivalent to the hyperbolic length of this circle, independently of R.

CHAPTER 2

The fixed point problem

Let N be a connected and orientable closed 3-manifold. Let S be a (not necessarily connected) closed, orientable, incompressible surface embedded in N. Suppose that the Euler characteristic of each component of S is strictly negative. Denote by M the complement in N of an open regular neighborhood of S. We say that M is obtained *by splitting* N *along* S (cf. §7). Since S is incompressible, M has incompressible boundary.

The boundary of M is made of the union of two copies S_1 and S_2 of S. There is an orientation reversing diffeomorphism $f : S_1 \to S_2$ such that N is diffeomorphic to the quotient of M by the relation: $x \equiv y$ if and only if $x \in S_1$, $y \in S_2$ and $y = f(x)$. Rather than to consider the diffeomorphism $f : S_1 \to S_2$, it is more convenient to introduce the map $\tau : \partial M \to \partial M$ defined by $\tau(x) = f(x)$ for $x \in S_1$ and $\tau(y) = f^{-1}(y)$ for $y \in S_2$. Then τ is an orientation reversing involution of ∂M which permutes the components by pairs. And N is diffeomorphic to M/τ, i.e. to the quotient space of M by the equivalence relation

$$x \simeq y \text{ if and only if } x \in \partial M \text{ and } y = \tau(x).$$

The core of the proof of Thurston's hyperbolization theorem is the following result.

Final gluing theorem. — *Let M be a hyperbolic manifold with incompressible boundary which is not an interval bundle. Let τ be an orientation reversing involution of ∂M which permutes the components by pairs. If M/τ is atoroidal, then it is hyperbolic.*

In this chapter, we prove that if a certain map from $\mathcal{T}(\partial M)$ to itself has a fixed point, then the conclusion of the Final gluing theorem holds (M is assumed to satisfy the hypothesis in the Final gluing theorem).

2.1 Maskit's combination theorem

We keep the same notations as in the introduction of this chapter. But we suppose that S is connected and we don't suppose anymore that N is closed. Then M has two or one components according to whether S separates N or not. *We suppose also that N is not an interval bundle (cf. §7).*

Suppose that M is hyperbolic. Then there exists a geometrically finite group G_1 (resp. two geometrically finite groups G_1 and G_2) such that we can identify M with $\overline{M}(G_1)$ (resp. with the disjoint union of $\overline{M}(G_1)$ and $\overline{M}(G_2)$) in the case that S does not separate N (resp. separates N). Maskit's combination theorem provides a sufficient condition on G_1 (resp. on G_1 and G_2) which implies that N is hyperbolic. Since $S' = S_1 \cup S_2$ is incompressible in M, Maskit's theorem (cf. §1) says that the images of $\pi_1(S_1)$ and $\pi_1(S_2)$ in G_1 (resp. in G_1 and in G_2) are quasi-Fuchsian groups. We still denote these images by $\pi_1(S_1)$, $\pi_1(S_2)$. For $i = 1, 2$, we set $N_i = M(\pi_1(S_i))$ and $\overline{N}_i = \overline{M}(\pi_1(S_i))$. Since $\pi_1(S_i)$ is quasi-Fuchsian, $\overline{N}_i$ is diffeomorphic to $S_i \times [0,1]$. Under this diffeomorphism, one component of $\partial\overline{N}_i$ gets identified with S_i and will be still denoted by S_i. The diffeomorphism f induces a homotopy equivalence $\widetilde{f} : N_1 \to N_2$. With these notations we have:

Maskit's combination theorem [Mas1].— *Assume that $\widetilde{f}$ is homotopic to an isometry $\mathfrak{I} : N_1 \to N_2$ whose extension to $\overline{N}_i$ satisfies $\mathfrak{I}(S_1) = \partial\overline{N}_2 - S_2$. Then N is hyperbolic.*

Remark.— This theorem would not hold if $\overline{M}(G_1)$ and $\overline{M}(G_2)$ were twisted interval bundles.

Proof.— We consider the case when S separates N, the proof in the other case being similar. When one of the manifolds $\overline{M}(G_1)$ and $\overline{M}(G_2)$ is diffeomorphic to a trivial product, there is nothing to prove. We will suppose during the proof that say, $\overline{M}(G_1)$ is not a twisted interval bundle over a closed surface. Hence, only $\overline{M}(G_2)$ might be an interval bundle over a closed surface. Recall that $\overline{M}(G_i)$ is an interval bundle if and only if $\pi_1(S_i)$ has index one or two in G_i, if and only if $\Omega(G_i)$ has two components, see §7.

For $i = 1, 2$, $\overline{M}(G_i)$ is diffeomorphic to $N_\delta(G_i)$. Consider the covering $p_i : N_i \to M(G_i)$. The map p_i extends continuously to a map $N_i \cup S_i \to M(G_i) \cup S_i$ whose restriction to S_i is an embedding.

Let f_i be the (unique) harmonic function on N_i such that $f_i(x)$ tends to 1 when x tends to S_i and to 0 when x tends to $\partial\overline{N}_i - S_i$ (the existence of such a function is obtained by solving the corresponding Dirichlet problem in $\mathbb{H}^3$).

Suppose that $\overline{M}(G_i))$ is not an interval bundle. Then f_i has the following two properties:

(i) the level surface $f_i^{-1}(1/2)$ maps injectively into $M(G_i)$ under the covering map p_i. This follows from the maximum principle for harmonic functions and from the hypothesis on the index of $\pi_1(S_i)$ in G_i.

(ii) $f_i^{-1}(1/2)$ is compact. The reason is that $f_i(x)$ tends to 0 or 1 as x tends to ∞ in $M(G_i)$.

(iii) for $\delta > 0$, $f_i^{-1}(1/2)$ is contained in the interior of $N_\delta(G_i)$. This follows from the fact that $f_i^{-1}(1/2)$ is contained in the Nielsen core $N(G_i)$, by the maximum principle again.

Therefore, since p_i is a local homeomorphism, for each regular value c of f_i which is sufficiently close to $1/2$, $\Sigma_i = f_i^{-1}(c)$ is a compact surface embedded in the interior of $N_\delta(G_i)$ such that $p_i|\Sigma_i$ is an embedding.

Also, if c is chosen sufficiently close to $1/2$, $\Sigma_2 = f_2^{-1}(1-c)$ is an embedded compact surface in the interior of $N_\delta(G_2)$ which satisfies $p_2|\Sigma_2$ is an embedding.

When $\overline{M}(G_2)$ is an interval bundle, then (ii) and (iii) still hold, but (i) does not anymore. Let t be the deck transformation of the cover $N_2 \to M(G_2)$. Then t leaves $f_2^{-1}(1/2)$ invariant (inducing a degree two cover) and exchanges $f_2^{-1}([0,1/2[)$ with $f_2^{-1}(]1/2,1])$. Therefore, if $c < 1/2$ is a regular value of f_2 sufficiently close to $1/2$, $\Sigma_2 = f_2^{-1}(1-c)$ is a compact surface embedded in the interior of $N_\delta(G_2)$ such that p_2 restricts to $f_2^{-1}([1-c,1[)$ as an embedding. Since by hypothesis, the index of $\pi_1(S_1)$ in G_1 is greater than 2, we may suppose for this choice of c, that $\Sigma_1 = f_2^{-1}(c)$ is contained in the interior of $N_\delta(G_1)$ and that $p_1|\Sigma_1$ is an embedding.

By the maximum principle, no sum of components of Σ_i can be homologous to 0 in N_i. Therefore since Σ_i separates the two components of $\partial\overline{N_i}$, it is connected. Denote by H_i the submanifold of N_i bounded by Σ_i and whose closure in $\overline{N}_i$ contains S_i. The covering map p_i is an embedding when restricted to Σ_i and to the end of H_i approaching S_i. When $\pi_1(S_i)$ has index greater than 2 in G_i, it follows that $p_i|H_i$ is an embedding. When $\overline{M}(G_2)$ is an interval bundle, the same conclusion holds by the choice of Σ_2.

The hypothesis of the theorem and the uniqueness of the functions f_1 and f_2 imply: $f_2 \circ \mathfrak{J} = 1 - f_1$. Therefore $\Sigma_2 = \mathfrak{J}(\Sigma_1)$ and $\mathfrak{J}$ induces an orientation reversing diffeomorphism from Σ_1 to Σ_2 (Σ_i is oriented as boundary of H_i).

Let us consider now the manifold N' obtained as the result of the gluing of $M_1' = \overline{N_\delta(G_1) - p_1(H_1)}$ and $M_2' = \overline{N_\delta(G_2) - p_2(H_2)}$ identifying $p_1(\Sigma_1)$ and $p_2(\Sigma_2)$ by the diffeomorphism $p_2 \circ \mathfrak{J} \circ p_1^{-1}$. Then N' admits an atlas with charts modelled on convex sets of $\mathbb{H}^3$ (since a neighborhood of $\partial N'$ is isometric to a neighborhood of certain components of the boundary of the disjoint union of $N_\delta(G_1)$ and $N_\delta(G_2)$) and with coordinate changes in $\mathrm{PSL}_2(\mathbb{C})$. Therefore N' is a hyperbolic manifold.

In order to prove the Maskit combination theorem, it remains to prove that N' is diffeomorphic to N. For this, we could invoke Waldhausen's theorem on homotopy equivalences between Haken manifolds (cf. [**Mor**]). But since only a small part of this theorem is necessary, we prefer to explain this point.

Suppose first that $p_1(\Sigma_1)$ is incompressible in N'. Then Σ_i is incompressible in N_i. Therefore, by a theorem of Stallings [**Stal**], $H_i \cup S_i$ is diffeomorphic to $S_i \times [0,1]$. Since $p_i|H_i$ is an embedding, this implies that M_i' is diffeomorphic to $\overline{M}(G_i)$. Under this diffeomorphism, $p_2 \circ \mathfrak{J} \circ p_1^{-1} : \Sigma_1 \to \Sigma_2$ is homotopic to

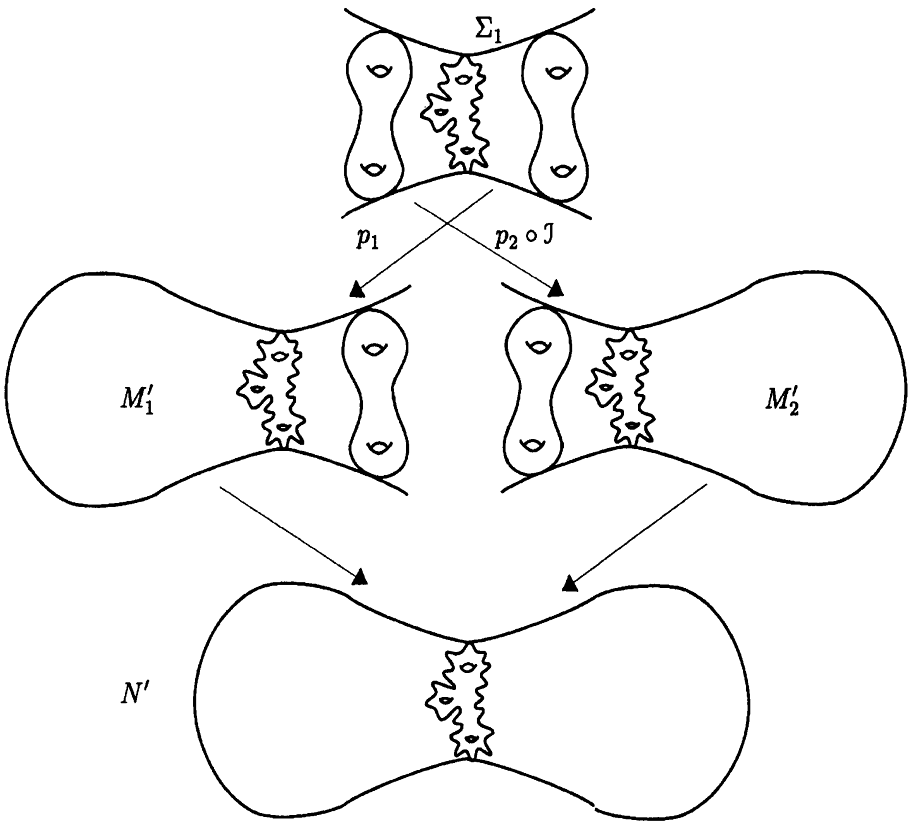

Figure 2.1

$f : S_1 \to S_2$. Thus, by a theorem of Nielsen [Nie], $p_2 \circ \mathfrak{J} \circ p_1^{-1}$ is isotopic to f. So N' is diffeomorphic to N.

When $p_1(\Sigma_1)$ is compressible in N', there exists, by Dehn's lemma, *a compression disc* for $p_1(\Sigma_1)$ in N', i.e. there is a disc D' embedded in N' which intersects $p_1(\Sigma_1)$ transversally and exactly along its boundary and such that the curve $\partial D'$ is not homotopic to 0 on $p_1(\Sigma_1)$ (cf. §7). Suppose that D' is contained, say in $M_1' \subset M(G_1)$. Then D' can be lifted isomorphically to a compression disc D for Σ_1 in N_1. Hence $p_1|(\Sigma_1 \cup D)$ is injective. Therefore, the restriction of p_1 to the union of H_1 and a regular neighborhood $N(D)$ of D is injective. The boundary of $H_1 \cup N(D)$ contains two or one components according to wether ∂D does or does not disconnect Σ_1. Let Σ_1' be the component of $\partial(H_1 \cup N(D))$ which is homologuous to Σ_1. If it exists, the other component, denoted by B_1, is homologuous to 0. It bounds therefore a compact (connected) manifold Z_1 in N_1. The surface Σ_1' cuts N_1 into two components, one of which, denoted by H_1', contains H_1. We prove that $p_1|H_1'$ is an embedding. Clearly, H_1' equals the union along B_1 of $H_1 \cup N(D)$ and Z_1. The surface $p_1(B_1)$ cuts $M(G_1)$ into two components: one is unbounded and contains $p_1(H_1)$, the other, denoted by T_1, is compact. The image $p_1(Z_1)$ is compact. Since p_1 is an open map, $p_1(Z_1)$ contains T_1. Since p_1 is an open map and since $M(G_1)$ is not compact, $p_1(Z_1)$ equals T_1. It follows that p_1 is an embedding when restricted to Z_1 and therefore also when restricted to H'_1. The surface $\Sigma'_2 = \mathcal{I}(\Sigma'_1)$ cuts N_2 into two components, one of which, denoted by H'_2, is contained in H_2. Certainly p_2 restricts to H'_2 as an embedding since it does already on H_2. Also, for some constant $\delta' > 0$, $p_1(D)$ and $p_2(D)$ are contained respectively in $N_\delta(G_1)$ and $N_\delta(G_2)$. Therefore, up to replacing in the original description of N', $p_1(\Sigma_i)$ by $p_1(\Sigma'_i)$ and δ by δ', we obtain a hyperbolic manifold N'' (diffeomorphic to N').

The genus of $p_1(\Sigma'_1)$ is strictly smaller than the genus of $p_1(\Sigma_1)$ since ∂D is not homotopic to 0 on $p_1(\Sigma_1)$. Hence this process ends up, after a finite number of steps, with the case when the gluing surface is incompressible, bringing a hyperbolic manifold diffeomorphic to N. □

2.2 The skinning map

The skinning map, introduced by Thurston, allows us to formulate the hypothesis of Maskit's combination theorem as the existence of a fixed point for a certain map defined on a Teichmüller space. In this section, we define this map and we enunciate the Fixed point theorem.

We keep the same notations and hypothesis as for the definition of the Ahlfors-Bers map. For $1 \le i \le k$, Γ'_i is a quasi-Fuchsian group by Maskit's theorem (cf. §1). Thus $\Omega(\Gamma'_i)$ has two connected components, Ω_i and $\overline{\Omega}_i$. Let Γ_i be a Fuchsian group such that S_i is conformally equivalent to $\mathbb{H}^2/\Gamma_i$. For each i, there is a quasi-conformal homeomorphism f_i of $\overline{\mathbb{C}}$ such that:

(i) $f_i(\mathbb{H}^2) = \Omega_i$;

(ii) $f_i \circ \gamma \circ f_i^{-1} = \gamma'$, for all $\gamma \in \Gamma_i$.

Set $\mathcal{T}(\partial M) = \times_i \mathcal{T}(\Gamma_i)$ and $\mathcal{T}(\overline{\partial M}) = \times_i \mathcal{T}(\overline{\Gamma_i})$. Let $s \in \mathcal{T}(\partial M)$ and let $(\rho, \widetilde{\varphi})$ be the quasi-conformal deformation of G such that $s = \partial(\rho, \widetilde{\varphi})$. Then $\rho(\Gamma'_i)$ is conjugated to Γ_i by the quasi-conformal homeomorphism $\widetilde{\varphi} \circ f_i$. This defines a point in $\mathcal{GF}(\Gamma_i)$. By the Ahlfors-Bers theorem again, $\mathcal{GF}(\Gamma_i)$ is parametrized by the product $\mathcal{T}(\Gamma_i) \times \mathcal{T}(\overline{\Gamma_i})$. The first Ahlfors-Bers coordinate of $(\rho|\Gamma'_i, \widetilde{\varphi} \circ f_i)$ is s_i. We denote the second coordinate by s'_i.

Definition. — *The skinning map associated to* M is the map $\sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\overline{\partial M})$ defined by

$$\sigma(s) = (s'_1, ... s'_k).$$

The Riemann surface s_i can be interpreted as the "outside" structure on the component S_i, whether s'_i is the "inside" structure, i.e. the one which appears when one takes off the "skin" of the manifold M. Another way is to consider the covering of M having fundamental group $\pi_1(S_i)$. This covering is homeomorphic to $S_i \times \mathbb{R}$. Any quasi-conformal deformation of G with Ahlfors-Bers parameters equals to $(s_1, \cdots, s_k)$ can be lifted to a quasi-conformal deformation of this covering. The parameters of this deformation are (s_i, s'_i).

Notation.— Let $s \in \mathcal{T}(\partial M)$. We will denote $\sigma(s)$ by $\sigma(\partial M^s)$, thinking of $\sigma(\partial M^s)$ as a hyperbolic metric on $\overline{\partial M}$ (in the same way as we think of s as a hyperbolic metric on ∂M, cf. §1).

A reformulation of Maskit's combination theorem.

We use now the skinning map to formulate in a different way the hypothesis of Maskit's combination theorem. We keep the same notations as in the statement of this theorem. Consider first the particular case when S separates N into two components.

Let σ_i be the skinning map associated to M_i. Let $(s_i, z_i) \in \mathcal{T}(\partial M_i)$ where s_i denotes the coordinate on the factor S_i and z_i denotes the coordinates on the components of ∂M_i others than S_i. Fix z_1 and z_2 and denote by $\sigma_i^1(s_i)$ the coordinate of $\sigma_i(s_i, z_i)$ on the factor $\overline{S_i}$. Consider the quasi-conformal deformation $M_i^{(s_i,z_i)}$ of M_i. Then N_i is deformed to the point with Ahlfors-Bers coordinates $(s_i, \sigma_i^1(s_i))$. The hypothesis of Maskit's combination means exactly that

$$(1) \qquad f^*(s_1) = \sigma_2^1(s_2) \quad \text{and} \quad f^* \circ \sigma_1^1(s_1) = s_2.$$

When (1) is satisfied, Maskit's combination theorem asserts that $M_1^{(s_1,z_1)}$ and $M_2^{(s_2,z_2)}$ can be "glued together", yielding a hyperbolic manifold diffeomorphic to N. (1) can be stated in a more symmetric way. The diffeomorphism τ from $S_1 \cup S_2$ to itself defined by $\tau(x,y) = (f^{-1}(y), f(x))$. induces a map $\tau^* : \mathcal{T}(\overline{S_1 \cup S_2}) \to \mathcal{T}(S_1 \cup S_2)$. It is straightforward to check that (i) means precisely that $\tau^*(\sigma_1^1(s_1), \sigma_2^1(s_2)) = (s_1, s_2)$, or in other terms, that (s_1, s_2) is a fixed point of the composition of τ^* with (σ_1^1, σ_2^1).

This particular case extends similarly to the case when S is connected but does not separate N.

Consider now the situation of the Final gluing theorem, i.e. when S is not necesarily connected. Then M is the disjoint union of hyperbolic manifolds with incompressible boundary $M_1, \cdots M_\ell, \cdots, M_p$.

Definition.— The Teichmüller space $\mathcal{T}(\partial M)$ (resp. $\mathcal{T}(\overline{\partial M})$) is defined as the product of the Teichmüller spaces $\mathcal{T}(\partial M_i)$ (resp. $\mathcal{T}(\overline{\partial M_i})$). *The skinning map* $\sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\overline{\partial M})$ is the product of the skinning maps σ_i associated to M_i.

Let τ be an orientation reversing diffeomorphism of ∂M which permutes the components by pairs. Then τ induces a map $\tau^* : \mathcal{T}(\overline{\partial M}) \to \mathcal{T}(\partial M)$.

With these notations, the following theorem is merely an extension to this general case of the Maskit combination theorem in the formulation given above. Its proof follows from the original statement by induction on the number of components of S.

Theorem.— *If $\tau^* \circ \sigma$ has a fixed point, then M/τ is hyperbolic.*

Using that, the Final gluing theorem becomes equivalent to the following:

Thurston's fixed point theorem.— *Let M be a hyperbolic manifold with incompressible boundary which is not an interval bundle. Let τ be an orientation reversing* *involution of ∂M which permutes the components by pairs. If M/τ is atoroidal, then $\tau^* \circ \sigma$ has a fixed point.*

We conclude this section by computing the skinning map associated to a connected hyperbolic manifold M which is an interval bundle. There are only two possibilities for M up to diffeomorphism. It is either the product of a closed orientable surface S with the interval $[0,1]$ or *the* twisted interval bundle over a non-orientable surface T (cf. §7).

In the first case, we can identify S with the quotient $\mathbb{H}^2/\Gamma$ for some Fuchsian group Γ. Then $\mathcal{T}(\partial M) = \mathcal{T}(\Gamma) \times \mathcal{T}(\overline{\Gamma})$ and the skinning map is given by $\sigma((x,y)) = (y,x)$. In the second case, ∂M is identified with the orientation cover S of T. The deck transformation t of this cover reverses the orientation on S and induces therefore a map $t^* : \mathcal{T}(\Gamma) \to \mathcal{T}(\overline{\Gamma})$. By definition, we have: $\sigma = t^*$.

The computation in these two cases shows that σ is an isometry when $\mathcal{T}(\partial M)$ and $\mathcal{T}(\overline{\partial M})$ are endowed with their respective Teichmüller distances. Hence when M is diffeomorphic to an interval bundle, σ is an isometry. However if M is connected and is not an interval bundle, then σ is contracting. We will understand why it is so soon in this chapter, when we compute the derivative and the coderivative of σ. There is another way to prove this contraction property, by using the Teichmüller theorem which describes *the extremal quasi-conformal map* between two homeomorphic Riemann surfaces (cf. [**Mor**]).

2.3 The derivative and the coderivative of σ

McMullen's proof of Thurston's fixed point theorem entails a detailed analysis of the derivative of $\tau^* \circ \sigma$. The derivative and coderivative of σ at a point $s \in \mathcal{T}(\partial M)$ are expressed in terms of the geometry of the skinned surface $\sigma(\partial M^s)$. Like a leopard skin, $\sigma(\partial M^s)$ is covered by spots.

The leopard spots.

Same hypothesis and notations as for the definition of the skinning map. Let $s \in \mathcal{T}(\partial M)$.

Definition. — The image of a component $\widetilde{U}$ of $\Omega(G^s) \cap \widetilde{\varphi}(\widetilde{\Omega}_i)$ in the surface $\widetilde{\varphi}(\widetilde{\Omega}_i)/\rho(\Gamma'_i)$ is called a *spot.*

Notation. — Let $U \subset \sigma(\partial M^s)$ be a spot covered by a component $\widetilde{U}$ of $\Omega(G^s)$. Let $\Gamma'_{\widetilde{U}}$ be the stabilizer of $\widetilde{U}$ in G^s. Then U is conformally equivalent to the quotient $\widetilde{U}/\Gamma'_{\widetilde{U}} \cap \rho(\Gamma'_i)$. Hence U covers the component $\widetilde{U}/\Gamma'_{\widetilde{U}}$ of ∂M^s (cf. Figure 2.2). We denote this component by X_U.

Remark. — The quasi-conformal homeomorphism $\widetilde{\varphi}$ projects to a homeomorphism between $\sigma(\partial M)$ and $\sigma(\partial M^s)$ which maps the spots on $\sigma(\partial M)$ to the spots on $\sigma(\partial M^s)$. In particular, the topological configuration of the spots on $\sigma(\partial M^s)$ does not depend on s. Since $\widetilde{\varphi}$ conjugates G with G^s, the topological types of the covers $U \to X_U$ associated to the spots do not depend on s either.

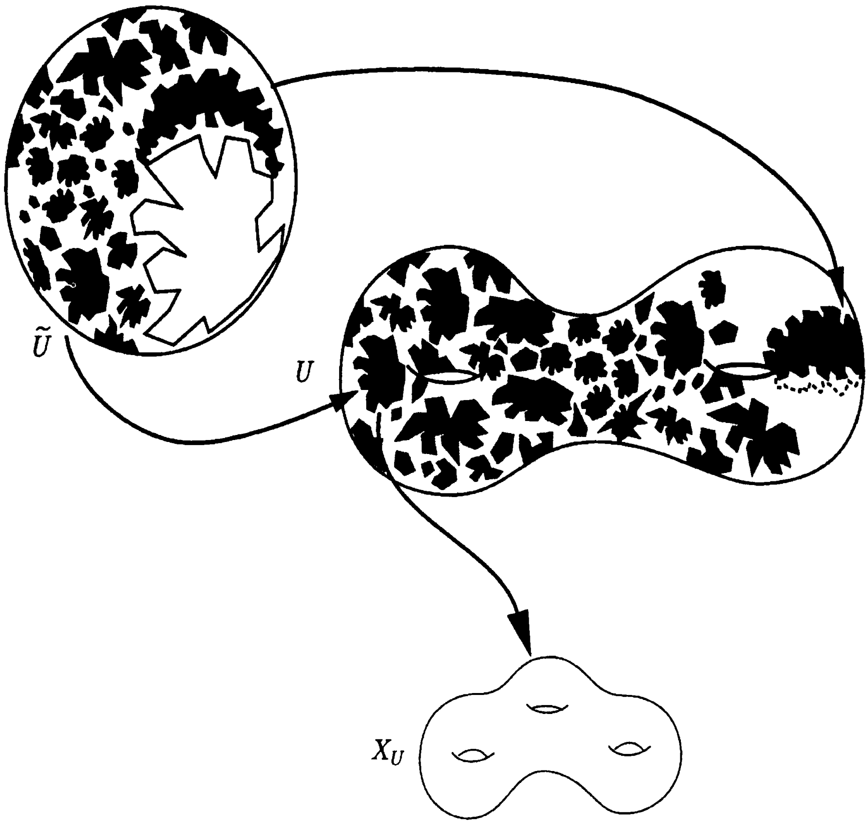

Figure 2.2

Since the components of $\Omega(G)$ are simply connected, the inclusion of each spot in $\sigma(\partial M)$ induces an injection on the fundamental group.

The coderivative of σ.

Notation.— For a spot U contained in $\sigma(\partial M^s)$, we denote by Θ_U the Theta operator Θ_{U/X_U} associated to the cover $U \to X_U$.

Let $\phi \in \mathcal{Q}(\sigma(\partial M^s))$. Since the restriction ϕ_U of ϕ to U is integrable, we can define a holomorphic quadratic differential $\Theta_U \phi_U$. It is an element of $\mathcal{Q}(\partial M^s)$ with norm less than or equal to $||\phi_U||$ (cf. §1). Thus, we can sum the differentials $\Theta_U \phi_U$ when U varies over all the spots contained in $\sigma(\partial M^s)$. This defines an element of $\mathcal{Q}(\partial M^s)$, denoted by $\sum_U \Theta_U \phi_U$.

The derivative of σ.

The tangent space to $\mathcal{T}(\partial M)$ (resp. to $\mathcal{T}(\sigma(\partial M))$) at s is isomorphic to the quotient $\mathcal{B}(\partial M^s)/\mathcal{N}(\partial M^s)$ (resp. to $\mathcal{B}(\sigma(\partial M^s))/\mathcal{N}(\sigma(\partial M^s))$. Let $\mu \in \mathcal{B}(\partial M^s)$. Let $\tilde{\mu}(z)\overline{dz}/dz$ be the pull-back of μ under the covering $\Omega(G^s) \to \partial M^s$. By setting

$\bar{\mu} \equiv 0$ on $L(\Omega(G^s))$, we obtain an element $\bar{\mu} \in L^\infty(\mathbb{C})$ such that

$$\bar{\mu}(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \bar{\mu}(z),$$

for almost all $z \in \mathbb{C}$ and for all $\gamma \in G^s$. In particular for all i, the restriction of $\bar{\mu}$ to $\widetilde{\varphi}(\widetilde{\Omega}_i)$ is invariant under $\rho(\Gamma_i')$. It defines therefore a Beltrami form $\widetilde{\mu}_i$ on $\widetilde{\varphi}(\widetilde{\Omega}_i)/\rho(\Gamma_i')$. In this way, we obtain a Beltrami form $\widetilde{\mu} = (\widetilde{\mu}_i)$ on $\sigma(\partial M^s)$.

With the above notations, we have:

Proposition 2.1. —

(i) *$\mu \to \widetilde{\mu}$ induces a map from $\mathcal{B}(\partial M^s)/\mathcal{N}(\partial M^s)$ to $\mathcal{B}(\sigma(\partial M^s))/\mathcal{N}(\sigma(\partial M^s))$ which is the derivative of σ at s, and*

(ii) *the coderivative of σ at s is given by*

$$d_s^*\sigma\phi = \sum_U \Theta_U\phi_U.$$

Proof. — Let $\Pi : \mathcal{B}^1(\partial M) \to \mathcal{T}(\partial M)$, and $\overline{\Pi} : \mathcal{B}^1(\sigma(\partial M)) \to \mathcal{T}(\sigma(\partial M))$ be the projections defined in §1. Recall that Π (resp. $\overline{\Pi}$) induces an isomorphism between $\mathcal{B}(\partial M)/\mathcal{N}(\partial M)$ (resp. $\mathcal{B}(\sigma(\partial M))/\mathcal{N}(\sigma(\partial M))$) and the tangent space to $\mathcal{T}(\partial M)$ (resp. $\mathcal{T}(\sigma(\partial M))$) at ∂M (resp. $\sigma(\partial M)$). Choose a local differentiable section ℓ of Π, defined in an open neighborhood $\mathcal{V}$ of ∂M in $\mathcal{T}(\partial M)$ (cf. §1). It follows from the definition of σ that we have, for $s \in \mathcal{V}$: $\sigma(s) = \overline{\Pi} \circ \widetilde{\ell(s)}$. This implies that the map $\mu \to \widetilde{\mu}$ projects to a map from $\mathcal{B}(\partial M)/\mathcal{N}(\partial M)$ to $\mathcal{B}(\sigma(\partial M))/\mathcal{N}(\sigma(\partial M))$ which is the derivative of σ at ∂M. The differentiablity of σ at an arbitrary point follows from this special case, by using the naturality of the differentiable structure on $\mathcal{T}(\partial M)$ (cf. §1).

Let $\mu \in \mathcal{B}(\partial M^s)$. Denote by $\widetilde{\mu}_U$ the restriction of $\widetilde{\mu}$ to U, and by $\langle\cdot,\cdot\rangle_U$ the pairing between holomorphic quadratic differentials and Beltrami forms on U (cf. §1). By definition, $\widetilde{\mu}$ vanishes in the complement of the spots. Thus, for any $\phi \in \mathcal{Q}(\sigma(\partial M^s))$, we have

$$\langle\widetilde{\mu}, \phi\rangle = \sum_U \langle\widetilde{\mu}_U, \phi_U\rangle_U,$$

where the sum carries over all the spots $U \subset \sigma(\partial M^s)$. By definition, $\widetilde{\mu}_U$ is the pull-back of μ under the covering $U \to X_U$. Since the pull-back operator on Beltrami forms is the adjoint of the Theta operator, we have

$$\langle\widetilde{\mu}_U, \phi_U\rangle_U = \langle\mu, \Theta_U\phi_U\rangle_{X_U}.$$

Therefore

$$\langle\widetilde{\mu}, \phi\rangle = \langle\mu, \sum_U \Theta_U\phi_U\rangle.$$

This proves Proposition 2.1 (ii). □

Proposition 2.1 shows the relation between σ and the Theta operators associated to the spots. It implies that, for $s \in \mathcal{T}(\partial M)$ and $\phi \in \Omega(\sigma(\partial M^s))$, we have

$$||d_s^*\sigma\phi|| \leq \sum_U ||\Theta_U|| ||\phi_U||.$$

In particular $||d_s^*\sigma|| \leq \sup_U ||\Theta_U||$. In order to study more precisely the contraction properties of σ, we will first establish related contraction properties for the operators Θ_U. We describe now the topological type of the covers associated to the spots.

The leopard spots and the topology of M.

If M is an interval bundle over a closed surface, each component of $\sigma(\partial M^s)$ is a spot and the cover associated to it is trivial: then $d^*\sigma$ is clearly an isometry (in that case, we had already noticed that σ was an isometry).

Therefore, we assume in the rest of this section that M is not an interval bundle. Since this topological type of the covers associated to the spots does not depend on s, we may suppose $M^s = M$.

Definition. — Let S be a compact surface contained in a connected surface X. We say that S is *incompressible* if it is connected and if $\pi_1(S)$ injects into $\pi_1(X)$. A cover $Y \to X$ between connected surfaces is *geometric* if $\pi_1(Y)$, viewed as a subgroup of $\pi_1(X)$, is equal to the fundamental group of a proper incompressible surface $S \subset X$. In this case we say that *the cover* $Y \to X$ *is associated to* S.

