SURVEYS IN DIFFERENTIAL GEOMETRY, 1995
Vol. 2 ©1995, International Press

Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy mininimizing maps

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Introduction. The question of what can be said about the structure of the singular set of minimal surfaces arises naturally from the work of the pioneers in the field of geometric measure theory/geometric calculus of variations, including De Giorgi [**6**], Reifenberg [**21**], Federer [**8**], [**9**], Almgren [**1**], [**2**], and Allard [**3**]. During the 1960's and 70's these authors established a partial regularity theory and existence theory for minimal submanifolds. An analogous theory for energy minimizing maps between Riemannian manifolds was later established by Schoen and Uhlenbeck [**24**] and (in case of image contained in a single coordinate chart) by Giaquinta & Giusti [**11**], and similar questions about the structure of the singular set of such minimizing maps naturally arise from their work.

In recent years some progress has been made on these questions, and this paper has two main aims: First, we want to make a brief survey of these recent results and, second, we want to give a proof of the fact that the singular set of a minimal submanifold in a "multiplicity one class" $\mathcal{M}$ (see the discussion in §1 below for the terminology) locally decomposes into a finite union of locally m-rectifiable locally compact subsets, where m is the maximum dimension of singularities which can occur in the class $\mathcal{M}$. The proof of this, given in §7, exactly parallels the proof of the corresponding result (described in Theorem 2 below and first proved in [**32**]) for the singular set of energy minimizing maps into a real-analytic target; thus the reader will see that the proof given in §7 follows almost exactly, step by step, the proof of the main theorem of [**32**] (to the extent that even the labelling system is almost identical).

The methods used in the proofs of all the recent results on the structure of the singular set (as presented in Theorems 1–7 below) are a mixture of geometric measure theory and PDE methods. The PDE methods involve in part ideas originating in quasilinear elliptic theory, developed by C. B. Morrey, E. De Giorgi, O. Ladyzhenskaya, N. Ural'tseva, J. Moser, and others, principally during the period from the late 1930's to the mid 1970's.

A precise outline of the present paper is as follows:

§1: Basic definitions, and a survey of known results.

§2: Basic properties of mulitiplicity one classes of minimal surfaces.

§3: A rectifiability lemma and gap measures for certain subsets of $\mathbf{R}^n$.

Partially supported by NSF grant DMS-9207704 at Stanford University; part of the work described here was carried out during visits to the Pure Mathematics Department, University of Adelaide, and ETH, Zürich. The author is grateful for the hospitality of these institutions.

1 Basic Definitions and a Survey of Known Results. k, ℓ, m, n will denote fixed positive integers with $n = \ell + m \geq 2$, and $k \geq 0$. n will be the dimension of the minimal submanifolds or the domain of the energy miminizing maps under consideration. In the case of the minimal submanifolds, k will be the codimension, and ℓ will be the "cross-sectional" dimension of the cylindrical tangent cones, as described below, and in the case of the energy minimizing maps ℓ is the dimension of the domain of the cross-section of the approprate "cylindrical tangent maps" again as described below; in the energy minimizing setting we always take $k = 0$.

$B^q_\rho(z)$ denotes the open ball with center z and radius ρ in $\mathbf{R}^q$; $B_\rho(z)$, B_ρ will often be used as an abbreviation for $B^{n+k}_\rho(z)$, $B^{n+k}_\rho(0)$ respectively.

$\eta_{z,\rho}$ will denote the map $x \mapsto \rho^{-1}(x - z)$. Thus $\eta_{z,\rho}$ translates z to the origin and homotheties by a factor ρ^{-1}.

$\mathcal{H}^j$ will denote j-dimensional Hausdorff measure.

First we consider energy minimizing maps:

N will denote a smooth compact Riemannian manifold, which for convenience we assume is isometrically embedded in some Euclidean space $\mathbf{R}^p$; of course this involves no loss of generality because of the Nash embedding theorem.

$W^{1,2}(\Omega; \mathbf{R}^p)$ will denote the space of $\mathbf{R}^p$ functions $u = (u^1, \dots, u^p)$ such that each u^j and its first order distribution derivatives $D_i u^j$ are in $L^2(\Omega)$; the energy of such a map is

$$\mathcal{E}(u) = \int_\Omega |Du|^2,$$

where $|Du|^2 = \sum_{i=1}^n \sum_{j=1}^p (D_i u^j)^2$. If Ω is equipped with a smooth Riemannian metric $\sum g_{ij} dx^i \otimes dx^j$ (so that (g_{ij}) is positive definite and each g_{ij} is smooth), then the corresponding energy $\mathcal{E}^{(g)}$ is defined by

$$\mathcal{E}^{(g)}(u) = \int_\Omega \sum_{i,j=1}^n g^{ij}(x) D_i u \cdot D_j u \sqrt{g}\, dx,$$

$$(g^{ij}) = (g_{ij})^{-1}, \qquad \sqrt{g} = \det(g_{ij}).$$

For a measurable subset $A \subset \Omega$,

$$\mathcal{E}_A(u) = \int_A |Du|^2.$$

$W^{1,2}_{loc}(\Omega; \mathbf{R}^p)$ denotes the set of $u \in L^2_{loc}(\Omega; \mathbf{R}^p)$ such that $u \in W^{1,2}(\widetilde{\Omega}; \mathbf{R}^p)$ for every bounded $\widetilde{\Omega}$ with closure contained in Ω (i.e., for every open $\widetilde{\Omega} \subset\subset \Omega$).

$W^{1,2}(\Omega; N)$ will denote the set of functions $u \in W^{1,2}(\Omega; \mathbf{R}^p)$ such that $u(x) \in N$ for a.e. $x \in \Omega$, and $W^{1,2}_{loc}(\Omega; N)$ denotes the set of $u \in W^{1,2}_{loc}(\Omega; \mathbf{R}^n)$ with $u(x) \in N$ for a.e. $x \in \Omega$.

$u \in W^{1,2}_{loc}(\Omega; N)$ is said to be energy minimizing in Ω if

$$\mathcal{E}_{\tilde{\Omega}}(u) \leq \mathcal{E}_{\tilde{\Omega}}(v),$$

whenever $\widetilde{\Omega} \subset\subset \Omega$ and $v \in W^{1,2}_{loc}(\Omega; N)$ satisfies $v = u$ a.e. in $\Omega \setminus \widetilde{\Omega}$. For any such energy minimizing map we define the regular and singular sets, $\operatorname{reg} u$ and $\operatorname{sing} u$, by

$$\begin{aligned} \operatorname{reg} u &= \{z \in \Omega : u \text{ is } C^\infty \text{ in a neighbourhood of } z\}, \\ \operatorname{sing} u &= \Omega \setminus \operatorname{reg} u. \end{aligned}$$

Notice that by definition $\operatorname{reg} u$ is open, and hence $\operatorname{sing} u$ is automatically relatively closed in Ω.

If $u \in W^{1,2}(\Omega; N)$ is energy minimizing, then for any $\widetilde{\Omega} \subset\subset \Omega$ the energy $\mathcal{E}_{\tilde{\Omega}}$ is evidently stationary in the sense that

$$\frac{d}{ds}\mathcal{E}_{\tilde{\Omega}}(u_s)|_{s=0} = 0, \tag{1.1}$$

whenever the derivative on the left exists, provided $u_0 = u$ and $u_s \in W^{1,2}_{loc}(\Omega; N)$ with $u_s(x) \equiv u_0(x)$ for $x \in \Omega \setminus \widetilde{\Omega}$ and $s \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. In particular, by considering a family $u_s = \Pi(u + s\zeta)$ where Π denotes nearest point projection of an $\mathbf{R}^p$-neighbourhood of N onto N and $\zeta \in C^\infty_c(\Omega; \mathbf{R}^p)$, we obtain the system of equations

$$\Delta_{\mathbf{R}^n} u + \sum_{j=1}^n A_u(D_j u, D_j u) = 0, \tag{1.2}$$

(weakly in Ω), where $\Delta_{\mathbf{R}^n} u = (\Delta_{\mathbf{R}^n} u^1, \ldots, \Delta_{\mathbf{R}^n} u^p)$, and A_z denotes the second fundamental form of N at any point $z \in N$.

On the other hand if $u_s(x) = u(x + s\zeta(x))$, where $\zeta \in C^\infty_c(\Omega; \mathbf{R}^p)$, then 1.1 implies the integral identity

$$\begin{gathered} \textstyle\int_\Omega \sum_{i,j=1}^n (\delta_{ij}|Du|^2 - 2D_i u \cdot D_j u) D_i \zeta^j = 0, \\ \zeta = (\zeta^1, \ldots, \zeta^n) \in C^\infty_c(\Omega; \mathbf{R}^n). \end{gathered} \tag{1.3}$$

Notice that 1.3 implies (for a.e. ρ such that $B_\rho(z) \subset \Omega$)

$$\begin{aligned} &\int_{B_\rho(z)} \sum_{i,j=1}^n (\delta_{ij}|Du|^2 - 2D_i u \cdot D_j u) D_i \zeta^j \\ &\qquad = \int_{\partial B_\rho(z)} \sum_{i,j=1}^n (\delta_{ij}|Du|^2 - 2D_i u \cdot D_j u) \eta_i \zeta^j \end{aligned} \tag{1.4}$$

for any $\zeta = (\zeta_1, \dots, \zeta_n) \in C^\infty(\overline{U}; \mathbf{R}^n)$, where $\eta = |x-z|^{-1}(x-z)$ is the outward pointing unit normal for $\partial B_\rho(z)$. In particular $\zeta(x) \equiv x - z$ implies

$$(1.5) \qquad (n-2)\int_{B_\rho(z)} |Du|^2 = \rho \int_{\partial B_\rho(z)} (|Du|^2 - 2|u_{R_z}|^2) \quad \text{a.e. } \rho,$$

provided $\overline{B}_\rho(z) \subset \Omega$, where $u_{R_z} = (|x-z|^{-1}(x-z)\cdot D)u$. This can be written

$$\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(z)} |Du|^2\right) = 2\int_{\partial B_\rho(z)} \frac{|R_z u_{R_z}|^2}{R_z^n}, \quad R_z = | x - z |,$$

whence by integration

$$(1.6) \qquad \rho^{2-n}\int_{B_\rho(z)} |Du|^2 - \sigma^{2-n}\int_{B_\sigma(z)} |Du|^2 = 2\int_{B_\rho(z)\setminus B_\sigma(z)} \frac{|R_z u_{R_z}|^2}{R_z^n}$$

for any $0 < \sigma < \rho$ with $\overline{B}_\rho(z) \subset \Omega$. Notice in particular this implies

$$(1.7) \qquad \rho^{2-n}\int_{B_\rho(z)} |Du|^2 \text{ is an increasing function of } \rho,$$

so the limit

$$(1.8) \qquad \Theta_u(z) \equiv \lim_{\rho\downarrow 0} \rho^{2-n}\int_{B_\rho(z)} |Du|^2$$

exists at every point $z \in \Omega$. Θ_u is called the density function of u. Letting $\sigma \downarrow 0$ in 1.6 we obtain

$$(1.9) \qquad \rho^{2-n}\int_{B_\rho(z)} |Du|^2 - \Theta_u(z) = 2\int_{B_\rho(z)} \frac{|R_z u_{R_z}|^2}{R_z^n},$$

and by using 1.5 we have the alternative identity

$$(1.10) \qquad \begin{aligned} 2\int_{B_\rho(z)} \frac{|R_z u_{R_z}|^2}{R_z^n} &= (n-2)^{-1}\rho^{3-n}\int_{\partial B_\rho(z)} (|Du|^2 - 2|u_{R_z}|^2) - \Theta_u(z) \\ &\le (n-2)^{-1}\rho^{3-n}\int_{\partial B_\rho(z)} |Du|^2 - \Theta_u(z). \end{aligned}$$

We also want to consider "multiplicity one classes" of minimal submanifolds here, the theory of singularities of which are entirely analogous to the theory for energy minimizing maps. First we introduce the basic terminology.

$\mathcal{M}$ will denote a set of smooth n-dimensional minimal submanifolds, each $M \in \mathcal{M}$ is assumed properly embedded in $\mathbf{R}^{n+k}$ in the sense that for each $x \in M$ there is $\sigma > 0$ such that $M \cap \overline{B}_\sigma(x)$ is a compact connected embedded smooth submanifold with boundary contained in $\partial B_\sigma(x)$. We also assume that for each $M \in \mathcal{M}$ there is a corresponding open set $U_M \supset M$, such that $\mathcal{H}^n(M \cap K) < \infty$

for each $M \in \mathcal{M}$ and each compact $K \subset U_M$, and such that M is stationary in U_M in the sense that

$$(1.1') \qquad \int_M \operatorname{div}_M \Phi \, d\mu = 0.$$

whenever $\Phi = (\Phi^1, \ldots, \Phi^{n+k}) : U_M \to \mathbf{R}^{n+k}$ is a C^∞ vector field with compact support in U_M. Here $d\mu$ denotes integration with respect to ordinary n-dimensional volume measure (i.e., n-dimensional Hausdorff measure) on M, and $\operatorname{div}_M \Phi$ is the "tangential divergence" of Φ relative to M. Thus

$$\operatorname{div}_M \Phi = \sum_{j=1}^{n+k} (e_j \cdot \nabla^M) \Phi^j,$$

where $e_1, \ldots, e_{n+k}$ is the standard basis for $\mathbf{R}^{n+k}$, and ∇^M denotes tangential gradient operator on M, so that if $f \in C^1(U)$ then $\nabla^M f(x) = P_x\big(\operatorname{grad}_{\mathbf{R}^{n+k}} f(x)\big)$, with P_x the orthogonal projection of $\mathbf{R}^{n+k}$ onto the tangent space $T_x M$ for any $x \in M$.

We assume that the $M \in \mathcal{M}$ have no removable singularities: thus if $x \in \overline{M} \cap U_M$ and, there is $\sigma > 0$ such that $\overline{M} \cap \overline{B}_\sigma(z)$ is a smooth compact connected embedded n-dimensional submanifold with boundary contained in $\partial B_\sigma(z)$, then $z \in M$. Subject to this agreement, the (interior) singular set of M (relative to U_M) is then defined by

$$\operatorname{sing} M = U_M \cap \overline{M} \backslash M,$$

and the regular set $\operatorname{reg} M$ is just M itself. (We give examples of such M in 1.12 below.)

The monotonicity and density results for energy minimizing maps given in 1.5–1.10 have analogues for such stationary minimal submanifolds; viz. using analogous arguments (starting with 1.1′ rather than 1.3—see e.g. [**25**] for the detailed arguments) we have the identity

$$(1.7') \qquad \rho^{-n}|M \cap B_\rho(z)| - \sigma^{-n}|M \cap B_\sigma(z)| = \int_{M \cap B_\rho(x) \backslash B_\sigma(x)} \frac{|(x-z)^\perp|^2}{|x-z|^{n+2}}$$

for any $x \in \overline{M} \cap U_M$ for all σ, ρ with $0 < \sigma < \rho < R$, provided $B_R(x) \subset U_M$. In particular

$$\rho^{-n}|M \cap B_\rho(z)| \textit{ is increasing,}$$

and the density function

$$(1.8') \qquad \Theta_M(z) \equiv \lim_{\rho \downarrow 0} (\omega_n r^n)^{-1} |M \cap B_\rho(z)|$$

exists for all $z \in \overline{M}$. (Of course the density is identically equal to 1 on M, because M is a smooth n-dimensional submanifold.)

Letting $\sigma \downarrow 0$ in 1.7′ we obtain

$$(1.9') \qquad \omega_n^{-1} \int_{M \cap B_\rho(z)} \frac{|(x-z)^\perp|^2}{|x-z|^{n+2}} = \omega_n^{-1} \rho^{-n} |M \cap B_\rho(z)| - \Theta_M(z)$$

for all $z \in \overline{M}$ and $\rho \in (0, R)$, provided $B_R(z) \subset U_M$, where $(x - z)^\perp = p_{(T_x M)^\perp}(x - z)$ (i.e., $(x - z)^\perp$ is the orthogonal projection of $x - z$ onto the normal space of M at x). By multiplying through by ρ^n and differentiating with respect to ρ we also get the following analogue of 1.10:

$$\omega_n^{-1} \int_{M \cap B_\rho(z)} \frac{|(x-z)^\perp|^2}{|x-z|^{n+2}} = (n\omega_n)^{-1} \rho^{1-n} \int_{M \cap \partial B_\rho(z)} |\nabla |x - z|| - \Theta_M(z). \tag{1.10'}$$

We assume here also that the class $\mathcal{M}$ is closed under appropriate homotheties, rigid motions, and weak limits—we shall call such a class a "multiplicity one class"; more precisely, we assume:

1.11(a) $M \in \mathcal{M} \Rightarrow q \circ \eta_{x,\rho} M \in \mathcal{M}$ and $q \circ \eta_{x,\rho} U_M = U_{q \circ \eta_{x,\rho} M}$ for each $x \in U_M$, each $\rho \in (0, 1]$, and for each orthogonal transformation q of $\mathbf{R}^{n+k}$.

1.11(b) If $\{M_j\} \subset \mathcal{M}$, $U \subset \mathbf{R}^{n+k}$ with $U \subset U_{M_j}$ for all sufficiently large j, and $\sup_{j \geq 1} \mathcal{H}^n(M_j \cap K) < \infty$ for each compact $K \subset U$, then there is a subsequence $M_{j'}$ and an $M \in \mathcal{M}$ such that $U_M \supset U$ and $M_{j'} \to M$ in U in the measure-theoretic sense that $\int_{M_{j'}} f(x)\, d\mathcal{H}^n(x) \to \int_M f(x)\, d\mathcal{H}^n(x)$ for any fixed continuous $f : \mathbf{R}^{n+k} \to \mathbf{R}$ with compact support in U.

(Notice that 1.11(b) is a strong restriction, in that it precludes, in particular, the possibility of getting varifolds with multiplicity greater than one on a set of positive measure as the varifold limit of a sequence $M_j \subset \mathcal{M}$ with each $U_{M_j} \supset U$ for some fixed open U; for this reason we refer to such a class as a multiplicity one class.)

1.12 Examples. In view of later applications, we should mention here a couple of important classes $\mathcal{M}$ which satisfy the conditions imposed above. One such class consists of the interior regular sets of the mod 2 minimizing currents described as follows: If T is an n-dimensional locally rectifiable multiplicity one current in $\mathbf{R}^{n+k}$, if $\operatorname{spt}_2 \partial T$ denotes the mod 2 support of ∂T, if T is mod 2 minimizing in $\mathbf{R}^{n+k}$ (in the sense that for each bounded open $U \subset \mathbf{R}^{n+k}$ the mass of $T \llcorner U$ is $\leq$ the mass of $S \llcorner U$ for any multiplicity one current S such that support of $T - S$ is a compact subset of U and such that $T - S$ has zero mod 2 boundary in U), and if $\operatorname{reg}_2 T$ is the mod 2 regular set of T defined in the usual way as the set of all $x \in \operatorname{spt} T \setminus \operatorname{spt}_2 \partial T$ such that T is mod 2 equivalent in a neighbourhood of x to multiplicity one integration over a smooth properly embedded n-dimensional submanifold containing x, then the collection $\mathcal{T}_2$ of all such sets $M = \operatorname{reg}_2 T$ is a class $\mathcal{M}$ satisfying all the conditions imposed above, provided we take $U_M = \mathbf{R}^{n+k} \setminus \operatorname{spt}_2 \partial T$. Indeed by the Allard theorem $\operatorname{spt} T \setminus (\operatorname{reg} T \cup \operatorname{spt}_2 \partial T)$ has $\mathcal{H}^n$-measure zero, and it follows that $M = \operatorname{reg}_2 T$ satisfies 1.1, and, using the notation introduced above in our discussion of the general class $\mathcal{M}$, we have $\operatorname{sing} M = \operatorname{spt} T \setminus (\operatorname{reg} T \cup \operatorname{spt}_2 \partial T)$, which coincides with the usual definition of the (interior) singular set of such mod 2 minimizing currents T. The property 1.11(b) (plus an existence theory) is true by the compactness theorem for flat chains mod p (see e.g. [8]).

Another such class is the collection $\mathcal{T}_3 = \{\operatorname{reg}_3 T\}$ of the interior regular sets of n-dimensional multiplicity one currents T which are mod 3 minimizing in $\mathbf{R}^{n+k}$ (defined analogously to the mod 2 case); if $M = \operatorname{reg}_3 T$ then M satisfies 1.1 with $U_M = \mathbf{R}^{n+k} \setminus \operatorname{spt}_3 \partial T$, and $\operatorname{sing} M = \operatorname{spt} T \setminus (\operatorname{reg} T \cup \operatorname{spt}_3 \partial T)$. Again the property 1.11(b) (plus an existence theory) is true by the compactness theorem for flat chains mod p.

Notice that these classes $\mathcal{T}_2$, $\mathcal{T}_3$ have dim $\operatorname{sing} M \leq (n-2)$, $(n-1)$ respectively by **[9]**, **[25]**.

A third class which has the form of $\mathcal{M}$ above is the collection $\mathcal{T}_1$ of all submanifolds M of the form $M = \operatorname{reg} T$, where T is an n-dimensional oriented boundary of least area in some open $U = U_T \subset \mathbf{R}^{n+k}$, in the usual sense that $T = \partial [\![V]\!]$ in U (in the sense of currents) for some measurable $V \subset U$ and $T \llcorner U$ has mass $\leq$ than the mass of $S \llcorner U$, for any multiplicity one locally rectifiable current S in $\mathbf{R}^{n+k}$ with support $S - T$ equal to a compact subset of U and with $\partial(S - T) = 0$ in U. In this case, with $M = \operatorname{reg} T$, we take $U_M = U$, $\operatorname{sing} M = U \cap \operatorname{spt} T \setminus (\operatorname{reg} T \cup \operatorname{spt} \partial T)$, and the singular set satisfies dim $\operatorname{sing} M \setminus \operatorname{spt} \partial T \leq n - 7$ (see e.g. **[9]** or **[25]** or **[10]**). The property 1.11(b) in this case is discussed in e.g. **[10]**, **[8]** or **[25]**.

