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Homology cobordism and the simplest perturbative Chern-Simons 3-manifold invariant

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1 Introduction. Witten predicted [**18**] that certain products of a certain 2-form could be integrated over products of a compact, oriented 3-manifold to give differential invariants of the 3-manifold. These predicted invariants were first constructed by Axelrod and Singer [**2, 3**] in the case where the 3-manifold has the rational homology of S^3. (A similar prediction in [**18**] for computing Jones' knot invariants had been partially realized by Bar Natan [**4**].) Subsequently, Kontsevich [**9**] gave an alternative realization of Witten's proposed invariants, with the same constraint on the homology of the 3-manifold. (Presumably, the invariants of Axelrod/Singer and of Kontsevich are the same, but the author has not seen a proof that such is the case.) Note that the invariants of Axelrod/Singer and Kontsevich have only been calculated for the 3-sphere (where they vanish).

The Axelrod/Singer and Kontsevich invariants are formally related to the 3-manifold invariants of Reshitikin and Turaev [**14**]. (The relationship here is presumed analogous to that between Jones, HOMFLY and other knot invariants and the knot invariants of Vassiliev [**16**], [**17**]; see [**6**], [**5**], [**10**].) There is no theorem at present which describes the precise relationship between these various 3-manifold invariants. Such a theorem would be useful in light of the fact that the invariants of Reshitikin and Turaev can be explicitly computed; they have been computed in closed form for lens spaces [**8**] and Seifert fibered 3-manifolds [**13**].

This is the first of two articles focusing solely on the simplest of the invariants of Kontsevich, an invariant, I_2, which assigns a number to a 3-manifold M (as constrained above) by integrating the cube of a certain real valued 2-form over $M \times M$. Of particular concern here is the value of I_2 on the 3-manifold boundaries of a 4-dimensional spin cobordism which has the rational homology of S^3. The results in this article, together with those in the sequel [**15**], prove that $I_2(M) = I_2(M')$ when M and M' are the boundary components of an oriented, spin 4-manifold W for which:

1. The intersection form on W's second homology (mod torsion) is conjugate to a direct sum of metabolic pairs.
2. The inclusions of M and M' into W induce injections of $H_1(\cdot;\mathbb{Z}/2)$.

(1.1)

(A metabolic pair is a symmetric, 2×2 matrix with zero's on the diagonal.)

In particular, the preceding result implies that $I_2(M) = 0$ when M has the integral homology of S^3. These results are restated and proved in [**15**].

This article makes a large step on the way to (1.1); the main theorem here, Theorem 2.9, states (in part) that $I_2(M) = I_2(M')$ when M and M' are the boundaries of an oriented, spin 4-manifold W for which the inclusions of M and M' into W induce

1) Isomorphisms on $H_p(\cdot;\mathbb{Q})$ for $p = 0, ..., 4$.
2) Injective maps on $H_1(\cdot;\mathbb{Z}/2)$.

(1.2)

In the course of proving Theorem 2.9, $I_2(S^3)$ is shown to vanish. Thus, even without the sequel [**15**], the main theorem here can be used, in principle, to show that I_2 vanishes for certain 3-manifolds. (It is possible that $I_2 \equiv 0$ for all M!)

The author hopes that the constructions in this article will prove useful in studying the full set of invariants of Axelrod/Singer and Kontsevich, and this accounts, in part, for the length of the presentation. (The constructions here play a crucial role in [**15**].)

Before beginning the story, the author wishes to thank Robion Kirby and Paul Melvin for their comments concerning this work, and also for their encouragement and support. A debt is owed as well to Dror Bar-Natan for sharing his knowledge of knot invariants.

This article is organized as follows: The definition of I_2 and the main theorem (Theorem 2.9) are given in the next section. The remaining sections (3-11) are occupied with constructions that are needed for the main theorem's proof. Section 3 is a digression to present certain facts from Morse theory. Section 4 studies the homological constraints which arise in the proof. Sections 5-10 contain the construction of a solution to the homological constraints. The final aspects of the proof of the main theorem are provided in Section 11.

2 The definition and properties of $I_2(M)$. The purpose of this section is to give a definition of Kontsevich's invariant, $I_2(\cdot)$, for compact, oriented 3-manifolds that have the rational homology of S^3. This section also contains the paper's main theorem about the equality of I_2 for a pair of 3- manifolds which occur as the boundary components of a certain kind of 4-dimensional cobordism.

a) Topological considerations.

Let M be a compact, oriented 3-manifold with the rational homology of S^3. Fix a point $p_0 \in M$. Let $\Delta \subset M \times M$ denote the diagonal. Define the subspace

$$\Sigma \equiv \Delta \cup (p_0 \times M) \cup (M \times p_0). \tag{2.1}$$

Lemma 2.1 describes the cohomology of $M \times M - \Sigma$. Before reading Lemma 2.1, be forewarned that a regular neighborhood of Σ in $M \times M$ is a neighborhood of Σ which strongly deformation retracts (rel Σ) onto Σ. It is an exercise to show that such neighborhoods exist. Also, in Lemma 2.1, cohomology is computed with real ($\mathbb{R}$) coefficients.

LEMMA 2.1. *Let Σ be as defined in (2.1). Then*

1) $H^2((M \times M) - \Sigma) \approx \mathbb{R}$.
2) *Let $N \subset M \times M$ be a regular neighborhood of Σ. Then, restriction gives an isomorphism $H^2((M \times M) - \Sigma) \approx H^2(N - \Sigma)$.*
3) *Let $i : \mathbb{R}^3 \to N$ be an embedding which intersects $\Sigma - (p_0, p_0)$ transversely in a single point, $i(0)$. Then $i^* : H^2(N - \Sigma) \approx H^2(\mathbb{R}^3 - 0)$ is an isomorphism.*
4) $H^1((M \times M) - \Sigma) \approx H^1(N - \Sigma) \approx 0$.

Proof. For the first assertion, use Meyer-Vietoris to prove that $(M \times M) - ((p_0 \times M) \cup (M \times p_0))$ has the rational homology of $\mathbb{R}^6$. Then, use Meyer-Vietoris again to compute the cohomology of the remainder when Δ is deleted. In fact, this calculation with Meyer-Vietoris shows that $M \times M - \Sigma$ has the rational cohomology of $S^3 \times S^2$.

Prove the second assertion using the Meyer-Vietoris exact sequence for the cover of $M \times M$ by N and $M \times M \cup \Sigma$. (The Kunneth formula gives $H^2(M \times M) = 0$, while restriction injects $H^3(M \times M)$ into $H^3(\Sigma)$.)

The third and fourth assertions are left as exercises with Meyer-Vietoris. □

The cohomology of $(M \times M) - \Sigma$ with rational coefficients is isomorphic to its DeRham cohomology.

b) An invariant.

Let C denote the set of pairs (N, φ) where N is a regular neighborhood of Σ, and where $\varphi : N \to \mathbb{R}^3$ is a smooth map with the property that $\varphi^{-1}(0) = \Sigma$. Define an equivalence relation on C as follows: Say that (N_0, φ_0) and (N_1, φ_1) are equivalent if there is a regular neighborhood $N_2 \subset N_0 \cap N_1$ and a smooth map

$$\Phi : [0,1] \times N_2 \to \mathbb{R}^3 \tag{2.2}$$

which obeys $\Phi(0, \cdot) = \varphi(0)$ and $\Phi(1, \cdot) = \varphi_1$ and $\Phi^{-1}(0) = [0,1] \times \Sigma$.

Let c denote the set of equivalence classes in C.

Now, change gears somewhat and pick a smooth, closed 2- form, μ, on $\mathbb{R}^3 - 0$ whose integral over the standard unit 2-sphere is equal to 1. For example,

$$\mu = (4\pi)^{-1}|x|^{-3}(x_1\, dx_2\, dx_3 + x_2\, dx_3\, dx_1 + x_3\, dx_1 dx_2) \tag{2.3}$$

Let $\varphi \in C$. According to Lemma 2.1, there exists a smooth, closed 2-form on $M \times M - \Sigma$ which agrees with $\varphi^*\mu$ on N. Fix such a form and call it ω_φ.

PROPOSITION 2.2. *Let* $(\mathrm{N}, \varphi) \in C$ *and choose* ω_φ *as described above. Then the following integral converges:*

$$I_2 \equiv \int_{M \times M - \Sigma} \omega_\varphi \wedge \omega_\varphi \wedge \omega_\varphi \tag{2.4}$$

Furthermore, I_2 *is independent of the choice of* ω_φ *to extend* $\varphi^* \mu$, *and it is independent of the choice of* μ. *Infact,* I_2 *depends only on the equivalence class of* (N, φ) *in* c .

Proof. The integral converges because the integrand has compact support on $M \times M - N$. Indeed, $\omega_\varphi \wedge \omega_\varphi$ vanishes on N because ω_φ on N is the pull back of a form on S^2.

Now, suppose that (N_0, φ_0) and (N_1, φ_1) define the same equivalence class in c . Suppose that μ_0 and μ_1 are different choices for μ in Proposition 1.2. Suppose that ω_0 and ω_1 are closed 2-forms on $M \times M - \Sigma$ which extend $\varphi_0^* \mu_0$ and $\varphi_1^* \mu_1$ from N_0 and N, respectively.

Let $N_2 \subset N_0 \cap N_1$ and $\Phi : [0,1] \times N_2 \to \mathbb{R}^3 - 0$ give the equivalence between (N_0, φ_0) and (N_1, φ_1). There are no obstructions to finding a 2-form μ on $[0,1] \times (\mathbb{R}^3 - 0)$ which is closed and restricts to $0 \times (\mathbb{R}^3 - 0)$ as μ_0 and to $1 \times (\mathbb{R}^3 - 0)$ as μ_1. Meanwhile, Lemma 2.1 insures that there are no obstructions to extending $\Phi^* \mu$ to $[0,1] \times (M \times M - \Sigma)$ as a closed 2-form ω.

With ω defined, compute

$$0 \equiv \int_{[0,1] \times (M \times M - \Sigma)} d(\omega \wedge \omega \wedge \omega) \tag{2.5}$$

using Stokes' theorem to express 0 (i.e. Equation (2.5)) as a sum of three terms. (Note that the integrand in (2.5) is compactly supported away from $[0,1] \times \Sigma$ since ω is pulled back from a 4- dimensional manifold on $[0,1] \times N_2$.) The three terms alluded to above are as follows: The first term is the contribution to Stokes' theorem from $\{1\} \times (M \times M - \Sigma)$; it is the integral in (2.4) as computed using the data with subscript "$_1$".

The second term is the contribution to Stokes theorem from $\{0\} \times (M \times M - \Sigma)$; it is the integral in (2.4) as computed using the data with subscript "$_0$".

To write down the third term which contributes to the Stokes' theorem computation of (2.5), one must first fix $N \subset N_2$, a smooth, oriented, codimension 1 submanifold that separates $\Sigma \subset M \times M$ from $M \times M - N_2$. With N understood, here is the third contribution to (2.5):

$$\delta I_2 \equiv \int_{[0,1] \times N} \omega \wedge \omega \wedge \omega \tag{2.6}$$

Note that (2.6) is zero because ω on $[0,1] \times N$ equals $\Phi^* \mu$, the pull-back of a 2-form on a 4-dimensional manifold.

Thus, the number I_2 is the same, whether computed using the data with subscript "$_0$" or with subscript "$_1$". □

c) Singular framings.

The previous subsection introduced the set c of equivalence classes of pairs (N, φ), where N is a regular neighborhood of Σ, and where $\varphi : N \to \mathbb{R}^3$ has $\Sigma = \varphi^{-1}(0)$. The purpose of this subsection is to describe a fiducial set of such classes. However, a preliminary, digression is required to define the notion of a singular framing of T^*M. The digression has four parts.

Part 1 of the digression introduces the standard framing of $T^*\mathbb{R}^3$, $dx = (dx_1, dx_2, dx_3)$. Part 1 also introduces the framing $\underline{\delta}$ of $T^*(\mathbb{R}^3 - 0)$ which is given at x $\in \mathbb{R}^3$ by

$$\underline{\delta} \equiv -dx + 2 \mid x \mid^{-2} < x,\, dx > x, \tag{2.7}$$

where $< x, dx > \equiv \Sigma_{i=1}^3 x_i dx_i$.

Part 2 of the digression makes the remark that a framing (such as $\underline{\delta}$) can be changed to a different framing using a matrix in $GL(3, \mathbb{R})$. Indeed, if $g \equiv (g_{ij})$ is a such a matrix, and if $\zeta \equiv (\zeta_1, \zeta_2, \zeta_3)$ is a framing, then $g\,\zeta$ is the framing given by $(g\,\zeta)_i = \Sigma_{j=1}^3\, g_{ij}\, \zeta_j$.

Part 3 of the digression defines the notion of a singular framing:

DEFINITION 2.3. *A singular frame, ζ, for T^*M is an oriented trivialization of $T^*(M - p_0)$ which has the following property: Let $\varphi : \mathbb{R}^3 \to M$ be an orientation preserving embedding (coordinate system) with $\varphi(0) = p_0$. There should exist an element $g \in GL(3, \mathbb{R})$, with positive determinant, and such that*

$$\lim_{r \to 0} \sup_{|x| = r} \mid \varphi^* \zeta - g\, \underline{\delta} \mid (x) = 0. \tag{2.8}$$

(Note: Let ζ be a frame for $T^*(M - p_0)$. Suppose that φ and φ' are two coordinates systems as in Definition 2.3 and that there exists g which makes (2.8) true for φ. It is an exercise to show that there will exist g' which makes (2.8) true for φ'.)

Roughly, a singular frame for T^*M is a frame which looks like $\underline{\delta}$ in some coordinate system centered at p_0.

Part 4 of the digression defines a homotopy class of singular framing of T^*M. Measure the distance between singular frames using the C^0 norm on sections of $T^*(M - p_0)$. Then, give the set of C^0 norms the induced metric topology. A homotopy class of singular frames is just a path component of the space of singular frames.

LEMMA 2.4. *The set c of homotopy classes of singular frames for T^*M is naturally a principal bundle over a point for the abelian group $\pi_0 \equiv (Maps(M;\, SO(3))$.*

Proof. Since $\pi_2(SO(3)) = 0$, the set of homotopy classes of singular frames is in $1 - 1$ correspondence with the set of homotopy classes of honest framings of T^*M. Fix a frame of T^*M and then the space of framings of T^*M can be identified with the space of maps from M into $SO(3)$. Thus, the set of homotopy classes of singular frames is in $1-1$ correspondence with the group π_0.

The action of π_0 on the set of homotopy classes of singular frames comes about as follows: Let $\zeta \equiv (\zeta_1, \zeta_2, \zeta_3)$ be a singular frame, and let $g \equiv (g_{ij})_{i,j=1}^3$ be a map from M to $SO(3)$. Then $g\,\zeta$ is the frame whose i'th component is given by $\Sigma_{j=1}^3 g_{ij}\,\zeta_j$. It is left to the reader to check that the aforementioned action is free and transitive.

By the way, the group π_0 is naturally isomorphic to an abelian extension of $H^1(M;\mathbb{Z}/2)$ by $\mathbb{Z}$. The projection,

$$w : \pi_0 \to H^1(M;\mathbb{Z}/2), \tag{2.9}$$

of a homotopy class $[g]$ of map $g : M \to SO(3)$ is the cohomology class of the pull-back by g of the generator of $H^1(SO(3);\mathbb{Z}/2)$. Any two maps which have the same pull-back of said generator differ by a map which lifts to a map from M to S^3. The homotopy class of such maps to S^3 is classified by assigning to a map its degree, an integer.

There is also a homomorphism,

$$p_\infty : \pi_0 \to \mathbb{Z}, \tag{2.10}$$

which is defined on a class $[g]$ by taking the generator $H^3(SO(3);\mathbb{Z})$ and evaluating its pull-back on M's fundamental class. Note that classes in π_0 which lift to map M into S^3 are sent by p_∞ into $2\,\mathbb{Z}$.

End the digression. □

PROPOSITION 2.5. *A homotopy class, $[\zeta]$, of singular frames for T^*M canonically defines an equivalence class, $c_\zeta \equiv c$ of pairs (N,φ), where N is a regular neighborhood of Σ and where $\varphi : N \to \mathbb{R}^3$ is a smooth map with $\varphi^{-1}(0) = \Sigma$. Furthermore, when $(N,\varphi) \in c_\zeta$, then φ^* is an isomorphism between $H^2(\mathbb{R}^3 - 0)$ and $H^2(N-\Sigma)$.*

Propositions 2.2 and 2.5 define a map, I_2, from the set, c, of equivalence classes of canonical frames to $\mathbb{R}$.

PROPOSITION 2.6. *The map $I_2 : c \to \mathbb{R}$ is equivariant under the action of π_0 on c when the π_0 action on $\mathbb{R}$ is defined by sending a pair $([g],r) \in \pi_0 \times \mathbb{R}$ to $2^{-1}\,p_\infty[g] + r$.*

The propositions in this subsection are proved below.

d) A canonical singular frame.

Note that I_2 above is not a numerical invariant of M as it is apriori defined on the quotient, $\underline{c}$, of the set of homotopy classes of singular frames by the action of the kernel of the homomophism p_∞. However, (as suggested by Kevin Walker) one can apply an observation of Atiyah to produce a canonical element in c, and then apply I_2 to this canonical element to produce the numerical invariant, $I_2(M)$.

The definition of $I_2(M)$ requires the following four part digression: Part 1 of the digression recalls the observation of Atiyah **[1]** that a compact, oriented 3-manifold has a canonical homotopy class of framing of $T^*M \oplus T^*M$. Atiyah calls this 2-frame the *canonical* 2-frame. It will be denoted here by A.

A frame Ξ for $T^*M \oplus T^*M$ which defines A is characterized by two conditions. The first condition requires that Ξ differ from a product frame (ζ, ζ) by a map from M into $SO(6)$ which lifts to a map M into Spin(6).

To describe the second characterizing condition, remark first that when M is the boundary of a compact, oriented 4-manifold, X, with boundary, then the frame Ξ defines a framing of $(T^*X \oplus T^*X)|_M$ since $T^*X|_M \approx T^*M \oplus \underline{\epsilon}$ with $\underline{\epsilon}$ being the trivial line bundle. Remark second that a framing Ξ of $(T^*X \oplus T^*X)|_M$ has a relative first Pontrjagin number $p_1(T^*X \oplus T^*X, \Xi)$. (This number is defined to be the first Pontrjagin number of an $\mathbb{R}^6$-bundle over the space X/M obtained by crushing M to a point. The bundle in question is trivial near M, and is isomorphic to $T^*X \oplus T^*X$ away from M, with Ξ defining the isomorphism.)

With $p_1(T^*X \oplus T^*X, \Xi)$ understood, here is the 2nd condition that characterizes Atiyah's canonical 2-frame: For any X as above,

$$p_1(T^*X \oplus T^*X, \Xi) = 6 \ \text{signature}(X) \tag{2.11}$$

(Note that the Hirzebruch signature theorem insures that when (2.11) holds for one X as above, it holds for all such X.)

Part 2 of the digression serves as a reminder that every compact, oriented 3-manifold is the boundary of some compact, oriented, spin 4-manifold with boundary **[18]**. Furthermore, the signature mod(8) of such a bounding 4-manifold is an invariant of M.

Part 3 of the digression remarks that any map from a 3- manifold into Spin(6) deforms into Spin(3) and has a natural degree. Part 4 of the digression describes the relationship between singular framings for T^*M and framings of $T^*M \oplus T^*M$:

PROPOSITION 2.7. *Let M be a compact, oriented 3-manifold with the rational homology of S^3.*

1) *A homotopy class of singular frame $[\zeta]$ for T^*M naturally defines a pair, $([\zeta_-], [\zeta_+])$, of homotopy classes of honest frames for T^*M. Here, $\zeta_+ = g\,\zeta_-$ where $g : M \to SO(3)$ is a map with $p_\infty(g) = 2$ and which lifts to S^3.*
2) *The assignment $[\zeta] \to ([\zeta_-], [\zeta_+])$, above, induces a natural, injective map θ from $\underline{c}$ into the set of homotopy classes of framings of $T^*M \oplus T^*M$.*
3) *If M bounds a compact, oriented, spin 4-manifold whose signature is zero mod(4), then Atiyah's canonical 2-frame is in the image of θ.*

4) In general, there is a map g from M to Spin(6) with non-negative degree and such that $g\,A$ is in the image of θ.

This proposition is also proved below.

End the digression and consider the following definition of $I_2(M)$:

DEFINITION 2.8.*Let M be a compact, oriented 3-manifold which has the rational homology of S^3. Let A denote the canonical 2-frame of Atiyah.*

a) If M bounds a spin 4-manifold with signature 0 mod(4), then define $c_M \in \underline{c}$ to be $\theta^{-1}(A)$.

b) In general, define $c_M \in c$ so that $\theta(c_M) = g\,A$, where g is a map from M to Spin(6) whose degree is non-negative and minimal among the set of all g such that $g\,A \in$ Image(θ).

c) Define $I_2(M)$ to be the value of Proposition 2.6's homomorphism I_2 on c_M.

e) $I_2(\cdot)$ and cobordisms.

The main purpose of this article is to prove that $I_2(M)$ is an invariant of a certain type of cobordism. A precise statement requires a two part digression.

For Part 1 of the digression, consider a pair, M_0 and M_1, of compact, oriented 3-manifolds. A 4-manifold with boundary, W, will be called an oriented, rational homology, spin cobordism between M_0 and M_1 when the following requirements are met: First, W is oriented and spin. Second, W's boundary is the disjoint union of M_0 and M_1. Third, let $i_{0,1} : M_{0,1} \to W$ denote the inclusions as boundary components. These inclusions, plus the given orientation of W, orient M_0 and M_1. This boundary orientation of M_1 should agree with its given orientation, but the boundary orientation of M_0 should disagree with M_0's given orientation. Fourth, the inclusions of M_0 and M_1 into W should induce isomorphisms on the rational homology.

End Part 1 of the digression and start Part 2. Let W be a compact, oriented, spin 4-manifold with boundary and let M be a component of ∂W. Let $K(M;W)$ denote the cokernel of the restriction induced homomorphism $H^1(W;\mathbb{Z}/2) \to H^1(M;\mathbb{Z}/2)$. The purpose of Part 2 of the digression is to define a homomorphism

$$l_W : c :\to K(M;W). \tag{2.12}$$

The definition of l_W takes five steps: Step 1 remarks that T^*W is isomorphic to the trivial bundle because W is spin and not a compact manifold. In particular, T^*W has a frame, ζ. For Step 2, let ψ be a singular frame for T^*M and let ψ' be any honest frame for T^*M which agrees with ψ on the compliment of a ball around p_0. Note that ψ' defines a frame for $T^*W|_M \approx T^*M \oplus \underline{\epsilon}$ by adding an appropriately oriented frame, ϵ, for the trivial real line bundle ϵ. Step 3 observes that there is a unique map $g : M \to SO(4)$ which is characterized by the equation $(\psi', \epsilon) = g\,(\zeta|_M)$. Step 4 observes that pull-back by g defines an element $l(\psi', g) \in H^1(M;\mathbb{Z}/2)$. Finally, Step 5 observes that $l(\psi', g)$ depends only on the pair (ψ, g), and that the image of $l(\psi', g)$ in $K(M; W)$ depends only on the homotopy class of the singular frame f and on the 4-manifold W. This image is l_W.

End the digression. Here is the main theorem in this article:

THEOREM 2.9. *Let M_0 and M_1 be compact, oriented 3-manifolds with the rational homology of S^3. Let W be an oriented, rational homology, spin cobordism between M_0 and M_1.*

1) *If the inclusions of both M_0 and M_1 into W induce injective maps on $H_1(\cdot; \mathbb{Z}/2)$, then $I_2(M_0) = I_2(M_1)$.*
2) *More generally, if both the canonical homotopy class of singular frames for M_0 and for M_1 (as defined in Proposition 2.8) are represented by $c \in c$ with $l_W(c) = 0$, then $I_2(M_0) = I_2(M_1)$.*
3) *$I_2(S^3) = 0$ and so $I_2(M) = 0$ if M and S^3 are cobordant by a spin cobordism with the rational homology of S^3.*

The proof of Theorem 2.9 occupies Sections 3-11 of this article, but see Subsections 2*i* for the proof that $I_2(S^3) = 0$ and see Subsection 2*k* for an outline of the strategy for the proof of the rest of the theorem.

f) Proof of Proposition 2.5.

The proof of Proposition 2.5 uses the Pontrjagin-Thom construction; and here is a short digression to outline how it works: Let Y be a smooth manifold and let $Z \subset Y$ be a smooth submanifold of codimension p with trivial normal bundle $N \subset TY|_Z$. Note that a trivialization of Z's co-normal bundle, N^*, defines a unique homotopy class of maps from a regular neighborhood of Z in Y to R^p which have Z as the inverse image of 0.

Indeed, a trivialization, ζ, of N^* defines a (fiberwise) linear map $\underline{\varphi}_\zeta : N \to \mathbb{R}^p$ which has zero as a regular value, and which has the zero section as the inverse image of 0.

Now, call a map $e : N \to Y$ an *exponential* map if it has the following two properties:

1) e's restriction to the zero section, $Z_0 \subset N$, is the identity.
2) e's differential along Z_0 induces the canonical identification between

$$(TN|_{Z_0})/TZ_0 \quad \text{and} \quad (TY|_Z)/TZ \equiv N. \tag{2.13}$$

It is not hard to show that exponential maps exist.

Because of (2.13), an exponential map defines a diffeomorphism between a neighborhood of the zero section in N with a neighborhood of Z in Y. (Use the inverse function theorem.) Furthermore, any two exponential maps are homotopic through exponential maps. (The zero'th and first order terms in the Taylor's expansion off the zero section of an exponential map are fixed completely by (2.12).)

Fix an exponential map $e : N \to Y$. As remarked, e defines a diffeomorphism between a regular neighborhood, N, of Z in Y with a regular neighborhood of the zero section in N. Thus

(2.14) $$\varphi_{\zeta,e} \equiv \underline{\varphi}_{\zeta}\, e^{-1} : N \to \mathbb{R}^p$$

is a map with Z as the inverse image of zero.

Because any two exponential maps are homotopic through exponential maps, one can conclude that the coframe ζ, all by itself, defines an equivalence class, c_ζ, of pair (N, φ) where N is a regular neighborhood of Z in Y, and where $\varphi : N \to \mathbb{R}^p$ is a smooth map with $\varphi^{-1}(0) = Z$. (The equivalence relation between such pairs is analgous to the equivalence relation in Proposition 2.2.)

By the way, this construction has an inverse. Let $Z \subset Y$ be an embedded submanifold, and suppose there is a neighborhood $N \subset Y$ of Z and a map $\varphi : N \to \mathbb{R}^p$ with 0 a regular value and with $Z = \varphi^{-1}(0)$. Let $dx \equiv (dx_i)_{i=0}^p$ denote the standard basis for $T^*\mathbb{R}^p$. Then $(\varphi^* dx)|_Z$ defines a framing for Z's conormal bundle in Y.

End the digression.

The plan of proof of Proposition 2.5 is as follows: Step 1 presents an essentially canonical map, Φ_0, from a neighborhood of (p_0, p_0) to $\mathbb{R}$ with the correct inverse of 0. Step 2 finds canonical conormal framings for $p_0 \times (M - p_0)$ and for $(M - p_0) \times p_0$ to be used with the Pontrjagin-Thom construction to extend Φ_0 along neighborhoods of $p_o \times M$ and $M \times p_0$. Step 3: The given singular framing of T^*M will then be used to define a conormal framing for $\Delta - (p_0, p_0)$ and, finally, the Pontrjagin-Thom construction will be used to turn this conormal framing into an extension of Φ_0 to a neighborhood of Δ.

Step 1: Choose an oriented frame for $T^*M|_{p_0}$ and fix an exponential map from $T^*M|_{p_0}$ into M. This identifies a neighborhood of p_0 in M with a ball of some non-zero radius in $\mathbb{R}^3$. To avoid complicated notation, assume the radius is 1. This identification will be made implicitly in what follows. Use $x \equiv (x_1, x_2, x_3)$ to denote the Euclidean coordinates in $\mathbb{R}^3$.

Note that the coordinate system so defined depends on a choice of frame for $T^*M|_{p_0}$ and also on an exponential map. As described above, the dependence on the choice of exponential map is inconsequential if one is interested only in a homotopy class of map from a neighborhood of Σ to $\mathbb{R}^3$. The choice of frame for $T^*M|_{p_0}$ is also inconsequential because two such frames differ by the action of an element in the component of the identity matrix in $GL(3; \mathbb{R})$.

The preceding coordinate system identifies a neighborhood, $N_0 \times N_0$, of (p_0, p_0) with a neighborhood of $(0,0)$ in $\mathbb{R}^3 \times \mathbb{R}^3$. Make this identification implicit in what follows.

A point in $\mathbb{R}^3 \times \mathbb{R}^3$ will be written as (x, y). With this understood, $p_0 \times M$ intersects $N_0 \times N_0$ as the subspace where $x \equiv 0$, while $M \times p_0$ intersects $N_0 \times N_0$ as the subspace where $y = 0$. The diagonal Δ intersects $N_0 \times N_0$ as the subspace where $x - y = 0$.

Define Φ_0 on $N_0 \times N_0$ by the formula

(2.15) $$\Phi_0(x, y) \equiv |\, x \,|^2\, y - |\, y \,|^2\, x$$

(Dror Bar-Natan described Φ_0 to the author.) Here are the salient features of Φ_0:

LEMMA 2.10. *The map Φ_0 has the following properties:*
1) $|\Phi_0(x,y)| = |x||y||x-y|$.
2) *Φ_0 is a submersion on* $\mathbb{R}^3 \times \mathbb{R}^3 - (0,0)$.
3) $d\Phi_0|_{x=0} = -|y|^2\,dx$.
4) $d\Phi_0|_{y=0} = |x|^2\,dy$.
5) *Let $w \equiv x+y$ and let $u = x-y$. Then*

$$d\Phi_0|_{u=0} = 2\,|w|^2\,(-du + 2\,|w|^{-2}\,\langle w, du\rangle\, w), \tag{2.16}$$

where $\langle w, du\rangle \equiv \Sigma_{i=1}^3 w_i\, du_i$.

Proof. All remarks are exercises with multivariable calculus. □

Step 2: The coordinate system on N_0 identifies $N_0 \times M$ with $B \times M$, where $B \subset \mathbb{R}^3$ is a neighborhood of the origin. Map $N_0 \times M$ to B by first using the preceding identification and then projecting onto B. Call the preceding map φ. Note that 0 is a regular value of φ and that $p_0 \times M = \varphi^{-1}(0)$. Thus, $-\varphi^* dx$ defines a frame for the conormal bundle of $p_0 \times M$ in $M \times M$.

Now, restrict attention to $N_0 \times N_0 \subset N_0 \times M$. The map Φ_0 of Lemma 2.10 maps $p_0 \times N_0$ to zero, and is a submersion along $p_0 \times N_0$ except at $p_0 \times p_0$. Note that $\Phi_0^* dx$ and $-\varphi^* dx$ differ along $p_0 \times N_0$ only by multiplication of a scalar function. (This is Assertion 3 of Lemma 2.10.) In the coordinates of Lemma 2.10, $\Phi_0^* dx = -|y|^2\, dx$ while $\varphi^* dx = -dx$. Note also that the former frame is defined for $|y| < 1$, while the latter is defined for $|y| > 0$. Take a favorite, positive function h on $[0,\infty)$ which obeys
1) $h(t) = t$ for $t < 1/2$
2) $h(t) = 1$ for $t \geq 3/4$.

(2.17)

Use $-h(|y|^2)\,dx$ to interpolate between $\Phi_0^* dx$ where $|y| < 1/2$ and between $\varphi^* dx$ where $|y| > 1$ and so construct a coframe for $p_0 \times M$ on the compliment of $p_0 \times p_0$ which agrees with $\Phi_0^* dx$ on $p_0 \times N_0$ and with $-\varphi^* dx$ on $p_0 \times (M - N_0)$. Use the Pontrjagin-Thom construction (as described above) to extend this coframe to a smooth map from a neighborhood of $p_0 \times M$ in $M \times M$ to $\mathbb{R}^3$ which agrees with Φ_0 on a neighborhood of $p_0 \times p_0$.

Given that the coordinate system about p_0 is fixed, the above extension will be canonical up to homotopy.

As for $M \times p_0$, introduce the switch map,

$$\Theta : M \times M \to M \times M \tag{2.18}$$

which interchanges the factors. Obviously, the switch map interchanges $p_0 \times M$ with $M \times p_0$. Use the switch map to extend Φ_0 along a neighborhood of $M \times p_0$. □

Step 3: To extend the map Φ_0 to a neighborhood of Δ, the strategy will be to find a coframe, ζ, for the normal bundle of $\Delta - p_0$ which agrees with $\Phi_0^* dx$ on $(\Delta - p_0) \cap N_0 \times N_0$. Given such a frame ζ, one can copy the arguments from Step 2, above, to extend Φ_0 along Δ. As in Step 2, the extension will be unique up to homotopy.

Now, Δ is canonically diffeomorphic to M, while the conormal bundle of Δ in $M \times M$ is canonically isomorphic to T^*M. (The isomorphism here is given by $\pi_R^* - \pi_L^*$, where $\pi_{R,L} : M \times M \to M$ are the projections onto the right (R) and left (L) factors.) So, the coframe ζ is a frame for $T^*(M - p_0)$ with a prescribed form near p_0. The constraint on ζ near p_0 comes from the requirement that ζ agree with $\Phi_0^* dx$. The latter is given in (2.16). Thus, up to the scalar factor, ζ should be a singular frame for T^*M in the sense of Definition 2.3, but with the matrix g in Definition 2.3 equal to the identity.

If $g \equiv g_\zeta$ in Definition 2.3 is not the identity for the given frame ζ, then replace ζ by $g_\zeta \, \zeta$; this last frame will obey (2.8) with g the identity matrix.

Thus, a singular frame for T^*M can be used to build a map from a neighborhood of Σ to $\mathbb{R}^3$ with all of the requisite properties. That said map is unique up to homotopy follows from two already mentioned facts: First, positive determinant matrices in $GL(3, \mathbb{R})$ form the path component of the identity. Second, the map in (2.14) is insensitive (modulo homotopies) to the choice of exponential map.

g) Proof of Proposition 2.6.

To consider the behavior of I_2 when π_0 acts on c, remember that the action of π_0 is generated as follows: Let ζ be a singular frame for T^*M. Let $g : M \to SO(3)$ be a smooth map which equals the identity near to p_0. Then $[g] \in \pi_0$ acts on the class $[\zeta]$ of the frame ζ to give the class of the singular frame $g\,\zeta$.

To compare the value of Φ_2 on $[\zeta]$ and on $[g\,\zeta]$, construct the map $\varphi_\zeta : N \to \mathbb{R}^3$ as described in Proposition 2.5. One could construct $\varphi_{g\,\zeta}$ too, but here is a shortcut to this map: Introduce the neighborhood $N_0 \subset N$ as before. Then, define a smooth map ψ from N_0 to $SO(3) \times \mathbb{R}^3$ as follows:

1) If $p \in (B_0 \times M) \cup (M \times B_0)$, then set $\psi(p) \equiv (1, \varphi_\zeta(p))$.
2) If $p \in N_\Delta$, then set $\psi(p) \equiv (g(p), \varphi_\zeta(p))$.

(2.19)

Let $m : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ denote the map which describes the group action. Let $p : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ denote the projection. With m and with ψ understood, one can take $\varphi_{g\,\zeta}$ as follows:

$$\varphi_{g\,\zeta} = m \circ \psi \qquad (2.20)$$

Equation (2.20) can be exploited with the help of the following observation: On $SO(3) \times (\mathbb{R}^3 - 0)$, the 2-forms $m^* \omega_0$ and $p^* \omega_0$ are cohomologous, so find a 1-form α_0 on $SO(3) \times (\mathbb{R}^3 - 0)$ such that $m^*\omega_0 = p^*\omega_0 + d\alpha_0$. Make α_0 restrict to 0 on $1 \times (\mathbb{R}^3 - 0)$. With α_0 understood, notice that $\varphi^*_{g\,\zeta}\omega_0 = \varphi^*_\zeta\omega_0 + \psi^* d\alpha_0$ on $N_0 - \Sigma$.

Let ω and ω' be closed 2-forms on $M \times M - \Sigma$ which extend $\varphi^*_\zeta\omega_0$ and $\varphi^*_{g\,\zeta}\omega_0$, respectively. There is no obstruction to extending α from $N_0 - \Sigma$ as a smooth 1-form α on $M \times M - \Sigma$ such that $\omega' = \omega + d\alpha$ (See Lemma 1.1.) Then, $\Phi_2[g\,\zeta] - \Phi_2[\zeta]$ is given by

$$\Phi_2[g\,\zeta] - \Phi_2[\zeta] = \int_{M\times M-\Sigma} (3\,d\alpha \wedge \omega \wedge \omega + 3\,d\alpha \wedge d\alpha \wedge \omega + d\alpha \wedge d\alpha \wedge d\alpha) \tag{2.21}$$

Here is how to evaluate the right side of (2.21): Let $S^2 \subset \mathbb{R}^3$ denote a ball of small radius about 0 and which is in the image of φ_ζ on $N_\Delta - \pi^{-1}(B_0)$ and which is transversal to φ_ζ on $N_\Delta - \pi^{-1}(B_0)$. (One can assume, with no loss of generality, that such a 2-sphere exists.) Look carefully at the proof of Proposition 2.5. Introduce S to denote $\varphi_\zeta^{-1}(S^2)$. This is a two sphere bundle over $N_\Delta - \pi^{-1}(B_0)$. Because α_0 vanishes on $B_0 \times M$ and on $M \times B_0$, Stokes theorem equates the right hand side of (2.21) with

$$\Phi_2[g\,\zeta] - \Phi[\zeta] = \int_S \varphi^*(\alpha_0 \wedge d\alpha_0 \wedge p^*\omega_0). \tag{2.22}$$

(One uses here the fact that $\omega_0 \wedge \omega_0 = 0$ and that $2\,d\alpha_0 \wedge p^*\omega_0$ is equal to $-d\alpha_0 \wedge d\alpha_0$.)

Introduce $i : SO(3) \to SO(3) \times S^2$ to be the inclusion as a fiber of the projection, p, to S^2. Then, the integral in (2.22) can be evaluated by pushing the integrand forward to $SO(3) \times S^2$, and then by doing the S^2 integration first. The result is

$$\Phi_2[g\,\zeta] - \Phi_2[\zeta] = p_\infty[g] \int_{SO(3)} i^*(\alpha_0) \wedge di^*(\alpha_0). \tag{2.23}$$

The factor of $p_\infty[g]$ in (2.23) is due to the fact that g^* maps the fundamental class of M to $p_\infty[g]$ times the fundamental class of $SO(3)$.

Now, the integral in (2.23) is simply a Hopf invariant, and can be computed to be equal to 1/2. (Restrict $m : SO(3) \times S^2 \to S^2$ to $m_0 : SO(3) \times n \to S^2$, where n is the north pole. Then $di^*\alpha_0 = m_0^*\omega_0$. But, m_0 defines a principal $SO(2)$ bundle with Euler class equal to 2, so $2\,i^*\alpha_0$ will integrate to 1 over the fiber of m_0, while ω_0 integrates to 1 over the base. Thus, the value of the integral in (2.23) is equal to 1/2.) So, $\Phi_2[g\,\zeta] = 2^{-1}\,p_\infty[g] + \Phi_2[\zeta]$ as claimed. □

h) Proof of Proposition 2.7.

A singular frame, ζ, for T^*M defines a 2-frame for TM as follows: Choose a coordinate system around p_0 and introduce the matrix g as in (2.8). Then

$\zeta' \equiv g\,\zeta$ obeys (2.8) with g replaced by the identity. Thus, on a small ball about p_0, ζ' differs from the frame $\underline{\delta}$ of (2.7) by a small amount, and so there is a canonical homotopy of ζ' so that it agrees with $\underline{\delta}$ on a small ball, B, about p_0.

Now, $\pi_2(SO(3))$ vanishes, so there is no obstruction to deforming $\underline{\delta}$ inside B so that the result, $\underline{\delta}'$, agrees with $\underline{\delta}$ near B's boundary, and agrees with the constant framing dx on an even smaller ball, $B' \subset B$. Such a deformation would change ζ' to an honest framing of TM. Unfortunately, $\pi_3(SO(3)) \approx \mathbb{Z}$, so a topologist might argue that there is no canonical way to get an honest framing from ζ'.

However, the frame $\underline{\delta}$ is rather special (Kevin Walker pointed this out to the author); restrict it to the 2-sphere boundary $S^2 \subset B$. Compare $\underline{\delta}$ with the constant framing to define a map from $S^2 \to SO(3)$. Lift this map to $S^3 = SU(2)$ and one has the inclusion of S^2 in S^3 as the equator which is invariant under multiplication by ± 1 on S^3. And, there are precisely two canonical deformations to a point of this equatorial embedding of S^2 in S^3. Indeed, consider taking the family of two spheres of decreasing latitude starting from $\pi/2$ and going to 0. Or, take the family of increasing latitude, starting from $\pi/2$ and going to π.

Thus, there are two canonical ways to obtain an honest framing from the singular framing ζ. Denote these two honest framings by $\zeta_\pm$. By construction, $\zeta_+ = g\,\zeta_-$ where $g : M \to SO(3)$ has degree 2 and lifts to map M into S^3. This exhibition of $\zeta_\pm$ completes the proof of the first assertion of Proposition 2.7.

To prove Assertion 2 of Proposition 2.7, take $\zeta_\pm$ above and produce the 2-framing $\Theta_\zeta \equiv \zeta_+ \oplus \zeta_-$. (Note that $\zeta_+ \oplus \zeta_-$ is homotopic to $\zeta_- \oplus \zeta_+$ as a framing of $TM \oplus TM$.) Clearly, if ζ_0 and ζ_1 define the same equivalence class of singular framing for T^*M, then Θ_{ζ_0} and Θ_{ζ_1} will be homotopic as framings of $TM \oplus TM$ and so define the same 2-framing of TM.