Examples. — The universal cover of any surface is geometric: it is associated to a disc. A non-trivial finite cover of a geometric cover is not geometric.

Proposition 2.2. —

(i) *Let $U \subset \sigma(\partial M)$ be a spot. Then the cover $U \to X_U$ is geometric.*

(ii) *All but finitely many spots in $\sigma(\partial M)$ are simply connected.*

Proof. — We keep the same notations that we used to define the spots. Let S_i be the component of ∂M such that U is contained in $\sigma(S_i)$. To simplify the notations we denote Ω_i by Ω, $\widetilde{\Omega}_i$ by $\widetilde{\Omega}$ and Γ_i' by Γ'. Then Γ' is a quasi-Fuchsian group and $\Omega(\Gamma') = \Omega \cup \widetilde{\Omega}$. Let $\widetilde{U} \subset \widetilde{\Omega}$ be a component of the preimage of U. Denote by Γ the stabilizer of $\widetilde{U}$ in G. Since $U = \widetilde{U}/\Gamma \cap \Gamma'$ and $X_U = \widetilde{U}/\Gamma$, the cover $U \to X$ corresponds to the subgroup $\Gamma \cap \Gamma'$ of $\pi_1(X_U) \simeq \Gamma$. Therefore, (i) is equivalent to say that $\Gamma \cap \Gamma'$ is the fundamental group of a proper incompressible surface of X_U.

Since Γ' (resp. Γ) is quasi-Fuchsian, the frontier of Ω (resp. $\widetilde{U}$) in $\overline{\mathbb{C}}$ equals the Jordan curve $L(\Gamma')$ (resp. $L(\Gamma)$). Consider the closed set $F = L(\Gamma) \cap L(\Gamma')$, which is invariant under $\Gamma' \cap \Gamma$. We note first that F is a proper subset of $L(\Gamma)$. For if F were equal to $L(\Gamma)$, then $\Omega(G)$ would be the disjoint union of Ω and $\widetilde{U}$. This would imply that Γ' is a subgroup of G of index at most 2, i.e. that M is an interval bundle (cf. §7).

Let $\gamma \in \Gamma$. We have $\gamma(\widetilde{\Omega}) \cap \widetilde{\Omega} = \emptyset$ if $\gamma \notin \Gamma \cap \Gamma'$ and $\gamma(\widetilde{\Omega}) = \widetilde{\Omega}$ if $\gamma \in \Gamma \cap \Gamma'$. This implies that F *and* $\gamma(F)$ *are not linked on* $L(\Gamma)$, i.e. that no pair of points in

F alternates on $L(\Gamma)$ with a pair of points in $\gamma(F)$. We consider now distinct cases according to the cardinality of F.

1) $\sharp F \geq 2$.

Let $C(F)$ denote the convex hull of F in $\widetilde{U} \cup L(\Gamma)$ for the hyperbolic metric on $\widetilde{U}$.

Lemma 2.3.— *Under the covering $\widetilde{U} \to X_U$, the frontier of $C(F)$ in $\widetilde{U}$ maps to a disjoint union of embedded closed geodesics.*

Proof.— Let $\widetilde{g}$ be a geodesic in the frontier of $C(F)$ in $\widetilde{U}$. Since, for $\gamma \in \Gamma$ the endpoints of $\widetilde{g}$ do not alternate on $L(\Gamma)$ with the endpoints of $\gamma(\widetilde{g})$, $\widetilde{g}$ maps to a geodesic g on X_U without transverse self-intersections. Since two distincts translates of F are not linked, two distincts translates of $C(F)$ intersect atmost along their frontiers; therefore, $\partial C(F)$ maps to a disjoint union of embedded geodesics. Suppose for a contradiction that for some geodesic $\widetilde{g} \subset \partial C(F)$, g is not a closed curve. Then g is not compact. Hence there is an infinite sequence (γ_k) of elements of Γ such that the geodesics $\gamma_k(\widetilde{g})$ are distincts and that their endpoints accumulate on two distincts points of $L(\Gamma)$. Since the endpoints of $\widetilde{g}$ are contained in the frontier of F in $L(\Gamma)$, the components $\gamma_k(\Omega)$ of $\Omega(G)$ are all distincts for sufficiently large k. Then $\Omega(G)$ has an infinity of components with diameter bounded from below by a non-zero constant. This is impossible (cf. §1). □

1a) $\sharp F = 2$.

Then $C(F)$ is a geodesic and, by Lemma 2.3 its projection is an embedded closed curve. Let $\gamma \in \Gamma$ be the element represented by this geodesic. Then γ leaves also Ω invariant. For, if this is not true, the components $(\gamma)^n(\Omega)$ of $\Omega(G)$ are all distinct and their diameter is bigger than the diameter of F. This is impossible (cf. §1). Thus, $\gamma \in \Gamma'$ and $\Gamma \cap \Gamma'$ is equal to the cyclic group generated by γ. This proves Proposition 2.2 (i) in this case.

1b) $\sharp F > 2$.

In this case, $C(F)$ has non-empty interior. Let Σ' be the projection of $C(F)$ to X_U. By Lemma 2.3 and since X_U is compact, the projection of $\partial C(F)$ to X_U is the disjoint union of a finite number of embedded closed geodesics. Therefore, Σ' is a compact connected surface with geodesic boundary (however, the projection of some components of $\partial C(F)$ is maybe contained in the interior of Σ'). The complement in Σ' of an open regular neighborhood of the projection of $\partial C(F)$ defines an incompressible surface S. Let $\widetilde{S}$ be the component of the preimage of S that is contained in $C(F)$. Since $\widetilde{S}$ is contained in the interior of $C(F)$, we have, for $\gamma \in \Gamma$: $\gamma(\widetilde{S}) = \widetilde{S}$ if $\gamma \in \Gamma \cap \Gamma'$ and $\gamma(\widetilde{S}) \cap \widetilde{S} = \emptyset$ if $\gamma \notin \Gamma \cap \Gamma'$. Therefore $\Gamma \cap \Gamma'$ equals $\pi_1(S)$ (up to conjugacy). This proves Proposition 2.2 (i) in this case.

2) $F = \emptyset$.

Then, for any non-zero $\gamma \in \Gamma'$, we have $\gamma(\widetilde{U}) \cap \widetilde{U} = \emptyset$. Hence $\widetilde{U}$ is homeomorphic to U which is therefore simply connected. Thus the cover $\widetilde{U} \to X_U$ is geometric.

3) $F = \{f\}$.

We show that this cannot happen in our case — when G does not contain parabolic elements (if G contained parabolics elements, it could happen that F were reduced to one point; a slight modification of the next argument could prove however Proposition 2.2 (i) in this situation as well).

Let $g \subset \widetilde{\Omega}/\Gamma'$ be the projection of the Jordan arc $L(\Gamma) - \{f\}$. Suppose that g is compact. Then g is a closed curve which is not homotopic to 0 since $L(\Gamma) - \{f\}$ is not compact. Thus g is homotopic to a closed geodesic, and any of its lifts to $\widetilde{\Omega}$ accumulates to the two fixed points of some hyperbolic element of Γ' (since G does not contain parabolics elements). This contradicts the fact that $L(\Gamma) - \{f\}$ accumulates to f.

Thus g is non-compact. Since $\widetilde{\Omega}/\Gamma'$ is compact, there exists a sequence of distincts elements $\gamma_k \in \Gamma'$ such that $\gamma_k(L(\Gamma) - \{f\})$ accumulates to a point $p \in \widetilde{\Omega}$. Then the domains $\gamma_k(\widetilde{U})$ are distinct components of $\Omega(G)$ and their diameter is bigger than the distance from p to $L(\Gamma')$. This excludes the case 3) and finishes the proof of Proposition 2.2 (i).

To prove (ii), recall that, for any spot $U \subset \widetilde{\Omega}/\Gamma'$, $\pi_1(U)$ maps injectively into Γ'. In particular, each spot is homeomorphic to the interior of a compact surface with boundary. Only finitely many spots can have strictly negative Euler characteristic, since such a spot contributes at least -1 to the Euler characteristic of $\widetilde{\Omega}/\Gamma'$. To exclude the presence of an infinity of spots which are homeomorphic to annuli, we argue by contradiction. Then there are also infinitely many spots $U_i \subset \widetilde{\Omega}/\Gamma'$ which are homotopic to the same simple closed curve c. These spots can be lifted in $\widetilde{\Omega}$ to distincts components of $\Omega(G)$ which have the same endpoints as some lift $\widetilde{c}$ of c. Since their diameter is bigger than the distance between the two endpoints of $\widetilde{c}$, we obtain a contradiction. This finishes the proof of (ii). □

The next result can be proven with the same arguments as Proposition 2.2.

Corollary 2.4.— *Let $\widetilde{\Omega}$ and $\widetilde{U}$ be components of $\Omega(G)$ with stabilizers Γ' and Γ respectively. Let $\gamma \in \Gamma$ be a hyperbolic element which has one fixed point in $L(\Gamma) \cap L(\Gamma')$. Then $\gamma \in \Gamma' \cap \Gamma$.* □

Acylindricity.

The hypothesis that all spots are simply connected will introduce an important dichotomy in the proof of Thurston's fixed point theorem. We now show that this situation reflects a topological property of M, namely that M *is acylindrical.*

Definition.— Let A denote the annulus $\mathbb{S}^1 \times [0,1]$. Let M be a compact 3-manifold. A continuous map $f : (A, \partial A) \to (M, \partial M)$ is *essential* if it induces an injective map on $\pi_1(A)$ and on $\pi_1(A, \partial A)$. The image $f(A)$ is *an essential annulus.*

We say that M is *acylindrical* if it does not contain any essential annulus.

Fact 2.5.— *The manifold $M = \overline{M}(G)$ is acylindrical if and only if all the spots contained in $\sigma(\partial M)$ are simply connected.*

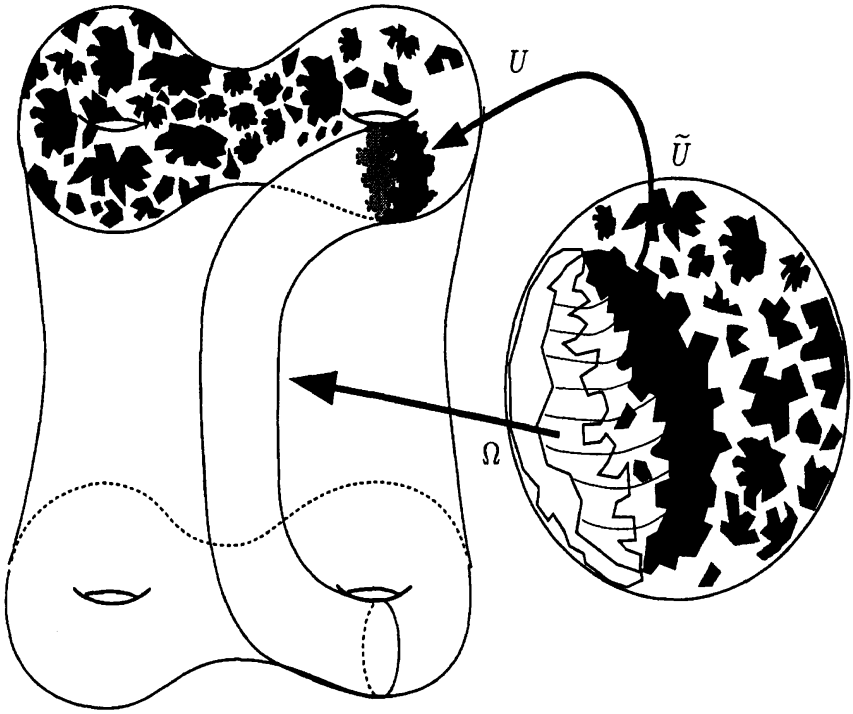

Figure 2.3

Proof. — Suppose that M is not acylindrical. Let $f : A \to M$ be an essential map. Let f^* be the map induced by f on the fundamental group. Then $f^*(\pi_1(A))$ is a cyclic group generated by a hyperbolic element g. Let $\widetilde{f} : \widetilde{A} \to \widetilde{M}$ be a lift of f to the universal cover. Since f induces an injection of $\pi_1(A, \partial A)$, the two components of $\widetilde{f}(\partial\widetilde{A}))$ are contained in distincts components Ω and Ω' of $\Omega(G)$. Then, the intersection of the closures of Ω and Ω' in $\overline{\mathbb{C}}$ contains the two fixed points of a conjugate of g. It follows from the proof of Proposition 2.2 that the image of Ω' on the component of $\sigma(\partial M)$ which is covered by $\widetilde{\Omega} = \overline{\mathbb{C}} - \overline{\Omega}$ is a spot that is not simply connected.

Conversely let $U \subset \sigma(\partial M)$ be a spot which is not simply connected. Suppose that U is contained in the component $\sigma(S_i)$ of $\sigma(\partial M)$. Let $\widetilde{U} \subset \widetilde{\Omega}_i$ be a component of the preimage of U. Since U is not simply connected, and since $\pi_1(U)$ maps injectively into $\pi_1(S_i) = \Gamma'_i$, $\widetilde{U}$ is invariant by a non-zero element of Γ'_i. Then the intersection of $\widetilde{U}$ with $L(\Gamma'_i)$ contains the two fixed points of this element and it is easy to construct an essential map from A into M (cf. Figure 2.3). □

CHAPTER 3

Holomorphic quadratic differentials

In this chapter, we study triples (X, x, ϕ) where X is a connected Riemann surface, $x \in X$ and $\phi \in \Omega(X)$. We won't describe the topology on the set of these triples in all its generality. The reader is referred to **[McM1]** for the general definition. We will rather explain, in an elementary way, what it means for a sequence of such triples to converge, so that the proof of the main theorem, Theorem 3.1, can be reduced to classical compactness theorems on holomorphic functions.

3.1 Compactness properties of holomorphic quadratic differentials

Definition. — *A pointed Riemann surface* is a pair (X, x) where X is connected Riemann surface and $x \in X$.

We recall first how the space of pointed compact hyperbolic Riemann surfaces can be compactified (**[Thu1]**,**[Mu]**).

Limits of pointed Riemann surfaces.

Consider a sequence (X_i, x_i) of pointed compact hyperbolic Riemann surfaces with fixed topological type. The behaviour of this sequence, viewed as a sequence of pointed metric spaces where X_i is endowed with the hyperbolic metric depends on *the injectivity radius of* X_i *at* x_i. This is the largest radius of an open embedded hyperbolic ball centered at x_i. It is denoted by $\mathrm{inj}(x_i)$. Since the topological type of X_i is fixed, the hyperbolic volume of X_i is constant: in particular $\mathrm{inj}(x_i)$ is bounded from above by a constant depending only on $\chi(X_i)$. We distinguish two cases according to whether $\mathrm{inj}(x_i)$ is bounded from below by a non-zero constant or not.

a) $\operatorname{inj}(x_i)$ *is bounded from below away from* 0.

Let us identify X_i with the quotient of $\mathbb{D}^2$ by a Fuchsian group Γ_i in such a way that the origin $0 \in \mathbb{D}^2$ maps to x_i. By definition, *the Dirichlet domain of* Γ_i *with respect to* 0 is the set of points closer to 0 than to any of its translates by non-zero elements of Γ_i. It is a finite sided convex polygon $\mathcal{D}_i$. By an Euler characteristic argument, the number of sides of $\mathcal{D}_i$ is bounded independently of i. The hypothesis on $\operatorname{inj}(x_i)$ implies that $\mathcal{D}_i$ contains a ball centered at 0 of radius independent of i. Therefore, up to extracting a subsequence, $(\mathcal{D}_i)$ converges to a finite sided polygon (having perhaps some vertices on the circle at infinity) which is the Dirichlet domain of a discrete group Γ. This group has finite covolume and is called *a geometric limit of* Γ_i. Let $X = \mathbb{D}^2/\Gamma$ and let x be the projection of $0 \in \mathbb{D}^2$. We say that (X_i, x_i) *converges to* (X, x). One can prove that this convergence is equivalent to the convergence of Γ_i to Γ, for *the Chabauty topology*, i.e. for the Hausdorff topology on closed subsets of $\mathrm{PSL}_2(\mathbb{R})$ [CEG].

b) $\operatorname{inj}(x_i)$ *tends to* 0.

We use the formulas for the hyperbolic metric on an annulus (cf §1). Let $\varepsilon \leq \varepsilon(2)$. Suppose that x_i belongs to a component of $X_i^{]0,\varepsilon]}$ which is a Margulis tube around a geodesic g_i. Consider the geometric cover of X_i with fundamental group isomorphic to the cyclic group generated by g_i. This cover can be identified conformally with $A_{e^{-R_i},e^{R_i}}$, for $R_i = \pi^2/\ell(g_i)$. By the Margulis lemma, the ε-thin part of $A_{e^{-R_i},e^{R_i}}$ embeds under the covering $A_{e^{-R_i},e^{R_i}} \to X_i$, and in particular, the annulus $A_{e^{-\rho_i},e^{\rho_i}}$ for

$$\rho_i = \frac{2R_i}{\pi}\cos^{-1}\frac{\ell(g_i)}{\varepsilon}.$$

Let $\widetilde{x}_i$ be the lift of x_i which is contained in $A_{e^{-\rho_i},e^{\rho_i}}$. Denote by Y_i the image of $A_{e^{-R_i},e^{R_i}}$ by the homothety of ratio $1/\widetilde{x}_i$. Up to extracting a subsequence, Y_i converge to the annulus $A_{0,\infty} = \mathbb{C}^*$ and the hyperbolic metrics on Y_i, rescaled by the factor $1/\operatorname{inj}(x_i)$, converge to a flat complete metric on $\mathbb{C}^*$. We say that (X_i, x_i) *converges to* $(\mathbb{C}^*, 1)$.

This normalization allows us, by looking at an appropriate cover of X_i —which is either the universal cover $\mathbb{D}^2$ or the annulus Y_i— to compare charts around $x_i \in X_i$ when X_i varies.

Definition. — Let (X_i, x_i) be a sequence of pointed compact hyperbolic Riemann surfaces which converges to (X, x). Let $\phi_i \in \mathcal{Q}(X_i)$ and let ϕ be a holomorphic quadratic differential defined on X. If $\operatorname{inj}(x_i)$ does not tend to 0, let $\widetilde{\phi}_i(z)dz^2$ (resp. $\widetilde{\phi}(z)dz^2$) be the pull-back of ϕ_i (resp. ϕ) to $\mathbb{D}^2$. If $\operatorname{inj}(x_i)$ tends to 0, let Y_i be the covering of X_i associated to the fundamental group of that component and let $\widetilde{\phi}_i(z)dz^2$ (resp. $\widetilde{\phi}(z)dz^2$) be the pull-back of ϕ_i (resp. ϕ) to Y_i. We say that ϕ_i *converges uniformly to* ϕ if $\widetilde{\phi}_i$ converges to $\widetilde{\phi}$ uniformly over compact sets (in $\mathbb{D}^2$ or in $\mathbb{C}^*$).

Theorem 3.1. — *Let* (X_i, x_i) *be a sequence of pointed compact hyperbolic Riemann surfaces with a fixed topological type which converges to* (X, x). *Let* $\phi_i \in \mathcal{Q}(X_i)$

with $\phi_i \neq 0$. *Then there exists constants* c_i *and a non-zero holomorphic quadratic differential* ϕ *on* X *such that* $(c_i\phi_i)$ *converges uniformly to* ϕ *up to extracting a subsequence.*

Remark.— One important feature of this theorem is to produce a non-zero limit, merely by applying a homothety to ϕ_i. The existence of a limit for a sequence (ϕ_i), when $||\phi_i||$ is bounded can be shown by a more elementary argument. It follows from the precompacity of a sequence of holomorphic functions with bounded L^1-norm. However, this is not sufficient to guarantee that the limit is non-zero, even if $||\phi_i|| = 1$.

Remark.— The limit ϕ produced by Theorem 3.1 is not necessarily integrable. The basic example of a non-integrable limit appears in Lemma 3.4.

In order to prove Theorem 3.1, we construct first non-zero holomorphic quadratic differentials θ_i on X_i and θ on X such that (θ_i) converges uniformly to θ as i tends to infinity.

Proposition 3.2.— *Let* (X_i, x_i) *be a sequence of pointed compact hyperbolic Riemann surfaces with a fixed topological type which converges to* (X, x). *Then there exists non-zero* $\theta_i \in \mathcal{Q}(X_i)$ *and a non-zero holomorphic quadratic differential* θ *on* X, *such that* (θ_i) *converges uniformly to* θ, *up to extracting a subsequence.*

Proof.— We consider two cases according as X is a finite volume hyperbolic surface or is an annulus.

1) X *is a finite volume hyperbolic surface.*

We begin with the following lemma.

Lemma 3.3.— *Suppose that* (X_i, x_i) *converge to* (X, x) *where* X *is a finite volume hyperbolic surface. Then* $\Theta_{\mathbb{D}^2/X_i}$ *converge weakly to* $\Theta_{\mathbb{D}^2/X}$, *i.e. for any* $P \in \mathcal{Q}(\mathbb{D}^2)$, $\Theta_{\mathbb{D}^2/X_i}P$ *converges uniformly to* $\Theta_{\mathbb{D}^2/X}P$.

Proof.— Assume that X_i and X are uniformized by Fuchsian groups Γ_i and Γ acting on $\mathbb{D}^2$, in such a way that $0 \in \mathbb{D}^2$ projects to x_i and x respectively. Let $P \in \mathcal{Q}(\mathbb{D}^2)$. Fix a compact set $K \subset \mathbb{D}^2$. For each $r < 1$, there is a compact neighborhood C_r of $\mathrm{Id} \in \mathrm{PSL}_2(\mathbb{R})$ such that, for any $g \notin C_r$, $g(K)$ is contained outside of the disc Δ_r of radius r. We may assume that the frontier of C_r in $\mathrm{PSL}_2(\mathbb{R})$ is disjoint from Γ so that $\Gamma_i \cap C_r \to \Gamma \cap C_r$ as i tends to ∞. We can choose r so that the $|P|$-mass of $\mathbb{D}^2 - D_r$ is arbitrarily small. Therefore, for i sufficiently large, the difference $|\Theta_{\mathbb{D}^2/X}P(z) - \Theta_{\mathbb{D}^2/X_i}P(z)|$ can be made arbitrarily small over K: this follows directly from Cauchy's formula. □

We use now the fact that $\Theta_{\mathbb{D}^2/X}$ is surjective when X has finite volume. The proof can be sketched as follows. For a general Riemann surface X, the image of the restrictions of the polynomials to $\mathbb{D}^2$ is dense in $\mathcal{Q}(X)$, when $\mathcal{Q}(X)$ is endowed with the Weil-Peterson scalar product (cf. [**Kr**]). If the hyperbolic volume of X is finite, $\mathcal{Q}(X)$ is a finite dimensional vector space (cf. §1). It follows that $\Theta_{\mathbb{D}^2/X}$ is surjective in this case. For a composition of covers of Riemann surfaces $Z \to Y \to X$, we have: $\Theta_{Z/X} = \Theta_{Y/X} \circ \Theta_{Z/Y}$. Hence if X has finite volume, the operator $\Theta_{Y/X}$ is surjective.

Lemma 3.3 and the surjectivity of $\Theta_{\mathbf{D}^2/X}$ yield a proof of Proposition 3.2 as follows.

1a) *X is different from the thrice punctured sphere.*

In this situation, the vector space $\Omega(X)$ is non-trivial. By the above, it contains a non-zero element of the form $\Theta_{\mathbf{D}^2/X}P$. By Lemma 3.3, $(\Theta_{\mathbf{D}^2/X_i}P)$ converges uniformly to $\theta = \Theta_{\mathbf{D}^2/X}P$. For all sufficiently large i, $\theta_i = \Theta_{\mathbf{D}^2/X_i}P$ is non-zero. This proves Proposition 3.2.

1b) *X is the thrice punctured sphere.*

Then the space of integrable holomorphic quadratic differentials is trivial and the reasonning above cannot be applied. The proof we will give could be extended with a minor modification to the case when X has finite volume but is not compact. However, we present it only when X is the thrice punctured sphere: any puncture on X corresponds to a closed geodesic $g_i \subset X_i$ whose length tends to 0, and since X is a thrice punctured sphere, we can choose such a geodesic g_i which gives rise to a single puncture on X.

The cover of X_i associated to the cyclic subgroup of Γ_i representing the curve g_i can be conformally identified with $A_{e^{-R_i},e^{R_i}}$ (cf. §1). Let $\varepsilon \leq \varepsilon(2)$. By Margulis lemma, the annulus $A_{e^{-\rho_i},e^{\rho_i}}$ embeds under the covering map into X_i, for

$$\rho_i = \frac{2R_i}{\pi} \cos^{-1}\left(\frac{\ell(g_i)}{\varepsilon}\right).$$

Without loss of generality we may assume, (up to applying an inversion through the unit circle) that the image of the circle of radius e^{ρ_i} remains at a bounded distance from x_i on X_i. Denote by Y_i the image of $A_{e^{-R_i},e^{R_i}}$ by the homothety of ratio $e^{-\rho_i}$, i.e. the annulus $A_{e^{-R_i-\rho_i},e^{R_i-\rho_i}}$ (the unit circle has now length ε). Since $\ell(g_i)$ tends to 0, R_i tends to infinity; therefore $R_i - \rho_i \simeq 2\pi/\varepsilon$. Up to extracting a subsequence, (Y_i) converges to the annulus $Y = A_{0,e^{2\pi/\varepsilon}}$, and the hyperbolic metrics on Y_i converge to the hyperbolic metric on Y. The annulus Y is also the covering of X corresponding to the puncture we have selected.

Consider the holomorphic quadratic differential $\phi = dz^2/z^2$ on $\mathbb{C}^*$. Since ϕ is integrable on Y_i, we can apply the operator Θ_{Y_i/X_i} to it, obtaining a differential $\theta_i = \Theta_{Y_i/X_i}\phi \in \Omega(X_i)$. However ϕ is not integrable on Y and therefore $\Theta_{Y/X}\phi$ cannot be defined in the classical way. Nevertheless, ϕ is integrable in the complement of the unit disc. By the Margulis lemma, the punctured unit disc embeds under the covering $Y \to X$. Let B be a small ball contained in X. The preimage of B in Y consists of disjoint copies of B. At most one of them is contained in the punctured unit disc, all the others are in $A_{1,e^{2\pi/\varepsilon}}$. Therefore in the series defining $\Theta_{Y/X}\phi$, the sum of the terms coming from $A_{1,e^{2\pi/\varepsilon}}$ converges uniformly over B (by Cauchy's formula) and gives an integrable holomorphic quadratic differential. The term coming from the punctured unit disc contributes as a differential with a double pole at the puncture. This allows us to define $\theta = \Theta_{Y/X}\phi$. It is a holomorphic quadratic differential on X, which is non-zero because it has a pole of order 2 at the puncture. The following lemma implies Theorem 3.1 in this case.

Lemma 3.4. — *(θ_i) converges uniformly to θ.*

For this lemma to be true, it is necessary that g_i gives rise to a single puncture in X. In the case when g_i gives rise to two punctures, (θ_i) converges to the sum of the two differentials constructed in the same way as θ for each of these punctures.

Proof. — We will prove that the pull-back of θ_i to Y_i converges to the pull-back of θ to Y. This will imply the lemma. We denote by the same letter π the covering maps from Y to X or from Y_i to X_i. Let $K \subset Y$ be a compact set, which we choose to be a lift of a small ball $\pi(K)$ embedded in X. Since $A_{e^{-2\rho_i},1}$ embeds into X_i (as consequence of the Margulis lemma), at most one component of $\pi^{-1}(\pi(K))$ is contained in $A_{e^{-2\rho_i},1}$. Since the metrics on Y_i converge to the metric on Y, the distance on Y_i between K and the circle of radius 1 is bounded from above independently of i. Now, we analyze the contribution of the various components of $\pi^{-1}(\pi(K))$ to $\Theta_{Y_i/X_i}\phi$.

Fact 3.5. — *The hyperbolic distance on Y_i between the circle of radius $e^{-2\rho_i}$ and any component of $\pi^{-1}(\pi(K))$ contained in $A_{e^{-R_i-\rho_i},e^{-2\rho_i}}$ tends to ∞ with i.*

Proof. — If not, then the image on X_i of the circle of radius $e^{-2\rho_i}$ is at a bounded distance from $\pi(K)$ and thus at a bounded distance from x_i too. By the normalization of Y_i, the projection of the unit circle is at a finite distance from K. But since g_i gives rise to a single puncture in X, only one of the two boundary components of $A_{e^{-2\rho_i},1}$ can stay within a bounded distance of x_i. This contradiction proves Fact 3.5. □

The formula for the hyperbolic distance on Y_i between two circles shows that the components of $\pi^{-1}(\pi(K))$ contained in $A_{e^{-R_i-\rho_i},e^{-2\rho_i}}$ are confined inside an annulus $A_{e^{-R_i-\rho_i},e^{-r_i-\rho_i}}$ where $R_i - r_i$ tends to 0 when i tends to ∞. The ϕ-mass of this strip is equal to $2\pi(R_i - r_i)$ and so tends to 0. In particular, the ϕ-mass of $\pi^{-1}(\pi(K)) \cap A_{e^{-R_i-\rho_i},e^{-2\rho_i}}$ tends to 0 when i tends to ∞.

Since ϕ is integrable near the exterior end of Y, we can select, for all $\eta > 0$, a radius $r < e^{R_i-\rho_i}$ such that the ϕ-mass of $A_{r,e^{R_i-\rho_i}}$ is smaller than η for all i sufficiently large. In particular, the ϕ-mass of $\pi^{-1}(\pi(K)) \cap A_{r,e^{R_i-\rho_i}}$ is smaller than η, for all i sufficiently large.

The components of $\pi^{-1}(\pi(K)) \cap A_{e^{-2\rho_i},r}$ are K itself, and finitely many others which intersect $A_{1,r}$. As in Lemma 3.3, it follows now from Cauchy's formula that the pull-back of θ_i to Y_i converges to the pull-back of θ uniformly in the interior of K. This ends the proof of Lemma 3.4. □

2) $X = \mathbb{C}^*$.

We keep the same notations as for the description of the convergence of (X_i, x_i). The annulus Y_i is identified with an annulus $A_{e^{-2R_i+R'_i},e^{R'_i}}$ for a certain number R'_i, in order for the lift $\tilde{x}_i$ of x_i to be the point 1. As in the case of the thrice punctured sphere, we can apply Θ_{Y_i/X_i} to $\phi = dz^2/z^2$, and define a holomorphic quadratic differential θ_i on X_i.

Lemma 3.6.— *(θ_i) converges uniformly to $\phi = dz^2/z^2$.*

Proof.— Let $K \subset \mathbb{C}^*$ be a compact set. The length of the unit circle for the hyperbolic metric of Y_i of the unit circle is equivalent to $\mathrm{inj}(x_i)$. It follows from the formula for the length of the circles on Y_i that the injectivity radius tends to 0 uniformly over K. In particular, K is contained in $Y_i^{]0,\varepsilon]}$ for all i sufficiently large, and the hyperbolic distance between K and $\partial Y_i^{]0,\varepsilon]}$ tends to ∞ with i. By the Margulis lemma, $Y_i^{]0,\varepsilon]}$ maps injectively to X_i. Therefore K is the single component of $\pi^{-1}(\pi(K))$ contained in $Y_i^{]0,\varepsilon]}$, and the hyperbolic distance between $\partial Y_i^{]0,\varepsilon]}$ and the components of $\pi^{-1}(\pi(K))$ others than K tends to ∞ with i. It follows that the components of $\pi^{-1}(\pi(K))$ which are near the exterior end (say) are confined inside an annulus $A_{e^{R_i'-r_i},e^{R_i'}}$ such that $R_i' - r_i$ tends to 0 when i tends to infinity. The same holds for the components which are near the interior end. Therefore, as in the case of the thrice punctured sphere, the pull-back of θ_i to the interior of K converges to ϕ uniformly in the interior of K. This proves Lemma 3.6. □

This completes the proof of Proposition 3.2. □

Proof of Theorem 3.1.— Let (θ_i) be the sequence constructed in Proposition 3.2. The ratio $f_i = \phi_i/\theta_i$ is a rational function $X_i \to \overline{\mathbb{C}}$. The degree d of f_i is independent of i, by the Hurwitz formula. The preimage $E_i = f_i^{-1}\{0, 1, \infty\}$ has cardinality less than $3d$. Denote by $\widetilde{E}_i$ the preimage of E_i in the universal cover $\mathbb{D}^2$ or in the annuli Y_i, according to the nature of $\mathrm{inj}(x_i)$. Let $\widetilde{f}_i$ be the lift of f_i to the corresponding cover $\mathbb{D}^2$ or Y_i. Let K be an arbitrary compact set contained in $\mathbb{D}^2$ if $\mathrm{inj}(x_i)$ is bounded away from 0, or in $\mathbb{C}^*$ if $\mathrm{inj}(x_i)$ tends to 0. If $K \subset \mathbb{D}^2$, the cardinality of $K \cap \widetilde{E}_i$ is bounded only in terms of d and of the "degree" of the restriction to K of the covering map $\pi : \mathbb{D}^2 \to X$, i.e. the maximal cardinality of $\pi^{-1}(z) \cap K$ of a point $z \in \pi(K)$. If $K \subset \mathbb{C}^*$, the covering map $\pi : Y_i \to X_i$ restricts to K as an embedding (cf. proof of Lemma 3.6); hence, the cardinality of $K \cap \widetilde{E}_i$ is bounded independently of i. Therefore, up to extracting a subsequence, the sets $\widetilde{E}_i$ converge to a discrete set $\widetilde{E}$ contained in $\mathbb{D}^2$ or in $\mathbb{C}^*$. By the Montel theorem, the rational functions $\widetilde{f}_i$ converge to a holomorphic function $\widetilde{f}$ uniformly on compact sets in $K - \widetilde{E}$, up to passing to a subsequence. The degree of $\widetilde{f}$ is finite because the degree of $\widetilde{f}_i$ over K is bounded independently of i. Hence by the Picard theorem, $\widetilde{f}$ extends across $\widetilde{E}$ to a meromorphic function. Thus, the functions $\widetilde{f}_i$ converge to a meromorphic function $\widetilde{f}$ uniformly on compact sets.