We now want to state the main theorems about the singular sets of energy minimizing maps and minimal submanifolds. To do this we first need to recall the definition of rectifiability of subsets of Euclidean space:

A subset $A \subset \mathbf{R}^n$ is said to be m-rectifiable if $\mathcal{H}^m(A) < \infty$, and if A has an approximate tangent space a.e. in the sense that for $\mathcal{H}^m$-a.e. $z \in A$ there is an m-dimensional subspace L_z such that

$$\lim_{\sigma \downarrow 0} \int_{\eta_{z,\sigma}(A)} f \, d\mathcal{H}^m = \int_{L_z} f \, d\mathcal{H}^m, \quad f \in C_c^0(\mathbf{R}^n),$$

where, here and subsequently, $\eta_{z,\sigma}(x) \equiv \sigma^{-1}(x - z)$ and $\mathcal{H}^m$ is m-dimensional Hausdorff measure. The above definition of m-rectifiability is well-known (see e.g. **[25]**) to be equivalent to the requirements that $\mathcal{H}^m(A) < \infty$ and that $\mathcal{H}^m$-almost all of A is contained in a countable union of embedded m-dimensional C^1-submanifolds of $\mathbf{R}^n$.

A subset $A \subset \mathbf{R}^n$ is said to be locally m-rectifiable if it is m-rectifiable in a neighbourhood of each of its points. Thus for each $z \in A$ there is a $\sigma > 0$ such that $A \cap \{x : |x - z| \leq \sigma\}$ is m-rectifiable. Similarly A is locally compact if for each $z \in A$ there is $\sigma > 0$ such that $A \cap \{x : |x - z| \leq \sigma\}$ is compact.

Now we give a brief survey of the known results about the the structure of the set of singularities of energy minimizing maps and minimal submanifolds in multiplicity one classes.

First we discuss energy minimizing maps:

In the theorems concerning energy minimizing maps we continue to let Ω denote an arbitrary subset of $\mathbf{R}^n$, equipped with the standard Euclidean metric, but the reader should keep in mind that all theorems readily generalize to the case where Ω is equipped arbitrary smooth Riemannian metric $g_{ij} \, dx^i \otimes dx^j$.

The most general result presently known concerning the structure of the singular set of energy minimizing maps is the following, which was proved (for $\Omega \subset \mathbf{R}$ equipped with arbitrary Reimannian metric) in [**32**]:

Theorem 1. *If u is an energy minimizing map of Ω into a compact real-analytic Riemannian manifold N, then, for each closed ball $B \subset \Omega$, $B \cap \operatorname{sing} u$ is the union of a finite pairwise disjoint collection of locally $(n-3)$-rectifiable locally compact subsets.*

Remarks. (1) Notice that being a finite union of locally m-rectifiable subsets is slightly weaker than being a (single) locally m-rectifiable subset, in that if $A = \cup_{k=1}^{Q} A_k$, where each A_k is locally m-rectifiable, there may be a set of points y of positive measure on one of the A_ℓ such that $\mathcal{H}^m((\cup_{k\neq\ell} A_k) \cap B_\sigma(y)) = \infty$ for each $\sigma > 0$. (This is possible because A_k has locally finite measure in a neighbourhood of each of its points, but may not have locally finite measure in a neighbourhood of points in the closure $\overline{A}_k$ and this may intersect A_ℓ, $\ell \neq k$.)

(2) It is also proved in [**32**] that $\Theta_u(z)$ is a.e. constant on each of the sets in the finite collection referred to in the above theorem, and that $\operatorname{sing} u$ has a (unique) tangent plane in the Hausdorff distance sense at $\mathcal{H}^m$-almost all points $z \in \operatorname{sing} u$, and u itself has a unique tangent map at $\mathcal{H}^m$-almost all points of $\operatorname{sing} u$. (See the discussion of [**32**] for terminology.)

There is an important refinement of Theorem 1 in case

$$\dim \operatorname{sing} u \leq m \tag{1.13}$$

for all energy minimizing maps into N.

In this case the conclusion of Theorem 1 holds with m in place of $n-3$:

Theorem 2. *If u, N are as in Theorem 1, $m \leq n-3$ is a non-negative integer, and (1.13) holds, then for each closed ball $B \subset \Omega$, $B \cap \operatorname{sing} u$ is the union of a finite pairwise disjoint collection of locally m-rectifiable locally compact subsets.*

Remarks. (1) As for Theorem 1, again $\Theta_u(z)$ is constant a.e. on each of the sets in the finite collection referred to in the statement, $\operatorname{sing} u$ has a tangent space in the Hausdorff distance sense, and also u has a unique tangent map, at $\mathcal{H}^m$-almost all points of $\operatorname{sing} u$.

In [**26**], [**28**] there are also results about singular sets (albeit for special classes of energy minimizing maps and stationary minimal surfaces), which, unlike the results here, were proved using "blowup methods". In particular we have

Theorem 3. *If $N = S^2$ with its standard metric, or N is S^2 with a metric which is sufficiently close to the standard metric of S^2 in the C^3 sense, then $\operatorname{sing} u$ can be written as the disjoint union of a properly embedded $(n-3)$-dimensional $C^{1,\mu}$-manifold and a closed set S with $\dim S \leq n-4$. If $n = 4$, then S is discrete and the $C^{1,\mu}$ curves making up the rest of the singular set have locally finite length in compact subsets of Ω.*

For further discussion and proofs, we refer to [**27**].

There is an analogue of Theorem 2 which applies to an arbitrary submanifold M in a mulitiplicity one class $\mathcal{M}$ of stationary minimal submanifolds:

Here and subsequently we let

(1.13′) $$m = \max\{\dim \operatorname{sing} M : M \in \mathcal{M}\};$$

this maximum exists and is an integer $\in \{0, \dots, n-1\}$, as shown in the discussion following 2.7 below.

Theorem 4. *Suppose $\mathcal{M}$ is a multiplicity one class of stationary minimal surfaces as in 1.11, suppose m is as in 1.13′, and $M \in \mathcal{M}$. Then for each $x \in \operatorname{sing} M$ there is a neighbourhood U_x of x such that $\operatorname{sing} M \cap U_x$ is a finite union of locally m-rectifiable locally compact subsets.*

1.14 Remark. Analogous to the remarks after Theorems 1, 2 we have in addition that $\Theta_M(z)$ is constant a.e. on each of the sets in the finite collection referred to in the statement of the theorem, $\operatorname{sing} M$ has a tangent space in the Hausdorff distance sense, and also M has a unique tangent map, at $\mathcal{H}^m$-almost all points of $\operatorname{sing} M$.

We give the detailed proof of Theorem 4 and Remark 1.14 in §7 below; as we pointed out in the introduction, the proof involves only very minor technical modifications of the proof of Theorem 2 given in [**32**].

In view of the examples in 1.12, we thus have in particular the following:

Theorem 5. *(i) If M is the regular set of an n-dimensional mod 2 mass minimizing current in $\mathbf{R}^{n+k}$ (n, $k \geq 2$ arbitrary), then the singular set $\operatorname{sing} M$ is locally a finite union of locally $(n-2)$-rectifiable, locally compact subsets.*
(ii) If M is the regular set of an arbitrary n-dimensional mass minimizing current in $\mathbf{R}^{n+1}$, then $\operatorname{sing} M$ can locally be expressed as the finite union of locally $(n-7)$-rectifiable, locally compact subsets.

(Except for the local compactness result, part(i) of the above theorem is also proved in [**26**] by using "blowup" methods, which are quite different than the techniques used in the proof of Theorem 4.)

In addition to the above results, there are also more special results, proved using blowup techniques in [**26**], analogous to the results for energy minimizing maps described in Theorem 3. For example, we have the following:

Theorem 6. *Suppose the m of (1.13) is equal to $(n-1)$. If $M \in \mathcal{M}$, $\mathbf{C}^{(0)} = \mathbf{C}_0^{(0)} \times \mathbf{R} \in \mathcal{C} \cap \operatorname{Tan}_{x_0} M$ with $\mathbf{C}_0^{(0)}$ a 1-dimensional cone consisting of an odd number of rays emanating from 0, and $\Theta_{\mathbf{C}^{(0)}}(0) = \min_{\mathbf{C} \in \mathcal{T}} \Theta_{\mathbf{C}}(0)$, then there is $\rho > 0$ such that $\operatorname{sing} M \cap B_\rho(x_0)$ is a properly embedded $(n-1)$-dimensional $C^{1,\alpha}$ manifold.*

Theorem 7. *If $\mathbf{V}$ is an n-dimensional stationary integral varifold in some open set $U \subset \mathbf{R}^{n+k}$, and $x_0 \in U$ with $1 < \Theta_{\mathbf{V}}(x_0) < 2$, then $\operatorname{sing} \mathbf{V} \cap B_\rho(x_0)$ is the union of an embedded $(n-1)$-dimensional $C^{1,\alpha}$ manifold and a closed set of dimension $\leq n-2$. If $n = 2$ we have the more precise conclusion that there is $\rho > 0$ such that either $\operatorname{sing} \mathbf{V} \cap B_\rho(x_0)$ is a properly embedded $C^{1,\alpha}$ Jordan arc with endpoints in $\partial B_\rho(x_0)$ or else is a finite union of properly embedded locally $C^{1,\alpha}$ Jordan arcs of finite length, each with one endpoint at x_0 and one endpoint in $\partial B_\rho(x_0)$.*

For some special (but important) classes of minimal surfaces Jean Taylor [**33**] and Brian White [**35**] used methods based on the "epiperimetric" approach of Reifenberg, and, for the special classes to which they apply (for example for 2-dimensional "(M,ε,δ)-minimizing" surfaces), these methods yield a more complete description of the singular set than even that given in Theorem 6. (Theorem 6 refers only to the "top-dimensional" part of the singular set, so does not entirely subsume the results of [**33**], [**35**].)

2 Basic Properties of Multiplicity 1 Minimal Submanifolds. Here $\mathcal{M}$ continues to denote a multiplicity one class of n-dimensional minimal submanifolds in $\mathbf{R}^{n+k}$ and $M \in \mathcal{M}$ with U_M the corresponding open set as in 1.11. $\mathcal{C}$ will denote the set of all cones in $\mathcal{M}$; thus

$$\mathcal{C} = \{\mathbf{C} \in \mathcal{M} : U_{\mathbf{C}} = \mathbf{R}^{n+k} \ \ \textit{and}\ \eta_{0,\lambda}\mathbf{C} = \mathbf{C}\ \forall\,\lambda > 0\},$$

where $\eta_{0,\lambda}$ is the homothety $x \mapsto \lambda x$. $\mathcal{T}$ will denote the "cylindrical" elements of $\mathcal{C}$ with singular axis of dimension m; thus

$$\mathcal{T} = \{\mathbf{C} \in \mathcal{C} : \exists\ \textit{an } m\textit{-dimensional subspace } L_{\mathbf{C}} \subset \mathbf{R}^{n+k} \textit{ with } z + \mathbf{C} = \mathbf{C}\ \forall\, z \in L_{\mathbf{C}}\},$$

where m is as in 1.13′; notice that for technical reasons we include the case where $\mathbf{C}$ is an n-dimensional subspace, in which case $\operatorname{sing}\mathbf{C} = \emptyset$. In all other cases, $\operatorname{sing}\mathbf{C} = L_{\mathbf{C}}$.

An important consequence of the Allard regularity theorem ([**3**]—see also [**25**] for an alternative presentation of this theory) is that the singular set $\operatorname{sing} M$ of M can be characterized in terms of the density Θ_M as follows:

$$z \in \operatorname{sing} M \iff \Theta_M(z) > 1 + \alpha \iff \Theta_M(z) > 1, \tag{2.1}$$

where $\alpha = \alpha(\mathcal{M}) > 0$ is independent of M. (Of course it is true that $\Theta_M(z) \geq 1$ at all points of $U_M \cap \overline{M}$; this follows for example from the fact that $\Theta_M(z) \equiv 1$ on M together with the upper semicontinuity of $\Theta_M(z)$ decribed in 2.3 below.)

We shall often use the quantitative part of the main regularity theory of [**3**]. To state this, assume $z \in \overline{M}$, $B_\rho(z) \subset U_M$ and either (a) $\omega_n^{-1}\rho^{-n}|M \cap B_\rho(z)| < 1 + \alpha$ or (b) $\beta > 0$ is given and both $\rho^{-n}|M \cap B_\rho(z)| < \beta$ and $\inf_L \rho^{-n-2}\int_{M\cap B_\rho(z)} \operatorname{dist}(x,L)^2 < \alpha$, where the inf is taken over all n-dimensional affine spaces in $\mathbf{R}^{n+k}$. Then the main theorems of [**3**] tell us that if either hypothesis (a) with suitable $\alpha = \alpha(n,k) > 0$ or hypothesis (b) with suitable $\alpha = \alpha(n,k,\beta,\mathcal{M}) > 0$ implies $\overline{M} \cap B_{\rho/2}(z) \subset M$ and

$$\sup_{L\cap B_{\rho/2}(z)} \rho^{j-1}|D^j u| \leq C_j \alpha^{1/2}, \quad j \geq 0, \tag{2.2}$$

where C_j depends only on j, L is an n-dimensional affine space containing z, and $u : L \cap B_{3\rho/4}(z) \to L^\perp$ is such that $B_{3\rho/4}(z) \cap \operatorname{graph} u = B_{3\rho/4}(z) \cap M$. Indeed the conclusion subject to the first hypothesis (that $\omega_n^{-1}\rho^{-n}|M \cap B_\rho(z)| < 1+\alpha$)

is just one of the standard versions of the Allard theorem and the conclusion subject to the second hypothesis is easily checked to follow from the Allard theorem together with the compactness assumption 1.11(b) on $\mathcal{M}$.

By the monotonicity 1.7′ it is easy to check that Θ_M is an upper semicontinuous function:

$$\Theta_M(z) \geq \limsup_{z_j \to z} \Theta_{M_j}(z_j) \tag{2.3}$$

for any sequences of points $z_j \to z$ and submanifolds $M_j \in \mathcal{M}$ with $M_j \to M$ in the sense of 1.11(b). 2.2 and 2.3 will be used frequently in the sequel. For the present, notice that if we define

$$M_{z,\sigma} = \eta_{z,\sigma} M,$$

for any given $z \in U_M \cap \overline{M}$ and $\sigma \in (0,1]$, then, according to monotonicity 1.7′, we have that $M_{z,\sigma}$ has bounded area in any ball $B_R(0)$ as $\sigma \downarrow 0$, and hence the compactness assumption 1.11(b) implies that for any $\sigma_j \downarrow 0$ there is a subsequence $\sigma_{j'}$ such that $M_{z,\sigma_{j'}} \to \mathbf{C}$, where $\mathbf{C} \in \mathcal{M}$ with $U_{\mathbf{C}} = \mathbf{R}^{n+k}$. Any such $\mathbf{C}$ is called a tangent cone of M at z. Using the monotonicity 1.7′ it is easy to check (see [**3**] or [**25**]) that any such $\mathbf{C}$ is a cone with vertex at 0, so $\mathbf{C}$ is invariant under homotheties; that is,

$$\eta_{0,\sigma}\mathbf{C} \equiv \mathbf{C}, \quad \sigma > 0, \tag{2.4}$$

and also that $\Theta_M(z) = \Theta_{\mathbf{C}}(0)$. An important property of cones $\mathbf{C}$ in $\mathcal{M}$ is that

$$\Theta_{\mathbf{C}}(0) = \max\{\Theta_{\mathbf{C}}(z) : z \in \overline{\mathbf{C}}\}, \tag{2.5}$$

and the set of points z where equality is attained form a linear subspace $L_{\mathbf{C}}$ (possibly the trivial subspace $\{0\}$), and we also have the translation invariance

$$z + \mathbf{C} \equiv \mathbf{C}, \quad z \in L_{\mathbf{C}}. \tag{2.6}$$

These facts are easily checked by using 1.7′ (with $\mathbf{C}$ in place of M) and 2.4. We emphasize that 2.5 and 2.6 automatically hold for $\mathbf{C} \in \mathcal{M}$ which are cones; i.e., which have $U_{\mathbf{C}} = \mathbf{R}^{n+k}$ and $\eta_{0,\sigma}\mathbf{C} = \mathbf{C}$.

There is another way in which such cones $\mathbf{C} \in \mathcal{C}$ arise, which is a slight variant of the idea of tangent cone. We let $M \in \mathcal{M}$, $z \in \overline{M}$ and take arbitrary sequences $z_j \to z$ with $\Theta_M(z_j) \geq \Theta_M(z)$, $\sigma_j \downarrow 0$. Then (again using monotonicity to justify the local boundedness of the area) by the compactness assumption of 1.11(b) we can take a subsequence $\{j'\} \subset \{j\}$ such that $M_{z_{j'},\sigma_{j'}}$ converges to $\mathbf{C}$, which again satisfies the invariance 2.4. We shall call such a map a pseudo tangent cone of M at z. Notice that if $z_j = z$ for each j, then this procedure is the same as the procedure above for constructing tangent cones, hence the terminology "pseudo tangent map". (The proof that 2.4 holds in this case forms part of the argument in the proof of Lemma 2.16 below.)

From now on m is defined by

$$m = \sup\{\dim\ \operatorname{sing} M : M \in \mathcal{M}\}. \tag{2.7}$$

Notice by Federer's dimension reducing argument or by the more refined method of Almgren (see the discussion following 2.17 below) it is automatic that m is an integer, and that $m \le n-1$. Indeed to begin we can define $m_0 = \max \dim L_{\mathbf{C}}$ over all cones $\mathbf{C} \in \mathcal{C}$ with $\operatorname{sing} \mathbf{C} \neq \emptyset$. Since clearly $\operatorname{sing} \mathbf{C} \subset L_{\mathbf{C}}$ (by 2.6), we must then have $m \ge m_0$. The fact that $m \le m_0$ follows direct from 2.16 below (see Remark (3) following 2.16, keeping in mind that the proof of 2.16 used only that $\dim L_{\mathbf{C}} \le m$ for each cone $\mathbf{C} \in \mathcal{M}$ with $\operatorname{sing} \mathbf{C} \neq \emptyset$). Thus the m of 2.7 automatically satisfies

$$m \in \{0, \dots, n-1\}, \qquad m = \max \dim L_{\mathbf{C}},$$

where the maximum is over $\mathbf{C} \in \mathcal{C}$ with $\operatorname{sing} \mathbf{C} \neq \emptyset$.

Of course if $\mathbf{C} \in \mathcal{C}$ with $\operatorname{sing} \mathbf{C} \neq \emptyset$, and $\dim L_{\mathbf{C}} = m$ (i.e., $\dim L_{\mathbf{C}}$ maximal), then we must have that $\operatorname{sing} \mathbf{C} = L_{\mathbf{C}}$, because otherwise by the homogeneity 2.4 and the translation invariance 2.6 $\operatorname{sing} \mathbf{C}$ would contain some $(m+1)$-dimensional half-space, contradicting 2.7. Cleary then, letting q be an orthogonal transformation of $\mathbf{R}^n$ which takes $L_{\mathbf{C}}$ to $\{0\} \times \mathbf{R}^m$, such $\mathbf{C}$ must satisfy, for $(x, y) \in \mathbf{R}^{\ell+k} \times \mathbf{R}^m = \mathbf{R}^{n+k}$ ($\ell = n - m \ge 1$),

$$q(\mathbf{C}) \equiv \mathbf{C}_0 \times \mathbf{R}^m, \tag{2.8}$$

where $\mathbf{C}_0$ is a minimal cone in $\mathbf{R}^{\ell+k}$ with

$$\mathbf{C}_0 \cap S^{\ell+k-1} = \Sigma, \tag{2.9}$$

with Σ a compact $(\ell-1)$-dimensional embedded submanifold of $S^{\ell-1+k}$ (or a finite set of points if $\ell = 1$); thus for $\ell \ge 2$ we have in particular that

$$\mathbf{H}_\Sigma \equiv 0, \tag{2.10}$$

where $\mathbf{H}_\Sigma$ is the mean-curvature of Σ as a submanifold of $S^{\ell-1+k}$; thus for $\ell \ge 2$ Σ is a compact minimal submanifold of $S^{\ell+k-1}$.

For given $\beta > 0$ we define

$$\mathcal{T}_\beta = \{\mathbf{C} \in \mathcal{T} : \Theta_{\mathbf{C}}(0) \le \beta\}, \tag{2.11}$$

where $\mathcal{T}$ denotes the set of cylindrical $\mathbf{C} \in \mathcal{C}$ as defined at the beginning of this section. Using 2.2, 2.7, and the compactness 1.11(b), it is easy to check that the set $\mathcal{T}_\beta$ is sequentially compact in $\mathcal{M}$ with respect to convergence as in 1.11(b), and that

$$\sum_{j=0}^{3} \sup_{\Sigma} |D^j A_{\mathbf{C}_0}| \le C, \quad \mathbf{C} \in \mathcal{T}_\beta, \quad j \ge 0, \tag{2.12}$$

where $C = C(\ell, k, \beta)$, $\mathbf{C}_0$, Σ are as in 2.8, 2.9, and $A_{\mathbf{C}_0}$ is the second fundamental form of $\mathbf{C}_0$.

