Metabolic cobordisms and the simplest perturbative Chern-Simons 3-manifold invariant

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1 Introduction. The perturbative, Chern-Simons three-manifold invariants were predicted by Witten [8] and then defined by Axelrod and Singer ([1] and [2]). Kontsevich [4] has given a second realization of Witten's predictions; and the simplest of Kontsevich's invariants is the subject of this article, and its prequel, [7].

As described in the first article, [7], the simplest of Kontsevich's perturbative Chern-Simons invariants is defined for compact, oriented 3-manifolds M which have the rational homology of S^3. The invariant, $I_2(M)$, is computed by an integral

$$(1.1) \qquad I_2(M) = \int_{M \times M} \omega \wedge \omega \wedge \omega,$$

where ω is a closed 2-form on $M \times M$ with a prescribed singularity on

$$(1.2) \qquad \Sigma_M \equiv \Delta_M \cup (p_0 \times M) \cup (M \times p_0).$$

Here, $\Delta_M \subset M \times M$ is the diagonal, and $p_0 \in M$ is a chosen base point.

The reader is referred to Section 2 of [7] and Definition 2.8 of [7] for the details. (This article is a sequel to [7].) Suffice it to say that $H^2_{DeRham}(M \times M - \Sigma_M) \approx \mathbb{R}$, and ω is a generator of this group. In particular, ω restricts to every linking 2-sphere around Σ_M as 2-form with total volume 1. None-the-less, ω is constrained so that $\omega \wedge \omega = 0$ near Σ_M, thus insuring that the integral in (1.1) is well defined.

a) Cobordisms and I_2.

Let M_0 and M_1 be a pair of compact, oriented, 3-manifolds with the rational homology of S^3. An oriented cobordism, W, between M_0 and M_1 is a compact, oriented 4-manifold with boundary; and that boundary should be the disjoint union of M_0 and M_1. Furthermore, the induced boundary orientation (using the outward pointing normal) should be correct for M_1 and wrong for M_0. If W is also a spin manifold, then the cobordism is called a spin cobordism.

Theorem 2.9 in [7] gives a set of conditions on the spin cobordism W which imply $I_2(M_0) = I_2(M_1)$. In particular, one condition in [7]'s Theorem 2.9 required that W have the rational homology of S^3. It is the purpose of this

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article to greatly relax this condition. The relaxed conditions are stated below in Theorem 1.2. Here is a corollary of Theorem 1.2:

THEOREM 1.1.   *The invariant $I_2(\cdot)$ equals zero on a 3-manifold with the integral homology of S^3.*

The full statement of Theorem 1.2 requires the following digression to introduce some necessary terminology. To start the digression, introduce $\mathrm{tor}(H_2) \subset H_2(W;\mathbb{Z})$ to denote the torsion sub-group. Next, recall that there is a natural, symmetric, bilinear form on $\underline{H}_2(W;\mathbb{Z}) \equiv H_2(W;\mathbb{Z})/\mathrm{tor}(H_2)$, this being the intersection pairing. This form is non-degenerate, but not in general unimodular. (It is unimodular if M_0 and M_1 have the integral homology of S^3.) The intersection form, β, on $\underline{H}_2(W;\mathbb{Z})$ will be called equivalent to a sum of metabolics if it is conjugate under $GL(\cdot,\mathbb{Z})$ to

$$(1.3) \qquad\qquad\qquad \oplus_a H(m_a)$$

where $H(m)$ for $m \in \mathbb{Z}$ is the symmetric, 2×2 matrix with zero on the diagonal and with m in the off diagonal entries. For example, the compliment of a pair of disjoint, open balls in $S^2 \times S^2$ is a spin cobordism between S^3 and S^3 whose intersection form is conjugate to either $H(1)$ or to $H(-1)$, depending on the orientation.

End the digression.

THEOREM 1.2.   *Let M_0 and M_1 be compact, oriented, 3-manifolds with the rational homology of S^3. Let W be an oriented, spin cobordism between M_0 and M_1. Suppose that:*
  1) The intersection form of W is equivalent to a sum of metabolics.
  2) The inclusions of both M_0 and M_1 into W induce injective maps on
      $H_1(\cdot;\mathbb{Z}/2)$.
  Then $I_2(M_0) = I_2(M_1)$.

Theorem 1.2 is an immediate corollary to Theorem 1.3, below. The statement of Theorem 1.3 requires the following 2-part digression. For Part 1 of the digression, consider a compact, oriented 3-manifold M with the rational homology of S^3. Fix a point $p_0 \in M$. Then, introduce from Definition 2.3 in [7] the notion of a singular frame for T^*M. (This is a frame for $T^*(M - p_0)$ with a prescribed singularity at p_0.) As in Lemma 2.4 of [7], let c denote the set of homotopy classes of singular frames for T^*M. Define an equivalence relation on c by declaring ζ and ζ' to be equivalent when $\zeta = g \cdot \zeta'$ where g is a degree zero map from M to $SO(3)$. Let $\underline{c}$ denote the resulting set of equivalence classes. Finally, use Definition 2.8 of [7] to identify a canonical element $c_M \in \underline{c}$.

For Part 2 of the digression, let W be a compact, oriented 4-manifold with boundary and suppose that M, as above, is a component of ∂W. Let $K(M;W)$ denote the cokernel of the restriction homomorphism $H^1(W;\mathbb{Z}/2) \longrightarrow H^1(M;\mathbb{Z}/2)$. As in (2.12) of [7], introduce the homomorphism $l_W : c \longrightarrow K(M;W)$.

End the digression.

THEOREM 1.3. *Let M_0 and M_1 be compact, oriented, 3-manifolds with the rational homology of S^3. Let W be an oriented, spin cobordism between M_0 and M_1. Suppose that:*

    1) The intersection form of W is equivalent to a sum of metabolics.

    2) Both c_{M_0} and c_{M_1} are represented by c with $l_W(c) = 0$.

Then $I_2(M_0) = I_2(M_1)$.

b) Proof of Theorem 1.1.

Orient S^3 as the boundary of the unit ball in $\mathbb{C}^2$. Let P denote the Poincaré homology sphere. This is a quotient of S^3 by a certain finite subgroup of $SO(4)$, and as such, inherits an orientation from S^3. (Note that P is an integral homology sphere.)

Note that P is spin cobordant to S^3, and there is such a spin cobordism with signature divisable by 8 (See, e.g., [**3**]). (The signature of a 4-manifold is the number of positive eigenvalues minus the number of negative eigenvalues in the intersection form after diagonalizing the latter over $\mathbb{Q}$.)

Let M be a compact, oriented 4-manifold with the integral homology of S^3. There is an oriented, spin cobordism between M and S^3, and the index of such a cobordism is automatically divisible by 8 (again, see [**3**].) Thus, there is a spin cobordism between M and S^3 or between M and P with signature divisable by 16. Let W denote the afore-mentioned cobordism. If the signature of W is not zero, then the connect sum of W with some number of $K3$ surfaces (with the appropriate orientations) gives a new spin cobordism between M and S^3 or P which has signature zero. This will now be assumed.

The intersection form of W is a signature zero, unimodular, even, symmetric matrix. (The form is even if $\beta(x, x) \in 2\mathbb{Z}$ for all x.) Such a form is equivalent over $Gl(\cdot, \mathbb{Z})$ to a sum of metabolics as in (1.3) with all $m_a = \pm 1$ (once again, see [**3**]). Thus, the first condition of Theorem 1.2 is met with $M_0 = M$ and with $M_1 = S^3$ or P. The second condition of Theorem 1.2 is also met because both M_0 and M_1 have trivial homology. It follows that $I_2(M) = I_2(S^3)$ or else $I_2(M) = I_2(P)$. Now $I_2(S^3) = 0$ (Theorem 2.9 in [**7**]) so Theorem 1.1 will follow with a demonstration that $I_2(P) = 0$.

With the preceding understood, let $\underline{P}$ denote the Poincaré homology sphere with the opposite orientation. Then $I_2(\underline{P}) = I_2(P)$ since $P \cup P$ bounds a spin cobordism with index divisible by 16. However, unless $I_2(P) = 0$, this conclusion is incompatible with

LEMMA 1.4. *Let M be a compact, oriented 3-manifold with the integral homology of S^3. Let $\underline{M}$ denote the manifold M with reversed orientation. Then $I_2(M) = -I_2(\underline{M})$.*

Proof. The form ω for the computation of $I_2(M)$ is constructed after choosing a singular frame, ζ, for T^*M in the class of c_M. (See Proposition 2.5 in [**7**].) The frame $\zeta \equiv (\zeta_1, \zeta_2, \zeta_3)$ must be compatible with the orientation of M. With this point understood, then $\zeta' \equiv (-\zeta_1, \zeta_2, \zeta_3)$ will be an oriented, singular frame for $T^*\underline{M}$, which is in the class $c_{\underline{M}}$ (see Proposition 2.7 and Definition 2.8 of

[7].) The form $-\omega$ will result from the constructions in [7] using the frame ζ'. Meanwhile, the orientation of $M \times M$ is not sensitive to the choice of orientation for M. $\square$

c) An outline for proving Theorem 1.3.

Here is a 4 step program for proving Theorem 1.3:

Step 0: Start with a cobordism between M_0 and M_1 which obeys the conditions of Theorem 1.3. Appeal to Proposition 3.2 in [7] to find a cobordism W between M_0 and M_1 which decomposes as $W_1 \cup W_2 \cup W_3$ with the following properties:

(1.4)

1) $\partial W_1 = M_0 \cup M_0'$, $\partial W_2 = M_0' \cup M_1'$, and $\partial W_3 = M_1' \cup M_1$, where M_0' and M_1' are compact, oriented 3-manifolds with the rational homology of S^3.

2) $W_{1,2,3}$ are oriented, spin manifolds.

3) Both W_1 and W_3 have the rational homology of S^3. Meanwhile W_2 has vanishing first and third Betti numbers; and W_2 has a proper Morse function with no index 1 or index 3 critical points.

4) Both c_{M_0} and c_{M_1} are represented by elements in the kernel of ι_W.

Step 1: Construct a compact, oriented 7-manifold with boundary $Z \subset W \times W$. The boundary of this Z is the disjoint union of $M_0 \times M_0, M_1 \times M_1$ and some number of copies of $S^3 \times S^3$. Note that

(1.5) $$Z \equiv Z_1 \cup Z_2 \cup Z_3,$$

where $Z_i \subset W_i \times W_i$ are compact submanifolds with boundary. The Z_i are described in Section 3g of [7]. (The boundary of $Z_1, \partial Z_1$, is the disjoint union of $M_0 \times M_0$ and $M_0' \times M_0'$ plus copies of $S^3 \times S^3$. Meanwhile ∂Z_2 contains $M_0' \times M_0', M_1' \times M_1'$ and copies of $S^3 \times S^3$. Finally, ∂Z_3 is the disjoint union of $M_1' \times M_1', M_1 \times M_1$ plus copies of $S^3 \times S^3$. In addition, Z_1 and Z_2 are glued together along $M_0' \times M_0'$, while Z_2 and Z_3 are glued together across $M_1' \times M_1'$.)

Step 2: Inside Z, find an oriented, dimension 4 subvariety Σ_Z with ∂Z being the disjoint union of $\Sigma_{M_0}, \Sigma_{M_1}$ and copies of Σ_{S^3}, one in each $S^3 \times S^3$ boundary component. Note that

(1.6) $$\Sigma_Z = \Sigma_{Z_1} \cup \Sigma_{Z_2} \cup \Sigma_{Z_3},$$

where $\Sigma_{Z_{1,3}}$ are described in [7] (see Eq. (4.10) in [7] and consider Sections 4d, 4e and 10 in [7].) The subvariety $\Sigma_{Z_2} \subset Z_2$ is constructed here, and the details of the construction of Σ_{Z_2} account for most of the length of this article.

Step 3: Find a closed 2-form, ω_Z, on $Z - \Sigma_Z$ with the following properties:

(1.7)

1) The 2-form ω_Z should restrict to $M_0 \times M_0 - \Sigma_{M_0}$ as the 2-form used in (1.1) for computing $I_2(M_0)$. Its restriction to $M_1 \times M_1 - \Sigma_{M_1}$ should give the 2-form used in (1.1) for computing $I_2(M_1)$.

2) The 2-form ω_Z should restrict to each $S^3 \times S^3 - \Sigma_{S^3}$ as the 2-form used for (1.1) for computing $I_2(S^3) = 0$.

3) The triple wedge product $\omega_Z \wedge \omega_Z \wedge \omega_Z$ should vanish near Σ_Z.

Step 4: Given that ω_Z exists as prescribed above, use Stokes' theorem as in (2.28) of [7] to prove that $I_2(M_0) = I_2(M_1)$.

d) Issues in the construction of Z, Σ_Z and ω_Z.

Compare the outline above with the outline in Section 2k of [7] for the proof of [7]'s Theorem 2.9 and note the similarity between the two strategies. The construction of $Z_2 \subset W_2 \times W_2$ mimics the construction of Z_1 and Z_3 in [7]. In all cases $Z.$ is given as $F^{-1}(0)$ for a function $F \equiv (\pi_R^* f - \pi_L^* f)$ on $W. \times W.$ which is constructed from an appropriately chosen function $f : W. \longrightarrow [0,1]$. Here $\pi_{R,L} : W. \times W. \longrightarrow W.$ are the left and right projections. (As in [7], the extra $S^3 \times S^3$ boundary components of $Z.$ are in 1-1 correspondence with the critical points of the function f.)

The construction of Σ_{Z_2} here also mimics the construction of Σ_{Z_1} and Σ_{Z_3} in [7] in that all are given as $\Sigma_Z. = \Delta_Z. \cup E_L \cup E_R \cup E_- \cup E_+$. Here, as in [7], $\Delta_Z. \subset W. \times W.$ is the diagonal, $E_{L,R}$ are sections for the projections $\pi_{L,R}$, and $E_\pm$ are certain subvarieties which are constructed with the help of a pseudo-gradient vector field for the function f. As in [7], the variety Σ_Z will restrict to each component $M \times M \subset \partial Z$ as Σ_M.

Furthermore, the variety Σ_{Z_2} is constructed so that the conclusions of Lemma 4.1 hold for Σ_Z. This lemma gives necessary and sufficient conditions for $Z_2 - \Sigma_{Z_2}$ to admit a closed 2-form, ω_{Z_2}, whose restriction to each component $M \times M \subset \partial Z_2$ is a generator of $H^2(M \times M - \Sigma_M)$. The analogous closed forms, ω_{Z_1} and ω_{Z_3} are constructed on $Z_1 - \Sigma_{Z_1}$ and $Z_3 - \Sigma_{Z_3}$ in Section 10 of [7]. The form ω_Z in (1.7) is defined so that its restriction to $Z. = Z_1, Z_2,$ and Z_3 is equal to the corresponding $\omega_Z.$.

The construction of a 2-form ω_{Z_2} as above which satisfies the first two requirements of (1.7) follows essentially the same plan as used in Section 10 of [7] to construct ω_{Z_1} and ω_{Z_3}. The most difficult requirement to satisfy is the third requirement of (1.7). The new difficulty, not present in [7], is the fact that the restriction map $\iota^* : H^2(Z_2) \longrightarrow H^2(\Sigma_{Z_2})$ is not surjective when W has rational H^2.

Indeed, the arguments in [7] can be followed with minor modifications to construct a 2-form, $\underline{\omega}_{Z_2}$, on the compliment of Σ_{Z_2} in a neighborhood of $\Sigma_{Z_2} \cup \partial Z_2$ which obeys the first two conditions of (1.7) and which has square zero near Σ_{Z_2}. However, Meyer-Vietoris shows that there is an obstruction in $\mathrm{coker}(\iota^*)$

to extending this form ω_{Z_2} over $Z_2 - \Sigma_{Z_2}$. (There is no such extension problem for Z_1 and Z_3; see Lemma 4.2 in [7].)

There is no argument at present which demonstrates that the coker(ι^*) obstruction is zero. But all is not lost because the application of Stokes' theorem in (2.28) of [7] requires less than the vanishing of $\omega_{Z_2} \wedge \omega_{Z_2}$ near Σ_{Z_2}. The application of Stokes' theorem requires only the vanishing of $\omega_{Z_2} \wedge \omega_{Z_2} \wedge \omega_{Z_2}$.

With this fact understood, remark that $\underline{\omega}_{Z_2} - \mu$ has cube zero if $\underline{\omega}_{Z_2}$ and μ both have square zero. Thus, the issue is framed as follows: Can the obstruction in coker(ι^*) be killed by adding a closed form μ to $\underline{\omega}_{Z_2}$, where μ has square zero, is smooth near Σ_{Z_2} and has support on Z_2? As demonstrated in Section 6, the answer to this question is yes.

The construction of μ as above requires that care be taken with the construction of $E_\pm$ for Σ_{Z_2}. In particular, $H^2(E_\pm)$ must be controlled; as well as the kernel of a certain homomorphism

$$(1.8) \qquad\qquad \iota' : H_2(\Sigma_{Z_2}) \longrightarrow H_2(Z_2 - \Sigma_{Z_2}).$$

The control of ker(ι') requires arguments which do not appear in [7].

e) The remaining sections.

Here is a brief outline of the remainder of this article: Sections 2, 3, 4 and 5 concern themselves with the construction of $E_\pm$ for Σ_{Z_2}. In fact, Sections 2 and 3 are occupied with various preliminary constructions on the cobordism W_2, with the proper introduction of $E_\pm$ reserved for Section 5. Section 4 describes a preliminary version of $E_\pm$, while Section 5 constructs the final version from the preliminary version by ambient surgery in W_2.

The proof of Theorem 1.3 is completed in Section 6.

2 Morse Theory. This section serves as a preliminary digression to introduce certain Morse theoretic constructions that are used in the subsequent construction of $E_\pm$ and Σ_Z. The subject here is a compact, oriented, 4-dimensional cobordism between a pair of compact, oriented 3-manifolds with the rational homology of S^3. Given such a cobordism, one can find a second cobordism which is described by (1.4) and Proposition 3.2 in [7]. The whole of the discussion in [7] concentrates on the factors W_1 and W_3 (which have the rational homology of S^3); and the discussion here will concentrate on the factor W_2.