To prove that the map Θ is injective on $\underline{c}$, consider first a 4- manifold X which bounds M. Let Ξ, Ξ' be frames for $T^*M \oplus T^*M$. Then Ξ is homotopic to Ξ' only if $p_1(TX \oplus TX; \Xi) = p_1(TX \oplus TX; \Xi')$.

With the preceding understood, let ζ and ζ' be a pair of singular frames for T^*M. Then $\zeta = g\,\zeta'$ where $g : \to SO(3)$. A direct computation reveals that

$$p_1(TX \oplus TX; \Theta_{g\zeta}) = p_1(TX \oplus TX; \Theta_\zeta) + 4\,p_\infty[g]. \tag{2.24}$$

This shows that the map Θ is injective and completes the proof of Assertion 2 of Proposition 2.7.

To prove Assertion 3 of Proposition 2.7, consider a compact, oriented 4-manifold, X, with boundary M. Suppose that X is connected, spin, and that X's signature is even. Define

$$\chi_0(X) \equiv \sum_{i=1}^{3} (-1)^i \ \dim(H^i(X; \mathbb{R})). \tag{2.25}$$

Now χ_0 is even because M is a rational homology sphere, and because the signature of X is assumed even. With this understood, then one can assume, with no loss of generality that $\chi_0 = 0$. (Indeed, if χ_0 is initially positive, connect sum with $\chi_0/2$ copies of $S^1 \times S^3$ to obtain a compact, oriented, spin 4-manifold with $\chi_0(\cdot) = 0$. If $\chi_0 < 0$ initially, connect sum with $\chi_0/2$ copies of $S^2 \times S^2$ to obtain a compact, oriented, spin 4-manifold with $\chi_0(\cdot) = 0$.)

Choose an embedding of the closed unit 4-ball in $B \subset X$ and let X_0 denote the compliment of this 4-ball. This X_0 has two boundary components, one M and the other S^3. Since $\chi_0(X) = 0$, the boundary splittings, $T^*X_0 \mid_M = T^*M \oplus \underline{\epsilon}$ and $T^*X_0 \mid_{S^3} = T * S^3 \oplus \underline{\epsilon}$, extend over X_0 as a splitting $T^*X_0 = V \oplus \underline{\epsilon}$. Here, V is an oriented 3-plane bundle over X_0. Furthermore, because X_0 is a spin manifold, V is a spin 3- plane bundle, and so trivial; $V \approx \oplus_3 \underline{\epsilon}$.

Let h be a singular frame for T^*S^3, and construct the frames $h_\pm$ for T^*S^3 as instructed above. The fact that V is trivial implies that the frame h_+ for $V \mid_{S^3}$ extends over X_0; so does h_-. Restrict these frames to $M \subset \partial X_0$ to give frames, $\zeta_\pm$, for $T^*M^3 = V \mid_{M^3}$. Notice that $\zeta_+ = g\,\zeta_-$ for some $g : M \to SO(3)$ which lifts to a map into S^3.

The point of the preceding is this: The extendability of $h_\pm$ over X_0 implies $p_1(TB \oplus TB, \Theta_h) = p_1(TZ \oplus TZ, \Theta_\zeta)$. Now, for S^3, one can compute rather explicitly that $p_1(TB \oplus TB; \Theta_h) = 0 \bmod 8$. Indeed, one can compute with the following choice for $h_\pm$: Take h_+ to be the Lie group framing given by the left-invariant 1-forms, and take h_- to be the Lie group framing given by the right invariant 1-forms. For $h_\pm$ as above, $p_1(TB \oplus TB; \Theta_h) = 0$.

Since, $p_1(TZ \oplus TZ; \Theta_\zeta) = 0 \bmod (8)$, Atiyah's canonical framing will be realized by Θ_ζ if Z's signature is 0 mod(4).

Assertion 4 of Proposition 2.7 follows from Proposition 1 in **[1]**. □

i) $I_2(S^3)$.

The purpose of this subsection is to provide a proof of the assertion in Theorem 2.9 that $I_2(S^3) = 0$. (Dror Bar-Natan taught the author this proof.) This fact is needed later and so, for future reference, is stated as

LEMMA 2.11. *$I_2(S^3) = 0$.*

Proof. The strategy has two parts: Part 1 extends the map Φ_0 in (2.15) to define a map, Φ_1, from $S^3 \times S^3$ to $\mathbb{R}^3$ with inverse image Σ_{S^3}. Given that such an extension exists, take φ in Proposition 2.2 to be the restriction of Φ_1 to a regular neighborhood of Σ_{S^3} and use $\omega_\varphi \equiv \Phi_1^*\mu$ in (2.4). The integrand is then zero so the integral in (2.4) is zero.

Part 2 observes that $d\Phi_1 \mid_{S^3 \times p_0}$ and $d\Phi_1 \mid_{p_0 \times S^3}$ define linear maps from the respective normal bundles of $S^3 \times p_0$ and $p_0 \times S^3$ to $\mathbb{R}^3$. Part 2 also observes that $d\Phi_1|_\Delta$ defines a linear map from the normal bundle of $\Delta \equiv \Delta_{S^3}$ into $\mathbb{R}^3$. With these facts understood, the fact that $I_2(s^3) = 0$ is established by demonstrating the following two points:

1) $d\Phi_1|_{S^3 \times p_0}$ and $d\Phi_1|_{p_0 \times S^3}$ define respective normal bundle framings of $(S^3 - p_0) \times p_0$ and $p_0 \times (S^3 - p_0)$ which are homotopic (rel neighborhoods of $p_0 \times p_0$) to the normal bundle framings which are respectively

induced by the projections, $\pi_{R,L}$, on the right and left factors of S^3 in $S^3 \times S^3$.

2) $d\Phi_1|_\Delta$ defines a normal bundle framing of $\Delta_{S^3} - (p_0, p_0)$ which is gives Definition 2.8's canonical singular framing of T^*S^3.

Part 1: To exhibit the map Φ_1, agree first to use stereographic projection from the north and south poles of S^3 to cover S^3 by two coordinate patches, U_s and U_n. (So, the north pole is the origin in U_n and the south pole is the origin in U_s). Agree to take the point p_0 to equal the south pole.

Define Φ_1 by

1) On $U_s \times U_s : \Phi_1(x,y) \equiv (|x|^2+1)^{-1}(|y|^2+1)^{-1}(|x|^2y - |y|^2x)$.
2) On $U_n \times U_s : \Phi_1(x,y) \equiv (|x|^2+1)^{-1}(|y|^2+1)^{-1}(y + |y|^2x)$.
3) On $U_s \times U_n : \Phi_1(x,y) \equiv (|x|^2+1)^{-1}(|y|^2+1)^{-1}(-|x|^2y - x)$.
4) On $U_n \times U_n : \Phi_1(x,y) \equiv (|x|^2+1)^{-1}(|y|^2+1)^{-1}(x - y)$.

(2.26)

It is left to the reader to verify that Φ_1 is consistently defined, and that $\Phi_1^{-1}(0) = \Sigma_{S^3}$.

Part 2: Using line 1 or line 2 of (2.26), one finds $d\Phi_1|_{S^3 \times p_0}$ to be proportional to the pull-back by π_R of the constant frame on $\mathbb{R}^3$. Likewise, using line 1 or line 3 of (2.26), one finds that $d\Phi_1\ |_{p_0 \times S^3}$ is proportional to the pull-back by π_L of the constant frame.

As for $d\Phi_1\ |_\Delta$, compute on $U_n \times U_n$ to find that $d\Phi_1\ |_\Delta$ is equal to $(\pi_R^* - \pi_L^*)(-(1+\mid x \mid^2)^{-2}\,dx)$. Remember that $S^3 - p_0 = U_n$ and thus $-(1+\mid x\mid^2)^{-2}\,dx$ is a framing for T^*U_n which extends as a singular framing for T^*S^3. The image of this framing under the map Θ of Proposition 2.7 is the pair (ζ_-, ζ_+) where $\zeta_\pm$ are the Lie group framings given by the left and right invariant 1-forms on S^3 as $SU(2)$.

Let $B \subset \mathbb{R}^4$ denote the unit 4-ball. It is left as an exercise to check that (ζ_-, ζ_+) stabilizes to give a framing of $(TB \oplus TB)|_B$ which extends over B. □

k) Outline of Theorem 2.9's proof.

Here is a simplified outline for a proof of Theorem 2.9: One would like to find a smooth, oriented, 7-dimensional manifold Z whose boundary is the disjoint union of $M_1 \times M_1$ and $M_0 \times M_0$, with the latter oriented in reverse. This Z should contain an oriented, dimension 4 subvariety, Σ_Z, (a union of submanifolds) whose boundary is the union of Σ_{M_1} and Σ_{M_0}, with the latter oriented in reverse. (Here, Σ_M is defined in (2.1) as the union of submanifolds. Each submanifold has its fundamental class; these are oriented so that the inclusion $\Sigma_M \to M \times M$ sends $[\Delta_M] - [M \times p_0] - [p_0 \times M]$ (a linear combination of fundamental classes) to zero in $H_3(M \times M; \mathbb{R})$.)

Given a singular frame for M_1 and an appropriate singular frame for M_0, one would like to find a smooth 2-form ω_Z on $Z - \Sigma_Z$ and then use Stokes' theorem to compare the integrals in (2.4) for M_0 and for M_1.

Unfortunately, this simplified strategy has not been realized. However, there is a modification of this strategy which can be carried out. Here are the steps:

Step 1: Find a smooth, oriented manifold Z with boundary, and suppose that ∂Z is the disjoint union of $M_1 \times M_2$ and $M_0 \times M_0$ (with the latter oriented in reverse) plus some number of copies of $S^3 \times S^3$. Label these extra boundary components by a finite set, crit.

Step 2: Inside Z, find an oriented, dimension 4 subvariety Σ_Z with $\partial\Sigma_Z = \Sigma_{M_0} \cup \Sigma M_0 \cup_{p\in\text{crit}} (\Sigma_{S^3})_p$.

Step 3: Make sure that there is a closed 2-form, ω_Z, on $Z - \Sigma_Z$ with the choice of a singular frame for M_0 and an "appropriate" singular frame for M_1. (See Step 4, below, for the definition of "appropriate".) Require the following:

(2.27)

1) The 2-form ω_Z should restrict to $M_0 \times M_0 - \Sigma_{M_0}$ as the 2-form used in (2.4) to compute I_2 for M_0. It's restriction to $M_1 \times M_1 - \Sigma_{M_1}$ should give the 2-form used in (2.4) for the computation of I_2 for M_1.
2) The 2-form ω_Z should restrict to each copy of $S^3 \times S^3$ as the 2- form used for (2.4) with the singular frame that gives $I_2(S^3)$ (see Lemma 2.11).
3) The wedge product $\omega_Z \wedge \omega_Z$ should vanish near Σ_Z.

Step 4: If the canonical frame (Definition 2.8) was chosen for M_0, check that the "appropriate" frame for M_1 is M_1's canonical frame.

Step 5: Given that ω_Z exists as prescribed above, use Stokes' theorem to prove that the values of I_2 in (2.4) for M_0 and for M_1 agree:

$$0 \equiv \int_Z d(\omega_Z \wedge \omega_Z \wedge \omega_Z) = \int_{M_1\times M_1} \omega_Z \wedge \omega_Z \wedge \omega_Z - \int_{M_0\times M_0} \omega_Z \wedge \omega_Z \wedge \omega_Z + \int_{\partial \underline{N}_Z} \omega_Z \wedge \omega_Z \wedge \omega_Z \,. \tag{2.28}$$

Here, $\underline{N}_Z$ is the closure of a regular neighborhood of Σ_Z in Z (with smooth boundary) on which $\omega_Z \wedge \omega_Z = 0$. Thus, the last term in (2.28) vanishes.

Section 3 begins the proof of Theorem 2.9 with the definition of the space Z. Later sections construct Σ_Z and carry out the remaining steps of the proof. The construction of ω_Z is completed in Section 11.

3 Morse theory. This subsection reviews various Morse theoretic constructions which will be used in subsequent sections. The final two subsections define and explore the space Z for Section 2k's first step in the proof of Theorem 2.9.

Suppose here that M_0 and M_1 are both compact, connected, oriented 3-manifolds whose rational homology is the same as that of S^3. Let W be an oriented, spin cobordism between M_0 and M_1. Assume that W is connected. By surgery on embedded circles, one can modify W so that $H^1(W;\mathbb{R}) = H^3(W;\mathbb{R}) = 0$. This will be assumed as well.

a) Good Morse functions.

A function $f : W \to [0,1]$ will be called a good Morse function if the criteria below are met:

1) $M_0 = f^{-1}(0)$ and $M_1 = f^{-1}(1)$.
2) $df\,|_{M_0}$ and $df\,|_{M_1}$ are never zero.
3) The critical points of f are non-degenerate.
4) There are no critical points of index 0 or 4.
5) Let c be an index i critical point. Then $|\, f(c) - i/4 \,| < 1/100$.
6) If c, c' are distinct critical points of f, then $f(c) \neq f(c')$.

(3.1)

It is shown in [**12**] (for example) that W has *good Morse functions*.

When $f : W \to [0,1]$ is a good Morse function, use $\operatorname{crit}(f) \subset W$ to denote the set of critical points of f. Use $\operatorname{crit}_k(f) \subset \operatorname{crit}(f)$ to denote the set of critical points of index k.

Let $p \in \operatorname{crit}_k(f)$. There is an almost canonical coordinate system on a neighborhood of p. This coordinate system is an embedding $\psi_p : \mathbb{R}^4 \to W$ with the following properties:

1) $\psi_p(0) = p$.
2) There exists $\delta > 0$ such that $\psi_p^* f$ restricts to the radius δ ball about p as the function

$$\psi_p^* f = -x_1^2 \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_4^2. \tag{3.2}$$

(See, e.g. [**12**].) These coordinates will be called *Morse coordinates.* The image of $\mathbb{R}^4$ under ψ_p^* will be called U_p.

b) Pseudo-gradient vector fields.

Aside from a Morse function and Morse coordinates, the standard machinery for Morse theory requires the choosing of a *pseudo-gradient vector field* for f. This is a vector field, v, on W with the property that $v(f) > 0$ on $W - \operatorname{crit}(f)$. Also, require of v that it have the following form near $p \in \operatorname{crit}_k(f)$: The pushforward by $(\psi_p)^{-1}$ of v should restrict to a small ball about the origin in $\mathbb{R}^4$ to equal

$$-x_1\partial_1 \cdots - x_k\partial_k + x_{k+1}\partial_{k+1} + \cdots + x_4\partial_4. \tag{3.3}$$

A pseudo-gradient vector field for f will often be called a *pseudo- gradient*, for short.

A gradient flow line of a pseudo-gradient v is a map γ, of a closed interval, I, into W with the following properties:

1) $I = [a, b]$ with $-\infty \leq a < b \leq +\infty$.
2) If $a = -\infty$, then $\gamma(a) \in \text{crit}(f)$; and if $b = +\infty$, then $\gamma(b) \in \text{crit}(f)$.
3) If $a > -\infty$, then $\gamma(a) \in M_0$; and if $b < +\infty$, then $\gamma(b) \in M_1$.
4) $\gamma_*(\partial_t) = v \mid_{\gamma(t)}$ for all $t \in I$

(3.4)

(Here, ∂_t differentiates the coordinate t to give 1.)

If γ is a gradient flow line of a pseudo-gradient, v, say that γ begins at $\gamma(a)$ and ends at $\gamma(b)$.

There is a great deal of flexibility in the choice of a pseudo- gradient. And, there are specific constraints which can be imposed on a pseudo-gradient which simplify some subsequent constructions.

c) The Morse complex.

With the help of good Morse function f and an appropriate pseudo gradient, v, one can define a finite dimensional complex whose homology is naturally isomorphic to the relative homology $H^*(W, M_0; \mathbb{Z})$. (See, e.g. [**12**].) The complex is written

$$0 \longrightarrow C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \longrightarrow 0 \tag{3.5}$$

To describe the $\{C_k\}$ in (3.5), it is necessary to first digress to review the construction of ascending and descending disks: As described in [**12**], one can use v to define, for each $p \in \text{crit}_k(f)$, a pair of open subsets, $B_{p-} \subset \text{int}(W)$ and $B_{p+} \subset \text{int}(W)$, which are embedded disks of dimension k and $4 - k$, respectively. Here, B_{p+} is the ascending disk from p, and B_{p-} is the descending disk from p.

As a subset, B_{p-} is the union $\{p\}$ with the set of points of $\text{int}(W) - \text{crit}(f)$ which lie on gradient flow lines which end at p. And, B_{p+} is the union of $\{p\}$ with the set of points in $W - \text{crit}(f)$ which lie on gradient flow lines which start at p.

Note that (3.4) implies that

1) $\psi_p(B_{p-} \cap U_p) = \{(x_1, \cdots, x_4) \in \mathbb{R}^4 : x_{k+1} = \cdots = x_4 = 0\}$,
2) $\psi_p(B_{p+} \cap U_p) = \{(x_1, \cdots, x_4) \in \mathbb{R}^4 : x_1 = \cdots = x_k = 0\}$.

(3.6)

These disks intersect at one point, p, and there transversally. Otherwise,

$$f \mid (B_{p-} - p) < f(p) < f \mid (B_{p+} - p) \tag{3.7}$$

As W is assumed oriented, an orientation for B_{p-} orients B_{p+} so that their intersection number, $[B_{p-}] \cdot [B_{p+}]$, is equal to $\{1\}$.

End the digression.

One defines C_k in (3.5) from the free $\mathbb{Z}$-module, $\underline{C_k}$, on the set of pairs

$$\{(p, \epsilon) : p \in \operatorname{crit}_k(f) \text{ and } \epsilon \text{ is an orientation for } B_{p-}\} \tag{3.8}$$

Indeed, set $C_k \equiv \underline{C}_k / \sim$, with the equivalence relation $(p, \epsilon) \sim -(p, -\epsilon)$. (The C_k for different choices of pseudo-gradient are canonically isomorphic.)

To define the operator ∂ in (3.5), it is necessary to make a two part digression.

Part 1 of the digression introduces some constraints on the pseudo-gradient v. These are described next.

DEFINITION 3.1. *Let f be a Morse function on W. A pseudo-gradient v will be called* good *if the following criteria are met:*

1) *If $p, q \in crit_k(f)$ and if $p \neq q$, then $B_{p+} \cap B_{q-} = \emptyset$.*
2) *If $p \in crit_k(f)$ and $q \in crit_{k+1}(f)$ then B_{p+} intersects B_{q-} transversely. Furthermore, $B_{p+} \cap B_{q-}$ is a finite union of gradient flow lines, the closure of each starts at p and ends at q.*
3) *If $p \in crit_1(f)$ and $q \in crit_3(f)$, then B_{p+} intersects B_{q-} transversally.*
4) *If $p_0 \in M_0$ and $p_1 \in M_1$ have been apriori specified, then require that p_0 start a gradient flow line with end at p_1.*

(By the way, because of their definitions in terms of v's flow lines, descending disks from distinct critical points do not intersect, and likewise, ascending disks.)

See [**12**] for a proof that good pseudo-gradients exist. Henceforth, assume that v is a good pseudo-gradient.

Part 2 of the digression considers the intersections of ascending and descending disks. Start the discussion with the introduction of $M_{k,k-1} \equiv f^{-1}(4^{-1}\, k - 1/8)$. Due to (3.1), one can conclude that df is nowhere zero along $M_{k,k-1}$, so this subspace is an embedded submanifold of W. Furthermore, $M_{k,k-1}$ is naturally oriented by using df to trivialize its normal bundle.

Because of (1) in Definition 3.1, each B_{p-} intersects $M_{k,k-1}$ in its interior as a $(k-1)$ sphere S_{p^-} which is oriented (by *df*) when B_{p-} is oriented. Likewise, B_{p+} intersects $M_{k+1,k}$ in a sphere, S_{p+}, of dimension $3-k$ which is oriented when B_{p-} is oriented.

Note that Definition 3.1 implies (in part) the following assertion: If $p \in \operatorname{crit}_k(f)$, then S_{p^-} has transversal intersection in $M_{k,k-1}$ with any S_{q^+} from $q \in \operatorname{crit}_{k-1}(f)$. With the preceding understood, use $[S_{p^-}] \cdot [S_{q^+}] \in \mathbb{Z}$ to denote the algebraic intersection number of S_{p^-} with S_{q^+} in $M_{k,k-1}$.

End the digression.

Here is the definition of the boundary map ∂ in (3.5):

$$\partial(p, \epsilon) \equiv \sum_{q \in C_{k-1}} ([S_{p^-}] \cdot [S_{q^+}])\, (q, \epsilon_q). \tag{3.9}$$

See [**12**] for a proof that (3.5) with ∂ as in (3.9) is a chain complex whose homology is isomorphic to $H * (W, M_0; \mathbb{Z})$.

Note that there is a dual complex to (3.5),

$$0 \longrightarrow C_3^* \xrightarrow{\partial^*} C_2^* \xrightarrow{\partial^*} C_1^* \longrightarrow 0 \tag{3.10}$$

which is defined using $-f$ and $-v$ when (3.5) is defined from the pair f and v. The homology of (3.10) computes $H^*(W, M_1; \mathbb{Z})$. Poincare' duality identifies $H^*(W, M_1; \mathbb{Z})$ with $H^{4-*}(W, M_0; \mathbb{Z})$, hence the duality between (3.10) and (3.5).

d) Factoring the cobordism.

The purpose of this subsection is to indicate how to factor the cobordism W into two simpler cobordisms. The following proposition summarizes:

PROPOSITION 3.2. *Let M_0, M_1 be compact, oriented 3-manifolds with the rational homology of S^3. Suppose that there is an oriented, spin cobordism, W', between M_0 and M_1. Then there exists an oriented, spin cobordism, W, between M_0 and M_1 which decomposes as $W = W_1 \cup W_1 \cup W_3$, where*

1) *$\partial W_1 = -M_0 \cup M_0', \partial W_2 = -M_0' \cup M_1'$, and $\partial W_3 = -M_1' \cup M_1$, where M_0' and M_1' are compact, oriented 3-manifolds with the rational homology of S^3.*
2) *$W_{1,2,3}$ are oriented, spin manifolds.*
3) *Both W_1 and W_3 have the rational homology of S^3. Meanwhile, W_2 has vanishing first and third Betti numbers.*
4) *W_1 and W_3 have a good Morse functions with no index 3 critical points. Meanwhile, W_2 has a good Morse function without index 1 and index 3 critical points.*
5) *If W' has the rational homology of S^3, then W above can be assumed to have the rational homology of S^3. And, one can assume that $M_0' = M_1'$ and that W_2 is the product cobordism.*
6) *Let $l_{W'}$ and l_W be as given in (2.12). Suppose that c_{M_0} or c_{M_1} (as in Definition 2.8) is represented by c in $ker(l_{W'})$. Then $l_W(c) = 0$ too.*
7) *The intersection forms of W and W' are conjugate by an element of $Gl(\cdot, \mathbb{Z})$.*

In particular, Assertions 5 and 6 of the preceding proposition allow one to prove Theorem 2.9's statements concerning 4- dimensional spin cobordisms with the rational homology of S^3 between a pair of 3-manifolds with the rational homology of S^3 by restricting to the following special case:

Special Case: Let M_0, M_1 be compact, oriented 3-manifolds with the rational homology of S^3. Let W be an oriented, spin cobordism between M_0 and M_1. Assume that W has the rational homology of S^3 and assume that W has a good Morse function f with no index 3 critical points.

(3.11)

The remainder of this subsection proves Proposition 3.2.

Proof of Proposition 3.2. First of all, let W' be the original spin cobordism between M_0 and M_1. Then, surgery on W' will produce an oriented, spin cobordism W which has vanishing first and third Betti numbers. The surgery removes tubular neighborhoods of embedded circles and replace them with copies of $B^2 \times S^2$. (Here, B^2 is the unit ball in $\mathbb{R}^2$.)

Given such W, find a good Morse function f on W and a good pseudo-gradient, v; and then define the complex in (3.5). Label the critical points of index 1 as $\{a_1, \cdots, a_r\}$, label those of index 2 as $\{b_1, \cdots, b_{r+s+t}\}$, and label the index 3 critical points as $\{c_1, \cdots, c_t\}$. Here, $s = \dim(H_2(W;\mathbb{R}))$. (Remember that W has, by assumption, vanishing rational homology in dimensions 1 and 3.) Fix orientations for the descending disks from all of these critical points. With this understood, this set of critical points defines a basis for the complex $\{C_k\}$ in (3.5).

Now, it is convenient to relable the basis for C_2 as follows: Since the map $\partial \oplus \partial^* : C_2 \to C_1 \oplus C_3$ is a surjection, one can relable the critical points $\{b_\alpha\}$ so that

$$\partial : \operatorname{Span}\{b_i\}^r{}_{i=1} \to C_1, \qquad (3.12)$$

$$\partial^* : \operatorname{Span}\{b_{r+s+i}\}_{i=1}^t \to C_3,$$

are both isomorphisms over $\mathbb{Q}$. At the same time, one can require that the projection of Span $\{b_{r+i}\}_{i=1}^s$ onto $C_2/(\partial^* C_1 \oplus \partial C_3)$ is an isomorphism.

With (3.12) understood, one can use 4.1 in [**12**] to find a new good Morse function f_{new} which has the following three properties: First, f_{new} agrees with f outside small neighborhoods of the points in $\operatorname{crit}_2(f)$. Second, f_{new} has the same critical points and pseudo- gradient as f. Third, there exists small $\epsilon > 0$ such that

1) $f_{\text{new}}(\{b_1, \cdots, b_r\}) \in (1/2 - 2\epsilon, 1/2 - \epsilon)$,
2) $f_{\text{new}}(\{b_{r+1}, \cdots, b_{r+s}\}) \in (1/2 - \epsilon, 1/2 + \epsilon)$,
3) $f_{\text{new}}(\{b_{r+s+1} \cdots, b_{r+s+t}\}) \in (1/2 + \epsilon, 1/2 + 2\epsilon)$

(3.13)

Note that (3.12), (3.13) indicate that $W = W_1 \cup W_2 \cup W_3$, where
1) $W_1 \equiv f^{-1}([0, 1/2 - \epsilon])$
2) $W_2 \equiv f^{-1}([1/2 - \epsilon, 1/2 + \epsilon])$
3) $W_3 \equiv f^{-1}([1/2 + \epsilon, 1])$

(3.14)

The boundaries of $W_{1,2,3}$ are compact, oriented 3-manifolds with the rational homology of S^3. This is guaranteed by (3.12). Meanwhile, the inclusion of any boundary component of $W_{1,3}$ into $W_{1,3}$ induces an isomorphism of rational homology. This is not true for W_2; this W_2 has the zero first and third Betti numbers, but the second Betti number of W_2 is equal to s.

Note that the function f_{new} can be used as a Morse function on $W_{1,2,3}$. On W_1, it has no critical points of index 3, on W_3 it has no critical points of index 1, while on W_2, it has only critical points of index 2.

The preceding remarks prove Assertions 1-5. To prove Assertion 6, suppose, for the sake of argument that c_{M_0} is represented by c in $\ker(l_{W'})$. Let ξ be a singular frame for T^*M_0 in the class c and let ξ' be a smooth frame for T^*M_0 which agrees with ξ on the complement of a ball about p_0. Write $T^*W' \mid_{M_0} \approx T^*M_0 \oplus \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$, is the trivial bundle, spanned by $df \mid_{M_0}$. With this understood, ξ' extends to a frame (ξ', df) for $T^*W' \mid_{M_0}$. Note that $l_{W'}(c)$ is the obstruction to extending this frame over W'. Likewise, $l_{W}(c)$ is the obstruction to extending (ζ', df) over W. With this understood remark that Assertion 6 will be proved by demonstrating that (ξ', df) extend over W if it extends over W'. This demonstration requires four steps.

Step 1: Fix a frame e' for T^*W' which extends (ξ', df).

Step 2: Let $\sigma \subset W'$ be an oriented, embedded circle whose fundamental class is a generator of $H_1(W'; \mathbb{Z})/\text{Torsion}$. Suppose a surgery is done on W' to kill the class generated by σ. Such a surgery will replace a tubular neighborhood of σ in W' with $B^2 \times S^3$. Because $\pi_2(SO(3)) = 0$, all framings of $T^*(B^2 \times S^2)$ are mutually homotopic. A framing of $T^*(B^2 \times S^2)$ restricts to the boundary where it can be written as $h\,e'$, where $h \equiv h(e')$ is a map from $S^1 \times S^2$ to $SO(4)$. If h lifts to $SU(2) \times SU(2)$, then the frame (σ', df) will also extend over the manifold which is obtained from W' by surgery on σ.

Step 3: With this last point understood, suppose that h does not lift as required. The strategy is to abandon e' and find a new extension, e'' for (σ', df) so that the resulting $h(e'')$ does lift to $SU(2) \times SU(2)$.

Step 4: To construct e'', let $s : \sigma \to S^1$ be a degree 1 map. Since the restriction map $H^1(W'; \mathbb{Z}) \to H^1(\sigma; \mathbb{Z})$ is surjective, the map s extends as a smooth map from W' to S^1. Since M_0 has vanishing first cohomology, the map s can be taken to map M_0 to point, $1 \in S^1$. Let $j : S^1 \to SO(4)$ be a map which generates $\pi_1(SO(4))$ and which takes 1 to the identity matrix. The composition $k \equiv j \circ s$ maps W' to $SO(4)$ and maps M_0 to the identity. Thus, $e'' \equiv k\,e'$ defines an extension of (ξ', df) over W', and $h(k\,e') = h(e)\,k^{-1}$ will lift to map $S^1 \times S^2$ into $SU(2) \times SU(2)$.

Thus, Assertion 6 follows from this last remark with Step 3.

As for Assertion 7, it is directly a consequence of the fact that W is obtained from W' by surgery on a set of circle generators of $H^1(W'; \mathbb{Z})/\text{Torsion}$. □

e) A basis theorem for the Special Case.

Assume here that W is a cobordism which satisfies the assumptions of (3.11). In particular, W has the rational homology of S^3, and also W has a good Morse function f with only index 1 and index 2 critical points. Fix a good pseudo-gradient v for f.

Introduce the complex in (3.5) for W. This is a 2-step complex, and the boundary map $\partial : C_2 \to C_1$ is an isomorphism over the rationals. Let $\{a_1, \cdots, a_r\}$

label the index 1 critical points and let $\{b_1, \cdots, b_r\}$ label the index 2 critical points. Orient the descending disks from these critical points so that these sets of critical points can be considered as a basis for $C_{1,2}$, respectively.

With the basis for $C_{1,2}$ chosen as above, the boundary maps in (3.5) are simply integer valued matrices. That is, $\partial b_i = \Sigma_j S_{i,,j}\, a_j$, where $S \equiv \{S_{i,,j}\}$ is an integer valued, $r \times r$ matrix.

Here is a useful observation: The precise form of the matrix S is determined by the choice of good pseudo-gradient v. With this fact understood, one can ask whether there is a choice of peudo- gradient for f which gives a "nice" matrix S. The answer to this question is given by Milnor's basis theorem (Theorem 7.6 in [**12**]):

PROPOSITION 3.3. *Let W be a cobordism which satisfies (3.11). Then W has a good Morse function, f, with no index 3 critical points and with the following additional properties: There exists a labeling, $\{a_1, \cdots, a_r\}$ and $\{b_1, \cdots b_r\}$, for the respective index 1 and index 2 critical points of f. And, there exists a good pseudo- gradient for f and a choice of orientations for the descending disks from f's critical points. And, this data is such that*

1) *For all $i \in \{1, \cdots, r\}$,*
2) *$\partial b_i = \Sigma_j S_{i,j}\, a_j$, where $S \equiv \{S_{i,j}\}$ is an upper triangular, integral matrix with positive entries along the diagonal.*
3) *For all $i \in \{1, \cdots, r-1\}$, one has $f(a_i) > f(a_{i+1})$ and $f(b_i) > f(b_{i+1})$.*

(3.15)

The remainder of this subsection is occupied with proving this proposition.

Proof of Proposition 3.3. Start with a good pseudo-gradient, v, for f. Fix orientations for the descending disks so that the boundary operator in (3.5) can be represented as a matrix, T, so that $\partial b_i = \Sigma_j T_{i,j}\, a_j$. Note that the matrix T is integral and invertible over the rationals.

Now, a fundamental result in algebra (see, e.g. [**11**]) states that there exists a unimodular, integral matrix V such that $V\,T \equiv T'$ has only zeros below the diagonal. Let $\underline{b} \equiv \{\underline{b}_i \equiv \Sigma_j V_{i,j}\, b_j\}$. This is a new basis for C_2, and $\partial \underline{b} = V\,T\,a = T'\,a$.

With V and T' understood, appeal to Theorem 7.6 in [**12**] to find a pseudo-gradient for f, v', for which the resulting descending disks represent the basis b for C_2. For this pseudo-gradient, the boundary operator in (3.5) is given by the matrix T'. By changing the orientations of the descending disks if necessary, one can change the signs of the diagonal elements of T' so that they are all positive. Call the resulting matrix S.

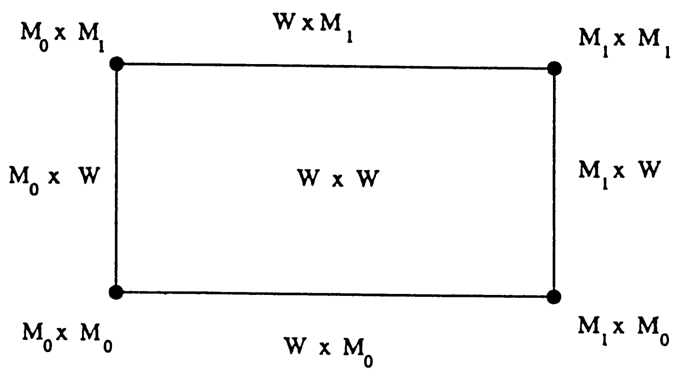
The given arrangement of the critical values of f can be insured by making an appropriate, small perturbation. □

By the way, if the boundary $\partial C_2 \to C_1$ is an isomorphism over Z, then the matrix S in Propostion 3.3 can be taken to be a diagonal matrix.

As a last remark, note that the matrix for the adjoint complex, $\partial^* : C_1 \to C_2$, will be the transpose of the matrix S in Proposition 3.3. This matrix, S^T, will be lower triangular. □

f) Morse theory on $W \times W$

The manifold $W \times W$ is a manifold with boundaries and corners. Here it is:



(3.16)

The reader is invited to formalize a "manifold with boundaries and corners", but the picture above should be self explanatory.

The good Morse function f on W can be used to illuminate (3.16) near the corners. To do so, one must note first that Properties 1 and 2 in (3.1) make it possible to use the pseudo gradient to give W its product structure near ∂W. To be precise, there is a diffeomorphism,

$$\lambda_0 : f^{-1}([0,1/8)) \to M_0 \times [0,1/8) \tag{3.17}$$

which restricts to $f^{-1}(0)$ as the identity and which has $\lambda_0^* f$ given by projection to $[0,1/8)$. There is a corresponding

$$\lambda_1 : f^{-1}((7/8,1]) \to M_1 \times (7/8,1]. \tag{3.18}$$

Using (3.17), a neighborhood of $M_0 \times M_0$ in $W \times W$ is mapped by $\lambda_0 \times \lambda_0$ to

$$(M_0 \times [0,1/8)) \times (M_0 \times [0,1/8)) \approx M_0 \times M_0 \times [0,1/8) \times [0,1/8). \tag{3.19}$$

Of course, $\lambda_0 \times \lambda_1$, $\lambda_1 \times \lambda_0$ and $\lambda_1 \times \lambda_1$ give similar structure to the other corners of $W \times W$.

With a good Morse function, f, chosen for W, introduce the function $F : W \times W \to [-1,1]$ which sends (x,y) to

$$F(x,y) \equiv f(y) - f(x). \tag{3.20}$$

This is a function with properties that are listed in the next lemma. The lemma's statement uses the following notation: First, introduce the projections, $\pi_L : W \times W \to W$ and $\pi_R : W \times W \to W$ which send (x,y) to x and to y, respectively. Second, when v is a vector field on W, introduce the vector fields v_L and v_R on $W \times W$ which are defined so that

(3.21)

1) $d\pi_L\, v_L = v$ and $d\pi_R\, v_L = 0$;
2) $d\pi_L\, v_R = 0$ and $d\pi_R\, v_R = v$.

LEMMA 3.4. *Let f be a good Morse function for W and let v be a good pseudo-gradient for f. Then, the function F of (3.20) has only non- degenerate critical points. Furthermore:*

1) *$crit_n(f) = \cup_k(crit_{4+k-n}(f) \times crit_k(f))$.*
2) *The vector field $v_R - v_L$ is a pseudo-gradient for F which obeys 1- 3 of Definition 3.1.*
3) *The pseudo-gradient $v_R - v_L$ gives the following descending and ascending disks for $(p,q) \in crit_{4+k-n}(f) \times crit_k(f) \subset crit_n(f)$:*

$$B_{(p,q)-} = B_{p+} \times B_{q-}, \tag{3.22}$$

$$B_{(p,q)+} = B_{p-} \times B_{q+}.$$

4) *The pseudo-gradient $v_R - v_L$ is nowhere tangent to a boundary or a corner in (3.16).*

Proof. The proofs of these assertions are left as exercises. But, for Assertion 3, note for example that near $M_0 \times M_0$, $(\lambda_0 \times \lambda_0)^{-1}$ (Of (3.19)) pulls back F to send the point $((x,t),(y,s))$ in $(M_0 \times [0,1/8)) \times (M_0 \times [0,1/8))$ to

$$((\lambda_0 \times \lambda_0)^{-1})^* F((x,t),(y,s)) = s - t. \tag{3.23}$$

Note, by the way, that (3.22) indicates how to orient $B_{(p,q)-}$ given orientations for B_{p-} and B_{q+}. And, with orientations to the descending disks $\{B_{(p,q)-} : (p,q) \in \text{crit}(F)\}$, one can consider the analog to the chain complex C in (3.5) as constructed for $W \times W$ using the function F and the pseudo-gradient $v_R - v_L$. The following lemma describes the homology of this complex. □

LEMMA 3.5. *The analog of the chain complex C in (3.5) as constructed for $W \times W$ using F and the pseudo-gradient $v_R - v_L$ gives a chain complex, C^F, which is canonically isomorphic to $C^* \otimes C$, where C^* is the complex in (3.10). The homology of the complex C^F is canonically isomorphic to $H*(W \times W, (W \times M_0) \cup (M_1 \times W); \mathbb{Z})$.*

Notice that the relative homology above is that of the square in (3.16) relative to the union of its bottom and right sides.

Proof. This follows from Lemma 3.4 and (3.22). □

g) The space Z.

As outlined in Section $2k$, the first step to proving Theorem 2.9 is to construct an oriented, 7-dimensional manifold Z whose boundary is the disjoint union of $M_0 \times M_0, M_1 \times M_1$ and some number of copies of $S^3 \times S^3$. The purpose of this subsection is to construct such a Z using the cobordism W and a good Morse function f on W. To begin, construct F from f as in (3.16). Use F to define

$$\underline{Z} \equiv F^{-1}(0) = \{(x,y) \in W \times W : f(x) = f(y)\}. \tag{3.24}$$

This subspace $\underline{Z}$ plays a central role in subsequent parts of the story, and the purpose of this subsection is to describe some of $\underline{Z}$'s properties.

To begin, note that both $M_0 \times M_0$ and $M_1 \times M_1$ lie in $\underline{Z}$ since f is constant on M_0 and also on M_1. Near these corners, $\underline{Z}$ is a manifold with boundary given by the disjoint union of $M_0 \times M_0$ and $M_1 \times M_1$. See (3.19).

Unfortunately, $\underline{Z}$ is not a manifold everywhere unless f has no critical points. This is because 0 is not a regular value of the function F. Fortunately, the singularities of $\underline{Z}$ are not hard to describe; they occur at the points of $\text{crit}(F) \cap \underline{Z}$, that is, points of the form $(p,p) \subset W \times W$ where $p \in \text{crit}(f)$. (Remember that the critical points of f are assumed to have distinct critical values.) Furthermore, the neighborhoods of these critical points are relatively easy to describe. The picture is given in the following lemma. The lemma introduces the notion of a *cone* on a manifold N. This is the space which is obtained by taking $[0,1) \times N$ and crushing $\{0\} \times N$ to a point.

LEMMA 3.6. *Let f be a good Morse function on W. Let $p \in crit_k(f)$. Then, a neighborhood of (p,p) in $\underline{Z}$ is naturally isomorphic to the cone on $S^3 \times S^3$. In fact with ψ_p and $U_p = \psi_p(\mathbb{R}^4)$ as in (3.2), then the map $(\psi_p \times \psi_p)^{-1}$maps $\underline{Z} \cap (U_p \times U_p)$ to a subset of $\mathbb{R}^4 \times \mathbb{R}^4$ which intersects a ball neighborhood of $(0,0)$ as the set of (x,y) which obey*

$$y_1^2 + \cdots + y_k^2 + x_{k+1}^2 + \cdots + x_4^2 = x_1^2 + \cdots + x_k^2 + y_{k+1}^2 + \cdots + y_4^2. \tag{3.25}$$

Warning: *As indicated by (3.25), the cone on $S^3 \times S^3$ here is not induced by the obvious product structure on $W \times W$. The product structure which induces (3.25) is the product structure in*

(3.26) $$B_{(p,p)-} \times B_{(p,p)+},$$

with $B_{(p,p)\pm}$ as in (3.22).

Proof. Equation (3.25) is an immediate consequence of (3.2). □

The manifold (with boundary) Z in Section 2k will be found inside $\underline{Z}$; it is obtained by excising from $\underline{Z}$ a small ball about each of the singular points (p,p) for $p \in \text{crit}(f)$. More precisely, one fixes some small $r > 0$. Then, the intersection of Z with $U_p \times U_p$ is mapped by $\psi_p \times \psi_p$ to the set of (x,y) which obey

(3.27) $$| x |^2 + | y |^2 \geq r.$$

With (3.27) understood, $\partial Z \cap U_p \times U_p$ is mapped by $\psi_p \times \psi_p$ to the set of (x,y) which obey

1) $y_1^2 + \cdots + y_k^2 + x_{k+1}^2 + \cdots + x_4^2 = r,$
2) $x_1^2 + \cdots + x_k^2 + y_{k+1}^2 + \cdots + y_4^2 = r.$

(3.28)

As the precise value of r here is immaterial (as long as r is small), the precise value will not be specified.