Finally we need a Łojasiewicz type inequality for the area functional on $(\ell-1)$-dimensional minimal submanifolds of $S^{\ell-1+k}$; we begin by noting that, according to Łojasiewicz [**20**], if f is a real-analytic function on some open set

U of some Euclidean space $\mathbf{R}^Q$, then for each critical point $y \in U$ of f (i.e., each point y where $\nabla f(y) = 0$) there is $\alpha \in (0,1]$ and C, $\sigma > 0$ such that

$$|f(x) - f(y)|^{1-\alpha/2} \le C|\nabla f(x)| \tag{2.13}$$

for every point $x \in B_\sigma(y)$. There is an infinite dimensional analogue of this inequality which applies to the area functional $\mathcal{A}_\Sigma$ over a given compact $(\ell-1)$-dimensional submanifold Σ of $S^{\ell-1+k}$. Specifically, for any such $\Sigma \subset S^{\ell-1+k}$ let $\mathcal{A}_\Sigma$ denote the area functional over Σ defined by

$$\mathcal{A}_\Sigma(\psi) = \mathcal{H}^{\ell-1}(G_\Sigma(\psi)),$$

where ψ is a C^3 section of the normal bundle over Σ, and $G_\Sigma(\psi)$ means the "spherical graph" of ψ defined by

$$G_\Sigma(\psi) = (1+|\psi|^2)^{-1/2}(\omega + \psi(\omega)), \quad \omega \in \Sigma.$$

Then there is $\alpha = \alpha(\Sigma) \in (0,1]$, $C = C(\Sigma)$, and $\sigma = \sigma(\Sigma) > 0$ such that

$$|\mathcal{A}_\Sigma(\psi) - \mathcal{A}_\Sigma(0)|^{1-\alpha/2} \le C\|\mathcal{Q}_\Sigma(\psi)\|_{L^2}, \tag{2.14}$$

whenever $|\psi|_{C^3} < \sigma$, where $\mathcal{Q}_\Sigma$ is the Euler-Lagrange operator of the functional $\mathcal{A}_\Sigma$. Thus $\mathcal{Q}_\Sigma$ is characterized by being an operator taking C^2 sections of the normal bundle over Σ to C^0 sections of this bundle such that

$$-\frac{d}{ds}\mathcal{A}(\psi + s\eta)|_{s=0} = \langle \mathcal{Q}_\Sigma(\psi), \eta\rangle_{L^2},$$

where the inner product is the usual L^2 inner product given by $\langle f, g\rangle_{L^2} = \int_\Sigma f \cdot g$. For the proof of 2.14 (based on the Liapunov-Schmidt reduction to reduce to the finite dimensional case 2.13) we refer to **[30]** or **[31]**.

Now take any $M \in \mathcal{M}$, $z_0 \in \operatorname{sing} M$, and define

$$S_+ = \{z \in M : \Theta_M(z) \ge \Theta_M(z_0)\}. \tag{2.15}$$

Then we have the following lemma:

2.16 Lemma. *If $M \in \mathcal{M}$, m is as in 2.7, and S_+ is as in 2.15, then for each $\epsilon > 0$ there is $\rho_0 = \rho_0(\epsilon, z_0, M) > 0$ such that S_+ has the following affine approximation property in $\overline{B}_{\rho_0}(z_0)$: For each $\sigma \in (0, \rho_0]$ and each $z \in S_+ \cap \overline{B}_{\rho_0}(z_0)$ there is an m-dimensional affine subspace $L_{z,\sigma}$ containing z with*

$$S_+ \cap B_\sigma(z) \subset \text{ the } (\epsilon\sigma)\text{-neighbourhood of } L_{z,\sigma}.$$

Remarks. (1) The conclusion here might be termed a "half Reifenberg" property; Reifenberg's topological disc theorem requires such an hypothesis together with the reverse inclusion $L_{z,\sigma} \cap B_\sigma(z) \subset$ the $(\epsilon\sigma)$-neighbourhood of S_+, in which case $S_+ \cap \overline{B}_{\rho_0}(z_0)$ is a topological disc.

(2) On the other hand one should keep in mind that even the full Reifenberg condition will not ensure any rectifiability properties for S_+, as is shown for example by the "Koch curves" in fractal geometry. For a fuller discussion of this, we refer to [**31**].

(3) But the reader should also keep in mind that the ϵ-approximation property of the above lemma does imply, as one can easily check by using successively finer covers of S_+ by balls (see [**28**] for the detailed argument), that $\mathcal{H}^{m+\beta(\epsilon)}(S_+ \cap B_{\rho_0}(z_0)) = 0$, where $\beta(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$, and hence $\dim S_+ \leq m$ by the definition of Hausdorff dimension.

Proof of Lemma 2.16. If the lemma is false, then there is $\epsilon > 0$, $z_0 \in \operatorname{sing} u$, $\rho_k \downarrow 0$, $\sigma_k < \rho_k$, and $z_k \in B_{\rho_k}(z_0) \cap S_+$ such that

(1) $\quad B_1(0) \cap \eta_{z_k,\sigma_k} S_+ \not\subset$ *ϵ-neighbourhood of any m-dimensional subspace.*

Choose $R_k \downarrow 0$ with $R_k/\rho_k \to \infty$. Then by monotonicity 1.7′ we have, for all $\rho \in (0, R_k]$ and all $k = 1, 2, \ldots$,

$$\begin{aligned}\Theta_M(z_k) &\leq \omega_n^{-1}\rho^{-n}|M \cap B_\rho(z_k)| \leq \omega_n^{-1}R_k^{-n}|M \cap B_{R_k}(z_k)| \\ &\leq \omega_n^{-1}R_k^{-n}|M \cap B_{R_k+\rho_k}(z_0)|.\end{aligned}$$

In terms of the rescaled submanifolds $M_k = \eta_{z_k,\sigma_k} M$ this implies

$$\Theta_M(z_k) \leq \omega_n^{-1}\rho^{-n}|M_k \cap B_\rho(0)| \leq \omega_n^{-1}R_k^{-n}|M \cap B_{R_k+\rho_k}(z_0)|$$

for every $\rho \in (0, R_k/\sigma_k)$ and all sufficiently large k (depending on ρ). Since $\rho_k/R_k \to 0$ we have $R_k^{-n}|M \cap B_{R_k+\rho_k}(z_0)| \to \Theta_M(z_0)$, and since $\Theta_M(z_k) \geq \Theta_M(z_0)$ by hypothesis, we then obtain

$$\Theta_M(z_0) \leq \omega_n^{-1}\rho^{-n}|M_k \cap B_\rho(0)| \leq \Theta_M(z_0) + \epsilon_k, \tag{2}$$

where $\epsilon_k \to 0$ as $k \to \infty$. In particular the M_k have uniformly bounded area on any fixed ball in $\mathbf{R}^{n+k}$, so by the compactness of 1.11(b) there is a $\mathbf{C} \in \mathcal{M}$ and a subsequence $M_{k'}$ such that $M_{k'} \to \mathbf{C}$ locally on $\mathbf{R}^{n+k}$ in the sense of 1.11(b). But then (2) guarantees

$$\omega_n^{-1}\rho^{-n}|\mathbf{C} \cap B_\rho(0)| \equiv \Theta_M(z_0), \quad \forall \rho > 0,$$

and by the monotonicity formula 1.9 applied to $\mathbf{C}$ we thus conclude that

$$x^\perp = 0, \quad x \in \mathbf{C},$$

and hence (by the argument of [**3**] or [**25**]) that $\mathbf{C}$ is a cone:

$$\mathbf{C} \in \mathcal{C}. \tag{3}$$

Now let $\alpha = \Theta_M(z_0) (\equiv \Theta_{\mathbf{C}}(0))$. By 2.6 and 2.7,

$$L_{\mathbf{C}} = \{z \in \mathbf{R}^n : \Theta_{\mathbf{C}}(z) = \alpha\} \tag{4}$$

is a subspace of dimension $\leq m$, and hence is contained in an m-dimensional subspace L of $\mathbf{R}^n$. Then (with $\epsilon > 0$ arbitrarily given), by the upper semicontinuity 2.3 of Θ we see immediately that this implies

$$\{z \in \overline{B}_1(0) : \Theta_{M_{k'}}(z) \geq \alpha\} \subset L_\epsilon \tag{5}$$

for all sufficiently large k', where L_ϵ denotes the ϵ-neighbourhood of L. Indeed otherwise there would be a subsequence $\{\tilde{k}\} \subset \{k'\}$ and $x_{\tilde{k}} \in \overline{B}_1(0)\backslash L_\epsilon \to x \in \overline{B}_1(0)\backslash L_\epsilon$ and with $\Theta_{M_{\tilde{k}}}(x_{\tilde{k}}) \geq \alpha$. But then by the upper semi-continuity 1.13 we have $\Theta_{\mathbf{C}}(x) \geq \alpha$ with $x \in \overline{B}_1(0)\backslash L_\epsilon$, which contradicts (4). Thus (5) is established. But evidently (5) contradicts (1), so the lemma is proved.

Now let S_j be the set of $z_0 \in \operatorname{sing} M$ such that the conclusion of 2.16 holds with $\rho_0 = 1/j$. Then for each $w \in S_j$, there is a sequence $\{w_\ell\}_{\ell=1,2,\ldots} \subset B_{\frac{1}{2j}}(w) \cap S_j$ with $\Theta_M(w_\ell) \to \inf\{\Theta_M(z) : z \in B_{\frac{1}{2j}}(w) \cap S_j\}$; if this inf is attained at some $w_* \in B_{\frac{1}{2j}}(w) \cap S_j$, then we select $w_\ell = w_*$ for each ℓ. Thus

$$
\begin{aligned}
B_{\frac{1}{2j}}(w) \cap S_j &= \cup_{\ell=1}^\infty \{z \in B_{\frac{1}{2j}}(w) \cap S_j : \Theta_M(z) \geq \Theta_M(w_\ell)\} \\
&\subset \cup_{\ell=1}^\infty \{z \in B_{\frac{1}{j}}(w_\ell) \cap S_j : \Theta_M(z) \geq \Theta_M(w_\ell)\},
\end{aligned}
$$

which has $\mathcal{H}^{m+\beta(\epsilon)}$-measure zero by 2.16 and Remark (3) above, because $w_\ell \in S_j$ for every ℓ. In view of the arbitrariness of w and the fact that $\operatorname{sing} M = \cup_j S_j$ by 2.16, we have $\mathcal{H}^{m+\beta(\epsilon)}(\operatorname{sing} M) = 0$, and therefore

$$
\dim\ \operatorname{sing} M \leq m,
$$

since $\epsilon > 0$ was arbitrary.

Now let $\operatorname{sing}_* M$ denote the set of $z \in \operatorname{sing} M$ such that $\Theta_M(z) = \Theta_{\mathbf{C}}(0)$ for some $\mathbf{C} \in \mathcal{C}$ as in 2.4 with $\dim L_{\mathbf{C}} = m$. Then we can apply exactly the same argument as in the above lemma and the subsequent discussion, with $\operatorname{sing} M \backslash \operatorname{sing}_* M$ in place of $\operatorname{sing} M$; notice that at each stage of the discussion we obtain affine spaces of dimension $m-1$ instead of affine spaces of dimension m. Thus in place of the conclusion $\dim\ \operatorname{sing} M \leq m$ we have

$$
\dim\,(\operatorname{sing} M \backslash \operatorname{sing}_* M) \leq m-1. \tag{2.17}
$$

We shall use this fact in §7 below. By a slightly different argument, involving only the use of tangent cones rather than pseudo-tangent cones as in the above proof, one can prove a refinement of 2.17. Viz., $\dim S^{(j)} \leq j$ for each $j = 0, \ldots, m$, where $S^{(j)}$ is the set of $z \in \operatorname{sing} M$ such that all tangent cones $\mathbf{C}$ of M at z have $\dim L_{\mathbf{C}} \leq j$. We shall not need this refinement here, so we shall not discuss the proof, which is given in [**26**] by modifying the corresponding argument in [**1**] for area minimizing currents.

3 A Rectifiability Lemma, and Gap Measures on Subsets of $\boldsymbol{R}^n$.

Let $B_{\rho_0}(x_0)$ be an arbitrary ball in $\mathbf{R}^n$, and suppose that $S \subset \overline{B}_{\rho_0}(x_0)$ is closed, that ϵ, $\delta \in (0,1)$ with $\epsilon < \delta/8$ (in the applications below we always have $\epsilon << \delta$), and that S has the ϵ-approximation property satisfied for S_+ in 2.16. Thus for each $y \in S$ and each $\rho \in (0, \rho_0]$ we assume

(3.1) $S \cap B_\rho(y) \subset$ *the* $(\epsilon\rho)$*-neighbourhood of some* m*-dimensional affine space* $L_{y,\rho}$ *containing* y.

In all that follows we assume that $L_{y,\rho}$, corresponding to each $y \in S$ and $\rho \le \rho_0$, is chosen. Then, relative to such a choice, we have the following definition.

Definition. With the notation in 3.1 above, we say S has a δ-gap in a ball $B_\rho(y)$ with $y \in S$ if there is $z \in L_{y,\rho} \cap \overline{B}_{(1-\delta)\rho}(y)$ such that $B_{\delta\rho}(z) \cap S = \emptyset$.

With notation as in the previous definition, we recall the general rectifiability lemma established in [**32**], which gives sufficient conditions for an arbitrary subset of $\mathbf{R}^n$ to be m-recifiable, as defined in the introduction. This rectifiability lemma will be crucial in our later proof of Theorem 4.

The reader should keep in mind that the results here will be applied in $\mathbf{R}^{n+k}$ (rather than $\mathbf{R}^n$) in the proof of Theorem 4.

3.2 Lemma (Rectifiability Lemma). *For any $\delta \in (0, \frac{1}{32})$, there is $\epsilon_0 = \epsilon_0(m, n, \delta) \in (0, \frac{\delta}{16})$ such that the following holds. Suppose $\epsilon \in (0, \epsilon_0]$, $\rho_0 > 0$, $x_0 \in S \subset \overline{B}_{\rho_0}(x_0)$, and S has the ϵ-approximation property 3.1 above. Suppose further that, for each $x_1 \in S$ and $\rho_1 \in (0, \rho_0]$, either S has a $\frac{\delta}{20}$-gap in $B_{\rho_1}(x_1)$ or there is an m-dimensional subspace $L(x_1, \rho_1)$ of $\mathbf{R}^n$ and a family $\mathcal{F}_{x_1,\rho_1}$ of balls with centers in $S \cap \overline{B}_{\rho_1}(x_1)$ such that the following 2 conditions hold:*

(a) $$\textstyle\sum_{B \in \mathcal{F}_{x_1,\rho_1}} (\operatorname{diam} B)^m \le \epsilon \rho_1^m,$$

and

(b) $$S \cap B_\rho(y) \subset \text{ the } \epsilon\rho\text{-neighbourhood of } y + L(x_1, \rho_1)$$

for every $y \in S \cap B_{\rho_1/2}(x_1) \backslash (\cup \mathcal{F}_{x_1,\rho_1})$ and every $\rho \in (0, \rho_1/2]$ such that S has no δ-gaps in any of the balls $B_\tau(y)$, $\rho \le \tau \le \rho_1/2$. Then S is m-rectifiable.

3.3 Remarks: (1) It is important, from the point of view of the application which we have in mind, that the property (b) need only be checked on balls $B_\rho(y)$ such that S has no δ-gap in any of the balls $B_\tau(y)$, $\rho \le \tau \le \rho_1/2$.

(2) Notice that if S does not have a $\frac{\delta}{20}$-gap in $B_{\rho_1}(x_1)$ (so that the first alternative hypothesis of the lemma does *not* hold), then, provided ϵ is sufficiently small relative to δ, no ball $B_\tau(y)$ for $\tau \in [\frac{\rho_1}{17}, \frac{\rho_1}{2}]$ and $y \in S \cap B_{\rho_1/2}(x_1)$ can have a δ-gap, so in particular condition (b) always has non-trivial content in this case.

(3) In order to establish the Theorem 4 we are going to show that this lemma can be applied with sets S of the form $S = \overline{B}_\rho(y) \cap \{x \in \operatorname{sing} M : \Theta_M(x) \ge \Theta_M(y)\}$ with suitable $y \in \operatorname{sing} M$ and with ρ sufficiently small. Notice that Lemma 2.16 of §2 already establishes the weak ϵ-approximation property, which is required before the above lemma can be used. Most importantly, we are able in the discussion of §§4–6 to get much more control on $\operatorname{sing} M$ in balls which do not have δ-gaps. This is the key point which makes it possible to check the hypothesis (b) and hence to prove the main theorems stated in the introduction.

For the proof of 3.2 (which is based on a covering lemma), we refer to [**32**].

Next we want to establish the existence of a certain class of Borel measures on subsets $S \subset \mathbf{R}^n$ having the ϵ-approximation of 3.1 above.

Let $\epsilon > 0$ and $m \in \{1, 2, \dots, n-1\}$, $B_\rho(z) = \{x \in \mathbf{R}^n : |x - z| < \rho\}$, and let $0 \in S \subset \overline{B}_1(0)$ be an arbitrary non-empty closed subset of $\mathbf{R}^n$ with the ϵ-approximation property 3.1.

With the affine spaces spaces $L_{z,\rho}$ fixed as in 3.1, and assuming $\epsilon \le \delta/2 \in (0, \frac{1}{64})$, we make the following definition.

3.4 Definition. If $z \in S$ and $\rho \in (0, \frac{1}{4}]$, the ball $B_\rho(z)$ is said to be a δ-bad ball for S if either there is $w \in L_{z,\rho} \cap B_{(1-\delta)\rho}(z)$ such that $B_{\delta\rho}(w) \cap S = \emptyset$, or $\|(L_{z,\rho} - z) - L_{0,1}\| \ge \delta/2$.

Now we are going to define a family of subsets $\{T_\rho\}_{\rho \in (0, \frac{1}{4}]}$ as follows:

3.5 Definition. For $\rho \in (0, \frac{1}{4}]$, we define T_ρ to be the union over all balls $B_\rho(z)$ such that $z \in S$ and no ball $B_\sigma(z)$, $\sigma \in [\rho, \frac{1}{4}]$, is a δ-bad ball for S.

3.6 Remarks. **(1)** Of course the sets T_ρ depend on S and δ, but for convenience this is not indicated by the notation. Notice also that $T_\rho \subset S_\rho$, where $S_\rho = \{x \in \mathbf{R}^n : \operatorname{dist}(x, S) < \rho\}$; intuitively one should think of T_ρ as being some sort of refinement or reduction of S_ρ, taking into account δ-gaps and δ-tilts.

(2) It is possible to check the following properties direct from the definition of the T_ρ:

(a) The σ-neighbourhood of $T_\rho \subset T_{\rho+\sigma}$ for each $\rho, \sigma > 0$ with $\rho + \sigma < \frac{1}{4}$ (so that in particular we have $\operatorname{dist}(T_\rho, \mathbf{R}^n \backslash T_{\rho+\sigma}) \ge \sigma$).

(b) $\forall\, z \in S \backslash T_\rho$, $\rho \in (0, \frac{1}{4}]$, there is $\sigma(z) \ge \rho$ such that $B_{\sigma(z)}(z)$ is a δ-bad ball for S.

(c) The $\frac{\rho}{4}$-neighbourhood of $T_\rho \backslash T_{\frac{\rho}{2}}$ is contained in $T_{2\rho} \backslash T_{\frac{\rho}{4}}$, $\rho \in (0, \frac{1}{8}]$.

Notice that, taking $\rho = 2^{-\ell}$ and $\sigma = 2^{-k} - 2^{-\ell}$ in (a) we have in particular that

(d) $\operatorname{dist}(T_{2^{-\ell}}, \mathbf{R}^n \backslash T_{2^{-k}}) \ge 2^{-k-1}$ for $\ell \ge k+1$, $k \ge 2$.

Proof of (a). Take any $w \in \sigma$-neighbourhood of T_ρ. Then by definition of T_ρ there is a $z \in S$ such that $w \in B_{\rho+\sigma}(z)$, where no $B_\tau(z)$ is a δ-bad ball, $\tau \ge \rho$. Thus $w \in T_{\rho+\sigma}$ by definition of $T_{\rho+\sigma}$.

Proof of (b). Suppose $z \in S \backslash T_\rho$. Then some $B_\sigma(z)$, $\sigma \ge \rho$, must be a δ-bad ball, otherwise $B_\rho(z) \subset T_\rho$ by definition, contradicting the hypothesis that $z \notin T_\rho$.

Proof of (c). By (a), the $\frac{\rho}{4}$-neighbourhood of $T_{\frac{\rho}{4}}$ is contained in $T_{\frac{\rho}{2}}$, and hence the $\frac{\rho}{4}$-neighbourhood of $\mathbf{R}^n \backslash T_{\frac{\rho}{2}}$ is contained in $\mathbf{R}^n \backslash T_{\frac{\rho}{4}}$. Also, again by (a), the $\frac{\rho}{4}$-neighbourhood of T_ρ is contained in $T_{2\rho}$. The combination of these inclusions then gives (c) as claimed.

3.7 Lemma. *There is $\delta_0 = \delta_0(m, n) \in (0, \frac{1}{16}]$ such that if $0 < \epsilon \le \frac{\delta}{2} \le \frac{\delta_0}{2}$, and S, $\{T_\sigma\}_{\sigma \in (0, \frac{1}{4}]}$ are as introduced above, then there is a Borel measure μ on S with the properties $\mu(S) = 1$ and, for each $\sigma \in (0, \frac{1}{16}]$,*

$$C^{-1}\rho^m \le \mu(B_\rho(z) \cap S) \le C\rho^m, \quad \rho \in [\delta^{1/2}\sigma, \tfrac{1}{16}], \quad z \in T_\sigma \cap S,$$

where $C = C(n,m)$. The measure μ has the general form

$$\mu = C_1\delta^{m/2}\sum_{k=2}^{\infty}2^{-mk}\sum_{j=1}^{Q_k}[\![z_{k,j}]\!] + C_2\mathcal{H}^m \llcorner T_0,$$

where $[\![z]\!]$ denotes the unit mass (Dirac mass) supported at z, $T_0 = \cap_{\rho>0}T_\rho$, C_1, C_2 depend only on n, m, and the $z_{k,j} \in S\cap T_{2^{-k}}\backslash T_{2^{-k-1}}$, $j = 1,\dots,Q_k$, $k \ge 2$, with

$$S\cap T_{2^{-k}}\backslash T_{2^{-k-1}} \subset \cup_{\ell=\max(k-2,2)}^{k+1}\cup_{j=1}^{Q_\ell} B_{\delta^{1/2}2^{-\ell}}(z_{\ell,j}), \quad k\ge 2.$$

For the proof of this lemma, we refer to [**32**].