With this last point understood, let $M_{0,1}$ be a pair of compact, oriented 3-manifolds, both with the rational homology of S^3. In this section, W will denote a compact, oriented, 4-dimensional cobordism from M_0 to M_1 which has a good Morse function with only index 2 critical points. (Section 3a in [7] defines a "good" Morse function.)

a) Algebraic considerations.

The intersection form, β, for W is a bilinear form on $H_2(W; \mathbb{Z})$ which is non-degenerate and symetric. Suppose that this form is conjugate over $\mathrm{Gl}(\cdot, \mathbb{Z})$ to a form which appears in (1.3).

Concerning the homology of W, remember that W has a good Morse function with only index 2 critical points. The relative homology exact sequence gives

$$(2.1) \qquad \begin{aligned} 0 \longrightarrow H_2(W; \mathbb{Z}) \longrightarrow H_2(W, M_0; \mathbb{Z}) \longrightarrow H_1(M_0; \mathbb{Z}) \\ \longrightarrow H_1(W; \mathbb{Z}) \longrightarrow 0, \end{aligned}$$

and the group $H_2(W, M_0; \mathbb{Z})$ is freely generated. This implies that $H_2(W; \mathbb{Z})$ is freely generated too. Let $\{\sigma_i\}_{i=1}^r$ be a given basis for $H_2(W, M_0; \mathbb{Z})$, and $\{\tau_i\}_{i=1}^r$ for $H_2(W; \mathbb{Z})$. Then the image of $\{\tau_i\}$ in $H_2(W, M_0; \mathbb{Z})$ is given as $\{\tau_j \equiv \Sigma_j S_{i,j} \cdot \sigma_j\}$ for some integer valued matrix $(S_{i,j})$. The matrix $(S_{i,j})$ is invertible over $\mathbb{Q}$; it is unimodular if and only if $H_1(M_0; \mathbb{Z}) = 0$.

Given the basis $\{\tau_i\}$ for $H_2(W; \mathbb{Z})$, then one can find a basis $\{\sigma_i\}$ for $H_2(W, M_0; \mathbb{Z})$ for which the matrix $(S_{i,j})$ is upper triangular (see, e.g., [5]) with positive diagonal entries. That is,

$$(2.2) \qquad S_{i,j} = 0 \quad \text{if} \quad i > j \quad \text{and} \quad S_{i,i} > 0.$$

Note: If $H_1(M_0; Z) = 0$, then the basis $\{\sigma_j\} \equiv \{\tau_j\}$ is allowed.

Let $f : M \longrightarrow [0, 1]$ be the good Morse function with only index 2 critical points. One can arrange for such an f to have one critical level, $f^{-1}(1/2)$. A choice of pseudo-gradient for the function f defines the descending 2-disks, $\{B_{p^-} : p \in crit(f)\}$, from the critical points of f. Each B_{p^-} is an embedded 2-disks in $f^{-1}([0, 1/2])$ to which f restricts with a single maximum, p. Orient these disks and they give a basis for $H_2(W, M_0; \mathbb{Z})$.

Using Milnor's basis theorem (Theorem 7.6 in [6]), one can find:

$$(2.3) \qquad \begin{array}{ll} 1) & \text{A good Morse function } f \text{ on } W \text{ with critical value} \\ & 1/2 \text{ and only index 2 critical points.} \\ 2) & \text{A labeling} \{b_1, ..., b_r\} \text{ of } crit(f). \\ 3) & \text{A pseudo-gradient, } v, \text{ for } f. \end{array}$$

And, these are such that the given basis $\{\sigma_i\}_{i=1}^r$ for $H_2(W, M_0; \mathbb{Z})$ is given by

$$(2.4) \qquad \{\sigma_i \equiv [B_{p^-}] : p = b_i\}_{i=1}^r.$$

Here, $[B_{p^-}] \in H_2(W, M_0; \mathbb{Z})$ is the fundamental class for an appropriate choice of orientation for B_{p^-}.

b) Factoring the cobordism.

It proves convenient to factor the cobordism W into a linear chain of simpler cobordisms. The following proposition describes the process:

PROPOSITION 2.1. *Let $M_{0,1}$ be a pair of compact, oriented 3-manifolds, each with the rational homology of S^3. Let W be a compact, connected, oriented 4 dimensional cobordism between M_0 and M_1. Assume that the intersection*

form for W obeys (1.3) and assume that W has a good Morse function with only index 2 critical points. Then W can be decomposed as

$$(2.5) \qquad\qquad W \equiv \cup_{j=1}^{n} W_j,$$

where $W_j \subset W$ is a compact 4-dimensional submanifold with two boundary components, F_j and F_{j+1}, which are embedded, 3-dimensional submanifolds of W. These have the following properties:

1) For each j, F_j has the rational homology of S^3.

2) $F_{n+1} = M_0$ and $F_1 = M_1$.

3) For each j, $W_j \cap W_{j-1} = F_j$.

4) For each j, $H_2(W_j;\mathbb{Z}) \approx \oplus_2\mathbb{Z}$ and the intersection matrix is conjugate by $GL(2;\mathbb{Z})$ to $H(m_j)$ for some $m_j \in \mathbb{Z} - \{0\}$.

5) For each j, W_j has a good Morse function which has only two critical points, both with index 2.

The remainder of this subsection is occupied with the proof of this proposition.

Proof. The first step is the construction of the W_j, and the second step verifies their properties.

Step 1: Because of (1.3), the number r of critical points of f must be even. Given this point, fix small $\epsilon > 0$ and modify f slightly so that for $j \in \{1,...,r/2\}$, the critical points (b_{2j}, b_{2j-1}) have critical value $1/2 - \epsilon \cdot (j/r)$. Thus, (b_r, b_{r-1}) have the smallest critical value, while (b_2, b_1) have the largest critical value.

Set $F_{r/2+1} \equiv M_0$, and for $j \in \{2,...,r/2\}$, let

$$(2.6) \qquad\qquad F_j \equiv f^{-1}(1/2 - \epsilon \cdot (j - 1/2)/r).$$

Note that each F_j is a smooth, oriented submanifold which splits W into two pieces. For $2 \le j \le r/2$, let $V_j \subset W$ denote the closure of the component of $W - F_j$ which contains M_0.

Set $W_{r/2} \equiv V_{r/2}$ and for $1 < j < r/2$, set

$$(2.7) \qquad\qquad W_j \equiv V_j - \text{int}(V_j + 1).$$

For $j = 1$, define $W_1 \equiv W - \text{int}(V_2)$ and define $F_1 \equiv M_1$.

Step 2: Consider now the properties of the $\{W_j\}$ and $\{F_j\}$: First of all, Assertions 2 and 3 of Proposition 2.1 follow by construction. As for Assertion 5, note that W_j is a submanifold with boundary in W which contains no critical points of f on its boundary, and which contains only the critical points b_{2j}, b_{2j-1} of f in its interior. Thus, a rescaling of f on W_j will yield a good Morse function on W_j to verify Assertion 5 of Proposition 2.1.

The proofs of Assertions 1 and 4 of Proposition 2.1 require a digression to construct representative cycles for the generators $\{\tau_i\}$ of $H_2(W;\mathbb{Z})$. The cycle for a given τ_i will be the fundamental class of a submanifold $T_i \subset W$.

To start the digression, remember that $H_2(W;\mathbb{Z})$ is assumed to have a basis $\{\tau_j\}_{j=1}^{r}$ in which the intersection form is given by (1.3). And, remember that

the image of τ_i in $H_2(W, M_0; \mathbb{Z})$ is given by $\Sigma_{b \in \text{crit}(f)} S_{i,j(b)} \cdot \sigma_{j(b)}$, where $\sigma_{j(b)}$s shorthand for $[B_{b-}]$ with $b \equiv b_j$. (This introduces the indexing function $j(\cdot)$: $\text{crit}(f) \longrightarrow \{1, ..., r\}$ which is defined so that $j(b) \equiv j$ when $b \equiv b_j$.) Finally, remember that the Morse function f and its pseudo-gradient have been assumed chosen so that the matrix $(S_{i,j})$ obeys (2.2).

The submanifold representative T_i, for τ_i can be recovered from $(S_{i,j})$ and $\{B_{b-} : b \in \text{crit}(f)\}$ by the following construction: Let $M_{1/4} \equiv f^{-1}(1/4)$. Note that $M_{1/4}$ is diffeomorphic to M_0. Note as well that $C_{b-} \equiv B_{b-} \cap M_{1/4}$ is an embedded circle which is naturally oriented given that B_{b-} is oriented. Thus, C_{b-} determines a homology class, $[C_{b-}] \in H_1(M_{1/4}; \mathbb{Z})$. Meyer-Vietoris (Eq. (2.1)) implies that

$$(2.8) \qquad \mu_i \equiv \Sigma_{b \in \text{crit}(f)} S_{i,j(b)} \cdot [C_{b-}] = 0 \in H_1(M_{1/4}; \mathbb{Z}).$$

Construct push-offs of each C_{b-} by taking a push-off copy of the corresponding B_{b-} and intersecting with $M_{1/4}$. Let $\underline{\mu}_i \subset M_{1/4}$ denote the oriented 1-dimensional submanifold which is the union, indexed by $b \in \text{crit}(f)$, of $|S_{i,j(b)}|$ push-off copies of C_{b-}, oriented correctly when $S_{i,j(b)} > 0$ and oriented incorrectly otherwise. According to (2.8), this $\underline{\mu}_i$ bounds an oriented surface with boundary, $R_i \subset M_{1/4}$, which is such that $\text{int}(R_i) \cap \underline{\mu}_i = \emptyset$.

With R_i understood, represent τ_i by the fundamental class of a subvariety T_i' which is defined to be the union of R_i with the union, indexed by $b \in \text{crit}(f)$, of $|S_{i,j(b)}|$ push-off copies of B_{b-}, oriented correctly if $S_{i,j(b)} > 0$ and oriented incorrectly otherwise.

Smooth the corners of T_i' near $\underline{\mu}_i$ to obtain an embedded surface, $T_i \subset W$. This T_i is naturally oriented and its fundamental class represents the class τ_i.

End the digression.

To return to the proof of Proposition 2.1, and, in particular, the proof of Assertion 1. By construction, $H_2(V_j, M_0; \mathbb{Z})$ is generated by $\{[B_{b-}] : j(b) \geq 2j-1\}$ and thus is a free group. Since M_0 is a rational homology sphere, $H_2(M_0; \mathbb{Z}) = 0$ and therefore (1.3) (with V_j replacing W) asserts that $H_2(V_j; \mathbb{Z})$ is also free; by construction, its generators are $\{[T_i] : i \geq 2j - 1\}$. The intersection form of V_j is the restriction of the form for W to $\{[T_i] : i \geq 2j - 1\}$. This is a sum as in (1.3) and is non-degenerate over $\mathbb{Q}$. The non-degeneracy of the intersection form of V_j over $\mathbb{Q}$ implies that F_j is a rational homology sphere.

To prove Assertion 4, note that $H_2(W_j, F_{j+1}; \mathbb{Z})$ is freely generated by $\{[B_{b-} \cap W_j] : j(b) = 2j-1 \text{ or } 2j\}$. Since F_{j+1} is a rational homology sphere, $H_2(F_{j+1}; \mathbb{Z}) = 0$ and so the (W_j, F_{j+1}) analog of (1.3) implies that $H_2(W_j; \mathbb{Z})$ is free of rank 2.

Furthermore, the intersection form on $H_2(W_j; \mathbb{Z})$ must be non-degenerate because the boundary of W_j has no rational homology. In fact, the inclusion of W_j into V_j induces an injection $H_2(W_j; \mathbb{Z}) \longrightarrow H_2(V_j; \mathbb{Z})$ with image the generators $[T_{2j-1}]$ and $[T_{2j}]$. This implies the statement in Assertion 5 concerning the intersection form on $H_2(W_j; \mathbb{Z})$.

Here is why $H_2(W_j; \mathbb{Z})$ injects into $H_2(V_j; \mathbb{Z})$: One must prove that the submanifolds $\{T_{2j}, T_{2j-1}\}$ are homologous to submanifolds which lie in W_j.

This happens if T_{2j} and T_{2j-1} have zero intersection number with all B_{b_-} for $b = b_i$ and $i > 2j$. Indeed, if $T \equiv T_{2j}, T_{2j-1}$ has zero intersection number as described, then the intersection points of T with each such B_{b_-} can be paired as $\pm$ pairs. (One point with positive intersection number, and one with negative.) Then, surgery on these embedded S^0's in T will yield a new surface, T', (with larger genus) which is homologous to T and which has no intersection with B_{b_-} when $b \equiv b_i$ and $i > j$. (Mimic the tubing construction in Section 5d of [7].) The pseudo-gradient flow can then be used to isotope this T' into W_j.

With the preceding understood, the lemma follows with the realization that the intersection number of T, as above, with B_{b_-}, as above, is a linear functional of the entries of the matrix $(S_{i,j})_{i>j}$. And, this is, by assumption, the zero matrix. $\square$

c) Z and $W \times W$.

This subsection describes $Z \subset W \times W$, a submanifold with boundary. For the most part, the discussion here mirrors the discussion in Section 3g, h of [7] where an analogous Z is defined.

The stage is set with the following Definition:

DEFINITION 2.2. Let M_0, M_1 be compact, oriented 3-manifolds with the rational homology of S^3.

· A simple type cobordism: A cobordism W between M_0 and M_1 is of simple type if the following criteria are met:
  1) W is oriented and connected.
  2) W has a good Morse function with only two critical points, both of index 2.
  3) $H_2(W; \mathbb{Z}) \approx \mathbb{Z}^2$, and the intersection form of W is conjugate over $GL(2; \mathbb{Z})$ to $H(m)$ for some integer $m \neq 0$.

· A simple type Morse function: Let W be a cobordism of simple type. Let $f : W \longrightarrow [0, 1]$ be a function and let v be a pseudo-gradient for f. Then (f, v) are of simple type if the following criteria are met:
  1) $f_{-1}(0) = M_0$ and $f_{-1}(1) = M_1$.
  2) $df \neq 0$ near ∂W.
  3) f has only two critical points, (b_1, b_2), both with index 2.
  4) $15/16 < f(b_2) < 1/2 < f(b_1) < 17/16$.
  5) There are integers $m_1 > 0, m_2 > 0$ and $m_{1,2}$; and there are orientations of the descending disks from b_1 and b_2 such that

(2.9) $\qquad \sigma_1 \equiv m_1 \cdot [B_{b_1-}] + m_{1,2} \cdot [B_{b_2-}]$ and $\sigma_2 \equiv m_2 \cdot [B_{b_2-}]$

generate the image in $H_2(W, M_0; \mathbb{Z})$ of $H_2(W; \mathbb{Z})$.
  6) The pseudo-gradient v is good in the sense of Definition 3.1 in [7].

With the stage set, assume below that W is a cobordism of simple type, and that (f, v) are a pair of Morse function and pseudo-gradient on W which are also of simple type.

As in Section 3g of [**7**], introduce

$$(2.10) \qquad \underline{Z} \equiv \{(x,y) \in W \times W : F(x,y) \equiv f(y) - f(x) = 0\}.$$

Define $Z \subset \underline{Z}$ by intersecting the latter with the compliment in $W \times W$ of (open) small radius balls about (b_1, b_1) and (b_2, b_2). That is, mimick the constructions in Sections 3i and 3h of [**7**].

Some properties of Z are listed below:

A manifold: Z is a manifold with boundary,

$$(2.11) \qquad \partial Z = (M_0 \times M_0) \cup (M_1 \times M_1) \cup (S^3 \times S^3)_{b_1} \cup (S^3 \times S^3)_{b_2};$$

here $(S^3 \times S^3)_b$ is the link around Z's singularity at (b,b). (See Section 3h of [**7**].)

Orientation: The manifold $\text{int}(Z)$ is naturally oriented using the orientation from $W \times W$ along with dF to trivialize the normal bundle to $\text{int}(Z)$ in $W \times W$. Orient the various components of (2.11) as described in Section 3h of [**7**].

Homology: The rational homology of Z is described by

LEMMA 2.3. *Let W be as described above. Then the following hold:*
1) $H_0(Z) \approx R$.
2) $H_1(Z) \approx 0$.
3) The inclusion $Z \subset W \times W$ induces $H_2(Z) \approx H_2(W \times W) \approx \mathbb{R}^4$.
4) There is a surjection

$$(2.12) \qquad 0 \longleftarrow H_3(Z) \longleftarrow L_- \oplus L_+ \oplus H_3(\partial Z).$$

Here, $L_\pm \approx \mathbb{R}$ are freely generated by embedded 3-spheres in Z as described in Equations (3.32) and (3.33) of [7].

Proof. Mimic the proof of Lemma 3.7 in [**7**]. $\square$

3 Constructing T_1 and T_2. The constructions in [**7**] aside, the proof of Theorem 1.3 is mostly occupied with constructions on $W_2 \times W_2$, where W_2 is described in (1.4). The previous subsection introduced a factorization of such a W_2 as a sequence of cobordisms of simple type, each with a Morse function f and pseudo-gradient v of simple type. (See Proposition 2.1 and Definition 2.2.) The required constructions for W_2 in (1.4) can be reduced to a series of identical constructions, one on each simple type cobordism factor in (2.5).

With the preceding as motivation, this section will restrict attention to a cobordism W of simple type with a Morse function f and pseudo-gradient v which are of simple type also. The purpose of this section is to describe a very useful pair of 2-dimensional submanifolds of W, T_1 and T_2, whose fundamental classes generate $H_2(W)$ and give the intersection form $H(m)$. Thus, this section serves as a second digression before the construction of $E_\pm$.

a) Reconstructing T_2.