Up to scaling $\widetilde{f}_i$ by a constant, we can ensure that $\widetilde{f}$ is not identically 0 or ∞. To see this, consider a point $x_n \in X_n - E_n$ which tends to a point in $X - E$. Then the functions $f_n' = \widetilde{f}_n/f_n(x_n)$ converge to a meromorphic function uniformly over compact sets by the argument above. This function is not identically 0 nor ∞ by construction. Hence the functions $\widetilde{\phi}_n/\widetilde{f}_n(x_n)$ converge uniformly on compact sets in $\mathbb{D}^2$ (or in $\mathbb{C}^*$), to a non-zero holomorphic function $\widetilde{\phi}$.

When $\mathrm{inj}(x_i)$ is bounded away from 0, $\widetilde{\phi}(z)dz^2$ is invariant under Γ by continuity. So $\widetilde{\phi}(z)dz^2$ induces a non-zero holomorphic quadratic differential ϕ on X. This concludes the proof of Theorem 3.1. □

3.2 Applications of Theorem 3.1

Let X be a hyperbolic Riemann surface. Let ϕ be a holomorphic quadratic differential. In a conformal chart around a point in X, the hyperbolic metric can be written $\lambda(z)|dz|$ and ϕ can be written $\phi(z)dz^2$. Then, the quantity

$$\langle\phi(z)\rangle = |\phi(z)|\lambda^{-2}(z)$$

is independent of the chart and it defines a function on X.

Definition.— We call $\langle\phi\rangle(z)$ *the hyperbolic norm of* ϕ *at the point* z.

The quantity $\langle\phi\rangle(z)$ can be viewed also as the Riemannian norm of the tensor ϕ on the hyperbolic surface X.

Definition.— *The systole of* X is the length of the shortest closed geodesic of X.

This is well defined since a hyperbolic surface X has finitely many closed geodesics shorter than $\varepsilon(2)$ (cf. §1). When X is compact, one can prove that a lower bound $\varepsilon > 0$ for the systole gives an upper bound for the diameter of X which is a function of ε and of $\chi(X)$.

Let Z be the set of zeroes of a non-zero $\phi \in \mathcal{Q}(X)$. If X has genus g, the cardinality of Z is smaller than $4g-4$ (cf. [**Ga**]). Denote by $Z(r)$ the neighborhood of radius r of Z on X and let

$$m(r) = \inf_{z\in X-Z(r)} \langle\phi(z)\rangle.$$

Proposition 3.7 [Di].— *Let* $\varepsilon > 0$. *Let* X *be a compact connected hyperbolic Riemann surface with systole bigger than* ε *and let* $\phi \in \mathcal{Q}(X)$ *with* $||\phi|| = 1$. *Let* $r < \varepsilon/2$. *Then,* $m(r) \geq m$ *where* $m > 0$ *is a function of* r, ε *and* $\chi(X)$.

Proof.— We argue by contradiction. Suppose that there exist compact hyperbolic Riemann surfaces X_i with fixed topological type whose systole is bigger than ε, and ϕ_i in $\mathcal{Q}(X_i)$ with $||\phi_i|| = 1$, such that

$$m_i(r) = \inf_{z\in X_i-Z_i(r)} \langle\phi_i(z)\rangle$$

tends to 0, when i tends to ∞.

We keep the same notations as for the description of the convergence of pointed Riemann surfaces. Let $x_i \in X_i$. Since $\mathrm{inj}(x_i) \geq \varepsilon$, the pointed surfaces (X_i, x_i) converge to a hyperbolic surface (X, x), up to extracting a subsequence. Since the systole of X_i is bigger than ε, the diameter of X_i is bounded from above independently of i. Therefore the diameter of $\mathcal{D}_i$ is bounded independently of i and X is compact. By Theorem 3.1 there exist constants c_i and a non-zero

$\phi \in \mathcal{Q}(X)$ such that $(c_i\phi_i)$ converges uniformly to ϕ: this means that the functions $c_i\widetilde{\phi}_i(z)$ converge to $\widetilde{\phi}(z)$, where $c_i\widetilde{\phi}_i(z)dz^2$ (resp. $\widetilde{\phi}(z)dz^2$) is the pull-back of ϕ_i (resp. ϕ) to $\mathbb{D}^2$. Choose a ball $B \subset \mathbb{D}^2$ of finite radius which contains $\mathcal{D}$ in its interior. It meets only a finite number of translates of $\mathcal{D}$ by Γ. Hence, the number of the translates of $\mathcal{D}_i$ by elements of Γ_i that meet B is less than a constant C independent of i. Therefore, we have

$$\int_B |\widetilde{\phi}(z)||dz|^2 = \lim_{i\to\infty} \int_B |c_i\widetilde{\phi}_i(z)||dz|^2 \leq \lim_{i\to\infty} C|c_i|.$$

It follows that $|c_i|$ is bounded from below by a non-zero constant.

For all i sufficiently large, $\mathcal{D}_i$ is contained in B. Therefore

$$\int_B |\widetilde{\phi}||dz|^2 = \lim_{i\to\infty} \int_B |c_i\widetilde{\phi}_i||dz|^2 \geq \lim_{i\to\infty} |c_i|,$$

so that $|c_i|$ is bounded also from above. This means that we can choose all of the constants c_i equal to 1.

Since Z is discrete, we may suppose that ∂B does not intersect the set of zeroes $\widetilde{Z}$ of $\widetilde{\phi}$. Then $\widetilde{Z}_i \cap B$ converges to $\widetilde{Z} \cap B$, and $\widetilde{Z}_i(r) \cap B$ converges to $\widetilde{Z}(r) \cap B$. Therefore, if we write the Poincaré metric of $\mathbb{D}^2$ in the form $\lambda(z)|dz|$, we have

$$m_i(r) = \inf_{z \in B - \widetilde{Z}_i(r)} |\widetilde{\phi}_i(z)|\lambda(z)^{-2}.$$

So the uniform convergence of $\widetilde{\phi}_i$ to $\widetilde{\phi}$ over B implies that $m_i(r)$ tends to $m(r)$ as i tends to ∞. Since $m(r) \neq 0$, this is a contradiction. □

Notation. — Let X be a hyperbolic Riemann surface. We denote by $B(x,r)$ the hyperbolic ball of radius r centered at x.

Proposition 3.8 [McM2]. — *Let X be a connected compact hyperbolic Riemann surface and let $x \in X$. Let $\alpha \geq 1$. Then, for any non-zero $\phi \in \mathcal{Q}(X)$*

$$\frac{\int_{B(x,\alpha r)} |\phi|}{\int_{B(x,r)} |\phi|} \leq c(\alpha),$$

where the constant $c(\alpha) < \infty$ is a function of $\chi(X)$ and of α.

Proof. — Observe that multiplying the differential ϕ by a non-zero constant does not affect the ratio of the ϕ-masses that we are considering. To prove Proposition 3.8, we argue by contradiction. Then there is a sequence of pointed compact hyperbolic Riemann surfaces (X_i, x_i) with fixed topological type, non-zero $\phi_i \in \mathcal{Q}(X_i)$, and balls $B(x_i, r_i)$, constant such that the ratio of the ϕ_i-masses of $B(x_i, r_i)$ and $B(x_i, \alpha r_i)$ tends to ∞ with i. We keep the same notations as for the description of the convergence of pointed Riemann surfaces.

1) $\mathrm{inj}(x_i)$ *is bounded from below.*

Under the covering map $\mathbb{D}^2 \to X_i$, $B(x_i, r_i)$ is isomorphic to $B(0, r_i) \subset \mathbb{D}^2$ and $B(x_i, \alpha r_i)$ is the image of $B(0, s_i) \subset \mathbb{D}^2$. In particular, for any $\phi \in \mathcal{Q}(X_i)$, the ϕ-mass of $B(x_i, r_i)$ equals the $\widetilde{\phi}$-mass of $B(0, r_i)$ and the ϕ-mass of $B(x_i, \alpha r_i)$ is less than or equal to the $\widetilde{\phi}$-mass of $B(0, \alpha r_i)$, where $\widetilde{\phi}$ denotes the pull-back of ϕ to $\mathbb{D}^2$.

By Theorem 3.1, up to passing to a subsequence and up to multiplying ϕ_i by a non-zero constant, we can suppose that (ϕ_i) converges uniformly to a holomorphic quadratic differential $\phi \neq 0$. Since $B(x_i, r_i)$ is embedded, and since the area of X_i is constant, r_i is bounded from above. If r_i admits a non-zero lower bound, we may suppose that it converges to $r > 0$. Then the $\widetilde{\phi}_i$-mass of $B(0, r_i)$ and $B(0, \alpha r_i)$ converge respectively to the $\widetilde{\phi}$-mass of $B(0, r)$ and $B(0, \alpha r)$. Both are non-zero, since $\phi \neq 0$. We obtain a contradiction in this case. When r_i tends to 0, suppose that $\widetilde{\phi}$ vanishes exactly up to the order n at 0. Then after a change of variables, we find

$$\frac{\int_{B(0,\alpha r_i)} |\widetilde{\phi}_i|}{\int_{B(0,r_i)} |\widetilde{\phi}_i|} \to (\alpha)^{2n}.$$

This gives a contradiction.

2) $\mathrm{inj}(x_i)$ *tends to* 0.

In our normalizations, the rescaled hyperbolic metrics on Y_i tend to the flat metric on $\mathbb{C}^*$ and $1 \in \mathbb{C}^*$ maps to x_i. Denote by $B_i(1, r)$ the ball of radius r centered at 1 for the rescaled hyperbolic metric on Y_i and by $B_\infty(1, r)$ this ball for the limit flat metric. Then, under the covering $Y_i \to X_i$, $B_i(1, r_i / \mathrm{inj}(x_i))$ maps isomorphically to $B(x_i, r_i)$, and $B(x_i, \alpha r_i)$ is the image of $B_i(1, \alpha r_i / \mathrm{inj}(x_i))$. Since $B(x_i, r_i)$ is embedded, $r_i \leq \mathrm{inj}(x_i)$. As in 1), we are led to consider two cases according to whether the ratio $r_i / \mathrm{inj}(x_i)$ admits a non-zero lower bound or tends to 0. Observe that for any $R > 0$, there are non-zero constants a and b such that, for all $r \leq R$ each ball $B_\infty(1, r)$ is contained in the euclidean ball of radius ar and contains the euclidean ball of radius br. The same result holds for the balls $B_i(1, r)$ when i is sufficiently large because of the convergence of the rescaled metrics on Y_i. Using this property and applying the same arguments as in 1), we obtain a contradiction. □

Proposition 3.9 [McM2].— *Let* $0 < \varepsilon \leq \varepsilon(2)$. *Let* X *be a compact hyperbolic Riemann surface with systole smaller than* $\varepsilon/2$. *Let* $\phi \in \mathcal{Q}(X)$ *with* $||\phi|| = 1$. *Then the* ϕ*-mass of a component of* $X^{]0,\varepsilon]}$ *is bigger than a constant* $\zeta > 0$ *which is a function of* ε *and* $\chi(X)$.

Proof.— The number of components of $X^{]0,\varepsilon]}$ is less than $3g - 3$, where g is the genus of X. In order to prove Proposition 3.9 by contradiction, we may therefore suppose that there is a sequence of Riemann surfaces X_i with the same topological type, with systole smaller than $\varepsilon/2$ and $\phi_i \in \mathcal{Q}(X_i)$ with $||\phi_i|| = 1$ such that the ϕ_i-mass of $X_i^{]0,\varepsilon]}$ tends to 0 as i tends to ∞. The number of components of $X_i^{[\varepsilon,\infty[}$ is smaller than $|\chi(X_i)|$. Let Z_i be one of these components with the property that its ϕ_i-mass is bigger than some number $\nu > 0$, for all i sufficiently large. Let $x_i \in Z_i$. Since $\mathrm{inj}(x_i) \geq \varepsilon$ the sequence (X_i, x_i) converges, up to extracting a subsequence, to (X, x) where X is a hyperbolic Riemann surface. By assumption, $X^{]0,\varepsilon/2]}$ is non-empty. By Theorem 3.1, there exist constants c_i such that $(c_i\phi_i)$ converges uniformly to a non-zero holomorphic quadratic differential ϕ defined on X. As the ϕ_i-mass of Z_i is more than ν, c_i can be chosen equal to 1 (cf. the proof of Proposition 3.7). Since $\phi \neq 0$, the ϕ_i-mass of $X_i^{]0,\varepsilon]} - X_i^{]0,\varepsilon/2]}$ does not tend to 0. This is a contradiction. □

The next result asserts that the pairing $\langle\phi, \mu\rangle$ between a unit norm holomorphic quadratic differential and a unit norm Beltrami form can be estimated from a certain local data.

Definition. — Let X be a compact hyperbolic Riemann surface. Let E be a measurable set contained in X. For a non-zero $\phi \in \mathcal{Q}(X)$ and for a Beltrami form $\mu \in \mathcal{B}(X)$, define *the efficiency of the pairing between ϕ and μ over E* to be the ratio

$$e(E) = \frac{\langle\phi, \mu\rangle_E}{||\phi||_E},$$

where $\langle\phi, \mu\rangle_E = \Re(\int_E \phi(z)\mu(z)|dz^2|)$, and where $||\phi||_E$ is the ϕ-mass of E.

When $||\mu|| \leq 1$, we have $e(E) \leq 1$. Equality holds if and only if the restriction of μ to E equals $\overline{\phi}/|\phi|$ a. e.

Proposition 3.10 [McM2]. — *Let X be a compact hyperbolic Riemann surface. Let $\phi \in \mathcal{Q}(X)$ with $||\phi|| = 1$ and let $\mu \in \mathcal{B}(X)$ with $||\mu|| \leq 1$ Let $E \subset X$ be a measurable subset of ϕ-mass bigger than m for some $m > 0$. Suppose that each point of E is the center of an embedded hyperbolic ball on which the efficiency of the pairing between ϕ and μ is less than $1-\alpha$, for some $\alpha > 0$. Then $\langle\phi, \mu\rangle \leq 1 - cm\alpha$, where $c > 0$ depends only on $\chi(X)$.*

Proof. — By a Vitali type argument, we can extract from the family of balls provided by the hypothesis, a family of disjoint balls $\{B_i\}$ such that the balls $5B_i$ cover E. Then by Proposition 3.8, we have:

$$(1) \qquad \int_E |\phi| \leq \int_{\sqcup 5B_i} |\phi| \leq c(5) \int_{\sqcup B_i} |\phi|.$$

Since the balls B_i are disjoint we have

$$\langle\phi, \mu\rangle = ||\phi||_{\complement \sqcup B_i} \frac{\langle\phi, \mu\rangle_{\complement \sqcup B_i}}{||\phi||_{\complement \sqcup B_i}} + \sum_i ||\phi||_{B_i} \frac{\langle\phi, \mu\rangle_{B_i}}{||\phi||_{B_i}}.$$

Using (1) we obtain

$$\langle\phi, \mu\rangle \leq 1 - ||\phi||_{\sqcup B_i} + (1-\alpha)||\phi||_{\sqcup B_i} \leq 1 - \alpha m/c(5).$$

□

CHAPTER 4

The volume form on open Riemann surfaces

Definition. — *An open Riemann surface* is a connected hyperbolic Riemann surface of finite topological type but of infinite hyperbolic volume.

Equivalently an open Riemann surface is conformally equivalent to $\mathbb{D}^2$, to an annulus, or to the quotient of $\mathbb{D}^2$ by a Fuchsian group Γ with $\Omega(\Gamma) \neq \emptyset$.

Let Y be a hyperbolic Riemann surface. *A 1-form of type* $(1,0)$ *on* Y is a smooth 1-form η which can be written in any complex chart on the form $\eta(z)dz$. In any complex chart, the hyperbolic metric can be written on the form $\lambda(z)|dz|$. So, when η is a 1-form of type $(1,0)$ on Y, the quantity $|\eta(z)|\lambda(z)^{-1}$ is independent of the chart.

Definition. — The function $\langle\eta\rangle : Y \to \mathbb{R}$ defined by $\langle\eta\rangle(z) = |\eta(z)|\lambda(z)^{-1}$ is called *the hyperbolic norm of* η.

Thus $\langle\eta\rangle$ is the norm of the 1-form η when Y is endowed with its hyperbolic metric. The norm of any tensor on Y can be similarly defined. For instance, when ω is a 2-form on Y, $\langle\omega\rangle$ equals the ratio ω/dv where dv is the hyperbolic volume form on Y.

Theorem 4.1 [Di]. — *Let* $\varepsilon > 0$. *Let* Y *be an open hyperbolic Riemann surface whose systole is bigger than* ε. *Then there is a 1-form* η *of type* $(1,0)$ *such that*

(i) $\overline{\partial}\eta = dv$, *and*

(ii) $||\langle\eta\rangle||_\infty$ *is finite and bounded by a constant* C *which is a function of* ε *and* $\chi(Y)$.

Remark. — This theorem applies to give an isoperimetric inequality for domains in Y. If $\mathcal{K} \subset Y$ is a compact domain with smooth boundary, a direct application of the Stokes formula gives

$$\text{Area}(\mathcal{K}) \leq \frac{1}{C}\ell(\partial\mathcal{K}).$$

This means that the hyperbolic metric of Y satisfies a linear isoperimetric inequality with a constant depending only on the systole of Y and on $\chi(Y)$.

Proof. — We consider only the case when Y has no cusps.

Suppose first that Y is conformally equivalent to a disk or to an annulus. On $\mathbb{D}^2$, the hyperbolic metric is given by

$$ds = \frac{2}{1-|z|^2}|dz|.$$

Hence the 1-form

$$\eta = \frac{-2i\bar{z}}{1-|z|^2}dz$$

satisfies $\overline{\partial}\eta = dv$ and $\langle\eta\rangle(z) = |z|$. Therefore $||\langle\eta\rangle||_\infty = 1$ and η satisfies the conclusions of Theorem 4.1.

Suppose that Y is conformally equivalent to

$$A_{e^{-R},e^R} = \{z \in \mathbb{C}, \quad e^{-R} < |z| < e^R\}.$$

The hyperbolic metric on this annulus is

$$ds = \frac{\pi}{2R|z|\cos(\pi\frac{\log|z|}{2R})}|dz|.$$

Hence the 1-form

$$\eta = \frac{-i\pi}{2Rz}\tan(\pi\frac{\log|z|}{2R})dz$$

satisfies $\overline{\partial}\eta = dv$. Moreover for all $z \in A_{e^{-R},e^R}$, we have

$$\langle\eta\rangle(z) = \sin(\pi\frac{\log|z|}{2R}) \leq 1.$$

Therefore η is the required differential. We note also that $||\langle\eta\rangle||_\infty = 1$ is independent of the systole of Y.

In the other cases, since Y has no cusps, the Nielsen core of Y is a compact surface Y_0 with geodesic boundary. The surface Y equals the union of Y_0 and a collection of half-infinite annuli. For an open Riemann surface such Y a lower bound on the systole does not guarantee that Y remains in a compact set of metrics, since the length of ∂Y_0 could tend to infinity. For instance, imagine a pair of pants tending closer and closer to a bikini. The strategy to prove Theorem 4.1 is to use the explicit solution constructed above in the half-infinite annuli which are components of $Y - Y_0$ and then to extend it over Y_0 using the Green's function on Y.

The Green's function on a Riemann surface X.

Definition. — Let X be a Riemann surface. *A Green's function of* X is a positive function $G(.,.)$ on $X \times X$ – diagonal which satisfies:

(i) $\Delta G(.,y) = \delta_y$, i.e. for x not equal to y, the function $x \to G(x,y)$ is harmonic and in a holomorphic chart around y, the function

$$x \to G(x,y) + \frac{1}{2\pi} \log |x-y|$$

is harmonic, and

(ii) G is minimal among all positive functions satisfying (iii).

In $\mathbb{D}^2$, the Green's function equals

$$-\frac{1}{2\pi} \log \tanh(\frac{d(x,y)}{2}),$$

where d is the hyperbolic distance. But there does not always exist a Green's function on a given Riemann surface (cf. **[Ah3]**, **[Nic]**, **[Ts]**). For instance on a hyperbolic Riemann surface of finite volume, it does not. But, when a Green's function exists, it is unique. Let Y be an open Riemann surface isomorphic to the quotient of $\mathbb{D}^2$ by a Fuchsian group Γ. Consider the positive function on $\mathbb{D}^2 \times \mathbb{D}^2$ – diagonal

$$\text{(1)} \qquad \widetilde{G}(x,y) = -\frac{1}{2\pi} \sum_{\gamma \in \Gamma} \log \tanh(\frac{d(x,\gamma(y))}{2}).$$

This series is invariant under Γ and therefore induces a function G on $Y \times Y$ – diagonal.

Lemma 4.2.— *Let $Y = \mathbb{D}^2/\Gamma$ be an open Riemann surface. Then for all $\alpha > 0$, the series* (1) *converges uniformly for all pairs (x,y) such that $d(x,\Gamma(y)) \geq \alpha$. The function G is the Green's function of Y.*

Proof.— For $d(x,\gamma(y)) \geq \alpha$, we have:

$$-\log \tanh(\frac{d(x,\gamma(y))}{2}) \leq C(\alpha) e^{-d(x,\gamma(y))}$$

for some constant $C(\alpha)$. By applying the triangle inequality, we see that the convergence of $\sum_{\gamma \in \Gamma} e^{-d(0,\gamma(0))}$ implies the convergence of the series (1). From the formula of the hyperbolic metric in $\mathbb{D}^2$, we obtain that the general term in the second series is equivalent to $(1 - |\gamma(0)|^2)$. A direct computation based on the fact that the Möbius transformations preserve the cross-ratio implies that for any $\theta \in \partial \mathbb{D}^2$ we have (cf. **[Nic]**)

$$|(\gamma^{-1})'(\theta)| = (1 - |\gamma(0)|^2)|\theta - \gamma(0)|^{-2}.$$

Since Y is an open Riemann surface, $\Omega(\Gamma) \neq \emptyset$. Let $I \subset \Omega(\Gamma)$ be a small interval which is disjoint from all its translates by non-trivial elements of Γ. The total euclidean length of the union of the translates of I is less than the length of the circle. Therefore

$$\sum_{\gamma \in \Gamma} \int_I |\gamma'(\theta)| d\theta \leq 2\pi.$$

Since $|\gamma'(\theta)| \geq 1/4(1 - |\gamma^{-1}(0)|^2)$, the convergence of the series (1) follows.

The function G defined by (1) is positive and satisfies property (i) of the Green's function. To check that it satisfies (ii) also, it suffices to prove that for any $y \in Y$, $G(x,y)$ tends to 0 as x tends to ∞ in Y. Let Y_0 be the Nielsen core of Y. Then Y_0 is a compact surface with geodesic boundary and its preimage $\widetilde{Y}_0$ in $\mathbb{D}^2$ is a closed convex set. Saying that x tends to ∞ in Y means that the distance $d(x,Y_0)$ tends to ∞. Let $\widetilde{x} \in \mathbb{D}^2$ be any point in the preimage of x. Denote by $\widetilde{x}_0$ the nearest point projection of $\widetilde{x}$ on $\widetilde{Y}_0$. Let $y \in Y$ and let $\widetilde{y} \in \mathbb{D}^2$ be a point in its preimage. Then an elementary distance estimate gives, for all $\gamma \in \Gamma$,

$$|d(\widetilde{x},\gamma(\widetilde{y})) - d(\widetilde{x},\widetilde{x}_0) - d(\widetilde{x}_0,\gamma(\widetilde{y}))| \leq c,$$

for a constant c independent of x and of γ. Therefore by the proof of (i), $G(x,y)$ tends to 0 as $d(x,Y_0) = d(\widetilde{x},\widetilde{x}_0)$ tends to ∞. □

The surface Y is the union along the boundary of Y_0 and a finite number of half-infinite annuli called $A'_1,\cdots,A'_k,\cdots,A'_\ell$. Denote by γ_k the geodesic $\partial Y_0 \cap A'_k$. Consider the covering $\pi_k : A_k \to Y$ with fundamental group equal to that of γ_k. The annulus A_k can be identified conformally with $A_{e^{-R_k},e^{R_k}}$, where R_k satisfies $\ell(\gamma_k) = \pi^2/R_k$ (cf. §1). In particular, the existence of a lower bound on $\ell(\gamma_k)$ is equivalent to the existence of an upper bound on R_k. The restriction of π_k to one of the two halves of A_k bounded by the unit circle is an embedding. Up to applying an automorphism of A_k we can assume that the identification of A_k with $A_{e^{-R_k},e^{R_k}}$ is such that π_k restricts to the "inner" half-annulus $A_{e^{-R_k},1}$ as an embedding. In this way A'_k gets identified with $A_{e^{-R_k},1}$.

Let η_k denote the restriction to A'_k of the 1-form constructed above that solves Theorem 4.1 for A_k. Choose a decreasing smooth function λ on $[-1,1]$ such that

(i) $\lambda(x) = 1$ for $x \leq -1/2$, and

(ii) $\lambda(x) = 0$ for $x \geq 0$.

Consider the 1-form η' on Y which vanishes on Y_0 and which is defined on A'_k by $\eta'|A'_k = \lambda_k\eta_k$, where

$$\lambda_k(z) = \lambda(\frac{\log|z|}{R_k}).$$

The 1-form η' is smooth and satisfies $\bar{\partial}\eta' = dv$ on the union of the annuli $A_{e^{-R_k},e^{-R_k/2}}$. Thus the 2-form

$$dv_0 = dv - \bar{\partial}\eta' = (1 - \sum \lambda_k)dv - \sum \bar{\partial}\lambda_k \wedge \eta_k$$

has compact support in Y. Therefore we can define a function h on Y by

$$h(z) = 4\int_Y G(z,w)dv_0(w). \tag{2}$$

The 2-form $\bar{\partial}\partial h$ equals dv_0. Therefore, $\eta = \eta' + \partial h$ is the required 1-form if we can show that its hyperbolic norm is bounded by a function of ε and $\chi(Y)$. Since the hyperbolic norm of η' on Y is easily seen to be bounded only in terms of λ, it suffices to study the L^∞-norm of $\langle\partial h\rangle$.

Lemma 4.3.— *There exist positive constants C_1 and C_2 such that if u is a C^2 function on $\mathbb{D}^2$, one has*

$$||\langle \partial u\rangle||_\infty \leq C_1||u||_\infty + C_2||\langle \bar\partial\partial u\rangle||_\infty.$$

Proof.— Using the invariance of $\langle . \rangle$ under conformal automorphisms of $\mathbb{D}^2$, it suffices to prove that, for appropriate constants C_1 and C_2, the right-hand side is bigger than $\langle \partial u\rangle(0)$. For z contained in the disc $1/2\mathbb{D}^2$ we have

$$|\Delta u| \leq \frac{4}{(1-|z|^2)^2}||\langle \bar\partial\partial u\rangle||_\infty \leq C||\langle \bar\partial\partial u\rangle||_\infty.$$

Green's formula on $1/2\mathbb{D}^2$ gives

$$u(z) = \int_{1/2\mathbb{D}^2} G(z,w)\Delta u(w)dw + \int_{|w|=1/2} \frac{\partial G}{\partial \nu}(z,w)u(w)dw.$$

where G denotes the Green's function on $1/2\mathbb{D}^2$. By differentiating this expression twice, we obtain Lemma 4.3. □

By Lemma 4.3 (applied to the lift of h to $\mathbb{D}^2$), in order to obtain the required bound for $||\langle \partial h\rangle||_\infty$, it suffices to bound $||\langle \bar\partial\partial h\rangle||_\infty$ and $||h||_\infty$. In fact it is easy to bound $||\langle \bar\partial\partial h\rangle||_\infty$. The computation of $\bar\partial\lambda_k$ shows $||\langle \bar\partial\lambda_k \wedge \eta_k\rangle||_\infty \leq c$, where c depends only on λ. It follows that $||\langle dv_0\rangle||_\infty \leq c+1$.

In order to estimate $||h||_\infty$, we split the formula (2) into the sum of three integrals:

$$(3)\quad \frac{1}{4}h(z) = \sum_k \int_{A'_k} G(z,w)(1-\lambda_k(w))dv(w)$$
$$-\sum_k \int_{A'_k} G(z,w)\bar\partial\lambda_k \wedge \eta_k + \int_{Y_0} G(z,w)dv(w).$$

To get a bound on each of these terms we use the following estimate on the circular averages of the lift of G to A_k (cf. **[Di]**).

Lemma 4.4.— *Let $\widetilde{G}_k$ be the lift of G to A_k. Let $\widetilde{z} \in A_k$. Then, for all r such that $-R_k < \log r < R_k$, we have*

$$\frac{1}{2\pi}\int \widetilde{G}_k(\widetilde{z}, re^{i\theta})d\theta \leq \alpha_k(\log r + R_k),$$

where the α_k are positive constants which satisfy $\sum \alpha_k = 1$.

Proof.— By definition $\widetilde{G}_k(\widetilde{z},\widetilde{w}) = G(\pi_k(\widetilde{z}), \pi_k(\widetilde{w}))$. Therefore $\widetilde{G}_k(\widetilde{z},\widetilde{w})$ is a harmonic function of $\widetilde{w}$ in the complement of the preimage of $\pi_k(\widetilde{z})$. This preimage is a discrete set in A_k which contains at most one point in A'_k.

Let u be a harmonic function on a circular annulus. It is well-known that the average of u over the circle of radius r is an affine function of $\log r$ (**[Ah3]**, p. 164). Let $\widetilde{z} \in A_k$ and set

$$\psi_k(r) = \frac{1}{2\pi}\int \widetilde{G}_k(\widetilde{z}, re^{i\theta})d\theta.$$

The minimality property of the Green's function implies that, for any $z \in Y$, $G(z,w)$ tends to 0 as w tends to ∞ in Y. Therefore since A'_k embeds under the covering $\pi_k : A_k \to Y$, $\widetilde{G}_k(\widetilde{z},\widetilde{w})$ tends to 0 as $\log|\widetilde{w}|$ tends to $-R_k$. Thus $\psi_k(r)$ tends to 0 when $\log r$ tends to $-R_k$. Set $z = \pi_k(\widetilde{z})$ and let $\widetilde{z}_k$ be the lift of z to A_k which has the smallest modulus. Then we have

$$\psi_k(r) = \alpha_k(\log r + R_k) \quad \text{for} \quad -R_k < \log r < \log|\widetilde{z}_k|.$$

Since $\widetilde{G}_k(\widetilde{z},.)$ is positive on A_k, we have $\alpha_k > 0$. There is an interpretation of α_k. Consider the 1-form $*d\widetilde{G}_k(\widetilde{z},.)$, which is the "complex conjugate" of $d\widetilde{G}_k(\widetilde{z},.)$ ([**Ah3**] p. 164). Then α_k is the period of $*d\widetilde{G}_k(\widetilde{z},.)$ along a circle of radius smaller than $|\widetilde{z}_k|$. The union of the projections to Y of those circles for all the annuli A_k is null-homologous on Y. In particular the sum of the periods of $*d\widetilde{G}_k(\widetilde{z},.)$ equals the period of $*dG(z,.)$ on a small circle around the single singularity of $G(z,.)$, i.e. the point z. In a conformal chart around z, $G(z,w)$ is equivalent to $-\frac{1}{2\pi}\log|z-w|$. Therefore this period is equal to 1. So we have

$$\sum \alpha_k = 1.$$

In particular each α_k is less than 1. Consider the change of ψ_k when $\log r$ crosses the value $\log|\widetilde{z}|$, corresponding to the first singularity of $\widetilde{G}_k(\widetilde{z},.)$. It is a continuous function of r. Just after crossing the circle of radius $|\widetilde{z}|$ it becomes again an affine function of $\log r$. The slope of this new function equals the period of $*d\widetilde{G}_k(\widetilde{z},.)$ along a circle of radius slightly bigger than $\log|\widetilde{z}|$. The same homological argument as before implies that this new period is equal to $\alpha'_k = \alpha_k - \nu$, where ν is the number of points of smallest modulus in $\pi_k^{-1}(z)$. Hence $\alpha'_k \leq 0$. This argument can be used across the entire annulus A_k, showing that ψ_k is a piecewise affine function of $\log r$. The slope of ψ_k decreases by a positive integer each time the circle of radius $\log r$ contains a point in the preimage of z. This completes the proof of Lemma 4.4. □

Let us go back to the estimate of each term in (3). On the annulus $\{w \in A_k | -R_k/2 < \log|w| < R_k/2\}$, the hyperbolic volume element dv is bounded by

$$\frac{C}{R_k^2} dr d\theta$$

where C is a constant depending only on an upper bound on R_k, or equivalently on a lower bound on ε. Hence, each integral in the first term of (3) is dominated by

$$\frac{C}{R_k^2}\int_{e^{-R_k/2}}^{1} \widetilde{G}_k(\widetilde{z}, re^{i\theta}) dr d\theta, \tag{4}$$

where $\widetilde{z}$ is any point in the preimage of z. The estimate $\frac{1}{2\pi}\int \widetilde{G}_k(\widetilde{z}, re^{i\theta}) d\theta \leq \log r + R_k$ (Lemma 4.4) implies that (4) is bounded above only in terms of R_k. We remarked earlier that there exists a constant c depending only on λ, such that $||\langle\bar{\partial}\lambda_k \wedge \eta_k\rangle||_\infty \leq c$. Therefore, by the same argument, the second term of (3) is bounded from above. However the last term of (3) is of different nature. It is an integral over the Nielsen core Y_0. The following lemma allows us to replace it by an integral over a certain annulus.