3.8 Remarks. **(1)** It is important for later application that C does not depend on δ, nor indeed on S. Of course one has to keep in mind that if the set S is very badly behaved (like a Koch curve for example), then the sets T_ρ can all reduce to the empty set for sufficiently small ρ, in which case the lemma has correspondingly limited content.

(2) As part of the proof given in [**32**], it is shown that T_0 is contained in the graph of a Lipschitz function defined over $\{0\}\times\mathbf{R}^m$ and with Lipschitz constant $\le C\delta$, so automatically $\mathcal{H}^m \llcorner T_0$ has total measure $\le C$.

4 Area Estimates for Submanifolds in $\mathcal{M}$. Here we continue to assume that $M \in \mathcal{M}$. Points in $\mathbf{R}^{n+k}$ will be denoted $(x,y)\in\mathbf{R}^{\ell+k}\times\mathbf{R}^m$, and we continue to use the notation $r = |x|$ and $\omega = |x|^{-1}x \in S^{\ell+k-1}$ for $x\in\mathbf{R}^{\ell+k}\backslash\{0\}$.

We are often going to use the variables $(r,y) = (|x|,y)$ corresponding to a given point $(x,y)\in\mathbf{R}^{\ell+k}\times\mathbf{R}^m$, and it will be convenient to introduce the additional notation

$$B_\rho^+ = \{(r,y) : r>0,\, r^2+|y|^2<\rho^2\},\quad B_\rho^+(y_0) = \{(r,y) : r>0,\, |y-y_0|^2<\rho^2\}$$

for given $y_0\in\mathbf{R}^m$ and $\rho>0$.

Also,

$$B_\rho^M(y) = M\cap B_\rho(y),$$

and we let ν_r, ν_y be defined on M by

$$\nu_r(x,y) = p_{T^\perp_{(x,y)}M}(|x|^{-1}x,0), \qquad \nu_{y^j} = p_{T^\perp_{(x,y)}M}(e_{\ell+k+j}),\ j=1,\dots,m,$$

where $p_{T^\perp_{(x,y)}M}$ denotes orthogonal projection of $\mathbf{R}^{n+k}$ onto the normal space $T^\perp_{(x,y)}M$. Notice that we thus have

$$\nu_r^2 = 1-|\nabla^M r|^2, \qquad \nu_y^2 \equiv \sum_{j=1}^m \nu_{y^j}^2 = \sum_{j=1}^m |p_{T^\perp_{(x,y)}M}(e_{\ell+k+j})|^2 \equiv \sum_{j=1}^m(1-|\nabla^M y^j|^2).$$

In particular, if e is any vector in $\{0\}\times\mathbf{R}^m$, then

$$|p_{T^\perp_{(x,y)}M}(e)|^2 \le |e|^2\nu_y^2.$$

The main inequality of this section is given in the following theorem:

4.1 Theorem. *If $\zeta \in (0, \frac{1}{4})$, $\beta > 0$ then there are $C = C(\beta, k, n) > 0$, $\eta = \eta(\beta, k, n, \zeta) > 0$ and $\alpha = \alpha(\beta, k, n) \in (0,1)$ such that the following holds: If $\rho^{-n}|B_\rho^M(0)| \le \beta$, $0 \in \overline{M}$, $\omega_n^{-1}\rho^{-n}|B_\rho^M(0)| - \Theta_M(0) < \eta$ and $\rho^{-n-2}\int_{B_\rho^M(0)} r^2(\nu_r^2 + \nu_y^2) < \eta$, then there is $\mathbf{C} \in \mathcal{T}$ with $\operatorname{sing}\mathbf{C} \subset \{0\} \times \mathbf{R}^m$, satisfying*

$$\rho^{-n-2}\int_{B_\rho^M(0)} \operatorname{dist}((x,y),\mathbf{C})^2 < \zeta,$$

$$\Theta_M(0) - \zeta \le \Theta_{\mathbf{C}}(0) \le \omega_n^{-1}\rho^{-n}|B_\rho^M(0)| + \zeta \quad \textit{and}$$

$$\int_{B^+_{\rho/2}} r^2 \left||M(r,y)| - r^{\ell-1}|\Sigma|\right| \, dr dy \le C\int_{B_\rho^M(0)} r^2(\nu_r^2 + \nu_y^2)$$
$$+ C\rho^n\left(\rho^{-n}\int_{B_\rho^M(0)\setminus\{(x,y)\,:\,|x|<\rho/2\}} r^2(\nu_r^2+\nu_y^2)\right)^{1/(2-\alpha)},$$

where $M(r,y) = M \cap S_{r,y}$.

In proving Theorem 1 we shall need three lemmas, each of which is of some independent interest. The first of these gives some important general facts about $\mathbf{C} \in \mathcal{T}$; we use the notation of 2.11, and define

$$\mathcal{T}_\beta^{(0)} = \{\Sigma : \Sigma \textit{ is a compact } (\ell-1)\textit{-dimensional embedded minimal submanifold of } S^{\ell+k-1} \textit{ with } \{(\lambda\omega, y) : \lambda > 0,\, y \in \mathbf{R}^m,\, \omega \in \Sigma\} \in \mathcal{T}_\beta\}. \tag{4.2}$$

If Σ is a compact $(\ell-1)$-dimensional embedded minimal submanifold of $S^{\ell+k-1}$, and if ψ is a C^j section of the normal bundle of $\mathbf{C}$ over Σ (we write $\psi \in C^j(\Sigma; \mathbf{C}^\perp)$), then we continue to let $G_\Sigma(\psi)$ denote the "spherical graph" defined in §2 and $\mathcal{A}_\Sigma(\psi)$ the corresponding area functional as in 2.15. Notice that if $|\psi|_{C^j}$ is small enough (depending on Σ), and if $j \ge 1$, then $G_\Sigma(\psi)$ will be an embedded C^j-submanifold of $S^{\ell+k-1}$.

Under suitable circumstances, we can also express appropriate parts of $M \in \mathcal{M}$ as a spherical graph taken off a cone $\mathbf{C} \in \mathcal{C}$. specifically, if $\Omega \subset \mathbf{C}$ is open and if u is a C^j section of the normal bundle of $\mathbf{C}$ over Ω (we write $u \in C^j(\Omega; \mathbf{C}^\perp)$) with $\sum_{j=0}^3 r^{j-1}|D^j u| \le \gamma$, with γ sufficiently small depending only on $\mathbf{C}$ (and not depending on the domain Ω), then we can define the spherical graph $G_{\mathbf{C}}(u)$ (analogous to 2.14) by

$$G_{\mathbf{C}}(u) = \{(1 + |x|^{-2}|u(x,y)|^2)^{-1/2}((x,y) + u(x,y))\};$$

$G_{\mathbf{C}}(u)$ is then an embedded C^j-submanifold of $\mathbf{R}^{n+k}$. We can also define the area functional $\mathcal{A}_{\mathbf{C}}(u)$ (analogous to 2.15) over $\mathbf{C}$ for such $u \in C^1(\Omega; \mathbf{C}^\perp)$ by

$$\mathcal{A}_{\mathbf{C}}(u) = |G_{\mathbf{C}}(u)|.$$

Then we have the following:

4.3 Lemma. *For each $\beta > 0$, $\mathcal{T}_\beta^{(0)}$ is compact in the sense that if $\Sigma_j \in \mathcal{T}_\beta^{(0)}$, then there is a subsequence converging in the Hausdorff distance sense to an element $\Sigma \in \mathcal{T}_\beta^{(0)}$. Also, there is $\zeta_1 = \zeta_1(\beta, n, k) \in (0, \frac{1}{4}]$ such that, if Σ_1, $\Sigma_2 \in \mathcal{T}_\beta^{(0)}$ and Σ_2 can be expressed as a spherical graph $G_{\Sigma_1}\psi$ of a C^3 function ψ taken off Σ_1 with $|\psi|_{C^3(\Sigma_1)} < \zeta_1$, then $|\Sigma_1| = |\Sigma_2|$. Furthermore there are constants $\zeta_2 = \zeta_2(\beta, n, k) \in (0, \frac{1}{4}]$ and $\alpha = \alpha(\beta, n, k) \in (0, 1)$ such that if $\Sigma_1 \in \mathcal{T}_\beta^{(0)}$ and if Σ_2 (not necessarily in $\mathcal{T}_\beta^{(0)}$) can be expressed as a spherical graph $G_{\Sigma_1}\psi$ of a C^3 section ψ of the normal bundle of Σ_1 with $|\psi|_{C^3(\Sigma_1)} < \zeta_2$, then*

$$\big||\Sigma_1| - |\Sigma_2|\big|^{2-\alpha} \le \int_{\Sigma_1} |\mathcal{Q}_{\Sigma_1}\psi|^2,$$

where $\mathcal{Q}_{\Sigma_1}$ denotes the minimal surface operator on Σ_1 (i.e., $\mathcal{Q}_{\Sigma_1}(\psi)$ is the Euler-Lagrange operator of the area functional $\mathcal{A}(\psi) \equiv |G_{\Sigma_1}(\psi)|$ of spherical graphs over Σ_1).

Remark. Thus we have a uniform Łojasiewicz inequality for a whole C^3 neighbourhood of $\mathcal{T}_\beta^{(0)}$, and also, by the first part of the above lemma, the area is constant on the connected components of of $\mathcal{T}_\beta^{(0)}$, and there are only finitely many values of the area corresponding to $\Sigma \in \mathcal{T}_\beta^{(0)}$.

Proof of Lemma 4.3. The compactness of $\mathcal{T}_\beta^{(0)}$ is a direct consequence of the estimates of 2.12 and the compactness 1.11(b) for $\mathcal{M}$. Next suppose there is no such ζ_1. Then there must be sequences Σ_j, $\widetilde{\Sigma}_j$ in $\mathcal{T}_\beta^{(0)}$ converging in the Hausdorff distance sense to a common limit $\Sigma \in \mathcal{T}_\beta^{(0)}$ but with

$$(1) \qquad |\Sigma_j| \ne |\widetilde{\Sigma}_j| \quad \forall\, j.$$

According to the Łojasiewicz inequality of 2.14 we have $\alpha = \alpha(\Sigma) \in (0, 1)$ and $\sigma = \sigma(\Sigma) > 0$ such that

$$(2) \qquad \big||G_{\Sigma_1}(\psi)| - |\Sigma_1|\big|^{1-\alpha/2} \le C\|\mathcal{Q}_{\Sigma_1}(\psi)\|_{L^2(\Sigma_1)}, \quad |\psi|_{C^3(\Sigma_1)} < \sigma.$$

Therefore for all sufficiently large j we can apply this with $\mathrm{graph}_{\Sigma_1}(\psi) = \Sigma_j$, $\widetilde{\Sigma}_j$ in order to deduce that $|\Sigma_1| = |\Sigma_2|$, thus contradicting (1).

Now if the inequality of the lemma fails, then there are sequences $\Sigma_j \in \mathcal{T}_\beta^{(0)}$ and $\psi_j \in C^3$ sections of the normal bundle of $\mathbf{C}_j$ over Σ_j, with $\mathbf{C}_j$ the cone determined by Σ_j, with Σ_j converging to a given $\Sigma \in \mathcal{T}_\beta^{(0)}$ and with $|\psi_j|_{C^3}$ but such that

$$(3) \qquad \big||G_{\Sigma_j}(\psi_j)| - |\Sigma_j|\big|^{1-\alpha_j/2} > j\|\mathcal{Q}_{\Sigma_j}(\psi_j)\|_{L^2(\Sigma_j)},$$

where $\alpha_j \downarrow 0$ as $j \to \infty$. Thus $|\Sigma_j| = |\Sigma|$ for all sufficiently large j by the first part of the proof above, and (3) contradicts (2), because $\|\mathcal{Q}_{\Sigma_j}(\psi_j)\|_{L^2(\Sigma_j)}$ is geometrically the L^2-norm of the mean curvature vector of $G_{\Sigma_j}(\psi_j)$ integrated over Σ_j and (since Σ_j is approaching Σ in the C^1-norm) this is proportional to the L^2 norm of the mean curvature vector of $\widetilde{\Sigma}_j = G_{\Sigma_j}(\psi_j)$ when $\widetilde{\Sigma}_j$ is expressed as a spherical graph taken off Σ.

4.4 Lemma. *Let $\sigma \in (0,1]$ and $\beta > 0$. There is $\eta = \eta(n,k,\beta) \in (0,1)$ such that if $B^M_{7\sigma/8}(0)\backslash\{(x,y) : |x| \le \sigma/16\} = G_{\mathbf{C}}u$ with $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}^m \in \mathcal{T}_\beta$, u a $C^3(\mathbf{C} \cap B_{7\sigma/8}(0)\backslash\{(x,y) : |x| \le \sigma/16\}; \mathbf{C}^\perp)$ function and*

$$\sup_{\mathbf{C}\cap B_{7\sigma/8}(0)\backslash\{(x,y)\,:\,|x|\le\sigma/16\}} \sum_{j=0}^{3} \sigma^{j-1}|D^j u| \le \eta,$$

then

$$\sup_{B^+_{3\sigma/4}\backslash\{(r,y)\,:\,r\le\sigma/8\}} \int_\Sigma |\mathcal{Q}_{\mathbf{C}}(u)|^2\, d\mathcal{H}^{\ell-1} \le C\sigma^{-n-2} \int_{B^M_\sigma\backslash\{(x,y)\,:\,|x|\le\sigma/16\}} r^2(\nu_r^2+\nu_y^2)$$

and

$$\sup_{B^+_{3\sigma/4}\backslash\{(r,y)\,:\,r\le\sigma/8\}} \left|\nabla_{r,y}(r^{1-\ell}|M(r,y)|)\right| \le C\sigma^{-1-n} \int_{B^M_\sigma\backslash\{(x,y)\,:\,|x|\le\sigma/16\}} r^2(\nu_r^2 + \nu_y^2).$$

Here $\nabla_{r,y}$ means the gradient with respect to the variables $(r,y) \in B^+_\sigma$, $C = C(\beta,n,k)$, and $u(r,y)$ denotes the function on Σ defined by $u(r,y)(\omega) = u(r\omega,y)$, and $\Sigma = \mathbf{C}_0 \cap S^{\ell+k-1}$.

Proof. As discussed in §2, the Euler-Lagrange operator $\mathcal{Q}_\Sigma v$ for $v \in C^2(\Sigma;\mathbf{C}^\perp)$ is characterized by the integral identity

$$\frac{d}{ds}|G_\Sigma(v+s\zeta)|_{s=0} = -\int_\Sigma \mathcal{Q}_\Sigma(v)\cdot\zeta\, d\mathcal{H}^{\ell-1}, \quad v,\,\zeta \in C^2(\Sigma;\mathbf{C}^\perp),$$

so in particular

$$\nabla_{r,y}|M(r,y)| \equiv \nabla_{r,y}|G_\Sigma(u(r,y))| \equiv -\int_\Sigma \mathcal{Q}_\Sigma(u(r,y))\cdot\nabla_{r,y}u. \tag{1}$$

Also (see, e.g., the discussion of **[26]**) the Euler-Lagrange operator $\mathcal{Q}_{\mathbf{C}}$ of the area functional over $\mathbf{C}$ has the form

$$\mathcal{Q}_{\mathbf{C}}(v) = \Delta_{r,y}v + r^{-2}\mathcal{Q}_\Sigma v(r,y) + r^{-2}\mathcal{R}(v),$$

where $\Delta_{r,y}v = \frac{1}{r^{\ell-1}}\frac{\partial}{\partial r}\left(r^{\ell-1}\frac{\partial v(r,y)}{\partial r}\right) + \sum_{j=1}^m \frac{\partial^2 v(r,y)}{\partial(y^j)^2}$, and

$$|\mathcal{R}(v)| \le Cr^2(|\nabla v_r|+|\nabla v_y|)|\nabla v| + Cr^2(|v_r|+|v_y|)(r^{-1}|v|+|\nabla v|+r|\nabla^2 v|). \tag{2}$$

Notice that if $\mathcal{Q}_{\mathbf{C}}u = 0$ in some region $\Omega \subset \mathbf{C}$, then by definition,

$$r^{-2}\mathcal{Q}_\Sigma u(r,y) = -\Delta_{r,y}u - \mathcal{R}(u) \quad \text{on } \Omega. \tag{3}$$

We also recall that the linear operator

$$\mathcal{L}_u v = \frac{d}{dt}\mathcal{Q}_{\mathbf{C}}(u+tv)|_{t=0}$$

is a linear elliptic operator of the form

$$\mathcal{L}_u v = \Delta_{r,y} v + r^{-2} L_{\Sigma,u} v,$$

where $L_{\Sigma,u}$ is a linear elliptic self-adjoint operator on functions $v \in C^2(\Sigma; \mathbf{C}^\perp)$. In particular (using the notation introduced prior to 4.3) if $M = G_{\mathbf{C}} u$ with $u \in C^2(\Omega; \mathbf{C}^\perp)$ for some $\Omega \subset \mathbf{C}$, then since $M_t = M - te_{\ell+k+j}$ is a minimal surface for each t, and $M_t = \operatorname{graph}_{\mathbf{C}} u_t$, where $u_t(x,y) = u((x,y) + te_{\ell+k+j})$, then we have $\mathcal{Q}_{\mathbf{C}} u_t \equiv 0$ on a domain $\Omega_t = \Omega - te_{\ell+k+j}$, and hence $v = u_{y^j} \equiv \frac{d}{dt} u((x,y) + te_{\ell+k+j})|_{t=0}$ is a solution of

$$\mathcal{L}_u v = 0$$

for each $j = 1, \ldots, m$. Also since $\widetilde{M}_t = (1+t)M$ is a minimal surface for each t with $|t| < 1$, and $\widetilde{M}_t = \operatorname{graph} \tilde{u}_t$, where $\tilde{u}_t(x,y) = (1+t)^{-1} u((1+t)(x,y))$, then we have similarly that $v = Ru_R - u \equiv \frac{d}{dt} \tilde{u}_t|_{t=0}$ is also a solution of this equation. But $Ru_R - u = ((x,y) \cdot D)u - u \equiv r^2(u/r)_r + \sum_{j=1}^m y^j u_{y^j}$, so we have the equations

$$\text{(4)} \qquad \mathcal{L}_u(u_{y^j}) = 0, \quad \mathcal{L}_u(r^2(u/r)_r) = -\mathcal{L}_u\Big(\sum_{j=1}^m y^j u_{y^j}\Big) = -2\Delta_y u.$$

Notice that the operator $\mathcal{L}_u w$ has the form

$$\Delta_{r,y} w + r^{-2} \Delta_\Sigma w + r^{-1} a \cdot \nabla^\Sigma w + r^{-2} b \cdot w$$

with $|a|$, $|b| \le C(n,k,\beta)$ on $B_{7\sigma/8}(0) \backslash \{(x,y) : |x| < \sigma/16\}$. Then the standard $C^{1,\alpha}$ Schauder theory for such linear operators ([**12**]) gives

$$\text{(5)} \qquad \sup_{B^{\mathbf{C}}_{7\sigma/9} \backslash \{(x,y) : |x| < \sigma/9\}} \sum_{j=0}^1 |\sigma^j D^j u_y|^2 \le C\sigma^{-n} \int_{B^{\mathbf{C}}_{8\sigma/9} \backslash \{(x,y) : |x| < \sigma/10\}} u_y^2,$$

where $B^{\mathbf{C}}_\sigma = \mathbf{C} \cap B_\sigma(0)$. By means equation (4) for $r^2(u/r)_r$ and again the $C^{1,\alpha}$ Schauder theory (this time using also (2) to estimate the sup norm of $\Delta_y u$), we deduce that $r^2(u/r)_r$ satisfies

$$\text{(6)} \qquad \begin{aligned} &\sup_{B^{\mathbf{C}}_{3\sigma/4} \backslash \{(x,y) : |x| < \sigma/8\}} \sum_{j=0}^1 |\sigma^j D^j (r^2(u/r)_r)|^2 \\ &\qquad \le C\sigma^{-n} \int_{B^{\mathbf{C}}_\sigma \backslash \{(x,y) : |x| < \sigma/16\}} r^2(u_r^2 + u_y^2). \end{aligned}$$

Next we note, by the notation introduced above, that $te_{\ell+k+j} + (x,y) + u_t((x,y)) \in M$ for all small $|t|$, and hence by differentiating with respect to t and setting $t = 0$ we have $e_{\ell+k+j} + u_{y^j}(x,y) \in T_{(x,y)+u(x,y)}M$, whence $(e_{\ell+k+j})^\perp = -(u_{y^j}(x,y))^\perp$, where $v^\perp$ means the orthogonal projection into the normal space of M at the point $(x,y) + u(x,y) \in M$. Since u is already normal to $\mathbf{C}$, and $\mathbf{C}$ is invariant under translations in the direction $e_{\ell+k+j}$ we also have

$$\text{(7)} \qquad \frac{1}{2}|u_{y^j}(x,y))| \le |(u_{y^j}(x,y))^\perp| = |(e_{\ell+k+j})^\perp| \le |u_{y^j}(x,y)|,$$

for (x, y) in the domain of u, provided the constant η of the lemma is chosen small enough (depending on n, k, β).