There is an alternative approach to defining Z. Here, Z is a "blow up" of $\underline{Z}$ at the points of the form $(p,p) \in \text{crit}(f)$. In this case, Z maps to $\underline{Z}$ by a map π. Each point in $\underline{Z} - \{(p,p) \in \text{crit}(F)\}$ has a single point in its inverse image. But, the inverse image of any point $(p,p) \in \text{crit}(F)$ is the corresponding $S^3 \times S^3 \subset \partial Z$. This blow up corresponds to resolving the cone point in $\underline{N} \equiv ([0,1) \times N)/(\{0\} \times N)$ with the tautological projection $\pi : [0,1) \times N \to N$.

h) Properties of Z.

With Z now defined, here are its salient features:

A manifold: Z is a manifold with boundary,

(3.29) $$\partial Z = (M_0 \times M_0) \cup (M_1 \times M_1) \cup_{p \in \text{crit}(f)} (S^3 \times S^3)_p.$$

Orientation: The manifold $\text{int}(Z)$ has a natural orientation. Indeed, $W \times W$ has a natural orientation. Then, $\text{int}(Z) \subset F^{-1}(0)$ is open, and $dF \neq 0$ on $\text{int}(Z)$, so the 1 form dF trivializes the normal bundle to $\text{int}(Z) \subset W \times W$. This serves to orient Z. The induced orientation on $M_1 \times M_1 \subset \partial Z$ agrees with its canonical orientation, but the induced orientation on $M_0 \times M_0 \subset \partial Z$ disagrees with the canonical orientation.

To orient $(S^3 \times S^3)_p$, use the inclusion of $W \approx \Delta_W \subset W \times W$ to orient Δ_W and hence Δ_Z. The boundary of Δ_Z intersects $(S^3 \times S^3)_p$ as $\Delta_{S^3}(\equiv (\Delta_{S^3})_p)$ Give $(\Delta_{S^3})_p$ the induced orientation from Δ_Z. Then, orient the left factor of S^3 in $(S^3 \times S^3)_p$ so that the composition of $\pi_L : \Delta_{S^3} \to S^3$ and then the inclusion $S^3 \to (S^3 \times \text{ point}) \subset (S^3 \times S^3)_p$ is orientation preserving. Orient the right factor analogously and use the product orientation to orient $(S^3 \times S^3)_p$. (Remark that the induced orientation on $(S^3 \times S^3)_p$ as a boundary component of Z agrees with this orientation if index p is odd, and it disagrees if index p is 2.)

Homology: The rational homology in dimensions 0-3 of Z is of some concern in subsequent sections. Consider

LEMMA 3.7. *Suppose that W has the rational homology of S^3. Then the rational homology of Z is as follows:*

1) $H_0(Z) \approx \mathbb{R}$.

2) $H_1(Z) \approx H_2(Z) \approx 0$.

3) There is a surjection

$$0 \leftarrow H_3(Z) \leftarrow L_- \oplus L_+ \oplus H_3(\partial Z). \tag{3.30}$$

Here L_- is freely generated over $\mathbb{R}$ by

$$\underline{L}_- \equiv \{B^4_{(p,q)-} \cap Z : (p,q) \in crit_4(F) \text{ and } F(p,q) > 0\}, \tag{3.31}$$

while L_+ is freely generated over $\mathbb{R}$ by

$$\underline{L}_+ \equiv \{B^4_{(p,q)+} \cap Z : (p,q) \in crit_4(F) \text{ and } F(p,q) < 0\}, \tag{3.32}$$

Note that the intersections which define $\underline{L}_\pm$ in (3.31), (3.32) are all embedded 3-spheres.

Also note that the inclusion of Z in $W \times W$ gives an isomorphism on π_1 and π_2.

Proof. Note first that $H_{0,1,2}(Z)$ and $H_{0,1,2}(\underline{Z})$ agree, and that

$$H_3(Z) \approx H_3(\underline{Z}) \oplus_{p \in \text{crit}(f)} H_3((S_3 \times S_3)_p) \tag{3.33}$$

This follows using Meyer-Vietoris for the cover of $\underline{Z}$ by the union of Z and the cones on the $(S^3 \times S^3)_p$ in ((3.25).

Next, pick $\epsilon > 0$, but small so that F has only critical points of the form (p, p) in $F^{-1}((-\epsilon, \epsilon))$. Let $V \equiv F^{-1}((-\epsilon, \epsilon))$ observe that V strongly deformation retracts into $\underline{Z}$. Thus, $H_i(V) \approx H_i(\underline{Z})$.

To compute $H_i(V)$, observe that $W \times W$ can be constructed from V by a sequence

$$V \equiv V_3 \subset V_4 \subset V_5 \subset V_6 \equiv W \times W, \tag{3.34}$$

where V_{k+1} is obtained from V_k by the attachment of disjoint handles, $(B^k \times B^{8-k})$'s, on disjointly embedded $(S^{k-1} \times B^{8-k})$'s in the boundary of V_k.

To be precise, V_4 contains all of F's index 4-critical points,

$$V_4 \equiv F^{-1}([-1/8, 1/8]); \tag{3.35}$$

and V_5 contains all index $3, 4$, and 5 critical points,

$$V_5 \equiv F^{-1}([-3/8, 3/8]). \tag{3.36}$$

The attaching 3-spheres for the handles that change V_3 to V_4 are given by (3.31), (3.32). Meanwhile, the attaching 4-spheres for the handles that change V_4 to V_5 are

$$\{B_{(p,q)-} \cap F^{-1}(1/8)\}_{(p,q)\in \mathrm{crit}_5(F)} \cup \{B_{(p,q)+} \cap F^{-1}(-1/8)\}_{(p,q)\in \mathrm{crit}_3(F)}. \tag{3.37}$$

The 5-spheres for the attachments that change V_5 to V_6 should be obvious.

The resulting Meyer-Vietoris sequences from (3.34) read, in part,

$$H_3(\underline{L}_+) \oplus H_3(\underline{L}_-) \to H_3(V_3) \to H_3(V_4) \to 0, \quad H_3(V_4) \approx H_3(V_5) \approx H_3(V_6). \tag{3.38}$$

The third assertion in Lemma 3.7 follows from (3.38) and (3.33). The other assertions follow by Meyer-Vietoris from (3.34)-(3.37). □

4 Homological constraints. In this section, M_0 and M_1 will both be oriented, 3- dimensional manifolds with the rational homology of S^3. And, W will be an oriented, connected, spin cobordism between M_0 and M_1. Let $f : W \to [0,1]$ be a good Morse function. Use f to construct the space Z as described in Sections $3g$ and $3h$. The proof of Theorem 2.9 is a five step affair which is outlined in Section $2k$. The manifold (with boundary) Z of Sections $3g, h$ realizes the first step in the proof. The next step in the proof is to construct a subvariety $\Sigma_Z \subset Z$ with various properties as outlined in Steps 2 and 3 of Section $2k$. The purpose of this section is to reformulate some of these requirements in a purely homological way.

a) The homology of Σ_M and $M \times M$.

In order to understand the homological constraints on Σ_Z, it proves useful to digress first with a homological interpretation of some of the constructions in Section 2. Return then to the milieu of Section 2 where M is a compact, oriented 3-manifold with the rational homology of S^3 and where $\Sigma_M \subset M \times M$ is defined by (2.1).

The inclusion $\Sigma_M \subset M \times M$ induces a surjective homomorphism on the respective rational homology groups in dimension 3, with a one dimensional kernel.

$$\sigma_M \equiv [\Delta_M] - [p_0 \times M] - [M \times p_0] \tag{4.1}$$

This σ_M bounds (rationally) in $M \times M$, and a bounding cyle defines a class,

$$\rho_M \in H_4(M \times M, \Sigma_M). \tag{4.2}$$

(Here, $H.(X, Y)$ for a space X and subspace $Y \subset X$denotes the relative homology with rational coefficients.) The Poincare dual of ρ_M is the generator of $H_2(M \times M - \Sigma_M)$ which figures so prominently in Section 1. End the digression.

b) Homological constraints on Σ_Z from ω_Z.

Return to the bordism milieu of the introduction. The subvariety Σ_Σ should have a physical boundary (as a cycle, for example) which is given by

$$\partial\Sigma_Z = \Sigma_{M_1} \cup \Sigma_{M_2} \cup_{p \in \mathrm{crit}(f)} (\Sigma_{S^3})_p, \tag{4.3}$$

where $(\Sigma_{S^3})_p$ is the obvious Σ_{S^3} in the boundary component $(S^3 \times S^3)_p$ of Z. Finding Σ_Z to satisfy (4.3) would satisfy Step 2 in Section $1h$.

However, there are certain cohomological constraints on a solution to (4.3) which must be satisfied before it can solve the constraints which are implicit in Step 3 of Section $2k$, and in particular, Parts 1 and 2 of (2.27). These are expressed by the following lemma:

LEMMA 4.1. *Let $\Sigma_Z \subset Z$ be a subvariety which obeys (4.3). Then, there is a closed 2-form, ω_Z, on $Z - \Sigma_Z$ which restricts to any component $Y \subset \partial Z - \partial\Sigma_Z$ to generate $H^2(Y)$ if and only if $H_4(\Sigma_Z, \partial\Sigma_Z)$ contains a class σ_Z which obeys:*

1) *The image of σ_Z in $H_4(Z, \partial Z)$ is zero.*
2) *$\partial\sigma_Z$ in $H_3(\partial\Sigma_Z)$ obeys*

$$\partial\sigma_Z = \sigma_{M_1} - \sigma_{M_0} + \sum_{p \in crit(f)} (\sigma_{S^3})_p. \tag{4.4}$$

(The absence of signs in the last term in (4.4) stems from the convention of Section 3h for orienting the right and left factors of S^3 in the boundary component $(S^3 \times S^3)_p \subset \partial Z$.)

The third constraint in (2.27) is the most difficult of all to satisfy. The strategy for satisfying the third constraint in (2.27) has two parts, one homological and the other geometric. For both parts, fix $N_Z \subset Z$, a regular neighborhood of Σ_Z. The homological issue is to characterize a closed 2-form on $N_Z - \Sigma_Z$ which is the restriction from $Z - \Sigma_Z$ of a closed 2-form ω_Z from Lemma 4.1. The geometric issue is to find such an ω which obeys $\omega \wedge \omega = 0$.

The following lemma resolves the homological issue:

LEMMA 4.2. *Suppose that Conditions 1 and 2 of Lemma 4.1 are obeyed. Let $N_Z \subset Z$ be a regular neighborhood of Σ_Z. A closed 2-form, ω, on $N_Z - \Sigma_Z$ is the restriction to $N_Z - \Sigma_Z$ of a closed 2-form ω_Z on $Z - \Sigma_Z$ as described in Lemma 4.1 if the following occur:*

1) *The connecting homomorphism from $H^2(N_Z - \Sigma_Z)$ to $H^3_{comp}(N_Z)$ sends ω to a multiple of the Poincare dual of $\sigma_Z \in H_4(N_Z, N_Z \cap \partial Z)$.*
2) *The restriction homomorphism $H^2(Z) \to H^2(\Sigma_Z)$ is surjective.*

This lemma is proved below. The last subsection in this section discusses the strategy for finding an appropriate ω near Σ_Z with $\omega \wedge \omega = 0$.

Proof of Lemma 4.1. To prove necessity, start with the observation that the cohomology class in $H^2(Z - \Sigma_Z)$ of the 2-form in question has Poincare dual

$$\rho_Z \in H_5(Z, \Sigma_Z \cup \partial Z). \tag{4.5}$$

The requirements in (2.27.1) and (2.27.2) concerning the restriction of ω_Z to ∂Z imply the following homological condition on $\partial \rho_Z$

$$\partial \rho_Z = \rho_{M_1} - \rho_{M_2} + \sum_{p \in \mathrm{crit}(f)} (\rho_{S^3})_p - \sigma_Z, \tag{4.6}$$

where

$$\sigma_Z \in H^4(\Sigma_Z, \partial \Sigma_Z) \tag{4.7}$$

is a class which obeys (4.4) (so that $\partial^2 \rho_Z$ will vanish).

To prove the sufficiency assertion of the lemma, start with ρ_Z as described. Represent σ_Z as a cycle on Σ_Z. By assumption, one has $\sigma_Z - \tau = \partial \rho_Z$, where τ is a 4-cycle on ∂_Z, and where ρ_Z is a 5-cycle on Z. Note that $\partial \tau$ is equal to the right side of (4.6) also. Thus,

$$\tau - (\rho_{M_1} - \rho_{M_2} + \sum_{p \in \mathrm{crit}(f)} (\rho_{S^3})_p) \tag{4.8}$$

has zero boundary, and so defines a class in $H_4(\partial Z)$. However, this group is zero ($H_4(\partial Z) \approx H^2(\partial Z) = 0$ (see Section 2). Thus, (4.6) holds for some 5-cycle

ρ_Z on Z. The Poincare dual of ρ_Z is a class in $H^2(Z - \Sigma_Z)$ with the required properties. □

Proof of Lemma 4.2. The question of extending a closed 2-form on $N_Z - \Sigma_Z$ over $Z - \Sigma_Z$ is described by part of the Meyer-Vietoris sequence for the cover of Z by $(Z - \Sigma_Z) \cup N_Z$. The relevent part is:

$$H^2(Z) \to H^2(Z - \Sigma_Z) \oplus H^2(\Sigma_Z) \to H^2(N_Z - \Sigma_Z) \to H^3(Z) \tag{4.9}$$

The last arrow in (4.9) factors through the inclusion induced map $H^3_{\text{comp}}(N_Z) \to H^3(Z)$. So, if the image of ω in $H^3_{\text{comp}}(N_Z)$ is Poincare dual to a multiple of σ_Z as a class in $H_4(N_Z, N_Z \cap \partial Z)$, then the image of ω in $H^3(Z)$ is zero if the image of σ_Z in $H_4(Z, \partial Z)$ is zero. This is the first condition in Lemma 4.1. Thus, under Condition 1 of Lemma 4.2, the class ω maps to zero in $H^3(Z)$.

When Condition 2 of Lemma 4.2 holds, then ω must be in the image of the restriction homomorphism from $H^2(Z - \Sigma_Z)$ because of the exactness of (4.9). □

c) Satisfying Lemma 4.1's constraints.

The second constraint in Lemma 4.1 will be satisfied by construction; as it is essentially a restatement of (4.3) with orientations taken into account. The first constraint in Lemma 4.1 is more subtle. Here is a strategy for finding a solution: The variety Σ_Z will be constructed from a union of varieties,

$$\Sigma_Z = \Delta_Z \cup E_L \cup E_R \cup E_- \cup E_+. \tag{4.10}$$

Each variety on the right side of (4.10) will carry a fundamental class. (Here, a variety is a union of embedded submanifolds. If the constituent submanifolds are oriented, then the variety has a fundamental class which is the sum (in the relevent homology group) of the fundamental classes of the constituent submanifolds.) And, for a particular integer $N > 0$, the class σ_Z will be given as

$$\sigma_Z = [\Delta_Z] - [E_L] - [E_R] - N^{-1}[E_-] - N^{-1}[E_+]. \tag{4.11}$$

In (4.10), (4.11), Δ_Z and $E_{L,R}$ are honest submanifolds; these will be defined in subsequent subsections. Meanwhile, $E_\pm$ will be honest varieties unless $N = 1$ in (4.11). The construction of $E_\pm$ is quite lengthy and starts in the next section with the completion in Section 10. But, see subsections 4f, g below.

With (4.11) understood, the first constraint of Lemma 4.1 will be solved with the help of Lemma 4.3, below. (The statement of this lemma reintroduces $\underline{L}_\pm$ from (3.31), (3.32).)

LEMMA 4.3. *Suppose that W has the rational homology of S^3. Let $V \subset Z$ be a union of dimension 4 submanifold with boundary such that $\partial V \subset \partial Z$. Suppose* *that each component of V carries a fundamental class. Then $[V] \in H_4(Z, \partial Z; \mathbb{R})$ vanishes if:*

1) $[\partial V] = 0$ in $H_3(\partial Z; \mathbb{R})$.

2) V has zero intersection number with any component $x \subset (\underline{L}_- \cup \underline{L}_+)$.
(The intersection number of V with an embedded, 3-dimensional submanifold of Z is defined to be the sum of the intersection numbers of the components of V.)

Proof. Poincare duality equates $H_4(Z, \partial Z)$ with $H^3(Z)$. Intersection theory makes this explicit, as the intersection pairing between $H_4(Z, \partial Z)$ and $H_3(Z)$ becomes, under Poincare duality, the dual pairing between $H^3(Z)$ and $H_3(Z)$. Now, use this fact with Assertion 3 of Lemma 3.7. □

d) The subspace Δ_Z Let $\Delta_W \subset W \times W$ denote the diagonal. Clearly, $\Delta_W \subset$

Z. Let Δ_Z denote the intersection of Δ_W with $Z \subset \underline{Z}$. (Alternately, if Z is thought of as the blow up of $\underline{Z}$, then Δ_Z can be defined as the inverse image of Δ_W under this blow up.) Note that Δ_Z is a submanifold with boundary in W, and

$$\partial \Delta_Z = \Delta_{M_0} \cup \Delta_{M_1} \cup_{p \in \mathrm{crit}(f)} (\Delta_{S^3})_p. \tag{4.12}$$

The orientation of W defines an orientation for Δ_W and thus for Δ_Z. The orientation of $(\Delta_{S^3})_p$ is induced from the orientation of Δ_Z in Section $3h$ as a boundary component of Δ_Z. With this understood, one has:

LEMMA 4.4. *Let $[\Delta_Z] \in H_4(Z, \partial Z)$ denote the fundamental class of Δ_Z. Then*

$$\partial[\Delta_Z] = -[\Delta_{M_0}] + [\Delta_{M_1}] + \sum_{p \in crit(f)} [(\Delta_{S^3})_p]. \tag{4.13}$$

as a class in $H_3(\partial Z)$.

Proof. This is left as an exercise.
As a final remark, note that

$$\Delta_Z \cap (L_- \cup L_+) = \emptyset. \tag{4.14}$$

This is a consequence of Condition 1 in Definition 3.1. □

e) The submanifolds $E_{R,L}$. By assumption (see 4 of Definition 3.1), there is a gradient flow line for the pseudo-gradient v which starts at p_0 and which ends at p_0. Let γ denote this line. Define

1) $E_R \equiv (\gamma \times W) \cap Z,$
2) $E_L \equiv (W \times \gamma) \cap Z.$

(4.15)

Here are the properties of these spaces:

LEMMA 4.5. *Both E_R and E_L are embedded submanifolds (with boundary) of Z. Also,*

1) $\partial E_R = (p_0 \times M_0) \cup (p_1 \times M_1)$.

2) $\partial E_L = (M_0 \times p_0) \cup (M_1 \times p_1)$.

3) Let π_L and π_R denote the respective right and left factor projections from $W \times W$ to W. Then $\pi_R : E_R \to W$ and $\pi_L : E_L \to W$ are both diffeomorphisms.

4) $E_R \cap \Delta_Z = E_L \cap \Delta_Z = E_R \cap E_L = (\gamma \times \gamma) \cap \Delta_Z$. *Furthermore, this subspace $(\gamma \times \gamma) \cap \Delta_Z$ has a neighborhood $U \subset Z$ with a diffeomorphism (of manifold with boundary) $\psi_U : U \approx [0,1] \times \mathbb{R}^3 \times \mathbb{R}^3$ which obeys*

(a) $\psi_U((\gamma \times \gamma) \cap \Delta_Z) = [0,1] \times (0,0)$.

(b) $\psi_U(E_R) = [0,1] \times \{0\} \times \mathbb{R}^3$.

(c) $\psi_U(E_L) = [0,1] \times \mathbb{R}^3 \times \{0\}$.

(d) $\psi_U(\Delta_Z) = [0,1] \times \Delta_{\mathbb{R}^3}$.

(e) $\psi_U(M_0 \times M_0) = \{0\} \times \mathbb{R}^3 \times \mathbb{R}^3$.

(f) $\psi_U(M_1 \times M_1) = \{1\} \times \mathbb{R}^3 \times \mathbb{R}^3$.

(g) The interchange map $(z, z') \to (z', z)$ on Z is mapped by ψ_U to $(t, x, y) \to (t, y, x)$.

5) Both E_R and E_L have empty intersection with the components of $\underline{L}_- \cup \underline{L}_+$ of (3.30), (3.31).

6) Orient E_R and E_L by π_L and π_R, respectively. Then

$$\partial[E_R] = -[p_0 \times M_0] + [p_1 \times M_1], \tag{4.16}$$

$$\partial[E_L] = -[M_0 \times p_0] + [M_1 \times p_1].$$

The remainder of this subsection is occupied with the proof of this lemma.

Proof. Since γ is a flow line of v, it has a parametrization

$$\underline{\gamma} : [0,1] \to W \tag{4.17}$$

with $(\underline{\gamma}^* f)(t) = t$. This implies that the function F of (3.16) restricts without critical points to $\gamma \times W$ and to $W \times \gamma$; therefore, both E_R and E_L are submanifolds of Z.

Assertions 1 and 2 of the lemma follow because γ is assumed to miss crit(f).

To prove the third assertion, use $\underline{\gamma}$ to view E_R as the graph of f in $[0,1] \times W$, where π_R restricts as the obvious projection to W. The proof of Assertion 3 for E_L is analogous.

To prove Assertion 4, note that γ, being embedded, has a neighborhood $U_\gamma \subset W$ with a diffeomorphism $\psi_\gamma : U_\gamma \to [0,1] \times \mathbb{R}^3$ which obeys $f(\psi_\gamma^{-1}(t,x)) = t$ and $\psi_\gamma(\underline{\gamma}(t)) = (t,0)$. (Use the implicit function theorem to construct such a ψ_γ .) Take $U = U_\gamma \times U_\gamma$ and take $\psi_U = \psi_\gamma \times \psi_\gamma$. The verification of (a)-(g) follow immediately. □

f) The varieties $E_\pm$.

With Δ_Z and $E_{R,L}$ defined in the preceding section, the solution σ_Z of (4.11) to Lemma 4.1's constraints is missing still $[E_-]$ and $[E_+]$. Indeed, the class $[\Delta_Z] - [E_R] - [E_L]$ is a class in $H_4(Z, \partial Z)$ whose boundary is equal to

$$-\sigma_{M_0} + \sigma_{M_1} + \sum_{p \in \mathrm{crit}(f)} [\Delta_{S^3}]_p, \tag{4.18}$$

which is only a part of the right side of (4.4).

As remarked earlier, the construction of $E_\pm$ is quite lengthy. To simplify matters, the decomposition given in Proposition 3.2 will be invoked to break the discussion into two parts so that the cobordism W can be assumed to obey the conditions of (3.11). That is, W will be assumed to have the rational homology of S^3 and W has a good Morse function with no index 3 critical points.

The construction of $E_\pm$ for W given by (3.11) is started in the next section with a digression to describe certain constructions on such W. The construction of $E_\pm$ for (3.11) is completed in Section 9.

With W understood to be given by (3.11), here is a rough description of $E_\pm$: Fix a good Morse function $f : W \to [0,1]$ with no index 3 critical points. Let $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ label the index 1 and index 2 critical points of f.

Now fix a good pseudo-gradient, v, for f, and fix orientations from the descending disks from crit(f) such that the conclusions of Proposition 3.3 hold. That is, with the orientations implicit, the points $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ define a basis for C_1 and C_2, respectively. And, with respect to this basis, the boundary map in (3.5), $\partial : C_1 \to C_2$, is represented by an upper triangular matrix, S, with positive entries on the diagonal.

A pair $E_\pm$, of subvarieties (with boundary) of Z will be constructed with $\partial E_\pm \subset \partial Z$. The variety E_- is obtained as the intersection with Z of a subvariety of $W \times W$; this subvariety is constructed by performing various surgeries on multiple copies of products of the ascending disks from points in crit$_1(f)$ with the descending disks from the points in crit$_2(f)$. Meanwhile, the variety E_+ is obtained as the intersection with Z of a different subvariety of $W \times W$. In this case, the subvariety is constructed by surgery on multiple copies of the product of the descending disks from crit$_2(f)$ with the ascending disks from crit$_1(f)$.

The varieties $E_\pm$ will be naturally oriented and seen to define classes $[E_\pm] \subset H_4(Z, \partial Z)$. The boundaries of these classes are

$$\partial[E_-] = N \sum_{p \in \mathrm{crit}(f)} [S^3]_{p^-}, \tag{4.19}$$

$$\partial[E_+] = N \sum_{p \in \mathrm{crit}(f)} [S^3]_{p^+},$$

where (4.19) has introduced the following shorthand: When p is a critical point of f, use $[S^3]_{p^-}$ to denote $[S^3 \times \text{point}] \in H_3((S^3 \times S^3)_p)$, and use $[S^3]_{p^+}$ to denote $[\text{point} \times S^3] \in H_3((S^3 \times S^3)_p)$,. Here, the orientations on $(S^3 \times \text{point})$ and $(\text{point} \times S^3)$ are defined in Section 3h. (The diagonal in $(S^3 \times S^3)_p$ is oriented as a component of the boundary of Δ_Z and then the right and left factors of S^3 in $(S^3 \times S^3)_p$ are oriented by using the canonical identification of S^3 with Δ_{S^3}.)

The $[E_\pm]$, of (4.19) will be constructed to have zero intersection pairing with the classes in $L_\pm$ of (3.30). This will insure that σ_Z of (4.11) satisfies both requirements of Lemma 4.1. (See Lemma 4.3.)

g) Constraints from $\omega_Z \wedge \omega_Z = 0$.

With Σ_Z in (4.10) constructed so that both requirements of Lemma 4.1 are satisfied, there is a 2-form on $Z - \Sigma_Z$ which is a candidate for the form ω_Z in Step 3 of Theorem 2.9's proof.

The issue then arises as to whether σ_Z can be found to satisfy the conditions in (2.27). The construction of a closed 2-form which satisfies the conditions of (2.27) is carried out in Section 10. However, to motivate some of the intervening contortions, here is a rough summary of the difficulties:

Remark 1: As long as $E_\pm$ in (4.10) have empty intersection with $M_0 \times M_0$ and with $M_1 \times M_1$, then there is no obstruction to finding ω_Z which obeys (2.27.1). (See Lemma 2.1)

Remark 2: The remaining requirements of (2.27) are harder to satisfy. In particular, the second requirement in (2.27) will require that for each $p \in \mathrm{crit}(f)$,

(4.20)

1) $E_- \cap (S^3 \times S^3)_p = S^3 \times x_p$
2) $E_+ \cap (S^3 \times S^3)_p = x_p \times S^3$.

This requirement and (4.19) are incompatible unless $N = 1$ or unless $E_\pm$ are singular. Together, (4.19), (4.20) force the use *subvarieties* for $E_\pm$ instead of submanifolds.

Given (4.20), the second constraint in (2.27) can also be satisfied. (See Lemma 2.1 again.)

Remark 3: The first condition of Lemma 4.2 is not easy to satisfy with a 2-form ω which obeys $\omega \wedge \omega = 0$. In the case where $N = 1$ in (4.11) (so $E_\pm$ are manifolds) the strategy will be to find a regular neighborhood $N_Z \subset Z$ of Σ_Z and a map

$$\varphi_Z : N_Z \to \mathbb{R}^3 \tag{4.21}$$

which obeys $\varphi_Z^{-1}(0) = \Sigma_Z$ and which pulls back the generator of $H^3_{\text{comp}}(\mathbb{R}^3)$ to a non-zero multiple of the Poincare dual in $H^3_{\text{comp}}(N_Z)$ to $\sigma_Z \in H_4(N_Z, N_Z \cap \partial Z)$. In this case,

$$\omega_Z \equiv \varphi_Z^{-1}(\mu) \tag{4.22}$$

with μ as in (2.3).

In the case where $N > 1$ in (4.11), the preceding strategy will be modified. When $N > 1$ in (4.11), then φ_Z, as in (4.21), will be defined only in a neighborhood of $\Delta_Z \cup E_L \cup E_R \subset \Sigma_Z$, and ω_Z will be defined near $\Delta_Z \cup E_L \cup E_R$ by (4.22). But, near the remainder of Σ_Z (i.e. near most of $E_\pm$), the form ω_Z will be defined somewhat differently. (The basic difference being that ω_Z will be defined *locally* as the pull-back of a closed 2-form from a space of dimension less than 4. However, the space in question will not always be S^2. In some places, the space will be the compliment in S^3 of $N + 1$ distinct points.)

This strategy for constructing a closed, square zero solution to Condition 1 of Lemma 4.2 requires $\Delta_Z, E_{L,R}$ and the constituent submanifolds of $E_\pm$ to have trivial normal bundles in Z. (See Remark 4, below.) The success of this strategy also requires that the mutual intersections of $Z, E_{L,R}$ and $E_\pm$ have a canonical form. (See Remark 5, below.)

Remark 4: The normal bundle of Δ_Z in Z is trivial if and only if Z is a *spin* manifold. Indeed, $H^4(\Delta_Z) = 0$ and therefore, an oriented 3-plane bundle over Z is classified by its 2nd Stieffel-Whitney class. Since Z has trivial normal bundle in $W \times W$, the Stieffel-Whitney classes of the normal bundle to Δ_Z in Z are the same as those of the normal bundle to Δ_Z in $W \times W$. The latter are the restrictions of the Stieffel-Whitney classes of the normal bundle of Δ_W in $W \times W$. And, this last normal bundle is naturally isomorphic to the tangent bundle of W. Finally remember that W is said to be spin if the 2nd Stieffel-Whitney class of its tangent bundle vanishes.

The normal bundles to $E_{L,R}$ are trivial, since they are isomorphic to the normal bundle to the path γ in W.

A constituent submanifold of E_+ (or E_-) has a normal bundle in Z. If care is taken in constructing $E_\pm$, then these normal bundles will be trivial too.

Remark 5: The construction of a square zero ω to satisfy the first condition of Lemma 4.2 seems to require that $E_\pm$ do not intersect each other or Δ_Z and $E_{L,R}$ in a complicated way. Infact, the $E_\pm$ that are finally constructed will have empty intersections with $E_{L,R}$, while

$$E_- \cap E_+ = E_- \cap \Delta_Z = E_+ \cap \Delta_Z = \cup_{i=1}^r v_i, \tag{4.23}$$

where $\{\epsilon_i\}_{i=1}^r$ are disjoint, embedded paths in Δ_Z. In fact, fix the label $i \in \{1, \cdots, r\}$ and let $a \equiv a_i$ and $b \equiv b_i$ be the i'th pair of index 1 and index 2 critical points of the Morse function f as described by Proposition 3.3. Then, v_i

will start at the point $(x_a, x_a) \in (S^3 \times S^3)_a$ and will end at the point $(x_b, x_b) \in (S^3 \times S^3)_b$.

Assertion 4 of Lemma 4.5 describes the intersections amongst $E_{L,R}$ and Δ_Z. Assertion 4 of Lemma 4.5 and (4.23) (with some conditions on normal bundle framings) insure that the intersection of $E_\pm$ with Δ_Z has the appropriate form for the construction of a square zero ω to satisfy the first condition of Lemma 4.2.

Remark 6: The second condition in Lemma 4.2 will be satisfied by taking care to construct $E_\pm$ to have vanishing H^2. Note that Z, Δ_Z and $E_{L,R}$ all have vanishing H^2. (See Lemmas 3.7, 4.4 and 4.5, respectively).

Care must also be taken to insure that $E_\pm$ do not intersect each other or Δ_Z and $E_{L,R}$ in a complicated way. Infact,

LEMMA 4.6. *Suppose that Σ_Z is given by (4.10) with Δ_Z and $E_{L,R}$ as described in Sections 4d and 4e, respectively. Suppose that $E_\pm \subset Z$ are varieties which have empty intersection with $E_{L,R}$ and which intersect each other and Δ_Z as described in (4.23). Suppose, in addition, that $H^2(E_\pm;\mathbb{Q}) = 0$. Then $H^2(\Sigma_Z;\mathbb{Q}) = 0$ and the homomorphism $H^2(Z;\mathbb{Q}) \to H^2(\Sigma_Z;\mathbb{Q})$ of Lemma 4.2 is surjective by default.*

Proof. Because the intersections of $\Delta_Z, E_{L,R}$ and $E_\pm$ with each other are a union of line segments (which have vanishing H^1), the Meyer-Vietoris exact sequence shows that $H^2(\Sigma_Z)$ is isomorphic to the direct sum of $H^2(\cdot)$ for $(\cdot) = \Delta_Z, E_{L,R}$ and $E_\pm$. By assumption $H^2(E_\pm) = 0$. Meanwhile, $H^2(\Delta_Z) \approx 0$, since Δ_Z is the compliment in $\Delta_W \approx W$ of a finite union of disjoint (open) 4-balls. And, $H^2(E_{L,R}) \approx 0$ since $E_{L.R} \approx W$. □

Remark 7: In summary, the construction of $E_\pm$ for the case of (3.11) will proceed with care taken with:

1) Normal bundle framings.
2) Intersections with $\Delta_Z, E_{L,R}$ and with each other.
3) Keeping $H^2(E_\pm)$ equal to zero.

(4.24)

5 Disk intersections for the Special Case. The construction of $E_\pm$ for W given by (3.11) starts in this section with a digression to describe certain constructions on such W. The constructions here serve to modify the ascending disks from index 1 critical points and also descending disks from index 2 critical points.

With W understood to be given by (3.11), begin the discussion by fixing a good Morse function $f : W \to [0,1]$ as described by Proposition 3.3. As in Proposition 3.3, let $a_1, \cdots, a_r$ label f's index 1 critical points and $b_1, \cdots, b_r$ label the index 2 critical points.

Fix a good pseudo-gradient, v, for f, and fix orientations from the descending disks from crit(f) so that the conclusions of Proposition 3.3 hold. That is, with the orientations implicit, the points $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ define a basis for C_1 and C_2, respectively. And, with respect to this basis, the boundary map in (3.5), $\partial : C_1 \to C_2$, is represented by an upper triangular matrix, S, with positive entries on the diagonal.

The matrix S gives a certain amount of algebraic information about the intersections of the descending disks from crit$_2(f)$ and the ascending disks from crit$_1(f)$. That is, the intersection of the descending disk from b_i and the ascending disk from a_j is a discrete set of flow lines which start at a_j and end at b_i. Each such flow line carries a sign, ± 1. And, the matrix element $S_{i,j}$ computes the sum of these ± 1's. In particular, Proposition 3.3 insures that the algebraic intersection number of the descending disk from b_i and the ascending disk from a_j is zero if $i > j$.

However, even when $i > j$, the point a_j may lie in the closure of the descending disk from b_i. This is an unpleasant fact which must be circumvented in order to facilitate certain constructions in the subsequent subsections. The purpose of this subsection is to modify the descending and ascending disks so as to make this eventuality irrelevent. The expense here is to replace the disk with a more complicated submanifold of W.

a) Past and future.

The purpose of this subsection is to introduce some terminology which will arise in the modification constructions below. To begin, focus on a subset $U \subset W$. Define the past of U, written past(U), as follows: A point$x \in$ past(U) if there is a gradient flow line $\gamma : [a, b] \to W$ and times $t, t' \in (a, b]$ with $t' \geq t$ and with

1) $\gamma(t) = x$,
2) $\gamma(t') \in U$.

(5.1)

Define the future of U, written fut(U), as the subset of points x in W for which there is a gradient flow line which obeys (5.1) but where $t, t' \in [a, b)$ and $t' \geq t$. Note that past$(U) \cap$ fut$(U) = U$.

For example, if $p \in W$ is not a critical point, then past(p) is the set of points which are hit before p on the gradient flow line through p. However, if $p \in$ crit(f), then past$(p) = B_{p-}$.

b) Tubing descending disks from crit$_2(f)$.

This subsection begins the modification process; it describes a construction, *tubing*, which modifies the descending disk from an index 2 critical point b_i so that the closure of the modified submanifold is disjoint from any index 1 critical point a_j for $j < i$.

To make the tubing construction, focus first on some index 2 critical point $b = b_i$ and a particular index 1 critical point $a = a_j$ for $j < i$. A descending disk from the index 2 critical point b will intersect a neighborhood, U_a of a in a finite set of components. Each of these components contains the intersection of U_a with a gradient flow line which starts at a and ends at b. To be precise, let $V \subset B_{b-} \cap U_a$ be a component. After a small isotopy, one can find Morse coordinates for U_a so that

$$\psi_a(V) = \{(x_1, x_2, x_3, x_4) : x_2 \geq 0 \text{ and } x_3 = x_4 = 0\}. \tag{5.2}$$

With (5.2) understood, the flow line between a and b which lies in V is given in the Morse coordinates by intersecting $\psi_a(V)$ with the ray $\{(x_1, x_2, x_3, x_4) : x_2 > 0 \text{ and } x_1 = x_3 = x_4 = 0\}$.

To consider the full intersection of B_{b-} with a neighborhood of a in W, it is convenient to first intersect $\psi_a(B_{a+})$ with a small radius sphere in $\mathbb{R}^4$ about the origin. Call the result S_+; in Morse coordinates, this S_+ is a small radius 2-sphere in the 3-plane where $x_1 = 0$. The intersection $B_{b-} \cap S_+$ is transverse, and is a finite number of points, $B_{b-} \cap S_+ = \{e_\alpha\}$. Because $j < i$ and the matrix S is upper triangular, the 2-sphere S_+ has zero algebraic intersection number with B_{b-}. This means that the points $\{e_\alpha\}$ of B_{b-}'s intersection with S_+ can be paired so that each pair contains one point with positive intersection number and one with negative intersection number. Write this pairing as

$$\{e_\alpha\} = \{\{e_1, e_2\}, \cdots, \{e_{2n-1}, e_{2n}\}\}. \tag{5.3}$$

Since S_+ is a 2-sphere, the two points of any pair can be connected by a path in S_+. These paths can be drawn so that paths coming from distinct pairs in $B_{b-} \cap S_+$ do not intersect. The paths should also be drawn to avoid intersections of S_+ with any other descending sphere from $\text{crit}_2(f)$. Let $\{\zeta_\mu\}_{\mu=1}^n$ be the set of paths just defined.

The value of f on S_+ is some constant, $f_0 > f(a)$. Then, introduce $M \equiv f^{-1}(f_0) \cap U_a$. This will be a smooth 3-manifold given by

$$\psi_a(M_+) = \{(x_1, \cdots, x_4 : -x_1^2 + x_2^2 + x_3^2 + x_4^2 = f_0\}. \tag{5.4}$$

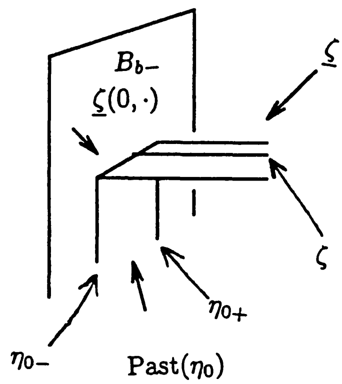
With M_+ understood, thicken each $\zeta \in \{\zeta_\mu\}$ to a thin ribbon in M_+; call this ribbon $\underline{\zeta} \approx I \times I$, where $I \equiv [0,1]$. (The ribbon should be thin so that it's only intersection with a descending disk from $\text{crit}_2(f)$ is with $\partial\underline{\zeta}$.) Thus parameterized, $I \times \{1/2\} = \zeta$, while $\partial I \times I$ is embedded in $B_{b-} \cap M_+$. To be explicit, parameterize as $\zeta(\tau) \in S_+$ for $\tau \in [0,1]$. Then, to a first approximation, $\underline{\zeta}$ should be parameterized by (τ, τ') as

$$\begin{aligned} &1) \quad x_1 = f_0^{1/2}\,\epsilon\,(2\,\tau' - 1), \\ &2) \quad (x_2, x_3, x_4) = (1 + \epsilon^2\,(2\,\tau' - 1)^2)^{1/2}\,\zeta(\tau)\} \end{aligned} \tag{5.5}$$

for small $\epsilon > 0$.

Let η_{0-} denote the past of $\underline{\zeta}(0,0)$; it is part of a gradient flow line which starts on M_0. Let η_{0+} denote the past of $\underline{\zeta}(0,1)$, part of another gradient flow line starting on M_0.

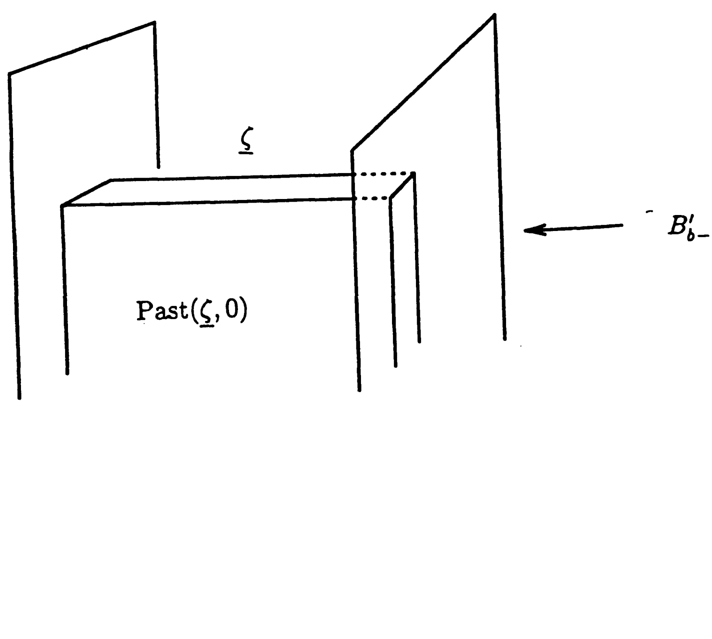
The union $\eta_0 \equiv \eta_{0-} \cup \zeta(0,\cdot) \cup \eta_{0+}$ is a piece-wise smooth curve in B_{b-}. Here is a picture of η_0 and past(η_0):



(5.6)

Let η_1 - denote the past of $\underline{\zeta}(1,0)$ and let η_{1+} denote the past of $\underline{\zeta}(1,1)$. Set $\eta_1 \equiv \eta_{1-} \cup \underline{\zeta}(1,\cdot) \cup \eta_{1+}$. This is a piece-wise smooth curve in B_{b-}.