The submanifold T_2 is obtained by smoothing the corners of a C^0 embedding of a smooth surface into W. This embedding can be obtained as follows:

Step 1: Let $V \subset W$ denote the set $\{x \in W : f(x) \geq 1/4\}$. To construct T_2, first introduce the number m_2 from (2.9) and take m_2 push-off copies of $B_{b_2-} \cap V$, all with the same orientation. Make these push-offs so that f restricts to each copy with only one critical point, a maximum. And require that said maximum be close to b_2 in the following sense: Use the Morse coordinates of (3.2) in [7] and the Euclidean metric on $\mathbb{R}^4$ to measure distance. With this understood, the distance from each such minimum to b_2 should be much less than the number r which is used in (3.29) of [7] to define the boundary of Z.

To be precise, work in the Morse coordinates of (3.2) in [7] near b_2. Choose m_2 distinct unit vectors $\{n_\alpha\}$ in the (x_3, x_4) plane. Then, choose $\epsilon > 0$ but with $\epsilon << r$. Define the α'th push-off of B_{b_2-} to be the set

$$(3.1) \qquad \{(x_1, x_2, x_3, x_4) : (x_3, x_4) = \epsilon \cdot n_\alpha\}.$$

Step 2: Use B'_{b_2-} to denote the resulting m_2 push-offs of B_{b_2}; this is an oriented, submanifold with boundary in V. It is important to realize that $\partial B'_{b_2-} \subset M_{1/4} \equiv f^{-1}(1/4)$ is a disjoint union of m_2 embedded, oriented circles. These circles bound an oriented, embedded surface with boundary $R_2 \subset M_{1/4}$ which intersects $\partial B'_{b_2-}$ as ∂R_2. Take such an R_2 which is connected and which has no compact components.

Set

$$(3.2) \qquad T'_2 \equiv B'_{b_2-} \cup R_2.$$

This is a (tame) C^0-embedding of a smooth surface; the embedding is smooth save for the corners along ∂R_2. However, these corners are right angle corners in a suitable coordinate system and can be smoothed without difficulty. The resulting smooth submanifold of W is T_2.

Step 3: The push-offs B'_{b_2-} can be constructed so that T'_2 has the following properties:

(3.3)

1) No pseudo-gradient flow line intersects T'_2 more than once.
2) No pseudo-gradient flow line is anywhere tangent to B'_{b_2-}.
2) T'_2 has empty intersection with B_{b_2-}.
3) The restriction of f to B'_{b_2-} has only index 2 critical points, and precisely one on each component.
4) Each component of B'_{b_2-} intersects B_{b_2+} transverally in exactly one point.

To satisfy (3.3), first note that the explicit description in (3.2) for B'_{b_2-} obeys (3.3). (This is because the vectors $\{n_\alpha\}$ in (3.1) are assumed to be distinct.)

Second, note that B'_{b_2-} can be made so that: (3.3) holds, $\partial B'_{b_2-}$ lies on the boundary of an embedded solid torus $N \subset M_{1/4}$, and past$(B'_{b_1-}) \cap M_{1/4}$ lies in the interior of N. Note that the core circle of N is $B_{b_2-} \cap M_{1/4}$. (Recall from Section 5a in [7] the definition of the past and future of a set U (written past(U) and fut(U), respectively). For example, past$(U) \subset W$ is the set of points which can be obtained from U by traveling along pseudo-gradient flow lines to decrease f.) The Morse coordinates in (3.1) extend over a neighborhood of B_{b_2-} in W, and with this understood, the tubular neighborhood N is described by

$$(3.4) \quad N \approx (x_1, x_2, x_3, x_4) : x_3^2 + x_4^2 \le \epsilon \quad \text{and} \quad x_1^2 + x_2^2 = x_3^2 + x_4^2 + c,$$

Here $c > 0$ is an appropriate constant.

Equation (3.3) follows by showing that $\partial B'_{b_2-}$ bounds an embedded surface with boundary in the compliment of int(N). And, such a surface exists because the class $\tau_2 \in H_2(W; \mathbb{Z})$ has zero self intersection number.

With the coordinates of (3.1) and (3.4) understood, the submanifold R_2 can be assumed to intersect a neighborhood of N as the set of (x_1, x_2, x_3, x_4) which obey:

$$(3.5) \quad
\begin{aligned}
&1) \quad (x_3, x_4) = t \cdot n_\alpha \quad \text{for} \quad t \ge \epsilon \quad \text{and some} \quad \alpha \in \{1, ..., m_2\}. \\
&2) \quad x_1^2 + x_2^2 = x_3^2 + x_4^2 + c.
\end{aligned}$$

Step 4: Henceforth, assume that T'_2 in (3.2) obeys (3.3) and (3.5). The corner in T'_2 at ∂R_2 can be smoothed to produce a smooth submanifold $T_2 \subset f^{-1}([1/4, 1])$ which contains R_2 as a submanifold. The manifold T_2 is obtained by flowing points in B'_{b_2-} near $M_{1/4}$ slightly into their past so that the result (also called B'_{b_2-}) is tangent to $M_{1/4}$ at $\partial B'_{b_2-}$ to infinite order. Note that T_2 can be so constructed to obey

$$(3.6)\quad
\begin{aligned}
&1) \quad \text{No pseudo-gradient flow line intersects } T_2 \text{ more than once.} \\
&2) \quad \text{No pseudo-gradient flow line is anywhere tangent to } T_2. \\
&3) \quad T_2 \text{ has empty intersection with } B_{b_2-}. \\
&4) \quad \text{Where } f > 1/4 + 1/100, \text{ the restriction of } f \text{ to } T_2 \text{ has only} \\
&\qquad \text{index 2 critical points.}
\end{aligned}$$

b) Constructing T'_1.

It is convenient to replace T_1 of the previous section with a different, though homologous submanifold.

The construction of the new version of T_1 requires first the construction of a piece-wise smooth submanifold $T'_1 \subset W$ which is defined in this subsection. Second, the construction of the new T_1 requires a modification of T'_1 to give a smooth submanifold, $T''_1 \subset W$. This T''_1 is described in the next subsection. The new version of T_1 is finally presented in Subsection 2f, below.

The construction of T'_1 is accomplished in the following steps:

Step 1: Let $M_{3/8} \equiv f^{-1}(3/8)$. Introduce the integers m_1 and $m_{1,2}$ from (2.9). Use B'_{b_1-} to denote the union of m_1 disjoint, push-off copies of $B_{b_1-} \cap$

$\text{fut}(M_{3/8})$ together with $|m_{1,2}|$ disjoint, push-off copies of $B_{b_2-} \cap \text{fut}(M_{3/8})$. Orient B'_{b_1-} by taking the given orientation for the push-offs of B_{b_1-} and, if $m_{1,2} > 0$, the given orientation for the push-offs of B_{b_2-}. However, if $m_{1,2} < 0$, use the opposite orientation for the push-offs of B_{b_2-}.

The function f should restrict to each of the m_1 push-off copies of B_{b_1-} to have only one critical point, a maximum. And this maximum should be close to b_1; its distance should be much less than the number r from (3.29) in [**7**] when distance is measured using the Euclidean metric on $\mathbb{R}^4$ in the Morse coordinates from (3.2) of [**7**]. Thus, use the Morse coordinates around b_1 and (3.1) to define the typical push-off of B_{b_1-}; in (3.1), use $\epsilon << r$ and use distinct $\{n_\alpha\}$.

Note that $\text{fut}(T_2)$ intersects B_{b_1-} in a finite set of arcs with one endpoint at b_1. With this understood, make the push-off copies of $B_{b_1-} \cap \text{fut}(M_{3/8})$ which comprise B'_{b_1-} such that each intersects B_{b_1+} transversally in a single point, and such that each intersects $\text{fut}(T_2)$ as a finite set of half-open arc with the following properties:

(3.7)

    1)   Each end-point lies on $\partial B'_{b_1-}$.

    2)   The closure of each arc has its second endpoint on the intersection with B_{b_1+}.

    3)   The function f restricts to each arc without critical points.

Meanwhile, the function f should restrict to each of the $|m_{1,2}|$ copies of B_{b_2-} to have only one critical point, a maximum. In this case, this maximum should have distance from b_2 much greater than r (of (3.29) in [**7**]) when measured with the Euclidean metric in the Morse coordinates of (3.2) in [**7**]. Thus, a push-off copy of B_{b_2-} in B'_{b_1-} should be given by (3.1) but with $\epsilon >> r$. Also, for these $|m_{1,2}|$ push-offs of B_{b_2-}, use a set of unit vectors $\{n'_\alpha\}$ in (3.2) that is disjoint from the set that was used to define B'_{b_2-}.

Require that

(3.8)
$$\text{fut}(T_2) \cap B = \emptyset$$

whenever $B \subset B'_{b_1-}$ is any of the set of $|m_{1,2}|$ push-off copies of $B_{b_2-} \cap \text{fut}(M_{3/8})$ in B'_{b_1-}. (See (3.5) and (3.6.3).)

Step 3: The boundary of B'_{b_1-} is a disjoint union of oriented circles in $M_{3/8}$. This union of circles defines a cycle which is null homologous in $M_{3/8}$. And, $\partial B'_{b_1-}$ bounds an oriented, embedded surface with boundary, $R'_1 \subset M_{3/8}$, which intersects $\partial B'_{b_1-}$ as $\partial R'_1$. Take R'_1 which is connected and which has no compact components.

With R'_1 and B'_{b_1-} understood, set

(3.9)
$$T'_1 \equiv B'_{b_1-} \cup R'_1.$$

c) Constructing T_1''.

Given that T_2 has already been constructed, it is desirable to modify T_1' by isotopy so that the result, T_1'', has the following property:

(3.10)
1) No pseudo-gradient flow line intersects T_1'' more than once where $f \leq \sup(f|T_2)$.

2) No pseudo-gradient flow line is anywhere tangent to T_1'' where $f \leq \sup(f|T_2)$.

If $T_1'' \equiv T_1'$ is given by (3.9), this condition may not hold. (Note that (3.10) holds separately for R_1' and for B_{b_1-}'.) However, (3.10) can always be achieved by redefining the push-offs in B_{b_1-}'.

The redefinition of B_{b_1-}' begins with the following observation: A component $B \subset B_{b_1-}'$ which is a push-off copy of B_{b_2-} needs a choice of ϵ in (3.1) for its definition. Let ϵ_1 denote the chosen value of ϵ. Likewise, each copy of B_{b_2-} comprising B_{b_2-}' needs a choice of ϵ in (3.1). Use ϵ_2 to denote the choice here. One is required to choose $\epsilon_1 >> \epsilon_2$. If ϵ_1 is, none-the-less, much less than $f(b_1) - f(b_2)$, then there are numbers $h_1 < h_2$ which are both greater than $\sup(f|T_2)$ but which are both significantly less than $f(b_1)$ (as measured in multiples of ϵ_1) and also less than the maximum of f on any of the components of B_{b_1-}'.

Note next that $\partial B_{1-}'$ is disjoint from the solid torus N in (3.4). This implies that $\partial R_1'$ has a collar $C \subset R_1'$ which is disjoint from N. Each component of C is an embedding in R_1' of $[0,1] \times S^1$; and the convention will be that $\{0\} \times S^1$ corresponds to a component of $\partial R_1'$. Fix such a collar with the following properties: A component which intersects a push-off of B_{b_2-} in B_{b_2-}' should be disjoint from $\text{fut}(T_2)$. (See (3.8).) And, a component which intersects a push-off of B_{b_1-} in B_{b_2-}' should intersect $\text{fut}(T_2)$ as an arc between said components two boundaries. (See (3.7).)

Define a re-imbedding of $C \cup B_{b_1-}'$ in W as follows: Move points of $C \cup B_{b_1-}'$ by an isotopy of W which pushes points along pseudo-gradient flow lines. The result of the isotopy should push B_{b_1-}' to where $f \geq h_1$ but it should leave the image of C where $f \leq h_1$. Thus, $\partial R_1'$ is pushed to $f^{-1}(h_1)$. The isotopy should keep fixed the compliment in C of a neighborhood of $\partial R_1'$ and it should fix points in B_{b_1-}' where $f \geq h_2$. It is not difficult to make this re-imbedding so that

(3.11)
1) The restriction of f to the image of C has no critical points where $f > 3/8$.

2) The restriction of f to the image of B_{b_1-}' has only index 2 critical points, and there is precisely one on each component.

The embedded image of $C \cup B_{b_1-}'$ gives a piecewise smooth embedding in W of a union of disks. Indeed, the embedded image of $C \cup B_{b_1-}'$ has a corner where the images of C and of B_{b_1-}' intersect, that is, along $f^{-1}(h_1)$. Choose

in advance a neighborhood of this corner, and the image of $C \cup B'_{b_1-}$ can be smoothed in the chosen neighborhood so that the result, B''_{b_1-}, has the following properties:

(3.12)

1) $B''_{b_1-} = B'_{b_1-}$ where $f < h_2$.

2) B''_{b_1-} agrees with the image of C where $f \geq h_1$.

3) The restriction of f to B''_{b_1-} has only index 2 critical points where $f > 3/8$; and there is precisely one on each component.

4) Each component of B''_{b_1-} is either a push-off copy of B_{b_1-}, or else one of B_{b_2-}.

5) A component of B''_{b_1-} which is a copy of B_{b_1-} intersects B_{b_1+} transversely in a single point. Such a component also intersects $\mathrm{fut}(T_2)$ in a finite set of half-open arcs with their boundaries on $M_{3/8}$. The closures of each half-open arc is an embedded arc whose other end-point is the intersection point with B_{b_1+}. Furthermore, f restricts to each half-open arc with no critical points where $f < 3/8$.

6) A component of B''_{b_1-} which is a copy of B_{b_2-} has empty intersection with $\mathrm{fut}(T_2)$.

7) No pseudo-gradient flow line is anywhere tangent to B''_{b_1-}.

Let $R''_1 \equiv R'_1 - C$ and define

(3.13)
$$T''_1 \equiv R''_1 \cup B''_{b_1-}.$$

This submanifold obeys (3.10). (Where $f \geq h_1, T''_1$ is obtained from R'_1 by flowing the latter along pseudo-gradient flow lines.)

d) Intersection of $\mathrm{fut}(T''_1)$ **with** T_2 **and** T''_1 **with** $\mathrm{fut}(T_2)$**.**

The intersection between $\mathrm{fut}(T''_1) \cap T_2$ is the union of a finite set of half-open arcs each of which has its endpoint at one of the points of $T''_1 \cap T_2$, and vice-versa. (Note that $T''_1 \cap T_2 = R''_1 \cap T_2$.) The closure of each half open arc is an embedded arc with its other end point where B'_{b_2-} intersects B_{b_2+}. There are at least m such arcs.

The intersection of T''_1 with $\mathrm{fut}(T_2)$ is more complicated. After perturbing T''_1 slightly, this intersection can be assumed to have the following form: It consists of a finite, disjoint set of closed arcs, half-open arcs, and open arcs in T''_1; and disjoint from these arcs, there is a finite set of disjoint, embedded circles. Each point of $T''_1 \cap T_2 (= R''_1 \cap B'_{b_2-})$ will be a boundary component for some arc, either half open or closed. (But, there may be more or less arcs than boundary components of arcs.)

The closure of a half-open arc will be a smooth arc whose other endpoint lies on $B''_{b_1-} \cap B_{b_1+}$ (and thus in a push-off copy of B_{b_1-} in B''_{b_1-}). The closure

of an open arc will also be a smooth arc, but with both of its endpoints in $B''_{b_1-} \cap B_{b_1+}$. To see that such is the case, introduce $\underline{R}_2 \equiv \text{fut}(R_2) \cap f^{-1}(7/16)$. This will intersect T''_1 in the m_2 push-off copies of B_{b_1-}. (In fact, its intersection number with the union of said m_2 copies is equal to m. See (5) and (6) of (3.12).) Each intersection point of $\underline{R}_2$ with T''_1 has one half-open arc component or one open arc component of $T''_1 \cap \text{fut}(T_2)$ passing through it. Furthermore, each half-open arc component intersects precisely one point of $T''_1 \cap \underline{R}_2$, while each open arc component intersects precisely two such points. Each half-open arc component intersects $f^{-1}([7/16, 1])$ as a push-off copy of a pseudo-gradient flow line for f in B_{b_1-} which ends in b_2; and each open arc component intersects $f^{-1}([7/16, 1])$ in a pair of such push-offs.

The circles in $T''_1 \cap \text{fut}(T_2)$ can be assumed to lie in the interior of R''_1. (See (5) and (6) of (3.12).)

It is important to note that there are at least m half-open arcs components of $T''_1 \cap \text{fut}(T_2)$; any less would be incompatible with the assumed value of m for $T''_1 \cdot T_2$. If a pair of points in $T''_1 \cap R_2$ are points on the same open arc, then these points will have opposite local intersection numbers for $T''_1 \cap T_2$.

A similar argument shows that for at least m of these arcs, both the intersection point in $B''_{b_1-} \cap \underline{R}_2$ and the endpoint in $R''_1 \cap B'_{b_2-}$ are points of positive local intersection number for $T''_1 \cap \underline{R}_2$ and for $R''_1 \cap T_2$, respectively.

With the preceding understood, fix one half-open arc,

$$(3.14) \qquad\qquad v^0 \subset (T''_1 \cap \text{fut}(T_2)),$$

which intersects $B''_{b_1-} \cap R_2$ at a point of positive local intersection number, and which ends in $R''_1 \cap B'_{b_2-}$ at a point with positive local intersection number.

e) Homology of T'_1 and T_2 and the linking matrix.

There is one additional constraint that must be imposed on T''_1; and this one also requires advanced knowledge of T_2. Suppose that T''_1 and T_2 have already been constructed.

The surface T_2 has some genus $g_2 \geq 0$. As such, its first homology has a basis which is represented by the fundamental class of a set, $\{\eta_{2\beta}\}_{\beta=1}^{2g_2} \subset \text{int}(R_2)$, of $2 \cdot g_2$ embedded, oriented circles. Take n_1 (from above) like oriented, push-off (in R_2) copies of each $\eta_{2\alpha}$. Together, these form a set $\{\rho'_{\beta_i}\}$, where β runs from 1 to $2g_2$, and where i runs from 1 to n_1.