By a similar argument using $\widetilde{M}_t$ and $\tilde{u}_t$ we obtain

$$(8)\quad ((x,y)+u(x,y))^{\perp} = -r^2((u/r)_r)^{\perp} - \sum_{j=1}^{m} y^j (u_{y^j})^{\perp} (= (-R^2(u/R)_R)^{\perp}).$$

Since $\mathbf{C}$ is invariant under homotheties of the x-variable, we whence have

$$\frac{1}{2}|r^2(u/r)_r| \le |((x,y)+u(x,y))^{\perp}| + |y|\,|u_y|.$$

But $((x,y)+u(x,y))^{\perp} = p_{T^{\perp}M}((x,0)+u(x,y)) + \sum_{j=1}^{m} y^j e^{\perp}_{\ell+k+j}$, so we also have $|((x,y)+u(x,y))^{\perp}| \le \sigma(|\nu_r|+m|\nu_y|)$ on $M \cap B_\sigma(0)$, where the right side is evaluated at the point $(\tilde{x}, y) = (x,y)+u(x,y)$—note that $|\tilde{x}| = \sqrt{r^2+|u(x,y)|^2}$ at this point $(\tilde{x}, y)$). Also, by definition of ν_y, $|e^{\perp}_{\ell+k+j}| \le \nu_y$ on M. Then (7) and (8) yield

$$(9)\quad |u_y| + |r(u/r)_r| \le C(m)(|\nu_r|+|\nu_y|).$$

Now (1), (2), (5), (6), and (9) evidently imply the inequalities of the lemma.

Next we have a lemma which gives important information about approximation of $M \in \mathcal{M}$ by $\mathbf{C} \in \mathcal{T}$.

4.5 Lemma. *Let $\beta > 1$, $\zeta > 0$. There are constants $\eta = \eta(\beta, \zeta, n, k) > 0$, $\alpha = \alpha(n,k,\beta) \in (0,1)$ such that if $\rho^{-n}|M \cap B_\rho(0)| \le \beta$, $0 \in \overline{M}$, and $\omega_n^{-1}\rho^{-n}|M \cap B_\rho(0)| - \Theta_M(0) < \eta$, then the inequality*

$$\rho^{-n-2}\int_{M \cap B_{3\rho/4}(0)\setminus\{(x,y)\,:\,|x|<\rho/2\}} r^2(\nu_r^2+\nu_y^2) < \eta$$

implies that there is a $\mathbf{C} \in \mathcal{T}$ with

$$\operatorname{sing}\mathbf{C} = \{0\}\times\mathbf{R}^m, \quad \Theta_M(0)-\zeta \le \Theta_{\mathbf{C}}(0) \le \Theta_M(0)+\eta,$$

and

$$u \in C^3(\mathbf{C}\cap B_{15\rho/16}(0)\setminus\{(x,y)\,:\,|x|\le\rho/16\};\mathbf{C}^{\perp})$$

with

$$M \cap B_{15\rho/16}(0)\setminus\{(x,y)\,:\,|x|\le\rho/16\} = G_{\mathbf{C}}u$$

and

$$\rho^{-n-2}\int_{M\cap B_{15\rho/16}(0)} \operatorname{dist}((x,y),\mathbf{C})^2 \le \zeta,$$

$$\sup_{B^{\mathbf{C}}_{7\rho/8}(0)\setminus\{(x,y)\,:\,|x|\le\rho/16\}} \sum_{j=0}^{3} \rho^{j-1}|D^j u|_{C^3} \le \zeta.$$

Also there is $\zeta = \zeta(n, k, \beta) \le \min(\zeta_1, \zeta_2)$, ζ_1, ζ_2 as in Lemma 4.3, such that, in addition to the above, we have

$$\sup_{B^+_{3\rho/4}\setminus\{(r,y)\,:\,r<\rho/4\}} \left|r^{1-l}|M(r,y)| - |\Sigma|\right| \le C\left(\rho^{-n}\int_{B^M_\rho\setminus\{(x,y)\,:\,|x|<\rho/8\}} r^2(\nu_r^2+\nu_y^2)\right)^{1/(2-\alpha)},$$

where $M(r,y)$ is as in 4.1.

Proof. First notice that the inequality

$$\sup_{B^{\mathbf{C}}_{7\rho/8}(0)\setminus\{(x,y)\,:\,|x|\le\rho/16\}} \sum_{j=0}^{3} \rho^j |D^j u|_{C^3} \le C\zeta \tag{1}$$

is implied by the other inequalities

$$\begin{gathered} \Theta_M(0) - \zeta \le \Theta_{\mathbf{C}}(0) \le \Theta_M(0) + \eta, \\ \rho^{-n-2}\int_{B^M_{15\rho/16}(0)} \operatorname{dist}((x,y),\mathbf{C})^2 < \zeta, \end{gathered} \tag{2}$$

together with the estimates of 1.12 and 2.12, so we only need to check (2). By rescaling it is enough to check (2) in case $\rho = 1$. If there is no such η for some given ζ, then there must exist a sequence $M^{(j)} \in \mathcal{M}$ with $|M^j \cap B_1(0)| \lneq \beta$, $|M^j \cap B_1(0)| - \Theta^{(0)}_{M^{(j)}} \to 0$, $0 \in \overline{M}^{(j)}$, and

$$\int_{M^{(j)}\cap B_{3/4}(0)\setminus\{(x,y)\,:\,|x|\le\frac12\}} ((\nu_r^{(j)})^2 + (\nu_y^{(j)})^2) \to 0$$

yet such that, for every $\mathbf{C} \in \mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\}\times\mathbf{R}^m$, at least one of the inequalities in (2) fails if $\rho = 1$ and $M = M^{(j)}$.

By the compactness 1.11(b) there is a subsequence (still denoted $M^{(j)}$) such that $M^{(j)} \to \mathbf{C}$, where $\nu_r^{\mathbf{C}} \equiv 0$ and $\nu_y^{\mathbf{C}} \equiv 0$ on $B_{5/8}(0)\setminus\{(x,y) : |x| \le \frac12\}$, $\Theta_{\mathbf{C}}(0) = |\mathbf{C}\cap B_1(0)|$(by (2.3)). The monotonicity 1.7′ yields that $\mathbf{C}$ extends to give an element of $\mathcal{T}$ with $\operatorname{sing}\mathbf{C} \subset \{0\}\times\mathbf{R}^m$. Since $M^{(j)} \to \mathbf{C}$ we have

$$\Theta_{\mathbf{C}}(0) \equiv |\mathbf{C}\cap B_1(0)| \le \liminf_{j\to\infty} |M^{(j)}\cap B_1(0)| \equiv \liminf_{j\to\infty} \Theta_{M^{(j)}}(0),$$

and from the upper-semicontinuity 2.3 of the density function we also know that

$$\Theta_{\mathbf{C}}(0) \ge \limsup_{j\to\infty} \Theta_{M^{(j)}}(0),$$

and hence (2) is satisfied with $\rho = 1$ and with $M^{(j)}$ in place of M for j sufficiently large. Evidently this is a contradiction, so the required inequalities (2) (and hence also (1)) must hold for some $\mathbf{C} \in \mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\}\times\mathbf{R}^m$, provided η is sufficiently small.

We now need to establish the final inequality of the lemma. By virtue of Lemma 4.3 we have that there is $\alpha = \alpha(n,k,\beta) \in (0,1)$ and $\zeta = \zeta(n,k,\beta) > 0$ such that (1) implies

$$\left|r^{1-l}|M(r,y)| - |\Sigma|\right|^{1-\alpha/2} \leq C\|\mathcal{Q}_\Sigma u(r,y))\|^2_{L^2(\Sigma)}$$

for each $(r,y) \in B^+_{7\rho/8}(0)$ with $r \geq \rho/16$, where $C = C(n,k,\beta)$. Then the required inequality holds by virtue of Lemma 4.4; notice that the hypothesis

$$\sup_{B^{\mathbf{C}}_{7\rho/8}(0)\setminus\{(x,y)\,:\,|x|\leq\rho/16\}} \sum_{j=0}^{3} \rho^j |D^j u| \leq \eta$$

required in Lemma 4.4 is satisfied (with $C\zeta$ in place of η) due to 2.12 and the inequality (1) above.

We shall need the following corollary of the above lemma later.

4.6 Corollary. *For any given $\zeta > 0$, $\beta > 1$ there is $\eta_0 = \eta_0(\zeta,\beta,n,k) > 0$ such that the following holds. Suppose $\mathbf{C} \in \mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\}\times\mathbf{R}^m$, $M \in \mathcal{M}$ with $\rho^{-n}|M\cap B_\rho(0)| \leq \beta$, $\omega_n^{-1}\rho^{-n}|M\cap B_\rho(0)| - \Theta_M(0) < \eta_0$, $0 \in \overline{M}$, and also*

$$\rho^{-n-2}\int_{B^M_\rho(0)\setminus\{(x,y)\,:\,|x|<\frac{2\rho}{5}\}} \operatorname{dist}((x,y),\mathbf{C})^2 < \eta_0.$$

Then

$$\rho^{-n-2}\int_{B^M_{3\rho/4}(0)} \operatorname{dist}((x,y),\mathbf{C})^2 < \zeta$$

and

$$\operatorname{sing} M\cap B_{\rho/2}(0) \subset \textit{ the } (\zeta\rho)\textit{-neighbourhood of } \{0\}\times\mathbf{R}^m.$$

Proof. By the regularity estimates 2.2 and 2.12 we have immediately that $M\cap B_{3\rho/4}(0)\setminus\{(x,y)\,:\,|x|\leq\frac{\rho}{2}\} = G_{\mathbf{C}}u$ for some $u \in C^2(B^{\mathbf{C}}_{3\rho/4}(0)\setminus\{(x,y)\,:\,|x|\leq\frac{\rho}{2}\};\mathbf{C}^\perp)$ with

$$\sum_{j=0}^{3}\rho^{j-1}|D^j u| \leq C\eta_0^{1/2} \text{ on } B_{3\rho/4}(0)\setminus\{(x,y)\,:\,|x|\leq\tfrac{\rho}{2}\}.$$

Since $\mathbf{C}\in\mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\}\times\mathbf{R}^m$, we have in particular that for any given $\zeta > 0$ the hypotheses of Lemma 4.5 above hold, provided $\eta_0 = \eta_0(\beta,\zeta,n,k) \in (0,\zeta)$ is sufficiently small. Thus that lemma yields

$$\rho^{-n-2}\int_{M\cap B_{3\rho/4}(0)} \operatorname{dist}((x,y),\widetilde{\mathbf{C}})^2 < \zeta$$

for suitable $\widetilde{\mathbf{C}} \in \mathcal{T}$ with sing $\widetilde{\mathbf{C}} \subset \{0\}\times R^m$ and hence, the the triangle inequality

$$\rho^{-n-2}\int_{\widetilde{\mathbf{C}}\cap B_{3\rho/4}(0)\setminus\{(x,y)\,:\,|x|<\frac{\rho}{4}\}} \operatorname{dist}((x,y),\mathbf{C})^2 < C\zeta$$

which leads to

$$\rho^{-n}\int_{M\cap B_{3\rho/4}(0)}\operatorname{dist}((x,y),\mathbf{C})^2\le C\zeta$$

since both $\mathbf{C}$, $\widetilde{\mathbf{C}}$ are cones. The first conclusion of the Corollary is now clear. To prove the final conclusion, we argue as follows:

Suppose $z\in\operatorname{sing}M$ with $\operatorname{dist}((x,y),\mathbf{C})\ge 2(\eta_0^{-1}\zeta)^{1/(n+2)}\rho$, where $\eta_0\in(0,\frac{1}{2}]$ is to be chosen shortly. Then, in consequence of the above proof,

$$\rho^{-n-2}\int_{M\cap B_{3\rho/4}(0)}\operatorname{dist}^2((x,y),\mathbf{C})\le\zeta,$$

which, together with the fact that $B_{\zeta_0\rho}(z)\subset B_{3\rho/4}(0)$, implies that

$$(\zeta_0\rho)^{-n-2}\int_{M\cap B_{\zeta_0\rho}(z)}\operatorname{dist}^2((x,y),\mathbf{C})\le C\eta_0,\quad C=C(n,\beta).$$

Thus by the monotonicity 1.7′, $\operatorname{dist}(z,\mathbf{C})\le C\eta_0^{1/2}\zeta_0\rho$. Since $\mathbf{C}\in\mathcal{T}_\beta$, we have bounds $|A_{\mathbf{C}}|\le Cr^{-1}$ on the second fundamental form of $\mathbf{C}$ at distance r from $\{0\}\times\mathbf{R}^m$. So by the regularity theorem, $z\in\operatorname{reg}M$, a contradiction.

Proof of Theorem 4.1. Let $\zeta=\zeta(n,k,\beta)$ and $\eta=\eta(n,k,\zeta,\beta)>0$ be as in Lemma 4.5. Then Lemma 4.5 implies that there is a $\mathbf{C}\in\mathcal{T}$ with $\mathbf{C}=\mathbf{C}_0\times\mathbf{R}^m$, $\Sigma=\mathbf{C}_0\cap S^{\ell+k-1}$ smooth compact, and a $u\in C^3(\mathbf{C}\cap B_{15\rho/16}(0)\backslash\{(x,y):|x|\le\rho/16\};\mathbf{C}^\perp)$ with

$$M\cap B_{15\rho/16}(0)\backslash\{(x,y)\,:\,|x|\le\rho/16\}=\operatorname{graph}_{\mathbf{C}}u$$

which

$$\text{(1)}\qquad \sup_{\mathbf{C}\cap B_{7\rho/8}(0)\backslash\{(x,y):|x|\le\rho/16\}}\sum_{j=0}^{3}\rho^{j-1}|D^ju(r,y)|_{C^3}\le\zeta,$$

and

$$\text{(2)}\qquad \begin{aligned}&\sup_{B^+_{3\rho/4}\backslash\{(r,y):r<\rho/4\}}\big||M(r,y)|-r^{\ell-1}|\Sigma|\big|\\&\le C\left(\rho^{-n}\int_{B^M_\rho\backslash\{(x,y):|x|<\rho/8\}}r^2(\nu_r^2+\nu_y^2)\right)^{1/(2-\alpha)}\end{aligned}$$

Notice that by (1) and the estimates 2.12 we then have

$$\text{(3)}\qquad \sup_{\mathbf{C}\cap B_{3\rho/4}(0)\backslash\{(x,y):|x|<\rho/16\}}\sum_{j=0}^{3}\rho^j|D^ju|\le C\beta.$$

For each $y\in\mathbf{R}^m$ with $|y|<\rho/2$ we let

$$\text{(5)}\qquad \begin{aligned}\sigma_y=\sup(\{0\}\cup\{\sigma\in(0,\rho/2]\,:\,\\ \sigma^{-n}\int_{B^M_\sigma(0,y)\backslash\{(x,y):|x|<\sigma/8\}}r^2(\nu_r^2+\nu_y^2)\ge\eta\}).\end{aligned}$$

Since $\rho^{-n}\int_{M\cap B_\rho(0)} r^2(\nu_r^2+\nu_y^2) < \eta$, automatically

$$\sup \sigma_y \leq 10^{-2}\rho \tag{6}$$

if η is sufficiently small (which we subsequently assume). By the "five times" covering lemma (see e.g. [8] or [25]) we can find a countable pairwise-disjoint collection $\{B_{4\sigma_{y_j}}(0,y_j)\}$ such that

$$\cup_{|y|\leq\rho/2,\,\sigma_y>0}\overline{B}_{4\sigma_y}(0,y) \subset \cup_j \overline{B}_{20\sigma_{y_j}}(0,y_j). \tag{7}$$

In particular, that (by definition of σ_y we have

$$\sigma^{-n}\int_{B^M_\sigma(0,y)\setminus\{(x,y)\,:\,|x|<\sigma/8\}} r^2(\nu_r^2+\nu_y^2) < \eta$$

for each $\sigma \in (\sigma_y, \rho/2]$, and so for exactly the same reasoning (involving the first part of Lemma 4.5 and 2.12) which we used to conclude (1), (3) above, and keeping in mind that $\sigma^{-n}|M\cap B_\sigma(0,y)| \leq \beta$ by the monotonicity 1.9′, by taking a smaller $\eta = \eta(n,k,\beta,\zeta)$ if necessary, we can deduce that

$$\sup_{B^{\mathbf{C}}_{7\sigma/8}(0,y)\setminus\{(x,y)\,:\,|x|\leq\sigma/16\}} \sum_{j=0}^{3}\sigma^{j-1}|D^j u| \leq C\beta$$

for any $\sigma \in (\sigma_y, \rho/2]$ and $|y| \leq \rho/2$. Hence by Lemma 4.4, for each $y_0 \in \mathbf{R}^m$ with $|y_0| \leq \rho/2$, and for all $\sigma \in [\sigma_{y_0}, \rho/2]$, we obtain

$$\begin{aligned}\sup_{B^+_{3\sigma/4}(y_0)\setminus\{(r,y)\,:\,r<\sigma/8\}} & r\left|\nabla_{r,y}(r^{1-\ell}|M(r,y)|)\right| \\ &\leq C\sigma^{-n}\int_{B^M_\sigma(0,y_0)\setminus\{(x,y)\,:\,|x|<\sigma/16\}} r^2(\nu_r^2+\nu_y^2).\end{aligned} \tag{9}$$

We now want to define a Whitney-type cover for $B^+_{\rho/2}(0)$, as follows. For $j \geq 2$ let $B_{\rho/2^{j+2}}(0,z_{j,k})$, $k = 1,\dots,Q_j$ be a maximal pairwise disjoint collection of balls with centers $(0,z_{j,k}) \in B_{\rho/2}(0)\cap\{0\}\times\mathbf{R}^m$. Then for $j \geq 2$:

$$\cup_{k=1}^{Q_j} B_{\rho/2^j}(0,z_{j,k}) \supset B_{\rho/2}(0)\cap\{(x,y) : |x| < \rho/2^{j+1}\} \tag{10}$$

for any point $(x,y) \in B_{\rho/2}(0)$, and

$$\#\{k : (x,y) \in B_{\rho/2^{j-5}}(0,z_{j,k})\} \leq C, \quad C = C(n), \tag{11}$$

for any $(x,y) \in \mathbf{R}^n$, where $\#A$ denotes the number of elements in the set A. Next let $\Omega_{1,1} = B^+_{\rho/2}(0)\setminus\{(r,y) : r < \rho/8\}$, $\widehat{\Omega}_{1,1} = B^+_\rho(0)\setminus\{(r,y) : r < \rho/16\}$, and $Q_1 = 1$, and define for $j \geq 2$ and $k = 1,\dots,Q_j$

$$\Omega_{j,k} = B^+_{\rho/2^j}(z_{j,k})\setminus\{(r,y) : r < \rho/2^{j+2}\}, \tag{12}$$

and

$$\widehat{\Omega}_{j,k} = B^+_{\rho/2^{j-1}}(z_{j,k})\backslash\{(r,y) : r < \rho/2^{j+3}\}. \tag{13}$$

Notice that all points $(r,y) \in \widehat{\Omega}_{j,k}$ satisfy $\rho/2^{j+3} \leq r < \rho/2^{j-1}$, and in particular

$$\widehat{\Omega}_{j,k} \cap \widehat{\Omega}_{i,\ell} = \emptyset, \quad |i-j| \geq 4,$$

so (11) it follows from that

$$\forall (r,y) \in B^+_\rho(0), \ \# \{(j,k) : (r,y) \in \widehat{\Omega}_{j,k}\} \leq C, \quad C = C(n). \tag{14}$$

Also, by (10),

$$\begin{aligned}\cup_{j=1}^\infty \cup_{k=1}^{Q_j} \Omega_{j,k} \supset B^+_{\rho/2}(0) \cap (\cup_{j=2}^\infty \{(x,y) : 2^{-j-2}\rho \leq r < 2^{-j-1}\rho\} \\ \cup \{(r,y) : r \geq \tfrac{1}{8}\}) \supset B^+_{\rho/2}(0).\end{aligned} \tag{15}$$

Now, by (6), $\Omega_{1,1}$ intersects no $B_{\sigma_y}(y)$, while for each (j,k) such that $\Omega_{j,k}$ does not intersect $B_{\sigma_{z_{j,k}}}(z_{j,k})$ we must have $\rho/2^{j+2} \geq \sigma_{z_{j,k}}$. Thus, in any case, if $\Omega_{j,k}$ does not intersect $B_{\sigma_{z_{j,k}}}(z_{j,k})$ we can apply (9) with $\sigma = \rho/2^{j-1}$, $y = z_{j,k}$ (so $y = 0$ in case $j = k = 1$), to deduce

$$\begin{aligned}\int_{\Omega_{j,k}} r \left|\nabla_{r,y}(r^{1-\ell}|M(r,y)|)\right| r^{\ell-1} dr dy \\ \leq C \int_{\widehat{\Omega}_{j,k}} \int_{S^{\ell-1}} r^2(\nu_r^2 + \nu_y^2)\, d\omega\, r^{\ell-1} dr dy.\end{aligned} \tag{16}$$

On the other hand if $\Omega_{j,k}$ does intersect $B^+_{\sigma_{z_{j,k}}}(z_{j,k})$, then $j \geq 2$ (by (6)) and $\sigma_{j,k} \geq 2^{-j-2}\rho$, so $\Omega_{j,k} \subset B^+_{4\sigma_{j,k}}(z_{j,k}) \subset \cup_i \overline{B}^+_{20\sigma_i}(y_i)$. Hence by summing in (16) and using (14), (15), we conclude that

$$\begin{aligned}\int_{B^+_{\rho/2}(0)\backslash(\cup_j B^+_{20\rho_j}(y_j))} r \left|\nabla_{r,y}(r^{1-\ell}|M(r,y)|)\right| r^{\ell-1} dr dy \\ \leq C \int_{B^M_\rho(0)} r^2(\nu_r^2 + \nu_y^2)\, dx dy.\end{aligned} \tag{17}$$

Notice also that using the monotonicity 1.7′ and the definition (5) of σ_y, we have that for each j

$$\sigma_{y_j}^{-n-2} \int_{B^M_{40\sigma_{y_j}}(0,y_j)} r^2 \leq C \leq C\eta^{-1} \sigma_{y_j}^{-n-2} \int_{B^M_{\sigma_{y_j}}(0,y_j)\backslash\{(x,y) : |x| < \sigma_j/8\}} r^2(\nu_r^2 + \nu_y^2).$$

Hence by summing on j, and using the disjointness of the $B_{\sigma_{y_j}}(0,y_j)$, we deduce that

$$\sum_j (\sigma_{y_j}^{n+2} + \int_{B^M_{40\sigma_{y_j}}(0,y_j)} r^2) \leq C \int_{B^M_\rho(0)} r^2(\nu_r^2 + \nu_y^2). \tag{18}$$

Now we want to use the collection $\{B_{40\sigma_{y_j}}(0, y_j)\}$ to construct a cut-off function. For each j, let $\zeta_j : (0, \infty) \times \mathbf{R}^m \to [0, 1]$ be a C^∞ function with $\zeta_j(r, y) \equiv 1$ outside $B^+_{40\sigma_{y_j}}(y_j)$, $\zeta_j(r, y) \equiv 0$ in $B^+_{20\sigma_{y_j}}(y_j)$ and with

$$\sup_{B^+_\rho} |\nabla \zeta_j| \le C/\sigma_{y_j}. \tag{19}$$

Now evidently, since the $\{B_{4\sigma_j}(y_j)\}$ are pairwise disjoint, at most a finite subcollection of the $B_{40\sigma_{y_j}}(0, y_j)$ can intersect a given compact subset of $\mathbf{R}^n \setminus (\{0\} \times \mathbf{R}^m)$, so we can define a smooth function $\zeta : (0, \infty) \times \mathbf{R}^m \to [0, 1]$ by

$$\zeta = \Pi_j \zeta_j.$$

By construction $\zeta \equiv 0$ on $\cup_j B^+_{20\sigma_{y_j}}(y_j) \supset \cup_{|y| \le \rho/2,\, \sigma_y > 0} B^+_{4\sigma_y}(y)$. In particular $\zeta(r_0, y_0) > 0 \Rightarrow r_0 > \sigma_{y_0}$ and hence

$$r_0^{-n-2} \int_{B^M_{r_0}(0, y_0) \setminus \{(x,y) \,:\, |x| < r_0/8\}} r^2(\nu_r^2 + \nu_y^2) \le \eta,$$

which (since $\frac{4}{3} \cdot \frac{7}{8} = \frac{7}{6}$) guarantees by Lemma 4.5 and the estimates 2.12 that u is smooth on each of the subsets $B_{7r_0/6}(0, y_0) \setminus \{(x, y) : |x| < r_0/2\}$, and hence in particular the function $r^{1-\ell}|M(r, y)| - |\Sigma|$ is smooth in a neighbourhood of (r_0, y_0). Thus

$$f(r, y) \equiv \zeta(r, y)(r^{1-\ell}|M(r, y)| - |\Sigma|)$$

is a smooth function of $(r, y) \in B^+_\rho(0)$.