With the preceding understood, here is a surgery on B_{b-}: Delete from B_{b-} the set past$(\eta_0)\cup$ past(η_1) to get a manifold with piecewise smooth boundary $\eta_0 \cup \eta_1$, and then glue on to this boundary image $(\underline{\zeta})\cup$ past$(\underline{\zeta}(\cdot,0))\cup$ past$(\underline{\zeta}(\cdot,1))$. Call the resulting space B'_{b-}. See the following picture:



(5.7)

The surgery just described is the *tubing* construction on a cancelling pair of intersection points of B_{b-} with S_+

Effect this tubing construction for all the pairs in (5.3) which comprise B_{b-}'s intersection with S_+. Because the surgeries are constructed using gradient flow lines, the surgeries from different pairs in (5.3) do not interfere (or intersect) each other.

After all n surgeries are performed, the result is a piecewise smooth submanifold of W whose closure misses the critical point a_j. This submanifold can be smoothed after a small perturbation and will henceforth will be assumed smooth.

Effect the same tubing construction for all pairs of intersection points for all a_j with $j < i$. Use B_{1b-} to denote the result of doing this surgery. (Note: Because the surgeries are defined using gradient flow lines, the surgeries which come from different index 1 critical points do not interfere nor intersect with each other.)

Finally, effect this same tubing construction for all b_i in $\text{crit}_2(f)$. Note that this can be done so that the resulting set of submanifolds $\{B_{1b-} : b = b_i\}_{i=1}^{r}$ are disjoint in W. (The point here is that the paths ζ and the ribbons $\underline{\zeta}$ in (5.5) have only the two boundary points of ζ as intersection points with descending disks from $\text{crit}_2(f)$. The rest of the tubing construction uses only gradient flow lines–and so won't create intersections with descending disks.)

c) Normal bundles.

Let $b = b_i \in \text{crit}_2(f)$. The submanifold $B_{b-} \subset W$ is oriented as the negative disk from the degree 2 critical point $b = b_i$. As an oriented submanifold of W, B_{b-} has a canonical trivialization of its normal bundle (up to homotopy). Simply flow the trivialization of the normal bundle of B_{b-} at b along B_{b-} using the pseudo-gradient v.

The preceding subsection described the construction of a submanifold B_{1b-} from B_{b-} by doing surgery on embedded arcs in B_{b-} with endpoints on $B_{b-} \cap M_0$. The resulting 2-dimensional submanifold can be seen to be orientable, and it inherits a canonical orientation from B_{b-}. (Note that each surgery that is performed on B_{b-} is constrained to lie in a 3-dimensional ball in W. One dimension of this ball is the pseudo-gradient flow direction, the other two dimensions can be parameterized by the ribbon coordinates on $\underline{\zeta}$ in (5.5).)

As B_{1b-} is not closed in W, the normal bundle to B_{1b-} will be a trivial bundle, and the claim is that there is an essentially canonical trivialization up to homotopy. The point is that in constructing B_{1b-} from B_{b-} one does a large number, say N, of essentially identical, non-interfering surgeries. So, one need only check that the canonical normal trivialization of B_{b-} extends over any one of these surgeries to give a normal trivialization of the postoperative manifold which agrees with the normal trivialization of B_{b-} away from the area of surgery. That such is the case is easy to check, since each individual surgery can be performed inside a 3- dimensional ball inside of W.

d) Tubing ascending disks from $\mathrm{crit}_1(f)$.

Let $a \equiv a_i \subset \mathrm{crit}_1(f)$. The closure of the ascending disk from a will typically intersect many of the points in $\mathrm{crit}_2(f)$. The purpose of this subsection is to modifies the ascending disk so that the closure of the resulting submanifold of W is disjoint from $\{b_j\}_{j>i}$. This modification procedure will also be called *tubing.*

To begin the tubing construction, focus attention first on an index 2 critical point $b \equiv b_j$ with $j > i$. Introduce the Morse coordinates, ϕ_b of (3.2), on a neighborhood of b. Let $S_- \subset \mathbb{R}^4$ denote a small radius circle in the (x_1, x_2)-plane, centered at the origin. This is a small radius circle in the inverse image by ϕ_b of the descending disk from b.

The submanifold B_{a+} intersects S_- transversely in a finite set. The algebraic intersection number here is zero because the transpose of the matrix S in Proposition 3.3 is lower triangular. Thus, the intersection points can be paired off so that the one member of each pair is a positive intersection point and the other member is negative. However, care must be taken in making this pairing.

The pairing process is an inductive one, where each induction step pairs off some non-zero number of points until all points have been paired. To describe the pairing process in an induction step, remark that the n'th induction step will pair up a non-zero number of points and leave a subset, $Y_{n+1} \subset S_-$ of points which have not yet been paired. Here, Y_{n+1} contains an even number of points, half are points of positive intersection and half are points of negative intersection. By definition, $Y_1 \equiv B_{a+} \cap S_-$.

Being points on the circle, the points of Y_{n+1} have a cyclic ordering. Break this ordering by choosing one point, e_1, as the fiducial point. With e_1 chosen, there is a decomposition of Y_{n+1} into subsets of consecutive points,

$$Y_{n+1} = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_J, \tag{5.8}$$

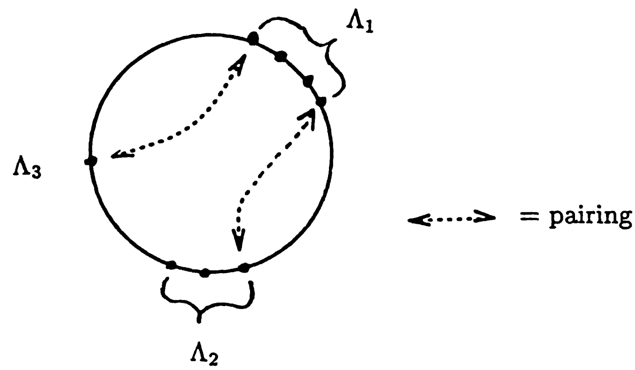
where the points in Λ_j can be described as follows:

(5.9)

1) The signs for all points in Λ_j agree, and for $j > 1$, disagree with those in Λ_{j-1}.
2) In clockwise order, Λ_1 starts with e_1.
3) The points in Λ_j follow clockwise those in Λ_{j-1}.

The $(n+1)$'st round of pairings is obtained by assigning the last point in Λ_j to the first point in Λ_{j+1}; doing this for $j = 1, \cdots, J-1$.

See the following diagram:



(5.10)

Let $P_{n+1} \subset Y_{n+1}$ denote the set of points just paired. This P_{n+1} contains at least two points (i.e., one pair) unless Y_{n+1} is already the empty set. If $Y_{n+1} \neq \emptyset$, then $Y_{n+2} \equiv Y_{n+1} - P_{n+1}$ contains strictly fewer points then Y_{n+1}. Thus, iterating the pairing process as just described will eventually pair up all the points of $B_{a+} \cap S_-$.

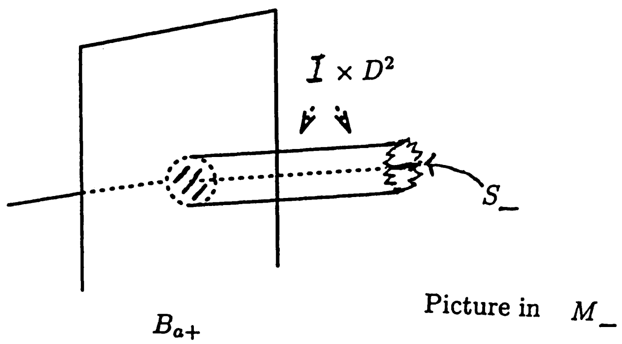
With the pairing process complete, turn to the tubing process. Start the discussion by considering the set P_1 of points which were paired on the first round. By construction, P_1 decomposes as a union of nearest neighbor pairs, with a pair, $\{e, e'\}$, in the decomposition composed of one point with positive intersection number and one with negative intersection number. Write $\{e, e'\}$ with e' clockwise from e on S_-. Let ζ denote a (closed) interval of S_- which sits clockwise between e and e'.

The value of the function f on S_- is some constant, f_0. One can assume, with no loss of generality, that $M_- \equiv f^{-1}(f_0)$ is a smooth submanifold of W near the critical point b. Indeed,

$$\psi_b(M_-) = \{(x_1, \cdots, x_4 : -x_1^2 - x_2^2 + x_3^2 + x_4^2 = f_0\}. \tag{5.11}$$

With M_- understood, thicken ζ to an embedded $I \times D^2$ inside M_-. Here, $I = [0,1]$ and D^2 is the standard 2-disk. The embedding sends $I \times \{0\}$ to ζ with $\{0\} \times \{0\}$ going to e and with $\{1\} \times \{0\}$ going to e'. Meanwhile, the embedding should embed $\partial I \times D^2$ into $M_- \cap B_{a+}$ as neighborhoods of $\{e, e'\}$ in M_-. Only $I \times \{0\}$ should intersect S_- and only $\partial I \times D^2$ should intersect B_{a+}.

Agree to identify $I \times D^2$ with its image inside M_-. See the following picture:

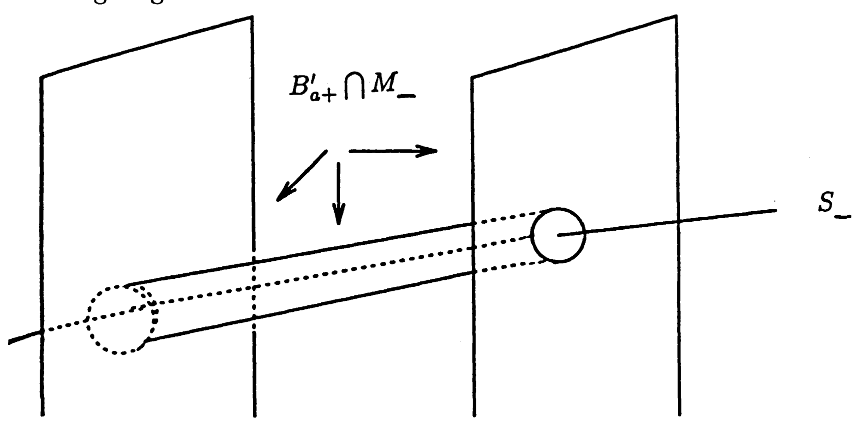


(5.12)

With $I \times D^2$ understood, perform the following surgery on B_{a+}: Delete fut$(\{0\} \times D^2) \cup$ fut$(\{1\} \times D^2)$ from B_{a+}. The closure of the resulting space has a new "boundary" which is piecewise smooth, being the union

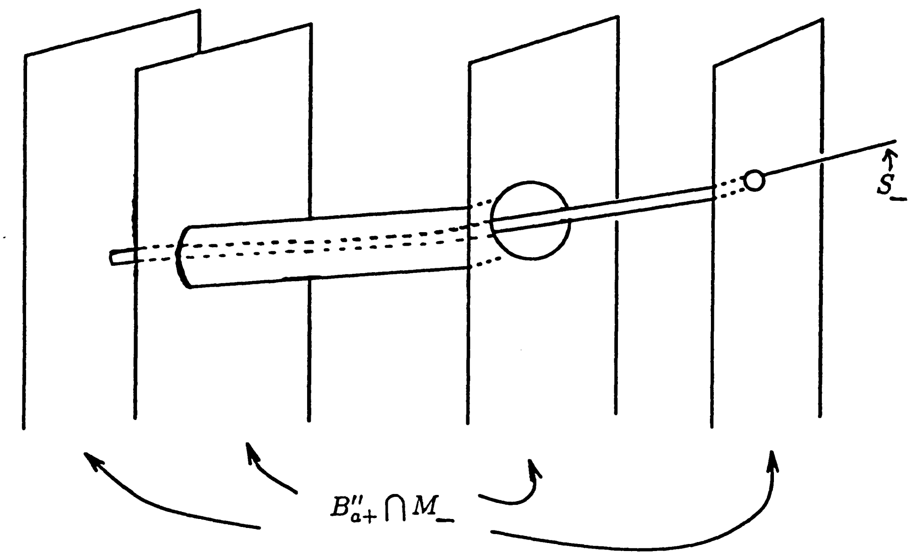
(5.13) $\quad \mathrm{fut}(\{0\} \times \partial D^2) \cup \mathrm{fut}(\{1\} \times \partial D^2) \cup (\{0\} \times D^2) \cup (\{1\} \times D^2).$

With this deletion complete, glue onto the boundary above the set fut$(I \times \partial D^2) \cup (I \times D_2)$. The result will be a piece-wise smooth manifold (in its interior) which can be smoothed to give a smooth manifold (in its interior) whose closure has two less intersections with S_-. Let B'_{a+} denote the resulting smoothed manifold. See the following diagram:



(5.14)

Notice that the B'_{a+} intersects S_- in the set Y_2. And, the subset P_1 of points paired on the second round consists, by construction, of nearest neighbor pairs on S_-. One can therefore repeat the preceding tubing construction to obtain a sub-manifold $B''_{a+} \subset W$ whose intersection with S_- is precisely the set Y_3. Clearly, an iteration of the tubing construction (as described above) will result, finally, with a sub-manifold of W with no intersections with S_-. See the following diagram:



(5.15)

One can make the same tubing constructions simultaneously at all of the points in $\{b_{i+1}, \cdots, b_r\}$. This is because tubings from distinct critical points will not interfere with each other. (They are defined by gradient flow lines so do not intersect each other in $W-$ crit(f).) Use B_{1a+} to denote the submanifold of W which results from doing these tubing surgeries near all of the points in the set $\{b_{i+1}, \cdots, b_r\}$.

Complete this tubing construction for each $a \in \text{crit}_1(f)$. The resulting set $\{B_{1a+} : a \in \text{crit}_2(f)\}$ may contain pairs which mutually intersect, but notice that these intersections will occur only in small ball-neighborhoods of the points of crit$_2(f)$.

e) Normal framings for B_{1a+}

The purpose of this subsection is to point out that the submanifolds $\{B_{1a+}\}$ all come with a canonical normal bundle framing. Indeed, B_{a+} is oriented and the tubing constructions do not destroy the orientability of B_{1a+} and it is left to the reader to check that B_{1a+} inherits a natural orientation from B_{a+}. An orientation for B_{1a+} induces one on the normal bundle of B_{1a+}. And, since $B_{1a+} \subset W$ has codimension 1, the act of orienting the normal bundle of B_{1a+} gives that bundle a trivialization.

f) The flow line γ.

There is a flow line, γ, which starts at $p_0 \in M_0$ and which ends at $p_1 \in M_1$. (See 4 of Definition 3.1.) One can assume that all B_{1a+} and B_{1b-} constructed here are disjoint from γ. Here is why: The flow γ must avoid all of the critical points of f. Since W is compact, there is an open subset of $U \subset W$ which contains crit(f) and is such that past(U) and fut(U) are disjoint from γ. Given such a set, one can assume without loss of generality that

$$(5.16) \qquad \begin{array}{ll} 1) & B_{1a+} \subset \text{fut}(U) \text{ for all } a \in \text{crit}_1(f). \\ 2) & B_{1b-} \subset \text{past}(U) \text{ for all } b \in \text{crit}_2(f). \end{array}$$

6 The first pass at $E_\pm$. This section will construct submanifolds $E_{1\pm} \subset Z$ which plug into (4.10), (4.11) to solve the constraints of Lemma 4.1 in the case when W is described by (3.11). (So W has the homology of S^3 and W has a good Morse function with no index 3 critical points.)

a) The submanifolds $Y_{i,j\pm}$.

As in the preceding section, fix a good Morse function f on W with no index 3 critical points, and label the index 1 and index 2 critical points of f as $\{a_1, \cdots, a_r\}$ and $\{b_1, \cdots, b_r\}$, respectively. For each $a \in$ crit$_1(f)$, construct the submanifold $B_{1a+} \subset W$ as directed in the preceding section. Also, for each $b \in$ crit$_2(f)$, construct the submanifold $B_{1b-} \subset W$ as directed in said section.

When $1 \leq i \leq j \leq r$, set $a \equiv a_i$ and set $b \equiv b_j$. Now define

$$(6.1) \qquad \begin{array}{ll} 1) & Y_{i,j-} \equiv Z \cap (B_{1a+} \times B_{1b-}) \\ 2) & Y_{i,j+} \equiv Z \cap (B_{1b-} \times B_{1a+}) \end{array}$$

These subspaces will be used to construct the varieties $E_\pm$ of (4.10). (See also (6.3) and Definition 6.4, below.)

Lemmas 6.1 and 6.2, below describes some of the salient features of $Y_{i,j\pm}$.

LEMMA 6.1. *Let $1 \leq i \leq j \leq r$, and set $a \equiv a_i$ and $b \equiv b_j$.*

1) *After (arbitrarily) small isotopies of B_{1a+} and B_{1b-}, the former the identity near a and the latter the identity near b, the subspaces $Y_{i,j\pm} \subset Z$ will be closed, embedded, dimension 4 submanifolds (with boundary).*
2) *$\partial Y_{i,j\pm} = \emptyset$ if $i \neq j$.*
3) *$\partial Y_{i,i\pm} \subset (S^3 \times S^3)_a \cup (S^3 \times S^3)_b$.*
4) *$\partial Y_{i,i-} \cap (S^3 \times S^3)_a$ is the disjoint union of embedded 3-spheres, each isotopic to ($S^3\times$ point). Likewise, $\partial Y_{i,i-} \cap (S^3 \times S^3)_b$ is the disjoint union of embedded 3-spheres, each isotopic to ($S^3\times$ point).*
5) *$\partial Y_{i,i+} \cap (S^3 \times S^3)_a$ is the disjoint union of embedded 3-spheres, each isotopic to (point $\times S^3$). Meanwhile, $\partial Y_{i,i+} \cap (S^3 \times S^3)_b$ is the disjoint union of embedded 3-spheres, each isotopic to (point $\times$ S^3).*
6) *All $Y_{i,j\pm}$ are orientable.*
7) *The product normal framings of B_{1a+} and B_{1b-} in W induce a framing of the normal bundles to $Y_{i,j\pm}$ in Z.*
8) *All $Y_{i,j\pm}$ have $H^2(Y_{i,j\pm}) = 0$.*

9) *All $Y_{i,j\pm}$ are disjoint from $E_{R,L}$ in (4.15).*

(Remark that Assertions 8 and 9 are not needed until one reaches the part of Theorem 2.9's proof where (2.27.3) must be verified.)

The proof of this lemma is deferred to Subsection 6c.

Since all the $Y_{i,j\pm}$ are orientable, one can consider their intersection numbers with the generators of $H_3(Z)$. These intersection numbers are computed in Lemma 6.2, below.

Lemma 6.2 uses the following notations and conventions: Let p and p' be critical points of f with the same index, and for which $f(p) > f(p')$. Consider $L_{(p,p')+} \equiv (B_{p-} \times B_{p'+}) \cap Z \in \underline{L}_+$. This is the boundary of the subset of $B_{p-} \times B_{p'+}$ where $F \leq 0$. (The latter is a manifold with boundary.) Orient $B_{p-} \times B_{p'+}$ with the product orientation and then agree henceforth to orient $L_{(p,p')+}$ with the induced orientation as the boundary of the subset where $F \leq 0$.

Consider now the 3-sphere $L_{(p',p)-} \equiv (B_{p'+} \times B_{p-}) \cap Z \in \underline{L}_-$. This is the boundary of the subset of $B_{p'+} \times B_{p-}$ where $F \geq 0$. (The latter is a manifold with boundary.) Orient $B_{p'+} \times B_{p-}$ with the product orientation and then agree henceforth to orient $\underline{L}_{(p',p)-}$ with the induced orientation as the boundary of the subset where $F \geq 0$.

LEMMA 6.2. *Add the following to the conclusions of Lemma 6.1: The submanifolds $\{Y_{i,j\pm}\}$ have transversal intersections with the 3- spheres in $\underline{L}_\pm$ and $Y_{i,j-} \cap \underline{L}_- = \emptyset$ and $Y_{i,j+} \cap \underline{L}_+ = \emptyset$, where $\underline{L}_\pm$ are given by (3.31), (3.32). Furthermore, the $\{Y_{i,j\pm}\}$ can be oriented so that*

1) *The intersection of $Y_{i,j-}$ with $L_{(p,p')+} \in \underline{L}_+$ is empty unless $p = a_i$ or $p' = b_j$. If $p = a_i$ and $p' = a_k$, then the intersection number is $-S_{j,k}$. If $p = b_k$ and $p' = b_j$, then this intersection number is $S_{k,i}$.*
2) *The intersection of $Y_{i,j+}$ with $L_{(p',p)-} \in \underline{L}_-$ is empty unless $p' = b_j$ or $p = a_i$. If $p' = b_j$ and $p = b_k$, then the intersection number is $S_{k,i}$. If $p = a_i$ and $p' = a_k$, then the intersection number is $S_{j,k}$.*
3)

$$\partial[Y_{i,i-}] = (S_{i,i})\,([S^3]_{a-} + [S^3]_{b-}), \tag{6.2}$$

$$\partial[Y_{i,i+}] = (S_{i,i})\,([S^3]_{a+} + [S^3]_{b+}).$$

Here, $S_{i,i} > 0$ is given in (3.15). (For $p = a$ or b, the classes $[S^3]_{p\pm}$ are defined subsequent to (4.19).)

The proof of this lemma is deferred to Subsection 6d, below.

b) The construction of $[E_\pm]$.

With the orientations of Lemma 6.2, the submanifolds $\{Y_{i,j\pm}\}$ of Lemma 6.1 will define homology classes in $H_4(Z, \partial Z)$ and linear combinations of these classes will produce classes $[E_\pm]$ which fit into (4.11) to solve the constraints of Lemma 4.1. To be precise here, introduce the matrix S of (3.15) and the integer valued matrix $T \equiv \det(S)\, S^{-1}$. Note that $T \equiv (T_{i,j})$ is upper triangular (when $i > j$, then $T_{i,j} = 0$) with $T_{i,i} = \det(S)/S_{i,i}$.

With T understood, introduce

$$(6.3) \qquad [E_{1-}] \equiv \sum_{i,j} T_{i,j}\,[Y_{i,j-}] \text{ and } [E_{1+}] \equiv \sum_{i,j} T_{j,i}\,[Y_{i,j+}].$$

(In (6.3), the sums are over all pairs i, j with $1 \le i \le j \le r$.) Here are the salient features of these classes:

LEMMA 6.3. *Define the classes $[E_{1\pm}]$ by (6.3). Then*

$$(6.4) \qquad \partial[E_{1-}] = \mathit{det}(S) \sum_{p \in \ \mathit{crit}(f)} [S^3]_{p-}.$$

$$\partial[E_{1+}] = \mathit{det}(S) \sum_{p \in \ \mathit{crit}(f)} [S^3]_{p+}.$$

Furthermore, $[E_{1\pm}]$ have zero intersection pairing with the classes which are generated by the 3-speres in $\underline{L}_{\pm}$ of (3.31) and (3.32).

It follows from this lemma that Lemma 4.1 is satisfied if the classes $[E_{\pm}]$ in (4.11) are set equal to $[E_{1\pm}]$ from (6.3). In this case, (4.11)'s integer N must equal $\det(S)$. (In later constructions, it proves convenient to take $[E_{\pm}]$ in (4.11) to be some multiple of $[E_{1\pm}]$ from (6.3).)

Proof. Consider first the properties of $[E_{1-}]$. It follows from Assertion 1 of Lemma 6.1 and Assertion 1 of Lemma 6.2 that $\partial[E_{1-}]$ obeys (6.4). This is because the boundary annihilates all terms in (6.3) save those for which $i = j$. Then, (6.4) follows from (6.2) and the fact that $T_{i,i} = \det(S)/S_{i,i}$.

According to Assertion 2 of Lemma 6.2, $[E_{1-}]$ is represented by the fundamental classes of submanifolds with empty intersection with the classes from $\underline{L}_{-}$. To study the intersection pairing between $[E_{1-}]$ and a class from $\underline{L}_{+}$, fix integers m and n with $1 \le m < n \le r$. Let $a \equiv a_m$ and let $a' \equiv a_n$. Consider the pairing between $[E_{1-}]$ and the class of $L_{(a,a')+}$. Using Assertion 3 of Lemma 6.2, one finds that this number is equal to

$$(6.5) \qquad \sum_k T_{m,k}\, S_{k,n},$$

which is zero because $m \ne n$ and T is proportional to S^{-1}.

Next, let $b \equiv b_m$ and let $b' \equiv b_n$ and consider the pairing between $[E_{1-}]$ and the class of $L_{(b,b')_+}$. Using Assertion 3 of Lemma 6.2 again, one finds that this pairing is equal to

$$(6.6) \qquad \sum k S_{m,k}\, T_{k,n},$$

which is also zero, because $m \neq n$ and $T = \det(S)\, S^{-1}$.

Thus Lemma 6.3 is proved for $[E_{1-}]$. The proof for $[E_{1+}]$ is analogous and is left to the reader. □

c) Proof of Lemma 6.1

Fix i and j such that $1 \leq i \leq j \leq r$ and let $a \equiv a_i$ and $b \equiv b_j$. For Assertion 1's proof, note that $Y_{i,j-} \cap \operatorname{int}(Z)$ will be a submanifold of $\operatorname{int}(Z)$ if F's restriction to $B_{1a+} \times B_{1b-}$ has zero as a regular value. This will follow if f's restriction to B_{1a+} has disjoint critical values from its restriction to B_{1b-}. With an arbitrarily small isotopy, of B_{1b-} near $f^{-1}(f(a))$, one can insure that $f(a)$ is not a critical value of f on B_{1b-}. Likewise, an arbitrarily small isotopy of B_{1a+} near where $f = f(b)$ will insure that $f(b)$ is not a critical value of f on B_{1a+}. With this understood, a small isotopy of B_{1a+} which is the identity near a will insure that the critical values of f on B_{1a+} are disjoint from those of f on B_{1b-}.

Argue as follows to prove that $Y_{i,i-}$ is closed: The closure of B_{1b-} in W adds only the descending disks from $\{a_k\}_{k \geq j}$. However, $f(B_{1a+}) \geq f(a_i) > f(\{a_k\}_{k>i})$ (see Assertion 2 of Proposition 3.2). Therefore, where $(\pi_L^* f) \leq 3/8$, the closure of $(B_{1a+} \times B_{1b-}) \cap Z$ adds nothing unless $i = j$, and then, only the point (a, a) is added.

Likewise, B_{1a+} is not closed in W, but its closure adds only ascending disks from $\{b_k\}_{k \leq i}$. (By construction the closure of B_{1a+} misses $\{b_k\}_{k>i}$.) However, $f(B_{1b-}) \leq f(b_j) < f(\{b_k\}_{k<j})$ because of Assertion 2 in Proposition 3.2. Therefore, where $(\pi_L^* f) \geq 3/8$, the closure of $(B_{1a+} \times B_{1b-}) \cap Z$ adds nothing except when $i = j$, in which case only the point (b, b) is added.

The preceding proves that $(B_{1a+} \times B_{1b-}) \cap Z$ is closed.

A similar argument proves that $Y_{i,j+}$ is closed.

Note that the preceding argument proves Assertions 2 and 3 also.

To prove Assertion 4, consider first the neighborhood U_a of a in W as described by the Morse coordinates (3.2). The submanifold B_{1b-} intersects this ball in at least $S_{i,i}$ components; and a typical component, say V, has the following form: There is a unit vector v with coordinates $(0, v_2, v_3, v_4)$ and

$$V = \{(x_1, t\,v_1, t\,v_2, t\,v_3) : x_1 \in \mathbb{R} \text{ and } t > 0.\}. \tag{6.7}$$

(The unit vectors (i.e. v) are distinct for distinct components of $B_{1b-} \cap U_a$.)

Equation (3.6) describes B_{1a+} near a since near a, it is identical to B_{a+}.

Consider next the neighborhood $U_a \times U_a$ of (a, a) in $W \times W$, and use the coordinates of (3.25). One sees that near (a, a), each component of $B_{1a+} \times B_{1b-}$ has the form $B_{1a+} \times V$, where $V \subset B_{1b-}$ is given by (6.7). Thus, the intersection $B_{1a+} \times V$ with Z near (a, a) is given by the set of points $((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) \in \mathbb{R}^4 \times \mathbb{R}^4$ where

1) $x_1 = 0$
2) $(y_2, y_3, y_4) = t\,v$ for $t > 0$,
3) $t^2 = y_1^2 + x_2^2 + x_3^2 + x_4^2$.

(6.8)

Note that this set intersects $(S^3 \times S^3)_a \subset \partial Z$ as $S^3 \times p_v$, where

$$p_v = (0, r\,v_2, r\,v_3, r\,v_4). \tag{6.9}$$

Equations (6.7) and (6.8) establish the first part of Assertion 3 concerning the intersection of $Y_{i,j-}$ with $(S^3 \times S^3)_a$.

An analogous argument shows that the intersection of $Y_{i,j-}$ with $(S^3 \times S^3)_b$ has the following form: The coordinate chart U_b describes a neighborhood of b in W. A component, V, of the intersection of B_{1a+} with U_b is given as

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : (x_1, x_2) = t\,v, \text{ for } t > 0 \text{ and } v \in \mathbb{R}^2 \text{ with } \mid v \mid = 1\}. \tag{6.10}$$

Use $U_b \times U_b$ to describe a neighborhood of (b, b) in W. The intersection of $B_{1a+} \times B_{1b-}$ with this neighborhood will be a union of components, each of the form $V \times B_{1b-}$ with V as above. With this understood, $V \times B_{1b-}$ intersects Z as the set of points in $\mathbb{R}^4 \times \mathbb{R}^4$ of the form $((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4))$ where

(6.11)

1) $(x_1, x_2) = t\,v$ for $t > 0$ and $\mid v \mid = 1$
2) $y_3 = y_4 = 0$
3) $t^2 = y_1^2 + y_2^2 + x_3^2 + x_4^2$.

The preceding equation demonstrates that $V \times B_{1b-}$ intersects $(S^3 \times S^3)_b \subset \partial Z$ as $S^3 \times p_v$, where $p_v = (r\,v_1, r\,v_2, 0, 0)$. The proof of Assertion 5 of the Lemma 6.1 follows essentially the same arguments which prove Assertion 4. The details for Assertion 5's proof are omitted.

Consider now the proof of Assertion 6: Both B_{1a+} and B_{1b-} are orientable (as described in the previous section), and so their product is orientable. Then, the restriction of dF to the product trivializes the normal bundle of $Y_{i,j-}$ in $B_{1a+} \times B_{1b-}$ and similarly that of $Y_{i,j+}$ in $B_{1b-} \times B_{1a+}$.

To prove Assertion 7, remark that both B_{1a+} and B_{1b-} were constructed with canonically trivial normal bundles. Thus, their product has a canonical (up to homotopy) trivialization of its normal bundle in $W \times W$. With this understood, remember that Z is cut out of $W \times W$ as part $F^{-1}(0)$, while $Y_{i,j-}$ is cut out of $B_{1a+} \times B_{1b-}$ as part of $F^{-1}(0)$, so the trivialization of $(B_{1a+} \times B_{1b-})$'s normal bundle in $W \times W$ defines, upon restriction to $F^{-1}(0)$, a trivialization of the normal bundle to $Y_{i,j-}$ in Z.

Once again, the argument for $Y_{i,j+}$ is analogous and omitted.

To prove Assertion 8, first remember that B_{1a+} and B_{1b-} are constructed from B_{a+} and B_{b-}, respectively by surgery. The surgery on B_{b-} occurs near where $f = 1/4$, while the surgery on B_{1a+} occurs near $f = 1/2$. This implies that $Y_{i,j-}$ can be seen as the result of a surgery on the 4-sphere which is the intersection of the descending disk from F's index 5 critical point (a, b) with $F^{-1}(1/8)$. The surgery is on embedded $S^0 \times B^4$'s in said 4-sphere. The number of these surgeries is the combined total of the surgeries which make B_{1a+} from B_{a+} and B_{1b-} from B_{b-}. Each such surgery increases the rank of $H_1(\cdot; Z)$ by one, but leaves $H_2(\cdot; Z) = 0$.

Assertion 9 follows from (4.15) and (5.16). □

d) Proof of Lemma 6.2.

Consider first that the 3-spheres in $\underline{L}_\pm$ do not come near the critical points (p,p) of F. This follows from Proposition 3.2. Therefore, an (arbitrarily) small isotopy of B_{1a+} or of B_{1b-} will result in transversal intersections between $Y_{i,j\pm}$ and any of the spheres in $\underline{L}_\pm$.

Remark next that the intersection of $Y_{i,j-}$ with some $L_{(p',p)-}$ is non-empty only if $B_{1a+} \cap B_{p'+} \neq \emptyset$ and also $B_{1b-} \cap B_{p-} \neq \emptyset$. The former is empty if p and p' have index 2, while the latter are empty if p and p' have index 1.

To prove Assertion 1, one should consider orienting $Y_{i,j-}$ as follows: Orient $B_{1a+} \times B_{1b-}$ with its product orientation. Then, note that $Y_{i,j-}$ is a codimension zero part of the boundary of the subset of $B_{1a+} \times B_{1b-}$ where $F \geq 0$. Give $Y_{i,j-}$ the induced boundary orientation. Use o to denote said orientation. With the orientation o, the intersection number between $Y_{i,j-}$ and some $L_{(p,p')+} \in \underline{L}_+$ is equal to the coefficient in front of (p,p') in the expression for the $\partial(a,b)$ in the complex C^F of Lemma 3.5. (Note that $B_{1a+} \times B_{1b-}$ is homologous to the descending 5-disk from (a,b).) The computation of this coefficient is straightforward and leads to Assertion 1. (The fact that the intersection in question is empty unless $p = a_i$ or $p' = b_j$ follows from the fact that when a and a' are index 1 critical points of f, then $B_{1a+} \cap B_{a'-} = \emptyset$ unless $a = a'$. Likewise, when b and b' are index 2 critical points of f, then $B_{1b-} \cap B_{b'+} = \emptyset$ unless $b = b'$.)

The proof of Assertion 2 is analogous. Here, the orientation o for $Y_{i,j+}$ is defined by considering $Y_{i,j+}$ as a codimension zero part of the boundary of the subset of $B_{1b-} \times B_{1a+}$ where $F \leq 0$.

Consider now the proof of Assertion 3. There is a proof along the lines of the proof of Assertion 1, but a direct proof is had by the following argument: Let $a \equiv a_i$ and $b \equiv b_i$. An intersection point, q, of $(B_{a+} \cap M_{3/8})$ with $(B_{b-} \cap M_{3/8})$ corresponds to one boundary component of $(B_{a+} \times B_{b-}) \cap Z$ in $(S^3 \times S^3)_a$ and, likewise, to one boundary component in $(S^3 \times S^3)_b$. (And vice-versa.) The orientation of these boundary components relative to the given orientations of $(S^3)_{a-}$ and to $(S^3)_{b-}$ will be found equal, but opposite the local intersection number at q of $(B_{a+} \cap M_{3/8})$ with $(B_{b-} \cap M_{3/8})$ in $M_{3/8}$.

Step 1: This step compares the local intersection number at q with the orientation of the corresponding boundary component of $(B_{a+} \cap B_{b-}) \cap Z$ in $(S^3 \times S^3)_a$. To begin, take the Morse coordinates near a of (3.2) so that $B_{a+} = \{x \equiv (x_1, x_2, x_3, x_4) : x_1 = 0\}$. Orient B_{a+} by $\partial_2\partial_3\partial_4 \in \Lambda^3 TB_{a+}$. A neighborhood, $U \subset M_{3/8}$ of $M_{3/8}$'s intersection with B_{a+} is isotopic to $\{x : -x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\}$ for some $R > r$. This U is oriented at $q \in (0, R, 0, 0) \in U$ by $-\partial_1\partial_3\partial_4$.

Now q lies in B_{a+}, but suppose that q is also a point of intersection B_{b-} and B_{a+}. Suppose further that the local intersection number at q of $(B_{a+} \cap M_{3/8})$

with $(B_{b-} \cap M_{3/8})$ is equal to $\epsilon = \pm 1$. Without loss of generality, B_{b-} can be assumed to intersect a neighborhood of U as $\{x : x_2 > 0 \text{ and } x_3 = x_4 = 0\}$. To obtain the correct intersection number at q, it is necessary to orient B_{b-} using $-\epsilon\, \partial_1 \partial_2$. (Note that $df = dx_2$ at q, so $(B_{a+} \cap M_{3/8})$ is oriented near q by $\partial_3 \partial_4$, while $(B_{b-} \cap M_{3/8})$ at q is oriented by $-\epsilon\, \partial_1$. Then, their intersection at q has local orientation $-\epsilon \partial_1 \partial_3 \partial_4$ which agrees or disagrees with the orientation $-\partial_1 \partial_3 \partial_4$ of $M_{3/8}$ depending on whether $\epsilon = \pm 1$.)

With the preceding understood, it follows that $B_{a+} \times B_{b-}$ is oriented near (q, q) by $-\epsilon \partial_2 \partial_3 \partial_4 \partial_1' \partial_2'$; here the prime indicates a vector field from the second factor of W in $W \times W$, while the absence of a prime indicates a vector field from the first factor of W. Now, at the point (q, q), the 1-form $dF = dx_2' - dx_2$; this implies that $\epsilon\, (\partial_2 + \partial_2')\partial_3 \partial_4 \partial_1'$ orients $(B_{a+} \times B_{b-}) \cap Z$ in $(S^3 \times S^3)_a$ near (q, q). Finally, the boundary of the component of $(B_{a+} \times B_{b-}) \cap Z$ in $(S^3 \times S^3)_a$ which corresponds to the point q is oriented by contracting this last frame with $-dx_2' - dx_2$ which yields $-\epsilon \partial_3 \partial_4 \partial_1'$. This disagrees with the orientation of $(S^3)_{a-}$ when $\epsilon = +1$ and it agrees with said orientation when $\epsilon = -1$.

Step 2: This step compares the local intersection number at q with the orientation of the corresponding component in $(S^3 \times S^3)_b$ of the boundary of $(B_{a+} \times B_{b-}) \cap Z$. To begin, take the Morse coordinates of (3.2) around b. Then, $B_{b-} = \{x : x_3 = x_4 = 0\}$. Orient B_{b-} by $\partial_1 \partial_2$. A neighborhood, U, of the point q in $M_{3/8}$ is isotopic to the subset given in Morse coordinates as $\{x : -x_1^2 - x_2^2 + x_3^2 + x_4^2 = -R^2\}$. Here $R >> r$ and q is the point $(0, R, 0, 0)$. The orientation of $M_{3/8}$ is determined from the fact that df at q is $-dx_2$. Thus, $\partial_1 \partial_3 \partial_4$ orients $M_{3/8}$.

Meanwhile, a neighborhood of q in B_{a+} can be assumed given by the set $\{x : x_1 = 0 \text{ and} x_2 > 0\}$. This part of B_{a+} is oriented by $\epsilon\, \partial_2 \partial_3 \partial_4$. (Thus, $(B_{a+} \cap M_{3/8})$ is oriented at q by $-\epsilon\, \partial_3 \partial_4$ while $-\partial_1$ orients $(B_{b-} \cap M_{3/8})$ at q. Their intersection gives $\epsilon \partial_1 \partial_3 \partial_4$ for the orientation of $M_{3/8}$ as it should.) The orientation for $B_{a+} \times B_{b-}$ near (q, q) is thus given by $\epsilon\, \partial_2 \partial_3 \partial_4 \partial_1' \partial_2'$.

The 1-form dF at (q, q) is given by $-dx_2' + dx_2$, and this means that $\epsilon\, (\partial_2 + \partial_2')\partial_3 \partial_4 \partial_1'$ orients the part near (q, q) of $(B_{a+} \cap Bb-) \cap Z$. With this last point understood, it follows that $-\epsilon\, \partial_3 \partial_4 \partial_1'$ orients the part of $\partial((B_{a+} \cap B_{b-}) \cap Z)$ in $(S^3 \times S^3)_b$ which corresponds to q. Note that this orientation disagrees with the given orientation of $(S^3)_{b-}$ when $\epsilon = +1$, but it agrees when $\epsilon = -1$. In particular, note that this anti-correlation with the local intersection number at q is the same as that for components of $\partial((B_{a+} \times B_{b-}) \cap Z)$ in $(S^3 \times S^3)_a$.

It follows from the preceding calculations that (6.2) holds if the orientation $-o$ is used on $\{Y_{i,j-}\}$.

A similar argument shows that the second line of (6.2) is correct if the $\{Y_{i,j+}\}$ are also oriented with $-o$. The details here are left to the reader.

e) Push-offs.

The next task is to provide a representative of each $[E_{1\pm}]$ as the fundamental class of a smoothly embedded submanifold (with boundary), $E_{1\pm} \subset Z$. Here, $\partial E_{1\pm} \subset \partial Z$.

The construction of $E_{1\pm}$ requires the introduction of a procedure, called *push-off*, for making copies of embedded submanifolds. The following digression described the push-off procedure.

Start the digression by considering the following abstract situation: Let X be a compact manifold with boundary, and let $Y \subset X$ be a compact submanifold with boundary, which intersects ∂X transversally as ∂Y. Let $N_Y \to Y$ denote the normal bundle to Y in X. (Note that N_Y restricts to ∂Y as the latter's normal bundle in ∂X.) Suppose that N_Y admits a section, s, which never vanishes. Let $e : N_Y \to X$ be an exponential map which maps $N_Y |_{\partial Y}$ to ∂X. (See (2.13).)