Lemma 4.5.— *Let $\varepsilon > 0$. Let Y be an open Riemann surface whose systole is bigger than ε. Then, there exists a constant D which depends only on ε and on $\chi(Y)$ such that*

(i) *any point of Y_0 is at distance smaller than D from the longest component, say γ_0, of ∂Y_0;*

(ii) *for $\rho_0 = \frac{2R_0}{\pi}\sin^{-1}(\tanh D)$, the restriction of the covering map $\pi_0 : A_0 \to Y$ to $A_{1,e^{\rho_0}}$ is onto.*

Proof. — (i) follows essentially from the fact that the area of Y_0 is finite, depending only on $\chi(Y_0)$. Another way to think about this result is by considering the Margulis decomposition of the compact hyperbolic surface obtained by doubling Y_0 along its boundary. (ii) is a restatement of (i) using the formula for the metric in A_0. □

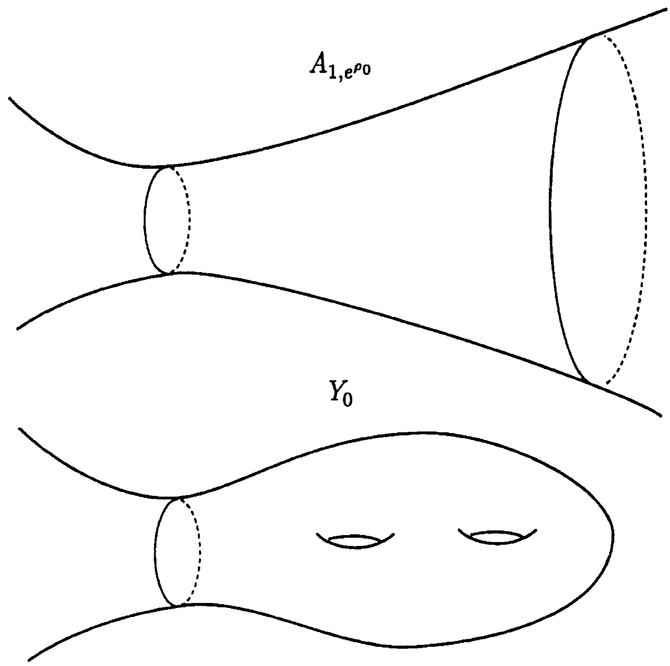

Figure 4.1

Using Lemma 4.5, the positivity of $\widetilde{G}_0$ and the fact that π_0 is a local isometry, we obtain

$$\int_{Y_0} G(z,w)dv(w) \leq \int_{A_{1,e^{\rho_0}}} \widetilde{G}_0(\widetilde{z},\widetilde{w})dv(\widetilde{w}),$$

for any point $\widetilde{z}$ in the preimage of z. On $A_{1,e^{\rho_0}}$ the hyperbolic volume element dv is bounded by $C/R_0^2 dr d\theta$ where C depends only on D, and therefore only on ε and $\chi(Y)$. So by Lemma 4.4 we have

$$\int_{A_{1,e^{\rho_0}}} \widetilde{G}(\widetilde{z},\widetilde{w})dv(\widetilde{w}) \leq \frac{C'}{R_0^2}\int_1^{e^{\rho_0}} (\log r + R_0)dr,$$

where C' depends only on ε and on $\chi(Y)$. An easy computation using the expression of ρ_0 gives that the integral is bounded from above in terms of R_0. This concludes the proof of Theorem 4.1. □

A primitive to the volume form in presence of short geodesics.

In the next chapter we will need a refinement of Theorem 4.1, which will allow us to deal with the case when Y contains geodesics shorter than the Margulis constant $\varepsilon(2)$. For $\varepsilon \leq \varepsilon(2)$, Y is the union of $Y^{]0,\varepsilon]}$ and $Y^{[\varepsilon,\infty[}$. Rather than finding solutions on Y to $\bar{\partial}\eta = dv$ which satisfy explicit bounds, we instead solve this equation on $Y^{[\varepsilon,\infty[}$. In fact, for the later applications, we only need to solve it on *the unbounded components of* $Y^{[\varepsilon,\infty[}$.

Definition. — Let Y be an open Riemann surface. *A component of* $Y^{[\varepsilon,\infty[}$ *is unbounded* if it is not compact (cf. Figure 4.2). Equivalently, a component of $Y^{[\varepsilon,\infty[}$ is unbounded if it is not entirely contained in Y_0.

Theorem 4.6. — *Let* $0 < \varepsilon \leq \varepsilon(2)$. *Let* Y *be an open Riemann surface. Then there exists a constant* C *depending only on* ε *and on* $\chi(Y)$, *such that on any unbounded component of* $Y^{[\varepsilon,\infty[}$ *there exists a 1-form* η *of type* $(1,0)$ *which satisfies*

(i) $\bar{\partial}\eta = dv$, *and*

(ii) $||\langle\eta\rangle||_\infty \leq C$.

Proof. — When Y is homeomorphic to a disc or to an annulus, the explicit solution described at the beginning of the proof of Theorem 4.1 has hyperbolic norm less than 1, independently of the length of the core geodesic.

In the other cases, let Y' be a component of $Y^{[\varepsilon,\infty[}$ which is unbounded. Suppose first that Y' is contained in a component of $Y - Y_0$, say A'_k. The 1-form constructed on A_k has norm less than 1. Since A'_k is isometrically embedded in A_k, the restriction of this form to Y' is a solution.

If Y' intersects Y_0, then each component of ∂Y_0 which intersects Y' is contained in Y' and so its length is bigger than ε. The surface Y' is the union of the compact surface $Y' \cap Y_0$ and a non-empty collection of half-infinite annuli, which are components of $Y - Y_0$. The proof of Theorem 4.6 follows exactly the same lines as that of Theorem 4.1. The restrictions of the 1-forms constructed on the annuli in $Y' - Y_0$ can be modified near the boundary of Y_0 to give a form η'. The 2-form $dv_0 = dv - \bar{\partial}\eta'$ has compact support in Y' so that we can define a function on Y' by the formula: $h(z) = 4\int_{Y'} G(z,w)dv(w)$, where G is the Green's function on Y. The proof that the form $\eta = \eta' + \partial h$ is the 1-form required in the theorem reduces to showing that h is bounded over Y'. The only difference between this proof and that of Theorem 4.1 appears in the estimate of the third integral in (3). The following lemma substitutes Lemma 4.5.

Lemma 4.7. — *Let* $\varepsilon > 0$. *Let* Y *be an open Riemann surface. Let* Y' *be a component of* $Y^{[\varepsilon,\infty[}$ *which intersects* Y_0. *Then there exists a constant* D' *which depends only on* ε *and on* $\chi(Y)$ *such that any point of* $Y' \cap Y_0$ *is at distance less than* D' *from the longest component of* ∂Y_0 *which is contained in* Y'.

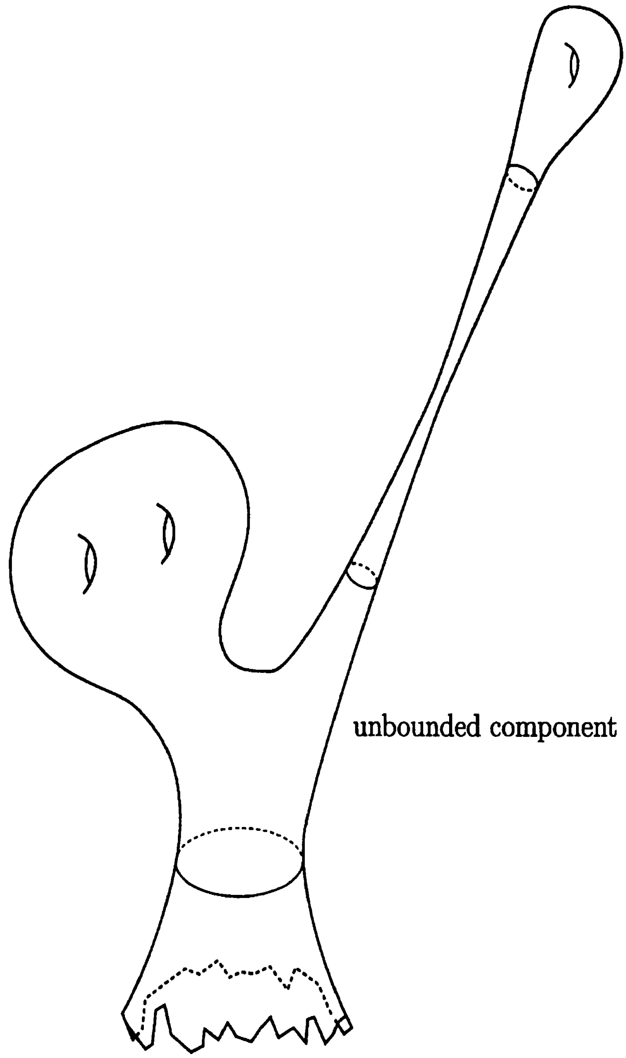

Figure 4.2

This concludes the proof of Theorem 4.6. □

CHAPITRE 5

Contraction properties of the Theta operator

Theorem 5.1. — *Let $\varepsilon > 0$. Let X be a compact hyperbolic Riemann surface whose systole is bigger than ε. Let $\pi : Y \to X$ be a geometric cover of X. There exists a constant $c = c(\varepsilon, \chi(X)) > 0$ such that $\|\Theta_{Y/X}\| \leq 1 - c$.*

This theorem is a special case of a theorem of McMullen [**McM1**]. Here the hypothesis "geometric" has replaced the original hypothesis "non-amenable". We present the proof that Barett and Diller have given [**BD**] using the result of [**Di**] that was described in the previous chapter. Theorem 5.1 remains true when X has finite volume. It can be proved using the same techniques.

Averaging operators.

We need to extend the averaging procedure of Poincaré series to other types of tensors. For instance, if F is a function on Y we can sum it (in certain cases) over the sheets of the cover $\pi : Y \to X$ by the formula

$$\theta F(x) = \sum_{y \in \pi^{-1}(x)} F(y).$$

The Cauchy formula implies that this series converges uniformly when F is holomorphic and integrable with respect to the hyperbolic volume on Y. More generally, this will hold when F is not necessarly holomorphic, but has modulus less than that of an integrable holomorphic function. A special case is the following. For $\phi \in \Omega(Y)$, we defined in §3 a function $\langle\phi\rangle$: it is the density of the measure $|\phi(y)||dy^2|$ with respect to the hyperbolic volume measure on Y. Then $\theta\langle\phi\rangle$ can be defined using the formula above and $\theta\langle\phi\rangle(x)$ is the density with respect to the hyperbolic volume on X of a measure which has the same total mass as the measure $|\phi(y)||dy^2|$ on Y.

The same averaging procedure can be applied to 1-forms. Let η be a 1-form of type $(1,0)$ on Y. Let U be a conformal chart around a point $x \in X$. For each component U_i of $\pi^{-1}(U)$, let $s_i : U \to U_i$ be a holomorphic section of π. On U_i, η can be written as $\eta_i(z)dz$. Then the series

$$\sum_i \eta_i(s_i(z))s_i'(z)dz$$

defines, in certain cases, a 1-form of type $(1,0)$ on U. This happens for instance when the hyperbolic norm $\langle\eta\rangle(y)$ is bounded by the modulus of an holomorphic function which is integrable on Y with respect to the hyperbolic volume. These local expressions paste together defining a 1-form on Y denoted by $\theta\eta$. In the next paragraphs we will commute the exterior differentiation with the averaging operator θ. This will be allowed since the series that we consider converge uniformly.

Proof of Theorem 5.1.— We denote by Θ the classical operator $\Theta_{Y/X}$ acting on integrable holomorphic quadratic differentials, and by θ the averaging operator introduced above, acting on other types of tensors. Since the cover $Y \to X$ is geometric, Y is an open Riemann surface and the results of §4 can be applied.

Let $\phi \in \mathcal{Q}(Y)$ with $||\phi|| = 1$. The pull-back $\pi^*(\Theta\phi)$ of $\Theta\phi$ to Y is a holomorphic quadratic differential (which, if not identically 0, is not integrable). Its zero set is exactly the preimage of the zero set Z of $\Theta\phi$. Assuming that $\Theta\phi$ does not vanish identically, we define a meromorphic function F on Y by $\phi = F\pi^*(\Theta\phi)$. Let $Z(r)$ denote the neighborhood of radius r of Z on X, and by $\widetilde{Z}(r)$ the preimage of $Z(r)$ in Y.

Lemma 5.2.— *For any $r > 0$, F is integrable in $Y - \widetilde{Z}(r)$ with respect to the hyperbolic volume.*

Proof.— Let $m(r) > 0$ be a lower bound for $\langle\Theta\phi\rangle(x)$ on $X - Z(r)$. On $Y - \widetilde{Z}(r)$, we have

$$|F(y)| \leq \frac{\langle\phi\rangle(y)}{m(r)}.$$

Hence, $\int_{Y-\widetilde{Z}(r)} |F|dv \leq 1/m(r)$. □

This observation means that we can define θF and $\theta|F|$ on $X - Z$. Since $\theta F \equiv 1$, $\theta|F| \geq 1$. As $||\phi|| = 1$, the integral of $\theta\langle\phi\rangle$ with respect to the hyperbolic volume is 1. Thus, we have

$$\begin{aligned} 1 - ||\Theta\phi|| &= \int_X \theta\langle\phi\rangle dv - \int_X |\Theta\phi| \\ &= \int_X (\theta|F| - 1)\langle\Theta\phi\rangle dv \\ &\geq \int_0^{r_0} m(r)(\int_{\partial Z(r)} \theta(|F|ds) - \ell(\partial Z(r)))dr. \end{aligned}$$

In the last line, we have bounded from below the integral over X of the positive function $(\theta|F| - 1)\langle\Theta\phi\rangle$ by its integral over $Z(r_0)$ for a sufficiently small constant r_0 which will be fixed later.

Let η be the 1-form on Y constructed in Theorem 4.1. Its hyperbolic norm is less than C. In particular, $\theta(F\eta)$ is well defined since F is integrable on Y. We have

$$\begin{aligned}
\int_{\partial Z(r)} \theta(|F|ds) &\geq \frac{1}{C}\left|\int_{\partial Z(r)} \theta(F\eta)\right| \\
&= \frac{1}{C}\left|\int_{X-Z(r)} \bar{\partial}\theta(F\eta)\right| \quad \text{by Stokes formula} \\
&= \frac{1}{C}\left|\int_{X-Z(r)} \theta(Fdv)\right| \quad \text{since } F \text{ is holomorphic} \\
&= \frac{1}{C}\left|\int_{X-Z(r)} dv\right| \\
&= \frac{1}{C}\text{Area}(X - Z(r)) \\
&= \frac{1}{C}(2\pi|\chi(X)| - \text{Area}(Z(r))).
\end{aligned}$$

The holomorphic quadratic differential $\Theta\phi$ has at most $4g-4$ zeroes, where g is the genus X. Hence, for $r \leq \varepsilon/2$, the area of $Z(r)$ is less than $c_1 r^2$ and the length of $\partial Z(r)$ is less than $c_2 r$ for constants c_1 and c_2 depending only on $\chi(X)$. It follows that for all r sufficiently small in terms of ε, $\int_{\partial Z_r} \theta(|F|ds) - \ell(\partial Z_r) \geq c$, where $c > 0$ depends only on ε and on $\chi(X)$.

If $||\Theta\phi|| \geq 1/2$, we know from Proposition 3.7 that $m(r)$ is bounded from below, independently of $\Theta\phi$, by a positive constant depending only on r, ε and $\chi(X)$. Recall that the constant C provided by Theorem 4.1 depends only on ε and $\chi(Y)$. Therefore, if $||\Theta\phi|| \geq 1/2$, $1 - ||\Theta\phi||$ is bounded from below by a constant which depends only on ε and $\chi(X)$.

When $||\Theta\phi|| \leq 1/2$, we obviously have $1 - ||\Theta\phi|| \geq 1/2$.

This concludes the proof of Theorem 5.1. □

Remark. — Geometric covers of a Riemann surface are examples of *non-amenable covers* , which were defined by McMullen's. Without going into the precise definition, let us just say that an example of amenable cover is given by a Galois cover whose automorphism group is amenable. The basic result of **[McM1]** is that the Theta operator associated to a non-amenable cover is strictly contracting. There is a link between McMullen's proof and the one of Barrett and Diller. The basic tool for the proof of Theorem 5.1 was the existence of a $\bar{\partial}$-primitive to the volume form, whose hyperbolic norm is bounded from above. As we noticed, this implied a linear isoperimetric inequality on Y. This is reminiscent of Fölner's criterium which characterizes non-amenable graphs as those which satisfy a linear isoperimetric inequality (cf. **[Gr]**).

Remark. — McMullen proved also a converse to his theorem, namely that the Theta operator associated to *an amenable cover* $Y \to X$, has norm 1 **[McM1]**: i.e. there is a sequence of elements ϕ_i in $\Omega(Y)$ with $||\phi_i|| = 1$ such that $||\Theta\phi_i||$ tends to

1. However, one should notice that, for any infinite cover $\pi : Y \to X$ there does not exist a $\phi \in \mathcal{Q}(Y)$ such that $||\phi|| = 1$ and $||\Theta\phi|| = 1$. Let us justify this observation in the case of a Galois cover. This won't never be used as such in the sequel, but is behind the argument that we will use for proving Proposition 6.3. If, for some $\phi \in \mathcal{Q}(Y)$, $||\phi|| = ||\Theta\phi|| = 1$, then there exists $\mu \in \mathcal{B}(X)$ with $||\mu|| = 1$ such that $\langle\Theta\phi, \mu\rangle = \langle\phi, \pi^*\mu\rangle = 1$ (cf. §1). It follows easily from the definition of the pairing that

$$\pi^*\mu = \frac{\overline{\phi}}{|\phi|}. \tag{1}$$

Let g be an element of the automorphism group of the cover $Y \to X$. As $\pi^*\mu$ is invariant under g, (1) implies $\phi(gz)g'(z)^2 = k\phi(z)$ for a constant $k \neq 0$. If g has infinite order, this equality contradicts that $||\phi||$ is finite.

The norm of $\Theta_{Y/X}$ in presence of short geodesics.

Let $0 < \varepsilon \leq \varepsilon(2)$. The surface X can be decomposed as the union of the ε-thin part $X^{]0,\varepsilon]}$ and the ε-thick part $X^{[\varepsilon,\infty[}$. Recall that the cover $\pi : Y \to X$ we consider is associated to a proper incompressible surface $S \subset X$.

Definition.— A component of $X^{]0,\varepsilon]}$ or of $X^{[\varepsilon,\infty[}$ is *liftable* if it can be isotoped into S.

Let Z be a component of $X^{]0,\varepsilon]}$ or of $X^{[\varepsilon,\infty[}$. If Z is liftable, $\pi^{-1}(Z)$ consists of a single isomorphic copy of itself, which is called *the lift of* Z, and a disjoint union of copies of the universal cover of Z. Note that the latter are necessarily contained in the unbounded components of $Y^{[\varepsilon,\infty[}$. If Z is not liftable, any component of $\pi^{-1}(Z)$ is an infinite cover of Z. Therefore, it is entirely contained in an unbounded component of $Y^{[\varepsilon,\infty[}$. So the only components of the preimage of $X^{]0,\varepsilon]}$ or of $X^{[\varepsilon,\infty[}$ which are compact are the lifts of the liftable components. All the others are contained in the unbounded components of $Y^{[\varepsilon,\infty[}$.

Definition.— *The ε-amenable part of the cover* $Y \to X$ is the union of the total preimage of $X^{]0,\varepsilon]}$ and of the lifts of the liftable components of $X^{[\varepsilon,\infty[}$. We denote it by $\mathcal{A}(X)^\varepsilon$.

The following result explains how the presence of short geodesics does influence the behaviour of of $||\Theta_{Y/X}||$.

Theorem 5.3.— *Let $\pi : Y \to X$ be a geometric cover of Riemann surfaces. Let $\varepsilon \leq \varepsilon(2)$. Let $\eta > 0$. There exists a $\delta > 0$ which depends only on η, ε and on $\chi(X)$, such that: for all $\phi \in \mathcal{Q}(Y)$ with $||\phi|| = 1$ and $||\Theta_{Y/X}\phi|| \geq 1 - \delta$, then the ϕ-mass of $\mathcal{A}(X)^\varepsilon$ is more than $1 - \eta$.*

Proof.— We argue by contradiction. Then, there is a sequence of isomorphic covers $Y_n \to X_n$, and $\phi_n \in \mathcal{Q}(Y_n)$ with unit norm, such that $||\Theta_{Y_n/X_n}\phi_n||$ tends to 1 and that the ϕ_n-mass of $\mathcal{A}(X_n)^\varepsilon$ is bounded away from 1. Observe that the topological type of a geometric cover associated to an incompressible surface $S \subset X$ only depends on the embedding of S into X; therefore we can assume that the covers $Y_n \to X_n$ have the same topological type. Denote Θ_{Y_n/X_n} by Θ_n. Since the number of components of $X_n^{[\varepsilon,\infty[}$ is bounded independently of n, there is some component K_n of $X_n^{[\varepsilon,\infty[}$ for which the ϕ_n-mass of the total union of the non-compact components of $\pi^{-1}(K_n)$ admits a lower bound $m > 0$. Let W_n denote the total union of the non-compact components of $\pi^{-1}(K_n)$. Let $x_n \in K_n$. Choose a uniformizing map $(\mathbb{D}^2, 0) \to (X_n, x_n)$. We can suppose, up to extracting a subsequence, that (X_n, x_n) converges to (K, x) where K is a hyperbolic surface of finite area, since $\mathrm{inj}(x_n) \geq \varepsilon$.

If K were compact, the injectivity radius of X_n would be bounded from below independently of n. Then, Theorem 5.1 would contradict the hypothesis of Theorem 5.3.

Therefore K is non-compact. Fix a positive number $\mu \leq \varepsilon$ smaller than the systole of K. For $\mu' \leq \mu$, let $K_n^{[\mu',\infty[}$ be the component of $X_n^{[\mu',\infty[}$ which contains K_n and let $W_n^{[\mu',\infty[}$ be the union of the non-compact components of $\pi^{-1}(K_n^{[\mu',\infty[})$. It is important to notice that $K_n^{[\mu',\infty[}$ is not necessarily diffeomorphic to K_n. However, since μ is smaller than the systole of K, for all $\mu' \leq \mu$, $K^{[\mu',\infty[}$ is diffeomorphic to $K^{[\mu,\infty[}$ and $K_n^{[\mu',\infty[}$ converges therefore to $K^{[\mu',\infty[}$.

In order to obtain a contradiction, we distinguish two cases according as $K_n^{[\mu,\infty[}$ does or does not lift to Y_n for sufficiently large n. When $K_n^{[\mu,\infty[}$ lifts, we will show that the ϕ_n-mass of $W_n^{[\mu,\infty[}$ tends to 0 when n tends to ∞. Since $W_n \subset W_n^{[\mu,\infty[}$, the ϕ_n-mass of W_n tends to 0 also. This contradicts our assumption that the ϕ_n-mass of W_n is bigger than m. When $K_n^{[\mu,\infty[}$ does not lift, we will contradict the hypothesis $||\Theta_n\phi_n|| \to 1$.

Let $0 < \mu' \leq \mu$ be a constant that will be fixed precisely later. As discussed already, we can associate to the cover $W_n^{[\mu',\infty[} \to K_n^{[\mu',\infty[}$ an averaging operator, which operates on integrable holomorphic quadratic differentials, and tensors of other types. We denote this operator by Θ'_n, when it is considered as acting on integrable holomorphic quadratic differentials and by θ'_n, when acting on tensors of other types.

1) $K_n^{[\mu,\infty[}$ *lifts to* Y_n *for sufficiently large* n.

Since $K_n^{[\mu',\infty[}$ is isotopic to $K_n^{[\mu,\infty[}$, it is liftable to Y_n also. The preimage of $K_n^{[\mu',\infty[}$ is the disjoint union of $W_n^{[\mu',\infty[}$ and the isomorphic lift $\widetilde{K}_n^{[\mu',\infty[}$ of $K_n^{[\mu',\infty[}$. Therefore in restriction to $K_n^{[\mu',\infty[}$, we have

$$\Theta_n\phi_n = \phi_n|\widetilde{K}_n^{[\mu',\infty[} + \Theta'_n\phi_n, \tag{1}$$

where we have identified the restriction $\phi_n|\widetilde{K}_n^{[\mu',\infty[}$ with its projection on $K_n^{[\mu',\infty[}$.

For any measurable subset $U \subset X_n$, we have $\int_U |\Theta_n \phi_n| \leq \int_{\pi^{-1}(U)} |\phi_n|$ and therefore

$$
\begin{aligned}
&1 - \|\Theta_n \phi_n\| \\
&= \int_{\pi^{-1}(K_n^{[\mu',\infty[})} |\phi_n| + \int_{Y_n - \pi^{-1}(K_n^{[\mu',\infty[})} |\phi_n| - \int_{K_n^{[\mu',\infty[}} |\Theta_n \phi_n| - \int_{X_n - K_n^{[\mu',\infty[}} |\Theta_n \phi_n| \\
&\geq \int_{\pi^{-1}(K_n^{[\mu',\infty[})} |\phi_n| - \int_{K_n^{[\mu',\infty[}} |\Theta_n \phi_n|.
\end{aligned}
$$

To obtain a contradiction, it suffices to bound from below the last expression in the above inequality. Applying the triangle inequality and (1), we find that it is bounded from below by $\int_{W_n^{[\mu',\infty[}} |\phi_n| - \int_{K_n^{[\mu',\infty[}} |\Theta'_n \phi_n|$. The hypothesis that $K_n^{[\mu,\infty[}$ lifts to Y_n has the following topological consequence.

Fact 5.4.— *Let $\mu' \leq \mu$. Then for all sufficiently large n, $W_n^{[\mu',\infty[}$ is contained in $Y_n^{[\mu,\infty[}$.*

Proof.— If this is false, some component W' of $W_n^{[\mu',\infty[}$ intersects a component Y' of $Y_n^{]0,\mu]}$. Since the cover $\pi : Y_n \to X_n$ is geometric, the restriction of π to each component of $Y_n^{]0,\mu]}$ is a homeomorphism. In particular, $W' \cap Y'$ is homeomorphic to its image by π. Since $\pi(W') = K_n^{[\mu',\infty[}$, $\pi(W' \cap Y')$ is the intersection of $K_n^{[\mu',\infty[}$ with a component of $X_n^{]0,\mu]}$. By our choice of μ, any component of $K^{]0,\mu]}$ is a cusp. Such a cusp is approximated in $X_n^{]0,\mu]}$ by an annulus whose core geodesic has length tending to 0. Therefore, for all sufficiently large n, the intersection of $K_n^{[\mu',\infty[}$ with any component of $X_n^{]0,\mu]}$ is an annulus such that the injectivity radius equals μ on one boundary component and μ' on the other. It follows that the component of $W_n^{[\mu,\infty[}$ which is contained in W' contains a boundary curve of $Y_n^{]0,\mu]}$. This is impossible because all components of $W_n^{[\mu,\infty[}$ are simply connected since $K_n^{[\mu,\infty[}$ lifts. □

Since the differentials $\Theta'_n \phi_n$ have norm less than 1, they converge uniformly to a holomorphic quadratic differential ϕ defined on $K_n^{[\mu',\infty[}$, up to possibly passing to a subsequence (cf. the remark after the statement of Theorem 3.1). But this does not guarantee that ϕ is non-zero, in contrast to Theorem 3.1. Therefore, we consider two subcases.

1a) $\phi = 0$.

The diameter of $K_n^{[\mu',\infty[}$ is bounded from above independently of n. Hence there exists a ball of fixed radius in $\mathbb{D}^2$ centered at 0 whose projection to X_n contains $K_n^{[\varepsilon',\infty[}$, for all n. Therefore the uniform convergence of $\Theta'_n \phi_n$ to 0 implies

$$\lim_{n \to \infty} \int_{K_n^{[\mu',\infty[}} |\Theta'_n \phi_n| = 0.$$

So we obtain a non-zero lower bound on $\int_{W_n^{[\mu',\infty[}} |\phi_n| - \int_{K_n^{[\mu',\infty[}} |\Theta'_n \phi_n|$ for all sufficiently large n. This contradicts the hypothesis of Theorem 5.3.

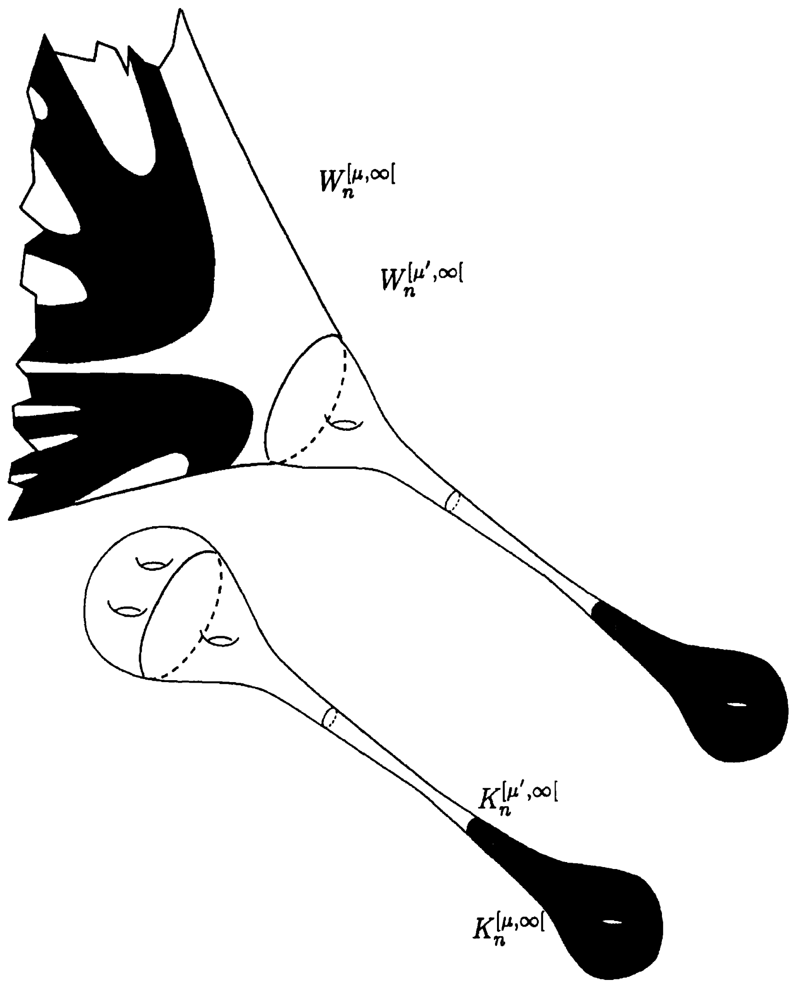

Figure 5.1

1b) $\phi \neq 0$.

The proof in this case is similar to that of Theorem 5.1. Denote by Z (resp. Z_n) the zero set of ϕ (resp. $\Theta'\phi_n$). Denote by $Z_n(r)$ the neighborhood of radius r of Z_n in $K_n^{[\mu',\infty[}$. Let $\langle\Theta'_n\phi_n\rangle$ be the hyperbolic norm of $\Theta'_n\phi_n$.

Fact 5.5. — *Let $\mu' \leq \mu$. For any $r > 0$, there is lower bound $m(r) > 0$ for $\langle\Theta'_n\phi_n\rangle$ over $K_n^{[\mu',\infty[} - Z_n(r)$ which is independent of n.*

This lower bound $m(r)$ depends on μ'.

Proof. — The proof is exactly the same as the one of Proposition 3.7: it follows directly from the convergence of $\Theta'_n\phi_n$ to ϕ. □

Let $\pi^*(\Theta'_n\phi_n)$ be the pull-back of $\Theta'_n\phi_n$ to $W_n^{[\mu',\infty[}$. The zero set of $\pi^*(\Theta'_n\phi_n)$ is the preimage $\tilde{Z}_n$ of Z_n. Let F_n be the meromorphic function on $W_n^{[\mu',\infty[}$ defined by $\phi_n = F_n \pi^*(\Theta'_n \phi_n)$. Then F_n is integrable on $W_n^{[\mu',\infty[} - \widetilde{Z}_n(r)$ (as in the proof of Lemma 5.2). Thus the averages $\theta'_n |F_n|$ and $\theta'_n F_n$ are well defined; also, $\theta'_n F_n \equiv 1$. Therefore, $\theta'_n |F_n| \geq 1$.

Suppose that $\mu' \leq \mu$ is chosen so that $\partial K^{[\mu',\infty[}$ is disjoint from Z. Then, for all n sufficiently large, $\partial K_n^{[\mu',\infty[}$ is disjoint from Z_n too. Following the proof of Theorem 5.1, we obtain

$$\int_{W_n^{[\mu',\infty[}} |\phi_n| - \int_{K_n^{[\mu',\infty[}} |\Theta'_n \phi_n| = \int_{K_n^{[\mu',\infty[}} \theta'_n \langle \phi_n \rangle dv - \int_{K_n^{[\mu',\infty[}} |\Theta'_n \phi_n|$$
$$= \int_{K_n^{[\mu',\infty[}} (\theta'_n |F_n| - 1) \langle \Theta'_n \phi_n \rangle dv$$

$$\geq \int_0^{r_0} m(r) \Big(\int_{\partial(K_n^{[\mu'+r,\infty[} - Z_n(r))} \theta'_n (|F_n| ds) - \ell(\partial(K_n^{[\mu'+r,\infty[} - Z_n(r))) \Big) dr.$$

In the formula above, $r_0 > 0$ is chosen for the moment small with respect to μ' and to the distance between Z and $\partial K^{[\mu',\infty[}$, but will be fixed more precisely later.