Next we note that since f is smooth on $\{(r, y) : r \in (0, \rho/2],\, |y| < \rho/2\}$, integrating by parts with respect to the r-variable gives

$$\begin{aligned} \int_{B^+_{\rho/2}(0)} |f| r^{\ell-1}\, dr dy &\le \int_{|y| < \rho/2,\, r < \rho/2} |f| r^{\ell-1}\, dr dy \\ &\le C \int_{|y| < \rho/2,\, \rho/4 < r < \rho/2} |f| r^{\ell-1}\, dr dy \\ &\quad + \ell^{-1} \int_{|y| < \rho/2,\, r < \rho/2} \left|\frac{\partial f}{\partial r}\right| r^\ell\, dr dy. \end{aligned} \tag{20}$$

We emphasize that this is valid even if f is not bounded near $r = 0$, because we can first prove (by integration by parts) an inequality as in (20) with $r_\epsilon = \max\{r - \epsilon, 0\}$ in place of r, and then let $\epsilon \downarrow 0$. Since $\zeta \equiv 1$ on $B^+_{\rho/2} \setminus \cup_{j=1}^Q B^+_{40\sigma_{y_j}}(y_j)$ and $D_k\zeta = \sum_i D_k\zeta_i \Pi_{j \ne i}\zeta_j$, we obtain, in view of (17), (19), (6) and the fact that $r < \rho/2$, $|y| < \rho/2 \Rightarrow \sqrt{r^2 + |y|^2} < 3\rho/4$,

$$\begin{aligned} &\int_{B^+_{\rho/2}(0)} \left| r^{1-\ell}|M(r, y)| - |\Sigma| \right| r^{\ell-1}\, dr dy \\ &\qquad \le C \int_{B^+_{3\rho/4}(0) \setminus \{(r,y) \,:\, r < \rho/4\}} \left| r^{1-\ell}|M(r, y)| - |\Sigma| \right| r^{\ell-1}\, dr dy \\ &\qquad\quad + C \int_{B^M_\rho(0)} r^2(\nu_r^2 + \nu_y^2) + C \sum_j \Big(\sigma_{y_j}^{n+2} + \int_{B^M_{40\sigma_{y_j}}(0, y_j)} r^2\Big), \end{aligned}$$

which proves the theorem, in consequence of (2) and (18).

5 L^2 estimates. Here we are going to use the area estimates of the previous section together with the monotonicity identities 1.7′, 1.9′ (and some variants of these) to obtain L^2 estimates for u. These will be needed in the next section for proving the decay properties of the deviation function introduced there.

M continues to denote an element of the multiplicity one class $\mathcal{M}$, and we assume that $\overline{B}_2(0) \subset U_M$ and that

$$\text{(5.1)} \qquad 0 \in \operatorname{sing} M, \quad \Theta_M(0) \geq \theta_0, \quad |B_2^M(0)| \leq \beta,$$

where β is a given constant and $\theta_0 \in \{\Theta_{\mathbf{C}}(0) : \mathbf{C} \in \mathcal{T}\}$. Notice that by monotonicity 1.7′ this implies

$$\text{(5.2)} \qquad \rho^{-n}|B_\rho^M(z)| \leq C\beta, \quad \forall \rho \in (0,1], \quad |z| \leq 1.$$

With θ_0 as in 5.1

$$\text{(5.3)} \qquad S_+ = \{z \in \overline{B}_1(0) : \Theta_M(z) \geq \theta_0\}.$$

S_+ will be assumed to satisfy a weak ϵ-approximation property, with $\epsilon \leq \epsilon_0 = \epsilon_0(n,k,\beta) > 0$ to be chosen, like that in 2.16; thus for each $\rho \in (0,1]$ and each $z \in S_+$ we assume that

$$\text{(5.4)} \qquad S_+ \cap B_\rho(z) \subset \textit{ the } (\epsilon\rho)\textit{-neighbourhood of } L_{z,\rho},$$

where $L_{z,\rho}$ is an m-dimensional affine space containing z. We henceforth fix these affine spaces $L_{z,\rho}$. We also here assume that, with $R_z(x,y) = |(x,y) - z|$,

$$\text{(5.5)} \qquad \sup_{z \in S_+} \int_{B_2^M(0)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \leq \epsilon, \qquad \sup_{z \in S_+} \omega_n^{-1}\rho^{-n}|B_\rho^M(z)| \leq \theta_0 + \epsilon,$$

where $\theta_0 \in \{\Theta_{\mathbf{C}}(0) : \mathbf{C} \in \mathcal{T}\}$ is as in 5.1, and $\rho \in (0, \frac{1}{4}]$ is given. (Of course by 1.7′, the latter inequality in 5.5 implies $\sup_{z \in S_+} \Theta_M(z) \leq \theta_0 + \epsilon$.)

Remark. We show in §7 below that for every given $\epsilon > 0$ and $w_0 \in \operatorname{sing} M$ with $\Theta_M(w_0) = \theta_0 \in \{\Theta_{\mathbf{C}}(0) : \mathbf{C} \in \mathcal{T}\}$, there is $\sigma > 0$ (depending on M, w_0, ϵ) such that all of the above conditions are satisfied, by Lemma 2.16 and monotonicity 1.7′–1.10′, with the rescaled surface $\eta_{w_1,\sigma_1}M$ in place of M for any $w_1 \in B_{\sigma/2}(w_0) \cap \{z : \Theta_M(z) \geq \Theta_M(w_0)\}$ and $\sigma_1 \leq \frac{1}{2}\sigma$. These facts are of crucial importance in the eventual applicability of the results of the present section.

We also here suppose that $z_0 \in S_+$, $\rho \in (0, \frac{1}{4}]$, $\gamma \in (0, \frac{1}{2}]$, and that there exist points $z_1, \ldots, z_m$ in $S_+ \cap B_\rho(z_0)$ such that

$$\text{(5.6)} \qquad \begin{aligned} &\{z_j - z_0\}_{j=1,\ldots,m} \textit{ are linearly independent and} \\ &\sum_{j=1}^m ((z_j - z_0)\cdot a)^2 \geq \gamma\rho^2|a|^2 \;\forall\, a \in L, \end{aligned}$$

where L is the m-dimensional linear space spanned by $z_1 - z_0, \dots, z_m - z_0$. Notice that this states that the $z_j - z_0$ are in "uniformly general position", up to the factor γ, in $B_\rho(z_0) \cap L$.

The main result of this section is the following:

5.7 Theorem. *There is $\epsilon_0 = \epsilon_0(n,k,\beta) > 0$ such that if 5.1, 5.4, 5.5, 5.6 hold with $\epsilon = \epsilon_0$, then for all $z \in S_+$*

$$\int_{B^M_{\rho/8}(z_0)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}$$
$$\leq C\rho^{-2}d^{-n}\int_{B^M_\rho(z_0)} \sum_{j=0}^m |((x,y)-z_j)^\perp|^2 + C\left(\frac{\rho^{n+2}}{d^{n+2}}\right)^{1-1/(2-\alpha)} \times$$
$$\times\Big[\int_{B^M_\rho(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|<\frac{\rho}{8}\}} \Big(\frac{1}{\rho^2 d^n}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2 + \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\Big)\Big]^{\frac{1}{(2-\alpha)}},$$

where $\alpha = \alpha(n,k,\beta) \in (0,1)$, $C = C(n,k,\beta,\gamma)$, and $d = \rho + |z - z_0|$.

We shall need the following three lemmas in the proof:

5.8 Lemma. *Suppose $L, z_0, \dots, z_m$ are as in 5.6 (although here we do not need to assume that $z_j \in S_+$). Then for any n-dimensional embedded surface M (we do not need $M \in \mathcal{M}$ here) we have*

$$C^{-1}(r_L^2\nu_{r_L}^2 + \rho^2|\nu_L|^2) \leq \sum_{j=0}^m |((x,y)-z_j)^\perp|^2 \leq C(r_L^2\nu_{r_L}^2 + \rho^2|\nu_L|^2) \text{ in } M,$$

where $C = C(m,n,\gamma)$, $r_L(w) \equiv \operatorname{dist}(w, z_0 + L)$, r_L = radial distance from $z_0 + L = |p_{L^\perp}((x,y) - z_0)|$, and where $|\nu_L|^2 = \|p_{T^\perp_{(x,y)}M} \circ p_L\|^2$, $r_L^2\nu_{r_L}^2 = |p_{T^\perp_{(x,y)}M}(p_{L^\perp}((x,y)-z_0))|^2$. If $\sigma \in (0,\rho]$ and $\zeta_0, \dots, \zeta_m$ are any other points in $B^M_\rho(z_0)$ with $\zeta_1, \dots, \zeta_m \in B_\sigma(\zeta_0)$ and with

$$\sum_{j=1}^m ((\zeta_j - \zeta_0)\cdot a)^2 \geq \gamma\sigma^2|a|^2, \quad a \in L,$$

then

$$C^{-1}(r_L^2\nu_{r_L}^2 + \sigma^2\nu_L^2) - C\sum_{j=0}^m \operatorname{dist}^2(\zeta_j, z_0 + L) \leq \sum_{j=0}^m |((x,y)-\zeta_j)^\perp|^2$$
$$\leq C(r_L^2\nu_{r_L}^2 + \sigma^2\nu_L^2) + C\sum_{j=0}^m \operatorname{dist}^2(\zeta_j, z_0 + L) \text{ on } B^M_\sigma(\zeta_0)$$

for suitable $C = C(n,m,\gamma)$. In particular, on $B^M_\sigma(\zeta_0)$,

$$\sum_{j=0}^m |((x,y)-z_j)^\perp|^2 \leq C(\rho/\sigma)^2\sum_{j=0}^m \big(|((x,y)-\zeta_j)^\perp|^2 + \operatorname{dist}^2(\zeta_j, z_0 + L)\big).$$

Remark. Notice that the inequality $\sum_{j=1}^{m}((z_j - z_0)\cdot a)^2 \geq \gamma\rho^2|a|^2, a \in L$, means that $z_0, \ldots, z_m$ must be in "uniformly general position" in $z_0 + L$ up to the factor γ; likewise the condition $\sum_{j=1}^{m}((\zeta_j - \zeta_0)\cdot a)^2 \geq \gamma\sigma^2|a|^2, a \in L$, requires that the nearest point projections $\zeta_0', \ldots, \zeta_m'$ of the ζ_j onto L should be in such uniformly general position in $B_\sigma(\zeta_0')$.

Proof of Lemma 5.8. By definition

$$(1)\quad (w - z_j)^\perp = (w - w')^\perp + (w' - z_j)^\perp = r_L\nu_{r_L} + p_{T_w^\perp M}(p_L(w - z_j)),$$

so in particular

$$(w - z_j)^\perp - (w - z_0)^\perp = (z_0 - z_j)^\perp,$$

and by the hypothesis we then have that on M

$$(2)\qquad \gamma\rho^2\nu_L^2 \leq C\sum_{j=0}^{m}((w - z_j)^\perp - (w - z_0)^\perp)^2.$$

On the other hand using (1) with $j = 0$ we also have on $B_{2\rho}^M(z_0)$ that

$$(3)\qquad C^{-1}r_L^2\nu_{r_L}^2 \leq |(w - z_0)^\perp|^2 + \rho^2\nu_L^2.$$

Combining (2) and (3) we then have

$$r_L^2\nu_{r_L}^2 + \rho^2\nu_L^2 \leq C\sum_{j=0}^{m}|(w - z_j)^\perp|^2$$

as claimed. Notice that the reverse inequality

$$C^{-1}\sum_{j=0}^{m}|(w - z_j)^\perp|^2 \leq (r_L\nu_{r_L})^2 + \rho^2\nu_L^2$$

follows directly from (1) on $B_{2\rho}^M(z_0)$.

Next notice (Cf. (1) above) that at any point $w \in B_\sigma^M(\zeta_0)$

$$(4)\qquad \begin{aligned}(w - \zeta_j)^\perp &= (w - w')^\perp + (w' - \zeta_j')^\perp + (\zeta_j' - \zeta_j)^\perp \\ &= r_L\nu_{r_L} + (p_L(w - \zeta_j))^\perp + (\zeta_j' - \zeta_j)^\perp.\end{aligned}$$

Taking differences in (4) we see that

$$(\zeta_j - \zeta_0)^\perp = -(\omega - \zeta_j)^\perp + (\omega - \zeta_0)^\perp + (\zeta_j' - \zeta_j)^\perp - (\zeta_0' - \zeta_0)^\perp.$$

Since $|\zeta_j' - \zeta_j| = \operatorname{dist}(\zeta_j, z_0 + L)$, by using the given hypothesis on the ζ_j we then see that on U

$$(5)\qquad \sigma^2\nu_L^2 \leq C\sum_{j=0}^{m}|(w - \zeta_j)^\perp|^2 + C\sum_{j=0}^{m}\operatorname{dist}^2(\zeta_j, z_0 + L).$$

Going back to (4) again we thus also conclude that on $B^M_\sigma(\zeta_0)$

$$r_L^2\nu_{r_L}^2 \le C\sum_{j=0}^m |(w-\zeta_j)^\perp|^2 + C\sum_{j=0}^m \operatorname{dist}^2(\zeta_j, z_0+L),$$

which proves the required upper inequality for $r_L^2 v_{r_L}^2 + \sigma^2|D_L v|^2$.

The reverse inequality follows directly from (4) and the triangle inequality.

The final inequality of the lemma is simply a matter of combining two of the previous inequalities, so this completes the proof of the lemma.

In the proof of Theorem 5.7 we shall want to apply the main area estimate established in Theorem 4.1 of §4, and this requires that we check the hypothesis that M is L^2-sufficiently close to some $\mathbf{C}\in\mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\}\times\mathbf{R}^m$ in the appropriate ball.

5.9 Lemma. *For any given $\zeta > 0$ there is $\epsilon_0 = \epsilon_0(n,k,\beta,\zeta) > 0$ such that if 5.1, 5.4, 5.5, 5.6 hold with $\epsilon\le\epsilon_0$, then*

$$\rho^{-n}\int_{B^M_\rho(z_0)} (r_L^2\nu_{r_L}^2 + \rho^2\nu_L^2) \le C\epsilon,$$

where the notation is as in 5.8, and $u\in C^3((z_0+\mathbf{C})\cap B_{2\rho/3}(z_0)\backslash\{(x,y) : \operatorname{dist}((x,y), z_0+L)\le\rho/16\}; \mathbf{C}^\perp)$,

$$\rho^{-n-2}\int_{B^M_{\rho/2}(z_0)} \operatorname{dist}^2((x,y), z_0+\mathbf{C}) \le \zeta,$$

$$\sup_{(z_0+\mathbf{C})\cap B_{2\rho/3}(z_0)\backslash\{(x,y)\,:\,\operatorname{dist}((x,y),z_0+L)\le\rho/16\}} \sum_{j=0}^3 \rho^{j-1}|D^j u|_{C^3} \le \zeta$$

for some $\mathbf{C}\in\mathcal{T}_{C\beta}$ with $\operatorname{sing}\mathbf{C} = L$, $\Theta_{\mathbf{C}}(0) = \theta_0$. Furthermore there is $\epsilon_0 = \epsilon_0(n,k,\beta) > 0$ such that if 5.1, 5.4, 5.5, 5.6 hold with $\epsilon\le\epsilon_0$, then for all $z\in S_+$

$$\begin{aligned}&\operatorname{dist}^2(z, z_0+L)\\ e &\le C\rho^{-n}\int_{\{(x,y)\in B^M_{3\rho/4}(z_0)\,:\,r_L\ge\rho/4\}} (r_L^2\nu_{r_L}^2 + (\rho+|z-z_0|)^2\nu_L^2 + |((x,y)-z)^\perp|^2)\\ &\le C\epsilon(\rho+|z-z_0|)^2\end{aligned}$$

Remark. It is not assumed that $|z-z_0|$ is small here; z_0, z are unrelated points in S_+.

Proof of Lemma 5.9. Evidently we can assume without loss of generality that L in 5.6 is $\{0\}\times\mathbf{R}^m$. To prove the first inequality, notice that by Lemma 5.8 above we have

$$r_0^2\nu_{r_0}^2 + \rho^2\nu_y^2 \le C\sum_{j=0}^m |((x,y)-z_j)^\perp|^2,$$

where $r_0 = |x - \xi_{z_0}|$, $r_0\nu_{r_0} = p_{T^\perp_{(x,y)}M}(x - \xi_{z_0}, 0)$, $z_0 = (\xi_{z_0}, \eta_{z_0})$. Integrating this inequality over the ball $B_\rho(z_0)$ and noting that 5.5 implies

$$(1) \qquad \rho^{-n-2}\int_{B^M_\rho(z_0)}\sum_{j=0}^{m}|((x,y) - z_j)^\perp|^2 \le C\epsilon,$$

we then have the first inequality as claimed.

In view of the first inequality, the first part of Lemma 4.5 guarantees that the second and third inequalities of the lemma hold for some $\mathbf{C} \in \mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\} \times \mathbf{R}^m$ and

$$(2) \qquad |\mathbf{C} \cap B_1(0)| \le C\beta, \quad \theta_0 - \zeta \le \Theta_{\mathbf{C}}(0) \le \theta_0 + \epsilon,$$

and

$$(3) \qquad \sup_{(z_0+\mathbf{C})\cap B_{2\rho/3}(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|\le\rho/16\}}\sum_{j=0}^{3}\rho^{j-1}|D^j u|_{C^3} \le \zeta.$$

We agree that ϵ and ζ are chosen smaller than the minimum distance between distinct elements of $\{\Theta_{\mathbf{C}}(0) : \mathbf{C} \in \mathcal{T}_\beta\}$. Then (2) gives $\Theta_{\mathbf{C}}(0) = \theta_0$. We next claim that (for ζ small enough in (3)), for any $\xi \in \mathbf{R}^{ltk}$,

$$(4) \qquad |\xi|^2 \le C\rho^{-n}\int_{(x,y)\in M\,:\,|y-\eta_{z_0}|<\rho/2,\,\rho/4<r_0<\rho/2}|(\xi,0)^\perp|^2,$$

where $C = C(n,k,\beta)$ is fixed (independent of ξ, u), provided $\epsilon_0 = \epsilon_0(n,k,\beta) > 0$ is small enough. Indeed otherwise by (2) and (3), after rescaling and translating so that $\rho = 1$ and $z_0 = 0$, we would have a sequence $M_j \in \mathcal{M}$, with $0 \in \operatorname{sing} M_j$, $\mathbf{C}_j \in \mathcal{T}_{C\beta}$ with $\operatorname{sing}\mathbf{C}_j = \mathbf{C}_j^{(0)} \times \mathbf{R}^m$, and points $\xi_j \in S^{l+k-1}$ such that

$$|\mathbf{C}_j \cap B_1(0)| \le \beta, \qquad \sup_{\mathbf{C}_j\cap B_{2/3}(0)\setminus\{(x,y)\,:\,|x|\le 1/16\}}\sum_{i=0}^{3}\rho^i|D^i u_j|_{C^3} \to 0 \text{ as } j \to \infty,$$

and

$$(5) \qquad \xi_j \to \xi \in S^{\ell+k-1}, \quad \int_{(x,y)\in M_j\,:\,|y|<\frac{1}{2},\frac{1}{4}<|x|<\frac{1}{2}}|(\xi_j,0)^\perp|^2 \to 0.$$

Notice we also have

$$(6) \qquad \liminf_{j\to\infty}\Theta_{\mathbf{C}_j}(0) > 1$$

by virtue of 2.1. Using 1.11(b) we can assume that $\mathbf{C}_j \to \mathbf{C}$ locally in the Hausdorff distance sense in $\mathbf{R}^{n+k}$, $M_j \to \mathbf{C}$ in $B_{2/3}(0)\setminus\{(x,y) : |x| \le 1/16\}$ and that $(\xi,0)^\perp \equiv 0$ on $\mathbf{C}$. But this, together with stationarity of $\mathbf{C}$, implies that $\mathbf{C}$ is invariant under translations in the direction of $(\xi, 0)$, which means

$\operatorname{sing}\mathbf{C}$ contains the line through 0 in the direction of $(\xi, 0)$, contradicting the fact that $\operatorname{sing}\mathbf{C} = \{0\} \times \mathbf{R}^m$. (Notice that $\mathbf{C}$ is not a linear subspace because $\Theta_{\mathbf{C}}(0) > 1$ by (6) and upper-semicontinuity 2.3.) Thus (4) is established.