Together, e and s and a real number $\lambda \neq 0$ define a map,

$$e(\lambda\, s(\cdot)) : Y \to X, \tag{6.12}$$

whose image is disjoint from Y. If λ has small absolute value, then the image, Y', of (6.12) will be an embedding of Y into X, where $\partial Y'$ is an embedding of ∂Y into ∂X. This image, Y', is called a *push-off* of Y. Here are some properties of the push-off:

$(\cdot)$ Y' is disjoint from Y, but smoothly isotopic to Y. (The obvious isotopy is to consider $\lambda \to 0$ in (6.12). This isotopy will isotope $\partial Y'$ to ∂Y in ∂X.)

$(\cdot)$ If Y comes with some apriori orientation, then Y' has a canonically induced orientation which makes $[Y] = [Y']$ in $H^*(X, \partial X)$.

$(\cdot)$ Let $V \subset X$ be a submanifold which intersects Y transversally. Then V will also intersect Y' transversally if λ in 6.12) has sufficiently small absolute value.

$(\cdot)$ Let $V \subset X$ be a closed submanifold with empty intersection with Y. Then $V \cap Y' = \emptyset$ if λ in (6.12) has sufficiently small absolute value.

$(\cdot)$ If Y has a framed normal bundle, then this framing naturally induces a framing of the normal bundle to Y'.

(6.13)

Note also that one can define any finite number of disjoint push-offs of Y by using different values of λ in (6.12). Alternately, one can use different sections $\{s_1, \cdot\}$ of N_Y with fixed λ as long as the $\{s_j\}$ are no-where vanishing and no two are anywhere equal.

In the sequel, assume the following conventions:

$(\cdot)$ Any pair of distinct push-offs of the same submanifold are mutually disjoint.

$(\cdot)$ Suppose that the normal bundle N_Y is trivial, and that an apriori trivialization has been specified. (Call it the *canonical* trivialization.) In this case, agree that all push-offs of Y will be defined by using for s in (6.12) a constant linear combination of basis vectors for the canonical trivialization.

$(\cdot)$ When the precise choice of exponential map or parameter λ or section s in (6.12) are irrelevent to subsequent discussions, their presence will not be explicitly noted. (But, keep the preceding convention on the section s when the normal bundle to Y has been trivialized.)

(6.14)

(The last two conventions in (6.14) allow one to speak of a push-off of Y with-out cluttering the conversation with a list of irrelevent (but necessary) choices.)

End the digression.

f) $E_{1\pm}$ as submanifolds.

The purpose of this last subsection is to define $[E_{1\pm}]$ of (6.3) as the fundamental class of a closed, embedded submanifold (with boundary) of Z. Consider first $[E_{1-}]$.

This $[E_{1-}]$ is a sum of fundamental classes of the $\{Y_{i,j-}\}$. The first observation is that each $Y_{i,j-} \cap Y_{n,m-} = \emptyset$ unless $m = j$. This is because the various $\{B_{1b-}\}_{b \in \text{ crit}_2(f)}$ are mutually disjoint. There may be non-empty intersections between $Y_{i,j-}$ and $Y_{k,j-}$ when $i \neq k$. These can be avoided if the following convention is used: Remember that each B_{1b-} has trivial normal bundle in W. And, remember that said normal bundle has a canonical trivialization up to homotopy. For each $b \in$ crit$_2(f)$, choose a trivialization of the normal bundle of B_{1b-} which is in the canonical homotopy class. Then, fix j and when $i < j$, define $Y_{i,j-}$ as in (6.1) but where B_{1b-} is replaced by a push-off copy. For each such i, use a *different* push-off copy. This will make $Y_{i,j-}$ disjoint form $Y_{k,j-}$ when $i \neq k$.

Now, generalize this process of separating the $\{Y_{i,j-}\}$ as follows: Reintroduce the matrix $T = (T_{i,j})$ which appears in (6.3). For each pair (i,j) with $1 \leq i \leq j \leq r$, let $a \equiv a_i$ and $b \equiv b_j$. Take $|\ T_{i,j}\ |$ distinct push-off copies of B_{1b-} and use these in (6.1) to define $|\ T_{i,j}\ |$ distinct push-off copies of $Y_{i,j-}$. It will prove convenient to require that all such push-off copies are disjoint from the flow line γ of Part 4 in Definition 3.1. (One can make all such copies in past(U), where $U \subset W$is an open subset which contains crit(f) and whose past and future are disjoint from γ . See (5.16).)

Since the various $\{B_{1b-}\}_{b \in \text{ crit}_2(f)}$ are mutually disjoint, one can make all of these push-offs so that each copy of $Y_{i,j-}$ is disjoint from each copy of $Y_{k,l-}$ when $(i,j) \neq (k,l)$.

With the preceding understood, consider:

PROPOSITION 6.4. *Define $E_{1-} \subset Z$ as an oriented submanifold of Z (with boundary) as the union over all pairs (i,j) (with $1 \leq i \leq j \leq r$)) of the $|\ T_{i,j}\ |$ push-offs of $Y_{i,j-}$ as defined above. Take these copies with the following orientations: Orient the copies of $Y_{i,j-}$ as in Lemma 6.2 if $T_{i,j} > 0$; and oriented them in reverse if $T_{i,j} < 0$. Define $E_{1+} \subset Z$ as a submanifold to be the image of E_{1-} under the switch map on $W \times W$ which sends (x,y) to (y,x). (This map preserves Z.) Then these oriented submanifolds can be assumed to have the following properties:*

1) The fundamental classes of $E_{1\pm}$ obey (6.3).

2) $E_{1\pm}$ *intersect* ∂Z *transversely in* $\partial E_{1\pm}$.

3) $E_{1\pm}$ *have empty intersection with* $M_0 \times M_0$ *and* $M_1 \times M_1$.

4) $E_{1\pm}$ *have trivial normal bundles in* Z*, and said normal bundles have canonical trivializations up to homotopy.*

5) $H^2(E_{1\pm}; \mathbb{Z}) = 0$.

6) $E_{1\pm}$ *have empty intersection with* $E_{R,L}$ *of (4.15).*

The proof of this proposition is left to the reader.

7 The second pass at $E_\pm$. Assume here that W obeys the constraints of (3.11). If $E_\pm$ in (4.10) is $E_{1\pm}$ of Proposition 6.4, then the resulting Σ_Z satisfies Steps 1 and 2 plus Part 1 of Step 3 in Section 2k's outline of the proof of Theorem 2.9. However, the completion of Step 3 requires modifications of $E_{1\pm}$. The problem is that $E_{1\pm}$ intersect the various $(S^3 \times S^3)_p \subset \partial Z$ too many times, and they intersect each other too many times, and they intersect Δ_Z too many times.

The change of $E_{1\pm}$ into $E_\pm$ is a multi-step process which begins in this section and ends in Section 10. Then, Section 11 constructs a 2-form ω_Z to satisfy (2.27). This section starts the process by modifying $E_{1\pm}$ to make a submanifold, $E_{2\pm}$, with simpler intersections with the $(S^3 \times S^3)_p \subset \partial Z$.

a) The submanifold E'_{1-}.

To begin the modification process, fix $i \in \{1, \cdots, r\}$ and, as usual, set $a \equiv a_i$ and $b \equiv b_i$. Make 2det(S) additional push-off copies of B_{1b-}. Make these copies so that they are disjoint from all other push-off copies of $\{B_{1b'-} : b' \in \text{crit}_2(f)\}$ which have so far been constructed. Use these 2det(S) push-off copies of B_{1b-} to make $2\det(S)$ copies of $B_{1a+} \times B_{1b-}$ and then $2\det(S)$ copies of $Y_{i,i-}$ as describe in (6.1). Orient the first det(S) of these $Y_{i,i-}$ canonically, and orient the remaining det(S) of these copies opposite to their canonical orientation. The first det(S) copies of $Y_{i,i-}$ (the ones with the canonical orientation) will be called the special $Y_{i,i-}$.

Define E'_{1-} to be the union of Proposition 6.4's E_{1-} with the (oriented) submanifold which is comprised of the union of the preceding 2det(S) copies of oriented $Y_{i,i-}$. Notice that this E'_{1-} still obeys the conclusions of Lemma 6.3 and Proposition 6.4.

b) Tubing near (a, a)

Consider now the intersection of E'_{1-} with $(S^3 \times S^3)_a$: As described in (6.8), (6.9), this intersection is given as

$$S^3 \times \Lambda'_a, \tag{7.1}$$

where $\Lambda'_a \subset S^3$ is a finite set of points. Each point in Λ'_a comes with a sign (± 1), and there are det(S) more plus signs than minus signs. This means that the set Λ'_a can be decomposed as $\Lambda_a \cup T_a$, where the points in T_a can be paired so that the signs in each pair add to zero.

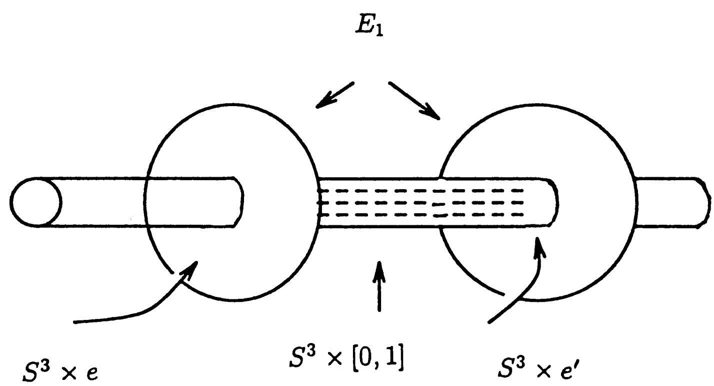
It proves useful to take some care in defining the set Λ_a. Here is how: To begin, note that the intersection of $B_{b-} \cap f^{-1}(3/8)$ with $B_{a+} \cap f^{-1}(3/8)$ is transversal, and has intersection number $S_{i,i}$. Pick a point in this intersection where the local intersection number is positive. Such a point lies on a gradient flow line, $\mu(\equiv \mu_i)$ which starts at a and ends at b. The intersection of $\mu \times \mu$ with $(S^3 \times S^3)_a$ is a point, $p_a \times p_a$, where $p_a \in S^3$. With p_a singled out, note that the intersection of any push-off copy of $Y_{i,i}$ with $(S^3 \times S^3)_a$ contains a unique $(S^3 \times p'_a)$ where p'_a is the push-off of p_a. (There is a canonical isotopy between the push-off copy and the original (shrink λ to zero in (6.12), and under this isotopy, p'_a moves to p_a.) In particular, each of the $\det(S)$ special copies of $Y_{i,i}$ defines such a point p'_a, and these $\det(S)$ points are the points that comprise Λ_a.

As remarked above, the points in T_a can be paired up so the signs of each pair sum to zero,

$$T_a = U_\alpha\{e_\alpha, e'_\alpha\} \tag{7.2}$$

For each pair $\{e_\alpha, e'_\alpha\}$ in (7.2), E_1- induces orientations on $S^3 \times e_\alpha$ and $S^3 \times e'_\alpha$, and these orientations are opposite.

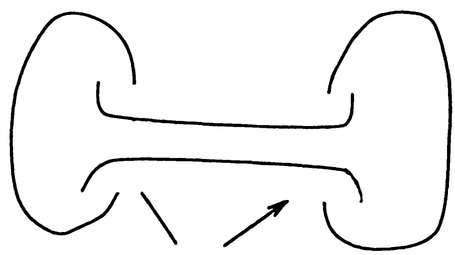
Now, for each pair, $\{e, e'\}$ on (7.2)'s right side, embed $[0,1]$ into S^3 to have boundary $\{e, e'\}$. (Do this in such a way that the embedded intervals from distinct pairs do not intersect.) The associated $S^3 \times [0,1]$ has boundary $(S^3 \times e) \cup (S^3 \times e')$ and the orientations here agree with those which are induced by E_{1-}. Hence, $S^3 \times [0,1] \subset (S^3 \times S^3)_a$ can be surgered to E_{1-} along their common boundaries, $(S^3 \times e) \cup (S^3 \times e')$. The result is a topological embedding in Z of a smooth, oriented manifold with boundary, where the boundary embeds (smoothly) in ∂Z. (The embedding has "corners", these being the components of $S^3 \times \{e_\alpha, e'_\alpha\}$ where the surgery took place.) The point is that this new manifold has two less boundary components then E'_{1-}. Here is a picture:



(7.3)

Make the preceding construction for each pair on the right side of (7.2). The result is a topological embedding of a surgered E'_{1-}. (The "corners" of the embedding are the components of $S^3 \times T_a$.) The embedding of this surgered E'_{1-} intersects $(S^3 \times S^3)_a$ in $S^3 \times \Lambda_a$ (where it is the same as E'_{1-}) and also in a copy of $S^3 \times [0,1]$ for each pair on the right in (7.2). (The copies of $S^3 \times [0,1]$ for distinct pairs will not intersect if one takes care to insure that the embedded $[0,1]$'s from different pairs do not intersect.)

Now note that the copies of $S^3 \times [0,1]$ can be isotoped normally off $(S^3 \times S^3)_a$ (push radially outward in the coordinates of Lemma 3.6 so that the resulting embedding of the surgered E'_{1-} intersects $(S^3 \times S^3)_a$ in $S^3 \times \Lambda_a$. And, note that all of the "corners" in the resulting embedding can be readily smoothed so that the result is an embedded submanifold of Z. The following diagram illustrates:



(7.4)

The preceding construction can be done at all $a \in \mathrm{crit}_1(f)$. The result is a submanifold, $E''_{1-} \subset W$. Note that E''_{1-} has a minimal number of intersections with any $(S^3 \times S^3)_a \subset \partial Z$ as its intersection is equal $S^3 \times \Lambda_a$, a set of $\det(S)$ push-off copies of $S^3 \times p_a$. Note also that E''_{1-} agrees with E'_{1-} away from $\{(S^3 \times S^3)_a\}_{a \in \mathrm{crit}_1(f)}$.

c) Tubing near (b, b)

Let $b \equiv b_i \in \mathrm{crit}_1(f)$. Consider now the intersection of E''_{1-} with $(S^3 \times S^3)_b$: As described in (6.8), 6.9), this intersection is given as

(7.5) $$S^3 \times \Lambda'_b,$$

where $\Lambda'_b \subset S^3$ is a finite set of points. Each point in Λ'_b comes with a sign (± 1), and there are $\det(S)$ more plus signs than minus signs. This means that the set Λ'_b can be decomposed as $\Lambda_b \cup T_b$, where the points in T_b can be paired so that the signs in each pair add to zero.

It proves useful to take some care in defining the set Λ_b. Here is how: The flow line $\mu (\equiv \mu_i)$ which starts at a and ends at b. The intersection of $\mu \times \mu$ with $(S^3 \times S^3)_b$ is a point, $p_b \times p_b$, where $p_b \in S^3$. With p_b singled out, note that the intersection of any push-off copy of $Y_{i,i}$ with $(S^3 \times S^3)_b$ contains a unique $(S^3 \times p'_b)$ where p'_b is the push-off of p_b. (There is a canonical isotopy between the push- off copy and the original (shrink λ to zero in (6.12), and under this isotopy, p'_b moves to p_b.) In particular, each of the $\det(S)$ special copies of $Y_{i,i}$

defines such a point p'_b, and these $\det(S)$ points are the points that comprise Λ_b.

As remarked above, the points in T_b can be paired up so the signs of each pair sum to zero,

$$T_b = U_\alpha\{e_\alpha, e'_\alpha\} \tag{7.6}$$

For each pair $\{e_\alpha, e'_\alpha\}$ in (7.5), E''_{1-} induces orientations on $S^3 \times e_\alpha$ and $S^3 \times e'_\alpha$, and these orientations are opposite. With this understood, one can repeat the tubing construction as described in the previous subsection (see (7.4), (7.5)) to surger E''_{1-} near (b, b) and then isotope the result to obtain an embedded submanifold of Z which intersects $(S^3 \times S^3)_b$ in $S^3 \times \Lambda_b$. Furthermore, this last construction can be done simultaneously near all (b, b) for $b \in \text{crit}_2(f)$. Use E_{2-} to denote the resulting submanifold of Z.

d) The intersection of $E_{2\pm}$ and $E_{R,L}$.

The next four subsections describe various properties of $E_{2\pm}$. The purpose of this subsection is to prove

LEMMA 7.1. *The submanifolds $E_{2\pm} \subset Z$ can be constructed as described above so that they do not intersect $E_{L,R}$ of (4.15).*

Proof. Let $U \subset W$ be an open neighborhood of $\text{crit}_1(f)$ and let $U' \subset W$ be an open neighborhood of $\text{crit}_2(f)$. Then $E_{2\pm}$ can be made (as described above) so that they are supported in Z's intersection with $(\text{fut}(U)\times \text{ past}(U'))$. The latter set is disjoint from $E_{R,L}$ if U and U' are not too big; this is because the flow line γ misses f's critical points. □

e) The intersection of $E_{2\pm}$ with Δ_Z:

Fix $i \in \{1, \cdots, r\}$ and let $a \equiv a_i$ and $b \equiv b_i$. By construction, E_{2-} intersects $(S^3 \times S^3)_a$ in $S^3 \times \Lambda_a$. It intersects $(S^3 \times S^3)_b$ in $S^3 \times \Lambda_b$. Here, Λ_a and Λ_b are sets of $\det(S)$ points.

Now, there is a natural way to pair the points in Λ_a with those in Λ_b and here it is: When $p \in \Lambda_a$ and $p' \in \Lambda_b$ are partners, then (p, p) and (p', p') are the endpoints of a transversal component of $E_{2-} \cap \Delta_z$ which is an embedding of $[0, 1]$.

Such a pairing exists for the following reasons: If $p \in \Lambda_a$, then $S^3 \times p$ is a component of the intersection of a push-off copy of $Y_{i,i-}$ with $(S^3 \times S^3)_a$. By design, there exists a unique $p' \equiv p'(p) \in \Lambda_b$ for which $S^3 \times p'$ is a component of the intersection of *the same* push- off copy with $(S^3 \times S^3)_b$. This is another definition of the pairing between Λ_a and Λ_b.

To finish the story, remark that the afore-mentioned push-off copy of $Y_{i,i-}$ is $(B_{1a+} \times B'_{1b-}) \cap Z$, where B'_{1b-} is a push-off copy of B_{1b-}. And, both p and p'

lie on a push-off copy, $\mu' \subset B_{1a+} \cap B'_{1b-}$, of a chosen flow line, $\mu(\equiv \mu_i)$, which starts at a and ends at b. Finally, $(\mu' \times \mu')$ intersects Z transversally in Δ_Z and $(\mu' \times \mu') \cap Z$ is an embedded interval in Δ_Z and a transversal component of $E_{2-} \cap \Delta_Z$.)

With the preceding understood, one sees that

$$E_{2-} \cap \Delta_Z = (\cup_{i=1}^r \Gamma_i) \cup C, \tag{7.7}$$

where Γ_i is the union of $\det(S)$ push-offs (in Δ_Z) of $(\mu_i \times \mu_i) \cap \Delta_Z$, and where $C \subset \operatorname{int}(\Delta_Z)$ is compact. Infact, after an (arbitrarilly small) isotopy of the push-offs of the $\{B_{1b-} : b \in \operatorname{crit}_2(f)\}$ (with support away from $\operatorname{crit}(f)$), one can arrange for the intersection in (7.7) to be transversal. In this case, C is a disjoint union of embedded circles in $\operatorname{int}(\Delta_Z)$.

f) Normal framings.

Consider now the normal bundle to E_{2-}. Of particular interest in subsequent sections is the fact (see Lemma 7.2, below) that E_{2-} has trivial normal bundle. Also of interest is the behavior of a framing of this normal bundle on ∂E_{2-} and along the components of $\{\Gamma_i\}$ from (7.7).

Two digressions are required before Lemma 7.2: The first digression defines the notion of a *product* framing of the normal bundle of a submanifold in Z: This is a framing of the normal bundle with the property that each basis vector is annihilated by the differential of either π_L or π_R. The same definition works to define the product framing of a submanifold of $W \times W$.

A second digression is required to set the stage for a discussion of the normal framing near ∂E_{2-} and $\{\Gamma_i\}$. To start, consider $i \in \{1, \cdots, r\}$. As usual, let $a \equiv a_i$ and $b \equiv b_i$. Let $\mu \subset \Gamma_i$ be a component and define p, p' by requiring $(p,p) \equiv \mu \cap (S^3 \times S^3)_a$ and $(p',p') \equiv \mu \cap (S^3 \times S^3)_b$. Associate to μ the subset of E_{2-}

$$(S^3 \times p) \cup (S^3 \times p') \cup \mu. \tag{7.8}$$

Note the following: Let $\mu' \subset \Gamma_i$ be any other component. Then, E_{2-} near the μ' version of (7.8) is naturally defined as a push-off of E_{2-} near the μ version of (7.8). (Near the μ-version of (7.8), E_{2-} is a push-off copy of an open neighborhood of $(B_{1a+} \times B_{1b-}) \cap Z$. And, near the μ'-version, E_{2-} is a different push-off copy of the same open neighborhood. Infact, each of these push-off copies is constructed as $B_{1a+} \times$ (push-off copy of B_{1b-}).

These last observations give a natural method of comparing a given normal framing of E_{2-} along the μ and μ' versions of (7.8). See (6.13).

End the second digression.

LEMMA 7.2. *The submanifold E_{2-} has trivial normal bundle in Z. Furthermore, the normal bundle to E_{2-} has a framing with the following properties:*

Let $i \in \{1, \cdots, r\}$ and let $\mu \in \Gamma_i$. Then the frame is a product frame along (7.8) and it restricts as a constant frame along $S^3 \times p$ and $S^3 \times p'$. Furthermore, let $\mu' \subset \Gamma_i$ be a different component. Then the push-off which identifies E_{2-} near the μ and μ' versions of (7.8) will identify the restriction of the frame to the μ and μ' versions of (7.8).

Proof. Because E_{2-} is constructed by surgering E_{1-} and the latter is a union of (6.1)'s $\{Y_{i,j-}\}$, the proof starts with a description of the normal bundles to (6.1)'s $\{Y_{i,j\pm}\}$. To begin, consider $i, j \in \{1, \cdots, r\}$ such that $i \leq j$. Let $a \equiv a_i$ and $b \equiv b_j$. Then $B_{1a+} \times B_{1b-} \subset W \times W$ has trivial normal bundle with a natural product framing. This implies that $Y_{i,j-}$ in (6.1) has a natural product framing of its normal bundle in Z. (See Lemma 6.1.)

Consider now $i = j$ and the induced normal framing of a component of $\partial Y_{i,i-}$. $\square$

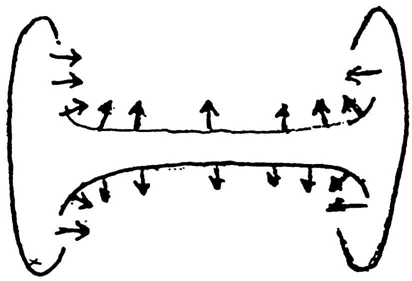
LEMMA 7.3. *Let c denote either a_i or b_i. Let $S^3 \times p$ be a component of $\partial Y_{i,i-} \cap (S^3 \times S^3)_c$. Then the product normal framing of $Y_{i,i-}$ in Z induces a product normal framing of $S^3 \times p$ in $(S^3 \times S^3)_c$ and this induced normal framing is homotopic through product framings to the constant normal framing as defined by choosing a fixed basis for $TS^3 \mid_p$ and using the projection π_R to write the normal bundle in question as $S^3 \times TS^3 \mid_p$.*

Proof. Consider first the case $c = a_i$. Here, p is described by (6.9). Think of the vector $v \equiv (v_2, v_3, v_4)$ as a point in the unit 2-sphere about the origin in the 3-plane spanned by the coordinates (y_2, y_3, y_4). With this understood, then (6.9) implies that a product normal frame to $Y_{i,i-}$ restricts to $(S^3 \times p) \subset \partial Y_{i.i-}$ to have the form $\partial_{x_1}, e_2, e_3)$, where the vectors $e_{2,3} \in TS^2 \mid_v$, and where ∂x_1 is tangent to the x_1 axis. In particular, this is a normal frame for $S^3 \times p$ in $(S^3 \times S^3)_c$. Furthermore, it is homotopic through product frames to the trivial frame because $\pi_3(SO(2)) = 1$. (In fact, the vectors $e_{2,3}$ depend only on the y_1 coordinate.)

Next, consider the case where $c = b_i$: Here, $S^3 \times p$ is described in (6.11). Think of the vector $v \equiv (v_1, v_2)$ in (6.11) as a point in the unit circle in the plane $x_3 = x_4 = 0$. Then, a product normal frame from $Y_{i,i-}$ restricts to $(S^3 \times p) \subset \partial Y_{i,i-}$ to have the form $(e_1, \partial_{y_3}, \partial_{y_4})$, where $e_1 \in TS^2 \mid_v$ and where $\partial_{y_{3,4}}$ are tangent to the y_3 and y_4 coordinate axis, respectively. This frame is evidently homotopic through product frames to the constant frame; simply homotope e_1 to a constant length vector.

End the digression. $\square$

To complete the proof of Lemma 7.2, remember that E_{2-} was constructed from E_{1-} by taking a pair, $S^3 \times e$ and $S^3 \times e'$, in the same boundary component and gluing to them a boundary $S^3 \times I$. Here I is an embedded interval in S^3 with boundary $\{e, e'\}$. According to Lemma 7.3, the induced normal framing on any boundary component is homotopic to the constant framing; and so there is no obstruction to connecting the normal framing on $S^3 \times e$ to the normal framing on $S^3 \times e'$ over the interval $S^3 \times I$. The following diagram illustrates the procedure:



(7.9)

The aforementioned argument shows that E_{2-} has a framing for its normal bundle. But, the argument above also shows that there is a framing for the normal bundle of E_{2-} which agrees with Lemma 7.3's product framing for E_{1-} near (7.8) for any $i \in \{1, \cdots, r\}$ and any $\mu \in \Gamma_i$. (Remember that near (7.8), E_{2-} and E_{1-} agree.) This last observation plus Lemma 7.3 imply the final two statements of Lemma 7.2.

g) Further properties.

Define E_{2-} as above. Then, define $E_{2+} \subset Z$ to be the image of E_{2-} under the switch map which sends $(x, y) \subset Z$ to (y, x). The following proposition lists the salient features of $E_{2\pm}$:

PROPOSITION 7.4. *Define $E_{2\pm}$ as above. These submanifolds can be constructed and oriented so that the following hold:*

1) *The fundamental classes of $E_{2\pm}$ obey (6.3).*
2) *$E_{2\pm}$ intersect ∂Z transversely in $\partial E_{2\pm}$*
3) *$E_{2\pm}$ have empty intersection with $M_0 \times M_0$ and $M_1 \times M_1$.*
4) *If $p \in$ crit(f), the intersection of E_{2-} with $(S^3 \times S^3)_p$ is $S^3 \times \Lambda_p$ where $\Lambda_p \subset S^3$ is a set of det(S) points. Similarly, the intersection of E_{2+} with $(S^3 \times S^3)_p$ is $\Lambda_p \times S^3$.*
5) *The normal bundle of E_{2-} are described by Lemma 7.2 and the normal bundle of E_{2+} is described by Lemma 7.2 if (7.8) is replaced by its switched version, $(p \times S^3) \cup (p' \times S^3) \cup \mu$.*
6) *$H^2(E_{2\pm}; \mathbb{Z}) = 0$.*
7) *$E_{2\pm}$ have empty intersection with $E_{R,L}$ of (4.15).*

Proof. The only assertion which is not already proved is Assertion 6. To prove Assertion 6 for E_{2-}, remark first that E_{1-} has vanishing H^2. (See Proposition 6.4.) Then, note that E_{2-} is constructed from E_{1-} by gluing various copies of $S^3 \times I$ onto boundary $(S^3 \times S^0)$'s. This sort of surgery will decrease H^0 or increase H^1, but it can not change H^2. □

8 The third pass at $E_\pm$. The submanifolds $E_{2\pm}$ of the preceding section intersect the diagonal as described in (7.7), with $C \subset \text{int}(\Delta_Z)$ being a finite union of embedded circles. The purpose of this subsection is to modify some number of like oriented, push-off copies of $E_{2\pm}$ so that the result, $E_{3\pm}$, intersects Δ_Z as in (7.7) but with $C = \emptyset$. To be precise, consider:

PROPOSITION 8.1. *There are oriented submanifolds (with boundary) $E_{3\pm} \subset Z$ with the following properties:*

1) *E_{3+} is the image of E_{3-} under the switch map on Z sending (x, y) to (y, x).*
2) *The fundamental classes $[E_{3\pm}]$ are equal to $N\,[E_{1\pm}]$ for some integer $N \geq 1$. Here, $[E_{1\pm}]$ are described by (6.3) and Lemma 6.3.*
3) *$E_{3\pm}$ have empty intersection with $M_0 \times M_0$ and $M_1 \times M_1$.*
4) *$E_{3\pm}$ have empty intersection with $E_{L,R}$ of (4.15).*
5) *If $p \in \mathit{crit}(f)$, then the intersection of E_{3-} with $(S^3 \times S^3)_p$ has the form $S^3 \times \Lambda_p$, where Λ_p is a set of N points. Similarly, the intersection of E_{3+} with $(S^3 \times S^3)_p$ is $\Lambda_p \times S^3$.*
6) *$E_{3-} \cap \Delta_Z = \cup_{i=1}^{r} \Gamma_i$, where $\Gamma_i \subset \Delta_Z$ is as follows: There is a flow line μ_i which starts at a_i and ends at b_i. With the canonical identification of Δ_W with W understood, Γ_i is the union of N like oriented, disjoint, push-off copies of a closed interval, $I \subset \mu_i$. And, each of these N push-offs of I starts in $(\Lambda_a \times \Lambda_a) \cap \Delta_Z$ and ends in $(\Lambda_b \times \Lambda_b) \cap \Delta_Z$.*
7) *Both $E_{3\pm}$ have trivial normal bundles in Z. The normal bundle of E_{3-} has a framing, ζ, which restricts to a product normal framing on a neighborhood of $(\cup_{i=1}^{r}\Gamma_i) \cup \{S^3 \times \Lambda_p\}_{p\in\, crit(f)}$. Furthermore, this framing ζ restricts to $\{S^3 \times \Lambda_p\}_{p\in\, crit(f)}$ as a constant framing. The normal bundle to E_{3+} in Z is described by applying the switch map to the preceding.*
8) *$H^2(E_{3\pm}; \mathbb{Q}) = 0$.*

(Compare with Proposition 7.4.)

The rest of this section is devoted to the construction of E_{3-}. The first subsection below ($8a$) introduces some of the basic tools. Subsections $8b - 8e$ apply the tools from $8a$ to the proof of Proposition 8.1. The final subsections, $8f - 8h$, contain the proofs of three propositions that are stated in $8a$.

a) Deleting circles.

In comparing Propositions 8.1 and 7.4, one sees that the essential difference between E_{2-} and E_{3-} is that the intersection of both are described by a form of (7.7), but that $E_{3-} \cap \Delta_Z$ has no compact components. With this understood, remark that E_{3-} will be constructed from some number of like oriented, disjoint, push-off copies of E_{2-} by surgery, with the point of the surgery to eliminate the unwanted compact components of the intersection with Δ_Z. Of course, this must be done so as not to destroy any of desired properties of E_{2-}–i.e., Assertions 2-5 and 7, 8 of Proposition 8.1.

In abstraction, the problem is to remove circles which are components of the transversal intersection between two four dimensional submanifolds inside a seven dimensional submanifold. Here is the model:

MODEL: Let X be a connected, oriented 7-manifold, and let $A, B \subset X$ be oriented, dimension 4 submanifolds which intersect transversally. Let $O \subset X$ be an open set and let $\sigma \equiv (A \cap B) \cap O$. Suppose that σ is compact; a disjoint union of oriented, embedded circles.

(8.1)

Given the model, here are the problems:

PROBLEM 1: Find an oriented, dimension 4 submanifold $A' \subset X$ with the following properties:

1) $A' \cap (B \cap O) = \emptyset$.
2) $A - (A \cap O) = A' - (A' \cap O)$.
3) $[A] = [A']$ in $H_4(X, X - O)$.

PROBLEM 2: Find A' as in Problem 1 with $H^2(A'; \mathbb{Q}) = 0$.

PROBLEM 3: Assuming that $A - (A \cap O)$ has trivial normal bundle, find A' solving Problems 1 and 2 with trivial normal bundle. And, given, apriori, a frame ζ for A' as normal bundle over $A - (A \cap O)$, extend ζ over A' as a normal bundle framing.

(8.2)

These three problems will arise a number of times in the subsequent two sections and will be solved under various assumptions on A, B and O.

The solution to Problem 1 begins with the following basic surgery result:

PROPOSITION 8.2. *Let X, A, B, and O be as described in (8.1) and in Problem 1 of (8.2). If the class, $[\sigma]$, of σ is zero in $H_1(B \cap O; \mathbb{Z})$, then there is a solution to Problem 1.*

Problem 2 can be solved when extra conditions are added:

PROPOSITION 8.3. *Let X, A, B, and O be as in (8.1) and Problems 1 and 2 of (8.2). Assume that*

a) $[\sigma] = 0$ *in* $H_1(B \cap O; \mathbb{Z})$.
b) The map $H^3_{comp}(A; \mathbb{Q}) \to H^3(A; \mathbb{Q})$ *is injective. And, assume either*
c) $H_1(\sigma; \mathbb{Q}) \to H_1(A; \mathbb{Q})$ *is injective, or else assume*
d) $B \cap O$ *is connected and* $[\sigma] \neq 0$ *in* $H_1(A; \mathbb{Q})$, *Then, there exists* $A' \subset X$ *which solves Problems 1 and is such that* $H^2(A'; \mathbb{Q}) \approx H^2(A; \mathbb{Q})$. *Thus, Problem 2 is solved by* A' *if* $H^2(A; \mathbb{Q}) = 0$.

Remark that Condition b of this proposition will be true automatically if $A \cap O$ is the interior of a manifold with boundary, $\underline{A}$, whose boundary, $\partial\underline{A}$, obeys $H^2(\partial\underline{A}; \mathbb{Q}) = 0$.

To solve Problem 3 of (8.2), it is necessary to digress first to define a $\mathbb{Z}/2$ valued invariant for homologically trivial, normally framed circles in an oriented 4-manifold with even intersection form. (This is invariant is well known to 4-manifold topologists.)

To start the digression, let B denote the oriented 4-manifold. To say that B's intersection form is even is to say that the self- intersection number of any embedded, orientable surface in B is an even number. (Note that B need not be compact.)

Let $\sigma \subset B$ be the finite union of disjointly embedded, oriented circles which represents the trivial element in $H_1(B; \mathbb{Z})$. The invariant in question, $\chi_{B,\sigma}(\cdot)$, assigns ± 1 to the various homotopy classes of framings of the normal bundle to σ in B. (If σ is a single circle, then there are precisely two normal framings up to homotopy since $\pi_1(SO(3)) \approx \mathbb{Z}/2$.)

To calculate $\chi_{B,\sigma}$, first choose an oriented surface with boundary, $R \subset B$, such that $\partial R = \sigma$. An oriented frame $\zeta \equiv (e_1, e_2, e_3)$ for the normal bundle to σ in B will be called an *adapted* frame when the vector e_3 is the inward pointing normal vector to R along ∂R.

LEMMA 8.4. *Let B, σ and R be as described above. Let ζ be an oriented, normal frame for $\sigma \subset B$. Then ζ is homotopic to an adapted frame.*

Proof. On a component, C, of σ, two normal frames differ by a map from S^1 to $SO(3)$. With this understood, note that $\pi_1(SO(3)) \approx \mathbb{Z}/2$, so there are two homotopy classes of normal frames along C. Two normal frames for which e_3 is the inward normal to R differ by a map from S^1 to $SO(3)$ which factors through a map from S^1 to $SO(2) \subset SO(3)$. With the preceding understood, the lemma follows because the induced homomorphism from $\pi_1(SO(2))$ to $\pi_1(SO(3))$ is surjective. □

The important feature of an adapted normal frame is that an adapted normal frame allows one to make an unambiguous definition of the mod(2) self-intersection number, $(R \cdot R)_2$, of R. Here is how: Take a section of R's normal bundle in B which agrees with e_1 on ∂R. Perturb the section away from ∂R so that it has transverse intersection with the zero section. Then, count the number of such intersection points mod(2).

One can also define $R \cdot R \in \mathbb{Z}$ by counting intersections with sign, but only the mod(2) intersection number is required for the definition of $\chi_{B,\sigma}$.

LEMMA 8.5. *If two adapted frames are homotopic in the space of* all *normal frames for σ, then the corresponding values of $(R \cdot R)_2$ agree.*

Proof. Adapted, normal frames to a given component $C \subset \sigma$ can be found which differ by a degree one map to $SO(2)$ and are such that the corresponding push-offs of R are identical save for a small open set near a point in C. With this understood, one need only check the lemma for the case where R is a planar 2-disk in $\mathbb{R}^4$. See, e.g Section 1.3 of [**7**]. □

It follows from Lemmas 8.4 and 8.5 that the surface R defines a map, $\chi_{B,\sigma}(\cdot)$, from the set of homotopy classes of normal frames of $\sigma \subset B$ to $\mathbb{Z}/2$. By definition, $\chi_{B,\sigma}(\zeta)$ assigns to ζ the number $(R \cdot R)_2$ that is computed by using an adapted frame which is homotopic to ζ.

Consider the dependence of $\chi_{B,\sigma}(\cdot)$ on the surface R:

LEMMA 8.6. *Suppose that B has even intersection pairing in its second homology. Then $\chi_{B,\sigma}(\cdot)$ is the same for all surfaces R bounding σ.*

Proof. Let ζ be a framing of the normal bundle to σ in B. Let $R_{1,2} \subset B$ be a pair of surfaces which bound σ. The task is to show that $R_1 \cdot R_1 = R_2 \cdot R_2$ mod(2).

One can assume, with no loss of generality, that ζ is adapted to R_1. Since $\pi_1(S^2) \approx 0$, the surface R_2 can be isotoped, with σ fixed, so that e_3 is the outward pointing normal vector to R_2. With this understood, R_1 and R_2 can be joined together along σ to obtain a C^1 immersion of a closed, oriented surface, R, in B. (The lack of smoothness occurs across σ.) The surface R may not be embedded because R_1 and R_2, though individually immersed, may intersect each other. Any way, with a small isotopy of R_1 (away from ∂R_1), the intersections of R_1 with R_2 can be made transverse.

An embedded surface in B has a well defined self-intersection number. An immersed surface has a well defined intersection number also. In this case,

$$R \cdot R = R_1 \cdot R_1 + R_2 \cdot R_2 - 2(R_1 \cdot R_2) \tag{8.3}$$

The number in (8.3) is the intersection number for the embedded surface that is obtained by resolving all of the double points of R.

Given that (8.3) is the self intersection number of an embedded surface in B, the assumptions in Lemma 8.5 require that (8.3) be an even number. Thus $R_1 \cdot R_1 = R_2 \cdot R_2$ mod(2) as required.

(Here is how to resolve a double point of an immersed surface: In local coordinates the transveral intersection of the two sheets of the surface is described by the zeros in $\mathbb{C}^2 = \mathbb{R}^4$ of the equation

$$z_1 \, z_2 = 0. \tag{8.4}$$

The resolution of the intersection point replaces the solution to (8.4) with the solution to the equation $z_1 \, z_2 = \epsilon$. Here, $\epsilon \in \mathbb{C}$ is small but not zero.) □

With the invariant $\chi_{B,\sigma}(\cdot)$ of Lemma 8.6 understood, end the digression.

Here is a solution to (8.2)'s third problem:

PROPOSITION 8.7. *Let X, A, B, and O be as in (8.1) and Problems 1 and 2 of (8.2). Assume that Conditions a, b and either c or c′ of Proposition 8.3 hold. Suppose that B has even intersection form and that $A \subset X$ has a trivial normal bundle. Let ζ be a given frame for A's normal bundle in X.*

1) *The restriction of ζ to $\sigma \equiv (A \cap B) \cap O$ defines a normal frame, $\zeta_\sigma \equiv e\,|_\sigma$, to σ in B.*
2) *If $\chi_{B,\sigma}(\zeta_\sigma) = 0$, then there is a solution, $A' \subset X$, to Problem 1 such that the normal frame ζ over $A - (A \cap O)$ extends over A'.*
3) *Thus, if $H^2(A; \mathbb{Q}) = 0$, then A' solves Problems 1-3 of (8.2).*

The proofs for Propositions 8.2, 8.3 and 8.7 are given in Subsections 8f-8h.

b) The proof of Proposition 8.1.

Let E'_{2-} denote the disjoint union of some number $N \geq 1$ disjoint copies of E_{2-}. The goal is to apply Propositions 8.2, 8.3 and 8.7 to remove the compact (circle) components, C, of the intersection of E'_{2-} with Δ_Z. With this goal understood, Proposition 8.2, 8.3 and 8.7 will be considered with the following identifications: Take

$$X = \operatorname{int}(Z), \quad A = \operatorname{int}(E'_{2-}), \quad B = \operatorname{int}(\Delta_Z). \tag{8.5}$$

Take O to be the compliment in $\operatorname{int}(Z)$ of the closure of a regular neighborhood of

$$\partial Z \cup (\cup_{i=1}^r \underline{\mu}_i) \cup E_R \cup E_L. \tag{8.6}$$

Here, $\underline{\mu}_i \equiv (\mu_i \times \mu_i) \cap \Delta_Z$ with μ_i as in Section 7*b*. This regular neighborhood should contain $\{\Gamma_i\}$ in (7.7) of $E_{2-} \cap \Delta_Z$, and it should also contain the push-off copies of $\{\Gamma_i\}$which comprise the interval components of $E'_{2-} \cap \Delta_Z$. Needless to say, O should contain the compact components of $E'_{2-} \cap \operatorname{int}(Z)$.

With this choice of X, A, B and O, the assertions of Proposition 8.1 will follow from Proposition 7.4 if the hypothesis of Propositions 8.2, 8.3 and 8.7 can be verified for a suitable N. (Remember that E'_{2-} is comprised of N push-off copies of E_{2-}.) Note: With regard to Proposition 8.7, the normal framing, ζ, of any push- off copy of $E_{2-} \subset E'_{2-}$ should be the normal framing of E_{2-} which is described by Lemma 7.2.