Consider the 1-form η_n that was constructed in Theorem 4.6 on the unbounded components of $Y_n^{[\mu,\infty[}$. Its hyperbolic norm is less than a constant C depending only on μ and on $\chi(Y)$. Since $W_n^{[\mu',\infty[} \subset Y_n^{[\mu,\infty[}$ (Fact 5.4), we obtain by imitating the argument used at the end of the proof of Theorem 5.1:

$$\int_{\partial(K_n^{[\mu'+r,\infty[} - Z_n(r))} \theta'_n (|F_n| ds) \geq \frac{1}{C} \int_{\partial(K_n^{[\mu'+r,\infty[} - Z_n(r))} \theta'_n |F_n \eta_n|$$
$$= \frac{1}{C} |\text{Area}(K_n^{[\mu'+r,\infty[} - Z_n(r))|.$$

If r_0 is sufficiently small, then for all $r \leq r_0$, the boundary of $K_n^{[\mu'+r,\infty[} - Z_n(r)$ is the disjoint union of $\partial K_n^{[\mu'+r,\infty[}$ and $\partial Z_n(r)$. When μ' and r tend to 0, the length of $\partial K_n^{[\mu'+r,\infty[}$ tends to 0. When r tends to 0, the length of $\partial Z_n(r)$ also tends to 0 . However since the cardinality of $Z_n \cap K_n^{[\mu',\infty[}$ might increase when μ' tends to 0, we first fix a choice of μ'. The hyperbolic area of $K_n^{[\mu,\infty[}$ admits a lower bound $\nu > 0$ which depends only on $\chi(X)$. Choose $\mu' \leq \mu/2$ such that $\ell(\partial K_n^{[2\mu',\infty[}) < \nu/2$ and such that $\partial K^{[\mu',\infty[} \cap Z = \emptyset$. Then, for all $r \leq \mu'$, we have

$$\text{Area}(K_n^{[\mu'+r,\infty[}) - \ell(\partial K_n^{[\mu'+r,\infty[}) > \frac{\nu}{2}.$$

With this choice of μ', the number of zeroes of $\Theta'_n \phi_n$ that are contained in $K_n^{[\mu',\infty[}$ is bounded above independently of n because it converges (counting multiplicity) to the number of zeroes of ϕ contained in $K^{[\mu',\infty[}$. Hence the area of $Z_n(r)$ and the length of its boundary are bounded from above by $c_1 r^2$ and $c_2 r$ respectively for constants c_1 and c_2 which are independent of n. Therefore, there is some $r_0 \leq \mu'$ such that, for all $r \leq r_0$, and for all sufficiently large n, we have

$$\int_{\partial(K_n^{[\mu'+r,\infty[} - Z_n(r))} \theta'_n (|F_n| ds) - \ell(\partial(K_n^{[\mu'+r,\infty[} - Z_n(r))) \geq \frac{\nu}{2C}.$$

Therefore

$$\int_0^{r_0} m(r)\Big(\int_{\partial(K_n^{[\mu'+r,\infty[}-Z_n(r))}\theta'_n(|F_n|ds)-\ell(\partial(K_n^{[\mu'+r,\infty[}-Z_n(r)))\Big)dr$$

is bounded away from 0 independently of n by Fact 5.5. This leads to a contradiction as in 1a).

2) *$K_n^{[\mu,\infty[}$ does not lift to Y_n for sufficiently large n.*

An important subcase to keep in mind occurs when Y_n is the universal cover of X_n. However this situation could be handled with the same methods as above. The main difference between the first case and second case occurs when $K_n^{[\mu,\infty[}$, although not liftable to Y_n, contains some boundary components which are liftable. Because of such curves, certain components of $W_n^{[\mu',\infty[}$ intersect $Y_n^{]0,\mu]}$ for $\mu'\leq\mu$ and Fact 5.4 is no longer true. We need to argue differently.

Since $K_n^{[\mu,\infty[}$ is not liftable, $\pi^{-1}(K_n^{[\mu,\infty[})=W_n^{[\mu,\infty[}$. Hence $\Theta_n=\Theta'_n$ and $\theta_n=\theta'_n$. Keeping the same notations as in 1), we assume that (X_n,x_n) converge to (K,x) where K is a hyperbolic surface of finite area. By Theorem 3.1, $(\Theta_n\phi_n)$ converges uniformly to a holomorphic quadratic differential ϕ on K. Since the $\Theta_n\phi_n$-mass of K_n is bounded away from 0, ϕ is non-zero (cf. 1a)). Let Z_n (resp. Z) be the zero set of $\Theta_n\phi_n$ (resp. ϕ). Let $\widetilde{Z}_n$ be the preimage of Z_n in Y_n and $\widetilde{Z}_n(r)$ be the neighborhood of radius r of $\widetilde{Z}_n$. Let F_n be the meromorphic function on Y_n defined by $\phi_n=F_n\pi^*(\Theta_n\phi_n)$. For any $r>0$, F_n is integrable on $Y_n-\widetilde{Z}_n(r)$ with respect to the hyperbolic volume. Its L^1-norm is smaller than $1/m(r)$, where $m(r)>0$ is a lower bound for $\langle\Theta_n\phi_n\rangle$ on $K_n-Z_n(r)$ (cf. Lemma 5.2).

Since Y_n covers X_n, its universal cover is naturally identified with $\mathbb{D}^2$. Let $\bar{F}_n$ be the lift of F_n to $\mathbb{D}^2$. Let $\bar{Z}$ (resp. $\bar{Z}_n$) be the preimage of Z (resp. Z_n) in $\mathbb{D}^2$.

Lemma 5.6. — *Up to extracting a subsequence, the functions $\bar{F}_n$ converge uniformly on compact subsets of $\mathbb{D}^2-\bar{Z}$ to a holomorphic function $\bar{F}$.*

Proof. — Let $\mathcal{K}\subset\mathbb{D}^2$ be a compact set. Under the covering $\mathbb{D}^2\to K$, the "degree" of the projection $\mathcal{K}\to K$ is finite, i.e. the cardinality of the preimage of any point of K which is contained in $\mathcal{K}$ is bounded, independently of that point. Hence since (X_n,x_n) tend to (K,x), the degree of the projections $\mathcal{K}\to X_n$ is bounded independently of n. The same property holds a fortiori for the projections $\mathcal{K}\to Y_n$. By the uniform convergence of $\Theta_n\phi_n$ to ϕ, $\bar{Z}_n$ converges to $\bar{Z}$ and $\bar{Z}_n(r)$ converges to $\bar{Z}(r)$ when n tends to ∞. It follows that for any $r>0$, the L^1-norm of $\bar{F}_n|\mathcal{K}-\bar{Z}(r)$ is bounded independently of n. By Cauchy formula, this implies that $\bar{F}_n$ converges uniformly to a holomorphic function $\bar{F}$ on compact sets in $\mathcal{K}-\bar{Z}$, up to passing to a subsequence. □

2a) *$\bar{F}$ is not constant.*

Since $\bar{F}$ is holomorphic and not constant, there exists a point $\bar{q}\in\mathbb{D}^2-\bar{Z}$ such that $\bar{F}(\bar{q})$ has a non-zero imaginary part. By the uniform convergence of $\bar{F}_n$ to $\bar{F}$, there exists positive numbers α, η and ρ so that for all q in the ball $B(\bar{q},\rho)$ and for all sufficiently large n, we have

(i) $|\bar{F}_n(q)| > \alpha > 0$, and

(ii) $0 < \eta < \arg \bar{F}_n(q) < \pi - \eta$.

Suppose that $\bar{q}$ is near the origin in $\mathbb{D}^2$, so that the projection B_n of $B(\bar{q}, \rho)$ on X_n is contained in $K_n^{[\mu,\infty[}$. If $\rho \leq \mu/2$, B_n is embedded for all sufficiently large n. Since $\bar{q} \notin Z$, the ball $B(\bar{q}, \rho)$ is at distance bigger than ρ from $\bar{Z}_n$ for any sufficiently small ρ and for all sufficiently large n. In particular B_n is contained in $K_n - Z_n(\rho)$. Let q_n denote the image of $\bar{q}$ in Y_n. For sufficiently large n, $B(q_n, \rho)$ maps homeomorphically to B_n, under the covering $Y_n \to X_n$.

Let x and y be complex numbers such that $x + y = 1$, $|x| \geq \alpha > 0$ and $0 < \eta \leq \arg(x) \leq \pi - \eta$. Then, we have $|x| + |y| - 1 \geq c(\eta, \alpha) > 0$. Hence, for all $z \in B(q_n, \rho)$

$$|F_n(z)| + |1 - F_n(z)| - 1 \geq c(\eta, \alpha).$$

Thus, for all sufficiently large n

$$\begin{aligned}\int_{\pi^{-1}(B_n)} |\phi_n| - \int_{B_n} |\Theta_n \phi_n| \geq & \int_{B_n} (\theta_n |F_n| - 1) \langle \Theta_n \phi_n \rangle dv \\ \geq & \int_{B(q_n, \rho)} (|F_n(z)| + |1 - F_n(z)| - 1) m(\rho) dv \\ \geq & c(\eta, \alpha) m(\rho) \int_{B_n} dv,\end{aligned}$$

where $m(\rho)$ is a lower bound of $\langle \Theta_n \phi_n \rangle$ on $K_n - Z_n(\rho)$. This is impossible as $||\Theta_n \phi_n||$ tends to 1.

2b) *$\bar{F}$ is a non-zero constant.*

Let $y_n \in Y_n$ be the image of $0 \in \mathbb{D}^2$ under the covering map $\mathbb{D}^2 \to Y_n$. Since y_n projects to x_n, $\mathrm{inj}(y_n) \geq \varepsilon$. Let $B(y_n, R)$ denote the ball in Y_n of radius R centered at y_n.

Lemma 5.7.— *For any sufficiently small positive r, the hyperbolic area of $B(y_n, R) - \widetilde{Z}_n(r)$ tends to ∞ with R, uniformly in n.*

Proof.— Since $K_n^{[\mu,\infty[}$ does not lift to Y_n, y_n is contained in an unbounded component of $Y_n^{[\mu,\infty[}$. Denote $B(y_n, R)^{[\mu,\infty[} = B(y_n, R) \cap Y_n^{[\mu,\infty[}$.

Suppose that y_n belongs to the Nielsen core Y_n' of Y_n. By Lemma 4.7, there is a geodesic $\gamma_n \subset \partial Y_n'$ whose length $\ell(\gamma_n)$ is bigger than μ, such that $d(y_n, \gamma_n) \leq C(\mu, \chi(Y))$. Thus, $B(y_n, R)$ contains a ball of radius $R - C(\mu, \chi(Y))$ centered at some point on γ_n. Since $\ell(\gamma_n) \geq \mu$, the volume of $B(y_n, R)^{[\mu,\infty[}$ is (much) bigger than $(R - C(\mu, \chi(Y)))\mu$, for large R and for all n.

Suppose that y_n belongs to component of $Y_n - Y_n'$ (which is an annulus). Since $\mathrm{inj}(y_n) \geq \mu$ a short computation shows that the volume of $B(y_n, R)^{[\mu,\infty[}$ is (much) bigger than $R\mu$.

In both cases, the volume of $B(y_n, R)^{[\mu,\infty[}$ tends to infinity with R, uniformly in n.

Any ball of radius μ in Y_n which is contained in the preimage of $K_n^{[\mu,\infty[}$ embeds into X_n. Therefore the cardinality of the intersection of $\widetilde{Z}_n$ with any ball of radius μ is smaller than the number of zeroes of $\Theta_n\phi_n$ and so is smaller than $4g-4$. Thus, for all sufficiently small $r>0$, the volume of $B(y_n,R)^{[\mu,\infty[}-\widetilde{Z}_n(r)$ tends to ∞ with R uniformly in n. This implies Lemma 5.7. □

Fix $r>0$ so that the conclusions of Lemma 5.7 are satisfied. For any R, $|F_n|$ is bounded from below over $B(y_n,R)-\widetilde{Z}_n(r)$ by $|\bar{F}|/2$ for all sufficiently large n because $|\bar{F}_n|$ tends uniformly on $B(0,R)-\bar{Z}_n(r)$ to the non-zero constant $|\bar{F}|$. By Lemma 5.7 the volume of $B(y_n,R)-\widetilde{Z}_n(r)$ is bigger than $3/(m(r)|\bar{F}|)$, for a sufficiently large R independent of n. This is impossible since the L^1-norm of F_n on $Y_n-\widetilde{Z}_n(r)$ is less than $1/m(r)$.

2c) $\bar{F}=0$.

We will apply essentially the same argument as in 1). We introduce first some notations. Let $\mu'\leq\mu$ to be fixed later. Recall that $K_n^{[\mu',\infty[}$ is isotopic to $K_n^{[\mu,\infty[}$, for all n sufficiently large. Restrict in what follows to those values of n. By hypothesis, $K_n^{[\mu',\infty[}$ cannot be lifted to Y_n. However, some components of $\partial K_n^{[\mu',\infty[}$ might be. We denote the union of these curves by ${}^S\partial K_n^{[\mu,\infty[}$.

For $\mu'<\mu$, each component of ${}^S\partial K_n^{[\mu,\infty[}$ cuts $K_n^{[\mu',\infty[}$ into two components, one of which is an annulus. We call ${}^SK_n^{[\mu,\mu']}$ the union of those annuli. Then the boundary of ${}^SK_n^{[\mu,\mu']}$ equals ${}^S\partial K_n^{[\mu,\infty[}\cup^S\partial K_n^{[\mu',\infty[}$. Since ${}^SK_n^{[\mu,\mu']}$ is isotopic into S, it can be lifted isomorphically to Y_n. The following is a generalization of Fact 5.4.

Fact 5.8. — *For all $\mu'<\mu$, $W_n^{[\mu',\infty[}$ is contained in the unbounded components of $Y_n^{[\mu,\infty[}$ except the isomorphic lift of ${}^SK_n^{[\mu,\mu']}$, which is contained in $Y_n^{]0,\mu]}$.*

Proof. — The preimage $W_n^{[\mu',\infty[}$ equals the union of $W_n^{[\mu,\infty[}$ and $\pi^{-1}(K_n^{[\mu',\mu]})$. Clearly, $W_n^{[\mu,\infty[}\subset Y_n^{[\mu,\infty[}$. Moreover since $K_n^{[\mu,\infty[}$ does not lift, $W_n^{[\mu,\infty[}$ is contained in the unbounded components of $Y_n^{[\mu,\infty[}$. Let A be a component of $K_n^{[\mu',\infty[}-K_n^{]\mu,\infty[}$. If $A\not\subset^S K_n^{[\mu,\mu']}$, it is not liftable to Y_n. Therefore $\pi^{-1}(A)$ is a disjoint union of discs and thus is disjoint from $Y_n^{]0,\mu]}$ (cf. Fact 5.4). If $A\subset^S K_n^{[\mu,\mu']}$, $\pi^{-1}(A)$ equals the isomorphic lift of A and a union of discs. Again only this isomorphic lift meets (is contained in) $Y_n^{]0,\mu]}$. This proves Fact 5.8. □

We need now to estimate $\int_{\partial(K_n^{[\mu',\infty[}-Z_n(r))}\theta_n(|F_n|ds)$. Let η_n be the 1-form on the unbounded components of $Y_n^{[\mu,\infty[}$ constructed in Theorem 4.6. Then $\langle\eta_n\rangle$ is bounded from above by a constant C independent of n. In contrary to 1), η_n is not defined over all $W_n^{[\mu',\infty[}$ but only on the complement of the isomorphic lift of ${}^SK_n[\mu',\mu]$. We need to modify the definition of θ_n to take these annuli into consideration.

A 1-form $\theta_n^S|F_n\eta_n|$ can be defined on $K_n^{[\mu',\infty[}$ by applying θ_n to the *discontinuous* 1-form defined by extending $|F_n\eta_n|$ by 0 in the complement of the unbounded

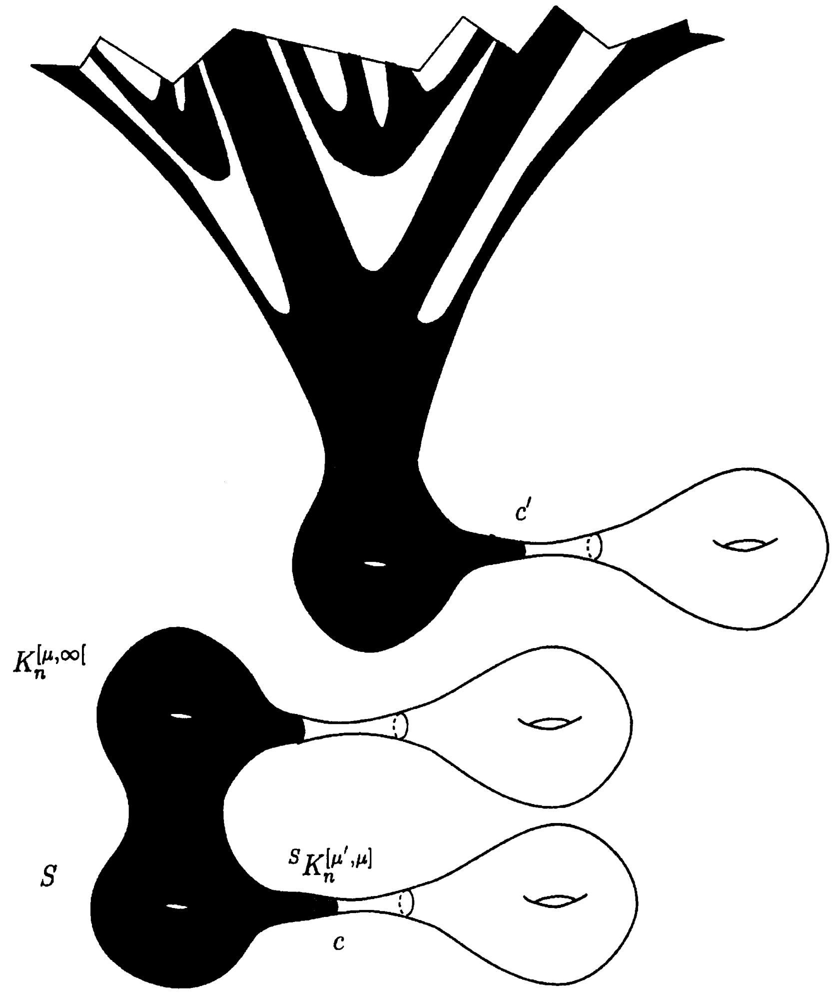

Figure 5.2

components of $Y_n^{[\mu,\infty[}$. Let c be a component of $\partial K_n^{[\mu',\infty[} -^S \partial K_n^{[\mu',\infty[}$. Since $\pi^{-1}(c)$ is contained in the unbounded components of $Y_n^{[\mu,\infty[}$, we have

$$\theta_n(|F_n|ds) \geq \frac{1}{C}\theta_n|F_n\eta_n| = \frac{1}{C}\theta_n^S|F_n\eta_n|.$$

More generally, if c is a component of $\partial K_n^{[\mu',\infty[}$, we have $\theta_n(|F_n|ds) \geq \frac{1}{C}\theta_n^S|F_n\eta_n|$.

A 1-form $\theta_n^S(F_n\eta_n)$ can be defined on $K_n^{[\mu',\infty[}$ using the same construction as for $\theta_n^S|F_n\eta_n|$. It is a form of type (1,0) in the complement of ${}^S\partial K_n^{[\mu,\infty[}$. Its discontinuity along ${}^S\partial K_n^{[\mu,\infty[}$ can be described as follows. Each component A of ${}^S K_n^{[\mu',\mu]}$ lifts to Y_n. Over A the covering map $\pi : Y_n \to X_n$ has an inverse π^{-1}. We denote by f_n the function defined on ${}^S K_n^{[\mu',\mu]}$ which equals $F_n \circ \pi^{-1}$ on each of his components. Then the 1-form $\theta_n^S(F_n\eta_n)$ has a jump discontinuity along the curves ${}^S\partial K_n^{[\mu,\infty[}$, which equals $f_n\eta_n$. We have

$$\bar\partial\theta_n^S(F_n\eta_n) = dv, \quad \text{on} \quad K_n^{[\mu',\infty[} - {}^S K_n^{[\mu',\mu]}, \quad \text{and}$$
$$\bar\partial\theta_n^S(F_n\eta_n) = (1-f_n)dv, \quad \text{on} \quad {}^S K_n^{[\mu',\mu]}.$$

Suppose, up to increasing μ slightly, that $\partial K^{[\mu,\infty[}$ is disjoint from Z. Then, $\partial K_n^{[\mu,\infty[}$ is disjoint from Z_n for all sufficiently large n. For all $\mu' < \mu$ such that $\partial K^{[\mu',\infty[}$ is disjoint from Z, for all sufficiently large n and for all sufficiently small r, we have:

$$\left|\int_{\partial(K_n^{[\mu',\infty[}-Z_n(r))} \theta_n^S(F_n\eta_n)\right|$$
$$= \left|\int_{\partial(K_n^{[\mu',\infty[}-Z_n(r))} \theta_n^S(F_n\eta_n) - \int_{{}^S\partial K_n^{[\mu,\infty[}} \theta_n^S(F_n\eta_n) + \int_{{}^S\partial K_n^{[\mu,\infty[}} \theta_n^S(F_n\eta_n)\right|$$
$$\geq \left|\int_{{}^S K_n^{[\mu',\mu]}-Z_n(r)} (1-f_n)dv + \int_{K_n^{[\mu',\infty[}-{}^S K_n^{[\mu',\mu]}-Z_n(r)} dv + \int_{{}^S\partial K_n^{[\mu,\infty[}} f_n\eta_n\right|$$
$$= \left|\int_{K_n^{[\mu',\infty[}-Z_n(r)} dv - \int_{{}^S K_n^{[\mu',\mu]}-Z_n(r)} f_n dv + \int_{{}^S\partial K_n^{[\mu,\infty[}} f_n\eta_n\right|.$$

Fix a μ' as in 1). Since (f_n) tends uniformly to 0 over ${}^S K_n^{[\mu',\mu]} - Z_n(r)$ the same inequalities as in 1) hold for all n sufficiently large. This gives a contradiction which finishes the proof of Theorem 5.3. □

CHAPTER 6

McMullen's proof of the Fixed point theorem

In this chapter we prove, following [McM3], the Fixed point theorem.

Thurston's fixed point theorem. — *Let M be a hyperbolic manifold with incompressible boundary which is not an interval bundle. Let τ be an orientation reversing involution of ∂M which permutes the components by pairs. Suppose that M/τ is atoroidal. Then $\tau^* \circ \sigma$ has a fixed point.*

Through all this chapter, ε will denote a strictly positive constant smaller than $\varepsilon(3)$.

In the first two sections, we denote by M a connected hyperbolic manifold with incompressible boundary which is not an interval bundle, and by G be a geometrically finite group such that M is diffeomorphic to $\overline{M}(G)$.

6.1 Consequences of the contraction properties of the Theta operators for the skinning map

In this section, we deduce two corollaries for the norm of $d^*\sigma$ from the results of §5. An immediate consequence of Theorem 5.1 is the following.

Proposition 6.1. — *Let $s \in \mathcal{T}(\partial M)$. Then the norm of $d^*\sigma$ at s is less than or equal to a constant $k < 1$ which is a function of the systole of s and of $\chi(\partial M)$.*

Proof. — Let $s \in \mathcal{T}(\partial M)$. Let $d_s^*\sigma$ denote the coderivative of σ at s. We have

$$d_s^*\sigma\phi = \sum_U \Theta_U \phi_U,$$

where U varies over all the spots contained in $\sigma(\partial M^s)$ (Proposition 2.1). Then

$$(1) \qquad ||d_s^*\sigma|| \leq \sup_U ||\Theta_U||$$

where the supremum is taken over all the spots $U \subset \sigma(\partial M^s)$.

Since M is not an interval bundle, all the spots $U \subset \sigma(\partial M^s)$ are proper surfaces (cf. §2). By Proposition 2.2, all the covers $U \to X_U$ are geometric and those which are not isomorphic to the universal cover are finite in number. Therefore, by Theorem 5.1, the norms of the various operators Θ_U are all smaller than a constant $k < 1$ which is a function of the systole of s and of $\chi(\partial M)$. By (1), the same k is also an upper bound for the norm of $d_s^*\sigma$. This proves Proposition 6.1. □

By Proposition 6.1, in order to study the contraction properties of σ over all $\mathcal{T}(\partial M)$, we need to understand what happens when s contains short geodesics. Theorem 5.3 gave a precise information on Theta operators in that situation. In order to apply this theorem to the skinning map, we introduce first another definition.

Let $s \in \mathcal{T}(\partial M)$. We denote by $\mathcal{A}(\sigma(\partial M^s))^\varepsilon$ the part of $\sigma(\partial M^s)$ which equals the disjoint union of the ε-amenable parts $\mathcal{A}(X_U)^\varepsilon$ of the covers $U \to X_U$ over all the spots $U \subset \sigma(\partial M^s)$.

Proposition 6.2.— *Let $0 < \eta < 1$. There exists $\delta > 0$ such that: for any $s \in \mathcal{T}(\partial M)$ and for any $\phi \in \mathcal{Q}(\sigma(\partial M^s))$ with $||\phi|| = 1$ and $||d_s^*\sigma\phi|| \geq 1 - \delta$, the ϕ-mass of $\mathcal{A}(\sigma(\partial M^s))^\varepsilon$ is bigger than $1 - \eta$.*

Proof.— Let $0 < \delta < 1$. Assume that $||\phi|| = 1$ and $||d_s^*\sigma\phi|| \geq 1 - \delta$. Then, since $d_s^*\sigma\phi = \sum \Theta_U\phi_U$, the ϕ-mass of the union of the spots U for which $||\Theta_U\phi_U|| \geq (1 - \sqrt{\delta})||\phi_U||$ is bigger than $1 - \sqrt{\delta}$. For any spot U such that $||\Theta_U\phi_U|| \geq (1 - \sqrt{\delta})||\phi_U||$, Theorem 5.3 asserts that the ϕ_U-mass of $\mathcal{A}(X_U)^\varepsilon$ is bigger than $(1 - c(\delta))||\phi_U||$ for a constant $c(\delta)$ which tends to 0 with δ and which depends only on ε and $\chi(X_U)$. Since there are only finitely many possibilities for the topology of the covers $U \to X_U$ (Proposition 2.2), $c(\delta)$ can be taken independent of U. Thus the ϕ-mass of $\mathcal{A}(\sigma(\partial M^s))^\varepsilon$ is bigger than $(1 - \sqrt{\delta})(1 - c(\delta))$. Proposition 6.2 directly from this. □

6.2 Inefficiency over the thin part

By Proposition 6.2, if $||d_s^*\sigma\phi||$ is near 1 for a unit norm ϕ, the ϕ-mass concentrates over the amenable part. In this section, we show that the ϕ-mass concentrates over *the ε-liftable part.*

Definition.— *The ε-liftable part* $\mathcal{L}(\sigma(\partial M^s))^\varepsilon$ is the union over the spots $U \subset \sigma(\partial M^s)$ of the lifts of the (liftable) components of $X_U^{]0,\varepsilon]}$ and of $X_U^{[\varepsilon,\infty[}$. Note that $\mathcal{L}(\sigma(\partial M^s))^\varepsilon$ is a compact surface that might be empty. We denote by $\mathcal{S}(\sigma(\partial M^s))^\varepsilon$ the union over all the spots $U \subset \sigma(\partial M^s)$ of the simply connected components in the preimage of $X_U^{]0,\varepsilon]}$. Hence $\mathcal{A}(\sigma(\partial M^s))^\varepsilon$ is the disjoint union of $\mathcal{L}(\sigma(\partial M^s))^\varepsilon$ and $\mathcal{S}(\sigma(\partial M^s))^\varepsilon$.

Proposition 6.3.— *There exists a constant $\varepsilon_M > 0$ depending only on $\chi(\partial M)$, such that for all $\varepsilon \leq \varepsilon_M$ and for all $\eta > 0$, there exists $\delta > 0$ such that: let $s \in \mathcal{T}(\partial M)$, let $\phi \in \mathcal{Q}(\sigma(\partial M^s))$ with $||\phi|| = 1$ and $||d_s^*\sigma\phi|| \geq 1 - \delta$, then the ϕ-mass of $\mathcal{L}(\sigma(\partial M^s))^\varepsilon$ is bigger than $1 - \eta$.*

Although this proposition looks very similar to Proposition 6.2, its proof is entirely different: it lies on the next result, whose proof is partially "3-dimensional".

Proposition 6.4.— *There exists a constant ε_M depending only on $\chi(\partial M)$ such that, for all $0 < \varepsilon \leq \varepsilon_M$, we have: let $s \in \mathcal{T}(\partial M)$, let $\phi \in \mathcal{Q}(\sigma(\partial M^s))$ with $||\phi|| = 1$, let $\mu \in \mathcal{B}(\partial M^s)$ with $||\mu|| = 1$, then every $x \in \mathcal{S}(\sigma(\partial M^s))^\varepsilon$ is the center of an embedded hyperbolic ball on which the efficiency of the pairing between ϕ and $d_s\sigma\mu$ is smaller than $1 - \alpha$, where the constant $\alpha > 0$ depends only on ε and on $\chi(\partial M)$.*

Proof.— We argue by contradiction. The contradiction will follow easily from Theorem 3.1 once we suitably normalize the limit set of G^s in $\overline{\mathbb{C}}$. Suppose that there exists a sequence (ε_i) tending to 0 and a sequence (s_i) in $\mathcal{T}(\partial M)$ such that the systole of s_i is smaller than ε_i, for which Proposition 6.4 fails. Denote to simplify $M^i = M^{s_i}$ and $G^i = G^{s_i}$. Then for each i, there is a point $x_i \in \mathcal{S}(\sigma(\partial M^i))^{\varepsilon_i}$ such that: there exists $\phi_i \in \mathcal{Q}(\sigma(\partial M^i))$ and $\mu_i \in \mathcal{B}(\partial M^i)$ with $||\phi_i|| = ||\mu_i|| = 1$, such that on any embedded hyperbolic ball $B_i \subset \sigma(\partial M^i)$ centered at x_i, the efficiency of the pairing between ϕ_i and $d_{s_i}\sigma\mu_i$ tends to 1.

For each i, choose a component Ω_i of $\Omega(G^i)$, with stabilizer Γ_i, such that x_i belongs to the component $\sigma(\Omega_i/\Gamma_i)$ of $\sigma(\partial M^i)$. Then $\Omega(\Gamma_i)$ is the disjoint union of $\Omega(\Gamma_i)$ and another component denoted by $\widetilde{\Omega}_i$. By definition, $\widetilde{\Omega}_i$ covers $\sigma(\Omega_i/\Gamma_i)$. Let $\widetilde{x}_i \in \widetilde{\Omega}_i$ be a point in the preimage of x_i. Let $\widetilde{U}_i$ be the component of $\Omega(G^i)$ which contains $\widetilde{x}_i$. The projection of $\widetilde{U}_i$ to $\sigma(\Omega_i/\Gamma_i)$ is the spot denoted by U_i which contains x_i. Since $x_i \in \mathcal{S}(\sigma(\partial M^i))^{\varepsilon_i}$, there is a non-zero element $\gamma_i \in G^i$ which stabilizes $\widetilde{U}_i$ and which moves $\widetilde{x}_i$ a distance smaller than ε_i for the hyperbolic metric on $\widetilde{U}_i$. Since the component of $\mathcal{S}(\sigma(\partial M^i))^{\varepsilon_i}$ which contains x_i is simply connected, $\gamma_i \notin \Gamma_i$ (cf. the proof of Proposition 2.2). Since G^i has no parabolic elements, γ_i is a hyperbolic isometry. Let α_i and ω_i be its two fixed points in $\partial\mathbb{D}^3 \simeq \overline{\mathbb{C}}$. With these notations, we have:

Lemma 6.5.— *Let $0 < \varepsilon_0 \leq \varepsilon(3)$ We can conjugate G^i inside* $\mathrm{PSL}_2(\mathbb{C})$ *so that, for all sufficiently large i, the following properties hold*

(i) $\alpha_i = 0$,

(ii) $\infty \in L(\Gamma_i)$ *and*

(iii) γ_i *moves the center of $\mathbb{D}^3$ a hyperbolic distance equal to ε_0.*

Proof.— By a conjugacy of G^i in $\mathrm{PSL}_2(\mathbb{C})$, it is easy to achieve (i) and (ii).

The element γ_i stabilizes $\widetilde{U}_i$. By Corollary 2.4, α_i (resp. ω_i) does not belong to $L(\Gamma_i)$ since $\gamma_i \notin \Gamma_i$. For the hyperbolic metric on $\widetilde{U}_i$, γ_i is a hyperbolic isometry with translation distance smaller than ε_i. By the Ahlfors lemma, the translation distance of γ_i in $\mathbb{H}^3$ is less than $2\varepsilon_i$. In particular, for all i sufficiently large, this translation distance is less than ε_0. The set of points in $\mathbb{D}^3$ which γ_i moves a distance exactly ε_0 is the boundary of the tube $n^{\varepsilon_0}(\gamma_i)$ (cf. §1). Since $\omega_i \neq \infty$, the boundary of $n^{\varepsilon_0}(\gamma_i)$ is pierced in exactly one point by the geodesic 0∞. Therefore, after conjugating G^i by a hyperbolic transformation fixing 0∞, we can achieve (iii). □

Denote by Δ_R the disc of radius R centered at $0 \in \mathbb{C}$.

Lemma 6.6. — *Let $R > 0$. If ε_0 is smaller than a constant which depends only on $\chi(\partial M)$ and on R, then in the normalization provided by Lemma 6.5, $\widetilde{\Omega}_i$ contains Δ_R, for all sufficiently large i.*

Proof. — Recall that $\gamma_i \notin \Gamma_i$. The convex hull $C(\Gamma_i)$ of $L(\Gamma_i)$ is the universal cover of $N(\Gamma_i)$, the Nielsen core of $M(\Gamma_i)$ (cf. §1). With the induced path metric, $\partial N(\Gamma_i)$ is isometric to a hyperbolic surface (cf. §1). Since ∂M is compact, $\partial N(\Gamma_i)$ is compact and there are only a finite number of possibilities for its topological type. Thus the injectivity radius of the induced metric on $\partial N(\Gamma_i)$ is bounded from above by a constant d depending only on $\chi(\partial M)$. Therefore any point of $\partial C(\Gamma_i)$ is moved a distance smaller than $2d$ by a non-zero element of Γ_i.