The pseudo-gradient flow pushes R_2 isotopically into $M_{3/8}$ as the submanifold $\text{fut}(R_2) \cap M_{3/8}$, and thus the circles $\{\rho'_{\beta_i}\}$ are pushed isotopically into $M_{3/8}$ as a set, $\{\rho_{\beta_i}\} \subset M_{3/8}$, of $2 \cdot n_1 \cdot g_2$ circles.

Fix the set of circles $\{\rho_{\beta_i}\}$ once and for all. These circles will be used to constrain R''_1; but a short digression is needed to define these new constraints.

Start the digression by observing that the surface T''_1 has some genus $g_1 \geq 0$ and so its first homology is represented by the fundamental class of a set of $2 \cdot g_1$ embedded, oriented circles, $\{\eta_{1\alpha}\}_{\alpha=1}^{2g_1} \subset int(R''_1) \subset M_{3/8}$. These generators should be chosen to be disjoint from the arc v^0 which is described in (3.14). (This is possible because v^0 is an arc with one endpoint on $\partial R''_1$ and the other in the interior of R''_1.)

The manifold $M_{3/8}$, being diffeomorphic to M_0, has the rational homology of S^3. This means, in particular, that some number $n_1 \geq 1$ of like oriented push-off copies (in R_1'') of each $\eta_{1\alpha}$ bounds an embedded surface with boundary, $S_\alpha \subset M_{3/8}$.

No generality is lost by assuming that R_1'' intersect each of the circles $\{\rho_{\beta_i}\}$ transversally. Likewise, there is no generality lost here by requiring that the $\{\eta_{1\alpha}\}$ which generate $H_1(T_1'')$ be disjoint from the set $\{\rho_{\beta_i}\}$.

Push-off, in R_1'', the n_1 copies of each $\eta_{1\alpha}$. Make these close to $\eta_{1\alpha}$ to insure that the push-off isotopy is disjoint from $\{\rho_{\beta_i}\}$. Find the submanifold with boundary $S_\alpha \subset M_{3/8}$ which intersects the n_1 push-off copies of $\eta_{1\alpha}$ as its boundary.

In general position, each such S_α will intersect each of the circles ρ_{β_i} transversally. So, there is a $2g_1 \times 2g_2$ matrix $A \equiv (A_{\alpha,\beta})$ where $A_{\alpha,\beta}$ is the sum of the intersection numbers between the surface S_α and the n_1 circles $\{\rho_{\beta_i}\}_{i=1}^{n_1}$. (Here, the index β is fixed.) The matrix A will be called the linking matrix between the set $\{\eta_{1\alpha}\}$ and the set $\{\rho_{\beta.}\}$. Note that the entries $\overline{A_{\alpha,\beta}}$ are divisible by the integer n_1, and that the definition of $A_{\alpha,\beta}$ requires the apriori choice of push-offs $\{\rho_{\beta_i}'\}$ of $\{\eta_{2\beta}\}$.

With the preceding understood, the point of this subsection is to remark that there is an isotopy of R_1'' in $M_{3/8}$ (rel $\partial R_1''$, the arc v^0, and $R_1'' \cap T_2$) to a surface $R*_1 \subset M_{3/8}$ so that the linking matrix A^* between the isotoped circles, $\{\eta*_{1\alpha}\}$, and $\{\rho_{\beta.}\}$ has all entries zero. In fact, this can be accomplished using finger moves to isotope $\eta_{1\alpha}$ to change its linking number with each $\rho_{\beta1}$ but leave unchanged the linking number with each $\rho_{\beta_i,>1}$. (Note that the linking number with ρ_{β_1} can be changed only by multiples of an integer which divides n_1, while the entries of the matrix $A_{\alpha,\beta}$ are divisible by n_1.)

Each such finger move changes R_1'' by an ambient isotopy which fixes the compliment of a small ball in R_1'' and which stretches the interior of this ball over a regular neighborhood of some arc in $M_{3/8}$. The ability to simultaneously change all entries of A to zero is based on the fact that the finger move isotopy moves R_1'' only in tubular neighborhoods of arcs.

Because each finger move changes R_1'' only in the neighborhood of a point, these finger can be made away from $\partial R_1''$, the path v^0. For the same reason, the finger moves can be done so as to leave $R_1'' \cap T_2$ unchanged.

With the preceding understood, it will be assumed in the sequel that there exist $n_1 \geq 0$ and a set of:

(3.15)
    1)   circles $\{\eta_{2\beta}\} \subset T_2$ which generate $H_1(T_2)$ for the homology of T_2,

    2) n_1 push-off copies, $\{\{\rho_{\beta_i}\}_{i=1}^{n_1}\}$, of $\{\eta_{2\beta}\}$,

    3)   circles $\{\eta_{1\alpha}\}$ which generate $H_1(T_1'')$,

with the property that the resulting linking matrix $A = (A_{\alpha,\beta})$ has all entries zero. Furthermore, $\{\eta_{1\alpha}\}$ will be assumed disjoint from v^0 of (2.14) and from $\mathrm{fut}(\{\rho_{\beta,i}'\})$.

f) Definition of T_1.

With T_1'' understood, the surface $T_1 \subset W$ can now be constructed by isotoping T_1'' into the future a small amount along pseudo-gradient flow lines.

This construction of T_1 is accomplished by the following steps:

Step 1: Find an embedding

(3.16) $$\varphi : f^{-1}([3/8, 1]) \longrightarrow f^{-1}([3/8, 1])$$

with the following properties:

(3.17)
> 1) φ is the end of an isotopy which moves points along pseudo-gradient flow lines.
> 2) φ is the identity where $f \geq 3/8 + 1/100$.
> 3) Let $\underline{M} \equiv \varphi(M_{3/8})$. Then $f|\underline{M} > 3/8$.
> 4) f restricts to $\varphi(v^0)$ with out critical points.
> 5) $\inf(f|\varphi(\{\eta_{1\alpha}\})) > \sup(f|\varphi(\{\rho_{\beta_i} \cap T_1''\}))$.
> 6) f restricts to $\varphi(B_{b_1-}'')$ with only index 2 critical points, one on each component.

To find such a φ, use the pseudo-gradient flow to construct a diffeomorphism

(3.18) $$f^{-1}([3/8, 7/16]) \approx M_{3/8} \times [3/8, 7/16],$$

where the pseudo-gradient flow lines are mapped to the lines $p \times [3/8, 7/16]$, and where f is given by projection onto the second factor. With respect to (3.18), the embedding φ sends (p, t) to $(p, g(p, t))$, where g is a smooth function. It is left to the reader to find g which makes (3.17) true. (Remark here that $\{\eta_{1\alpha}\}$ are disjoint from v^0 and from $\{\rho_{\beta_i}\}$.)

Step 2: With φ understood, define

(3.19) $$T_1 \equiv \varphi(T_1'').$$

Also, introduce $R_1 \equiv \varphi(R_1'')$. Here are some important properties of T_1:

1) No pseudo-gradient flow line intersects T_1 more than once where $f \leq \sup(f|T_2)$.

2) No pseudo-gradient is anywhere tangent to T_1 where

(3.20) $f \leq \sup(f|T_2)$.

3) $T_1 \cap T_2 = \text{int}(R_1) \cap B'_{b_2-}$.

4) Where $f \geq 3/8 + 1/100$, the restriction of f to T_1 has only index 2 critical points.

4 A start at Σ_Z. This section begins the construction of the subvariety Σ_{Z_2} as in (1.6). The plan is to factor the cobordism W_2 from (1.4) as a sequence of cobordisms of simple type (Definition 2.2), and to define a Σ. for each component, simple type cobordism in this factorization. Then, Σ_{Z_2} in (1.6) is defined to be the union of these Σ. for the constituent simple type cobordisms which comprise W_2.

With the preceding understood, assume in this section and in Section 5 that W, and the Morse function f and the pseudo-gradient v are of simple type, as defined in Definition 2.2. Use the definitions in Section 2c to define $Z \subset W \times W$. Sections 4 and 5 will construct a particular oriented, dimension-4 subvariety with boundary $\Sigma_Z \subset Z$. The boundary of Σ_Z will sit in ∂Z. Furthermore, Σ_Z will contain a class $\sigma_Z \in H_4(\Sigma_Z, \partial\Sigma)$ which obeys the conclusions of Lemma 4.1 in [7]. As in Section 4c and (4.10) of [7], the variety Σ_Z will be given as a union

(4.1) $$\Sigma_Z = \Delta_Z \cup E_L \cup E_R \cup E_- \cup E_+.$$

Here, Δ_Z is as described in Section 4d of [7], and $E_{L,R}$ are as described in Section 4e of [7]. (Remember: Δ_Z is the intersection of Z with the diagonal in $W \times W$. Meanwhile, E_R, E_L are the respective intersections of Z with $\gamma \times W$ and $W \times \gamma$; here $\gamma \subset W$ is the pseudo-gradient flow line which starts at $p_0 \in M_0$ and ends at $p_1 \in M_1$.)

a) A first pass at E_-.

Recall that the future of a set $U \subset W$ (written fut(U)) is the set of points in W which can be reached from U by traveling along a gradient flow line in the direction of increasing f.

Introduce

(4.2) $$E'_{1-} \equiv [(T_1 \times \text{fut}(T_2)) \cup (\text{fut}(T_1) \times T_2)] \cap Z.$$

Equations (3.6) and (3.20) ensure that $T_1 \times \text{fut}(T_2)$ and $\text{fut}(T_1) \times T_2$ intersect Z transversally, each as a smooth submanifold with boundary. These assertions are proved with the following fact: Let $U \subset W$ be a submanifold which intersects no pseudo-gradient flow line more than once, and which is nowhere tangent to a pseudo-gradient flow line. Then $\text{fut}(U) \subset W$ is a smooth submanifold with boundary, and that boundary is U.

b) E'_{1-} as a cycle.

To consider E'_{1-} as a cycle, it is necessary to understand first the boundaries of $(T_1 \times \text{fut}(T_2)) \cap Z$ and $(\text{fut}(T_1) \times T_2) \cap Z$. One finds

$$(4.3) \quad \partial[(T_1 \times \text{fut}(T_2)) \cap Z] = [(T_1 \times T_2) \cap Z] \cup [(T_1 \times \text{fut}(T_2)) \cap \partial Z],$$

and, likewise,

$$(4.4) \quad \partial[(\text{fut}(T_1) \times T_2) \cap Z] = [(T_1 \times T_2) \cap Z] \cup [(\text{fut}(T_1) \times T_2)) \cap \partial Z].$$

(The conditions in (3.6) and (3.20) are used here.)

It follows from (4.3), (4.4) that orientations exist for both $(\text{fut}(T_1) \times T_2) \cap Z$ and $(T_1 \times \text{fut}(T_2)) \cap Z$ such that $\partial[E'_{1-}]$ has support (as a cycle) in $(S^3 \times S^3)_{b_1} \cup (S^3 \times S^3)_{b_2}$. With the preceding understood, write

$$(4.5) \qquad\qquad E'_{1-} \cap \partial Z \equiv S_{b_1+} \cup S_{b_2-}.$$

where, $S_{b_1+} \subset (S^3 \times S^3)_{b_1}$ while $S_{b_2-} \subset (S^3 \times S^3)_{b_2}$. It is left as an exercise to prove that S_{b_1+} can be identified as being some number of push-off copies of the right-hand sphere, $(S^3)_{b_1+} \subset (S^3 \times S^3)_{b_1}$; while S_{b_2-} consists of some number of disjoint, push-off copies of $(S^3)_{b_2-} \subset (S^3 \times S^3)_{b_2}$. (See the proof of Lemma 4.1, below.)

The next task is to determine the homology classes of the cycles that S_{b_1+} and S_{b_2-} define.

LEMMA 4.1. *The components $(T_1 \times \text{fut}(T_2)) \cap Z$ and $(\text{fut}(T_1) \times T_2) \cap Z$ of E'_{1-} can be oriented so that as a cycle,*

$$(4.6) \qquad\qquad \partial[E'_{1-}] = m \cdot [S^3]_{b_1+} + m \cdot [S^3]_{b_2-}.$$

Proof. Orient E'_{1-} as follows: To begin, orient T_1 and T_2 to make their intersection number $[T_1] \cdot [T_2]$ equal to m. Let $o_{1,2} \in \Lambda^2 T(T_{1,2})$ denote the respective orientations. Next, orient $\text{fut}(T_1)$ and $\text{fut}(T_2)$ by using $-v \wedge o_{1,2}$, where v is the pseudo-gradient for f. (Note that v is tangent to $\text{fut}(T_{1,2})$ and is inward pointing along T_1 or T_2.) Orient $T_1 \times \text{fut}(T_2)$ as $\pi_L * o_1 \wedge \pi_R * (-v \wedge o_2)$ and orient $\text{fut}(T_1) \times T_2$ as $\pi_L * (v \wedge o_1) \wedge \pi_R * o_2$. Notice that the former is oriented using the product orientation, but the latter is oriented in reverse. This insures that the respective orientations which are induced on $T_1 \times T_2$ are, in fact, opposite.

Near b_1, T_1 is identified with m_1 like oriented, push-off copies of the descending disk B_{b_1-}. Using the Morse coordinates of (3.2) in [7], this descending disk is given by setting $x_3 = x_4 = 0$. And, one can assume, without loss of generality, that $o_1 = \partial_{x_1} \wedge \partial_{x_2}$. Here, the orientation for W can be assumed to be $o \equiv \partial_{x_1} \wedge \partial_{x_2} \wedge \partial_{x_3} \wedge \partial_{x_4}$.

Near $b_1, \text{fut}(T_2)$ is a union of some number of disjoint components. These components can be described as follows: The pseudo-gradient flow isotopes T_2

to where $f \geq 7/16$ in W. This isotopic image, $\underline{T}_2$, intersects T_1 transversally; in fact, $\underline{T}_2$ intersects T_1 in the m_1 push-off copies of B_{b_1-}. Each intersection of $\underline{T}_2$ with B_{b_1-} defines a component fut(T_2) near b_1, and likewise each intersection point of $\underline{T}_2$ with one of the m_1 push-off copies of B_{b_1-} defines a component of $T_1 \times$ fut(T_2) near (b_1, b_1). Thus, the intersection points of $\underline{T}_2$ with the m_1 push-off copies of B_{b_1-} are in 1-1 correspondence with the components of S_{b_1+}.

Using Morse coordinates of (3.2) in [**7**] near b_1, a typical component of fut(T_2) near b_1 is given by $\{x : x_1 = 0, x_2 > 0\}$. If this component corresponds to a positive intersection point of $\underline{T}_2$ with T_1, then this component can be assumed oriented by $-\partial_{x_2} \wedge \partial_{x_3} \wedge \partial_{x_4}$; here ∂_{x_2} is equal to v where x_1, x_3 and x_4 all vanish and $x_2 > 0$. Thus, the corresponding component of $T_1 \times$ fut(T_2) is oriented by

$$(4.7) \qquad -\partial_{x_1} \wedge \partial_{x_2} \wedge \partial_{y_2} \wedge \partial_{y_3} \wedge \partial_{y_4},$$

where

$$(4.8) \qquad x_1 = x_3 = x_4 = y_1 = y_3 = y_4 = 0 \quad \text{and} \quad y_2 > 0.$$

Here, the orientation for the intersection of $T_1 \times$ fut(T_2) with Z is given by contracting (4.7) with $-dy_2 + dx_2$. The resulting orientation is $\partial_{x_1} \wedge (\partial_{x_2} + \partial_{y_2}) \wedge \partial_{y_3} \wedge \partial_{y_4}$. The induced boundary orientation is given by contracting this with $-dx_2 - dy_2$; and the result is $\partial_{x_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$.

Meanwhile, $(S^3)_{b_1+} = \{(x, y) : y_1 = x_3 = x_4 = 0, x_2 = r\}$. At the point in (4.8), $(S^3)_{b_1+}$ is oriented by $\partial_{x_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$ also. Notice that this orientation is the same as that of the boundary of the given component of $T_1 \times$ fut(T_2), and this component, by assumption, corresponds to a positive intersection point between T_1 and $\underline{T}_2$.

To summarize the preceding, a component of S_{b_1+} is oriented the same as $(S^3)_{b_1+}$ if the corresponding intersection point between T_1 and $\underline{T}_2$ is positive; while it is oriented in reverse if the corresponding intersection point between T_1 and $\underline{T}_2$ is negative. This observation justifies the factor of m in the first term on the right side of (4.6) because the algebraic intersection number between T_1 and $\underline{T}_2$ is equal to that between T_1 and T_2, which is m.

Consider now the analogous calculation near (b_2, b_2). Here, the roles of T_1 and T_2 are interchanged. The intersection of T_1 and T_2 occur along B'_{b_2-}; the m_2 push-off copies of B_{b_2-}. Thus, the components of fut$(T_1) \times T_2$ near (b_2, b_2) are in 1-1 correspondence with the intersection points of T_1 and B'_{b_2-} as are the components of S_{b_2-}.

Use the Morse coordinates of (3.2) in [**7**] near b_2. A typical component of fut$(T_1) \times T_2$ near (b_2, b_2) is given as

$$(4.9) \qquad \{(x, y) : x_1 = y_3 = y_4 = 0, x_2 > 0\}.$$

If the component above corresponds to an intersection point of T_1 with B'_{b_2-} which has positive intersection number, then the orientation of (4.9) is given by $\partial_{x_2} \wedge \partial_{x_3} \wedge \partial_{x_4} \wedge \partial_{y_1} \wedge \partial_{y_2}$ at points where

$$(4.10) \qquad x_1 = x_3 = x_4 = y_1 = y_3 = y_4 = 0 \quad \text{and} \quad x_2 > 0.$$

The orientation for the intersection of (4.9) with Z is given by contracting its orientation with $-dy_2 + dx_2$. The resulting orientation at (4.10) is $\partial_{x_3} \wedge \partial_{x_4} \wedge \partial_{y_1} \wedge (\partial_{y_2} + \partial_{x_2})$. The boundary orientation is obtained by contracting again with $-dx_2 - dy_2$; the result is $\partial_{x_3} \wedge \partial_{x_4} \wedge \partial_{y_1}$. Note that this orientation equals the given orientation on $(S^3)_{b_2-}$.