Subsections 8c-8e verify that there exists $N \geq 1$ for which the hypothesis of these three propositions are satisfied. □

c) Removing circles in $E'_{2-} \cap \Delta_Z$.

The purpose of this subsection is to verify that there exists an integer $N_1 \geq 1$ which is such that the hypothesis of Proposition 8.1 can be verified when E'_{2-} is any multiple of N_1 push-off copies of E_{2-}.

The discussion begins with a digression to study the first homology of $B \cap O$. (Equations (8.5) and (8.6) define B and O.) The projection π_L (or π_R) identifies

B with $\operatorname{int}(W)$. This projection identifies $B \cap O$ with the compliment in W of a regular neighborhood of $\partial W \cup \operatorname{crit}(f) \cup (\cup_{i=1}^r \mu_i) \cup \gamma$, where γ is the flow line in 4 of Definition 3.1.

Now consider $\sigma \subset B \cap O$, a finite union of embedded, oriented circles. After a small isotopy, the circles in σ can be arranged to have empty intersection with the descending disks from $\operatorname{crit}_2(f)$. With this isotopy understood, the pseudo-gradient flow will isotope the circles in σ so that the resulting circles, σ_1, lies in the open submanifold $W_3 \equiv \{x \in W : 3/4 < f(x) < 1\}$. That is, $f(\sigma_1)$ is larger than any critical value of f.

The pseudo-gradient flow defines a diffeomorphism between W_3 and $M_1 \times (3/4, 1)$. By assumption, M_1 is a rational homology sphere, which means that the homology class, $[\sigma_1]$, of σ_1 is zero in $H_1(W_3 - W_3 \cap \gamma; \mathbb{Q})$. (Note that $W_3 \cap \gamma = p_1 \times (3/4, 1)$.) Alternately, one can conclude that

$$N_1\,[\sigma_1] = 0 \in H_1(W_3 - W_3 \cap \gamma, \mathbb{Z}) \tag{8.7}$$

for some integer $N_1 \geq 1$. This means that N_1 push-off copies of σ_1 bounds an embedded surface in $W_3 - W_3 \cap \gamma$. (Orient all N_1 push-off copies of σ_1 identically.)

Thus, N_1 push-off copies of σ will bound an embedded surface in $B \cap O$.

End the digression.

To verify Proposition 8.2's hypothesis for E'_{2-}, consider the discussion of the preceding subsection where σ is equal to C in (7.7). This choice of σ determines the integer N_1 in (8.7). If $N_1 = 1$ in (8.7), then Proposition 8.1 can be directly applied to $A \equiv E_{2-}$ so that the result, A', intersects $\Delta_Z \subset Z$ as described by (7.7) but with $C = \emptyset$.

However, the case $N_1 > 1$ in (8.7) can not be ruled out. In the case that $N_1 > 1$, let $m \geq 1$ and let E'_{2-} denote the disjoint union $m\,N_1$ disjoint, push-off copies of E_{2-}, all oriented as E_{2-}. (Use the normal framing of Section 7f when making these push-offs.)

With E'_{2-} understood, observe that

$$E'_{2-} \cap \Delta_Z = (\cup_{i=1}^r \Gamma_i) \cup C', \tag{8.8}$$

where C' in (8.8) is, by design, $m\,N_1$ disjoint, push-off copies of C from (7.7). In (8.8), each Γ_i is the union of $m\,N_1 \det(S)$ push-offs (in Δ_Z) of $\mu_i \equiv \Delta_Z \cap (\mu_i \times \mu_i)$.

By construction, the homology class of C' in $H_1(B \cap O; \mathbb{Z})$ is zero. (Because $[C'] = m\,N_1\,[C]$ and the class of C is N_1-torsion.)

With the preceding understood, then Proposition 8.1 can be applied with X, A, B and O as described by (8.5) and (8.6) so long as the number N is a multiple of N_1 in (8.7).

d) Constraining H^2.

Proposition 8.2 constructs a submanifold $A' \subset Z$ from some number $N \geq 1$ push-off copies of E_{2-}. (Here, N must be a multiple of N_1 from (8.7).) This

A' is constructed so that it misses $E_{L,R}$ and a form like E_{2-} near ∂Z. And, the intersection of A' with Δ_Z is the union $\cup_{i=1}^{r}\Gamma_i$, where Γ_i is the union of $N\det(S)$ push-off copies of the path $\underline{\mu}_i$. If Proposition 8.1's E_{3-} is this A', then A' will have to have vanishing 2nd cohomology. That is, A' must be a solution to Problem 2 in (8.2).

Proposition 8.3 will be used to solve Problem 2 in the case at hand; this is the subject of the present subsection.

The task here is to verify that the conditions of Proposition 8.2 can be met for $A \equiv E'_{2-} \cap X$ with E'_{2-} some number, N, of like- oriented, push-off copies of E_{2-}. (Note that Assertion 6 of Proposition 7.4 asserts that $H^2(A;\mathbb{Q}) = 0$.) Taking Conditions $a - c$ in order, remark that the previous subsection has established that Condition a is satisfied when N is divisable by a certain integer (N_1 of (8.7)).

Condition b is satisfied because of Assertion 4 of Proposition 7.4. That is, A has closure a manifold with boundary, and the boundary is a number of copies of S^3. Since $H^2(S^3) = 0$, the required injectivity holds.

Condition c is established by the following lemma:

LEMMA 8.8. *Let C denote the union of the compact components of $E_{2-}\cap\Delta_Z$. The inclusion of C into E_{2-} induces a monomorphism from $H_1(C;\mathbb{Z})$ into $H_1(E_{2-};\mathbb{Z})$.*

Proof. Remark that E_{2-} is obtained via ambient surgery (in Z) on various embedded $(S^0 \times B^4)$'s in disjoint unions of $\{Y_{i,j-} : i \le j \in \{1,\cdots,r\}\}$ (see (7.3)).

As remarked earlier, $Y_{i,j-}$ can be viewed as the result of ambient surgery (in $F^{-1}(1/8)$) on various embedded $(S^0 \times B^4)$'s in the 4-sphere which is the intersection of the descending 5-disk from (a_i, b_j) with $F^{-1}(1/8)$. For a given $S^0 \times B^4$, the $S^0 \times \{0\}$ is a pair of algebraically cancelling intersections of said descending 4-sphere with the ascending 4-disk from some (a_i, a_k) or (b_k, b_j) in $\mathrm{crit}_4(F)$.

Thus, E_{2-} is obtained from a disjoint union of embedded 4- spheres in $F^{-1}(1/8)$ by ambient surgery on embedded $(S^0 \times B^4)$'s. It follows from the preceding that $H_1(E'_{2-})$ is a summand of some number of $\mathbb{Z}$'s. And, it follows that a union, σ, of oriented, embedded circles in E_{2-} injects its first homology into $H_1(E_{2-};\mathbb{Q})$ if:

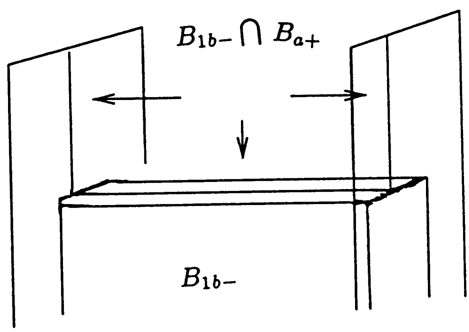
1) An added $I \times S^3$ which intersects σ has intersection number ± 1 with $\{\text{point}\} \times S^3$.
2) Each component of σ intersects at least one $I \times S^3$.

(8.9)

In the present circumstances, σ is the union of the compact components of $E'_{2-} \cap \Delta_Z$. To understand σ, remember that E_{2-} is constructed from E_{1-} by ambient surgery. The reader can check that this surgery is disjoint from any compact components of $E_{1-} \cap \Delta_Z$. Indeed, the surgery from E_{1-} to E_{2-} takes place on push-off copies of $\{Y_{i,i-}\}_{i=1}^{r}$, but the compact components of $E_{1-} \cap \Delta_Z$ are the components of the various push-offs of $\cup_{i<j}(Y_{i,j-} \cap \Delta_Z)$.

Thus, the compact components of $E_{2-} \cap \Delta_Z$are of two types: *A* Type 1 component is a compact component from $E_{1-} \cap \Delta_Z$. And, a Type 2 component is created by the surgery which changed E_{1-} into E_{2-}. (The latter are made in the surgery on the various push-off copies of $\{Y_{i,i-}\}_{i=1}^{r}$.)

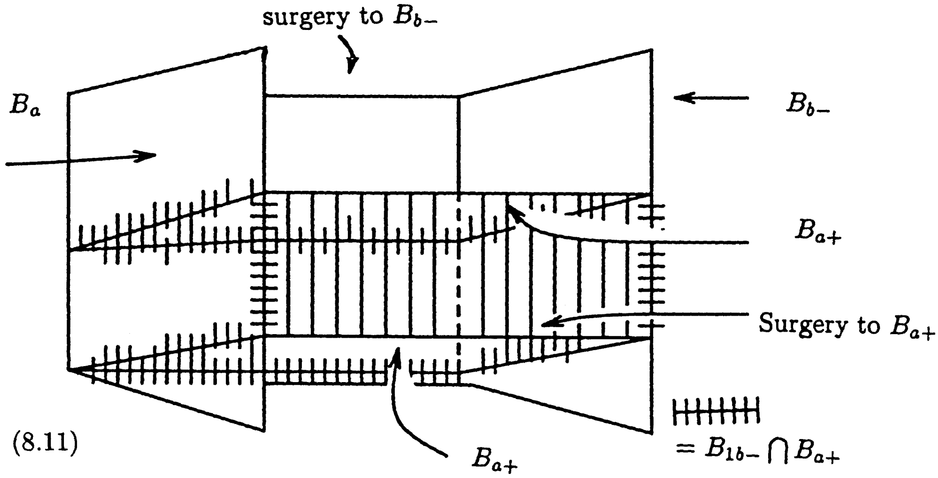
To understand the Type 1 components, use π_L or π_R to identify Δ_W with W and this intersection is identitified with $B_{1a+} \cap B_{1b-}$. (Here, $a \equiv a_i$ and $b \equiv b_j$.) To see the latter, start with $B_{a+} \cap B_{b-}$. This is a disjoint union of flow lines which start at a and end at b. The surgery which changes B_{b-} to B_{1b}-effects the intersection with B_{a+}. The effect is to surger the flow lines near a. See the following picture:



(8.10)

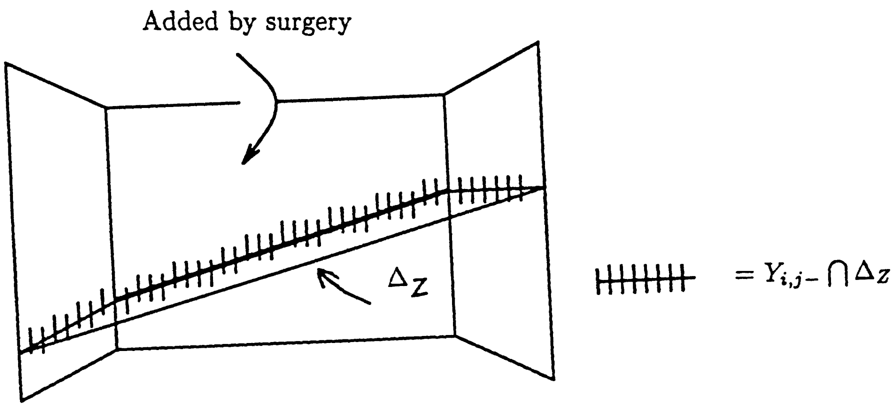
A similar picture occurs near b when B_{a+} is surgered to produce B_{1a+}. The resulting intersection $B_{1a+} \cap B_{1b-}$ differs from $B_{a+} \cap B_{b-}$ in that the ends of the flow lines in the latter have been tied together near a and near b to produce a compact intersection with some number, $n_{i,j}$, of components. (This $n_{i,j}$ is at least one, but no more than half of the number of components in $B_{a+} \cap B_{b-}$.)

See the following (very schematic) picture:



(8.11)

The effect of the preceding picture for the intersection with Δ_Z of one push-off copy of $Y_{i,j-}$ is as follows: Each surgery ties together an end of one flow line (for f's pseudo-gradient) in Δ_Z with the nearby end of a second flow line in Δ_Z the tie being across the associated $I \times S^3$. (Here, the canonical identification of Δ_W with W is taken implicitly.) See below:



(8.12)

Thus, each copy of $Y_{i,j-}$ in E_{2-} (for $i < j$) produces $n_{i,j}$ components in σ. And, (8.9) is satisfied for each such copy.

As (8.9) is obeyed for each copy of $Y_{i,j-}$, and as the surgeries on the different copies of $Y_{i,j}$ are independent, it follows that (8.9) is satisfied by the set of all Type 1 components. That is, (8.9) is satisfied by the union of the compact components of $E_{1-} \cap \Delta_Z$.

Consider now the Type 2 components. To understand these components, remember that E_{2-} was constructed from E_{1-} by ambient surgery on various push-off copies of $\{Y_{i,i-}\}$. The surgeries do not connect a push-off of $Y_{i,i-}$ with one of $Y_{j,j-}$ if $j \neq i$.

Fix $i \in \{1, \cdots, r\}$. The union of push-off copies of a given $Y_{i,i}$ intersects Δ_Z in the union of the corresponding push-offs of the set of flow lines which is $B_{a+} \cap B_{b-}$. (Use $a \equiv a_i$ and $b \equiv b_i$). As far as these copies of $Y_{i,i-}$ are concerned, E_{2-} is constructed from them by surgery on embedded $(S^0 \times B^4)$'s. The result surgery changes the afore mentioned intersection with Δ_Z; each such surgery near (a, a) ties the ends near (a, a) of two of flow line copies across the added $I \times S^3$. There is a similar effect near (b, b). See (8.12).

It follows from the preceding picture that (8.9) holds for all of the Type 1 flow lines also, and since the added $(I \times S^3)$'s which effect Type 1 flow lines are disjoint from Type 2 flow lines, the lemma is established in total. □

e) Prescribed framing.

The purpose of this subsection is to establish that the conditions of Proposition 8.7 can be met for X, A, B and O of (8.5) and (8.6) if the number N (of copies of E_{2-} in E'_{2-}) is an even multiple of the integer N_1 in (8.7). Here, the framing ζ of the normal bundle in Z to each push-off copy of $E_{2-} \subset E'_{2-}$ is described by Lemma 7.2.

To begin, observe first that B has vanishing rational homology in dimension 2, so the condition on B's intersection form is trivially satisfied. Also, as A is some number of push-off copies of E_{2-}, it has trivial normal bundle in Z.

Next, remember that an integer $N_1 > 1$ has already been found which has the following properties: If $N \geq 1$ is divisible by N_1 and if A is taken to be N push-off copies of E_{2-} (all like oriented), then σ is the boundary of an embedded, oriented surface $R \subset B$.

With the frame σ as described above, let $\zeta_\sigma \equiv \zeta\mid_\sigma$. The final question is the value of $\chi_{B,\sigma}(\zeta_\sigma)$. Here is the answer: If N is an *even* multiple of N_1, then $\chi_{B,\sigma}(\zeta_\sigma) = 0$. This assertion follows from the following lemma:

LEMMA 8.9. *Let X be an oriented 4-manifold with even intersection form. Let $\sigma_1, \sigma_2 \subset X$ be compact, oriented, embedded 1-manifolds which are disjoint. Let ζ_1 and ζ_2 be normal frames for $\sigma_{1,2}$, respectively. Let $\sigma = \sigma_1 \cup \sigma_2$ and let e by the normal frame for σ which is given by $\zeta\mid_{\sigma_{1,2}} \equiv \zeta_{1,2}$. Then $\chi_{B,\sigma}(\zeta) = \chi_{B,\sigma_1}(\zeta_1) + \chi_{B,\sigma_2}(\zeta_2)$.*

Proof. By assumption, σ_1 bounds an embedded, oriented surface, $R_1 \subset X$. Likewise, σ_2 bounds a similar surface, R_2. If R_1 and R_2 are in general position, then $R_2 \cap \sigma_2 = \emptyset$ and vice- versa. Meanwhile, R_1 will intersect R_2 transversally in a finite set of points. Resolve the double points in $R_1 \cup R_2$ (as in (8.4)) to obtain a compact, oriented, embedded surface, $R \subset X$, with boundary σ.

Now, no generality is lost by assuming that $\zeta_{1,2}$ are adapted frames for $R_{1,2}$, respectively. In this case, ζ will be an adapted frame for R. Then, $R \cdot R$ is given by (8.3), and the lemma follows. □

f) Proof of Proposition 8.2.

The proof starts with a digression for some constructions on a neighborhood of B in X. To start the digression, define $N \to B$ to be the normal bundle to B in X. Fix an exponential map,

$$e : N \to X \tag{8.13}$$

which maps $N\mid_\sigma$ into A. Put a smooth fiber metric on N with the property that e embeds the set

$$N' \equiv \{v \in N :\mid v \mid < 2\} \tag{8.14}$$

onto a neighborhood $N \subset X$ of B. Agree now to identify N with N' using e.

Introduce the 2-sphere bundle $S \to B$,

$$S \equiv \{v \in N :\mid v \mid = 1\}. \tag{8.15}$$

Identify S with its image by e in X. This S is the boundary of a tubular neighborhood of B in X,

$$T \equiv \{v \in N :\mid v \mid \leq 1\}. \tag{8.16}$$

End the digression.

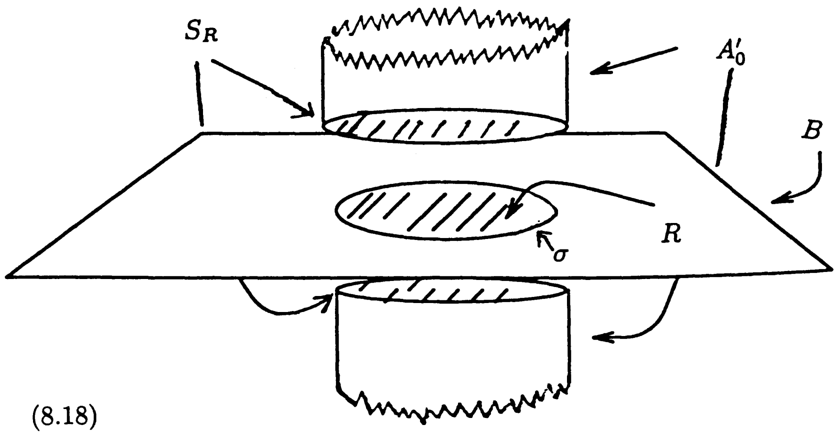
The proof proper of Proposition 8.2 starts by remarking that σ, by assumption, is the boundary of a smooth, oriented, embedded surface (with boundary), $R \subset B \cap O$. Find such an R for which $\text{int}(R)$ has no compact components. Because σ has codimension 3 in B, one can require that $\text{int}(R) \cap \sigma = \emptyset$. (The local model for R near σ is given by taking σ to be the line $x_1 = x_2 = x_3 = 0$ in $\mathbb{R}^4$, and R the half plane $x_1 = x_2 = 0$ with $x_3 \geq 0$.)

Let $S_R \equiv S\mid_R$. This is a smooth oriented 4-manifold (with boundary) which is embedded in X. The boundary of this 4-manifold is $S_\sigma \equiv S\mid_\sigma$. Note that $S_\sigma \subset A$ is the boundary of $T_\sigma \equiv T\mid_\sigma$, the embedded image of $\sigma \times B^3$ onto a tubular neighborhood of σ in A.

With the preceding understood, introduce the following surgery on A:

$$A_0' \equiv (A - \text{int}(T_\sigma)) \cup S_R. \tag{8.17}$$

Note that $A_0' \subset X$ is a C^0 embedding of a smooth, oriented manifold; the embedding has a corner at S_σ where S_R and $A - \text{int}(T_\sigma)$ overlap. See the following picture:



(8.18)

A neighborhood of this corner of A_0' is embedded in T_R. Smooth A_0' in T_R along the corner, and one obtains a smoothly submanifold, $A' \subset X$ which solves Problem 1 in (8.2). Indeed, the first two requirements are met by construction. As for the third, remark that $[T \,|_R]$ defines a 5-dimensional cycle in O whose boundary is $[A'] - [A]$. □

g) Proof of Proposition 8.3.

Consider first the proof of the proposition under the Assumption $a - c$. Let A' be as described above (see (8.17)). Then, $H^2(A')$ can be computed using the following homology exact sequences for the pairs (A, T_σ) and (A', S_R):

(8.19)

1) $H^1(A) \to H^1(T_\sigma) \to H^2(A, T_\sigma) \to H^2(A) \to H^2(T_\sigma)$.
2) $H^1(A') \to H^1(S_R) \to H^2(A', S_R) \to H^2(A') \to H^2(S_R) \to H^3(A', S_R)$.

(Use rational coefficients please.)

In Sequence 1, the first arrow is surjective because of Assumption c. And, $H^2(T_\sigma) \approx 0$ because T_σ is a tubular neighborhood of a disjoint union of circles. Thus

$$H^2(A, T_\sigma) \approx H^2(A). \tag{8.20}$$

To analyze the second sequence of (8.19), note that its first arrow is surjective. This is because R is path connected, thus forcing S_R to be a topologically trivial 2-sphere bundle. For the same reason, $H^2(S_R) \approx H^2(\partial T_\sigma)$. Meanwhile, excision identifies $H^*(A', S_R) \approx H^*(A, T_\sigma)$. Thus, with (8.20), the second sequence in (8.19) implies

(8.21) $$0 \to H^2(A) \to H^2(A') \to H^2(\partial T_\sigma) \to H^2(A, T_\sigma).$$

Poincare' duality plus Assumptions b and c of Proposition 8.3 imply that the last arrow in (8.21) is surjective, thus establishing an isomorphism between $H^2(A)$ and $H^2(A')$.

Now consider Proposition 8.3 with Assumptions a, b and c'. Remark here that when σ is connected (i.e. just one circle), then Assumption c' implies Assumption c. With this fact understood, here is the task ahead: Under Assumptions a, b and c', find an ambient (in O) surgery on A so that the result, $\underline{A}$, has the following properties:

(8.22)

1) $[\underline{A}] = [A]$ in $H_4(X, X - O; \mathbb{Z})$.
2) $H^2(\underline{A}) \approx H^2(A)$.
3) $H^3_{comp}(\underline{A}) \to H^3(\underline{A})$ is injective.
4) $\underline{A}$ intersects B inside O in a single compact component which is not trivial in $H_1(A, \mathbb{Q})$ but which bounds in $B \cap O$.

A solution to (8.22) will validate Proposition 8.3.

Here is an algorythm for constructing $\underline{A}$: To begin choose a pair of components, $C_{1,2} \subset \sigma$. Fix $p_1 \in C_1$ and $p_2 \in C_2$. By assumption, $B \cap O$ is path connected, so there is a path in B (a smoothly embedded interval), τ, which starts at p_1 and ends at p_2. Make sure that $\text{int}(\tau)$ has empty intersection with A. Also, arrange τ so that it is not tangent to A along its boundary. The choice of p_1 as the starting point and p_2 as the ending point orients τ.

Let $V \to \tau$ denote the normal bundle to τ in B. This is an oriented 3-plane bundle over τ. Note that $TC_1 \mid_{p_1} \subset V \mid_{p_1} \approx \mathbb{R}^3$ and also $TC_2 \subset V \mid_{p_2} \approx R^3$ are oriented lines. As S^2 is path connected, there is an oriented, dimension 1 sub-bundle $V_0 \subset V$ whose restriction to p_1 is TC_1 and whose restriction to p_2 is the line TC_2, but oriented in reverse.

With V_0 understood, remark that $V_0 \oplus N \to \tau$ is an oriented 4-plane sub-bundle of the normal bundle to τ in O. Also note that this bundle restricts to p_1 and as $TA \mid_{p_1}$, and it restricts to p_2 as the 4- plane $TA \mid_{p_2}$, but with its orientation reversed.

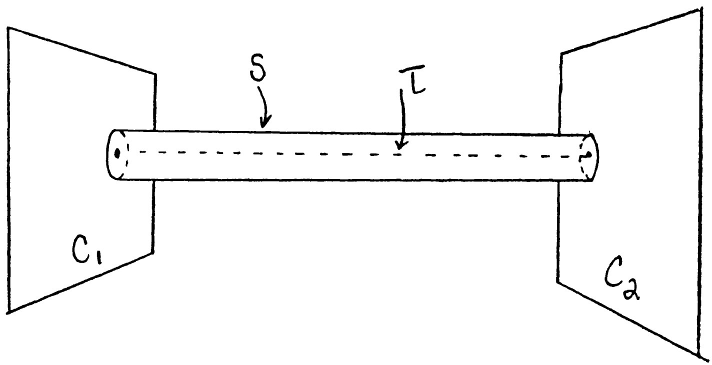
The normal bundle to τ in X is isomorphic to $V \oplus N$. Fix an exponential map $e_\tau : V \oplus N \to X$ which restricts to N as e in (8.13), which restricts to map V into B, and which maps $V_0 \mid_{p_{1,2}}$ into $C_{1,2}$, respectively. (Thus, $e_\tau \mid_V$ is an exponential map for τ in B.) Put a fiber metric on the bundle $V_0 \oplus N$ such that e_τ embeds the subspace of vectors v with norm less than 2. Let $S \subset V_0 \oplus N$ denote the radius 1 sphere bundle, and identify S with its embedded image under e_τ.

Let $T \subset V_0 \oplus N$ denote the radius one, 3-ball bundle. Identify T with its embedded image under e_τ. Note that $T \mid_{p_1}$ is a tubular neighborhood in A of p_1, while $T \mid_{p_2}$ is a tubular neighborhood of p_2 in A.

Now define the following surgery on A:

$$A_0 \equiv (A - \operatorname{int}(T\,|_{p_1} \cup T\,|_{p_2})) \cup S. \tag{8.23}$$

This is a C^0 embedding of a smooth manifold, A_1, in X. Here, the embedding is smooth away from $S\,|_{p_1} \cup S\,|_{p_2}$, where there is a corner. Near this corner, A_0 is embedded in $V_0 \oplus N$. Smooth out the corner in $V_0 \oplus N$ and the result is an embedding of A_1 into X:



(8.24)

With A_1 understood, consider its properties with respect to (8.22): First, the homology classes of A_1 and A agree are equal in $H_4(X, X - O)$. This is because A_1 and A_0 define the same class and the 5-manifold T defines a cycle in O with boundary $[A_0] - [A]$.

Second, $H^2(A_1) = 0$, because A_1 has been obtained from A by surgery on an embedded $S^0 \times B^4$ (i.e. $T\,|_{p_1} \cup T\,|_{p_2}$).

Third, $H^3_{\text{comp}}(A_1) \to H^3(A_1)$ is injective. Both are either equal to their A counter-parts, or are obtained from their A counter parts by the addition of 1 generator which is dual to the 3-sphere $S\,|_{p_1}$. (Prove this with Meyer-Vietoris.)

Fourth, the intersection of A_1 with B in O has one less component then that of A with B in O. This is because the surgery from A to A_1 has surgered C_1 to C_2 by removing an embedded $S^0 \times B^1$ from $C_1 \cup C_2$ (i.e. $(T \cap V_0)\,|_{p_1} \cup (T \cap V_0)_{p_1}$); the missing $S^0 \times B^1$ is replaced by $S \cap V_0$. (See (8.24).)

Note that the homology class in $H_1(B \cap O)$ of $\sigma_1 \equiv A_1 \cap B \cap O$ is the same as that of $\sigma \equiv A \cap B \cap O$. Indeed, σ_1 defines the same homology class as $\sigma_0 \equiv A_0 \cap B \cap O$, and $T \cap V_0$ as a 2-cycle in $B \cap O$ has boundary $[\sigma_0] - [\sigma_1]$.

If σ had only two components to start with, then set $\underline{A} \equiv A_1$ and stop, because (8.22) has just been verified for this $\underline{A}$. If σ had more than two components, iterate the preceding procedure by renaming $A_1 \equiv A$ and $\sigma_1 \equiv \sigma$. The iteration stops with $\underline{A}$ which obeys (8.22). □

h) Proof of Proposition 8.7.

The construction of A' from A for solving Problem 1 (and Problem 2) of (8.2) via Propositions 8.2 and 8.3 is described in the preceding two subsections. The construction involves two types of ambient surgeries in X. The first type of surgery gives the smoothing, A_1, of A_0 in (8.23). The second type of surgery gives the smoothing, A', of the A'_0 in (8.17), (8.18).

Type 1: Extending frames for surgery on $S^0 \times B^4$.

Consider the smoothing, A_1, of A_0 in (8.23). Let $U \subset A$ be an open neighborhood of $T\mid_{p_1} \cup T\mid_{p_2}$. □

LEMMA 8.10. *Let ζ be a normal frame for A in X. Then there is a normal frame for A_1 in X which agrees with ζ on $A - U$.*

Type 1: Extending frames over surgery on $S^1 \times B^3$.

The assumption here is that A' is obtained from A by smoothing the surgery A'_0 in (8.17). (See (8.18) too.) Let ζ be a normal frame for A in X, and let $U \subset A$ be a neighborhood of T_σ.

LEMMA 8.11. *The normal frame ζ on $A - U$ extends as a normal frame over the smoothing, A', of (8.17) if and only if $\zeta\mid_\sigma$ is homotopic to an adapted frame for which $(R \cdot R)_{mod(2)} = 0$.*

If B has even intersection pairing, then according to Lemma 8.6, the $\mathbb{Z}/2$ number $(R \cdot R)_{\mathrm{mod}(2)}$ is the invariant $\chi_{B,\sigma}(\zeta\mid_\sigma)$. Thus, Lemmas 8.10 and 8.11 with the constructions in the two preceding subsections prove Proposition 8.7.

The remainder of this section is occupied with the proofs of the preceding two lemmas.

Proof of Lemma 8.10. The strategy is to first define a normal frame, ζ_1, for int(S) in X. Having done so, the final step proves that there are no obstructions to connecting ζ_1 to ζ on $A - U$.

To construct ζ_1, first fix a normal frame, (e_1, e_2, e_3), for the normal bundle (V) of τ in B. One can arrange such a frame so that e_3 is tangent to the sub-line V_0. Then, (e_1, e_2) orient V/V_0.

Use the exponential map $e_\tau : V \oplus N \to X$ to identify a neighborhood of the zero section of $V \oplus N$ with a neighborhood of τ in X. With this identification understood, then the normal bundle to int(S) in X is spanned by $\zeta_1 \equiv (e_1, e_2, e'_3)$, where $e'_3 \in T(V_0 \oplus N)\mid_S$ restricts to the fiber over $x \in \tau$ as the inward pointing normal vector to $S\mid_x$ in $(V_0 \oplus N)\mid_x$.

With ζ_1 understood, consider connecting ζ_1 to ζ near ∂S. To make such a connection, introduce the tangent vector, v to τ. Let $\underline{v}$ denote a lift of v to $V \oplus N$.

At p_1 or p_2, the triple (e_1, e_2, v) defines a normal frame for A in X. The normal frame ζ for A in X can be homotoped inside U so that it agrees with (e_1, e_2, v) at p_1 and p_2 and equals $(e_1, e_2, \underline{v})$ on a neighborhood, U', of $T\mid_{p_{1,2}}$

with compact support in U. Thus, 2/3 of the normal frame ζ, i.e. (e_1, e_2), have been extended over A_1.

The compliment in the normal bundle to A_1 of the 2-plane span of (e_1, e_2) is an oriented line bundle which is framed by $\underline{v}$ on $U - U'$ and by e_3' on $\text{int}(S)$. There is no obstruction to framing this compliment by a frame which agrees with $\underline{v}$ on $U - U'$ and with e_3' on the compliment in S of an apriori specified neighborhood of ∂S. □

Proof of Lemma 8.11. The strategy for extending ζ on A - U as a normal frame for A' will be to construct a normal frame, ζ_1, for S_R in X, and then consider whether ζ and ζ_1 can be joined.

To construct ζ_1, introduce the normal 2-plane bundle, V, to R in B. Since R is oriented, V is an oriented bundle and so trivial because $\text{int}(R)$ has no compact components. Let e_1, e_2 be a frame for V.

Note that $V \oplus N \to R$ is the normal bundle to R in X. Choose an exponential map $e_R : V \oplus N \to X$ which restricts to N as e in (8.13). Use e_R to identify a neighborhood of the zero section in the bundle $V \oplus N$ with a neighborhood of R in X.

Let e_3 be the inward pointing normal vector to $S_R \subset T_R \subset N\,|_R$. Then the triple $\zeta_1 \equiv (e_1, e_2, e_3)$ span the normal bundle to S_R in X.

With the normal frame for S_R understood, consider its extension to a normal frame for A on the compliment in U of a neighborhood U' of T_σ. For this purpose, introduce v to denote the inward pointing normal vector field to R along σ. Lift v to a vector field $\underline{v}$ on $(V \oplus N)_\sigma$. Since A near σ is identified by (8.13) with a neighborhood of the zero section of $N\,|_\sigma$, it follows that a normal frame for A near σ is given by the triple $(e_1 e_2, \underline{v})$. Furthermore, there is no obstruction to joining this frame on the compliment of a neighborhood U' of T_σ with the frame $\zeta_1 \equiv (e_1, e_2, e_3)$ on the interior of S_R. (The pair (e_1, e_2) define 2/3 of the extension, and $\underline{v}$ and e_3 define the same orientation for the complimentary line.)

With the preceding understood, then one can conclude that the normal frame ζ extends from $A - U$ to A' if the restriction of ζ to σ is homotopic to the normal frame $(e_1, e_2, \underline{v})$. Now the latter frame is an adapted frame and, by construction, $R \cdot R = 0$ for $(e_1, e_2, \underline{v})$. Thus, $\zeta\,|_\sigma$ is homotopic to (e_1, e_2, σ) if and only if $\zeta\,|_\sigma$ is homotopic to an adapted frame for which the corresponding $R \cdot R$ is even. (See Lemma 8.5.) □

9 The fourth pass at $E_\pm$. The submanifold E_{3-} of the preceding section intersects Δ_Z as described by Assertion 6 of Proposition 8.1. Let $\Theta : Z \to Z$ denote the switch map which sends (x, y) to (y, x). Since $E_{3+} = \Theta(E_{3-})$, the intersections of E_{3-} with Δ_Z are also intersections of E_{3+} with Δ_Z. Unfortunately, there may be compact components to $E_{3-} \cap E_{3+}$ which occur in $Z - \Delta_Z$. Such extra components are troublesome and must be eliminated, and their elimination is the goal of this section.

As will be seen, surgery on $E_{3\pm}$ will result in oriented submanifolds (with boundary) $E_{4\pm} \subset Z$ which have the following properties:

PROPOSITION 9.1. *There are oriented, embedded submanifolds (with boundary) $E_{4\pm} \subset Z$ with the following properties:*

1) *There is an open neighborhood $U \subset Z$ of $\Delta_Z \cup \partial Z$ such that $E_{4+} \cap U$ and $E_{4-} \cap U$ are images of each other under the switch map on Z.*
2) *The fundamental classes $[E_{4\pm}]$ are equal to $N\,[E_{1\pm}]$ for some integer $N \geq 1$. Here, $[E_{1\pm}]$ are described by (6.3) and Lemma 6.3.*
3) *$E_{4\pm}$ have empty intersection with $M_0 \times M_0$ and $M_1 \times M_1$.*
4) *$E_{4\pm}$ have empty intersection with $E_{L,R}$ of (4.15).*
5) *If $p \in$ crit(f), then the intersection of E_{4-} with $(S^3 \times S^3)_p$ has the form $S^3 \times \Lambda_p$, where Λ_p is a set of N points. Similarly, the intersection of E_{4+} with $(S^3 \times S^3)_p$ is $\Lambda_p \times S^3$.*
6) *$E_{4-} \cap \Delta_Z = \cup_{i=1}^r \Gamma_i$, where $\Gamma_i \subset \Delta_Z$ is as follows: There is a flow line μ_i which starts at a_i and ends at b_i. With the canonical identification of Δ_W with W understood, Γ_i is the union of N like oriented, disjoint, push-off copies of a closed interval, $I \subset \mu_i$. And, each of these N push-offs of I starts in $(\Lambda_a \times \Lambda_a) \cap \Delta_Z$ and ends in $(\Lambda_b \times \Lambda_b) \cap \Delta_Z$. Likewise, $E_{4+} \cap \Delta_Z = \cup_{i=1}^r \Gamma_i$.*
7) *$E_{4-} \cap E_{4+} = \cup_{i=1}^r \Gamma_i'$, where $\Gamma_i' \subset Z$ is the union of Γ_i with $N-1$ like oriented, push-off copies of Γ_i in $Z - \Delta_Z$.*
8) *Both $E_{4\pm}$ have trivial normal bundles in Z. The normal bundle of E_{4-} has a framing, ζ, which restricts to a product normal framing on a neighborhood of $(\cup_{i=1}^r \Gamma_i) \cup \{S^3 \times \Lambda_p\}_{p \in \, crit(f)}$. Furthermore, this framing ζ restricts to $\{S^3 \times \Lambda_p\}_{p \in \, crit(f)}$ as a constant framing. The normal bundle to E_{4+} in Z has a framing which restricts to $E_{4+} \cap U$ as the image of ζ under the switch map.*
9) *$H^2(E_{4\pm}; \mathbb{Q}) = 0$.*

(Compare with Proposition 8.1. The only essential change is in $E_{4-} \cap E_{4+}$. But note that the integer N, the points $\{\Lambda_p\}$ and the line segments $\{\Gamma_i\}$ which appear here may be different from those which appear in Proposition 8.1.)

The rest of this section is devoted to the construction of $E_{4\pm}$.

a) $E_{3-} \cap E_{3+}$ on $Z - \Delta_Z$.

Isotope E_{3+} in $Z - (\Delta_Z \cup \partial Z)$ so that its intersection on $Z - \Delta_Z$ with E_{3-} is transversal. (Still use E_{3+} to denote the after isotopy submanifold.) This intersection is now a finite union of disjoint, embedded, oriented circles, σ, with $N-1$ like oriented, push-off copies of $\{\Gamma_i\}$ in $Z - \Delta_Z$. (These push-offs of $\{\Gamma_i\}$ are disjoint from σ.) There is a natural inclination to remove the circle components, σ, by apply the techniques from the previous section (Propositions 8.2, 8.3 and 8.7). However, either

$$[\sigma] = 0 \in H_1(E_{3\pm}; \mathbb{Q}) \tag{9.1}$$

or not; but $[\sigma]$ is never non-trivial in one and trivial in the other. (Because, before perturbing E_{3+}, one was the image of the other under the switch map.)

If (9.1) holds, then one can not (directly) apply Proposition 8.2 to remove the intersection σ. If (9.1) does not apply, then Proposition 8.2 can be applied, but not (directly) Proposition 8.3. So, whether or not (9.1) holds, some preliminary work must be done before the techniques from Section 8 can be employed.

Consider the case where (9.1) holds.

LEMMA 9.2. *Let $O \equiv Z - (\Delta_Z \cup \partial Z \cup E_L \cup E_R)$. There is an ambient surgery (in O) of E_{3-} which results in an oriented submanifold (with boundary) $E'_{3-} \subset Z$ with the following properties:*

1) Assertions 2-8 of Proposition 8.1 hold when E'_{3-} is substituted for E_{3-}.

2) There is a tubular neighborhood, $U_\Delta \subset Z$, of Δ_Z which intersects $E'_{3-} \cap E'_{3+}$ as $\cup_{i=1}^r \Gamma'_i$, where Γ'_i is the union of Γ_i with $N-1$ disjoint, like oriented, push-off copies of Γ_i.

3) $E'_{3-} \cap E'_{3+}$ intersects $Z - U_\Delta$ as a disjoint union, σ', of oriented circles which obey

(a) $[\sigma'] \neq 0 \in H_1(E'_{3-};\mathbb{Q})$.

(b) $[\sigma'] = 0 \in H_{-1}(E_{3+};\mathbb{Q})$.

(9.2)

(Note that there is no E'_{3+}; the symmetry under the switch map will be broken here.)

The proof of this lemma will be given shortly.

Consider the case where (9.1) is false. The goal here is to modify E_{3+} as described in the following lemma:

LEMMA 9.3. *Let $O \equiv Z - (\partial Z \cup \Delta_Z \cup E_L \cup E_R)$. There is an ambient surgery (in O) of some number $n \geq 1$ of like-oriented push-offs of E_{3+} which results in an oriented, embedded submanifold, (with boundary) $E'_{3+} \subset Z$ with the following properties:*

1) Let E'_{3-} denote the union of n like-oriented, push-off copies of E_{3-}. The Assertions 2-8 of Proposition 8.1 hold when $E'_{3\pm}$ are substituted for $E_{3\pm}$.

2) There is a tubular neighborhood, $U_\Delta \subset Z$, of Δ_Z which intersects $E'_{3-} \cap E'_{3+}$ is $\cup_{i=1}^r \Gamma'_i$, where Γ'_i is the union of Γ_i with $N-1$ disjoint, like oriented, push-off copies of Γ_i.

3) $E'_{3-} \cap E'_{3+}$ intersects $Z - U_\Delta$ as a disjoint union, σ', of oriented circles which obey

(a) $[\sigma'] \neq 0 \in H_1(E'_{3-};\mathbb{Q})$.

(b) $[\sigma'] = 0 \in H_{-1}(E_{3+};\mathbb{Q})$.

(9.3)

This lemma will also be proved shortly.

Proof of Proposition 9.1.

When (9.1) is true, take E'_{3-} from Lemma 9.2 and set $E'_{3+} \equiv E_{3+}$. When (9.1) is false, take $E'_{3\pm}$ from Lemma 9.3. Use σ' to denote the intersection in $O \equiv Z - (\partial Z \cup U_\Delta \cup E_L \cup E_R)$ between $E'_{3\pm}$. Use (9.2b) or (9.3b) in the respective cases, to find an integer $N_3 \geq 1$ with the property that $N_3\,[\sigma'] = 0 \in H_1(E_{3+};\mathbb{Z})$.