On the other hand we have, using the notation $z_0 = (\xi_{z_0}, \eta_{z_0})$, $z = (\xi_z, \eta_z)$,

$$(\xi_{z_0} - \xi_z, 0)^\perp = ((x,y) - z)^\perp - ((x,y) - z_0)^\perp - (0, y - \eta_z)^\perp,$$

and hence

$$|(\xi_{z_0} - \xi_z, 0)^\perp|^2 \leq 3|((x,y) - z)^\perp|^2 + 3|((x,y) - z_0)^\perp|^2 + 3|(0, y - \eta_z)^\perp|^2.$$

Integrating this identity over M and using (4) with $\xi = \xi_{z_0} - \xi_z$ yield

$$\begin{aligned} &|\xi_z - \xi_{z_0}|^2 \\ &\leq C\rho^{-n} \int_{|y - \eta_{z_0}| < \rho/2,\, \rho/4 < r_0 < \rho/2} (r_0^2 \nu_{r_0}^2 + (\rho + |z - z_0|)^2 \nu_y^2 + |((x,y) - z)^\perp|^2) \\ &\leq C\epsilon(\rho + |z - z_0|)^2 \end{aligned}$$

by 5.5 and Lemma 5.8, as claimed.

The third lemma is as follows:

5.10 Lemma. *For any $\mathbf{C} \in \mathcal{T}$ with $\operatorname{sing}\mathbf{C} = \{0\} \times \mathbf{R}^m$ and any Lipschitz ψ on B_ρ^+ with $\psi(r,y) \equiv 0$ for $r^2 + |y|^2 = \rho^2$, we have the identity*

$$\begin{aligned} \int_{B_\rho^M} (\ell + \nu_y^2)\psi - \ell \int_{\mathbf{C}} \psi = -\Big(\int_{B_\rho^M} r\psi_r - \int_{\mathbf{C}} r\psi_r\Big) \\ +2\int_{B_\rho^M} r\nu_r^2 \psi_r + \int_{B_\rho^M} r\nu_r \cdot \sum_{j=1}^m \nu_{y^j} \psi_{y^j}. \end{aligned}$$

Proof. We begin by recalling the identity 1.3, which is valid for any Lipschitz $\zeta = (\zeta^1, \ldots, \zeta^n) : \overline{B}_\rho \to \mathbf{R}^n$ with $\zeta = 0$ on ∂B_ρ. Taking $\zeta = \psi(r,y)(x,0)$ (where $r = |x|$) in this identity, we thus obtain

$$\int_{B_\rho^M} \sum_{i=1}^n \sum_{j=1}^{\ell+k} p^{ij} \delta_{ij} \psi = -\int_{B_\rho^M} \sum_{i=1}^n \sum_{j=1}^{\ell+k} p^{ij} x^j D_i[\psi(r,y)],$$

where (p^{ij}) is the matrix of the orthogonal projection of $\mathbf{R}^{n+k}$ onto $T_x M$. Since $D_i[\psi(r,y)] = r^{-1} x^i \psi_r$ for $i \leq \ell + k$, $D_i[\psi(r,y)] = D_{y^{i-\ell-k}}\psi$ for $i = \ell + k + 1, \ldots, n + k$, and

$$\sum_{i=1}^{\ell+k} p^{ii} = \sum_{i=1}^{n+k} p^{ii} - \sum_{i=\ell+k+1}^{n+k} p^{ii} = \ell + \sum_{i=\ell+k+1}^{n+k} (1 - p^{ii}) = \ell + \nu_y^2,$$

we have

$$\int_{B_\rho^M} (\ell + \nu_y^2)\psi = -\int_{B_\rho^M} r|\nabla r|^2 \psi_r + \int_{B_\rho^M} \sum_{j=1}^m \nu_r \cdot \nu_{y^j} \psi_{y^j}.$$

On the other hand by direct integration by parts in the r-variable we obtain

$$-\ell \int_{\mathbf{C}\cap B_\rho} \psi = \int_{\mathbf{C}\cap B_\rho} r\psi_r,$$

so by adding this to the previous inequality we conclude the identity claimed in the statement of the lemma.

Proof of Theorem 5.7. By rotating if necessary, we may assume that the subspace L of 5.6 is $\{0\} \times \mathbf{R}^m$. If $z_0 = (\xi_0, \eta_0)$, then $z_j = (\xi_0, \eta_j)$ for $j = 1, \ldots, m$. By the monotonicity inentity 1.10′, for any $\mathbf{C} \in \mathcal{T}$ such that $\Theta_C(0) = \theta_0$ and $\operatorname{sing} \mathbf{C} = \{0\} \times \mathbf{R}^m$, and for any $z \in S_+ \cap B_\rho(z_0)$, we have

$$\int_{B_\rho^M(z)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \le \frac{\rho^{1-n}}{n}\Big(\int_{M\cap \partial B_\rho(z)} |\nabla r|\, d\mathcal{H}^{n-1} - \mathcal{H}^{n-1}(\mathbf{C}\cap \partial B_\rho(z))\Big),$$

where we have used the fact that $\Theta_M(z) \ge \Theta_C(0) = \theta_0 = n^{-1}|\Sigma|$. Let $\psi : \mathbf{R} \to [0,1]$ satisfy $\psi(t) \equiv 0$ for $t \ge \rho$, $\psi(t) \equiv 1$ for $t \le \frac{(1+\theta)}{2}\rho$, $\psi' \le 0$ everywhere, and $|\psi'(t)| \le C(\theta)\rho^{-1}$. Multiplying each side of this inequality by $\psi(\rho)$ and integrating over $[\theta\rho, \rho]$, for any $\theta \in (0,1)$ we get

$$(1) \quad 2\int_{B_{\theta\rho}^M(z)} \frac{|((x,y)-z)^\perp|^2}{{R_z}^{n+2}} \le C\rho^{-n}\Big(\int_{B_\rho^M(z)} \psi(R_z) - \int_{(z+\mathbf{C})\cap B_\rho(z)} \psi(R_z)\Big).$$

On the other hand the identity of Lemma 5.10 above implies (after a translation taking z to 0)

$$\begin{aligned}
&\int_{B_\rho^M(z)} (\ell + \nu_y^2)\psi(R_z) - \ell \int_{(z+\mathbf{C})\cap B_\rho(z)} \psi(R_z) \\
&= \int_{B_\rho^M(z)} r_z^2 R_z^{-1} |\psi'(R_z)|\, |\nabla r_z|^2 - \int_{(z+\mathbf{C})\cap B_\rho(z)} r_z^2 R_z^{-1} |\psi'(R_z)| \\
&\quad + 2\int_{B_\rho^M(z)} r_z^2 R_z^{-1} |\psi'(R_z)| \nu_{r_z}^2 + 2\int_{B_\rho^M(z)} \sum_{j=1}^m (y^j/R_z) r_z \nu_{r_z} \cdot \nu_{y^j} \psi'(R_z),
\end{aligned}$$

where C depends on θ. Replacing ψ by ψ^2 and using the Cauchy-Schwarz inequality we obtain

(2)

$$\begin{aligned}
&\int_{B_\rho^M(z)} \psi(R_z) - \int_{(z+\mathbf{C})\cap B_\rho(z)} \psi(R_z) \\
&\le C\left(\int_{B_\rho^M(z)} r_z^2 R_z^{-1} \psi(R_z) |\psi'(R_z)|\, |\nabla r_z|^2 - \int_{(z+\mathbf{C})\cap B_\rho(z)} r_z^2 R_z^{-1} \psi(R_z) |\psi'(R_z)|\right) \\
&\quad + C\int_{B_\rho^M(z)} (R_z^{-1}\psi(R_z)|\psi'(R_z)| + (\psi'(R_z))^2) r_z^2 \nu_{r_z}^2.
\end{aligned}$$

On the other hand, for any non-negative continuous g on $B_\rho^M(z)$, the coarea formula tells us that

$$\int_{B_\rho^M(z)} gJ = \int_{B_\rho^+(0)} \Big(\int_{M(r,y)} g\, d\mathcal{H}^{\ell-1}\Big)\, dr dy,$$

where $J = \sqrt{\det(d\varphi \circ (d\varphi)^*)}$, $d\varphi : T_x M \to \mathbf{R}^{m+1}$ is the induced linear map of the transformation $\varphi : (x,y) \in M \to (r_z, y - \eta_z) \in (0,\infty) \times \mathbf{R}^m$, and

$$M(r,w) = \{(x,y) \in M : \varphi(x,y) = (r,w)\} \equiv \{(x,y) \in M : r_z = r,\ y - \eta_z = w\}.$$

Notice that then J is given explicitly by

$$J = \sqrt{\det(\nabla^M f_i \cdot \nabla^M f_j)_{i,j=0,1,\dots,m}},$$

where $f_0(x,y) = |x|$ and $f_j(x,y) = y^j$, $j = 1,\dots,m$. But $\nabla^M f_i \cdot \nabla^M f_j = Df_i \cdot Df_j - (Df_i)^\perp \cdot (Df_j)^\perp$, where Df is the full gradient of f on $\mathbf{R}^{n+k}$ and, at the point $(x,y) \in M$, $v^\perp$ means the orthogonal projection of v onto $(T_{(x,y)}M)^\perp$. Thus we deduce that

$$(\nabla^M f_i \cdot \nabla^M f_j)_{i,j=0,1,\dots,m} = I + (e_{ij})_{i,j=0,1,\dots,m},$$

where

$$|e_{ij}| \le C \sum_{j=0}^{m} |(Df_j)^\perp|^2 \equiv C(\nu_{r_z}^2 + \nu_y^2).$$

Hence we conclude that

$$1 \le J + C(\nu_{r_z}^2 + \nu_y^2),$$

so that using the above coarea formula in (2) and combining the resultant inequality with (1) yield

$$\begin{aligned}\int_{B^M_{\theta\rho}(z)} \frac{|((x,y)-z)^\perp|^2}{R_z^{\,n+2}} &\le C\rho^{-n-2} \int_{B^M_\rho(z)} r_z^2(\nu_{r_z}^2 + \nu_y^2) \\ &\quad + C\rho^{-n-2} \int_{B^+_\rho} \left| r^2 (r^{1-\ell} |M_z(r,y)| - |\Sigma|) \right| r^{\ell-1}\, dr\, dy,\end{aligned} \tag{3}$$

where $M_z(r,y) \equiv (M - z) \cap \{(x,y) : |x| = r\}$. Now by 5.5, 5.8 and 5.9 (with 2ρ in place of ρ) we have

$$\begin{aligned}&|\xi_z - \xi_{z_0}|^2 \\ &\quad \le C\rho^{-n} \int_{B_{2\rho}(z_0)\setminus\{(x,y) : |x-\xi_{z_0}| \le \rho/2\}} (r_0^2 \nu_{r_0}^2 + (\rho + |z - z_0|)^2 \nu_y^2 \\ &\qquad\qquad + |((x,y)-z)^\perp|^2) \le C\epsilon(\rho + |z - z_0|)^2.\end{aligned} \tag{4}$$

(Notice that for the present we need this only for the case $z \in S_+ \cap B_\rho(z_0)$, but in fact Lemma 5.9 shows that it is valid for all $z \in S_+$.) Since $r_z^2 \nu_{r_z}^2 = |(x - \xi_z, 0)^\perp|^2 = |(x - \xi_{z_0}, 0)^\perp + (\xi_z - \xi_{z_0}, 0)^\perp|^2 \le 2r_0^2 \nu_{r_0}^2 + 2|\xi_z - \xi_{z_0}|^2$, by (4), 5.2 and the first part of 5.9 obtain

$$
\begin{aligned}
&C^{-1}\int_{B^M_{2\rho}(z)} (r_z^2\nu_{r_z}^2+\rho^2\nu_y^2)\\
&\le \int_{B^M_{2\rho}(z)} (r_0^2\nu_{r_0}^2+\rho^2\nu_y^2)\\
&\quad+\int_{B^M_{2\rho}(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|<\rho\}} (r_0^2\nu_{r_0}^2+\rho^2\nu_y^2+|((x,y)-z)^\perp|^2)\\
(5)\qquad &\le \int_{B^M_{5\rho/2}(z_0)} (r_0^2\nu_{r_0}^2+\rho^2\nu_y^2)\\
&\quad+\int_{B^M_{2\rho}(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|<\rho/2\}} (r_0^2\nu_{r_0}^2+\rho^2\nu_y^2+|((x,y)-z)^\perp|^2)\\
&\le C\epsilon\rho^{n+2},
\end{aligned}
$$

assuming $z\in S_+\cap B_{\rho/2}(z_0)$. In particular with ϵ small enough we can apply the main area estimate 4.1 with 2ρ in place of ρ on the right side of (3), thus obtaining (after selecting $\theta=\frac{5}{6}$)

$$
\begin{aligned}
(6)\qquad \int_{B^M_{5\rho/6}(z)} \frac{|((x,y)-z)^\perp|^2}{R_z^{\,n+2}} &\le C\rho^{-n-2}\int_{B^M_{2\rho}(z)} r_z^2(\nu_{r_z}^2+\nu_y^2)\\
&\quad+C\left(\rho^{-n-2}\int_{B^M_{2\rho}(z)\setminus\{(x,y)\,:\,|x-\xi_z|<\rho\}} r_z^2(\nu_{r_z}^2+\nu_y^2)\right)^{1/(2-\alpha)}.
\end{aligned}
$$

Using (5) again on the right of (6) and also replacing ρ by $3\rho/4$ yield, for all $z\in S_+\cap B_{3\rho/8}(z_0)$,

(7)

$$
\begin{aligned}
&\int_{B^M_{\rho/4}(z_0)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \le C\rho^{-n-2}\int_{B^M_{2\rho}(z_0)} (r_0^2\nu_{r_0}^2+\rho^2\nu_y^2)\\
&+C\left(\rho^{-n-2}\int_{B^M_{2\rho}(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|<\rho/4\}} ((r_0^2\nu_{r_0}^2+\rho^2\nu_y^2)+|((x,y)-z)^\perp|^2)\right)^{\frac{1}{(2-\alpha)}}.
\end{aligned}
$$

Notice that here we have used the fact that $|\xi_z-\xi_{z_0}|\le\frac{\rho}{4}$ for $z\in S_+\cap B_\rho(z_0)$, by (4), and also used the inclusions $B_{3\rho/2}(z)\subset B_{2\rho}(z_0)$, $B_{\rho/4}(z_0)\subset B_{5\rho/8}(z)$ for $z\in B_{3\rho/8}(z_0)$.

Now we want to consider $|z-z_0|\ge 3\rho/8$. Then, with $z=(\xi_z,\eta_z)$, we have

$$
\begin{aligned}
|((x,y)-z)^\perp|^2 &= |(x-\xi_{z_0},0)^\perp+(0,y-\eta_z)^\perp+(\xi_{z_0}-\xi_z,0)^\perp|^2\\
&\le C(r_0^2\nu_{r_0}^2+|y-\eta_z|^2\nu_y^2+|\xi_z-\xi_{z_0}|^2)
\end{aligned}
$$

By integrating this inequality over the ball $B_{\rho/4}(z_0)$ (keeping in mind that we have the bound $\rho^{-n}|B^M_\rho(z_0)| \le C\beta$ by 5.2), and using (4) (with $\rho/2$ in place of ρ), we obtain

$$\begin{aligned}
&\int_{B^M_{\rho/4}(z_0)} |((x,y)-z)^\perp|^2 \\
&\le C\int_{B^M_\rho(z_0)} (r_0^2\nu_{r_0}^2 + d^2\nu_y^2) + C\int_{B^M_\rho(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|\le\rho/4\}} |((x,y)-z)^\perp|^2,
\end{aligned}$$

where $d = \rho + |z - z_0|$. Since $|z - z_0| \ge 3\rho/8$ this implies

$$\begin{aligned}
&\int_{B^M_{\rho/4}(z_0)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \\
&\quad\le Cd^{-n-2}\int_{B^M_\rho(z_0)} (r_0^2\nu_{r_0}^2 + d^2\nu_y^2) \\
&\quad\quad + Cd^{-n-2}\int_{B^M_\rho(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|\le\rho/4\}} |((x,y)-z)^\perp|^2 \\
&\quad\le Cd^{-n-2}\int_{B^M_\rho(z_0)} (r_0^2\nu_{r_0}^2 + d^2\nu_y^2) + C\left(\frac{\rho^n}{d^n}\right)^{1-(1/(2-\alpha))} \\
&\quad\quad\times\left(d^{-n-2}\int_{B^M_\rho(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|\le\rho/4\}} |((x,y)-z)^\perp|^2\right)^{1/(2-\alpha)},
\end{aligned}$$

where we have used the fact that $\int_{B^M_\rho(z_0)} |((x,y)-z)^\perp|^2 \le C\rho^n$ by 5.2. Using this and (7) for the case $|z - z_0| \ge 3\rho/8$ we thus have

$$\begin{aligned}
&\int_{B^M_{\rho/4}(z_0)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \\
&\le Cd^{-n-2}\int_{B^M_{2\rho}(z_0)} (r_0^2\nu_{r_0}^2 + d^2\nu_y^2) + C(\frac{\rho^n}{d^n})^{1-1/(2-\alpha)} \\
&\quad\times\left(\int_{B^M_{2\rho}(z_0)\setminus\{(x,y)\,:\,|x-\xi_{z_0}|\le\rho/4\}} (\frac{r_0^2\nu_{r_0}^2 + d^2\nu_y^2}{d^{n+2}} + \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}})\right)^{\frac{1}{(2-\alpha)}}
\end{aligned} \tag{8}$$

for every $z \in S_+$.

The proof is now completed by means the first conclusion of Lemma 5.8 (with $L = \{0\} \times \mathbf{R}^m$) in each of the integrals on the right side of this inequality (8) and then replacing ρ by $\rho/2$.

6 The deviation function ψ. Here we use the gap measures of §3 in order to construct a certain deviation function ψ, where $\psi(x,y)$ is the mean over $z \in S_+$ (S_+ as in §5) of the quantity $|(x,y)-z|^{-n-2}|((x,y)-z)^\perp|^2$ (which appears on the left of the main inequality 5.7) with respect to a gap measure constructed as in §3, with S_+ in place of S.

We continue to assume the hypotheses 5.1 (hence 5.2) and 5.3, 5.4, 5.5 of §5.
Let $\rho \in (0, \frac{1}{8}]$, $\delta \in (0, \frac{1}{16})$ (smaller than the $\delta_0(m, n)$ of Lemma 3.7), and let S^+_ρ, T^+_ρ, μ^+ correspond to S_ρ, T_ρ, μ of §3 with S_+ in place of S. By definition of T^+_ρ, $\operatorname{dist}(z, z_1 + L_{0,1}) \le C\delta\rho$ for $z_1 \in T_\rho \cap S_+$ and $z \in S_+ \cap B_\rho(z_1)$. Henceforth we assume without loss of generality that $L_{0,1} = \{0\} \times \mathbf{R}^m$, as we did in the proof of 3.7. Than

$$(6.1) \qquad \begin{aligned} &|\xi_z - \xi_{z_1}| \le C\delta\rho, \quad z_1 = (\xi_{z_1}, \eta_{z_1}) \in T_\rho \cap S_+, \\ &z = (\xi_z, \eta_z) \in B_\rho(z_1) \cap S_+, \ \rho \in (0, \tfrac{1}{8}]. \end{aligned}$$

Now define the deviation function ψ by

$$(6.2) \qquad \psi(x, y) = \int_{S_+} \frac{|((x, y) - z)^\perp|^2}{R_z^{n+2}}\bigg|_{(x,y)} d\mu^+(z).$$

Since for given $(x, y) \in \overline{B}_1(0) \setminus \operatorname{sing} M$, the integrand in 6.2 is an analytic function of $z \in S_+$, ψ is certainly well-defined on $\overline{B}_1(0) \setminus \operatorname{sing} M$.

The main result concerning this function is the following:

6.3 Theorem. *Suppose $\beta > 0$. Then there is $\delta_0 = \delta_0(n, k, \beta) > 0$ such that the following holds. If 5.1–5.5 hold, and $8\epsilon \le \delta \le \delta_0$, then for any $\rho \in (0, \frac{1}{16}]$ we have the estimate*

$$\int_{T^+_{\theta\rho}} \psi \le C \left(\int_{T^+_\rho \setminus T^+_{\theta\rho}} \psi \right)^{1/(2-\alpha)},$$

where $\alpha = \alpha(n, k, \beta) \in (0, 1)$ and $\theta = \theta(n, k, \beta) \in (0, \frac{1}{32}]$.