For $\nu > 0$, $n^{\nu}(\gamma_i)$ contains the neighborhood of radius $2d$ of $n^{e^{-2d}\nu}(\gamma_i)$ (cf. §1). Recall that, by the Margulis lemma, for any $\nu \leq \varepsilon(3)$, the tubes $n^{\nu}(g)$ corresponding to elements $g \in G^i$ which are not contained in the same cyclic subgroup are disjoint. Using this, we prove by contradiction that for any $\varepsilon \leq \varepsilon(3)$, $n^{e^{-2d}\varepsilon}(\gamma_i)$ is disjoint from $C(\Gamma_i)$. Suppose this is not the case. Then, $n^{e^{-2d}\varepsilon}(\gamma_i)$ must intersect $\partial C(\Gamma_i)$, since $\gamma_i \notin \Gamma_i$. We saw above that any point in $\partial C(\Gamma_i)$ is translated a distance less than $2d$ by some non-trivial element $g_i \in \Gamma_i$. Thus the tubes $n^{\varepsilon}(\gamma_i)$ and $n^{\varepsilon}(g_i \circ \gamma_i \circ g_i^{-1})$ have non-empty intersection. By Margulis lemma, $g_i \circ \gamma_i \circ g_i^{-1}$ and γ_i are contained in the same cyclic group. Then γ_i and g_i must have the same fixed points. Therefore $\gamma_i \in \Gamma_i$. This provides the required contradiction.

In our normalization, γ_i moves $0 \in \mathbb{D}^3$ a distance equal to ε_0 for sufficiently large i. Thus for any $K > 0$, the hyperbolic ball of radius K centered at $0 \in \mathbb{D}^3$ is contained in $n^{e^{K}\varepsilon_0}(\gamma_i)$. By the last observation, if we choose $\varepsilon_0 = \varepsilon_0(K)$ so that $e^{2d+K}\varepsilon_0 \leq \varepsilon(3)$, then this ball is disjoint from $C(\Gamma_i)$. Since $\infty \in L(\Gamma_i)$, this implies that $\widetilde{\Omega}_i$ contains $\Delta_{R(K)}$, where $R(K)$ depends only on K and tends to ∞ with it. This implies Lemma 6.6. □

Lemma 6.7. — *If ε_0 is smaller than a constant which depends only on $\chi(\partial M)$, then, for all sufficiently large i, Δ_1 is contained in $\widetilde{\Omega}_i$ and embeds in $\widetilde{\Omega}_i/\Gamma_i$.*

Proof. — By Lemma 6.6, for all $R > 0$, we can choose $\varepsilon_0 = \varepsilon_0(R)$ such that $\Delta_R \subset \widetilde{\Omega}_i$, for all i sufficiently large. If $R > 1$, then $\Delta_1 \subset \widetilde{\Omega}_i$ for all i sufficiently large.

We show now by contradiction that Δ_1 embeds in $\widetilde{\Omega}_i/\Gamma_i$, for all i sufficiently large (in the normalization of Lemma 6.6). If Δ_1 does not embed into $\widetilde{\Omega}_i/\Gamma_i$, then there is a non-zero $\delta_i \in \Gamma_i$ such that $\delta_i(\Delta_1) \cap \Delta_1 \neq \emptyset$. The two fixed points of δ_i are contained in $L(\Gamma_i)$ and in particular they are outside from Δ_R. Thus, for the natural metric on $\mathrm{PSL}_2(\mathbb{C})$ δ_i is $C(R)$-close to a parabolic isometry δ which fixes ∞ and moves $0 \in \mathbb{C}$ an euclidean distance less than 2. In particular, δ moves then $0 \in \mathbb{D}^3$ a hyperbolic distance smaller than 2. When R tends to ∞, $C(R)$ tends to 0. Thus we can choose R sufficiently large such that δ_i moves $0 \in \mathbb{D}^3$ a hyperbolic distance smaller than 3, for all i sufficiently large. But γ_i moves $0 \in \mathbb{D}^3$ a distance smaller than $\varepsilon_0 = \varepsilon_0(R)$ (depending only on $\chi(\partial M)$). Therefore, if ε_0

also satisfies $\varepsilon_0 e^3 \leq \varepsilon(3)$, the tubes $n^{\varepsilon_0 e^3}(\gamma_i)$ and $n^{\varepsilon_0 e^3}(\delta_i \gamma_i \delta_i^{-1})$ must intersect. Since $\delta_i \gamma_i \delta_i^{-1}$ and γ_i do not belong to the same cyclic subgroup of G^i, this is impossible by Margulis lemma. This ends the proof of Lemma 6.7. □

In the sequel, we fix ε_0 such that Lemma 6.7 and Lemma 6.6 for $R = 2$ are satisfied. In order to obtain a contradiction, we need to find, for each i, a hyperbolic ball centered at x_i on which the efficiency of the pairing between ϕ_i and $d_{s_i}\sigma\mu_i$ is bounded away from 1, for any non-zero holomorphic quadratic differential ϕ_i and for any Beltrami form μ_i of unit norm.

Note that up to extracting a subsequence, (γ_i) tends in $\mathrm{PSL}_2(\mathbb{C})$ to a parabolic isometry fixing 0. Since each γ_i moves $0 \in \mathbb{D}^3$ a distance exactly equal to ε_0, some subsequence of (γ_i) converges to a non-trivial element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$. Since the translation distance of γ_i tends to 0, γ is parabolic. By the normalization of Lemma 6.5, γ fixes $0 \in \mathbb{C}$.

Fact 6.8.— *The point $\widetilde{x}_i$ tends to 0 as i tends to ∞.*

Proof.— The hyperbolic metric on $\widetilde{U}_i$ can be written $\lambda_i(z)|dz|$. By the Koebe 1/4-lemma and since $0 \in \partial\widetilde{U}_i$, $\lambda_i(z) \geq 1/4d_{euc}(z, \partial\widetilde{U}_i) \geq 1/4|z|$. The hyperbolic distance between $\widetilde{x}_i$ and $\gamma_i(\widetilde{x}_i)$, for the hyperbolic metric on $\widetilde{U}_i$, tends to 0 as i tends to ∞. Therefore, if $\widetilde{x}_i$ would remain a bounded euclidean distance away from 0, the spherical distance between $\widetilde{x}_i$ and $\gamma_i(\widetilde{x}_i)$ would tend to 0. Then any accumulation point of $\widetilde{x}_i$ would be a fixed point of γ. Since γ is a parabolic isometry fixing $0 \in \mathbb{C}$, this is impossible. □

Let $\widetilde{B}_i$ denote the largest ball for the hyperbolic metric on $\widetilde{\Omega}_i$ which is centered at $\widetilde{x}_i$ and contained in the disc $\Delta = \Delta_1$.

Fact 6.9.— *Any limit of $\widetilde{B}_i$ for the Hausdorff topology on compact subsets of $\mathbb{C}$ contains a neighborhood of 0.*

Proof.— Let R_i be the radius of the largest Euclidean ball centered at 0 and contained in $\widetilde{\Omega}_i$. In our normalization, R_i is bigger than 2. By the Schwarz lemma, the conformal factor $\lambda_i(z)$ of the hyperbolic metric on $\widetilde{\Omega}_i$ is less than the one of the hyperbolic metric on Δ_{R_i} which equals

$$\frac{2R_i}{R_i^2 - |z|^2}.$$

As we noticed above, Koebe's $1/4$-theorem implies

$$\lambda_i(z) \geq \frac{1}{4d_{euc}(z, \partial\widetilde{\Omega}_i)}.$$

Therefore on Δ we have $\sup_\Delta \lambda_i \leq C \inf_\Delta \lambda_i$, for some constant C independent on i. Thus on Δ, the hyperbolic metric of $\widetilde{\Omega}_i$ is Lipschitz equivalent to the euclidean metric by a factor independent on i. Fact 6.9 follows from this and from Fact 6.8. □

Let B_i be the projection of $\widetilde{B}_i$ in $\widetilde{\Omega}_i/\Gamma_i$. By Lemma 6.7, the ball B_i is embedded. Suppose that the efficiency of the pairing between ϕ_i and $d_{s_i}\sigma\mu_i$ on B_i tends to 1. From (the proof of) Theorem 3.1 we know that, up to extracting a subsequence and up to multiplying the differentials ϕ_i by non-zero constants, their pull-back $\widetilde{\phi}_i(z)dz^2$ to $\widetilde{\Omega}_i$ converge uniformly over compact sets to a non-zero holomorphic quadratic differential $\widetilde{\phi}(z)dz^2$. Let $\widetilde{\mu}_i(z)\overline{dz}/dz$ be the pull-back of $d_{s_i}\sigma\mu_i$ to $\mathbb{C}$. We can suppose that $(\widetilde{\mu}_i)$ converges weakly to $\widetilde{\mu} \in L^\infty(\mathbb{C})$ with $||\widetilde{\mu}|| \leq 1$. The uniform convergence of $\widetilde{\phi}_i$ to $\widetilde{\phi}$ and the weak convergence of $\widetilde{\mu}_i$ to $\widetilde{\mu}$ imply that the efficiency of the pairing between $\widetilde{\phi}$ and $\widetilde{\mu}$ on any fixed ball which is contained in the limit of $\widetilde{B}_i$ equals 1. We observed already that, when i tends to ∞, γ_i tends to a parabolic isometry γ fixing 0. By continuity, $\widetilde{\mu}$ is invariant under γ. The transformation γ leaves a family of round balls containing 0 in their boundary invariant. By Fact 6.9, one of these balls, denoted by $\widetilde{B}$, is contained in the geometric limit of the balls $\widetilde{B}_i$. Then the efficiency of the pairing between $\widetilde{\phi}$ and $\widetilde{\mu}$ on $\widetilde{B}$ equals 1. Since $||\widetilde{\mu}|| \leq 1$, we have therefore on $\widetilde{B}$

$$\widetilde{\mu} = \frac{\overline{\widetilde{\phi}}}{|\widetilde{\phi}|}.$$

Since $\widetilde{\mu}$ is invariant under γ, $\widetilde{\phi}(\gamma(z)) = \kappa\widetilde{\phi}(z)\gamma'(z)^2$, for a constant $\kappa \neq 0$. But this is impossible since $\widetilde{\phi}$ is integrable on $\widetilde{B}$, as being holomorphic and defined on a larger domain. This concludes the proof of Proposition 6.4. □

Proof of Proposition 6.3.— Let ε_M be the constant provided by Proposition 6.4. Let $0 < \varepsilon \leq \varepsilon_M$. Let $\phi \in \mathcal{Q}(\sigma(\partial M^s))$ with $||\phi|| = 1$. Let m be the ϕ-mass of $\mathcal{S}(\sigma(\partial M^s))^\varepsilon$. From Proposition 6.4 and Proposition 3.10, we deduce that for any $\mu \in \mathcal{B}(\partial M^s)$ with $||\mu|| = 1$, $\langle\phi, d_s\sigma\mu\rangle \leq 1 - c\alpha m$, where $c > 0$ only depends on $\chi(\partial M)$. But $||d_s^*\sigma\phi|| = \sup_\mu\langle d_s^*\sigma\phi, \mu\rangle = \sup_\mu\langle\phi, d_s\sigma\mu\rangle$, where the supremum are both taken over all $\mu \in \mathcal{B}(\partial M^s)$ of unit norm. Thus $||d_s^*\sigma\phi|| \leq 1 - c\alpha m$. Therefore if $||d_s^*\sigma\phi|| \geq 1-\delta$, then the ϕ-mass of $\mathcal{S}(\sigma(\partial M^s))^\varepsilon$ is less than $\delta/c\alpha$. Since $\mathcal{A}(\sigma(\partial M^s))^\varepsilon$ is the disjoint union of $\mathcal{L}(\sigma(\partial M^s))^\varepsilon$ and $\mathcal{S}(\sigma(\partial M^s))^\varepsilon$, Proposition 6.3 follows from this estimate and from Proposition 6.2. □

6.3 Proof of Thurston's fixed point theorem

The proof of the Fixed point theorem amounts now essentially to translate Proposition 6.3 in terms of the topology of M.

In this section M is not necessarily connected. Denote by $M_1, ., M_\ell, ., M_p$ its components. For $s = (s_1, \cdots, s_\ell, \cdots, s_p) \in \mathcal{T}(\partial M)$, we denote by M^s the disjoint union of the manifolds $M_\ell^{s_\ell}$. Recall that σ equals the product of the skinning maps σ_ℓ associated to M_ℓ and that the norm on $\mathcal{Q}(\sigma(\partial M^s)) = \bigoplus_\ell \mathcal{Q}(\sigma(\partial M_\ell^s))$ is the sum of the norms on the spaces $\mathcal{Q}(\sigma(\partial M_\ell^{s_\ell}))$. Therefore, we have

$$(1) \qquad ||d_s^*\sigma|| = \sup_\ell ||d_{s_\ell}^*\sigma_\ell||.$$

Now, we distinguish two cases.

1) M *is acylindrical.*

Theorem 6.10. — *Let M be an acylindrical hyperbolic manifold with incompressible boundary. Then σ contracts the Teichmüller distance by a factor strictly less than 1.*

Proof. — Since σ is differentiable and since Teichmüller distance is induced by a Finsler metric, it suffices to prove that $||d_s\sigma||$ is bounded by some constant $k < 1$ independent of s. It is more convenient to prove the dual statement, namely that $||d_s^*\sigma||$ is bounded away from 1 independently of s. Since M is acylindrical, each of its components is acylindrical also. In view of (1), we may suppose that M is connected. Since M is acylindrical, each spot in $\sigma(\partial M^s)$ is simply connected. Therefore $\mathcal{L}(\sigma(\partial M^s))^\varepsilon$ is empty, for all s and for all ε. Therefore the norm of $d_s^*\sigma$ is bounded away from 1 independently of s. □

Proof of the Fixed point theorem in the acylindrical case. — Since τ^* is an isometry of the Teichmüller distance (cf. §1), Theorem 6.10 implies that $\tau^* \circ \sigma$ contracts the distance on $\mathcal{T}(\partial M)$ by a factor strictly less than 1. Since Teichmüller space is complete, $\tau^* \circ \sigma$ has a fixed point. □

As an application, we have the following.

Theorem 6.11. — *Let N be an acylindrical hyperbolic manifold with incompressible boundary. Then N carries a hyperbolic metric for which ∂N is totally geodesic.*

Proof. — This result is a consequence of the Fixed point theorem when M is the union of two copies of N having opposite orientations, and when τ is the natural identification between the two copies of ∂M. By Corollary 6.11, $\tau^* \circ \sigma$ has a fixed point $s \in \mathcal{T}(\partial M)$. Consider the hyperbolic manifold M^s corresponding to s via the Ahlfors-Bers isomorphism. Denote by $\mathcal{N}$ the Nielsen core of the component of M^s which has the same orientation as N. Let Γ' be the image in $\mathrm{PSL}_2(\mathbb{C})$ of the fundamental group of a component of $\partial\mathcal{N}$. Then Γ' is a quasi-Fuchsian group. Saying that s is a fixed point of $\tau^* \circ \sigma$ means precisely that the two components of $\Omega(\Gamma')$ are isometric by an orientation reversing map which extends to the identity on $L(\Gamma')$. Since the Ahlfors-Bers map is injective, Γ' is conjugated in $\mathrm{PSL}_2(\mathbb{C})$ to a Fuchsian group. It follows that Γ' leaves invariant a totally geodesic plane. Therefore $\partial\mathcal{N}$ is totally geodesic. Since $\mathcal{N}$ is diffeomorphic to N, this proves Theorem 6.11. □

2) M *is cylindrical.*

The hypothesis of Thurston's fixed point theorem is that M is not an interval bundle. Still some connected components of M can be cylindrical, like for instance interval bundles, some others may not. The hypothesis excludes only the case when *all components are interval bundles.* Let ε_M be the smallest of the constants ε_{M_ℓ} provided by applying Proposition 6.4 to the components M_ℓ of M which are not interval bundles. Recall that σ is an isometry when M is an interval bundle (§2).

Definition. — Denote by C the number of components of ∂M. Denote by S the maximal number of homotopy classes of disjoint simple closed curves in ∂M. Set $K = C + S$.

Theorem 6.12. — *Let M be a hyperbolic manifold which is not an interval bundle. Let τ be an orientation reversing involution of ∂M which exchanges the components by pairs. If $N = M/\tau$ is atoroidal, then $(\tau^* \circ \sigma)^K$ has a fixed point.*

Observe that this theorem does not claim that $(\tau^* \circ \sigma)^K$ contracts uniformly Teichmüller distance, like in the proof of the Fixed point theorem in the acylindrical case. However, it is sufficient.

Proof of Thurston's fixed point theorem. — Some component of M is not an interval bundle. By Proposition 6.1, the skinning map associated to this component contracts strictly Teichmüller distance. Since τ^* is an isometry and since K is bigger than the number of components of M, it follows that $(\tau^* \circ \sigma)^K$ contracts strictly Teichmüller distance. Therefore the fixed point provided by Theorem 6.12 is unique. Since $(\tau^* \circ \sigma)^K$ and $\tau^* \circ \sigma$ commute, this fixed point is also fixed by $\tau^* \circ \sigma$. □

Proof of Theorem 6.12. — Choose an arbitrary point $s^0 \in \mathcal{T}(\partial M)$. Let L be the Teichmüller distance between s^0 and $\tau^* \circ \sigma(s^0)$. Denote by $\mathcal{T}(\partial M)_L \subset \mathcal{T}(\partial M)$ the set of points which are moved a Teichmüller distance smaller than L by $\tau^* \circ \sigma$. Then $\mathcal{T}(\partial M)_L$ is a non-empty closed subset of $\mathcal{T}(\partial M)$ which has the following properties:

(i) it is invariant under $\tau^* \circ \sigma$ (because $\tau^* \circ \sigma$ decreases the Teichmüller distance), and

(ii) for any $s \in \mathcal{T}(\partial M)_L$, the Teichmüller geodesic between s and $\tau^* \circ \sigma(s)$ is contained in $\mathcal{T}(\partial M)_L$ (this is a consequence of the triangular inequality).

In order to prove Theorem 6.12, it suffices to establish that the norm of the derivative of $(\tau^* \circ \sigma)^K$ is bounded over $\mathcal{T}(\partial M)_L$ by a constant $c < 1$. This will imply that the path which equals the union of all the positive iterates by $(\tau^* \circ \sigma)^K$ of the geodesic joining s^0 to $(\tau^* \circ \sigma)^K(s^0)$ has finite Teichmüller length. Therefore, this path accumulates on a fixed point of $(\tau^* \circ \sigma)^K$.

To prove that the norm of the derivative of $(\tau^* \circ \sigma)^K$ at any point in $\mathcal{T}(\partial M)_L$ is less than a uniform constant $c < 1$, we argue by contradiction. Let $\delta > 0$ a constant to be fixed later. Then there is $s \in \mathcal{T}(\partial M)_L$ such that the coderivative $d_s^*(\tau^* \circ \sigma)^K$ has norm bigger than $1/(1+\delta)$. For $0 \leq k \leq K$, set $s^k = (\tau^* \circ \sigma)^k(s)$ and denote to simplify $M^k = M^{s^k}$. Hence, for $0 \leq k \leq K$, there exists $\phi_k \in \mathcal{Q}(\partial M^k)$ such that

(i) $\phi_k = d_{s^k}^*(\tau^* \circ \sigma)\phi_{k+1}$, and

(ii) $1 \leq ||\phi_k|| \leq 1 + \delta$.

Notation. — If X is a union of components of ∂M^k, we denote by $||\phi_k||_X$ the ϕ_k-mass of X.

Choose $\varepsilon > 0$ such that $\varepsilon \leq \varepsilon_M$ and $\log(\varepsilon) + 2KL \leq \log(\varepsilon(2))$. By Proposition 2.2, $\sigma(\partial M)$ contains only finitely many spots which are not simply connected. Let S' be twice the number of those spots and set

$$\eta = \frac{\zeta}{C(2S')^K}.$$

For all $k < K$ and for any component M_ℓ^k of M^k, we have:

$$0 \leq \|d^*\tau^*\phi_{k+1}\|_{\sigma(\partial M_\ell^k)} - \|\phi_k\|_{\partial M_\ell^k} \leq \|d^*\tau^*\phi_{k+1}\| - \|\phi_k\| \leq \delta.$$

By Proposition 6.3 and since $\phi_k = d^*\sigma(d^*\tau^*\phi_{k+1})$, we can choose δ sufficiently small so that for any component M_L^k which is not an interval bundle and whose ϕ_k-mass is more than η, the $d^*\tau^*\phi_{k+1}$-mass of $\sigma(\partial M_L^k) - \mathcal{L}(\sigma(\partial M_L^k))^{\varepsilon/2}$ is less than η.

Notation.— Let X be a compact hyperbolic surface and let $\gamma \subset X$ be a closed curve homotopic to a geodesic shorter than $\varepsilon/2$. We denote by $X(\gamma)$ the component of $X^{]0,\varepsilon]}$ which contains this geodesic.

By Proposition 3.9, there is a constant $\zeta \neq 0$ depending only on $\chi(\partial M)$ such that, if X is a component of ∂M^k which contains a closed geodesic shorter than $\varepsilon/2$, then the ϕ_k-mass of $X(\alpha)$ is bigger than $\zeta\|\phi_k\|_X$ for some geodesic $\alpha \subset X$ shorter than $\varepsilon/2$.

Fact 6.13.— *Let $0 \leq k \leq K-1$. Let α_k be a closed curve contained in the boundary of a component M_ℓ^k of M^k and suppose that α_k is homotopic to a geodesic shorter than $\varepsilon/2$. Set $\mu = \|\phi_k\|_{\partial M^k(\alpha_k)}$ and assume $\mu > \eta$. Then, $\partial M^k(\alpha_k)$ lifts to an annulus contained in $\sigma(\partial M_\ell^k)$ whose $d^*\tau^*\phi_{k+1}$-mass is bigger than $\mu' = (\mu - \eta)/S'$.*

Proof.— Since $\phi_k = d^*\sigma(d^*\tau^*\phi_{k+1})$ and since $d^*\sigma$ decreases the mass, the $d^*\tau^*\phi_{k+1}$-mass of the entire preimage of $\partial M_\ell^k(\alpha_k)$ in $\sigma(\partial M_\ell^k)$ is bigger than μ.

If M_ℓ^k is an interval bundle, this preimage consists of exactly one isomorphic lift of $\partial M_\ell^k(\alpha_k)$. The $d^*\tau^*\phi_{k+1}$-mass of this lift equals μ and thus is bigger than μ'.

If M_ℓ^k is not an interval bundle, the preimage of $\partial M_\ell^k(\alpha_k)$ equals the disjoint union of isomorphic lifts of $\partial M_\ell^k(\alpha_k)$ and simply connected components contained in $\mathcal{S}(\sigma(\partial M_\ell^k))^{\varepsilon/2}$. By our choice of δ, the $d^*\tau^*\phi_{k+1}$-mass of $\mathcal{S}(\sigma(\partial M_\ell^k))^{\varepsilon/2}$ is less than η. Thus, the $d^*\tau^*\phi_{k+1}$-mass of the union of the isomorphic lifts of $\partial M_\ell^k(\alpha_k)$ is bigger than $\mu - \eta$. Since the cover associated to any spot is geometric, each spot which is not simply connected contains at most two isomorphic lifts of $\partial M_\ell^k(\alpha_k)$ (cf. §2). Therefore the number of the isomorphic lifts of $\partial M_\ell^k(\alpha_k)$ is less than S' so that the $d^*\tau^*\phi_{k+1}$-mass of one of them is bigger than μ'. □

The geometric consequence of this result is the following.

Fact 6.14.— *Under the hypothesis of Fact 6.13, there is an essential annulus contained in M_ℓ^k which joins α_k to a curve $\gamma_k \subset \partial M_\ell^k$ which satisfies*

(i) *$\tau(\gamma_k)$ is homotopic to a geodesic shorter than $\varepsilon/2$ for the hyperbolic metric on ∂M^{k+1}, and*

(ii) *the* ϕ_{k+1}*-mass of* $\partial M^{k+1}(\tau(\gamma_k))$ *is bigger than* μ'.

Proof. — Let $\mathcal{A}_k$ be the isomorphic lift of $\partial M^k_\ell(\alpha_k)$ which is provided by Fact 6.14. The annulus $\mathcal{A}_k$ is isometric to $\partial M^k_\ell(\alpha_k)$ for the hyperbolic metric of the spot which contains it. Let $\beta_k \subset \mathcal{A}_k$ be the lift of α_k. Since the Poincaré metric decreases under inclusion, β_k is shorter than $\varepsilon/2$ for the hyperbolic metric of $\sigma(\partial M^k_\ell)$. For the same reason, $\mathcal{A}_k$ is entirely contained in $\sigma(\partial M^k_\ell)(\beta_k)$. In particular, the $d^*\tau^*(\phi_{k+1})$-mass of $\sigma(\partial M^k_\ell)(\beta_k)$ is bigger than μ'.

Let Y be the component of ∂M^k_ℓ such that $\beta_k \subset \sigma(Y)$. Let $\mathcal{Y}$ be the covering space of M^k_ℓ with fundamental group $\pi_1(Y)$. Then $\mathcal{Y}$ is naturally contained in the manifold $\overline{M}(\pi_1(Y))$. Since $\pi_1(Y)$ is a quasi-Fuchsian group, $\overline{M}(\pi_1(Y))$ has two boundary components and these are identified with Y and $\sigma(Y)$ respectively. One component of $\partial\mathcal{Y}$ is the canonical lift of Y, whereas the others components are the spots contained in $\sigma(Y)$. From the proof of Fact 2.5 (see also Figure 2.3) β_k can be homotoped inside $\mathcal{Y}$ to a curve $\gamma'_k \subset Y$. Let f be the composition of this homotopy with the covering map $\mathcal{Y} \to M^k_\ell$. Then f is a map from the annulus into M^k_ℓ whose image joins the image of β_k —i.e. α_k— to the image of γ'_k — a curve contained in Y which we denote by γ_k. The map f is essential. It is injective on the fundamental group since α_k is not homotopic to zero. It is injective on the relative fundamental group since the two components of its lift to $\mathcal{Y}$ are contained in distinct boundary components.

Now, the length of the geodesic homotopic to $\tau(\gamma_k)$ for the hyperbolic metric on $\partial M^{k+1} = \tau^* \circ \sigma(\partial M^k)$ equals the length of the geodesic homotopic to γ_k for the hyperbolic metric on $\sigma(\partial M^k)$. This is also the length of the geodesic homotopic to β_k for the metric on $\sigma(Y)$. Therefore $\tau(\gamma_k)$ is homotopic to a geodesic shorter than $\varepsilon/2$ for the metric of ∂M^{k+1}.

Similarly the ϕ_{k+1}-mass of $\partial M^{k+1}(\tau(\gamma_k))$ equals the $d^*\tau^*\phi_{k+1}$-mass of $\sigma(\partial M^k)(\beta_k)$, and so is bigger than μ'. This proves Fact 6.14. □

In order to obtain a contradiction, we will construct by induction a finite sequence of essential annuli contained in M. The boundary of these annuli will match up under τ to produce a π_1-injective map from a torus into M/τ. This will contradict atoroidality of M/τ.

The construction begins with the following result.

Fact 6.15. — *There exists* $0 \leq k \leq C-1$, *such that* ∂M^k *contains a closed geodesic* α *shorter than* $\varepsilon/2$ *and such that the* ϕ_k*-mass of* $\partial M^k(\alpha)$ *is bigger than* ζ/C.

Proof. — The ϕ_0-mass of some component X of ∂M^0 is at least $1/C$. Denote by V the component of M^0 which contains X.

Suppose that V is not an interval bundle over a closed surface. Since $\phi_0 = d^*\sigma(d^*\tau^*\phi_1)$ and since $d^*\sigma$ decreases the mass, the $d^*\tau^*\phi_1$-mass of the union of the spots on $\sigma(\partial M^0)$ which cover X is more than $||\phi_0||_X$. By the choice of δ and since $||\phi_0||_X \geq 1/C \geq \eta$, one of these spots must (intersect and therefore) contain a component of $\mathcal{L}(\sigma(\partial M^0))^{\varepsilon/2}$. Therefore ∂M^0 contains a closed geodesic shorter than $\varepsilon/2$. Fact 6.15 follows then fr

Suppose that V is an interval bundle. Since σ is an isometry in that case, the $d^*\tau^*\phi_1$-mass of $\sigma(X)$ equals $||\phi_0||_X$. The differential $d^*\tau^*\phi_1$ is defined on a boundary component of V endowed with the reversed orientation. This component is identified by τ with a component X^1 of ∂M^1. Since τ^* is an isometry, $||\phi_1||_{X^1} = ||\phi_0||_X \geq 1/C$. If X^1 lies in the boundary of a component of M^1 wich is not an interval bundle, the reasoning of the previous case can be applied. Then Fact 6.15 holds with $k = 1$. If X^1 lies in the boundary of a component of M^1 which is an interval bundle, we can repeat the same argument. Since M has less than C components and is not an interval bundle, there exists $k \leq C-1$ and a component X^k of ∂M^k such that

(i) $||\phi_k||_{X^k} \geq 1/C$ and

(ii) X^k is not contained in the boundary of a component of M^k which is an interval bundle.

The reasoning of the first case concludes then the proof of Fact 6.15. □

We are now ready for constructing essential annuli $A_i \subset M/\tau$. We start with the curve α_k provided by Fact 6.15, for some $0 \leq k \leq C-1$. Up to shifting the indices, we suppose that $k = 0$. Since $K = C + S$, we have now a sequence $M^0, \cdots, M^i, \cdots, M^R$ with $R \geq S+1$.

Recall that each M^i is identified with M by a diffeomorphism well defined up to isotopy. In what follows we use implicitely this identification.

By Fact 6.14, there exists an essential annulus $A_0 \subset M$ with $\partial A_0 = \alpha_0 \cup \gamma_0$. The curve α_0 is shorter than $\varepsilon/2$ for the hyperbolic metric ∂M^0 and $\alpha_1 = \tau(\gamma_0)$ is homotopic to a geodesic shorter than $\varepsilon/2$ for the hyperbolic metric ∂M^1. The ϕ_1-mass of $\partial M^1(\alpha_1)$ (and in particular the ϕ_1-mass of the boundary of the component of M^1 which contains α_1) is bigger than

$$\mu' = \frac{\zeta/C - \eta}{S'} \geq \frac{\zeta}{C(2S')^2}.$$

Since $K > 2$, Fact 6.14 can be applied again. On this way, we define by induction a sequence of curves (α_i) on ∂M, such that

(i) α_i is homotopic to a geodesic shorter than $\varepsilon/2$ for the hyperbolic metric ∂M^i,

(ii) there is an essential annulus $A_i \subset M$ whose boundary components are α_i and a curve γ_i,

(iii) for $i > 0$, $\alpha_i = \tau(\gamma_{i-1})$, and

(iv) the ϕ_i-mass of $\partial M^i(\alpha_i)$ is bigger than $\zeta/C(2S')^{i+1}$.

By (i) and (iv), the sequence (α_i) can be defined as long as $i+1 \leq K$ and $i \leq R$. Since $K \geq S+1$ and $R \geq S$, it is defined at least for all $i \leq S$. The length of α_i is less than $\varepsilon/2$ for the metric ∂M^i. By the triangular inequality, the Teichmüller distance between ∂M^i and ∂M^0 is smaller than KL. Thus, our choice of ε implies that the length of α_i for the metric ∂M^0 is smaller than $\varepsilon(2)$ (cf. §1). Therefore, by the Margulis lemma, the curves α_i can be homotoped to be disjoint. Hence, by the definition of S, their homotopy classes form a set of cardinality smaller than S. Therefore, two among the curves α_i, say α_ℓ and α_m must be homotopic on ∂M. This means that the result of gluing with τ the annuli A_i along their boundaries for $\ell \leq i < m$ can be closed up with an homotopy between α_ℓ and α_m to create a map from the torus T^2 into M. This map is π_1-injective. This follows from the fact that each annulus A_i is essential. This contradiction with the hypothesis that M/τ is atoroidal concludes the proof of Theorem 6.13. □

CHAPTER 7

Manifolds-with-corners

In this chapter, we describe the topological tools necessary to reduce the proof of Thurston's hyperbolization theorem to the Final gluing theorem.

Let M be a compact orientable 3-manifold (of class C^1).

Definition. — We say that M is *irreducible* if any 2-sphere embedded in M bounds a 3-ball. We say that M is *atoroidal* if the fundamental group of any component of M does not contain subgroups isomorphic to $\mathbb{Z} + \mathbb{Z}$.

In what follows, a surface is always a compact orientable surface.

Definition. — A surface $S \subset M$ is *properly embedded* when it is a submanifold of M and when $\partial S = \partial M \cap S$.

Definition. — Suppose that $S \subset M$ is either a properly embedded surface or the disjoint union of components of ∂M. We say that S is *incompressible* when any component S' of S satisfies

(i) if $\chi(S') \leq 0$, then $\pi_1(S')$ maps injectively into $\pi_1(M)$ and $\pi_1(S', \partial S')$ maps injectively into $\pi_1(M, \partial M)$,

(ii) if $\chi(S') = 1$, then S' is *an essential disc*, i.e. a disc whose boundary is not homotopic to 0 on ∂M.

Dehn's lemma allows to replace (i) above by the following more geometric conditions (cf. [**Hel**], [**Ja**])

(i) if $\gamma \subset S'$ is a curve which bounds *a compression disc*, i.e. a disc embedded in M whose interior is disjoint from S', then γ also bounds a disc in S',

(ii) if $k \subset S'$ and $k' \subset \partial M$ are properly embedded arcs such that $k \cup k'$ bounds *a ∂-compression disc* i.e. a disc embedded in M whose interior is disjoint from S', then there is an arc $k'' \subset \partial S'$ such that $k \cup k''$ bounds a disc in S'.