Take $2\,N_3$ push-off copies of E'_{3-}, all with the same orientation as E_{3-}, and let A denote the interior of the resulting union. Take $2\,N_3$ push-off copies of E'_{3+}, all oriented as E'_{3+}, and let B_0 denote the interior of the resulting union. Let $X \equiv int(Z)$ and let O be as before. Now, Proposition 8.2 can be invoked using B_0 for B, but not Proposition 8.3 because Assumption c has not been shown to hold, and because Assumption c' will be false because B_0 will not be connected. However, there is a surgery which remedies this problem: □

LEMMA 9.4. *Let B_0, A, X and O be as described above. Then, there is an ambient surgery in $O - A$ on some finite number of embedded $(S^0 \times B^4)$'s in B_0 such that the result, B is path connected. This surgery does not change either $H_1(\cdot;\mathbb{Z})$ or $H_2(\cdot;\mathbb{Z})$. Finally, if ζ is a normal frame for B_0 in X, and if $U \subset B_0$ is a neighborhood of the $(S^0 \times B^4)$'s, then $\zeta\,|_{B^0-U}$ extends smoothly over B as a normal frame for B in X.*

The proof of this lemma is given below.

With Lemma 9.4 understood, Propositions 8.2, 8.3 and 8.7 can be applied using X, O, A and B as described above. Use E_{4-} to denote the closure in Z of the promised solution, A', to Problems 1-3 in (8.2). Relable E_{4+} to denote the closure in Z of B. The pair $E_{4\pm}$ will satisfy the requirements of Proposition 9.1.

b) Making $[\sigma'] \neq 0 \in H_1(E'_{3-};\mathbb{Q})$.

This subsection is concerned with the construction of E'_{3-} of Lemma 9.2.

This construction requires a preliminary digression to introduce another surgery tool. The digression concerns the abstract model of (8.1). Proposition 9.5, below, summarizes the digression.

The statement of Proposition 9.5 requires the following remarks to set the stage: When $S \to A$ is an embedded 2-sphere, let $v_S \to S$ denote the normal bundle to S in A. Suppose that $P \subset X$ is an embedded 3-dimensional ball with boundary S. (The local model here takes S to be the plane $x_3 = \cdots = x_7 = 0$ in $\mathbb{R}^7$ and then P is the half plane $x_3 \geq 0, x_4 = \cdots = x_7 = 0$.). Use N_P to denote the normal bundle to P in X. The natural inclusion

$$0 \to v_S \to N_P\,|_S \tag{9.4}$$

plays an important role in Proposition 9.5.

PROPOSITION 9.5. *Let A, B, O, X and σ be as described in (8.1). Assume that:*

a) $H^3_{comp}(A;\mathbb{Q}) \to H^3(A;\mathbb{Q})$ *is injective.*

b) $[\sigma] = 0 \in H_1(A;\mathbb{Q})$. *Suppose that there exists an embedded 2-sphere* $S \subset (A \cap O) - \sigma$ *which bounds an embedded 3-ball* $P \subset O$, *and:*

c) *The fundamental class of* S *is homologically trivial in* $H_2(A;\mathbb{Q})$.

d) $int(P) \cap A = \emptyset$.

e) P *intersects* B *transversally.*

f) *With respect to some orientation on* $P, [P \cap B] \neq 0 \in H_0(P;\mathbb{Z})$.

g) $(N_P\, |_S)/\upsilon_S \to S$ *is a trivial 2-plane bundle.*
Let $U \subset X$ *be an open neighborhood of* P. *There exists an embedded, oriented, 4-dimensional submanifold* $A' \subset X$ *with the following properties:*

1) A' *intersects* B *transversally in* σ', *and* $[\sigma'] \neq 0 \in H_1(A';\mathbb{Q})$.

2) $H^2(A';\mathbb{Q}) = H^2(A,\mathbb{Q})$.

3) $H^3_{comp}(A';\mathbb{Q}) \to H^3(A';\mathbb{Q})$ *is injective.*

4) $A' = A$ *on* $X - U$ *and* $[A] = [A']$ *in* $H_4(X, X - O)$.

5) $[\sigma'] = [\sigma] \in H_1(B;\mathbb{Q})$. *Furthermore,*

6) *If* A *has trivial normal bundle in* X, *then so does* A'. *And, if* ζ *is a frame for the normal bundle to* A *in* X, *then* $\zeta\, |_{A-U}$ *extends over* A' *as a frame for the normal bundle to* A' *in* X.

This proposition will be proved in the next subsection. Consider now its application to (9.2).

Proof of Lemma 9.2. The lemma will be proved by applying Proposition 9.5. For this purpose, take X to be int(Z) and then define $O \equiv Z - (\partial Z \cup U_\Delta \cup E_L \cup E_R)$. Take $A \equiv$ int(E_{3-}) and, likewise, take $B \equiv$ int(E_{3+}). Given that the assumptions of Proposition 9.5 hold when (9.1) is true, one should take E'_{3-} to be the closure in Z of the submanifold A' of Proposition 9.5. As for the validity of the assumptions of Proposition 9.5, remark that Assumptions $a - b$ are satisfied by construction; see Proposition 8.1. (Assumption b holds since E_{3-} is a manifold with boundary whose boundary is a union of 3-spheres; and $H^2(S^3) = 0$.)

The remaining assumptions of Proposition 9.5 will be verified with the exhibition of a 2-sphere $S \subset A \cap O$ with the requisite properties. To find the appropriate 2-sphere, it is important to remember that E_{3-} was constructed from E'_{2-}. Here is a brief summary: The compact components of $E'_{2-} \cap \Delta_Z$ bound a connected, oriented, embedded surface (with boundary) $R \subset \Delta_Z$. Let $N \to \Delta_Z$ be the normal bundle. A fiber metric was chosen for N, and an exponential map $e : N \to Z$ was chosen so that e mapped the radius 2 ball fiber bundle in N diffeomorphically onto its image in Z. (This ball bundle in N was identified using e with a neighborhood of Δ_Z in Z.) Also, e was constrained to map $N\, |_{\partial R}$ into E'_{2-}.

Next, the radius 1 sphere bundle, $S_R \subset N\, |_R$ was introduced, as well as the radius 1 ball bundle, $T_R \subset N\, |_R$. Finally, E_{3-} was defined to be the result of smoothing the corner in the surgery

(9.5) $$(E'_{2-} - T_{\partial R}) \cup S_R.$$

By the way, no generality is lost by assuming that the surgery in (9.5) occured. Indeed, if $E_{2-} = E_{3-}$ and no such surgery occurs (in which case $E_{2-} \cap \text{int}(\Delta_Z)$ has no compact components), then there are isotopies of $E_{1\pm}$ so that the resulting submanifolds obey the conclusions of Proposition 9.1. Or, if $E_{2-} = E_{3-}$, then one could add two push-off copies of $Y_{1,1-}$ to E'_{1-} of Section 7*a*, one oriented positively and the other oriented negatively. Then, the tubing construction of E_{2-} will insure that $E_{2-} \cap \text{int}(\Delta_Z)$ has a compact component.

With (9.5) understood, remark that E_{3+} is obtained from $\Theta(E_{3-})$ by an isotopy. Let $R' \subset R$ denote the complement of a (small) collar of ∂R. Near R', E_{3-} is $S_R\mid_{R'}$ and this coincides with $\Theta(E_{3-})$ near R'. (Thus, $\Theta(E_{3-})$ and E_{3-} do not intersect transverally.) Choose the isotopy to obtain E_{3+} from $\Theta(E_{3-})$ so that E_{3+} near R' is a sphere bundle $S_{R+} \subset N\mid_{R'}$ of radius *greater* than 1.

Pick a point $x \in R'$ and let

(9.6) $$S_x \equiv S_R \mid_{R'} \subset E_{3-}.$$

This is a homologically trivial, embedded 2-sphere in E_{3-}. This S_x bounds the 3-ball $T_x \subset N\mid_x$ which is the unit ball in the fiber of N at x. Notice that this 3-ball has empty intersection with E_{3+} since the latter intersects the fiber at x in a sphere of radius larger than 1.

However, the ball T_x intersects Δ_Z transversally in a single point, namely x. This intersection with Δ_Z will now be traded for N transversal intersections with E_{3+}. (This is the same N as in Proposition 8.1.) The technique used here is called "connect summing with a transverse sphere" (see, e.g. Chapter 1 of [**7**]). This technique proceeds as follows: Fix $p \in \text{crit}(f)$ and then fix a point $q \in S^3 - \Lambda_p$. Observe that the sphere $S^3 \times q \subset (S^3 \times S^3)_p$ intersects Δ_Z transversally once (at $q \times q$), and it intersects E_{3+} transversally N times, at $q \times \Lambda_p$. Orient $S^3 \times q$ and all N intersections with E_{3+} will have the same sign. (See Assertion 5 of Proposition 8.1.) Because $q \notin \Lambda_p$, this $S^3 \times q$ will have empty intersection with E_{3-}.

Take this $S^3 \times q \subset (S^3 \times S^3)_p$ and push it off ∂Z so that it is an embedded submanifold, $Y \subset \text{int}(Z)$. Push if off only slightly, so that Y still intersects E_{3+} in the N points of the push-off of $q \times \Lambda_p$, and so that Y intersects Δ_Z in the push-off of $q \times q$. Also, do not let Y intersect E_{3-}.

Remark that N is an oriented vector bundle, and thus T_x is an oriented 3-ball. Orient Y so that its intersection number with Δ_Z is the opposite of that for $T_x \cap \Delta_Z$. Now, one can "tube" Y to T_x to obtain a new 3-ball, $P \subset Z$ with the following properties:

1) $\partial P = S_x$,
2) $P \cap (\Delta_Z \cup \partial Z \cup E_L \cup E_R) = \emptyset$,
3) $\text{int}(P) \cap E_{3-} = \emptyset$,

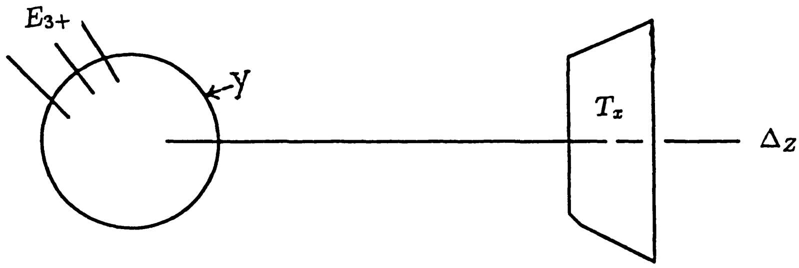
4) $P \cap E_{3+}$ is N distinct points, all homologically the same in $H_0(P)$.

(9.7)

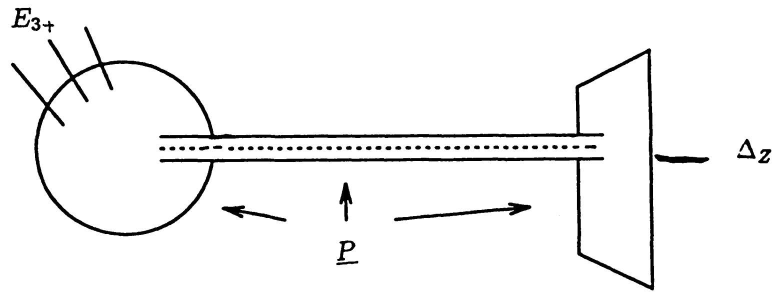
Abstractly, P is the connect sum of Y and T_x. Realize this connect sum in Z by choosing a path, τ, between x and $x' \equiv Y \cap \Delta_Z$ in $\text{int}(\Delta_Z)$. Make sure that τ avoids the paths $\{\Gamma_i\}_{i=1}^{r}$ of Assertion 6 in Proposition 8.1. Modify the exponential map $e : N \to Z$ so that e maps $N|_{x'}$ into Y. Let $S_\tau \subset N|_\tau$ denote the radius 1/8 sphere bundle. Let $T_x^0 \subset N|_x$ and $T_{x'} \subset N|_{x'}$ denote the radius balls of radius 1/8. Then Y is obtained by smoothing the corners of the surgery

$$((Y - T_{x'}^0) \cup (T_x - T_x^0)) \cup S_\tau. \tag{9.8}$$

Here is a picture:



(9.9)



(9.10)

The 3-ball P and the 2-sphere $S \equiv S_x$ satisfy Assumptions $c - f$ of Proposition 9.5. To apply Proposition 9.5 to prove Lemma 9.2, it is only necessary to check that Assumption f of Proposition 9.5 is satisfied. For this purpose, let $\pi : S_x \to x$ denote the projection. Then, the normal bundle to S_x in A is isomorphic to $\pi^*TR|_x$. Meanwhile, the normal bundle to P in X along S_x is the same as the normal bundle to T_x in X along S_x which is isomorphic to $\pi^*T\Delta_Z|_x$. Thus, the quotient $(N_P|_S)/v_S \approx \pi^*((T\Delta_Z|_x)/(TR|_x))$, which is trivial, as required. □

c) Proof of Proposition 9.5.

Since $[S]$ is homologically trivial in A, $v_S \to S$ is a trivial 2- plane bundle. Since P is a 3-ball, so $N_P \to P$ is a trivial 4-plane bundle. And, because $(N_P\,|_S)/v_S$ is a trivial bundle, there are no obstructions to extending v_S over P as a 2-plane subbundle of N_P. Use $v \to P$ to denote this 2-plane bundle.

Fix an exponentional map $e_P : N_P \to X$ with the following properties: First, e_P should restrict to v_S as a map into A. And, e_p should restrict to map $N_P\,|_{P\cap B}$ into B. Fix a fiber metric on N_p with the property that e_p embeds the interior of the unit ball bundle in N_p onto a neighborhood of P in X. Then, use e_p to implicitly identify the unit ball bundle in N_P with its e_p-image.

With the preceding understood, let $s_0 \subset v$ denote the 1-sphere bundle of radius 1/4. Let $t_0 \subset v$ denote the ball bundle of radius 1/4. (Thus, $\partial(t_0\mid_{\mathrm{int}(P)}) = s_0\mid_{\mathrm{int}(P)}$.)

Introduce the surgery

$$A_0 \equiv (A - t_0\,|_S) \cup s_0, \tag{9.11}$$

which is a C^0 embedding of a smooth, oriented 4-manifold, A' into X. This embedding is smooth away from the corner at $s_0\,|_S$ and it can be smoothed inside v to produce a smooth embedding of A' into X. Note that A' can be arranged to agree with A on the compliment of any apriori specified open neighborhood $U \subset X$ of P.

With A' understood, consider the various assertions of Proposition 9.5: To prove Assertion 1, consider that

$$\sigma' \equiv (A' \cap B) \cap O = s_0\,|_{P\cap B} \cup \quad \sigma. \tag{9.12}$$

This is a transversal intersection because P intersects B transversaly. The fact that $[\sigma'] \neq 0$ in $H_1(A';\mathbb{Q})$ follows via Meyer- Vietoris and the fact that $[P\cap B] \neq 0$ in $H_0(P)$. (Use the Meyer- Vietoris sequences for the decompositions $A = (A - s_0\,|_S) \cup (t_0\,|_S)$ and also $A' = (A - s_0\,|_S) \cup s_0$.)

Meyer-Vietoris also proves Assertion 2, namely $H^2(A';\mathbb{Q}) = H^2(A;\mathbb{Q})$. (Use the same sequences as above.)

Assertion 3 is true because one can interpret t_0 as a cycle, and this cycle obeys $\partial t_0 = [A] - [A']$.

Assertion 4 is true because the circles in $s_0\,|_{P\cap B}$ bound the discs $t_0\,|_{P\cap B}$.

To prove Assertion 5, the strategy will be to find a framing of the normal bundle to $\mathrm{int}(s_0)$ in X which is compatible with the framing ζ on $A - S$. To begin, introduce the notation $\underline{v}$ to denote the vector field along $s_0 \subset t_0$ which points radially inward on each 2- ball fiber of $t_0 \to P$. With $\underline{v}$ understood, the normal bundle to s_0 in X is isomorphic to $(N_P/v)\oplus$ Span$(\underline{v})$. The bundle $N_P/v \to P$ is a trivial 2-plane bundle (because P is a ball), and so it has a global frame, (e_1, e_2). Thus, $(e_1, e_2, \underline{v})$ is a frame for the normal bundle to $\mathrm{int}(s_0)$ in X.

Now consider the normal bundle to A along S. For this purpose, let e_3 denote the inward pointing tangent bundle to P along S. (So e_3 spans the normal bundle to S in P). Then, along S, the normal bundle to A in X is isomorphic to $(N_P/v)\,|_S \oplus \mathrm{Span}(e_3)$. Thus, $N_P\,|_S \oplus \mathrm{Span}(e_3) \to S$ is isomorphic to the normal bundle of S in X. Let $e_S : N_P\,|_S \oplus \mathrm{Span}(e_3) \to X$ be an exponential map which maps v_S into A and which maps $v\,|_S \quad \oplus \mathrm{Span}(e_3)$ into P. Use e_S to identify a neighborhood of S in X with a neighborhood of the zero section of the bundle $N_P\,|_S \oplus \mathrm{Span}(e_3) \to S$.

Because $\pi_2(SO(3)) = 0$, there are no obstructions to homotoping the given normal frame ζ in a neighborhood of S so that the restriction of ζ to said neighborhood is$(e_1\,|_S, e_2\,|_S, e_3)$. Thus, two thirds of the frame ζ can be extended over A' from $A - S$. As usual, there is no obstruction to homotoping $\underline{v}$ near $s_0\,|_S$ to equal e_3 on the compliment of a neighborhood of S in A. □

d) Making $[\sigma] = 0$ in $H_1(E_{3+})$.

This subsection is concerned with the construction of E'_{3+} of Lemma 9.3. The construction requires a preliminary digression to introduce a modified version of Proposition 8.2. Here is the scenario: As in Proposition 8.2, X is a smooth, oriented 7-manifold and A, B are oriented, 4-dimensional submanifolds of X. Let $O \subset X$ be an open set which contains a component, σ, of $A \cap B$.

PROPOSITION 9.6. *Let X, O, A, B and σ be as described above. Suppose that*

a) $[\sigma] \neq 0$ *in* $H_1(B;\mathbb{Q})$.

b) $[\sigma] = 0$ *in* $H_1(O;\mathbb{Q})$.

c) $H^3_{comp}(B;\mathbb{Q}) \to H^3(B;\mathbb{Q})$ *is injective.*

d) B has trivial normal bundle in X with a given normal framing ζ.

e) O is path connected.
Then there exists $n \neq 1$ and there exists an oriented, dimension 4 submanifold $B' \subset X$ which obeys:

1) Let B_0 denote the disjoint union of n distinct, like oriented push- off copies of B. Then $B' = B_0$ in $X - O$.

2) B' intersects A transversally, and $\sigma' \equiv (B' \cap A) \cap O$ is compact.

3) $[\sigma'] = 0$ *in* $H_1(B';\mathbb{Z})$.

4) $[\sigma'] = n\,[\sigma]$ *in* $H_1(A;\mathbb{Q})$.

5) $H^2(B';\mathbb{Q}) = H^2(B_0;\mathbb{Q})$.

6) $[B'] = n\,[B]$ *in* $H_4(X, X - O;\mathbb{Z})$.

7) B' has trivial normal bundle in X and the push-off normal framing, ζ, of B_0 extends from $B_0 - B_0 \cap O$ to a smooth normal framing of B' in X.

This proposition is proved below. Consider its application first.

Proof of Lemma 9.3. The strategy is to apply Proposition 9.6. For this purpose, set $X \equiv \mathrm{int}(Z)$ and $O \equiv Z - (\partial Z \cup U_\Delta \cup E_L \cup E_R)$. Set $A \equiv \mathrm{int}(E_{3-})$ and set $B \equiv \mathrm{int}(E_{3+})$. Assumption a of Proposition 9.6 holds under the assumption that (9.1) is false. Assumption *b* of Proposition 9.6 holds for the following reason: The vanishing of $H_1(Z;\mathbb{Q})$ is guaranteed by Lemma 3.7. The vanishing of $H_1(O;\mathbb{Q})$ then follows using Meyer-Vietoris. (Remember that Δ_Z and $E_{L,R}$ are codimension 3 in Z.) Assumption c of Proposition 9.6 holds because B is the interior of a manifold (E_{3+}) with boundary a disjoint union of 3-spheres. (Remember that $H^2(S^3) = 0$.)

Thus, the assumptions of Proposition 9.6 hold when (9.1) is false. Take E'_{3+} in Lemma 9.3 to be equal to the closure in Z of the submanifold B' of Proposition 9.6. □

Proof of Proposition 9.6. By assumption, there exists an integer n such that $n\,[\sigma] = 0 \in H_1(O;\mathbb{Z})$. Use the given framing of B's normal bundle in X to push-off n parallel copies of B, all with the same orientation as B. Let B_0 denote the union of these n disjoint submanifolds. Make these push-offs close to the original, so that $\sigma_0 \equiv (B_0 \cap A) \cap O$ will be equal to n like oriented, push-off copies of σ. Then $[\sigma_0] = 0$ in $H_1(O;\mathbb{Z})$. This means that σ_0 is the boundary of an oriented, embedded surface with boundary, $R \subset O$. Since A and B have codimension 3 in O, one can arrange R so that $\operatorname{int}(R) \cap A = \emptyset$ and $\operatorname{int}(R) \cap B = \emptyset$. One can also arrange R to be connected because O is connected.

Here is the local picture of R near a component of σ_0: Let C be a component of σ_0. Let v_C denote the normal bundle to C in R, an oriented line bundle. Then, the normal bundle to C in O splits as the direct sum $v_C \oplus L\mid_C \oplus TB_0\mid_C$, where $L \to \sigma_0$ is an oriented 2-plane bundle, and so trivial. (Thus, the normal bundle for B in X along σ_0 splits as $v_C \oplus L\mid_C$.) Choose a framing, ζ_B, for $TB_0\mid_C$ and also choose a framing (e_1, e_2) for L. If a frame, ζ, has been given for B's normal bundle in X, then choose (e_1, e_2) so that with the addition of an oriented frame, e_3, for v_C, the triple (e_1, e_2, e_3) defines an adapted frame to C which is homotopic to $\zeta\mid_C$. (See Lemma 8.4).

Choose an exponential map, e_C, from the normal bundle along C into X which maps the positive axis in v into R and which maps $TB_0\mid_C$ into B_0. Use this exponential map and the given framings to identify an open neighborhood of C in X with one of $C \times (0,0)$ in the triple product $C \times \mathbb{R}^3 \times \mathbb{R}^3$.

Then R near C is identified with the subspace of points (t,x,y)in $C \times R^3 \times R^3$ where $x_1 = x_3 = y = 0$ and $x_1 \geq 0$. Meanwhile, B_0 near C is identified with the subspace of points (t,x,y) with $x = 0$. And, A near C is identified with the subspace of points (t,x,y) where

$$x = \underline{A}(t)\,y + O(\mid y\mid^2), \tag{9.13}$$

where $\underline{A} : C \to Gl(3,\mathbb{R})$.

Because R is connected and has non-trivial boundary, the normal bundle $N_R \to R$ to R in X is isomorphic to the trivial 5- plane bundle. On σ_0, this bundle has a splits as $N_R \approx L \oplus TB_0\mid_{\sigma_0}$. □

LEMMA 9.7. *The subbundle $L \subset N_R\mid_{\sigma_0}$ extends over R as an oriented,* *2-plane subbundle, L_R, of N_R. Furthermore, given a framing (e_1, e_2) for L, there exists an extension $L_R \to R$ of L over which the framing (e_1, e_2) extends.*

Proof. Choose a trivialization for N_R to identify this bundle with $R \times \mathbb{R}^5$. Then $L \subset N_R\,|_{\sigma_0}$ is defined by a map from σ_0 into the space of oriented 2-planes in $\mathbb{R}^5$. The space of such oriented 2-planes deformation retracts onto $SO(5)/(SO(3)\times SO(2))$; thus, L is defined by a map, $\eta : \sigma_0 \to SO(5)/(SO(3)\times SO(2))$. Note that the choice of an oriented frame (e_1, e_2) for L defines a lift of η to a map $\underline{\eta} : \sigma_0 \to SO(5)/SO(3)$.

With the preceding understood, the question here is whether or not this map $\underline{\eta}$ can be extended over R. The answer is that $\underline{\eta}$ can be extended because $SO(5)/SO(3)$ is simply connected. □

With Lemma 9.7 understood, pick an extension, $L_R \to R$ of L over R over which the given frame (e_1, e_2) for L extends. Introduce $V_R \approx N_R/L_R$ and choose a splitting $N_R \approx L_R \oplus V_R$ with the property that $V_R\,|_{\sigma_0}$ agrees with $TB\,|_{\sigma_0}$.

Fix an exponential map $e_R : N_R \to X$ with the property that e_R on $N_R\,|_{\sigma_0}$ agrees with the restriction of $e\sigma_0$ to $L \oplus TB_0\,|_{\sigma_0}$. Choose a fiber metric on N_R so that e_R embeds the interior of the radius 2 ball bundle onto an open neighborhood of $\operatorname{int}(R)$ in X.

Let $\epsilon > 0$ and let $T_R \subset V_R$ denote the radius ϵ ball bundle. Let $S_R \subset T_R$ denote the sphere bundle of radius ϵ . Since V_R is oriented, both T_R and S_R are isomorphic to trivial fiber bundles.

With all of the above understood, define the surgery

$$B_1 \equiv (B_0 - \operatorname{int}(T_R\,|_{\sigma_0})) \cup S_R. \tag{9.14}$$

This B_1 is a C^0 embedding of a smooth, oriented manifold into X. The embedding fails to be smooth at the corner, $S_R\,|_{\sigma_0}$. Smooth B_1 inside V_R on a neighborhood of this corner to produce a smoothly embedded, oriented submanifold, $B' \subset X$.

The claim now is that B', as described above, will satisfy Assertions 1-7 of Proposition 9.6: Assertion 1 is true by construction. To prove Assertion 2, use (9.13) to see that σ' is a push-off of σ_0. Indeed, for small ϵ, the copy in σ' of the component C in (9.13) is given, to order ϵ^2, as the set of (t, x, y) with

$$x = (\epsilon\,(\sum_{1\le j\le 3} |\,\underline{A}^{-1})_{j,1}\,|^2)^{-1/2}, 0, 0) \tag{9.15}$$

and $y = \underline{A}^{-1}(x_1, 0, 0)$. Here $\underline{A}^{-1}_{i,j}$ are the components of the inverse to $\underline{A}$ in (9.13). Note that this identification of σ' confirms Assertion 4 as well.

To prove Assertion 3 of Proposition 9.6, observe first that (9.15) identifies σ' as a section of S_R over a push-off into $\operatorname{int}(R)$ of σ_0. Thus, σ' bounds in B' if the section in question, s, extends as a section over R of S_R. Now, S_R is a trivial 2-sphere bundle so isomorphic to $R \times S^2$. Such an isomorphism identifies the section s, with a map, also called s, from σ to S^2. Since $\pi_1(S^2) = 0$, any such map extends over R.

To prove Assertion 5, invoke the Meyer-Vietoris exact sequences for the decomposition $B_0 = (B_0 - T_R |_{\sigma_0}) \cup T_R |_{\sigma_0}$ and for the decomposition of B_1 in (9.14). (Note that B_1 and B' are homeomorphic.)

To prove Assertion 6, remark that B' and B_1 are C^0-isotopic by an isotopy with support in O, so $[B'] = [B_1]$ in $H_4(X, X - O; \mathbb{Z})$. Meanwhile, T_R, as a 5-cycle in O satisfies $\partial T_R = [B_1] - [B_0]$.

Finally, consider Assertion 7. To begin, remark that the normal bundle to S_R in X is isomorphic to $L_R \oplus \tau$, where $\tau \to S_R$ is the trivial line bundle which is S_R's normal bundle in V_R. Now, by construction, L_R has a frame, (e_1', e_2') which extends (e_1, e_2).

Meanwhile, the normal bundle, N_B, to B in X splits upon restriction to σ_0 as $L \oplus \upsilon_{\sigma_0}$, where υ_{σ_0} is the normal bundle to σ_0 in R, an oriented line. And, the frame ζ is homotopic in a neighborhood of σ_0 so that the result restricts to σ_0 as (e_1, e_2, e_3), where (e_1, e_2) frame L and where e_3 is the inward pointing normal vector to σ_0 in R. Thus, two thirds of the normal frame ζ can extended from the compliment of a neighborhood of σ_0 in B_0 over B'. As usual, there are no obstructions to extending the remaining third of ζ over B'.

e) Proof of Lemma 9.4.

If there are some number $q > 1$ components of B_0, label the components of B_0 as $\{B_{0,\alpha}\}_{\alpha=1}^{q}$. Pick $y_\alpha \in B_\alpha \cap (O - A)$ for each index $\alpha > 1$. Also, choose $q - 1$ distinct points $\{x_\alpha\}_{\alpha \geq 2} \subset B_{0,1}$. Since O is path connected, so $O - A$ will be path connected; and so one can find, for each $\alpha \geq 2$, a path ρ_α (an embedding of $[0, 1]$) which starts at y_α and ends at x_α. Choose the set $\{\rho_\alpha\}_{\alpha \geq 2}$ to be distinct. Now, for each $\alpha \geq 2$, mimick the surgery in (8.23) to make an ambient connect sum of $B_{0,\alpha}$ with $B_{0,1}$. Since the $\{\rho_\alpha\}$ are distinct, these connect sums can be made with out interfering with each other. Use B to denote the result, after smoothing near the corners. The verification that B does the job is left to the reader as an exercise. (For the framing issue, see Lemma 8.10.) □

10 The last pass at $E_\pm$. It is the purpose of this section to explain how to make $E_\pm$ from $E_{4\pm}$ of the preceding section. The metamorphasis from $E_{4\pm}$ to $E_\pm$ will be called *melding.*

This melding operation only changes $E_{4\pm}$ in a neighborhood of $\Delta_Z \cup \partial Z$, and the neighborhood in question can be as small as desired. In particular, in this neighborhood, $E_{4\pm}$ should be the image of E_{4-} under the switch map $\Theta : Z \to Z$ which sends (x, y) to (y, x). (See Assertion 1 of Proposition 9.1). Furthermore, in this neighborhood, E_{4-} (hence, E_{4+}) should consist, locally, of N parallel push-off copies (see Assertions 5 and 6 of Proposition 9.1).

The effect of the melding will be to push all of these parallel copies together on some smaller neighborhood of $\Delta_Z \cup \partial Z$. The cost of the melding is that $E_\pm$ will not be a manifold (unless $N = 1$ in Proposition 9.1).

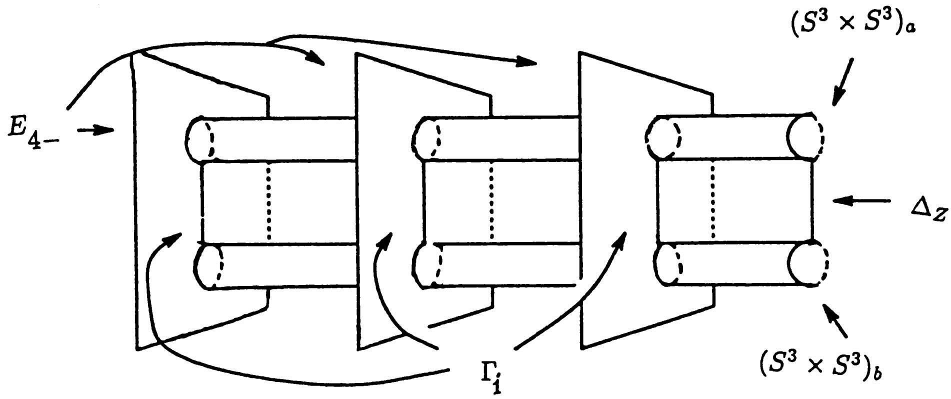
a) E_{4-} **near** $\Delta_Z \cup \partial Z$.

Consider E_{4-}. The following construction will be done r times, once for each pair in $\{(a_i, b_i)\}_{i=1}^r$. These r versions can be done simultaneously, so fix attention on one index i, and simplify notation by setting $a \equiv a_i$ and $b \equiv b_i$.

The set $\Gamma_i \subset E_{4-} \cap \Delta_Z$ is a set of N embedded intervals which connect the N components of $S^3 \times \Lambda_a = E_{4-} \cap (S^3 \times S^3)_a$ with the N components of $S^3 \times \Lambda_b = E_{4-} \cap (S^3 \times S^3)_b$. Near

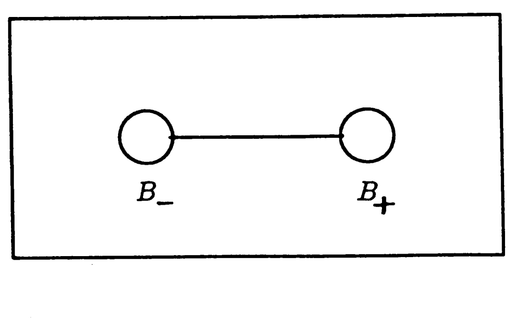
$$(S^3 \times \Lambda_a) \cup \Gamma_i \cup (S^3 \times \Lambda_b), \tag{10.1}$$

E_{4-} consists of N components (sheets), $\{Y_\alpha\}_{a=1}^N$. Here is a picture:



(10.2)

The sheets $\{Y_\alpha\}_{\alpha \geq 2}$ are push-off copies of Y_1. As for Y_1, it is an embedded image in Z of the compliment in the open, unit 4-ball of the interiors of a pair of disjoint 4-balls, $B_\pm$, of radius 1/8, respectively centered at $(\pm 1/4, 0, 0, 0)$. Note that the boundary of B_- is mapped to $S^3 \times p_a \subset S^3 \times \Lambda_a$, and the boundary of B_+ is mapped to the corresponding $S^3 \times p_b \subset S^3 \times \Lambda_b$. Furthermore, $Y_1 \cap \Delta_Z$ is the segment of the x_1 axis between $(\pm 1/8, 0, 0, 0) \in \partial B\pm$. Here is a picture of Y_1:



(10.3)

As remarked, the $\{Y_\alpha\}_{\alpha\geq 2}$ in (10.2) are push-off copies of Y_1. To be precise here, remember that Y_1 has a framing, $\zeta \equiv (e_1, e_2, e_3)$, to its normal bundle, N_{Y_1}, which is a product framing that restricts to both $(S^3 \times p_a)$ and $(S^3 \times p_b)$ as a constant framing. (See Assertion 8 of Propositio 9.1.) Fix an exponential map,

$$e : N_{Y_1} \to B, \tag{10.4}$$

which maps N_{Y_1}'s restriction to $(S^3 \times p_a)$ into $(S^3 \times S^3)_a$, and which likewise maps N_{Y_1}'s restriction to $(S^3 \times p_b)$ into $(S^3 \times S^3)_b$. Fix a metric on N_{Y_1} which makes the frame ζ orthonormal, and fix $\epsilon > 0$ such that (10.4) embeds the interior of the radius 2ϵ ball bundle onto a neighborhood of Y_1 in Z. Use e in (10.4) to identify the interior of this ball bundle with its image in Z.

With the preceding understood, the copy Y_α of Y_1 can be taken as the image of the section $s_\alpha : Y_1 \to N_{Y_1}$ that is given by

$$s_\alpha(x) = (\alpha - 1)\, N^{-1}\, \epsilon\, e_1. \tag{10.5}$$

b) The meld.

With the preceding picture E_{4-} near (10.1) understood, here is the meld: Fix a function $\beta : [0,1] \to [0,1]$ which has the following properties:

(10.6)

1) $\beta \equiv 1$ on $[5/8, 1]$.
2) $\beta \equiv 0$ on $[0, 1/2]$.
3) β is nondecreasing.

As described in (10.3), identify Y_1 with a subset of the unit ball about the origin in $\mathbb{R}^4$. Use x to denote the Euclidean coordinate in $\mathbb{R}^4$, and, by restriction, a point in Y_1. By the way, note that the assignment to $x \in Y_1$ of the number $\beta(|\, x\,|)$ defines a smooth function on Y_1 which vanishes in a neighborhood of

$$B_- \cup \{(x_1, 0, 0, 0) : -1/8 \leq x_1 \leq 1/8\} \cup B_+. \tag{10.7}$$

For $\alpha \geq 2$, define the deformation, Y'_α, of Y_α as follows: Y'_α is the image in N_Y of the section s'_α which sends $x \in Y_1$ to

$$s'_\alpha(x) \equiv (\alpha - 1)\, N^{-1}\, \beta(|\, x\,|)\, \epsilon\, e_1. \tag{10.8}$$

Notice that Y'_α agrees with Y_α on the compliment of a regular neighborhood of (10.1) in Z; but that Y'_α coincides with Y_1 on a smaller neighborhood of

(10.1). Here is a picture for $\alpha \geq 1$:

(10.9)

In (10.9), the shaded region marks where Y'_α and Y_1 intersect.

Here is a picture of all the $\{Y'_\alpha\}_{\alpha \geq 1}$:

(10.10)

Use E_- to denote the result of applying the preceding meld operation to E_{4-} in a neighborhood of (10.1) for each $i \in \{1, \cdots, r\}$.

As for E_+, remember that E_{4+} coincided with $\Theta(E_{4-})$ near each of the r versions of (10.2). This neighborhood can be assumed to include the regions that are depicted in (10.2). With this understood, set $E_+ \equiv E_{4+}$ outside of the Θ-image of the regions in (10.2), but inside the Θ-image of each region in (10.2), declare

$$E_+ \equiv \Theta(E_-). \tag{10.11}$$

c) Properties of $E_\pm$.

The following proposition describes some of the salient features of $E_\pm$:

PROPOSITION 10.1. *Construct $E_\pm \subset Z$ as described above. Then:*

1) *There is an open neighborhood $U \subset Z$ of $\Delta_Z \cup \partial Z$ such that $E_+ \cap U$ and $E_- \cap U$ are images of each other under the switch map on Z.*
2) *The fundamental classes $[E_\pm]$ are equal to $N[E_{1\pm}]$ for some integer $N \geq 1$. Here, $[E_{1\pm}]$ are described by (6.3) and Lemma 6.3.*
3) *$E_\pm$ have empty intersection with $M_0 \times M_0$ and $M_1 \times M_1$.*
4) *$E_\pm$ have empty intersection with $E_{L,R}$ of (4.15).*

5) *If* $p \in$ *crit*(f), *then the intersection of* E_- *with* $(S^3 \times S^3)_p$ *has the form* $S^3 \times x_p$, *where* x_p *is a single point. Similarly, the intersection of* E_+ *with* $(S^3 \times S^3)_p$ *is* $x_p \times S^3$.

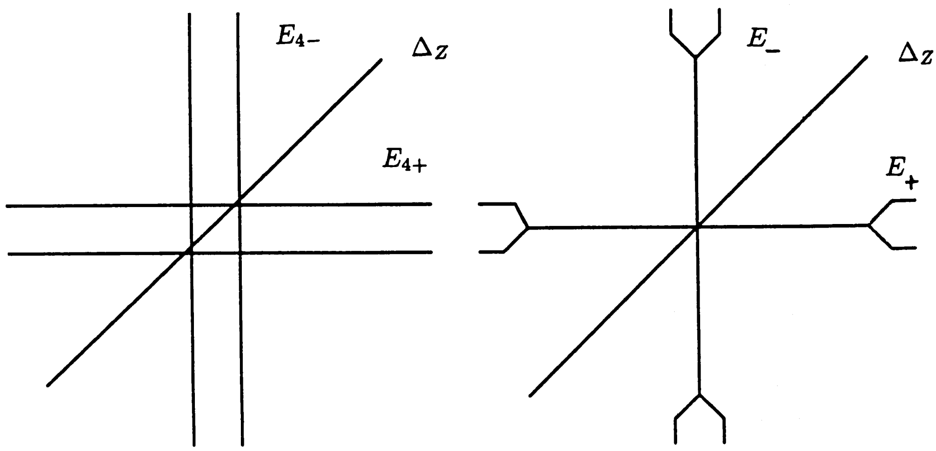
6) $E_\pm \cap \Delta_Z = \cup_{i=1}^r v_i$, *where* $v_i \subset \Delta_Z$ *is as follows: There is a flow line* μ_i *which starts at* a_i *and ends at* b_i. *With the canonical identification of* Δ_W *with* W *understood,* v_i *is a closed interval in a push-off copy of* μ_i. *And,* v_i *starts at* $(x_a, x_a) \in (S^3 \times S^3)_a$ *and* v_i *ends at* $(x_b, x_b) \in (S^3 \times S^3)_b$. *Here,* $a \equiv a_i$ *and* $b \equiv b_i$.

7) $E_- \cap E_+ = \cup_{i=1}^r v_i$.

8) $H^2(E_\pm; \mathbb{Q}) = 0$.

The proof is straightforward and left to the reader. (See Proposition 9.1. Also, use Meyer-Vietoris to compute $H^2(E_\pm)$.)

Here is a picture:



(10.12)

11 Completing the proof. The purpose of this last section is to complete the proof of Theorem 2.9. The strategy here will be as follows: Suppose that M_0 and M_1 are cobordant via a spin 4-manifold, W, with the rational homology of S^3. Factor the cobordism as in Assertion 5 of Proposition 3.2 into two pieces, $W_1 \cap W_3$. Both W_1 and W_3 are given by (3.11). Here, W_1 is a cobordism from M_0 to a rational homology sphere M, while W_3 is a cobordism from M to M_1. In both cases, the manifold with boundary, Z. ($\equiv Z_{1,3}$), has been defined, and Sections 4d, 4e and 10 describe the variety $\Sigma_Z. \subset Z$. (The latter require a choice of base point $p \in M$.) Let $Z \equiv Z_1 \cup Z_3$ and $\Sigma_Z \equiv \Sigma_{Z_1} \cup \Sigma_{Z_3}$, where the common boundary components in both cases are identified (these being $M \times M$ in the former and Σ_M in the latter).

With Z and Σ_Z understood, the proof plan from Section $2k$ will be complete with the completions of Steps 3 and 4 in Section $2k$. These steps are considered below.