The preceding is summarized as follows: A component of S_{b_2-} is oriented as $(S^3)_{b_2-}$ if the corresponding intersection point of T_1 and T_2 is positive; and the component is oriented negatively if the corresponding intersection point is negative. Thus, the factor of m in the second term on the right in (4.6) also follows from the fact that $T_1 \cdot T_2 = m$. $\square$

c) E_{1-} as a smoothing of E'_{1-}.

As defined by (4.2), E'_{1-} is the union of a pair of 4-dimensional submanifolds with boundary in Z which meet along a common boundary component which is $(T_1 \times T_2) \cap Z$. There are no obstructions to smoothing the crease along $(T_1 \times T_2) \times Z$ to obtain a smoothly embedded, oriented submanifold with boundary, $E_{1-} \subset Z$. The next few subsections will describe some additional properties of E_{1-}.

d) E_{1+}.

Introduce the switch map

$$(4.11) \qquad \Theta : W \times W \longrightarrow W \times W,$$

which interchanges the coordinates. This map preserves Z. Define $E'_{1+} \equiv \Theta(E'_{1-})$ and $E_{1+} \equiv \Theta(E_{1-})$. Thus,

$$(4.12) \qquad E'_{1+} = [(\text{fut}(T_2) \times T_1) \cup (T_2 \times \text{fut}(T_1))] \cap Z.$$

e) The intersection with Δ_Z.

Make the standard identification of $\Delta_W \subset W \times W$ with W (project on either right of left factor). This identifies Δ_Z with the compliment in W of the union of an open ball about b_1 and an open ball about b_2. And this identifies $E'_{1-} \cap \Delta_Z$ with the intersection of

$$(4.13) \qquad (T_1 \cap \text{fut}(T_2)) \cup (\text{fut}(T_1) \cap T_2)$$

with the compliment in W of said balls.

To begin the analysis of (4.13), note that $\text{fut}(T_1) \cap T_2$ is the union of a finite set of half-open arcs which start at the points of $T_1 \cap T_2$ (this is the same as $R_1 \cap B'_{b_2-}$). The closure of each of these half-open arcs is an embedded arc whose other endpoint is in $B'_{b_2-} \cap B_{b_2+}$. Remark that there are at least m such arcs.

The intersection of T_1 with $\text{fut}(T_2)$ is the image under the embedding φ in (3.16) of $T''_1 \cap \text{fut}(T_2)$. The latter is described in Section 3d.

It follows from the description in Section 3d of $T_1'' \cap \text{fut}(T_2)$ that the intersection of E_{1-} with Δ_Z is the disjoint union of some number of arcs and some number of circles. The end-points of the arcs lie $\partial E_{1-} \cap \Delta_Z$, that is, on $(S^3 \times S^3)_{b_1} \cup (S^3 \times S^3)_{b_2}$. It is important to note that there are at least m such arcs which join m points of $S_{b_1+} \cap (\Delta_{S^3})_{b_1}$ with m points of $S_{b_2-} \cap (\Delta_{S^3})_{b_2}$.

Furthermore, the proof of Lemma 4.1 shows that for at least m of these arcs, the one end point in S_{b_1+} and the other in S_{b_2-} lie in components which are oriented positively with respect to the given orientations of $(S^3)_{b_1+}$ and $(S^3)_{b_2-}$, respectively. In fact, there is an arc, $v \subset E_{1-} \cap \Delta_Z$, which connects a positively oriented component of S_{b_1+} with a positively oriented component of S_{b_2-}, and which is characterized as follows: Before smoothing E_{1-}' to E_{1-}, this v was an arc in E_{1-}' which intersected $T_1 \cap \text{fut}(T_2)$ as $\varphi(v^0) \cap \Delta_Z$, where v^0 is the half-open arc in (3.14).

f) Intersections with $E_{L,R}$.

The submanifold E_{1-} can be assumed to have empty intersection with $E_{L,R}$. Indeed, the flow line γ between $p_0 \in M_0$ and $p_1 \in M_1$ misses a small ball around b_1 and b_2; and a small perturbation of R_1 and R_2 will insure that γ misses these surfaces also.

g) Normal framings.

The claim here is that E_{1-} has trivial normal bundle in Z, and that there is a trivialization of said normal bundle which restricts to each component of S_{b_1+} and S_{b_2-} as the constant normal framing. (Recall from [7] that the constant framing of $S^3 \times$ point in $S^3 \times S^3$ is the normal framing which is given by $\pi_R * f$, where π_R maps $S^3 \times S^3$ onto the right factor of S^3, and f is a normal framing of the point.)

The establishment of this claim requires the following six steps.

Step 1: This first step identifies E_{1-}:

LEMMA 4.2. *The submanifold E_{1-} is diffeomorphic to the compliment in $T_1 \times T_2$ of a finite number of disjoint, open balls.*

Proof. The identification of E_{1-} starts with the identification

$$(4.14) \qquad [\text{fut}(T_1) \times T_2] \cap Z \approx (R_1 \times B_{b_2-}') - U,$$

where U is a finite set of disjoint, open balls. Meanwhile,

$$(4.15) \qquad \begin{aligned} &[T_1 \times \text{fut}(T_2)] \cap Z \\ &\approx [(R_1 \times R_2) \cup (B_{b_1-}' \times R_2) \cup (B_{b_1-}' \times B_{b_2-}')] - U'. \end{aligned}$$

Here, $R_1 \times R_2$ and $B_{b_1-}' \times R_2$ are attached along their common boundary component, $\partial B_{b_1-}' \times R_1$. Meanwhile, $(B_{b_1-}' \times R_2) \cup (B_{b_1-}' \times B_{b_2-}')$ are attached along their common boundary component, $B_{b_1-}' \times \partial B_{b_2-}'$. Finally, $U' \subset \text{int}(B_{b_1-}' \times R_2)$ is a finite, disjoint collection of open balls.

Remember that $(\operatorname{fut}(T_1) \times T_2) \cap Z$ and $(T_1 \times \operatorname{fut}(T_2)) \cap Z$ are attached along their common boundary to obtain E_{1-}. This common boundary is

$$(4.16) \qquad (T_1 \times T_2) \cap Z \approx [R_1 \times (S^1)^{m_2}] \cup [\partial R_1 \cup B'_{b_2-}],$$

where $(S^1)^{m_2} \approx (B'_{b_2-} \cap \underline{M})$. With (4.16) understood, one can see (4.14) and (4.15) as a decomposition of $T_1 \times T_2$ less some number of open balls by writing $T_1 \approx B'_{b_1-} \cup R_1$ and $T_2 \approx B'_{b_2-} \cup R_2$. $\square$

Step 2: The normal bundle to E_{1-} in Z is an oriented three-plane bundle, and since E_{1-} is not closed, this 3-plane bundle is classified by its 2nd Stieffel-Whitney class, w_2. This class is zero for the following reasons: First, $w_2(TW) = 0$ since W is assumed to be a spin manifold. Thus, $w_2(T(W \times W)) = 0$. Second, remark that $T(W \times W)|Z \approx TZ \oplus \mathbb{R}$, where $\mathbb{R}$ is the trivial, real line bundle. Thus, $w_2(TZ) = 0$. Restricted to $E_{1-}, TZ \approx TE_{1-} \oplus vE_{1-}$, where vE_{1-} is the normal bundle in question. Now, $T_1 \times T_2$ is a spin manifold, and therefore $w_2(E_{1-}) = 0$; so $w_2(vE_{1-}) = 0$ as claimed.

Step 3: Having established that E_{1-} has trivial normal bundle in Z, it remains yet to establish that this normal bundle has a trivialization which restricts to each component of ∂E_{1-} as the constant normal framing. Here is an outline of the argument:

a) Remember that E_{1-} is the image of an embedding of the compliment in $T_1 \times T_2$ of some number of open balls. With this understood, the proof establishes that this embedding extends as an embedding of $T_1 \times T_2$ into $W \times W$. This extension will be called $\hat{E}_{1-}$.

b) The proof establishes that the normal bundle in $W \times W$ to $\hat{E}_{1-}$ splits as $N \oplus \mathbb{R}$, where N is a trivial 3-plane bundle, and where $\mathbb{R}$ restricts to $E_{1-} \subset \hat{E}_{1-}$ as the normal bundle to Z in $W \times W$.

c) The proof establishes that N is a trivial 3-plane bundle over $\hat{E}_{1-}$.

d) Thus, N restricts to E_{1-} as vE_{1-}; and the restriction of a framing of N to E_{1-} gives a framing of vE_{1-} which is homotopic to the constant framing over each component of ∂E_{1-}.

Step 4: To esteblish Step 3a, above, remark that a component, C of ∂E_{1-} on $(S^3 \times S^3)_{b_1}$ has a neighborhood in E_{1-} which can be assumed to have the following form in coordinates from Lemma 3.6 in [7]:

$$(4.17) \qquad \{(x,y) : x_3 = x_4 = y_2 = 0 \quad \text{and} \quad x_1^2 + x_2^2 + y_3^2 + y_4^2 = y_1^2\},$$

where $y_1 \geq (r/2)^{1/2}$. Here, C is given by (4.17) with $y_1 = (r/2)^{1/2}$.

Note that C is the intersection with $(S^3 \times S^3)_{b_1}$ of a push-off of the ascending 4-ball from the critical point (b_1, b_1) for the function F on $W \times W$ which is given in (3.20) of [7]. Thus C bounds an embedded 4-ball in $W \times W$, for example, the ball $B \subset (W \times W - Z)$ which is given by

$$(4.18) \qquad \begin{array}{l} \{(x,y) : x_3 = x_4 = y_2 = 0, y_1 = (r/2)_{1/2} \\ \qquad \text{and} \quad x_1^2 + x_2^2 + y_3^2 + y_4^2 \leq r/2\}. \end{array}$$

Each boundary component of E_{1-} has its analogous B; and these can be taken to be mutually disjoint, being all push-off copies of a descending 4-ball for F from (b_1, b_1) or from (b_2, b_2). Glue these 4-balls to E_{1-} along their common boundaries and smooth the corner along ∂E_{1-} to obtain $\hat{E}_{1-}$, an embedding of $T_1 \times T_2$ into $W \times W$ which extends E_{1-}.

Step 5: To establish Step 3b, note that the normal bundle to $\hat{E}_{1-}$ in $W \times W$ splits as $\upsilon E_{1-} \oplus \mathbb{R}$ where $\mathbb{R}$ is spanned by a section of $T(W \times W)$ along E_{1-} which has positive pairing with dF. With this understood, consider the vector field $-\partial/\partial y_1$ in the coordinates of (4.17), (4.18). This vector field is nowhere tangent to $\hat{E}_{1-}$ and restricts to a neighborhood of B in $\hat{E}_{1-}$ to have positive pairing with the 1-form dF. Thus, $-\partial/\partial y_1$ extends the preceding splitting of the normal bundle of E_{1-} in $W \times W$ to a splitting of the normal bundle of $\hat{E}_{1-}$ in $W \times W$ as $N \oplus \mathbb{R}$, where $N = \upsilon E_{1-}$ over E_{1-}.

Step 6: The fact that υE_{1-} is trivial implies that $w_2(N) = 0$. Thus, N is the trivial bundle if N's first Pontrjagin class vanishes. This class is computed as follows: Since $p_1(T(T_1 \times T_2)) = 0$, it follows that $p_1(N)$ is the same as $p_1(T(W \times W))|\hat{E}_{1-}$. Thus, N is trivial if $p_1(T(W \times W))$ is trivial as a rational class. The latter is trivial because $p_1(T(W \times W)) \approx \pi_L * p_1(TW) + \pi_R * p_1(TW))$, and both these classes vanish because W is has non-trivial boundary.

h) A fiducial homotopy class of normal framing.

The previous subsection establishes that there are homotopy classes of normal framings for E_{1-} in Z which restrict to each component of ∂E_{1-} as the class of the constant normal framing. The purpose of this subsection is to describe a subset of such classes which behave nicely when restricted to a specific set of generators for $H_1(E_{1-})$.

To make this all precise, it proves useful to first digress to describe a set of generators of $H_1(E_{1-})$. To begin the digression, take the generators $\{\eta_{1\alpha}\}$ for $H_1(T_1'')$ and $\{\eta_{2\beta}\}$ for $H_1(T_2)$ as described in (3.15). Choose a point $x_1 \in R_1$ and a point $x_2 \in R_2$. Then, generators for $H_1(E_{1-})$ are given by

$$(4.19) \qquad \begin{aligned} &1) \quad \{s_{1\alpha} \equiv (\varphi(\eta_{1\alpha}) \times \mathrm{fut}(x_2)) \cap Z\}, \\ &2) \quad \{s_{2\beta} \equiv (x_1 \times \mathrm{fut}(\eta_{2\beta})) \cap Z\}. \end{aligned}$$

Fix generators $\{s_{1\alpha}, s_{2\beta}\}$ as above. End the digression.

Ideally, a normal frame for E_{1-} should restrict to these circles as a product normal frame, $e = (e_1, e_2, e_3)$, for $R_1 \times \mathrm{fut}(R_2)$ in $W \times W$ with the following properties:

$$(4.20) \qquad \begin{aligned} &1) \quad e_1 \;\; \text{is normal to} \;\; \varphi(M_{3/8}) \;\; \text{in} \;\; W \;\; \text{and} \;\; \langle df, e_1 \rangle < 0. \\ &2) \quad e_2 \;\; \text{is normal to} \;\; R_1 \;\; \text{in} \;\; \varphi(M_{3/8}). \\ &3) \quad e_3 \;\; \text{is normal to} \;\; \mathrm{fut}(R_2) \;\; \text{in} \;\; W \;\; \text{and} \;\; \langle df, e_3 \rangle = 0. \end{aligned}$$

LEMMA 4.3. *Given generators $\{s_{1\alpha}, s_{2\beta}\}$ for $H_1(E_{1-})$ as described in (4.19), there is a normal frame for E_{1-} in Z whose restriction to each component of*

∂E_{1-} *is a constant normal frame, and whose restriction to each* $s \in \{s_{1\alpha}, s_{2\beta}\}$ *is described by (4.20).*

Remark: A normal frame for E_{1-} which is described by Lemma 4.3 will be called a fiducial normal frame.

Proof. The restriction of a given normal frame ξ of E_{1-} to $s \in \{s_{1\alpha}, s_{2\beta}\}$ can be written as $g \cdot e$, where $g : s \longrightarrow SO(3)$. If g is null-homotopic, then, and only then can ξ be homotoped to a frame whose restriction to s is equal to e. With the preceding understood, note that a map $g : S^1 \longrightarrow SO(3)$ is classified by the class in $H^1(S^1; \mathbb{Z}/2)$ of the pull-back of the generator, σ, of the module $H^1(SO(3); \mathbb{Z}/2)$. Therefore, a normal frame ξ for E_{1-} (which is homotopic to a constant frame on each component of ∂E_{1-}) defines an element $\lambda(\xi) \in (\oplus_\alpha H^1(s_{1\alpha}; \mathbb{Z}/2)) \oplus (\oplus_\beta H^1(s_{2\beta}; \mathbb{Z}/2))$ which is the obstruction to deforming ξ to a fiducial frame. By the way, note that when $h : E_{1-} \longrightarrow SO(3)$, then $\lambda(h \cdot \xi) = \lambda(\xi) + i^* h^* \sigma$, where i is the inclusion map of $(\cup_\alpha s_{1\alpha}) \cup (\cup_\beta s_{2\beta})$ into E_{1-}.

To prove the lemma, take a normal frame ξ for E_{1-} and define a map $q : (\cup_\alpha s_{1\alpha}) \cup (\cup_\beta s_{2\beta}) \longrightarrow S^1$ as follows: If $s \in \{s_{1\alpha}, s_{2\beta}\}$ and $\lambda(\xi)$ has trivial summand in $H^1(s; \mathbb{Z}/2)$, then make $q|s$ the constant map. Otherwise, make $q|s$ a diffeomorphism to S^1 (a degree one map.) Because $\{s_{1\alpha}, s_{2\beta}\}$ generate $H_1(E_{1-})$, this map q extends as a map $q : E_{1-} \longrightarrow S^1$ which is trivial near ∂E_{1-}. Let $j : S^1 \longrightarrow SO(3)$ generate $H_1(SO(3))$ and set $h \equiv j \circ q$. Then $\lambda(h \circ \xi) = 0$ because of the equalities $i^* h^* \sigma = (j \circ q)^* \sigma = \lambda(\xi)$. $\square$

i) $H^2(E_{1-})$ **and** $H_2(E_{1-})$.

Lemma 4.2 implies that

$$(4.21) \qquad H^2(E_{1-}; \mathbb{R}) \approx H^2(T_1) \oplus H^2(T_1) \oplus (H^1(T_1) \otimes H^1(T_2)).$$

Of course, $H_2(E_{1-}; \mathbb{R})$ is isomorphic to (4.21), but the proof of the results in the introductory section requires a set of generators for $H_2(E_{1-}; \mathbb{R})$.

To give such generators, it is necessary to first choose orientations for T_1 and T_2 so that their intersection number equals m. Choose a point $p_1 \in B'_{b_1-}$ with $f(p_1) < f|T_2$. Also, choose a point $p_2 \in R_2$ which is on a gradient flow line which ends on M_1.

With these choices understood, then

$$(4.22) \qquad \begin{array}{ll} 1) & T_{1-} \equiv (T_1 \times \mathrm{fut}(p_2)) \cap Z, \\ 2) & T_{2-} \equiv p_1 \times (\mathrm{fut}(T_2) \cap f^{-1}(p_1)) \end{array}$$

are embedded submanifolds of E_{1-} each of whose fundamental class is a generator of $H_2(E_{1-})$.

To obtain the remaining generators, it is necessary to first choose embedded circles, $\{\eta_{1\alpha}\} \subset R_1''$ and $\{\eta_{2\beta}\} \subset R_2$ which generate the respective first homology of T_1'' and T_2. Equation (3.15) introduces an integer $n_1 \geq 1$ and then, for each β, a set $\{\rho'_{\beta_i}\}_{i=1}^{n_1}$ of n_1 like oriented, push-off copies (in R_2) of $\eta_{2\beta}$. Let $\eta'_{2\beta} \equiv \cup_i \rho'_{\beta_i}$. Orient this submanifold of $M_{1/4}$ by taking the given orientation of

each ρ'_{β_i}. For future applications, it should be assumed, as in Section 3e, that $\{\eta_{1\alpha}\}$ is disjoint from $\mathrm{fut}(\{\eta'_{2\beta}\})$.