Definition. — We say that $S \subset M$ is *a splitting surface* if S is an incompressible surface and if no component of S can be isotoped into ∂M.

Haken manifolds.

Definition. — A compact connected manifold M is *Haken* if it is irreducible and if it contains a splitting surface.

When M is Haken, it contains a splitting surface S and a new manifold M' can be formed by *splitting* M *along* S. This manifold M' is defined as the complement in M of an open regular neighborhood of S. It follows from Dehn's lemma that M' is irreducible. The boundary of any component of M' is non-empty. Therefore, M' is a 3-ball or is again Haken (cf. [**He1**], [**Ja**]).

We need now to distinguish a particular class of 3-manifolds.

Definition. — A manifold M is called *an handlebody* if it is diffeomorphic to the manifold obtained from the 3-ball by attaching g *1-handles* $I \times \mathbb{B}^2$ along $\partial I \times \mathbb{B}^2$. The integer $g = g(M)$ is called *the genus of* M.

Let M be an handlebody with $g(M) \geq 1$. Then M is Haken since it contains an essential disc, for instance $1/2 \times \mathbb{B}^2$. Furthermore the manifold obtained by splitting M along the surface which is the disjoint union of the g discs $1/2 \times \mathbb{B}^2$ contained in each 1-handle is diffeomorphic to $\mathbb{B}^3$.

Definition. — Let M be an handlebody. *A system of meridians for* M is a disjoint union of essential discs properly embedded in M such that the manifold obtained by splitting M along m is diffeomorphic to $\mathbb{B}^3$.

Clearly, each system of meridians for M contains g essential discs, if g is the genus of M.

Handlebodies are the most docile of all Haken manifolds: they can be immediatly reduced to the 3-ball. If M is a Haken manifold which is not an handlebody, then it contains a connected splitting surface which is not an essential disc (cf. [**Ja**, p. 59]). Furthermore, if $\partial M \neq \emptyset$, such a splitting surface exists which has non-empty boundary ([**Ja**, p. 35]). A connected splitting surface with these two properties is called *a special splitting surface.*

Definition. — Let M be a compact connected Haken manifold. *A partial hierarchy of length* n *for* M is a finite sequence $M_0, \cdots, M_n$ such that

(i) $M_0 = M$, and

(ii) for $k \leq n-1$, there is a special splitting surface $S_k \subset M_k$, such that M_{k+1} is obtained by splitting M_k along S_k.

If M is a Haken manifold which is not an handlebody, it admits a partial hierarchy of length at least 1.

Definition. — We define *the length of* M, as the largest integer n such that M admits a partial hierarchy of length n. We denote it by $\ell(M)$.

It is a basic observation due to Wolfgang Haken that $\ell(M)$ is always finite ([**Hak**], [**Ja**, p. 61]). Also, $\ell(M) = 0$ if and only if M is an handlebody. Another important property which follows from the defini if M is not an handlebody and if M' is a component of the manifold obtained by splitting M along an *arbitrary* special splitting surface, then $\ell(M') < \ell(M)$. So, if M is a Haken manifold, $\ell(M)$ measures the distance from M to handlebodies. This measure of the complexity of M simply with an integer goes back to Haken. It will be used during the proof of the Hyperbolization theorem for manifolds-with-corners.

Definition. — Let $S \subset M$ be a properly embedded orientable surface. Let $D \subset M$ be a compression disc for S and let $\gamma = \partial D$. Let $N(D)$ be an open regular neighborhood of (D, γ) in (M, S): the boundary of $N(D)$ consists of an annulus contained in S and two parallel copies of D. Then we define a new surface S' properly embedded in M as the union of $S - N(D)$ and these two discs. Similarly, let D be a ∂-compression disc for S and set $k = D \cap S$, $k' = D \cap \partial M$. Let $N(D)$ be an open regular neighborhood of (D, k) in (M, S). We define a new surface S' as the union of $S - N(D)$ and the two parralel copies of D contained in $N(D)$. We say (in both cases) that S' is obtained from S by *surgery along* D.

Fact 7.1. — *Let S be a splitting surface for M. Let D be a ∂-compression disc for S. Let S' be the surface obtained from S by surgery along D. If S is a special splitting surface, then one component of S' also. If M is an handlebody and if S is a system of meridians for M, then S' contains a system of meridians.*

Proof. — When S is a special splitting surface, then one component of S' is a disc and the other component, denoted by Σ, is homeomorphic to S. It follows from standard technics in 3-dimensional topology that Σ is a splitting surface for M. Since Σ is homeomorphic to S, it is a special splitting surface.

Let M be a handlebody and let S be a system of meridians for M. Let m be the component of S which intersects D. Then the surface obtained from m by surgery along D is the disjoint union of two discs. Since the union of these two discs is homologuous to m, one of them, called m', is non-separating. It follows that $S - \{m\} \cup \{m'\}$ is a system of meridians for M. □

Interval bundles.

Definition. — *An interval bundle* is an orientable manifold which fibers over a closed surf[illegible] with fiber diffeomorphic to $[0,1]$.

Up to d[illegible]morphism, there are two types of interval bundles: *the trivial product* $S \times [0,1]$ of a closed connected orientable surface S with the interval, and *the twisted product* $S \tilde{\times} [0,1]$ of S with the interval, i.e. the quotient of $S \times [0,1]$ by the relation: $(x, 0) \simeq (y, 0)$ if and only if $x = \tau(y)$, where τ is an orientation reversing fixed point free involution of S.

The following characterization of interval bundles was used in §2 (cf. **[Hel]**).

Theorem [Stal]. — *Let M be a connected irreducible manifold with non-empty boundary and let S be a component of M. If the index of $\pi_1(S)$ in $\pi_1(M)$ equals 1, then the inclusion of S into M extends to a diffeomorphism between $S \times [0,1]$ and M. If the index of $\pi_1(S)$ in $\pi_1(M)$ equals 2, then the inclusion of S extends to a diffeomorphism between $S \tilde{\times} [0,1]$ and M.*

7.1 Manifolds-with-corners

Definition. — A *manifold-with-corners* is a triple $(M, \mathcal{G}, \partial^0 M)$ such that

(i) M is a compact 3-manifold,

(ii) $\mathcal{G}$ is a trivalent graph contained in ∂M,

(iii) each component of $\partial M - \mathcal{G}$ equals the interior of its closure, and

(iv) each component of $\partial^0 M$ equals the closure of a component of $\partial M - \mathcal{G}$.

The closure of a component of $\partial M - \mathcal{G}$ which is not contained in $\partial^0 M$ is called *a mirror of* $(M, \mathcal{G}, \partial^0 M)$. The surface $\partial^0 M$ is called *the boundary of* $(M, \mathcal{G}, \partial^0 M)$.

From a differentiable structure on M, we can construct *a differentiable structure with corners on* M *associated to* $(M, \mathcal{G}, \partial^0 M)$, i.e. there is an atlas of class C^1 on M such that

(i) each point of M has a neighborhood isomorphic to an open set in $(\mathbb{R}^+)^3$, and

(ii) the points on edges (resp. the vertices) of $\mathcal{G}$ are exactly the points which have a neighborhood isomorphic to the neighborhood of a point of $(\mathbb{R}^+)^3$ with 2 (resp. 3) coordinates equal to 0.

When we are given a manifold-with-corners $(M, \mathcal{G}, \partial^0 M)$, we always think that M has such a differentiable structure with corners, i.e. that corners are really corners. This structure is unique. Any homeomorphism of M which preserves $\mathcal{G}$ is isotopic to a diffeomorphism: this follows essentially from [C].

Remark. — The definition of a manifold-with-corners can be made in any dimension. For instance *a surface-with-corners* having empty boundary is a pair (S, P) where S is a compact surface and $P \subset \partial S$ a finite set such that any component of $\partial S - P$ equals the interior of its closure. This is clearly equivalent to say that each component of ∂S either is disjoint from P or contains at least 2 points of P. In particular, let $(M, \mathcal{G}, \partial^0 M)$ be a manifold-with-corners and let P be the set of vertices of $\mathcal{G}$ contained in (the boundary of) $\partial^0 M$. Then $(\partial^0 M, P)$ is a surface-with-corners.

Definition. — Let $(M, \mathcal{G})$ be a manifold-with-corners having empty boundary. Let $S' \subset \partial M$ be a surface which is a union of mirrors of $(M, \mathcal{G})$. Let $\mathcal{G}'$ the trivalent graph obtained by erasing the edges of $\mathcal{G}$ which intersect the interior of S'. Then $(M, \mathcal{G}', S')$ is a manifold-with-corners. We say that $(M, \mathcal{G}', S')$ is obtained from $(M, \mathcal{G})$ by *erasing the mirrors contained in* S'.

Given a differentiable structure with corners on M associated to $(M, \mathcal{G})$, one can define a differentiable structure on M associated to $(M, \mathcal{G}', S')$ by "rounding the corners" contained in the interior of S'. This differentiable structure is unique up to diffeomorphism [**Do**].

Let $(M, \mathcal{G}, \partial^0 M)$ be a manifold-with-corners.

Definition. — Let $\gamma \subset \partial M$ be an embedded closed curve. We say that γ *intersects* $\mathcal{G}$ *tranversally* if γ is disjoint from the vertices of $\mathcal{G}$ and if γ intersects the edges of $\mathcal{G}$ transversally.

Definition. — Let $(M, \mathcal{G}, \partial^0 M)$ be a manifold-with-corners. We say that $(M, \mathcal{G}, \partial^0 M)$ has *incompressible boundary* when, if $\gamma \subset \partial M$ is an embedded closed curve which bounds a disc in M and satisfies

(i) $\gamma \cap \partial^0 M \neq \emptyset$, and

(ii) γ intersects $\mathcal{G}$ transversally in at most 3 points,

then γ bounds a disc in ∂M which has one of the forms described in Figure 7.1.

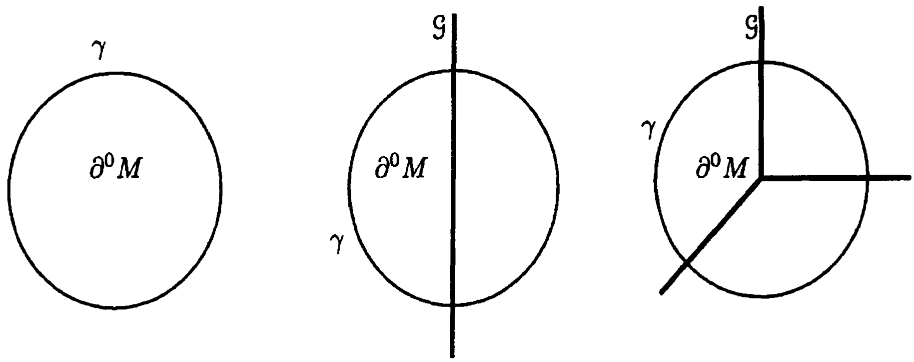

Figure 7.1

Definition. — Let $(M, \mathcal{G})$ be a manifold-with-corners having empty boundary. We say that $(M, \mathcal{G})$ *is irreducible* and *atoroidal* if the following conditions are satisfied:

(i) M is irreducible and atoroidal,

(ii) if A is an annulus properly embedded in M with $\partial A \cap \mathcal{G} = \emptyset$, and such that $\pi_1(A)$ maps injectively into $\pi_1(M)$, then A is parallel to an annulus contained in a mirror of $(M, \mathcal{G})$, and

(iii) if $\gamma \subset \partial M$ is an embedded closed curve which intersects $\mathcal{G}$ transversally in at most 4 points and which bounds a disc in M, then γ bounds a disc in ∂M whose intersection with $\mathcal{G}$ has one of the forms described in Figure 7.2.

Remark. — Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary. Then it follows from the definition that any mirror of $(M, \mathcal{G})$ has one of the following shapes:

(i) a n-gon with $n \geq 5$,

(ii) an annulus with at least one vertex of $\mathcal{G}$ in its boundary, or

(iii) a surface with strictly negative Euler characteristic.

Remark. — The present notion of a manifold-with-corners essentially coincides with that of an orbifold that happens to be modelled on the quotient of $\mathbb{R}^3$ by the group of eight elements generated by sign reversal of one of the 3 coordinates.

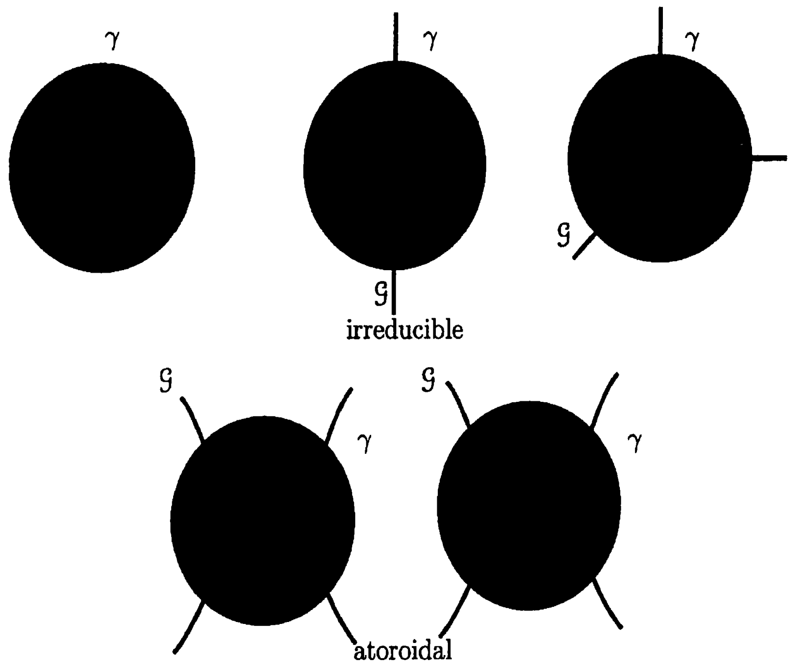

Figure 7.2

A manifold-with-corners is irreducible and atoroidal if and only if the corresponding orbifold is irreducible and atoroidal, i.e. does not contain spherical nor euclidean suborbifolds (cf. [Th1]).

Proposition 7.2. — *Let M be an irreducible and atoroidal 3-manifold. Then there is a graph $\mathcal{G} \subset \partial M$ such that $(M, \mathcal{G})$ is an irreducible and atoroidal manifold-with-corners having empty boundary.*

Proof. — We can suppose $\partial M \neq \emptyset$. Let $\mathcal{T}$ be a triangulation of ∂M. Refine $\mathcal{T}$ to a triangulation $\mathcal{T}'$ by modifying it inside each triangle as described on Figure 7.3 (this triangulation was shown to us by Emmanuel Giroux). Let $\mathcal{G}$ be 1-skeletton of the cellulation dual to $\mathcal{T}'$. Then $(M, \mathcal{G})$ defines a manifold-with-corners having empty boundary. Each mirror of $(M, \mathcal{G})$ is homeomorphic to a disc. To prove that $(M, \mathcal{G})$ is irreducible and atoroidal, it suffices to check property (iii). In fact, we will prove that any closed curve $\gamma \subset \partial M$ which intersects $\mathcal{G}$ in at most 4 points is as described on Figure 7.3. Such a curve γ gives rise to a closed path $\bar{\gamma}$ contained in the 1-skeletton of $\mathcal{T}'$ which follows at most 4 edges. From the way $\mathcal{T}'$ was defined, $\bar{\gamma}$ is contained in the union of at most 2 triangles of $\mathcal{T}$ which have a common edge or a common vertex. Then a new look at Figure 7.3 shows the existence of a disc bounded by γ which intersects $\mathcal{G}$ like on Figure 7.2. □

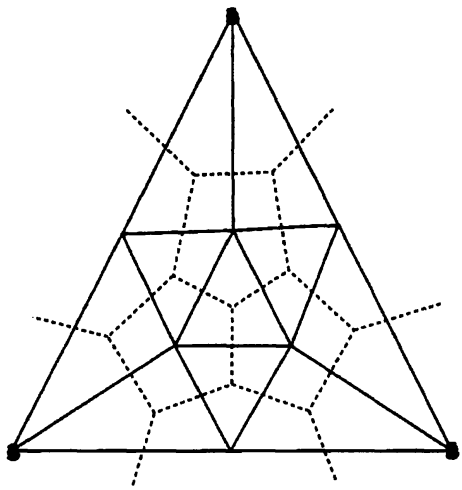

Figure 7.3

Splitting of a manifold-with-corners.

We now explain how the notion of splitting of a 3-manifold along a splitting surface can be extended to the context of manifolds-with-corners.

Let $(M, \mathcal{G})$ be a manifold-with-corners having empty boundary. Let S be a splitting surface for M such that ∂S intersects $\mathcal{G}$ transversally. .

Definition.— Let M_S be the manifold obtained by splitting M along S. Denote by S^+, S^- the two copies of S which are contained in ∂M_S and set $S' = S^+ \cup S^-$. There is an orientation reversing involution τ' of S' such that M is diffeomorphic to the quotient space M_S/τ' i.e. to the space of equivalence classes of the relation

$$x \simeq y \text{ if and only if } x \in S' \text{ and } y = \tau'(x).$$

Denote the quotient map by $\pi' : M_S \to M$. Let $\mathcal{G}_S \subset \partial M_S$ be the graph equal to $(\pi')^{-1}(\mathcal{G} \cup \partial S)$. Each vertex of $\mathcal{G}_S$ has valence 3. The closure of any component of $\partial M_S - \mathcal{G}_S$ equals either a component of S' or the closure of a component of $f - (f \cap \partial S)$ where f is a mirror of $(M, \mathcal{G})$. Thus $(M_S, \mathcal{G}_S, S')$ is a manifold-with-corners. We say that $(M_S, \mathcal{G}_S, S')$ is obtained by *splitting* $(M, \mathcal{G})$ *along* S.

Let $\mathcal{P}$ be the set of vertices of $\mathcal{G}_S$ which are contained in $\partial S'$. Then $(S', \mathcal{P})$ is a surface-with-corners having empty boundary. By construction, τ' preserves $\mathcal{P}$. Therefore τ' can be considered as an orientation reversing involution of $(S', \mathcal{P})$. Then, $(M, \mathcal{G})$ can be reconstructed from the data of $(M_S, \mathcal{G}_S)$ and τ'. The manifold M is diffeomorphic to the quotient M_S/τ' and $\mathcal{G}$ equals the complement in $\mathcal{G}_S/\tau'$ of the interior of the edges of $\mathcal{G}_S \cap \partial S'$.

Definition.— Let $(M, \mathcal{G})$ be a connected manifold-with-corners having empty boundary. *A good splitting surface for* $(M, \mathcal{G})$ is a splitting surface for M such that

(i) S is a system of meridians if M is an handlebody, and S is a special splitting surface if not,

(ii) ∂S intersects $\mathcal{G}$ transversally, and

(iii) $(M_S, \mathcal{G}_S, S')$ has incompressible boundary.

Proposition 7.3.— *Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary. Suppose that M is Haken. Then there is a good splitting surface for $(M, \mathcal{G})$.*

Proof.— Since M is Haken, it contains a splitting surface S which is a system of meridians if M is an handlebody, or a special splitting surface if it is not. We can assume furthermore that S intersects $\mathcal{G}$ transversally and minimizes $\sharp(\partial\Sigma \cap \mathcal{G})$ among all the splitting surfaces Σ such that

(i) Σ is a special splitting surface for M if M is not an handlebody, or is a system of meridians if M is an handlebody, and

(ii) Σ intersects $\mathcal{G}$ transversally.

We show now that $(M_S, \mathcal{G}_S, S')$ has incompressible boundary.

Let $\gamma \subset \partial M_S$ be a closed curve such that $\gamma \cap S' \neq \emptyset$, which intersects $\mathcal{G}_S$ transversally in at most 3 points and which bounds a disc D embedded in M_S. Observe that $\gamma \cap \partial S'$ contains an even number of points.

1) $\gamma \cap \partial S' = \emptyset$.

Then $\gamma \subset S'$ and γ bounds a disc in S' since S is incompressible in M.

2) $\gamma \cap \partial S' \neq \emptyset$.

If $\sharp(\gamma \cap \mathcal{G}_S) = 2$ (resp. 3), γ is the union of one arc $k \subset S'$ and one arc k' contained in a mirror f' of $(M_S, \mathcal{G}_S, S')$ (resp two arcs k'_1 and k'_2 contained in mirrors f'_1, f'_2 of $(M_S, \mathcal{G}_S, S')$). Then $\pi'(D)$ is a disc embedded in M and its boundary is the union of the two arcs $\pi'(k) \subset S$ and $\pi'(k') \subset \partial M$ (resp. $\pi'(k'_1 \cup k'_2)$). By Fact 7.1, the surface obtained from S by surgery along $\pi'(D)$ contains a splitting surface for M which is a system of meridians if M is a handlebody or a special splitting surface if not. We observe that $\partial\Sigma$ is obtained from ∂S by replacing an arc $\kappa \subset \partial D'$ by $\pi'(k')$ (resp. $\pi'(k'_1 \cup k'_2)$) and that $\kappa \cup \pi'(k')$ (resp. $\kappa \cup \pi'(k'_1 \cup k'_2)$) is a closed curve embedded in ∂M which bounds a disc in M.

2a) $\sharp(\gamma \cap \mathcal{G}_S) = 2$.

The mirror f' is obtained by splitting a mirror f of $(M, \mathcal{G})$ and $\pi'(k')$ is contained in f. If $\kappa \cap \mathcal{G} = \emptyset$, then $\kappa \cup \pi'(k')$ is contained in f. Since $(M, \mathcal{G})$ is irreducible and atoroidal, $\kappa \cup \pi'(k')$ bounds a disc contained in f. It follows that γ bounds a disc on ∂M_S which is as described in Figure 7.1. If $\kappa \cap \mathcal{G} \neq \emptyset$, then $\sharp(\partial\Sigma \cap \mathcal{G}) < \sharp(\partial S \cap \mathcal{G})$. This is impossible by the choice of S.

2b) $\sharp(\gamma \cap \mathcal{G}_S) = 3$.

Then f'_1 and f'_2 intersect along an edge of $\mathcal{G}_S$. Each mirror f'_i is obtained by splitting a mirror f_i of $(M, \mathcal{G})$ so that f_1 and f_2 share an edge e of $\mathcal{G}$ in common. We have $\pi'(k'_1 \cup k'_2) \subset f_1 \cup f_2$ and $\pi'(k'_1 \cup k'_2)$ intersects e in exactly one point. If $\kappa \cap \mathcal{G} = 0$ or 1 point, then $\kappa \cup \pi'(k'_1 \cup k'_2)$ intersects $\mathcal{G}$ in 1 or 2 points. Since $(M, \mathcal{G})$

is irreducible and atoroidal, $\kappa \cup \pi'(k'_1 \cup k'_2)$ bounds a disc contained in $f_1 \cup f_2$. It follows that γ bounds a disc contained in ∂M_S which intersects $\mathcal{G}_S$ like on Figure 7.1. If $\sharp(\kappa \cap \mathcal{G}) \geq 2$, then $\sharp(\partial\Sigma \cap \mathcal{G}) < \sharp(\partial S \cap \mathcal{G})$. This contradicts our choice of S. □

Proposition 7.4.— *Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary. Let S be a good splitting surface for $(M, \mathcal{G})$. Then there is a trivalent graph $\mathcal{G}'_S \subset \partial M_S$ such that*

(i) *$(M_S, \mathcal{G}'_S)$ is an irreducible and atoroidal manifold-with-corners having empty boundary,*

(ii) *S' is a union of mirrors of $(M_S, \mathcal{G}'_S)$, and*

(iii) *$(M_S, \mathcal{G}_S, S')$ is obtained from $(M_S, \mathcal{G}'_S)$ by erasing the mirrors contained in S'.*

The meaning of this lemma is that we can add to $\mathcal{G}_S$ edges which are contained in S' to obtain a graph $\mathcal{G}'_S$ such that $(M_S, \mathcal{G}'_S)$ is an irreducible and atoroidal manifold-with-corners having empty boundary.

Proof.— When $\partial S \neq \emptyset$, we begin by adding finitely many points to the set of vertices of $\mathcal{G}_S$ which sit on $\partial S'$, so that this new set of points forms the 0-skeletton of a triangulation of $\partial S'$ (i.e. so that there are at least 3 points per components). Denote by $\mathcal{V}$ this set of vertices. Choose now a triangulation $\mathcal{T}$ of S' which extends this triangulation of $\partial S'$. Refine then $\mathcal{T}$ to a new triangulation $\mathcal{T}'$ by modifying it inside each triangle as described on Figure 7.3. Let $\mathcal{E}$ be the 1-skeletton of the dual cellulation of $\mathcal{T}'$. Observe that each edge of $\mathcal{E}$ which is contained in $\partial S'$ contains at most one point of $\mathcal{V}$ in its interior, and in particular at most one vertex of $\mathcal{G}_S$. The union of the edges of $\mathcal{E}$ which intersect the interior of S' form a graph $\mathcal{E}'$. The vertices of $\mathcal{E}'$ which are contained in the interior of S' have valence 3 and those which are contained in $\partial S'$ have valence 1.

Define $\mathcal{G}'_S = \mathcal{G}_S \cup \mathcal{E}'$. By construction, $(M_S, \mathcal{G}'_S)$ is a manifold-with-corners which satisfies Proposition 7.4 (ii), (iii). Observe also that each mirror of $(M_S, \mathcal{G}'_S)$ which is contained in S' intersects $\partial S'$ in at most one edge, and that this edge contains at most one point of $\mathcal{G}_S$ in its interior.

We show now that $(M_S, \mathcal{G}'_S)$ is irreducible and atoroidal. If A is an annulus properly embedded in M_S such that $\pi_1(A)$ maps injectively into $\pi_1(M_S)$, then $\pi_1(A)$ maps injectively into $\pi_1(M)$, by Van Kampen and since S is incompressible. Therefore, there is a parallelism in M between A and an annulus contained in a mirror of $(M, \mathcal{G})$, since $(M, \mathcal{G})$ is irreducible and atoroidal. By incompressibility, S is disjoint from this parallelism. Hence, A is parallel in M_S to an annulus contained in a mirror of $(M_S, \mathcal{G}_S)$.

Let $\gamma \subset \partial M_S$ be a closed curve which bounds a disc embedded in M_S and which intersects $\mathcal{G}'_S$ transversally in at most 4 points. We consider distinct cases, according to the cardinality of $\gamma \cap \partial S'$.

1) $\gamma \subset S'$.

Then γ gives rise to a closed path contained in the 1-skeletton of $\mathcal{T}'$ which follows at most 4 edges. One deduces from the way the triangulation $\mathcal{T}'$ was defined that γ bounds a disc on ∂M_S which intersects $\mathcal{G}'_S$ (i.e. $\mathcal{E}'$) like on Figure 7.2.

2) $\gamma \cap S' = \emptyset$.

Then $\pi'(\gamma)$ is a closed curve which intersects $\mathcal{G}$ transversally in at most 4 points and which bounds a disc embedded in M. Since $(M, \mathcal{G})$ is irreducible and atoroidal, $\pi'(\gamma)$ bounds a disc on ∂M which intersects $\mathcal{G}$ like on Figure 7.2. Since S is a splitting surface, this disc is disjoint from ∂S. It follows that γ bounds a disc on ∂M_S which intersects $\mathcal{G}'_S$ (i.e. $\mathcal{G}_S$) like on Figure 7.2 (cf. Proposition 7.2).

3) $\sharp(\gamma \cap \partial S') = 2$.

Then $\gamma \cap S'$ is a single arc that we denote by k. We distinguish two subcases.

3a) $k \cap \mathcal{E}' = \emptyset$.

Then k is contained in a mirror f of $(M_S, \mathcal{G}'_S)$. By the construction of $\mathcal{E}$, f is simply connected and $e = f \cap \partial S'$ is a single 1-cell of $\mathcal{E}$. Therefore, k can be isotoped relatively to ∂k to an arc $\kappa \subset e$. By construction, e contains at most one vertex of $\mathcal{G}_S$. Therefore if γ' denotes the curve obtained from γ by replacing k by κ and slightly perturbed to be disjoint from S', then $\pi'(\gamma')$ is a closed curve which intersects $\mathcal{G}_S$ transversally in at most 3 points and which bounds a disc embedded in M. Since $(M, \mathcal{G})$ is irreducible and atoroidal, $\pi'(\gamma')$ bounds a disc on ∂M which intersects $\mathcal{G}$ like on Figure 7.2. This disc is disjoint from S. Therefore γ bounds also a disc contained in ∂M_S which intersects $\mathcal{G}'_S$ like on Figure 7.2.

3b) $k \cap \mathcal{E}' \neq \emptyset$.

Then $\sharp(\gamma \cap \mathcal{G}_S) \leq 3$. Since $(M_S, \mathcal{G}_S, S')$ has incompressible boundary, γ bounds a disc $\Delta \subset \partial M_S$ which intersects $\mathcal{G}_S$ like on Figure 7.1. Since $\sharp(k \cap \mathcal{G}'_S) \leq 4$, k gives rise to a path contained in the 1-skeletton of $\mathcal{T}'$ which follows at most 2 1-cells. From the way $\mathcal{T}'$ was defined, we deduce that Δ intersects $\mathcal{G}'_S$ as described on Figure 7.2.

4) $\sharp(\gamma \cap \partial S') = 4$.

Then $\gamma \cap S'$ is the union of two disjoints arcs k_1 and k_2 which are contained in mirrors of $(M_S, \mathcal{G}'_S)$. By the same reasoning as in 3a), k_i can be isotoped relatively to ∂k_i into an arc κ_i contained in an edge e_i of $\mathcal{E}$. Also, κ_i contains at most one vertex of $\mathcal{G}_S$ in its interior. Let γ' be the curve obtained from γ by replacing k_i by κ_i for $i = 1, 2$ and perturbed to be disjoint from S'. Then $\pi'(\gamma')$ is a closed curve which bounds a disc in M and which intersects $\mathcal{G}_S$ in at most 2 points. Since $(M, \mathcal{G})$ is irreducible and atoroidal, $\pi'(\gamma')$ bounds a disc on ∂M which intersects $\mathcal{G}$ like on Figure 7.2. It follows that γ bounds a disc contained in ∂M_S which intersects $\mathcal{G}'_S$ like in Figure 7.2. □

7.2 The mirrored manifold

Let $(M, \mathcal{G}, \partial^0 M)$ be a manifold-with-corners. We explain now a canonical procedure for associating to $(M, \mathcal{G}, \partial^0 M)$ a compact 3-manifold. This 3-manifold will carry an action by diffeomorphisms of a finite group which encodes the data of $(M, \mathcal{G}, \partial^0 M)$.

For a set $\mathcal{S}$, we denote by $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{S}}$ the set of maps from $\mathcal{S}$ to $\mathbb{Z}/2\mathbb{Z}$.

Definition. — Let $\mathcal{M}$ be the set of mirrors of $(M, \mathcal{G}, \partial^0 M)$. *The mirrored manifold* $\widehat{M}$ is the quotient of $M \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ by the equivalence relation $\mathcal{R}$ generated by

$(x, i) \sim (x, j)$ when x is contained in the mirror $f \in \mathcal{M}$ and the two maps i and j satisfy $i(f) = j(f) + 1$ and $i(f') = j(f')$ for $f' \neq f$.

By looking at these identifications in the neighborhood of point which is contained in a mirror of $(M, \mathcal{G})$ or in an edge of $\mathcal{G}$ or which is a vertex of $\mathcal{G}$, one checks that $\widehat{M}$ is a manifold with boundary (cf. Figure 7.4). Clearly the action of $H = (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ on $M \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ by translation on the second factor preserves the equivalence classes of $\mathcal{R}$. It projects therefore to an action by homeomorphisms of H on $\widehat{M}$. The group H is *the mirror group of* $\widehat{M}$. When $(M, \mathcal{G}, \partial^0 M)$ carries a differentiable structure with corners, then $\widehat{M}$ inherits a differentiable structure and the mirror group acts by diffeomorphisms of class C^1.

Remark. — If M is connected, then $\widehat{M}$ is connected. This is due to the fact that for any two maps i and j in $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$, there is a sequence $(i_k)_{k=1,\cdots,p}$ such that

(i) $i_1 = i$, $i_p = j$ and

(ii) for $k \leq p-1$ the maps i_k and i_{k+1} agree except on a single mirror.

It follows that if M is connected, the two copies $M \times i$ and $M \times j$ map to the same component of $\widehat{M}$ under the equivalence relation $\mathcal{R}$.

An important feature of the construction of $\widehat{M}$ is that one can recover $(M, \mathcal{G}, \partial^0 M)$ from the action of H on $\widehat{M}$. First the quotient space $\widehat{M}/H$ is clearly homeomorphic to M and $\partial\widehat{M}$ maps to a surface $\partial\widehat{M}/H$ contained in $\partial(\widehat{M}/H)$. The isotropy group of a point in $\widehat{M}$ under the action of H is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$ or $(\mathbb{Z}/2\mathbb{Z})^3$. The set of points in $\partial(\widehat{M}/H)$ whose isotropy group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ or $(\mathbb{Z}/2\mathbb{Z})^3$ form a trivalent graph $\mathcal{G}' \subset \partial(\widehat{M}/H)$. Each mirror of $(\widehat{M}/H, \mathcal{G}', \partial\widehat{M}/H)$ can be characterized as the closure of the set of points whose isotropy group is a given non-zero element in H. Therefore $(\widehat{M}/H, \mathcal{G}', \partial\widehat{M}/H)$ is a manifold-with-corners isomorphic to $(M, \mathcal{G}, \partial^0 M)$.