Completing Step 3 requires the construction of a 2-form ω_Z on $Z-\Sigma_Z$ which satisfies (2.27). This 2-form will be constructed first on the compliment of Σ_Z in a regular neighborhood $N_Z \subset Z$ of Σ_Z. The extension to $Z - \Sigma_Z$ will be made by appeal to Lemma 4.2.

Step 4 of Theorem 2.9's proof (from Section $2k$) will be completed during the construction of ω_Z.

Let $N_{Z_1} \equiv N_Z \cap Z_1$ and define N_{Z_3} analogously. The 2-form ω_Z on $N_Z - \Sigma_Z$ will be constructed first on $N_{Z_1} - \Sigma_{Z_1}$ and second on $N_{Z_3} - \Sigma_{Z_3}$. The case of $N_{Z_1} - \Sigma_{Z_1}$ is considered in Subsections $11a-h$ and that of $N_{Z_3} - \Sigma_3$ is considered in Subsection $11i$. These two constructions are matched in Subsection $11j$ where Step 4 of Section $2k$ is verified. Section $11k$ completes the proof of Theorem 2.9 with a discussion of the conditions in Lemma 4.2.

To avoid cumbersome notation, the subscript "$_1$" will be dropped in Subsections $a-h$. Thus, in these sections, Z will denote Z_1, Σ_Z will denote Σ_{Z_1}, etc.

a) Preliminary remarks.

Construction of ω_Z on $N_Z - \Sigma_Z$ is accomplished in two steps. The first step defines ω_Z near $\Delta_Z \cup E_L \cup E_R$ by using (4.22), but where φ_Z is a map which is defined only on a regular neighborhood, N', in Z of $\Delta_Z \cup E_L \cup E_R$. This φ_Z has $\varphi_Z^{-1}(0) = \Sigma_Z \cap N'$. The second step constructs ω_Z on $N_Z - N'$.

The construction of φ_Z occupies Subsections $b-f$, below. The construction of ω_Z on $N_Z - N'$ occupies Subsections g and h.

b) Near $E_L \cap E_R$.

The purpose of this subsection is to construct φ_Z near $E_L \cap E_R$. To be precise, a neighborhood $U \subset Z$ of $E_L \cap E_R$ will be described with a map

$$\varphi_U : U \to \mathbb{R}^3 \tag{11.1}$$

which obeys

$$\varphi_U^{-1}(0) = (E_L \cup E_R \cup \Delta_Z) \cap U. \tag{11.2}$$

Then, $\varphi_Z \mid U$ will be declared equal to φ_U.

A digression on framings begins the construction of φ_U. To start the digression, fix a frame for $TM\ |_p$. This frame can be thought of as a frame for the normal bundle to p in TM. Use the pseudo- gradient flow to extend this frame as a normal framing to the flow line $\gamma \subset Z$. This normal framing to γ induces a framing of $TM_0\ |_{p_0}$. End the digression.

Parameterize that flow line γ so that $f(\gamma(t)) = t$. Let $N_\gamma \to \gamma$ denote the normal bundle to γ in W and select an exponential map $e_\gamma : N_\gamma \to W$ which maps $N_\gamma\ |_{p_0,p}$ into M_0, M, respectively. Require that

$$(11.3) \qquad f \circ e_{\gamma(t)} \equiv t.$$

Use this exponential map and the normal framing with the afore- mentioned parameterization of γ to define a diffeomorphism, ψ_γ, from a neighborhood, $O_\gamma \subset W$ of γ onto $[0,1] \times B$, where $B \subset \mathbb{R}^3$ is a ball-neighborhood of the origin.

Using the preceding identification, build $\psi_U \equiv (u_\gamma \times u_\gamma)\ |_Z$ of the neighborhood $U \equiv Z \cap (O_\gamma \times O_\gamma)$ of $E_L \cap E_R$ with $[0,1] \times B \times B$. This U and ψ_U obey the conclusions of Assertion 4 in Lemma 4.5 except that $B \subset \mathbb{R}^3$ should everywhere replace $\mathbb{R}^3$ and $B \times B$ should everywhere replace $\mathbb{R}^3 \times \mathbb{R}^3$.

With the preceding understood, define φ_Z on U to be the composition of the map ψ_U with the map from $[0,1] \times \mathbb{R}^3 \times \mathbb{R}^3$ which sends (t,x,y) to $\Phi_0(x,y)$ with Φ_0 given by (2.15).

Note that $\varphi_Z\ |_U$ agrees with Proposition 2.5's map φ when restricted to a neighborhood of $p_0 \times p_0$ in $M_0 \times M_0$, or to a neighborhood of $p \times p$ in $M \times M$.

Note also, for reference below, that the map $\varphi_Z\ |_U$ is invariant under the switch map $\Theta : Z \to Z$ which sends (x,y) to (y,x).

c) Near $E_L \cup E_R$.

The purpose of this subsection is to construct φ_Z near $E_L \cup E_R$. To begin, remark that the normal bundle $N_R \to E_R$ to E_R in Z is naturally isomorphic to $\pi_L^* N_\gamma$. Thus, said normal bundle has a natural framing.

Take the dual to this natural framing to frame the dual bundle, N_R^* and choose an exponential map $e_R : N_R \to Z$. The framing of N_R^* and e_R together define a map, φ_R, from a neighborhood of E_R in Z into $\mathbb{R}^3$ which has E_R as the inverse image of zero. (See (2.14).) Choose this exponential map so that it sends $N_R\ |_{M_0 \times M_0}$ into $M_0 \times M_0$ and likewise sends $N_R\ |_{M \times M}$ into M. (The exponential map $e_\gamma : N_\gamma \to W$ of the preceding subsection induces such an exponential map in a natural way.)

On $E_R \cap U$, the differentials of the maps φ_R and φ_U are scalar multiples of each other, and so there is a homotopy of φ_R near U which has it agree with $\varphi_Z\ |_U$ on U and so extend $\varphi_Z\ |_U$ to map a neighborhood of E_R in Z to $\mathbb{R}^3$ with the correct inverse image of zero. See the Step 2 of the proof of Proposition 2.5 for the details.

With $\varphi_Z\ |_U$ now extended over E_R, extend it further over $E_R \cup E_L$ by using the switch map $\Theta : Z \to Z$. Use $\varphi_Z\ |_{RL}$ to denote this extended map.

Note that $\varphi_Z\ |_{R,L}$ can be made so that its restriction to a neighborhood of $(p_0 \times M_0) \cup (M_0 \times p_0)$ in $M_0 \times M_0$ agrees with the map φ for M_0 in Proposition 2.5. Likewise, its restriction to a neighborhood of $(p \times M) \cup (M \times p)$ in $M \times M$ can be arranged to agree with the analogous φ for M

d) Near $E_\pm \cap \Delta_Z$.

The intersections between $E_\pm$ and between these varieties and Δ_Z form a set of disjoint line segments, $\{v_i\}_{i=1}^r$. (Note that $E_\pm$ are manifolds near these line segments.) The purpose of this subsection is to define the map φ_Z near each v_i.

To start, fix $i \in \{1, \cdots, r\}$. Let $a \equiv a_i$ and $b \equiv b_i$. Note that v_i has end points $(x_a, x_a) \subset (S^3 \times S^3)_a$ and $(x_b, x_b) \subset (S^3 \times S^3)_b$. Also, the identification, using π_L or π_R, of Δ_Z with a subset of W identifies v_i with a subinterval in a pseudo-gradient flow line which starts at a and ends at b.

Remember that E_- is the result of melding E_{4-} of Proposition 9.1. This means, in particular, that $(S^3 \times S^3)_a \cup v_i \cup (S^3 \times S^3)_b$ has a neighborhood, $U_i \subset Z$, such that $E_- \cap U_i$ is the same point set as a component, Y, of $E_{4-} \cap \Delta_Z$. Meanwhile, $E_+ \cap U_i = \Theta(Y)$.

Remark next that E_{4-} has, according to Assertion 8 of Proposition 9.1, a special normal framing, ζ. And, E_{4+} has a special normal frame, ζ', which restricts to U_i as the image of ζ under the switch map Θ. The pair of frames (ζ, ζ') restrict to v_i to frame the normal bundle $N_i \to v_i$ of v_i in Z.

Notice that Θ fixes Δ_Z and the differential of Θ (denoted Θ^*) acts on N_i and interchanges Span(ζ) with Spanζ').

Fix an exponential map $e : N_i \to v_i$ with the following properties:

(11.4)

1) . e :Span$(\zeta) \to E_-$.
2) e :Span$(\zeta') \to E_+$.
3) At (x_a, x_a), e maps N_i into $(S^3 \times S^3)_a$.
4) At (x_b, x_b), e maps N_i into $(S^3 \times S^3)_b$.
5) $\Theta \circ e = e \circ \Theta^*$.

Together, the map e and the frames (ζ, ζ') define a map

$$\psi : v_i \times \mathbb{R}^3 \times \mathbb{R}^3 \to U_i \tag{11.5}$$

with the following properties:

(11.6)

1) There is an open ball $B \subset \mathbb{R}^3$ about 0 and ψ embedds $v_i \times B \times B$.
2) ψ is the identity on $v_i \times (0,0)$.
3) $\psi^{-1}(E_-) = \{(t, x, 0) \in v_i \times \mathbb{R}^3 \times \mathbb{R}^3\}$.
4) $\psi^{-1}(E_+) = \{(t, 0, y) \in v_i \times \mathbb{R}^3 \times \mathbb{R}^3\}$.
5) $\psi^{-1}(\Delta_Z) = \{(t, x, x) \in v_i \times \mathbb{R}^3 \times \mathbb{R}^3\}$.
6) $\psi((x_a, x_a) \times \mathbb{R}^3 \times \mathbb{R}^3) \subset (S^3 \times S^3)_a$.
7) $\psi((x_b, x_b) \times \mathbb{R}^3 \times \mathbb{R}^3) \subset (S^3 \times S^3)_b$.
8) $\psi(t, x, y) = \Theta(\psi(t, y, x))$.

Given the preceding, define the map φ_Z on a neighborhood of v_i in Z by declaring that

$$(\varphi_Z \circ \psi)(t, x, y) \equiv \Phi_0(x, y), \tag{11.7}$$

where Φ_0 is given in (2.15).

e) Near $(S^3 \times S^3)_p$.

The next step is to define the map φ_Z near Z's boundary components $\{(S^3 \times S^3)_p\}_{p\in \text{crit}(f)}$. So, fix $i \in \{1, \cdots, r\}$ and let p denote either a_i or b_i.

A neighborhood, $V_p \subset Z$, of $(S^3 \times S^3)_p$ is diffeomorphic to the product $(S^3 \times S^3)_p \times [0, 1)$ as a manifold with boundary. Furthermore, there is no difficulty in finding such a diffeomorphism so that

(11.8)

1) $E_- \cap V_p = (S^3 \times x_p) \times [0, 1)$.
2) $E_+ \cap V_p = (x_p \times S^3) \times [0, 1)$.
3) $\Delta_Z \cap V_p = \Delta_{S^3} \times [0, 1)$.
4) The switch map acts by $\Theta(x, y, t) = (y, x, t)$.

In V_p, the variety E_- is smooth and it agrees with a component, Y, of $E_{4-} \cap V_p$. Also, $E_+ \cap V_p = \Theta(E_- \cap V_p)$. Also, E_{4-} has the normal frame, ζ, which restricts to Y as a constant frame. And, E_{4+} has the normal frame ζ' which restricts to $\Theta(Y)$ as $\Theta * \zeta$.

Use these constant frames to define frames for the dual bundles to the normal bundles of Y and $\Theta(Y)$ in Z. Then, use these frames for the conormal bundles to extend $\varphi_Z \mid U_i$ of (11.7) to a neighborhood of $E_\pm \cap V_p$ by mimicking Step 2 in the proof of Proposition 2.5. (Exploit the product structure on V_p in (11.8).)

Meanwhile, T^*S^3 has its singular framing which gives (see Proposition 2.7, Definition 2.8 and Lemma 2.11) the canonical homotopy class of singular framing for which the value of $I_2(S^3)$ is zero. As in Step 3 of Proposition 2.5's proof, use this framing to obtain a singular framing of the normal bundle to $\Delta_{S^3} \times [0, 1)$ in Z.

(Remember that $(S^3 \times S^3)_p$ has two obvious projections to S^3, these are given by the product structure in (3.26) and are denoted $\pi_\pm$. To be explicit, introduce the coordinate sytem ψ_p of (3.2) and introduce $U_p \equiv \psi_p(\mathbb{R}^4)$. Note that $U_p \times U_p$ is a neighborhood of (p, p) in $W \times W$. With this understood, $\pi_\pm$ are the restrictions to $(S^3 \times S^3)_p$ of the maps from $U_p \times U_p$ to $\mathbb{R}^4$ which are given by

(11.9) $\pi_-(x, y) \equiv (y_1, x_2, x_3, x_4)$ and $\pi_+(x, y) \equiv (x_1, y_2, y_3, y_4)$

when $p \in \text{crit}_1(f)$; and by

(11.10) $\pi_-(x, y) \equiv (y_1, , y_2, x_3, x_4)$ and $\pi_+(x, y) \equiv (x_1, x_2, y_3, y_4)$

when $p \in \text{crit}_2(f)$. Then, the map $\pi_+^* - \pi_-^*$ identifies T^*S^3 with the normal bundle to Δ_{S^3} in $(S^3 \times S^3)_p$.)

As in Step 3 of the proof of Proposition 2.5, use the induced framing of the normal bundle to $\Delta_S^3 \times [0,1)$ in Z and the product structure of V_p as described in (11.8) to extend $\varphi_Z \mid U_i$ over a neighborhood of the intersection of $\Delta_Z \cap V_p$.

Note that this extension of φ_Z to a neighborhood of $\Sigma_Z \cap V_p$ has the property that its restriction to $(S^3 \times S^3)_p$ is equal to a map φ in Proposition 2.5 (for S^3) that gives $I_2 = 0$. See Lemma 2.11.

f) Near Δ_Z.

At this point, the map φ_Z is defined on a neighborhood of $E_{L,R}$ and on a neighborhood of each $\{(S^3 \times S^3)_p\}_{p\in \mathrm{crit}(f)}$ and on a neighborhood of each $\{v_i\}_{i=1}^r$. The purpose of this subsection is to extend φ_Z to a neighborhood of Δ_Z.

As a preamble, remark that φ_Z has already been defined near some parts of Δ_Z, specifically, near $\{v_i\}_{i=1}^r$ and near γ and near Δ_{S^3} in each $\{(S^3 \times S^3)_p\}_{p\in \mathrm{crit}(f)}$. The extension over a full neighborhood of Δ_Z is obtained by mimicking the constructions of Step 3 in the proof of Proposition 2.5.

To be more precise about this strategy, it is necessary to first introduce $N_Z^* \to \Delta_Z$, the dual to the normal bundle to Δ_Z in Z. A particular framing over $\Delta_Z - (\gamma \cup (\cup_{i=1}^r v_i))$ will be chosen for N_Z^*, and an exponential map, $e : N_Z \to Z$ as well. These give a map φ_Δ, as in (2.14), from a neighborhood in Z of $\Delta_Z - (\gamma \cup (\cup_{i=1}^r v_i))$ to $\mathbb{R}^3$. The singularities near $\gamma \cup (\cup_{i=1}^r v_i)$ of the chosen framing will be constrained so that φ_Δ can be readily homotoped near γ and $(\cup_{i=1}^r v_i)$ and each $\{\Delta_S^3 \subset (S^3 \times S^3)_p\}_{p=1}^r$ to match up with φ_Z where the latter is already specified.

The framing χ for N_Z^* over $\Delta_Z - (\gamma \cup (\cup_{i=1}^r v_i))$ will have the following form near γ: Write $\chi \equiv (e_1, e_2, e_3)$ and use the coordinates near γ that are given by the map ψ_U of Section 11a, above. Write $x = v + u$ and write $y = v - u$ so that $u \equiv 0$ signifies Δ_Z and $u = v = 0$ signifies γ. Then,

$$\chi\,|_{(t,v,v)} = 2\,\langle v, du\rangle\, v - |\, v\,|^2\; du, \tag{11.11}$$

where $\langle v, du\rangle \equiv \Sigma_{j=1}^3 v_j\, du_j$.

The singularity of χ near each v_i will have the form of (11.11) when the coordinates of (11.5), (11.6) are used.

Also, χ is constrained on the diagonal Δ_{S^3} in each $(S^3 \times S^3)_p$ so that the inverse of the natural identification $(\pi_+^* - \pi_-^*)$ between T^*S^3 and the conormal bundle of $\Delta_{S^3} \subset (S^3 \times S^3)_p$ sends χ to a singular coframe which gives S^3's canonical singular frame.

With the preceding understood, agree now to further constrain χ along Δ_{M_0} as follows: The inverse of the natural identification $(\pi_R^* - \pi_L^*)$ between T^*M_0 and the conormal bundle to $\Delta_{M_0} \subset M_0 \times M_0$ should send χ to a singular frame for T^*M_0. (See Definition 2.3).

This χ, as constrained above, will extend over the rest of Δ_Z when the following condition is met:

LEMMA 11.1. *Let*

$$T \equiv (\gamma \cup (\cup_{i=1}^{r} v_i) \cup (\cup_{p\in\ crit(f)} (\Delta_{S^3})_p) \cup \Delta_{M_0}) \subset \Delta_Z, \tag{11.12}$$

and let χ *be a frame for* $N_Z^* \to (\Delta_Z - (\gamma \cup (\cup_{i=1}^{r} v_i))$ *which is defined on a neighborhood of* T *as described above. If the homotopy class of* T^*M_0*'s singular frame* $(\pi_R^* - \pi_L^*)^{-1}(\chi \mid \Delta_{M_0})$ *is in* $ker(l_W)$ *(see (2.12), then the frame* χ *extends over* $\Delta_Z - T$ *as a frame for* N_Z^*.

Given that Lemma 11.1 is true, the extension of the map φ_Z near Δ_Z is obtained by using (2.14) and a singular frame χ as described above for which $(\pi_R^* - \pi_L^*)^{-1}(\chi \mid \Delta_{M_0})$ gives Proposition 2.7's canonical homotopy class of singular frame for T^*M_0. (Theorem 2.9 assumes that the canonical singular frame for M_0 is annihilated by the homomorphism $l_{W_1 \cup W_3}$; and this implies that this homotopy class is also annihilated by l_{W_1}.) The construction of φ_Z near Δ_Z using χ can be made with a straightforward appropriation of the arguments in Step 3 of the proof of Proposition 2.5. These final details are left to the reader.

Proof of Lemma 11.1. The singular framing χ has been defined near the point $p \in \Delta_M$ by (11.11). First, choose any extension, χ', of χ over the remainder of Δ_M so that $(\pi_R^* - \pi_L^*)^{-1}(\chi' \mid \Delta_{M_0})$ is a singular frame for T^*M as described in Definition 2.3.

Because W is a spin manifold, the bundle $N_Z^* \to \Delta_Z$ is a trivial bundle, so it has a framing, h. If $h_{1,2}$ are a pair of framings of N_Z^*, then $h_1 = g\, h_2$, where $g : \Delta_Z \to SO(3)$.

Let $U_0 \subset \Delta_Z$ be a regular neighborhood of $\Delta_{M_0} \cup \gamma \cup \Delta_M$. This U_0 can be taken so that the boundary of its closure in $\Delta_Z$is a submanifold which is diffeomorphic to the connect sum of M_0 with M. One can also take U_0 so that the extension χ' is defined on the boundary of its closure.

Fix $i \in \{1, \cdots, r\}$ and set $a \equiv a_i$ and $b \equiv b_i$. Let $U_i \subset \Delta_Z$ be a regular neighborhood of $(\Delta_{S^3})_a \cup v_i \cup (\Delta_{S^3})_b$. This U_i can be taken so that the boundary of its closure in Δ_Z is a submanifold which is diffeomorphic to S^3. One can also take U_i so that χ (and so χ') is defined over the boundary of its closure.

Let $X \equiv \Delta_Z - (U_0 \cup (\cup_i U_i))$. By construction, X is a smooth manifold with boundary, and χ' is defined over ∂X. Let h be a frame for N_Z^* over Δ_Z. Then $\chi' = g\,(h\,|_{\partial X})$ where $g : \partial X \to SO(3)$. Extending χ' over int(X) is the same as extending g. Obstruction theory shows that the map g will extend if:

(11.13)

1) $g_* : H_1(\partial X; \mathbb{Z}/2) \to H_1(SO(3); \mathbb{Z}/2)$ annihilates the kernel of the inclusion induced homomorphism $i^* : H_1(\partial X; \mathbb{Z}/2) \to H_1(X; \mathbb{Z}/2)$.
2) $g_* : H_3(\partial X; \mathbb{Z}) \to H_3(SO(3); \mathbb{Z})$ annihilates the kernel of the inclusion induced homomorphism $i^* : H_3(\partial X; \mathbb{Z}) \to H_3(X; \mathbb{Z})$.

These two conditions can be satisfied for some extension χ' of χ provided that the restriction of χ to $\Delta_{M_0} - p_0$ differs from h by a map $g_0 : (M_0 - p_0) \to SO(3)$ for which

$$g_{0*} : H_1(M_0; \mathbb{Z}/2) \to H_1(SO(3); \mathbb{Z}/2) \tag{11.14}$$

annihilates the kernel of $i_* : H_1(M_0; \mathbb{Z}/2) \to H_1(W; \mathbb{Z}/2)$. That is, if the invariant $l_W(\cdot)$ of (2.12) vanishes on the homotopy class of T^*M_0's singular frame $(\pi_R^* - \pi_L^*)^{-1}(\chi \mid \Delta_{M_0})$. This proves the lemma. □

g) ω_Z Near E_-.

The map φ_Z has now been defined near all of Σ_Z save for the compliment in $E_\pm$ of a neighborhood of Δ_Z. Let $N' \subset Z$ denote a regular neighborhood of $\Delta_Z \cup E_L \cup E_R$ over which φ_Z is defined. With this understood, define $\omega_Z \equiv \varphi_Z^* \mu$, where μ is the 2-form of (2.3). The task in this subsection and the next is to extend ω_Z over the rest of Σ_Z.

In order to accomplish this task, it proves useful to focus first on E_-. There are three distinguished regions of E_-. Regions 1 and 2 each consist of r components. Each such component is labeled by $i \in \{1, \cdots, r\}$. To describe the i'th component of Region 1 or 2, it proves convenient to return to the notation and coordinates that are used in Section 10 to describe the meld region in E_- near v_i and $(S^3 \times S^3)_a$ and $(S^3 \times S^3)_b$ for $a \equiv a_i$ and $b \equiv b_i$. In particular, return to (10.7) - (10.10).

With the preceding coordinates understood, the i'th component of *Region 1* is defined to be the compliment in the interior of the ball of radius $15/32$ in $\mathbb{R}^4$ of the balls $B_\pm$ of radius $1/8$ and center $\pm 1/4$, respectively.

The i'th component of *Region 2* is the transition region between the fully melded part of E_- and the part of E_- which agrees with E_{4-}. Here is a 5 step definition of the i'th component of Region 2: *Step 1*: Introduce the annulus $A \subset \mathbb{R}^4$ which is given as the compliment of the radius $13/32$ ball about the origin in the ball of radius 1 about the origin. This A is identified as an open subset of the sheet Y_1 of E_{4-}. (In Section 10, Y_1 is identified with the subset of $\mathbb{R}^4$ that is the compliment interior of the radius 1 ball of the balls $B_\pm$.) *Step 2*: In $\mathbb{R}^4$, intoduce the rays $\{r_\alpha\}_{\alpha=0}^N$ by

$$r_0 \equiv \{(t, y) : y = 0 \text{ and } t \leq 0\}. \tag{11.15}$$

$$\{r_\alpha \equiv \{(t, y) : y = N^{-1}\,(\alpha - 1)\,\epsilon\, e_1 \text{ and } t \geq 0\}.$$

Here, e_1 is the unit vector along the first axis in $\mathbb{R}^3$. *Step 3*: Take the function β of (10.6) and define a map ψ from $\mathbb{R}^4 \times \mathbb{R}^3$ into $\mathbb{R}^4$ by setting

$$\psi(x, y) = (\beta(\mid x \mid), y) \tag{11.16}$$

with β as in (10.6). Here, $\mathbb{R}^4$ is written as $\mathbb{R} \times \mathbb{R}^3$. *Step 4*: Introduce the projection

$$\pi_1 : \mathbb{R}^4 \times \mathbb{R}^3 \to \mathbb{R}^4. \tag{11.17}$$

Step 5: The i'th component of Region 2 is given as

$$\psi^{-1}(\cup_{\alpha=0}^{N} r_\alpha) \cap \pi_1^{-1}(A) \tag{11.18}$$

Region 3 contains the compliment in E_- of Regions 1 and 2. And, Region 3 intersects the i'th component of Region 2 in

$$\psi^{-1}(\cup_{\alpha=0}^{N} r_\alpha) \cap \pi_1^{-1}(A') \tag{11.19}$$

where A' is the compliment of the radius 7/8 ball in the interior of the radius 1 ball.

$\cdot\omega_Z$ in Region 3

With the preceding understood, Here is a four step definition of ω_Z in Region 3: *Step 1*: Note that E_- in Region 3 ($\equiv R_3$) agrees with E_{4-} which has framed normal bundle. Use the framing, ζ, from Assertion 8 of Proposition 9.1. *Step 2*: Choose an exponentional map to map said normal bundle into Z. *Step 3*: Define a map, φ, from a neighborhood of R_3 into $\mathbb{R}^3$ by using (2.14). The map φ will have $\varphi^{-1}(0) = R_3$. *Step 4*: Define ω_Z on Region 3 to be

$$\omega_Z \equiv N^{-1}\,\varphi^{-1}(\mu), \tag{11.20}$$

with μ as in (2.3) and with N as in Assertion 2 of Proposition 10.1.

ω_Z in Region 1

The definition of φ_Z in Region 1 is almost as simple: In each component of Region 1, E_- coincides with a sheet of E_{4-} and so has framed normal bundle. Use the framing ζ again. Again, pick an exponential map for the normal bundle, and define a map φ'_Z using (2.14).

This φ'_Z will not necessarily match up where φ_Z has already been defined, i.e. near each v_i and near each $(S^3 \times x_p) \subset (S^3 \times S^3)_p$. However, the differentials of φ_Z and φ'_Z differ at most by a scalar multiple along E_- where both are defined, so it is a straightforward proceedure to modify φ'_Z to match up with φ_Z where the two disagree. See the argument in Step 2 of the proof of Proposition 2.5.

With this matching complete, define ω_Z in Region 1 by

$$\omega_Z \equiv \varphi_Z^{-1}(\mu), \tag{11.21}$$

where μ is as specified in (2.3).

ω_Z in Region 2

Fix $i \in \{1, \cdots, r\}$ and consider the definition of ω_Z near the i'th component of Region 2, i.e. near (11.18). The 2-form ω_Z will be defined near (11.18) in $\mathbb{R}^4 \times \mathbb{R}^3$ as the pull-back via the map $\psi / \mid \psi \mid$ of a 2-form on $S^3 - \cup_{\alpha=0}^N (S^3 \cap r_\alpha)$. This 2-form μ_N, is given as follows: First, let $\{p_\alpha \equiv S^3 \cap r_\alpha\}_{\alpha=0}^N$. Then, employ:

LEMMA 11.2. *Let $N \geq 1$ be given as well as $N+1$ distinct points $\{p_\alpha\}_{\alpha=0}^N \subset S^3$. For each $\alpha \in \{0, \cdots, N\}$, let $B_\alpha \subset S^3$ be an embedded 3- ball with $B_\alpha \cap (\cup_{\alpha'} p_{\alpha'}) = p_\alpha$. Orient ∂B_α by the normal directed towards p_α. Let ω_α be a closed 2-form on $B_\alpha - p_\alpha$ with the following property:*

1) $\int_{\partial B_0} \omega_0 = 1$.

2) $\int_{\partial B_\alpha} \omega_\alpha = 1/N$ *if* $\alpha \geq 1$.

Then, there is a closed 2-form, μ_N, on $S^3 - \cup_{\alpha=0}^N p_\alpha$ which, for all α, restricts to $B_\alpha - p_\alpha$ as ω_α.

Proof. Use Meyer-Vietoris.

The lemma gives μ_N once suitable $\{\omega_\alpha\}_{\alpha=0}^N$ are specified. These should be chosen so that $(\psi / \mid \psi \mid)^* \mu_N$ agrees with ω_Z where Region 2 overlaps Regions 1 and 3. (The overlap with Region 1 determines ω_0 and the N components of the overlap with Region 3 determines $\{\omega_\alpha\}_{\alpha \geq 1}$. In this regard, remember that E_- in Region 2 is made by modifying the amounts of push-off of N sheets, $\{Y_\alpha\}_{\alpha=1}^N$, of E_{4-}. These push-offs are all parallel and in the e_3 direction with respect to the frame ζ of E_{4-}. On the sheet Y_α, the frame ζ is the push-off copy of the frame ζ on Y_1. Thus, the frame ζ on each sheet of (11.21) agrees with the constant frame (e_1, e_2, e_3) for $\mathbb{R}^3$.) The details here are left to the reader. □

h) ω_Z near E_+.

Where E_+ and E_{4+} differ, $E_+ = \Theta(E_-)$. In fact, E_+ can be divided into three regions, which are Region 3 and the image by Θ of Regions 1 and 2 in E_-. On Region 3, E_+ and E_{4+} agree. With this understood, the form ω_Z should be defined near the Θ image of the E_- regions 1 and 2 by pull-back using the map Θ. The definition of ω_Z on Region 3 of E_+ mimics the definition of ω_Z on Region 3 of E_- and the details are left to the reader.

i) The form ω_Z on Z_3.

Remember (from this section's introduction) that the original cobordism, W, between M_0 and M_1 was split in half as $W_1 \cup W_3$, and this resulted in a corresponding split of $Z = Z_1 \cup Z_3$. The preceding subsections defined ω_Z on $N_{Z_1} - Z_1$, and it is the purpose of this subsection to describe ω_Z on $N_{Z_3} - Z_3$.

But for an obvious change of notation, the construction of ω_Z on $N_{Z_3} - Z_3$ repeats the constructions of the previous subsections (a- h). (The notation change replaces $W \equiv W_1$ by $W \equiv W_3$ and (M_0, p_0) by (M_1, p_1).)

j) Continuity of ω_Z.

Let $Z \equiv Z_3 \cup Z_3$ as in the introduction to this section. Likewise, let $W \equiv W_1 \cup W_3$. Let $\Sigma_Z \equiv \Sigma_{Z_1} \cup \Sigma_{Z_3}$ and let N_Z be the corresponding union of N_{Z_1} and N_{Z_3}.

The continuity of ω_Z cross $(N_Z - \Sigma_Z) \cap (M \times M)$ is a concern because ω_Z has been separately constructed on $N_{Z_1} - \Sigma_{Z_1}$ and $N_{Z_3} - \Sigma_{Z_3}$. Let ω_{Z_3} denote the restriction of ω_Z to $N_{Z_1} - \Sigma_{Z_1}$ and let ω_{Z_3} denote the analogous restriction to $N_{Z_3} - \Sigma_{Z_3}$. The constructions in Subsections 11b, and 11c insure that ω_{Z_1} and ω_{Z_3} match up nicely near $p \times M$ and $M \times p$ in $M \times M$. At issue is the match between ω_{Z_1} and ω_{Z_3} near $\Delta_M \subset M \times M$.

Subsection f describes ω_{Z_1} and ω_{Z_3} near Δ_M. The form ω_{Z_1} is constructed with the help of a singular frame for M which is an extension over W_1 (see Lemma 11.1) of the canonical singular frame for M_0. Likewise, ω_{Z_3} is constructed near Δ_M with the help of a singular frame for M which is an extension over W_3 of the canonical singular frame for M_1. At issue is whether these two singular frames can be chosen to agree. The purpose of this subsection is to prove that these frames can be assumed equal under the assumptions of Theorem 2.9.

To begin, consider W_1 and remark that the singular frame in question which defines ω_{Z_1} near Δ_M comes from a frame χ_1 for the conormal bundle $N^*_{Z_1}$ for $\Delta_{Z_1} - (\gamma_1 \cup (\cap_{i=1}^{r_1} v_{1i}))$ as a submanifold of Z_1. Here, the notation is from Lemma 11.1 except that subscripts "1" now appear to signify subsets of Z_1. In particular, χ_1 has the prescribed singularity of (11.11) along $\gamma_1 \cup (\cup_{i=1}^{r_1} v_{1i})$.

Likewise, ω_{Z_3} is defined with the help of a frame, χ_3, for the conormal bundle $N^*_{Z_3}$ of $\Delta_{Z_3} - (\gamma_3 \cup (\cap_{i=1}^{r_3} v_{3i}))$ with the prescribed singularity of (11.11) along $\gamma_3 \cup (\cup_{i=1} r_3 v_{3i})$.

Theorem 2.9 assumes that (2.12)'s homomorphism l_W annihilates the canonical singular frame for M_0. This implies that the proof of Lemma 11.1 can be repeated with minor notational changes to prove that χ_1 extends over

$$\Delta_Z - ((\gamma_1 \cup (\cup_{i=1}^{r_1} v_{1i})) \cup (\gamma_3 \cup (\cup_{i=1}^{r_3} v_{3i}))). \tag{11.22}$$

as a frame for the conormal bundle N^*_Z for Δ_Z in Z. Likewise, Theorem 2.9 assumes that l_W annihilates the canonical singular frame for M_1; and so χ_3 extends over (11.22) also.

With the preceding understood, then the following lemma implies that χ_1 and χ_3 can be chosen to agree. Thus, the lemma below resolves the continuity issue.

LEMMA 11.3. *Let $\chi \equiv \chi_1$ or χ_3. If $(\pi^*_R - \pi^*_L)^{-1}(\chi \mid \Delta_{M_0})$ gives Definition 2.8's canonical homotopy class of singular frame for T^*M_0, then Definition 2.8's canonical homotopy class of singular frame for T^*M_1 is given by $(\pi^*_R - \pi^*_L)^{-1}(\chi \mid \Delta_{M_1})$.*

Proof. To start the argument consider the following three remarks: *Remark 1* is that the dual, N^*_W, to the normal bundle of $\Delta_W \subset W \times W$ is canonically isomorphic to T^*W. The isomorphism is $\pi_L^* - \pi_R^* : T^*W \to T^*(W \times W)\,|_\Delta$. (The image of the preceding map annihilates $T\Delta_W$.)

Remark 2 is that the restriction of N_W^* to Δ_Z has a natural line subbundle, namely $\mathrm{Span}(dF) \subset N_W$, where F is the function in (3.20). The quotient bundle is naturally isomorphic to N_Z^*. Thus, N_W^* splits over Δ_Z as

$$N_W^* \approx N_Z^* \oplus \mathrm{Span}(dF) \tag{11.23}$$

Remark 3 is the observation that, as in Proposition 2.7, there are, up to homotopy, two canonical, honest framings, (χ_+, χ_-), of $N_Z^* \to \Delta_Z$ which can be obtained from the singular framing χ. (Copy the construction of $\zeta_\pm$ in the proof of Proposition 2.7 and use the fact that the singularity of χ along any of the paths $\gamma_1 \cup \gamma_3$ or $\{v_{1i}\}_{i=1}^{r_1}$ or $\{v_{3i}\}_{i=1}^{r_3}$ are independent of the parameter along that path. See (11.11).) □

Given the following three remarks, it follows that χ with dF gives a framing of $N_W^* \oplus N_W^*$ over Δ_Z, namely $((\chi_+, dF), (\chi_-, dF))$.

LEMMA 11.4. *The framing $((\chi_+, dF), (\chi_-, dF))$ extends to a framing of $N_W^* \oplus N_W^*$ over Δ_W.*

Accept this lemma and here is how to finish the proof of Lemma 11.1's second assertion: It follows from Lemma 11.4 that the framing for $N_W^* \oplus N_W^*$ over Δ_{M_0} given by $((\chi_{1+}, dF), (\chi_{1-}, dF))$ extends over all of Δ_W to give a framing, $((\chi'_{1+}, dF), (\chi'_{1-}, dF))$, for $N_W^* \oplus N_W^*$ over Δ_{M_1}.

Suppose first that (χ_{1+}, χ_{1-}) gives Atiyah's canonical framing on the nose. (See, e.g. Assertion 3 of Proposition 2.7.) Then, because W has index zero, (χ'_{1+}, χ'_{1-}) must give Atiyah's canonical framing for M_1.

Now, suppose that $(\chi_{1+}, \chi_{1-}) = g\,A$, where g is a map from M_0 to Spin(6) with minimal, non-negative degree, and where A is a frame which gives Atiyah's canonical frame. There is no obstruction here to extending g over Δ_W as a map to Spin(6). With this extension understood, it follows that $g^{-1}\,(\chi'_{1+}, \chi'_{1-})$ must give Atiyah's canonical frame for M_1. (This is because W has signature zero.)

By definition, there exists a map h from M_1 of minimal non- negative degree such that $h\,g^{-1}\,(\chi'_{1+}, \chi'_{1-}) = (\chi_{3+}, \chi_{3-})$. This implies that $\mathrm{degree}(h\,g^{-1}) \leq 0$. (Note that the degrees of g's restrictions to M_0 and to M_1 must agree.)

Now, reverse the roles of M_0 and M_1 and also χ_1 and χ_3 in the preceding argument. The inevitable conclusion has $\mathrm{degree}(g\,h^{-1}) \leq 0$. However, $\deg(h\,g^{-1}) = -\deg(g\,h^{-1})$, so g and h must have the same degree. This implies the lemma in the general case.

Proof. The framing $((\chi_+, dF), (\chi_-, dF))$ is defined over each $\{(\Delta_{S^3})_p\}_{p\in\,\mathrm{crit}(f)}$ and the task is to extend this frame over the open 4-ball components in $\Delta_W - \Delta_Z$ which are bounded by the 3- spheres in $\{(\Delta_{S^3})_p\}_{p\in\,\mathrm{crit}(f)}$. For this purpose, fix $p \in \mathrm{crit}(f)$ and introduce the coordinate sytem ψ_p of (3.2) and let $U_p \equiv \psi_p(\mathbb{R}^4)$. Note that $U_p \times U_p$ is a neighborhood of (p,p) in $W \times W$. The projections $\pi_\pm$ of (11.9), (11.10) define a second product structure on $U_p \times U_p$.

Let τ denote the 1-form on $\mathbb{R}^4$ which is the exterior derivative of the square of the distance function from the origin. Then, note from (3.25) that

$$(11.24) \qquad (\pi_+^* - \pi_-^*)\tau = dF\,|_\Delta$$

Thus, the inverse of $(\pi_+^* - \pi_-^*)$ over $\Delta_{\mathbb{R}^4}$ is a map which identifies N_W^* with $T^*\Delta_{\mathbb{R}^4}$. This map identifies the frame $((\chi_+, dF), (\chi_-, dF))$ with a frame for $(T^*\mathbb{R}^4 \oplus T^*\mathbb{R}^4)\,|_{S^3}$ which gives Atiyah's canonical 2-frame (see Definition 2.8 and Lemma 2.11). Atiyah's canonical 2- frame for S^3 extends over the 4-ball, and so the image of this extension under the map $(\pi_+^* - \pi_-^*)$ extends $((\chi_+, dF), (\chi_-, dF))$. □

k) Verification of Lemma 4.2.

Define ω_Z on $N_Z - \Delta_Z$ as above. If ω_Z can be shown to extend to $Z - \Sigma_Z$ as a closed form, then (2.28) completes the proof of Theorem 2.9. The extension will be made by appealing to Lemma 4.2.

As previously remarked, the second condition of Lemma 4.2 follows from Proposition 10.1 so only Condition 1 of Lemma 4.2 is at issue.

To verify Condition 1 of Lemma 4.2, remark first that the image of the class of ω_Z in $H^3_{\text{comp}}(N_Z)$ is represented by a closed 3- form, η, which is obtained as follows: Let $\rho : N_Z \to \mathbb{R}$ be a smooth function which has compact support and which takes the value 1 near Σ_Z. Then

$$(11.25) \qquad \eta \equiv -d\rho \wedge \omega_Z.$$

Clearly, this form integrates to zero over any closed 3-cycle in N_Z. But, to decide whether η is Poincare dual to σ_Z, one must compute the integral of η over cycles which represent certain classes in $H_3(Z, Z - N_Z)$.

To identify the relevant cycles, remark that

$$(11.26) \qquad H_3(Z, Z - NZ) \approx H_3(Z_1, Z_1 - N_{Z_1}) \oplus H_3(Z_3, Z_3 - N_{Z_3}).$$

It is therefore permissable to concentrate on each factor in (11.26) separately. Since the arguments are the same for either factor in (11.26), the notation will be simplified starting in the next paragraph by using Z to denote *either* Z_1 or Z_3.

To begin the verification of Condition 1 of Lemma 4.2, remark that it is convenient to replace σ_Z by a homologous cycle. For this purpose, remark that an isotopy of $E_{4\pm}$ pushes this space into N_Z if it is not there already. With $E_{4\pm}$ in N_Z, note that the class σ_Z in $H_4(N_Z, N_Z - \Sigma_Z)$ is the same as $\sigma' \equiv [\Delta_Z] - [E_L] - [E_R] - N^{-1}([E_{4-}] - [E_{4+}])$. The latter is obviously a sum of classes which are represented as the fundamental classes of oriented submanifolds of N_Z which have trivial normal bundles. As such, there is an unambiguous intersection pairing between σ' and classes in $H_3(Z, Z - N_Z)$.

With the preceding understood, note that the Poincare' dual of σ' is characterized by the following fact: Its integral over a cycle in $H_3(Z, Z - N_Z)$ is the same as the intersection number of said cycle with σ'.

Since the Poincare dual of σ' is equal to the Poincare' dual of σ_Z, it is sufficient to consider the integral of the 3-form η over cycles with boundary in $Z - N_Z$ and compare the value of said integral with the intersection number of the cycle with σ'. This last task is straightforward because one can consider each of the constituent submanifolds (i.e., $\Delta_Z, E_{L,R}$ and $E_{4\pm}$) separately. The task is left to the reader as an exercise.

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