For each α, fix a set, $\eta'_{1\alpha} \subset R''_1$, of n_1, like oriented, push-off copies of $\eta_{1\alpha}$. Do not make a big push off: The push-off isotopy must not intersect $\mathrm{fut}(\{\eta'_{2\beta}\})$ nor should (3.17.5) fail with $\{\eta'_{1\alpha}\}$ replacing $\{\eta_{1\alpha}\}$.

The remaining generators of $H_2(E_{1-})$ can be taken to be the fundamental classes of

$$(4.23) \qquad T_{\alpha\beta-} \equiv (\varphi(\eta'_{1\alpha}) \times \mathrm{fut}(\eta'_{2\beta})) \cap Z.$$

j) Pushing off $H_2(E_{1-})$.

The second homology of E_L with real coefficients is generated by

$$(4.24) \qquad [T_{1L}] \equiv [(T_1 \times \gamma) \cap Z] \quad \text{and} \quad [T_{2L}] \equiv [(T_2 \times \gamma) \cap Z].$$

The second homology of E_R with real coefficients is generated by the corresponding $[T_{1R}] \equiv \Theta_*[T_{1L}]$ and $[T_{2R}] = \Theta_*[T_{2L}]$. The inclusion map from $E_L \cup E_R$ into Z identifies these four classes as generators of $H_2(Z)$. (Use real coefficients here and through out this subsection.)

The inclusion map of $E_{1-} \cup E_R \cup E_L$ into Z induces a homomorphism

$$(4.25) \qquad \iota : H_2(E_{1-} \cup E_L \cup E_R) \longrightarrow H_2(Z)$$

with the property that

$$(4.26) \qquad \begin{array}{ll} 1) & \iota \cdot ([T_{1-}] - [T_{1L}]) = 0, \\ 2) & \iota \cdot ([T_{2-}] - [T_{2R}]) = 0, \\ 3) & \iota \cdot [T_{\alpha,\beta-}] = 0. \end{array}$$

As discussed in Section 4h, the submanifold E_{1-} has a trivial normal bundle in Z with a fiducial homotopy class of framing which restricts to each component of ∂E_{1-} as the class of the constant normal framing. Choose a framing from such a homotopy class and use one of the frame vectors to push each of the submanifolds T_{1-}, T_{2-}, and $\{T_{\alpha,\beta-}\}$ into $Z - E_{1-}$.

If p_2 is chosen so that $\mathrm{fut}(p_2)$ is disjoint from T_1, then the submanifold T_{1-} is disjoint from $E_{1+} \cup \Delta_Z$. If p_1 is chosen to be disjoint from $\mathrm{fut}(T_2)$, then the submanifold T_{2-} is likewise disjoint from $E_{1+} \cup \Delta_Z$. As $\{\eta_{1\alpha}\}$ and $\mathrm{fut}(\{\eta'_{2\beta}\})$ are assumed to be disjoint, $\{T_{\alpha,\beta-}\}$ is disjoint from Δ_Z. And, because of (3.17.5), $\{T_{\alpha,\beta-}\}$ is disjoint from E_{1+}. Thus, T_{1-}, T_{2-} and all $\{T_{\alpha'\beta-}\}$ can be pushed off of E_{1-} into $Z - \Sigma_1$ where

$$(4.27) \qquad \Sigma_1 \equiv \Delta_Z \cup E_L \cup E_R \cup E_{1-} \cup E_{1+}$$

in an essentially canonical way.

Both E_L and E_R have a canonical homotopy class of normal bundle framing. The canonical homotopy class of normal framing for E_L is the class of the normal framing which is obtained by pulling back via the projection π_R a normal bundle

framing for the arc γ in W. Similarly, the canonical homotopy class of normal framing for E_R is obtained by pulling back via the projection π_L the same normal bundle framing for $\gamma \subset W$. Fix a framing in the canonical homotopy class for E_L's normal bundle and use one of the framing basis vectors to push T_{1L} and T_{2L} off of E_L into Σ_1. Then, push T_{1R} and T_{2R} off of E_R into Σ_1 by the analogous method.

These push-offs define a homomorphism

$$(4.28) \qquad \iota' : H_2(E_{1-} \cup E_L \cup E_R) \longrightarrow H_2(Z - \Sigma_1),$$

and the purpose of this subsection is to prove

LEMMA 4.4. *The classes* $([T_{1-}] - [T_{1L}]), ([T_{2-}] - [T_{2R}])$ *and* $\{[T_{\alpha,\beta-}]\}$ *generate the kernel of* ι'. *Thus, ker(ι') = ker(ι).*

Proof. The proof considers each of the three kinds of classes in turn.

Case 1: The class $[T_{1-}] - [T_{1L}]$. To begin, remark that there is a natural push-off, T'_{1-}, of T_{1-} into Σ_1 which is obtained by using (4.22.1) with p_2 replaced by a point $p'_2 \in M_{1/4} - R_2$ which is a push-off of p_2. This sort of push-off can be defined by a normal framing, (e_1, e_2, e_3), for E_{1-} which has the following property: Along $T_{1-} \subset (T_1 \times \mathrm{fut}(T_2)) \cap Z$, the frame is the restriction from $T_1 \times \mathrm{fut}(T_2)$ of a product frame, where

$$(4.29) \quad \begin{array}{ll} 1) & e_3 \text{ is normal to } \mathrm{fut}(T_2) \text{ in } W \text{ and } \langle df, e_3 \rangle = 0, \\ 2) & (e_1, e_2) \text{ is a normal frame for } T_1 \text{ in } W. \end{array}$$

The push-off T'_{1-} as described above is then obtained by pushing off T_{1-} along the normal vector e_3.

Now T_1 has trivial normal bundle (its self intersection number is zero), so there is a normal frame as in (4.29) for E_{1-} along T_{1-}. Furthermore,

LEMMA 4.5. *There is a fiducial normal frame from Section 4h whose restriction to* T_{1-} *is described by (4.29).*

This lemma is proved below; accept it for the time being to continue with the proof of Lemma 4.4 for $[T_{1-}] - [T_{1L}]$.

An acceptable push-off of T_{1L} is defined as follows: Take a point $p'_0 \epsilon M_{1/4}$ which is near too, but not equal to $\gamma \cap M_{1/4}$. A push-off of T_{1L} into $Z - \Sigma_1$ is $(T_1 \times \mathrm{fut}(p'_0)) \cap Z$.

Since R_2 and R_1 both are connected, and both have non-trivial boundaries, one can find a path μ in $M_{1/4}$ with one endpoint p'_2 and the other p'_0 and whose future is disjoint from T_1, T_2 and γ. With this understood, then $(T_1 \times \mathrm{fut}(\mu)) \cap Z$ is an isotopy in $Z - \Sigma_1$ between the push-offs of T_{1-} and T_{1L}.

Proof. Let ξ denote a normal frame from Section 4h. There is a map $g : T_{1-} \longrightarrow SO(3)$ such that $g \cdot (\xi|T_{1-})$ is described by (4.29). With this understood, the lemma follows if such a map g can be found which is null homotopic. Now, a map $g : T_{1-} \longrightarrow SO(3)$ is null homotopic if and only if the map lifts to a

map into S^3. The obstruction to such a lift is an element $\theta(g) \in H^1(T_{1-}; \mathbb{Z}/2)$ which is the pull-back by g of the generator of $H^1(SO(3); \mathbb{Z}/2)$. Note that $\theta(g_1 \cdot g_2) = \theta(g_1) + \theta(g_2)$. Store this information.

Consider now the homotopy classes of normal frames which have the form of (4.29). Given that e_3 is constrained to lie on a fixed side of R_2, these are in 1-1 correspondence with the homotopy classes of normal frames of T_1. The latter set is isomorphic (though not canonically) to the set of homotopy classes of maps from T_{1-} to $SO(2) \approx S^1$.

Meanwhile, a map $h : T_{1-} \longrightarrow S^1$ is distinguished up to homotopy by an invariant $\theta_1(h) \in H^1(T_{1-}; \mathbb{Z})$ which is the class of the pull-back by h of the generator of $H^1(S^1)$. Furthermore, let $j : SO(2) \longrightarrow SO(3)$ denote the usual inclusion. Then $j \circ h : T_{1-} \longrightarrow SO(3)$ and one has $\theta(j \circ h) = \theta_1(h) \bmod (2)$.

The lemma now follows from this last comment because $H^1(T_{1-}; \mathbb{Z}/2) \approx H^1(T_{1-}; \mathbb{Z}) \otimes \mathbb{Z}/2$.

Case 2: The class $[T_{2-}] - [T_{2R}]$. The argument for $[T_{2-}] - [T_{2R}]$ is essentially the same as the preceding one and will not be given.

Case 3: The classes $\{[T_{\alpha,\beta}]\}$. The first step is to consider the trivialization of the normal bundle of E_{1-} along a given $T_{\alpha,\beta}$.

LEMMA 4.6. *There are fiducial normal frames for E_{1-} (as defined in Section 4h) that restrict to $T_{\alpha,\beta}$ as the restriction of a normal frame $e = (e_1, e_2, e_3)$ which is a product normal frame for the submanifold $R_1 \times \text{fut}(R_2) \subset W \times W$ as described by (4.20).*

This lemma is proved below; accept it momentarily to continue the proof of Lemma 4.5.

Describe the push-off of $T_{\alpha,\beta-}$ into $Z - \Sigma_1$ as follows: Let $M_{3/8} \equiv f^{-1}(3/8)$. Push $\eta'_{1\alpha}$ into $M_{3/8}$ along the pseudo-gradient flow. Use $\underline{\eta}'_{1\alpha}$ to denote the resulting set of circles. Then

$$(4.30) \qquad \underline{T}_{\alpha,\beta-} \equiv \underline{\eta}'_{1\alpha} \times (\text{fut}(\eta'_{2\beta}) \cap M_{3/8})$$

is a acceptable push-off of $T_{\alpha,\beta-}$ into $Z - \Sigma_1$.

Now, remark that $\eta'_{1\alpha}$ bounds a smooth surface $S_\alpha \subset \varphi(M_{3/8})$, and this means that $\underline{\eta}'_\alpha$ bounds in $M_{3/8}$, the bounding surface, S_α, is obtained by flowing S_α along the pseudo-gradient flow lines into $M_{3/8}$. With the preceding understood,

$$(4.31) \qquad \underline{S}_\alpha \times (\text{fut}(\eta'_{2\beta}) \cap M_{3/8})$$

bounds $\underline{T}_{\alpha,\beta-}$ in Z.

Note that (4.31) is disjoint from E_{1-} and from E_{1+}. It is also disjoint from E_L, and it is disjoint from E_R if S_α is chosen to miss the point of intersection of γ with $\varphi(M_{3/8})$.

The intersection of (4.31) with Δ_Z is

$$(4.32) \qquad (S_\alpha \cap \text{fut}(\eta'_{2\beta})) \subset \Delta_Z.$$

This may be non-empty. However, by assumption, the linking matrix of Section 3e has all entries zero, which means that the intersection points in (4.32) can be paired so that the local intersection numbers (± 1) of the points in each pair cancel. The cancelling of these local signs in pairs implies that an ambient surgery in Z of the interior of (4.32) (remove ($S^0 \times B^3$) which intersect Δ_Z and replace with ($B^1 \times S^2$)'s which do not) will result in a submanifold with boundary in Z which is completely disjoint from Σ_1 and which bounds $\underline{T}_{\alpha,\beta-}$.

Proof. Upon restriction to $T_{\alpha,\beta}$, a fiducial normal frame ξ for E_{1-} (as described in Section 4h) has the form $g \cdot e$ for some $g : T_{\alpha,\beta} \longrightarrow SO(3)$. If g is null-homotopic, then ξ can be homotoped in a neighborhood of $T_{\alpha,\beta}$ to restrict to $T_{\alpha,\beta}$ as the restriction of e.

With the preceding understood, remark that a map g as above is null homotopic if and only if g lifts to a map into S^3. The obstruction to finding such a lift is $g^* \sigma \in H^1(T_{\alpha,\beta}; \mathbb{Z}/2)$, where σ generates $H^1(SO(3); \mathbb{Z}/2)$. Now, $T_{\alpha,\beta}$ is the disjoint union of push-off copies of an embedded torus, $(\varphi(\eta_{1\alpha}) \times \text{fut}(\eta_{2\beta})) \cap Z$. The first homology of this embedded torus is generated by the circles $(\varphi(\eta_{1\alpha}) \times \text{fut}(x_2)) \cap Z$ and $(x_1 \times \text{fut}(\eta_{2\beta})) \times Z$; here $x_1 \in \varphi(\eta_{1\alpha})$ while $x_2 \in \eta_{2\beta}$. This fact with Lemma 4.3 insures that $g^* \sigma$ is zero. $\square$

k) Intersections with $\underline{L}_{\pm}$.

Reintroduce the function F on $W \times W$ which assigns $f(y) - f(x)$ to a point (x, y). As remarked in Section 3 of [7], the critical points of F are the points (p, q) where p and q are critical points of f. The descending 4-ball from the point (b_2, b_1) intersects Z as an embedded 3-sphere which will be denoted by $S_{(2,1)}$. (In (3.32) of [7], this 3-sphere is denoted by $S_{(b_2,b_1)-}$; but such notation is not necessary here.) Likewise, the ascending 4-ball from (b_1, b_2) intersects Z as an embedded 3-sphere which will be denoted by $S_{(1,2)}$ (rather than $S_{(b_1,b_2)+}$ as in (3.33) of [7]).

The purpose of this subsection is to prove

LEMMA 4.7. *The intersection numbers of E_{1-} and of E_{1+} with $S_{(2,1)}$ add up to zero. The intersection numbers of E_{1-} and of E_{1+} with $S_{(1,2)}$ also add up to zero.*

Proof. To consider the case of $S_{(2,1)}$, note that the descending ball from (b_2, b_1) is $B_{b_2+} \times B_{b_1-}$. The intersection numbers of E_{1-} and of E_{1+} with $S_{(2,1)}$ (in Z) are minus the respective intersection numbers (in $W \times W$) of E_{1-} and of E_{1+} with $B_{b_2+} \times B_{b_1-}$.

Consider first the intersection number of E_{1-} with $B_{b_2+} \times B_{b_1-}$. There are no intersection points in $(\text{fut}(T_1) \times T_2) \cap Z$ because the intersection between B_{b_1-} and T_2 occurs near $f^{-1}(1/4)$, while on $\text{fut}(T_1), f \geq 3/8$.

As for $(T_1 \times \text{fut}(T_2)) \cap Z$, note that B_{b_2+} has intersection number $m_{1,2}$ with T_1; one intersection point is in each of the $|m_{1,2}|$ copies of B_{b_2-} which sit in B'_{b_1-}. Each of these intersection points can be assumed to have a different value of f, but all such values occur near $f(b_2)$. Meanwhile, B_{b_1-} has intersection

number m/m_1 with $\mathrm{fut}(T_2) \cap f^{-1}(f(b_2))$. This number is computed using the following facts:

(4.33)

1) The intersection number of T_1 with $\mathrm{fut}(T_2) \cap f^{-1}(f(b_2))$ is the same as that of T_1 with T_2.

2) T_1 intersects $\mathrm{fut}(T_2) \cap f^{-1}(f(b_2))$ only in the push-off copies of B_{b_1-} in $\varphi(B''_{b_1-})$.

Thus, $B_{b_2+} \times B_{b_1-}$ has intersection number $m \cdot m_{1,2}/m_1$ with E_{1-} (so $S_{(2,1)}$ has intersection number equal to $-m \cdot m_{1,2}/m_1$ with E_{1-}).

Now turn to the intersection number of $B_{b_2+} \times B_{b_1-}$ with E_{1+}. Here, there are no intersections in $(\mathrm{fut}(T_2) \times T_1) \cap Z$ because B_{b_2+} intersects $\mathrm{fut}(T_2)$ where $f > 7/16$, while B_{b_1-} intersects T_1 where f is approximately $3/8$.

On the other hand, B_{b_2+} has intersection number m_2 with T_2, once in each copy of B_{b_2-} that makes up B'_{b_2-}. Each such intersection takes place near b_2. Meanwhile, the intersection number between B_{b_1-} and $\mathrm{fut}(T_1) \cap^{-1}(b_2)$ is equal to $-m \cdot m_{1,2}/(m_1 \cdot m_2)$. This number is computed using the following facts:

(4.34)

1) T_1 has zero intersection number with itself.

2) A push-off copy of T_1 can be constructed which intersects T_1 as a push-off of B'_{b_1-} intersects $\mathrm{fut}(R_1) \cap f^{-1}(7/16)$.

3) T_2 has m intersections with T_1, one in each of the push-off copies of B_{b_2-} that comprise B'_{b_2-}.

Thus, $B_{b_2+} \times B_{b_1-}$ has intersection number $-m \cdot m_{1,2}/m_1$ with E_{1+} (so $S_{(2,1)}$ has intersection number $m \cdot m_{1,2}/m_1$ with E_{1+}).

The case for $S_{(1,2)}$ follows from the preceding computation because $S_{(1,2)} = \Theta(S_{(2,1)})$ while Θ interchanges E_{1-} with E_{1+}. $\square$

5 The Construction of $E_\pm$.

The previous section began the construction of Σ_Z in the case where W is a cobordism of simple type as described in Definition 2.2. (See (4.1).) This section will finish the construction of Σ_Z for such a cobordism. Indeed, (4.1) is missing only definitions of $E_\pm$; and this section will construct $E_\pm$ from $E_{1\pm}$ via ambient surgery in Z. The surgical techniques here are those from Sections 7-10 of [7].

a) Constructing E_{2-}: Push-offs and tubings.

Begin with E_{1-} of the preceding section. Using the fiducial normal framing at the end of Section 4h, make $2m$ disjoint, push-off copies of E_{1-} in addition to the original. Orient the first m copies as the original, and orient the last m copies in reverse. Use E'_{2-} to denote the resulting disjoint union.