Definition. — Let f be a mirror of $(M, \mathcal{G}, \partial^0 M)$. Let h_f be the element of H which has the f-coordinate as its single non-zero coordinate. Then h_f acts as an involution on $\widehat{M}$. Its fixed point set is exactly the image under $\mathcal{R}$ of $f \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$. This fixed point set can be described as follows. Let $\mathcal{S}$ be the set of mirrors of $(M, \mathcal{G}, \partial^0 M)$ which intersect f. Consider the natural inclusion of $(\mathbb{Z}/2\mathbb{Z})^{\mathcal{S}}$ into H where $i : \mathcal{S} \to \mathbb{Z}/2\mathbb{Z}$ is extended to a map $\mathcal{M} \to \mathbb{Z}/2\mathbb{Z}$ by the constant map 0 on $\mathcal{M} - \mathcal{S}$. Then the image of $f \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ under the equivalence relation $\mathcal{R}$ is made of disjoint homeomorphic copies of the image of $f \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{S}}$. This image is an embedded surface denoted by $\mathcal{F}$ and called *the surface in* $\widehat{M}$ *above* f.

Remark. — Suppose that $(M, \mathcal{G})$ is irreducible and atoroidal. From the description of the possible shapes for f, it follows that f is a hyperbolic surface-with-corners, i.e. that f has a hyperbolic metric such that the edges of $\mathcal{G}$ contained in ∂f are totally geodesic and that adjacent edges meet orthogonally. This implies that $\mathcal{F}$ is a hyperbolic surface and therefore that $\chi(\mathcal{F}) < 0$.

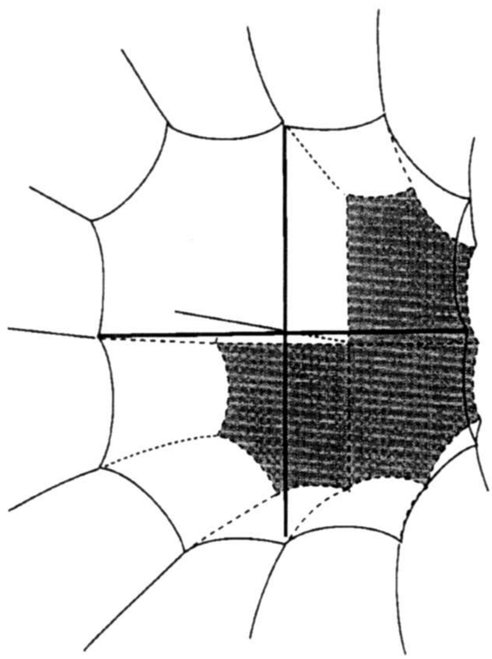

Figure 7.4

We describe now a relative version of the construction of the mirrored manifold.

The partially mirrored manifold.

Let $(M, \mathcal{G})$ be a manifold-with-corners having empty boundary. Let $S \subset \partial M$ be a splitting surface for $(M, \mathcal{G})$. Let $(M_S, \mathcal{G}_S, S')$ be the manifold-with-corners obtained by splitting $(M, \mathcal{G})$ along S. Then each mirror of $(M_S, \mathcal{G}_S, S')$ is contained in a unique mirror of $(M, \mathcal{G})$.

Definition. — We denote by $\widehat{M_S}$ the quotient space of $M_S \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ by the equivalence relation $\mathcal{R}_S$ generated by the relation $(x, i) \sim (x, j)$ if and only if

(i) x belongs to a mirror f' of $(M_S, \mathcal{G}_S, S')$ which is contained in the mirror $f \in \mathcal{M}$, and

(ii) the maps i and j satisfy $i(f) = j(f) + 1$ and $i(f'') = j(f'')$ for $f'' \neq f$.

$\widehat{M_S}$ is called *the partially mirrored manifold of* $(M_S, \mathcal{G}_S, S')$.

Then $\widehat{M_S}$ is a compact 3-manifold on which the group $H = (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ acts by homeomorphisms. The boundary of $\widehat{M_S}$ is the image of $S' \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ under the equivalence relation $\mathcal{R}_S$. Like in the case of $\widehat{M}$, $\widehat{M_S}$ is connected once M_S is. If we put a differentiable structure with corners on M associated to $(M, \mathcal{G})$, then M_S has a differentiable structure with corners associated to $(M_S, \mathcal{G}_S, S')$. Then $\widehat{M_S}$ is manifold with boundary of class C^1 on which H acts by diffeomorphisms, and, like for the case of the mirrored manifold, the manifold-with-corners $(M_S, \mathcal{G}_S, S')$ can be reconstructed from the action of H on $\widehat{M_S}$.

We will consider too other differentiable structures on $\widehat{M_S}$. Let $(M_S, \mathcal{G}'_S)$ be a manifold-with-corners having empty boundary such that S' is a union of mirrors of $(M_S, \mathcal{G}'_S)$. Suppose that $(M_S, \mathcal{G}_S, S')$ is obtained from $(M_S, \mathcal{G}'_S)$ by erasing the mirrors contained in S'. When M is endowed with a differentiable structure with corners associated to $(M_S, \mathcal{G}'_S)$, then $\widehat{M_S}$ inherits a differentiable structure with (maybe) corners on the boundary. Denote by $\widehat{M'_S}$ this differentiable manifold-with-corners. Then H acts by diffeomorphisms on $\widehat{M'_S}$ and $\widehat{M_S}$ is H-equivariantly diffeomorphic to the manifold obtained from $\widehat{M'_S}$ by rounding the corners.

Recall now that $(M, \mathcal{G})$ can be described as the quotient space $(M_S, \mathcal{G}_S, S')/\tau'$. In a similar way, we can describe $\widehat{M}$ as a quotient of $\widehat{M_S}$. Define for that an involution τ'' of $\partial M_S \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ by $\tau''(x, i) = (\tau'(x), i)$. Then τ'' preserves the classes of the equivalence relation $\mathcal{R}_S$ and commutes with each element in H. It induces therefore an orientation reversing involution τ of $\partial\widehat{M_S}$ which exchanges the components by pairs and which commutes with the action of H. It follows from the definitions of $\widehat{M_S}$ and τ, that $\widehat{M}$ is H-equivariantly diffeomorphic to $\widehat{M_S}/\tau$.

We justify now the terminology irreducible and atoroidal for a manifold-with-corners.

Proposition 7.5.— *Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary. Then $\widehat{M}$ is irreducible and atoroidal.*

Proof.— This proposition could be proven using elementary technics from 3-dimensional topology. But to simplify the exposition we will use "the Equivariant Dehn's lemma", "the Equivariant sphere theorem" and "the Equivariant torus theorem". Recall that $\widehat{M}$ carries an action of $H = (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$.

Suppose for a contradiction that $\widehat{M}$ is not irreducible. Then $\widehat{M}$ contains *an essential sphere*, i.e. an embedded sphere which does not bound a ball. By the Equivariant sphere theorem ([**MY**], [**Du**]) there is an essential sphere Σ which is disjoint or equal to any of its translates by elements of H. Let $H(\Sigma)$ denote the stabilizer of Σ in H. Then $H(\Sigma)$ is isomorphic to the trivial group, to $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$ or to $(\mathbb{Z}/2\mathbb{Z})^3$. A fundamental domain for the action of $H(\Sigma)$ on Σ is naturally contained in a copy $M \times i$ of M sitting in $\widehat{M}$. This fundamental domain is Σ or a disc whose boundary intersects $\mathcal{G}$ transversally in 0, 2 or 3 points. Since $(M, \mathcal{G})$ is an irreducible manifold-with-corners, this contradicts that Σ is essential.

In order to show that $\widehat{M}$ is atoroidal, we first prove that it is Haken. To see this, select a mirror $f \in \mathcal{M}$. Consider the involution h_f of $\widehat{M}$ which is associated to f and consider the surface $\mathcal{F}$ above f. The quotient $\widehat{M}/h_f$ is a manifold with boundary which naturally embeds in $\widehat{M}$. The group $H' = H/\langle h_f \rangle$ acts by diffeomorphism on $\widehat{M}/h_f$ and $\partial\widehat{M}$ equals the H'-orbit of $\mathcal{F}$. Suppose for a contradiction that $\mathcal{F}$ is not incompressible in $\widehat{M}$. Then by Van Kampen's theorem $\mathcal{F}$ is not incompressible in $\widehat{M}/h_f$ and by Dehn's lemma, there is an essential disc properly embedded in $\widehat{M}/h_f$. By the Equivariant Dehn's lemma [MY], there is such a disc D which is either disjoint or equal to any of its translates by H'. Let $H'(D)$ be the stabilizer of D' in H'. Then $H'(D)$ is isomorphic to the trivial group, to $\mathbb{Z}/2\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z})^2$. A fundamental domain for the action of $H'(D)$ on D is naturally contained in a copy $M \times i$ of M sitting in $\widehat{M}/h_f$. This fundamental domain is isomorphic to D or to a disc whose boundary intersects $\mathcal{G}$ transversally in 2 or 3 points. It follows from the irreducibility and the atoroidality of $(M, \mathcal{G})$ that D cannot be an essential disc. Therefore $\mathcal{F}$ is an incompressible surface in $\widehat{M}$ (which has a strictly negative Euler characteristic). Thus $\widehat{M}$ is Haken.

Let us show by contradiction that $\widehat{M}$ is atoroidal. Suppose that $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z} + \mathbb{Z}$. Since $\widehat{M}$ is Haken, the Torus theorem asserts that $\widehat{M}$ contains an incompressible torus ([F], [JS], [Joh]). By the Equivariant torus theorem ([BS], [JR], [MS]), there exists such a torus T which is disjoint or equal to any of its translates by elements of H. Let $H(T)$ be the stabilizer of T in H. Then $H(T)$ is either the trivial group or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z})^4$. A fundamental domain for the action of $H(T)$ on T is naturally contained in a copy of M sitting in $\widehat{M}$. This fundamental domain is either a torus, or an annulus with boundary disjoint from $\mathcal{G}$, or a disc whose boundary intersects $\mathcal{G}$ transversally in 4 points. Again, the irreducibility and the atoroidality of $(M, \mathcal{G})$ imply that T cannot be incompressible. □

The next proposition shows an important property of a good splitting surface.

Definition. — We say that a 3-manifold M is *fibered* if it contains a splitting surface S such that the manifold obtained by splitting M along S is an interval bundle over a (not necessarily connected) closed surface.

Note that a fibered manifold is not necessarily fibered over the circle.

Let $(M, \mathcal{G})$ be a connected irreducible and atoroidal manifold-with-corners with empty boundary. Let S be a good splitting surface for $(M, \mathcal{G})$. Let $(M_S, \mathcal{G}_S, S')$ be the manifold-with-corners obtained by splitting $(M, \mathcal{G})$ along S. Let $\widehat{M}_S$ be the partially mirrored manifold of $(M_S, \mathcal{G}_S, S')$.

Proposition 7.6. — *With these notations, we have:*

(i) *the boundary of $\widehat{M}_S$ is incompressible;*

(ii) *if M is not fibered over the circle or if $\partial M \neq \emptyset$, then $\widehat{M}_S$ is not an interval bundle over a closed surface.*

Proof. — The proof of (i) is similar to that of Proposition 7.5.

To prove (ii) suppose first that $\partial M = \emptyset$. Then since M is not fibered, the manifold obtained by splitting M along any embedded surface is not an interval bundle.

Suppose now that $\partial M \neq \emptyset$. Then $\partial S \neq \emptyset$, since S is a good splitting surface. Let f be a mirror of $(M, \mathcal{G})$ such that $f \cap \partial S \neq \emptyset$. Let $\mathcal{F}$ be the surface above f. Since $(M, \mathcal{G})$ is irreducible and atoroidal, the Euler characteristic of $\mathcal{F}$ is strictly negative. Let f' be the union of the mirrors of $(M_S, \mathcal{G}_S)$ which are contained in f. Let $\mathcal{S}$ be the set of mirrors of $(M, \mathcal{G})$ which intersect f. Let $\mathcal{F}'$ be the image of $f' \times (\mathbf{Z}/2\mathbf{Z})^{\mathcal{S}}$ under the equivalence relation $\mathcal{R}_S$. Then $\mathcal{F}'$ is homeomorphic to the complement in $\mathcal{F}$ of a non-empty disjoint union of annuli. In particular $\chi(\mathcal{F}') < 0$ and $\partial\mathcal{F}' \neq \emptyset$. The arguments used for proving the incompressibility of $\mathcal{F}$ and the fact that $(M_S, \mathcal{G}_S, S')$ has incompressible boundary show that $\mathcal{F}'$ is incompressible in $\widehat{M_S}$. Suppose for a contradiction that $\widehat{M_S}$ were an interval bundle. Then, by a theorem of Stallings [**Stal**], each component of $\mathcal{F}'$ would be an annulus. This is impossible since $\chi(\mathcal{F}') < 0$. □

7.3 Hyperbolic manifolds-with-corners

Let $(M, \mathcal{G}, \partial^0 M)$ be a manifold-with-corners. The definition of a hyperbolic metric on a manifold (cf. §1) can be made in exactly the same way if M has a differentiable structure with corners. For $(M, \mathcal{G}, \partial^0 M)$ to be hyperbolic, we require two more properties.

Definition. — We say that $(M, \mathcal{G}, \partial^0 M)$ *is hyperbolic* if M (with its differentiable structure with corners) has a hyperbolic metric such that

(i) each mirror of $(M, \mathcal{G}, \partial^0 M)$ is totally geodesic,

(ii) any two distincts mirrors or components of $\partial^0 M$ which share a common edge meet orthogonally.

Observe that along the components of ∂M which are not mirrors M is locally convex (from the definition of a hyperbolic metric).

We have:

Fact 7.7. — *Let $(M, \mathcal{G})$ be a manifold-with-corners having empty boundary. If $(M, \mathcal{G})$ is hyperbolic, then (the differentiable manifold) M is hyperbolic.*

Proof. — There exists a geometrically finite group G and an isometric embedding of M into $M(G)$ which realizes M as a convex of $M(G)$ (cf. §1). For sufficiently small δ, $N_\delta(G)$ is contained in M. Then the retraction r_δ (cf. §1) allows to construct an homeomorphism between M and the hyperbolic manifold $N_\delta(G)$. It follows from the Cerf theorem and from the uniqueness of the rounding that (the differentiable manifold) M is diffeomorphic to $N_\delta(G)$. □

Definition. — Let H be a finite group of diffeomorphisms of a manifold M. We say that M *is H-equivariantly hyperbolic* if M carries an *H-equivariant hyperbolic metric,* i.e. a hyperbolic metric for which H acts by isometries.

Let $(M, \mathcal{G}, \partial^0 M)$ be a manifold-with-corners and let $\mathcal{M}$ be its set of mirrors. Let $H = (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ be the mirror group of $\widehat{M}$.

Fact 7.8. — *The manifold-with-corners $(M, \mathcal{G}, \partial^0 M)$ is hyperbolic if and only if $\widehat{M}$ is H-equivariantly hyperbolic.*

Proof. — Suppose that $(M, \mathcal{G}, \partial^0 M)$ is hyperbolic. Since the mirrors of $(M, \mathcal{G})$ are totally geodesic and meet orthogonally, the "product" hyperbolic metric on $M \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ is invariant under the equivalence relation $\mathcal{R}$. It defines therefore a hyperbolic metric on $\widehat{M}$. Therefore $\widehat{M}$ is hyperbolic. By construction H acts by isometries. Therefore $\widehat{M}$ is H-equivariantly hyperbolic.

Conversely, suppose that $\widehat{M}$ has a H-equivariant hyperbolic metric. The fixed point set of a reversing orientation diffeomorphism of a hyperbolic 3-manifold is a totally geodesic submanifold of codimension 1 which meets the boundary orthogonally. Furthermore, if two distinct orientation reversing isometries of an hyperbolic manifold commute, the components of their fixed point sets are disjoint or intersect orthogonally. It follows then from the way the manifold-with-corners $(M, \mathcal{G}, \partial^0 M)$ can be reconstructed from the action of H on $\widehat{M}$, that $(M, \mathcal{G}, \partial^0 M)$ is hyperbolic. □

Let $(M, \mathcal{G})$ be a manifold-with-corners having empty boundary. Let S be a splitting surface for $(M, \mathcal{G})$. Let $(M_S, \mathcal{G}'_S)$ be a manifold-with-corners having empty boundary such that $S' \subset \partial M_S$ is a union of mirrors of $(M_S, \mathcal{G}'_S)$. Suppose that $(M_S, \mathcal{G}_S, S')$ is obtained from $(M_S, \mathcal{G}'_S)$ by erasing the mirrors that are contained in S'. With these notations, we have:

Fact 7.9. — *If $(M_S, \mathcal{G}'_S)$ is hyperbolic, then*

(i) *$(M_S, \mathcal{G}_S, S')$ is hyperbolic, and*

(ii) *$\widehat{M}_S$ is H-equivariantly hyperbolic.*

Proof. — We begin by proving (ii). Recall that $\widehat{M}_S$ is H-equivariantly diffeomorphic to the manifold obtained by rounding the corners of $\widehat{M}'_S$. When M_S carries a hyperbolic metric arising from a hyperbolic metric on $(M_S, \mathcal{G}'_S)$, the "product" hyperbolic metric on $M_S \times (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$ is invariant under the equivalence relation $\mathcal{R}_S$. It defines therefore a H-equivariant hyperbolic metric on the differentiable manifold with corners $\widehat{M}'_S$. There is a geometrically finite group G such that $N_\delta(G)$ is naturally embedded in $\widehat{M}'_S$ (cf. the proof of Fact 7.7). The group H preserves $N_\delta(G)$ and commutes with the retraction r_δ. It follows that $N_\delta(G)$ is H-equivariantly homeomorphic (and hence diffeomorphic, by uniqueness of the rounding) to the manifold obtained from $\widehat{M}'_S$ by rounding the corners. Therefore, $\widehat{M}_S$ is H-equivariantly hyperbolic.

Since $\partial N_\delta(G)$ is invariant under H, the fixed point set of any non-trivial isometry in H is either disjoint from $\partial N_\delta(G)$ or intersects $N_\delta(G)$ orthogonally. It follows then from the way that $(M_S, \mathcal{G}_S, S')$ can be reconstructed from the action of H on $\widehat{M}_S$ that $(M_S, \mathcal{G}_S, S')$ is also hyperbolic (cf. Fact 7.8). □

CHAPTER 8

Proof of Thurston's hyperbolization theorem

Hyperbolization theorem for manifolds-with-corners. — *Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary. If M is Haken, then $(M, \mathcal{G})$ is hyperbolic.*

In this chapter, we prove this theorem when M is not fibered. The major step of the proof is the following result.

Gluing theorem. — *Let $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners having empty boundary such that M is not fibered. Let S be a good splitting surface for $(M, \mathcal{G})$ and let $(M_S, \mathcal{G}_S, S')$ be the manifold-with-corners obtained by splitting $(M, \mathcal{G})$ along S. Suppose that $(M_S, \mathcal{G}_S, S')$ is hyperbolic. Then $(M, \mathcal{G})$ is hyperbolic.*

Proof. — We may suppose that M is connected. Let $\mathcal{M}$ be the set of mirrors of $(M, \mathcal{G})$ and set $H = (\mathbb{Z}/2\mathbb{Z})^{\mathcal{M}}$. Let $\widehat{M}$ be the mirrored manifold of $(M, \mathcal{G})$ and let $\widehat{M}_S$ be the partially mirrored manifold of $(M_S, \mathcal{G}_S, S')$. The group H acts by diffeomorphisms on $\widehat{M}$ and on $\widehat{M}_S$. There is an orientation reversing involution τ' of S' which permutes the components and such that $(M, \mathcal{G})$ is diffeomorphic to the quotient space $(M_S, \mathcal{G}_S, S')/\tau'$. This involution τ' lifts to an orientation reversing involution τ of $\partial\widehat{M}_S$ which permutes the components by pairs and which commutes with the action of H. Also $\widehat{M}$ is H-equivariantly diffeomorphic to the quotient space $\widehat{M}_S/\tau$.

By hypothesis and by Fact 7.9, $\widehat{M}_S$ is hyperbolic. Since S is a good splitting surface, Proposition 7.6 says that $\widehat{M}_S$ has incompressible boundary and that $\widehat{M}_S$ is not an interval bundle. Since $(M, \mathcal{G})$ is irreducible and atoroidal, $\widehat{M}$ is atoroidal (Proposition 7.5). Let

$$\tau^* : \mathcal{T}(\overline{\partial\widehat{M}_S}) \to \mathcal{T}(\partial\widehat{M}_S)$$

be the map induced by τ. Let $\sigma : \mathcal{T}(\partial\widehat{M}_S) \to \mathcal{T}(\overline{\partial\widehat{M}_S})$ be the skinning map associated to $\widehat{M}_S$. By Thurston's fixed point theorem (§6), $\tau^* \circ \sigma$ has a fixed point. This implies that $\widehat{M} = \widehat{M}_S/\tau$ is hyperbolic (cf. §2).

In order to prove that $(M, \mathcal{G})$ is hyperbolic, we need to prove that $\widehat{M}$ is H-equivariantly hyperbolic (Fact 7.8). We introduce first some notations. Since $\widehat{M}_S$ is H-equivariantly hyperbolic, there exists a geometrically finite group G (resp. geometrically finite groups $G_1, \cdots, G_p$, when S is separating) with an isometric action of H on $M(G)$ (resp. on the disjoint union of $M(G_1), \cdots, M(G_p)$) such that $\widehat{M}_S$ is H-equivariantly diffeomorphic to $\overline{M}(G)$ (resp. the disjoint union of $\overline{M}(G_1), \cdots, \overline{M}(G_p)$). To each $s \in \mathcal{T}(\partial\widehat{M}_S)$, it corresponds via the Ahlfors-Bers map a quasi-conformal deformation $(\rho, \widetilde{\varphi})$ of G (resp. quasi-conformal deformations $(\rho_i, \widetilde{\varphi}_i)$ of G_i for $i = 1, \cdots, p$) such that $\partial(\rho, \widetilde{\varphi}) = s$ (resp. $(\partial(\rho_i, \widetilde{\varphi}_i) = s)$. We denote by $\widehat{M}_S^s$ the manifold $\overline{M}(\rho(G))$ (resp. the disjoint union of $\overline{M}(\rho_1(G_1)), \cdots, \overline{M}(\rho_p(G_p))$).

The group H acts on $\partial\widehat{M}_S$ by diffeomorphisms but, since the elements of H don't all preserve the orientation, H does not act in a direct way on $\mathcal{T}(\partial\widehat{M}_S)$. An element $h \in H$ induces a map h^* from $\mathcal{T}(\partial\widehat{M}_S)$ to itself or from $\mathcal{T}(\partial\widehat{M}_S)$ to $\mathcal{T}(\overline{\partial\widehat{M}_S})$ according as h preserves or not the orientation (cf. §1). We define an action of H on $\mathcal{T}(\partial\widehat{M}_S)$ as follows. Let $s \in \mathcal{T}(\partial\widehat{M}_S)$. For an element $h \in H$ which preserves the orientation, we set $h'(s) = h^*(s)$. If $h \in H$ reverses the orientation, we set $h'(s) = \overline{h^*(s)}$, where $\overline{h^*(s)}$ is the complex conjugated of $h^*(s)$ (cf. §1). Since h^* commutes with the complex conjugation (cf. §1), $h \to h'$ defines an action of H on $\mathcal{T}(\partial\widehat{M}_S)$. Denote by $\mathcal{T}(\partial\widehat{M}_S)_H$ the fixed point set of this action. Since H acts by isometries on $\partial\widehat{M}_S$, the point $\partial\widehat{M}_S \in \mathcal{T}(\partial\widehat{M}_S)$ is fixed by H. Hence $\mathcal{T}(\partial\widehat{M}_S)_H \neq \emptyset$.

Fact 8.1. — *The map $\tau^* \circ \sigma$ leaves $\mathcal{T}(\partial\widehat{M}_S)_H$ invariant.*

Proof. — Consider an element $h \in H$ which preserves the orientation. From the definition of σ, it follows that σ and h^* commute. Therefore, since τ and h commute, $\tau^* \circ \sigma$ and $h' = h^*$ commute also.

For an element $h \in H$ which reverses the orientation, the same conclusion holds. One needs only to observe that the skinning map $\bar{\sigma}$ for the hyperbolic manifold $\widehat{M}_S$ with the reversed orientation satisfies $\bar{\sigma}(\bar{s}) = \sigma(s)$. This implies Fact 8.1. □

Since $\mathcal{T}(\partial\widehat{M}_S)_H$ is a closed non-empty subset of $\mathcal{T}(\partial\widehat{M}_S)$ and since $\tau^* \circ \sigma$ is contracting, its fixed point belongs to $\mathcal{T}(\partial\widehat{M}_S)_H$. Let $s_0 = \partial(\rho, \widetilde{\varphi})$ be this fixed point. Then H acts on $\partial\widehat{M}_S^{s_0}$ by isometries. Since the action of H on $\partial\widehat{M}_S^{s_0}$ extends to an action on $\widehat{M}_S^{s_0}$ by diffeomorphisms, it follows from the injectivity of the Ahlfors-Bers map that it extends also to an action by isometries. Also, $\widetilde{\varphi}$ induces a quasi-conformal homeomorphism $\varphi : \partial\widehat{M}_S \to \partial\widehat{M}_S^{s_0}$ which conjugates the actions of H. Denote by Φ the natural extension of φ (cf. §1). Then $\Phi : \widehat{M}_S \to \widehat{M}_S^{s_0}$ is a homeomorphism. By naturality, since φ conjugates the isometric actions of H on $\partial\widehat{M}_S$ and $\partial\widehat{M}_S^{s_0}$, Φ conjugates the isometric actions of H on $\widehat{M}_S$ and on $\widehat{M}_S^{s_0}$. The uniqueness of the differentiable structure on the manifold-with-corners $\widehat{M}_S/H$

implies that $\widehat{\Phi}$ is isotopic to a diffeomorphism which conjugates the actions of H. Therefore, $\widehat{M}_S$ is H-equivariantly diffeomorphic to $\widehat{M}_S^{s_0}$.

In order to show that $\widehat{M}_S/\tau$ is H-equivariantly hyperbolic, we need to recall the proof of Maskit's combination theorem given in §2. Using harmonic functions, we constructed a codimension 0 submanifold N contained in the interior of $\widehat{M}_S^{s_0}$. By definition, N is invariant under the action of H. Since s_0 is a fixed point of $\tau^* \circ \sigma$, τ induces an orientation reversing isometry $\mathfrak{I}$ of ∂N (cf. §2). Since $s_0 \in \mathcal{T}(\partial\widehat{M}_S)_H$, $\mathfrak{I}$ commutes with the action of H. Therefore the quotient space $M' = N/\mathfrak{I}$ is a hyperbolic manifold on which H acts by isometries. We need to show that M' is H-equivariantly diffeomorphic to $\widehat{M}_S$. Like in §2, we distinguish two cases.

1) ∂N *is incompressible in* M'.

To conclude then, we need an equivariant version of Stallings theorem, which in our case goes as follows. By construction, ∂N divides $\widehat{M}_S^{s_0}$ into two manifolds. Let V be the one which contains $\partial\widehat{M}_S^{s_0}$. Consider the product action of H on $\partial\widehat{M}_S^{s_0} \times I$ where H acts on the standard way on $\partial\widehat{M}_S$ and acts by the identity on the second factor. The quotient $(\partial\widehat{M}_S^{s_0} \times I)/H$ can be viewed then as a manifold-with-corners, naturally isomorphic to the product of a surface-with-corners (S', P) by I (its boundary is $S' \times \{0\} \cup S' \times \{1\}$ and its mirrors are squares, product of mirrors of (S', P) by I). From the proof of Maskit's combination theorem, V embeds H-equivariantly in $\widehat{S}' \times I$ in such a way that $\partial\widehat{M}_S^{s_0}$ maps to $\widehat{S}' \times \{1\}$. Then the surface-with-corners $\partial N/H$ embeds in $(S', P) \times I$. Its boundary $\partial(\partial N/H)$ is contained in the reunion of the mirrors and $\partial N/H$ divides $(S', P) \times I$ into two manifolds, one of which being V/H. Since ∂N is incompressible, $(\partial N/H, \partial(\partial N/H))$ is an incompressible surface in the pair $(S', \partial S') \times I$. Therefore by Stallings theorem, $\partial N/H$ is isotopic to $S' \times \{1/2\}$. In particular, $\partial N/H$ is homeomorphic to S' and each component of $\partial(\partial N/H)$ intersects as many mirrors as the corresponding component of $\partial S'$. Therefore the Euler characteristic of every component of ∂N is smaller than the Euler characteristic of $\widehat{S}'$ with equality if and only if each component of $\partial(\partial N/H)$ intersects each mirror in a single arc. Since each component of ∂N is homeomorphic to $\widehat{S}'$, this must be the case. Since the mirrors are squares, it follows that we can suppose that the isotopy in Stallings theorem respects the mirrors. Thus N is H-equivariantly diffeomorphic to $\widehat{M}_S$ and therefore to $\widehat{M}_S^{s_0}$ also. Under this identification τ and $\mathfrak{I}$ are homotopic diffeomorphisms and they commute with the action of H. By the equivariant version of Nielsen theorem (which can also be deduced in this special case from the classical version), τ and $\mathfrak{I}$ are H-equivariantly isotopic. This implies that M' and $\widehat{M}_S/\tau$ are H-equivariantly diffeomorphic.

2) ∂N **is not incompressible in** M'.

Then there is a compression disc for ∂N in M'. By the equivariant Dehn's Lemma, there is such a disc D which is either disjoint or equal to any of its translates by H. By doing equivariant surgery to ∂N along the H-orbit of D, we define a H-invariant manifold N' contained in the interior of $\widehat{M}_S^{s_0}$ with an involution $\mathfrak{I}'$ of $\partial N'$, such that M' is H-equivariantly diffeomorphic to the quotient space $N'/\mathfrak{I}'$ (cf. §2).

If $\partial N'$ is incompressible in M', then the reasoning of case 1) can be applied. If not, we do equivariant surgery again. This process stops after a finite number of steps. □

Proof of the Hyperbolization theorem for manifolds-with-corners. —

The proof is by induction on $\ell(M)$.

If $\ell(M) = 0$, M is a handlebody. Let S be a good splitting surface for $(M, \mathcal{G})$ (S is a system of meridians). Let $(M_S, \mathcal{G}_S, S')$ be the manifold-with-corners obtained by splitting $(M, \mathcal{G})$ along S. Then M_S is diffeomorphic to the 3-ball. Let $(M_S, \mathcal{G}'_S)$ be the manifold-with-corners with empty boundary provided by Proposition 7.4. Since $(M_S, \mathcal{G}'_S)$ is irreducible and atoroidal, the intersection of two distinct mirrors of $(M_S, \mathcal{G}'_S)$ is either empty or a single edge. Therefore $(M_S, \mathcal{G}'_S)$ can be viewed as a polyhedron whose faces are in correspondance with the mirrors of $(M_S, \mathcal{G}'_S)$). Saying that $(M_S, \mathcal{G}'_S)$ is hyperbolic means that this polyhedron can be realized in $\mathbb{H}^3$ with all dihedral angles equal to $\pi/2$. This turns out to be a special case of a theorem of Andreev which provides sufficient conditions on a polyhedron with prescribed dihedral angles to be realizable in $\mathbb{H}^3$. Here is a formulation of Andreev's theorem, when the prescribed dihedral angles all equal $\pi/2$ and in our particular case (recall that any vertex of $\mathcal{G}'_S$ has valence 3).

Theorem ([An], [Thl]). — *Let $\mathcal{P}$ be a polyhedron distinct from the tetrahedron or from the triangular prism. Then $\mathcal{P}$ can be realized in $\mathbb{H}^3$ with all dihedral angles equal to $\pi/2$ if and only if*

(i) *every cycle of faces of length 3 surrounds a vertex of $\mathcal{P}$, and*

(ii) *every cycle of faces of length 4 surrounds an edge of $\mathcal{P}$.*

Definition. — A *cycle of faces of length k of $\mathcal{P}$* is a finite sequence of faces $(f_i)_{i=0,\cdots,k}$ with $f_0 = f_k$ and such that any two successive faces share exactly an edge in common.

By the construction of $\mathcal{G}'_S$, the polyhedron $(M_S, \mathcal{G}'_S)$ has too many faces to be isomorphic to a tetrahedron or to a triangular prism. Also the hypothesis of Andreev's theorem are satisfied since $(M_S, \mathcal{G}'_S)$ is irreducible and atoroidal. Hence $(M_S, \mathcal{G}'_S)$ is hyperbolic. By Fact 7.9, $(M_S, \mathcal{G}_S, S')$ is hyperbolic. Since $(M, \mathcal{G})$ is irreducible and atoroidal and since $\partial M \neq \emptyset$ (cf. Proposition 7.6), the Gluing theorem implies that $(M, \mathcal{G})$ is hyperbolic. This proves the initial step of the induction.

Let now $(M, \mathcal{G})$ be an irreducible and atoroidal manifold-with-corners such that M is a Haken and suppose the theorem true for all manifolds-with-corners $(M', \mathcal{G}')$ with $\ell(M') < \ell(M)$. Let S be a good splitting surface for $(M, \mathcal{G})$. Let $(M_S, \mathcal{G}_S, S')$ be the manifold obtained by splitting $(M, \mathcal{G})$ along S. Since the length of any component of M_S is smaller than $\ell(M)$ (cf. §7) the manifold-with-corners $(M_S, \mathcal{G}'_S)$ provided by Proposition 7.4 is hyperbolic. By Fact 7.9, $(M_S, \mathcal{G}_S, S')$ is hyperbolic. Since M is not fibered, the Gluing theorem implies that $(M, \mathcal{G})$ is hyperbolic also.

This proves the Hyperbolization theorem for manifold-with-corners. □

Thurston's hyperbolization theorem. — *Let M be an irreducible and atoroidal manifold. If M is Haken, then M is hyperbolic.*

Proof. — By Proposition 7.2, there is a graph $\mathcal{G} \subset \partial M$ such that $(M, \mathcal{G})$ is an irreducible and atoroidal manifold-with-corners. If M is Haken, $(M, \mathcal{G})$ is hyperbolic by Thurston's hyperbolization theorem for manifold-with-corners. By Fact 7.7, M is hyperbolic. □

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