The boundary of E_{1-} is described in (4.5) and (4.6). That is, it is a union of 3-spheres which are push-off copies of $(S^3)_{b_1+}$ or of $(S^3)_{b_2-}$ in $(S^3 \times S^3)_{b_1}$ or in $(S^3 \times S^3)_{b_2}$, respectively. As described at the end of Section 4e, there is an

arc component v of $E_{1-} \cap \Delta_Z$ that connects a positively oriented component $S_1 \subset \partial E_{1-} \cap (S^3 \times S^3)_{b_1-}$ with a positively oriented component $S_2 \subset \partial E_{1-} \cap (S^3 \times S^3)_{b_2}$.

Each of the first $m - 1$ push-off copies of E_{1-} contains a push-off copy of S_1. Let $\{S_{1\alpha}\}_{\alpha=0}^{m-1}$ denote this set of push-offs. Here, S_{10} is the original S_1 in the original copy of E_{1-}. Use $\{S_{2\alpha}\}_{\alpha=0}^{m-1}$ to denote the corresponding copies of S_2 (with S_{20} denoting the original), and let $\{v_\alpha\}_{\alpha=0}^{m-1}$ denote the corresponding copies of v.

Note that the components of $(\partial E_{2-}' \cap (S^3 \times S^3)_{b_1}) - \{S_{1\alpha}\}$ can be paired up so that each pair contains one positively oriented sphere and one negatively oriented sphere. The spheres in each pair should be tubed to each other as described in Section 7b of [7] (see (7.3) in [7]). Note: As $m \geq 1$, there is at least one pair to tube here.

Likewise, the components of $(\partial E_{2-}' \cap (S^3 \times S^3)_{b_2}) - \{S_{2\alpha}\}$ can be paired so that each pair contains one positively oriented sphere and one negatively oriented sphere. The spheres in each pair should be tubed to each other as described in the same Section 7b of [7]. There is at least one pair to tube here too.

Use E_{2-} to describe the submanifold (with boundary) of Z that results. By construction,

$$(5.1) \qquad \partial E_{2-} = \{S_{1\alpha}\}_{\alpha=0}^{m-1} \cup \{S_{2\alpha}\}_{\alpha=0}^{m-1}.$$

Note as well (see Section 7e of [7]) that after a small perturbation, the intersection of E_{2-} with Δ_Z will be transversal, and given by

$$(5.2) \qquad E_{2-} \cap \Delta_Z = \{v_\alpha\}_{\alpha=0}^{m-1} \cup C,$$

where $C \subset \text{int}(\Delta_Z)$ is a disjoint union of embedded circles.

One can argue as in Section 7f of [7] that E_{2-} has trivial normal bundle in Z with a framing which restricts to each component of ∂E_{2-} as the constant normal framing.

Meyer-Vietoris (as used in the proof of Assertion 6 of Proposition 7.4 in [7]) shows that $H^2(E_{2-}) \approx H^2(E_{2-}')$.

Define $E_{2+} \equiv \Theta(E_{2-})$ and define Σ_2 as in (4.27) with $E_{2\pm}$ instead of $E_{1\pm}$. Define the homorphism $\iota : H_2(E_{2-} \cup E_L \cup E_R) \longrightarrow H_2(Z)$ from the inclusion into Z, and define $\iota' : H_2(E_{2-} \cup E_L \cup E_R) \longrightarrow H_2(Z - \Sigma_2)$ by analogy with (4.28) using the homotopy class of normal frame for E_{2-} which is inherited (as in Section 7f of [7]) from the canonical homotopy class of normal frame for E_{1-}. Then

$$(5.3) \qquad \qquad \ker(\iota) = \ker(\iota'),$$

just as in Lemma 4.4. To prove (5.3), note first that (5.3) holds for E_{2-}' since E_{2-}' is the disjoint union of some number of push-off copies of E_{1-}. Next, remark that $E_{2-} = E_{2-}'$ except near ∂Z. Finally, note that the homologies which prove Lemma 4.4 for E_{1-} are made away from ∂Z.

As a final comment about E_{2-}, remark that the tubing can be done in such a way that E_{2-} has empty intersection with $E_{L,R}$; and it can be done so that the tubing avoids the spheres $S_{(2,1)}$ and $S_{(1,2)}$ of Lemma 4.7. In any event, the fundamental class $[E_{2-}]$ in $H_4(Z, \partial Z)$ will equal $[E_{1-}]$.

b) Constructing E_{3-}: Removing circles.

The goal here is to take some number N_1 of like oriented, push-off copies of E_{2-} and do surgery on the circles in its intersection with Δ_Z. The goal is to obtain a manifold E_{3-} with the following properties:

PROPOSITION 5.1. *There is an oriented submanifold (with boundary) $E_{3-} \subset Z$ and an integer $N \geq 1$ with the following properties:*

1) *The fundamental class $[E_{3-}]$ in $H_4(Z, \partial Z)$ is equal to $m^{-1} \cdot N \cdot [E_{1-}]$, and, in particular, obeys*

$$\partial[E_{3-}] = N \cdot [S^3]_{b_1+} + N \cdot [S^3]_{b_2-}.$$

2) *The boundary of E_{3-} is a submanifold of ∂Z, given by*

$$\partial E_{3-} = (\cup_{\alpha=1}^N S_{1\alpha}) \cup (\cup_{\alpha=1}^N S_{2\alpha}),$$

   where each $S_{1\alpha}$ is a push-off copy of $(S^3 \times \ point) \subset (S^3 \times S^3)_{b_1}$, while each $S_{2\alpha}$ is a push-off copy of $(point \times S^3) \subset (S^3 \times S^3)_{b_2}$.

3) *E_{3-} has empty intersection with $M_0 \times M_0$ and with $M_1 \times M_1$.*

4) *E_{3-} has empty intersection with E_L and with E_R.*

5) *E_{3-} has transversal intersection with Δ_Z and*

$$E_{3-} \cap \Delta_Z = \cup_{\alpha=1}^N v_\alpha,$$

   where $\{v_\alpha\}$ are all push-off copies of an arc. Furthermore, for each α, v_α has one end point on $S_{1\alpha} \cap \Delta_Z$ and the other on $S_{2\alpha} \cap \Delta_Z$.

6) *E_{3-} has trivial normal bundle in Z, and this normal bundle has a fiducial frame ζ which restricts to each $S_{1\alpha}$ and each $S_{2\alpha}$ as the constant normal frame.*

7) *E_{3-} is obtained from the disjoint union, E_{2-}', of some number N_1 of like oriented, push-off copies of E_{2-} by ambient surgery in Z on embedded circles in $E_{2-}' \cap \Delta_Z$. This surgery naturally identifies $H^2(E_{3-}; \mathbb{Q}) \approx \oplus N_1 H^2(E_{2-}'; \mathbb{Q})$.*

8) *Define $E_{3+} \equiv \Theta(E_{3-})$ and define Σ_3 as in (4.27) with $E_{3\pm}$ instead of $E_{1\pm}$. Define the homomorphism $\iota : H_2(E_{3-} \cup E_L \cup E_R) \longrightarrow H_2(Z)$ from the inclusion into Z, and define $\iota' : H_2(E_{3-} \cup E_L \cup E_R) \longrightarrow H_2(Z - \Sigma_3)$ by analogy with (4.28) using the homotopy class of normal frame for E_{3-} from Assertion 6, above. Then $\ker(\iota) = \ker(\iota')$.*

9) *The intersection numbers of E_{3-} and E_{3+} with the sphere $S_{(1,2)}$ (of Lemma 4.7) sum to zero; and the same is true for the intersection numbers of E_{3-} and E_{3+} with $S_{(2,1)}$.*

Proof. The submanifold E_{3-} is constructed by mimicking the proof of Proposition 8.1 in [**7**]. To be brief, the first step is to invoke Propositions 8.3 and 8.7 in [**7**]. Copy the arguments in Sections 8c, 8d and 8e of [**7**] to verify that the assumptions of Propositions 8.3 and 8.7 can be met with the following choices of A, B, X and O:

(5.4)

1) A is the interior of some number N_1 of like oriented, push-off copies of E_{2-}.
2) $B = \text{int}(\Delta_Z)$.
3) $X = \text{int}(Z)$.
4) O is the compliment in $\text{int}(Z)$ of the closure of a regular neighborhood of $\partial Z \cup v \cup E_L \cup E_R$.

Here, $v \subset \Delta_Z \cap E_{1-}$ is described in Section 5a, above; and it is assumed that O does not contain the $N_1 \cdot m$ push copies of v which are the arc components of $A \cap \Delta_Z$. (Note that there is a basis (i.e., $[T_1]$ and $[T_2]$) for B's second homology in which the B's intersection form is a 2×2 matrix with zero's on the diagonal. A symmetric, bilinear form with this property is even.)

In proving Assertion 7, note that the union of the circles in $E'_{2-} \cap \Delta_Z$ is homologically non-trivial because the construction of E_{2-} required at least one pair of tubings near each of $(S^3 \times S^3)_{b_{1,2}}$.

The proof for Assertion 8 of Proposition 5.1 is as follows: The assertion holds with E'_{3-} replacing E_{3-} everywhere since E'_{3-} is a union of push-off copies of E_{2-}. Meanwhile, the surgery which changes E'_{3-} to E_{3-} takes place in a regular neighborhood of Δ_Z, and the homologies which prove that $\ker(\iota) = \ker(\iota')$ can be made with support away from Δ_Z. (See the proof of Lemma 4.4.)

Assertion 9 of Proposition 5.1 follows from Assertion 1 and Lemma 4.7. $\square$

c) Constructing $E_{4\pm}$; straightening $E_{3+} \cap E_{3-}$.

The intersection of E_{3+} with E_{3-} can be something of a mess. After small perturbations of $E_{3\pm}$, this intersection has the form

$$(5.5) \qquad\qquad E_{3+} \cap E_{3-} = \Gamma \cup C,$$

where $\Gamma \subset Z$ is the union of $\Gamma_1 \equiv \cup_{\alpha=1}^N v_\alpha$ with some $N - 1$ like oriented, push-offs of Γ_1 into $Z - \Delta_Z$. These push-offs can be assumed as close to Δ_Z as desired. Meanwhile, $C \subset \text{int}(Z) - \Delta_Z$ is a disjoint union of embedded circles.

By the way, (5.5) can be established using (3.17.4).

Argue as in Section 9 in [**7**] to prove that ambient surgery on a pair, E'_{3-}, of like oriented push-offs of E_{3-}, with ambient surgery on a pair, E'_{3+}, of like oriented push-offs of E_{3+} will result in submanifolds $E_{4\pm}$ with the following properties:

PROPOSITION 5.2. *There are connected, oriented submanifolds (with boundary) $E_{4-} \subset Z$ and $E_{4+} \subset Z$ and an integer $N \geq 1$ with the following properties:*

1) *The fundamental classes $[E_{4\pm}]$ in $H_4(Z, \partial Z)$ equal $m^{-1} \cdot N \cdot [E_{1\pm}]$ and furthermore obey*

$$\partial[E_{4-}] = N \cdot [S^3]_{b_1+} + N \cdot [S^3]_{b_2-},$$

$$\partial[E_{4+}] = N \cdot [S^3]_{b_1-} + N \cdot [S^3]_{b_2+}.$$

2) Let $\Theta : Z \longrightarrow Z$ denote the switch map. Near $\partial Z \cup \Delta_Z$,

$$E_{4+} = \Theta(E_{4-}).$$

3) The boundary of E_{4-} is a submanifold of ∂Z, given by

$$\partial E_{4-} = (\cup_{\alpha=1}^N S_{1\alpha}) \cup (\cup_{\alpha=1}^N S_{2\alpha}),$$

where each $S_{1\alpha}$ is a push-off copy of $(S^3 \times \text{point}) \subset (S^3 \times S^3)_{b_1}$, while each $S_{2\alpha}$ is a push-off copy of $(\text{point} \times S^3) \subset (S^3 \times S^3)_{b_2}$.

4) $E_{4\pm}$ have empty intersection with $M_0 \times M_0$ and with $M_1 \times M_1$.

5) $E_{4\pm}$ have empty intersection with E_L and with E_R.

6) $E_{4\pm}$ have transversal intersection with Δ_Z, and

$$E_{4-} \cap \Delta_Z = E_{4+} \cap \Delta_Z = \cup_{\alpha=1}^N v_\alpha,$$

where $\{v_\alpha\}$ are all push-off copies of an arc. Furthermore, for each α, v_α has one end point on $S_{1\alpha} \cap \Delta_Z$ and the other on $S_{2\alpha} \cap \Delta_Z$.

7) E_{4-} has transversal intersections with E_{4+}. Furthermore,

$$E_{4-} \cap E_{4+} = \Gamma,$$

where Γ is the union of $\Gamma_1 \equiv \cup_{\alpha=1} N v_\alpha$ and some $N-1$ like oriented, push-off copies of Γ_1 in $Z - \Delta_Z$.

8) $E_{4\pm}$ have trivial normal bundles in Z, and these normal bundles have frames $\zeta_\pm$ with properties which include: The frames $\zeta_\pm$ restrict to each boundary component as the constant frame. Furthermore, where Assertion 2 holds, $\zeta_+ = \Theta * (\zeta_-)$.

9) $E_{4\pm}$ are obtained from the union, $E'_{3\pm}$, of one or possibly two like oriented, push-off copies of $E_{3\pm}$ by ambient surgery in $\text{int}(Z - \Delta_Z)$ on the circles in $E'_{3-} \cap E'_{3+}$. These surgeries naturally identify $H^2(E_{4\pm}; \mathbb{Q}) \approx \oplus H^2(E'_{3\pm}; \mathbb{Q})$.

10) Define Σ_4 as in (4.27) with $E_{4\pm}$ instead of $E_{1\pm}$. Define the homomorphism $\iota : H_2(E_{4-} \cup E_{4+} \cup E_L \cup E_R) \longrightarrow H_2(Z)$ from the inclusion into Z, and define $\iota' : H_2(E_{4-} \cup E_{4+} \cup E_L \cup E_R) \longrightarrow H_2(Z - \Sigma_4)$ by analogy with (4.28) using the homotopy class of normal frame for $E_{4\pm}$ from Assertion 7, above. Then $\ker(\iota) = \ker(\iota')$.

11) The intersection numbers of E_{4-} and E_{4+} with the sphere $S_{(1,2)}$ (of Lemma 4.7) sum to zero; and the same is true for the intersection numbers of E_{4-} and E_{4+} with $S_{(2,1)}$.

The fact that Z is path connected implies that $E_{4\pm}$ can be constructed to be path connected. See Lemma 8.10 in [7] and its proof.

Remark that the last assertion of Proposition 5.2 follows from Assertion 1 and Lemma 4.7. The argument for Assertion 9 proceeds as follows: Since $E'_{4\pm}$ are

disjoint unions of push-off copies of $E_{3\pm}$, Assertion 9 holds if $E'_{4\pm}$ everywhere replace $E_{4\pm}$. Now, $E_{4\pm}$ is constructed by surgery on $E'_{4\pm}$; and these surgeries can be performed away from the generators of H_2. Furthermore, the surgeries take place in a regular neighborhood of a surface with boundary or 3-ball in Z, and so can be performed away from the homologies which establish Assertion 9 for $E'_{4\pm}$. (See the proof of Lemma 4.4.)

d) The meld construction and $E_\pm$.

This section constructs $E_\pm$ from $E_{4\pm}$ using the meld operation of Section 10 in [7]. In this regard, note that the behavior of E_{4-} near $\partial Z \cup \Delta_Z$ is described by (10.2-5) in [7] modulo notation.

To be precise, there is a regular neighborhood $U \subset Z$ of $\partial Z \cap \Delta_Z$ such that $E_{4-} \cap U$ is a set $\{Y_\alpha\}_{\alpha=1}^N$ (with N from Proposition 5.2), where $\{Y_{\alpha \geq 2}\}$ are disjoint, like oriented push-off copies of Y_1. Meanwhile, Y_1 is the image of a proper embedding into U of the space in (10.3) of [7]; this being the compliment in the open unit 4-ball of the open balls $B_\pm$ of radius $1/8$ and centers $(\pm 1/4, 0, 0, 0)$. Note here that the boundary of B_+ is mapped diffeomorphically onto $(S^3)_{b_1+} \subset (S^3 \times S^3)_{b_1}$, and the boundary of B_- is likewise mapped onto $(S^3)_{b_2-} \subset (S^3 \times S^3)_{b_2}$. Meanwhile, the arc along the x-axis between $(\pm 1/8, 0, 0, 0)$ is mapped to the arc $v \subset E_{4-} \cap \Delta_Z$.

The $\{Y_{\alpha \geq 2}\}$ are described by (10.4) and (10.5) in [7].

The melded space, E_-, is then described by (10.8) in [7]. (See also (10.9) and (10.10) in [7].)

As for E_+, the neighborhood U can be chosen to be invariant under the switch map (4.11) and such that $E_{4+} \cap U = \Theta(E_{4-} \cap U)$. With this understood, define $E_+ \cap (Z - U) \equiv E_{4+} \cap (Z - U)$, and define $E_+ \cap U \equiv \Theta(E_- \cap U)$.

Note that

(5.6)
$$
\begin{aligned}
&1) \quad [E_\pm] = m^{-1} \cdot N \cdot [E_{1\pm}] \quad \text{in} \quad H_4(Z, \partial Z; \mathbb{Z}). \\
&2) \quad \partial[E_-] = N \cdot [S^3]_{b_1+} + N \cdot [S^3]_{b_2-} \\
&\qquad \text{and} \quad \partial[E_+] = N \cdot [S^3]_{b_1-} + [S^3]_{b_2+}. \\
&3) \quad H^2(E_\pm; \mathbb{Q}) \approx H^2(E_{4\pm}; \mathbb{Q}).
\end{aligned}
$$

6 Completing the proof   The purpose of this last section is to complete the proof of Theorem 1.3 along the lines that were outlined in Section 1c. Thus, suppose that M_0 and M_1 are compact, oriented 3-manifolds with the rational homology of S^3. Assume that M_0 and M_1 are spin cobordant by a cobordism whose intersection form is equivalent to a sum of metabolics (see (1.3)). As descibed in (1.4), one can find such a cobordism which factors as $W_1 \cup W_2 \cup W_3$, where W_1 and W_3 have the rational homology of S^3, and where W_2 has a good Morse function with only index 2 critical points. As in Proposition 2.1, the cobordism W_2 can be factored as $\cup_{j=1}^n W_{2,j}$, where each $W_{2,j}$ is a cobordism of simple type (Definition 2.2) between a pair, F_{j-1} and F_j, of rational homology spheres. Here, $F_0 \equiv M'_0$ and $F_n \equiv M'_1$.

Define $Z_2 \equiv \cup_j Z_{2,j}$, where each $Z_{2,j} \subset W_{2,j} \times W_{2,j}$ is defined as in Section 2c. The identification of $F_j \times F_j \subset Z_{2,j}$ with $F_j \times F_j \subset Z_{2,j+1}$ is left implicit here. Use this Z_2 in (1.5)

Fix base points in each F_j. Then define $\{\Sigma_{Z_N}\}$ as in (4.1). With this understood, set $\Sigma_{Z_2} \equiv \cup_j \Sigma_{Z_N}$ after making the implicit boundary identifications. Use this Σ_{Z_2} in (1.6).

Step 3 of the outline in Section 1c constructs a closed 2-form ω_Z on the compliment of Σ_Z which obeys $\omega_Z \wedge \omega_Z \wedge \omega_Z = 0$ near Σ_Z. The construction of ω_Z proceeds by first constructing a closed 2-form $\underline{\omega}_Z$ on the compliment of Σ_Z in a regular neighborhood N_Z of Σ_Z in Z. The form $\underline{\omega}_Z$ will be built so that it satisfies Condition 1 of Lemma 4.2 in [7]. Also, $\underline{\omega}_Z \wedge \underline{\omega}_Z = 0$.

The form will then be extended over the compliment of Σ_Z of a neighborhood of $\partial Z \cup \Sigma_Z$ so that its pull-back to any boundary component $M \times M - \Sigma_M$ is a form which computes $I_2(M)$. Here $M \times M$ is any of $M_0 \times M_0$, $M_1 \times M_1$ or any $(S^3 \times S^3)_{b_{1j}}, (S^3 \times S^3)_{b_{2j}}$.

The next question is whether the form $\underline{\omega}_Z$ so constructed can be extended over $Z - \Sigma_Z$. The author does not know when such is the case. However, it is shown below that there is a closed 2-form μ on N_Z which obeys $\mu \wedge \mu = 0$, which vanishes near ∂Z, and is such that $\omega_Z \equiv \underline{\omega}_Z - \mu$ extends over $Z - \Sigma_Z$ as a closed form. Note that such a form will satisfy the third condition in (1.7).

The form μ will vanish near Σ_{Z_1} and near Σ_{Z_3}. Furthermore, μ will be written as $\mu = \Sigma_{j=1} n \mu_{2,j}$, where $\mu_{2,j}$ has compact support in the interior of $Z_{2,j}$. With this understood, the construction of $\mu_{2,j}$ can be made independently for each factor $Z_{2,j}$ which comprises Z_2.

a) $\underline{\omega}_Z$ near Σ_Z and ∂Z.

The construction of the closed 2-form $\underline{\omega}_Z$ on the compliment of Σ_Z in a regular neighborhood $N_Z \subset Z$ of $\Sigma_Z \cup \partial Z$ proceeds by mimicking the constructions in Sections 11a - 11i of [7] which construct ω_Z near Σ_Z when the cobordism between M_0 and M_1 has the rational homology of S^3. The conditions in Theorem 1.3 that W be spin and that the canonical frame be represented by c in the kernel of the homomorphism ι_W arise here.

The verification of Condition 1 of Lemma 4.2 for ω_Z proceeds as in Section 11k of [7], and the reader is referred there. (But note Assertion 10 of Proposition 5.2.)

b) The obstruction from cobordisms of simple type.

At this point, the proof of Theorem 1.3 must diverge from the proof of Theorem 2.9 in [7] because the restriction homomorphism $H^2(Z) \longrightarrow H^2(\Sigma_Z)$ will not generally be surjective. (Use real coefficients here and throughout this section.) Thus, the second part of Lemma 4.2 in [7] can not be invoked. This failure of surjectivity obstructs the extension of $\underline{\omega}_Z$ to $Z - \Sigma_Z$.

This extension obstruction will be studied by using the fact that restriction to the $Z_{2,j}$ defines isomorphisms $H^2(Z) \approx \oplus_j H^2(Z_{2,j})$ and $H^2(\Sigma_Z) \approx \oplus_j H^2(\Sigma_{Z_{2,j}})$. (Meyer-Vietoris proves these assertions.) These direct sum decompositions imply that the obstruction to extending ω_Z over $Z - \Sigma_Z$ can be understood by restricting attention to $Z_2 - \Sigma_{Z_2}$ and even further, by restricting attention to $Z_{2,j}$. More precisely, the obstructions to extending $\underline{\omega}_Z$ can be

understood by restricting attention to the very special case of a cobordism of simple type (as in Definition 2.2).

With the preceding understood, agree, for the remainder of Section 11, to restrict attention to a particular cobordism of simple type. Simplify notation by using W now to denote this simple type cobordism. Then, $Z \subset W \times W$ and $\Sigma_Z \subset Z$ are defined accordingly.

With Z as just redefined, note that the extension obstruction may well exist because $\mathrm{rank}(H^2(\Sigma_Z)) \geq 10$ while $\mathrm{rank}(H^2(Z)) = 4$. Indeed, Lemma 2.3 describes $H^2(Z)(\approx \mathbb{R}^4)$, while Meyer-Vietoris with Proposition 5.2 find

$$(6.1)\; H^2(\Sigma_Z) \approx H \equiv H^2(\Delta_Z) \oplus H^2(E_L) \oplus H^2(E_R) \oplus H^2(E_-) \oplus H^2(E_+).$$

In fact, the restriction map from Z to Σ_Z maps $H^2(Z)$ injectively into $H^2(E_L) \oplus H^2(E_R)$.

c) Analyzing the obstruction.

Let W be a cobordism of simple type and let $Z \subset W \times W$ and let $\Sigma_Z \subset Z$ be defined accordingly. Let $N_Z \subset Z$ be a regular neighborhood of Σ_Z. Introduce

$$(6.2)\qquad \begin{aligned} i_1^* &: H^2(Z - \Sigma_Z) \longrightarrow H^2(N_Z - \Sigma_Z) \quad \text{and} \\ i_2^* &: H^2(N_Z) \longrightarrow H^2(N_Z - \Sigma_Z) \end{aligned}$$

to denote the pull-back homomorphisms. One can conclude from the Meyer-Vietoris exact sequence that

$$(6.3)\qquad \underline{\omega}_Z = i_1^*\alpha + i_2^*\beta,$$

and the purpose of the subsequent arguments is to prove

PROPOSITION 6.1. *Equation (6.3) can be solved with a closed 2-form β on N_Z which obeys $\beta \wedge \beta = 0$ and which vanishes near ∂Z.*

Remark that the lemma implies that $\omega_Z \equiv \underline{\omega}_Z - i_2^*\beta$ extends over $Z - \Sigma_Z$ (as α in (6.3)) and it obeys $\omega_Z \wedge \omega_Z \wedge \omega_Z = 0$ near Σ_Z as required.

d) Strategy for the proof of Proposition 6.1.

The proof of Proposition 6.1 starts with the remark that the various framings that were introduced in the construction of ω_Z can be used to construct a homomorphism

$$(6.4)\qquad j : H_2(N_Z) \longrightarrow H_2(N_Z - \Sigma_Z)$$

with the property that the composition of j with i_2 (the dual of i_2^* in (6.2)) gives the identity. Indeed, each of $\Delta_Z, E_{L,R}$ and $E4\pm$ have natural trivializations of their normal bundles. And, these trivializations can be used to push-off the generating cycles for the homology groups in question. (For $E_{L,R}$, see the proof of Lemma 4.4, and see Assertion 9 of Proposition 5.2 for $E_{4\pm}$.) In this regard, note that an application of Meyer-Vietoris shows that the dimension 2 homology

of Σ_Z can be represented by submanifolds in $\Delta_Z, E_{L,R}$ and in the smooth parts of $E_\pm$; and these submanifolds can be assumed to be disjoint from $E_\pm \cap \Delta_Z$ and from $E_{L,R} \cap \Delta_Z$.

The homomorphism j has the property that

(6.5) $$\langle \omega_Z, j(\cdot) \rangle = 0.$$

(This is because j is defined by the same homotopy class of normal framing which is used to define ω_Z.)

Put (6.5) aside for the moment to consider the composition

(6.6) $$H_2(\Sigma_Z) \xrightarrow{j} H_2(N_Z - \Sigma_Z) \xrightarrow{i_1} H_2(Z - \Sigma_Z),$$

which will be denoted by ι'. (The arrow i_1 in (6.6) is induced by the inclusion.) Define $Q \subset H^2(\Sigma_Z)$ by the exact sequence

(6.7) $$0 \longrightarrow Q \longrightarrow H^2(\Sigma_Z) \longrightarrow \ker(\iota')^* \longrightarrow 0.$$

Note that the restriction induced monomorphism $H^2(Z) \longrightarrow H^2(\Sigma_Z)$ factors through Q.

If the quotient $Q/H^2(Z)$ is zero, then it follows from (6.7) that (6.3) can be solved with $\beta \equiv 0$. If the quotient $Q/H^2(Z)$ is one dimensional, and if a generator can be represented by a form β with $\beta \wedge \beta = 0$, then Proposition 6.1 follows.

Thus, the proof of Proposition 6.1 will proceed with a proof that the dimension of $Q/H^2(Z)$ is one or less. The proof will end by finding a generator (when $\dim(Q/H^2(Z)) = 1$) which is represented in Q by a form with square zero (see (6.10), below).

By the way, the following lemma will be the principle tool for finding closed forms with square zero:

LEMMA 6.2. *Let X be an oriented 4-manifold, and let $R \subset X$ be a compact, oriented, embedded surface. Suppose that R has zero self-intersection number. Given an open neighborhood $O \subset X$ of R, there is a closed 2-form μ with $\mu \wedge \mu = 0$ which is supported in O and which represents the Poincaré dual to R in $H^2_{comp}(X)$.*

Proof. The surface R has trivial normal bundle. Use this fact as in (6.12) of [7] to define a fibration from a neighborhood of R in X to the unit disk in $\mathbb{R}^2$ which sends R to the origin. Use such a map to pull-back from said unit disk a 2-form with compact support in the interior and with total mass equal to one. Set μ equal to this pull-back. □

e) The dimension of $Q/H^2(Z)$.

Here is the answer to the dimension question:

LEMMA 6.3. *The dimension of $Q/H^2(Z))$ is one or zero.*

Proof. The inclusion of Σ_Z into Z induces the homomorphism $\iota : H_2(\Sigma_Z) \longrightarrow H_2(Z)$. Then, the dimension of $Q/H^2(Z)$ is equal to the dimension of $\ker(\iota)/\ker(\iota')$.

To prove that $\ker(\iota)/\ker(\iota')$ has dimension 1 or less, consider an integral class $\sigma \in H_2(\Sigma_Z)$ with $\iota \cdot \sigma = 0$, but with $\iota' \cdot \sigma \neq 0$. Since $\iota \cdot \sigma = 0$, there is a bounding 3-cycle $\tau \subset Z$. The cycle τ is a sum of singular simplices; and these simplices can be chosen to have the following property: Each is a smooth map from the standard 3-simplex into Z which is transversal to each of $\Delta_Z, E_L, E_R, E_{4\pm}$ on the interior of every codimension $p = 0, \cdots, 3$ face of the standard simplex. (Thus, the boundary of the standard simplex is mapped into the compliment of Σ_Z.) With this understood, it makes sense to speak of the intersection number of τ with each of $\Delta_Z, E_L, E_R, E_{4\pm}$. Note that the intersection number between τ and $E_{4\pm}$ can be assumed to be divisible by the integer N of Proposition 5.2. This can be achieved by replacing σ with $N \cdot \sigma$.

Observe now that intersections of τ with any of $E_L, E_R, E_{4\pm}$ can be removed by changing τ to τ', where τ' has extra intersections with Δ_Z. For example, one can add to τ some multiple of $[p_0 \times M_0]$ to remove the intersection points with E_L at the expense of adding such points to Δ_Z. Likewise, adding to τ multiples of $[S^3]_{b_1-}$ will remove intersections with E_{4-} and add intersections with Δ_Z. Note that all of τ's intersections with E_{4-} can be transferred to Δ_Z because E_{4-} is connected, and because τ's intersection number with Δ_Z is assumed divisible by the integer N from Proposition 5.2. The cases for E_R and $E_{4\pm}$ are analogous. (See, e.g., (9.9a,b) in [7].)

It follows from the preceding that $\ker(\iota)/\ker(\iota')$ is at most one-dimensional. This is because any element in this quotient can be represented by a closed 2-cycle which bounds τ as above, whose intersections with Σ_Z lie in Δ_Z only. Given two such elements, a non-trivial linear combination would be represented by a closed 2-cycle which bounds τ as above with absolutely no intersections with Σ_Z. Such a linear combination would be zero in $\ker(\iota)/\ker(\iota')$. $\square$

f) $\ker(\iota)$.

The final step in the proof of Proposition 6.1 is to consider the possible generators of $Q/H^2(Z)$ in the case where this group has dimension 1.

A generator of this group is represented by a class $\ell \in H^2(\Sigma_Z)$ which annihilates the kernel of ι', but which is non-zero on a class $\sigma \in H_2(\Sigma_Z)$ which is annihilated by ι but not by ι'. The analysis of ℓ proceeds by considering various possibilities for $\ker(\iota)/\ker(\iota')$. Remark that if this group has dimension 1, then it can be represented in $\ker(\iota)$ by some generator.

In $H_2(\Delta_Z) \oplus H_2(E_L) \oplus H_2(E_R)$ sits a two-dimensional subspace of $\ker(\iota)$. An element in $\ker(\iota) \cap (H_2(\Delta_Z) \oplus H_2(E_L) \oplus H_2(E_R))$ has the form

$$(6.8) \qquad\qquad \sigma = \sigma_\Delta - \sigma_L - \sigma_R,$$

where σ. sits in the summand with the corresponding label. Here, each σ. pushes forward to W as the same class $\sigma_0 \in H_2(W)$. Then, two generators of the kernel of ι in $H_2(\Delta_Z) \oplus H_2(E_L) \oplus H_2(E_R)$ are given by σ as above with $\sigma_0 = [T_1]$ and with $\sigma_0 = [T_2]$.

The remaining generators of the kernel of ι can be taken to have the form

$$(6.9) \quad \begin{array}{ll} 1) & \sigma_{+1} - \sigma_{R1} \quad \text{and} \quad \sigma_{+2} - \sigma_{L2}, \\ 2) & \sigma_{-1} - \sigma_{L1} \quad \text{and} \quad \sigma_{-2} - \sigma_{R2}, \\ 3) & \{\lambda_{\pm c}\}. \end{array}$$

Here, $\sigma_{\pm 1,2} \in H_2(E_\pm)$, while $\sigma_{L1,2} \in H_2(E_L)$ and $\sigma_{R1,2} \in H_2(E_R)$ project to $H_2(W)$ as multiples of $[T_{1,2}]$, respectively. Meanwhile, $\{\lambda_{\pm c}\} \in H_2(E_\pm)$ is a finite set of classes, and each is represented by a push-off of some $T_{\alpha,\beta\pm}$ as described in (4.23).

LEMMA 6.4. *The classes in (6.9) are annihilated by ι'.*

Proof. This follows from Assertion 9 of Proposition 5.2.

With the preceding lemma understood, it follows that a generator of $\ker(\iota)/\ker(\iota')$ is described by (6.8).

g) If $\sigma = \sigma_\Delta - \sigma_L - \sigma_R$ is not annihilated by ι'.

In this case, there exists σ as above with either $\sigma_0 = [T_1]$ or $\sigma_0 = [T_2]$. For arguments sake, assume $\sigma_0 = [T_1]$. Let $\beta_{\Delta_1} \in H^2(E_\Delta)$ be the pull-back by the map π_L to W of the Poincaré dual to $[T_2]$. Then β_{Δ_1} pairs non-trivially with σ_L and so with σ. Let $\beta_{R2} \in H^2(E_R)$ be the pull-back by π_R of the Poincaré dual to $[T_1]$. Note that β_{R2} pairs trivially with σ.

It follows that there is $c \in \mathbb{R}$ such that $\beta' \equiv \beta_{\Delta_1} + c \cdot \beta_{R2}$ annihilates the $\ker(\iota)$ in $H_2(\Delta_Z) \oplus H_2(E_L) \oplus H_2(E_R)$. This β' will have trivial pairing with the classes in (6.9.1), and it will have trivial pairing with $\sigma_{-1} - \sigma_{L1}$ in (6.9.2), but unless $c = 0$, it will pair non-trivially with $\sigma_{-2} - \sigma_{R2}$ in (6.9.2). However, note that the Poincaré dual, $\beta_{-2} \in H_2(E_-)$, to σ_{-1} pairs trivially with σ_{-1} and non-trivially with σ_{-2}. And so, there is a real number c' such that

$$(6.10) \qquad \beta \equiv \beta_{\Delta_1} + c \cdot \beta_{R2} + c' \cdot \beta_{-2}$$

annihilates all of the classes in (6.9.2). Note that β will also annihilate the classes in (6.9.3).

By appeal to Lemma 6.2, each of $\beta_{\Delta_1}, \beta_{R2}$, and β_{-2} can be represented by a closed form with square zero and with support away from ∂Z. (This is because $[T_1]$ and $[T_2]$ are classes with square zero in W.) Furthermore, Lemma 6.2 insures that these forms can be constructed to have disjoint supports. Thus, β will vanish near ∂Z and have square zero as required.

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