Proof. The proof is based on the L^2 estimates of the previous section. As mentioned above, we assume

$$(1) \qquad L_{0,1} = \{0\} \times \mathbf{R}^m.$$

Take $\rho \in (0, \frac{1}{8}]$. If $T^+_\rho = \emptyset$, then we have nothing further to prove, so assume that $T^+_\rho \ne \emptyset$, and take an arbitrary point $w_0 \in T^+_\rho \cap S_+$. By definition of $T^+_\rho (\subset T^+_{2\rho})$, there is a point $\tilde{w}_0 \in B_\rho(w_0) \cap S_+$ such that

$$(2) \qquad B_{2\rho}(\tilde{w}_0) \cap S_+ \subset \{w : \operatorname{dist}(w, \tilde{w}_0 + \{0\} \times \mathbf{R}^m) < 2\delta\rho\}$$

and

$$(3) \qquad B_{3\delta\rho}(w) \cap S_+ \ne \emptyset \quad \forall\, w \in (\tilde{w}_0 + \{0\} \times \mathbf{R}^m) \cap B_{2\rho}(\tilde{w}_0),$$

So that

$$(4) \qquad B_\rho(w_0) \cap S_+ \subset \text{ the } (6\delta\rho)\text{-neighbourhood of } w_0 + \{0\} \times \mathbf{R}^m,$$

and

$$(5) \qquad B_{6\delta\rho}(w) \cap S_+ \ne \emptyset \quad \forall\, w \in (w_0 + \{0\} \times \mathbf{R}^m) \cap B_\rho(w_0).$$

Also since any $w \in B_\rho(w_0) \cap S_+$ is in $T^+_{2\rho} \cap S_+$, by Lemma 3.7 we know that

$$C^{-1}\sigma^m \le \mu^+(B_\sigma(w) \cap S_+) \le C\sigma^m, \tag{6}$$

$\forall\, \sigma \in [4\delta^{1/2}\rho, \frac{1}{16}]$ and for any $w \in (w_0 + \{0\} \times \mathbf{R}^m) \cap B_\rho(w_0)$, where $C = C(m,n)$. Now let $w_1, \dots, w_m$ be any points in $(w_0 + \{0\} \times \mathbf{R}^m) \cap B_\rho(w_0)$ such that

$$\sum_{j=1}^{m} ((w_j - w_0) \cdot a)^2 \ge \frac{\rho^2}{2}|a|^2 \quad \forall\, a \in \{0\} \times \mathbf{R}^m. \tag{7}$$

Let $\theta \in [8\delta^{1/2}, \frac{1}{64}]$ be arbitrary for the moment. (We choose $\theta = \theta(n,k,\beta)$ below.) In view of (5) and (6), for each $j \in \{0, \dots, m\}$ we can select points $z_j \in B_{\rho/32}(w_j) \cap S_+$ such that

$$\begin{aligned}
&\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|<\theta\rho/8\}} \frac{|((x,y)-z_j)^\perp|^2}{R_{z_j}^{n+2}} \\
&\le C\mu^+(B_{\rho/8}(w_j))^{-1} \\
&\quad \times \int_{B^M_{\rho/8}(w_j)} \Big(\int_{B_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|<\theta\rho/8\}} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \Big)\, d\mu^+(z) \\
&\le C\rho^{-m} \int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|<\theta\rho/8\}} \psi(x,y)\, dx dy.
\end{aligned} \tag{8}$$

Here we have used the general principle that for any Borel set $U \times V \subset B_\rho(w_0) \times S_+$ and any $\Gamma > 0$ we have

$$\begin{aligned}
\int_U \frac{|((x,y)-\zeta)^\perp|^2}{R_\zeta^{n+2}}
&\le \Gamma\mu^+(V)^{-1} \int_V \Big(\int_U \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\, d\mathcal{H}^n(x,y) \Big)\, d\mu^+(z) \\
&= \Gamma\mu^+(V)^{-1} \int_U \Big(\int_V \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\, d\mu^+(z) \Big)\, d\mathcal{H}^n(x,y) \\
&\le \Gamma\mu^+(V)^{-1} \int_U \psi(x,y)\, d\mathcal{H}^n(x,y)
\end{aligned} \tag{9}$$

for all $\zeta \in V$ except for a Borel set $E \subset V$ with $\mu^+(E) \le \Gamma^{-1}\mu^+(V)$. (This implies that if U_1, U_2 are two subsets of $B_\rho(w_0)$, and $\Gamma > 2$, then there exists at least one point $\zeta \in V$ such that we simultaneously have (9) with each of the choices $U = U_1$, $U = U_2$.)

Also, since $|z_j - w_j| \le \rho/32$, by (7) we obtain

$$\sum_{j=0}^{m} ((z_j - z_0) \cdot a)^2 \ge \frac{\rho^2}{8}|a|^2, \quad a \in L, \tag{10}$$

where L is the linear subspace spanned by $z_j - z_0$, $j = 1, \dots, m$. On the other hand, automatically L satisfies

$$\|L - \{0\} \times \mathbf{R}^m\| \le C\delta \tag{11}$$

by virtue of (2), (10) and the fact that $z_0, \dots, z_m \in S_+ \cap B_{2\rho}(w_0)$.

Similarly, for arbitrary given $w \in (w_0 + (\{0\} \times \mathbf{R}^m)) \cap B_\rho(w_0)$, and any set $\zeta_0^0, \dots, \zeta_m^0 \in B_{\theta\rho}(w) \cap (w_0 + \{0\} \times \mathbf{R}^m)$ with

$$\sum_{j=0}^{m}((\zeta_j^0 - \zeta_0^0) \cdot a)^2 \ge \frac{\theta^2\rho^2}{2}|a|^2, \quad a \in \{0\} \times \mathbf{R}^m, \tag{12}$$

we can again use the general principle (9). This time we in fact use (9) with the choices $U = B_{\theta\rho}(w)$ and $U = B_\rho(w_0)\backslash\{(x,y) : |x - \xi_{w_0}| \le \rho/8\}$, in each case taking $V = B_{\theta\rho/4}(\zeta_j^0) \cap S_+$. Then, keeping in mind (5), (6), the fact that $\theta \ge 8\delta^{1/2}$ and the remark immediately following (9), we can select $\zeta_j \in B_{\theta\rho/8}(\zeta_j^0) \cap S_+$ such that for each $j = 0, \dots, m$

$$\begin{aligned} &\int_{B^M_{\theta\rho}(w)} \frac{|((x,y) - \zeta_j)^\perp|^2}{R_{\zeta_j}^{n+2}} \le C(\theta\rho)^{-m} \int_{B^M_{\theta\rho/4}(w)} \psi, \\ &\int_{B^M_\rho(w_0)\backslash\{(x,y)\,:\,|x-\xi_{w_0}|\le\rho/8\}} \frac{|((x,y) - \zeta_j)^\perp|^2}{R_{\zeta_j}^{n+2}} \\ &\qquad\qquad \le C(\theta\rho)^{-m} \int_{B^M_\rho(w_0)\backslash\{(x,y)\,:\,|x-\xi_{w_0}|\le\rho/8\}} \psi, \end{aligned} \tag{13}$$

where $C = C(n,k,\beta)$. (We emphasize that the choice of ζ_j depends on w, but C only depends on n, k, β.) Since $|\zeta_j - \zeta_j^0| \le \theta\rho/8$, from (11) and (12) it also follows that

$$\sum_{j=0}^{m}((\zeta_j - \zeta_0) \cdot a)^2 \ge \frac{\theta^2\rho^2}{8}|a|^2, \quad a \in L. \tag{14}$$

In view of (10) and (14) we can apply Lemma 5.8 in order to conclude that

$$\sum_{j=0}^{m} |((x,y) - z_j)^\perp|^2 \le C\theta^{-2} \sum_{j=0}^{m} |((x,y) - \zeta_j)^\perp|^2 + C\theta^{-2}\,\mathrm{dist}^2(\zeta_j, z_0 + L) \tag{15}$$

on $B^M_{\theta\rho}(w)$, so that

$$\begin{aligned}
&\int_{B^M_{\theta\rho}(w)} \sum_{j=0}^m |((x,y)-\zeta_j)^\perp|^2 \\
&\quad\le C\theta^{-2}\int_{B^M_{\theta\rho}(w)} \sum_{j=0}^m |((x,y)-\zeta_j)^\perp|^2 \\
&\qquad + C\theta^{-2}|B^M_{\theta\rho}(w)|\operatorname{dist}^2(\zeta_j, z_0+L) \\
&\quad\le C\theta^{-2}(\theta\rho)^{n+2}\int_{B^M_{\theta\rho}(w)} \sum_{j=0}^m \frac{|((x,y)-\zeta_j)^\perp|^2}{R_{\zeta_j}^{n+2}} \qquad (16)\\
&\qquad + C\theta^{-2}(\theta\rho)^n\operatorname{dist}^2(\zeta_j, z_0+L) \\
&\quad\le C\theta^{-2}(\theta\rho)^{\ell+2}\int_{B^M_{\theta\rho}(w)}\psi + C\theta^{-2}(\theta\rho)^n\operatorname{dist}^2(\zeta_j, z_0+L)
\end{aligned}$$

by the first inequality in (13).

Hence 5.8 and 5.9 together with (8) and the second inequality in (13) yield (17)

$$\begin{aligned}
&\operatorname{dist}^2(\zeta_j, z_0+L) \\
&\le C\rho^2\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\le\frac{\theta}{8}\}}\sum_{j=0}^m\Big(\frac{|((x,y)-z_j)^\perp|^2}{R_{z_j}^{n+2}}+\frac{|((x,y)-\zeta_j)^\perp|^2}{R_{\zeta_j}^{n+2}}\Big) \\
&\le C\rho^2(\theta\rho)^{-m}\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\le\theta\rho/8\}}\psi.
\end{aligned}$$

By combining (16) and (17) we conclude

$$\begin{aligned}
&\int_{B^M_{\theta\rho}(w)} \sum_{j=0}^m |((x,y)-z_j)^\perp|^2 \\
&\quad\le C\theta^\ell\rho^{\ell+2}\int_{B^M_{\theta\rho}(w)}\psi + C\theta^{\ell-2}\rho^{\ell+2}\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\le\theta\rho/8\}}\psi \qquad (18)
\end{aligned}$$

for each $w\in(w_0+(\{0\}\times\mathbf{R}^m))\cap B_\rho(w_0)$.

The presence of the factor $\theta^\ell\le\theta$ in the first term here is crucial, as we shall see below.

Now we are going to use the main L^2-estimate from Theorem 5.7 with $\rho/2$ in place of ρ. Since $|z_0-w_0|<\rho/32|$ and hence $B_{\rho/16}(z_0)\supset B_{\rho/32}(w_0)$, or $B_{\rho/2}(z_0)\subset B_\rho(w_0)$, we have

$$\int_{B^M_{\rho/32}(w_0)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}$$

(19)
$$\leq C\rho^{-2}d^{-n}\int_{B^M_\rho(w_0)}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2 + C\left(\frac{\rho^n}{d^n}\right)^{1-1/(2-\alpha)}$$

$$\times\Big(\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\leq\frac{\rho}{16}\}}\big(\frac{1}{\rho^2 d^n}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2 + \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\big)\Big)^{\frac{1}{(2-\alpha)}}$$

for each $z \in S_+$.

Then we want to integrate this with respect to the measure μ^+. First notice that, using the notation $\mu^+(A) = \mu^+(A \cap S_+)$,

(20)
$$\begin{aligned}\int_{S_+}\frac{1}{d^n}\,d\mu^+ &\leq \frac{\mu^+(B_\rho(w_0))}{\rho^n} + \sum_{j=1}^\infty \frac{\mu^+(B_{(j+1)\rho}(w_0)) - \mu^+(B_{j\rho}(w_0))}{(j\rho)^n}\\ &\leq C\rho^{-\ell} + C\rho^{-n}\sum_{j=1}^\infty \mu^+(B_{(j+1)\rho}(w_0))(j^{-n}-(j+1)^{-n})\\ &\leq C\rho^{-\ell} + C\rho^{-\ell}\sum_{j=1}^\infty j^{-(\ell+1)} \leq C\rho^{-\ell},\end{aligned}$$

where we have used summation by parts and the fact that $\mu^+(B_{(j+1)\rho}(w_0)) \leq Cj^m\rho^m$ by virtue of Lemma 3.5. Thus integrating in (19) and using the Hölder inequality and (20) we deduce that

(21)
$$\int_{B^M_{\rho/32}(w_0)}\psi \leq C\rho^{-\ell-2}\int_{B^M_\rho(w_0)}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2 + C(\rho^m)^{1-1/(2-\alpha)}$$
$$\times\left(\int_{B_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\leq\rho/16\}}(\rho^{-\ell-2}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2+\psi)\right)^{\frac{1}{(2-\alpha)}}.$$

If we select points $w_1,\dots,w_Q$ (with $Q = Q(m,\theta)$) in $B_{3\rho/4}(w_0)\cap(w_0+(\{0\}\times\mathbf{R}^m))$ such that $\{B_{\theta\rho/16}(w_j)\}$ are pairwise disjoint and $\{B_{\theta\rho/4}(w_j)\}$ cover the $\frac{\theta\rho}{8}$-neighbourhood of $B_{3\rho/4}(w_0)\cap(w_0+(\{0\}\times\mathbf{R}^m))$, then (21) implies

(22)
$$\int_{B^M_{\rho/32}(w_0)}\psi \leq C\rho^{-\ell-2}\sum_{i=1}^Q\int_{B^M_{\theta\rho/4}(w_i)}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2$$
$$+C\rho^{-\ell-2}\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\leq\frac{\theta\rho}{8}\}}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2 + C(\rho^m)^{1-\frac{1}{(2-\alpha)}}$$
$$\times\left(\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\leq\frac{\theta\rho}{8}\}}\rho^{-\ell-2}\sum_{j=0}^m |((x,y)-z_j)^\perp|^2+\psi\right)^{1/(2-\alpha)}.$$

Now we use (18) with w_i in place of w, estimating the terms $\int_{B^M_{\theta\rho/4}(w_i)} \sum_{j=0}^m |((x,y)-z_j)^\perp|^2$ on the right. At the same time we can use (8) in the remaining terms on the right. Thus from (22) it follows that

$$\begin{aligned}\int_{B^M_{\rho/32}(w_0)} \psi \le C\theta \sum_{j=1}^{Q} \int_{B^M_{\theta\rho/4}(w_j)} \psi + C_1 \int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\le \frac{\theta\rho}{8}\}} \psi \\ +C_1(\rho^m)^{1-\frac{1}{(2-\alpha)}} \left(\int_{B^M_\rho(w_0)\setminus\{(x,y)\,:\,|x-\xi_{w_0}|\le \frac{\theta\rho}{8}\}} \psi\right)^{\frac{1}{(2-\alpha)}},\end{aligned} \tag{23}$$

where $C = C(n,k,\beta)$ is independent of θ and $C_1 = C_1(\theta,n,k,\beta)$. On the other hand, we have $B_\rho(w_0)\cap S_+ \subset \{(x,y) : |x-\xi_{z_0}| < 4\delta\rho\}$ by (4), and $\rho^m \le C\mu^+(B_\rho(w_0))$ by Lemma 3.7, where we continue to use the convention $\mu^+(A) = \mu^+(A\cap S_+)$. Hence, assuming $4\delta \le \theta/16$, we see that (23) implies

$$\begin{aligned}\int_{B^M_{\rho/32}(w_0)} \psi \le C\theta \int_{B^M_\rho(w_0)} \psi + C_1 \int_{B^M_\rho(w_0)\setminus S^+_{\theta\rho/16}} \psi \\ +C_1(\mu^+(B^M_\rho(w_0)))^{1-\frac{1}{(2-\alpha)}} \left(\int_{B^M_\rho(w_0)\setminus S^+_{\theta\rho/16}} \psi\right)^{\frac{1}{(2-\alpha)}},\end{aligned} \tag{24}$$

where $S^+_\sigma = \{(x,y) : \operatorname{dist}((x,y),S_+) < \sigma\}$. Notice that this was all valid starting with an arbitrary $w_0 \in T_\rho \cap S_+$. Now choose a maximal pairwise-disjoint collection $\{B_{\rho/128}(p_k)\}_{k=1,\dots,P}$ with $p_k \in S_+\cap T^+_{\rho/4}$. Then $\cup B_{\rho/32}(p_k)$ covers all of the $\frac{\rho}{64}$ neighbourhood of $S_+\cap T^+_{\rho/4}$. By Remark 3.4(2)(a) of §3 we have also that $\cup B_\rho(p_k)$ is contained in $T^+_{2\rho}$. Since any point of $T^+_{2\rho}$ lies in at most $C(n)$ of the balls $B_\rho(p_k)$, replacing w_0 by p_k in (24) and summing over k yield

$$\begin{aligned}\int_{T^+_{\theta\rho}} \psi &\le C\theta \int_{T^+_{2\rho}} \psi + C\int_{T^+_{2\rho}\setminus S^+_{\theta\rho/16}} \psi \\ &\quad +C_1(\mu^+(T^+_{2\rho}))^{1-1/(2-\alpha)} \left(\int_{T^+_{2\rho}\setminus S^+_{\theta\rho/16}} \psi\right)^{1/(2-\alpha)} \\ &\le C\theta \int_{T^+_{\theta\rho}} \psi + C\theta \int_{T^+_{2\rho}\setminus T^+_{\theta\rho}} \psi + C\int_{T^+_{2\rho}\setminus T^+_{\theta\rho/16}} \psi \\ &\quad +C_1(\mu^+(T^+_{2\rho}))^{1-1/(2-\alpha)} \left(\int_{T^+_{2\rho}\setminus S^+_{\theta\rho/16}} \psi\right)^{1/(2-\alpha)} \\ &\le C\theta \int_{T^+_{\theta\rho}} \psi + C\int_{T^+_{2\rho}\setminus T^+_{\theta\rho/16}} \psi \\ &\quad +C_1(\mu^+(T^+_{2\rho}))^{1-1/(2-\alpha)} \left(\int_{T^+_{2\rho}\setminus S^+_{\theta\rho/16}} \psi\right)^{1/(2-\alpha)},\end{aligned}$$

so that, in consequence of $\mu^+(S_+) = 1$ and $S_\sigma^+ \supset T_\sigma^+$,

$$\int_{T^+_{\theta\rho/16}} \psi \le C\left(\int_{T^+_{2\rho}\setminus T^+_{\theta\rho/16}} \psi\right)^{1/(2-\alpha)},$$

provided $\theta = \theta(n, k, \beta) \in [4\delta^{1/2}, \frac{1}{64}]$ is chosen to satisfy $C\theta \le \frac{1}{2}$. By changing a notation (taking $\theta/16$ to 2θ) and replacing ρ by $\rho/2$, we finally obtain the required inequality.

7 Proof of Theorem 4.

Let $\beta > 1$ and let $\theta_0 \in \{\Theta_{\mathbf{C}}(0) : \mathbf{C} \in \mathcal{T}_\beta\}$ be arbitrary, and suppose

$$w_0 \in \operatorname{sing} M \text{ with } \Theta_M(w_0) = \theta_0. \tag{7.1}$$

Recall that, by the monotonicity identity, for each $\epsilon \in (0,1)$ there exists $\sigma_0 = \sigma_0(\epsilon, u, w_0) > 0$ such that

$$\Theta_M(w_0) \le \sigma^{-n}|B^M_\sigma(w_0)| \le \Theta_M(w_0) + \epsilon, \quad \sigma \in (0, \sigma_0]. \tag{7.2}$$

Also, by monotonicity 1.9′ we have the identity

$$\omega_n^{-1}\int_{B^M_\rho(z)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} = \omega_n^{-1}\sigma^{-n}|B^M_\sigma(z)| - \Theta_M(z) \tag{7.3}$$

for each z, r such that $\overline{B}_\rho(z) \subset U_M$. Since $B_\sigma(z) \subset B_{(1+\epsilon)\sigma}(w_0)$ for any $z \in B^M_{\epsilon\sigma}(w_0)$, from 7.2 we deduce that

$$\begin{aligned}\omega_n^{-1}\int_{B^M_\sigma(z)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} &= \omega_n^{-1}\sigma^{-n}|B^M_\sigma(z)| - \Theta_M(z)\\ &\le \omega_n^{-1}(1+\epsilon)^n((1+\epsilon)\rho)^{-n}|B^M_{(1+\epsilon)\rho}(w_0)| - \Theta_M(z)\\ &\le C(n)\beta\epsilon,\end{aligned}$$

$z \in B^M_{\epsilon\sigma}(w_0)$, $\sigma \le \sigma_0/2$, provided that $\Theta_M(z) \ge \Theta_M(w_0)$ and that $\sigma_0 = \sigma_0(M, w_0, \epsilon) > 0$ is sufficiently small. Let

$$S_+ = \{z \in \overline{B}_{\epsilon\sigma_0/2}(w_0) : \Theta_M(z) \ge \theta_0\},$$

take $w_1 \in S_+ \cap \overline{B}_{\epsilon\sigma_0/4}(w_0)$, $\sigma_1 \in (0, \epsilon\sigma_0/4]$ and define

$$\widetilde{M} = \eta_{w_1,\sigma_1} M, \tag{7.4}$$

where $\eta_{w_1,\sigma_1}(x,y) \equiv \sigma_1^{-1}((x,y) - w_1)$. Then the above inequality gives

$$\omega_n^{-1}\int_{B^{\widetilde{M}}_\rho(z)} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}} \le \omega_n^{-1}\rho^{-n}|B^{\widetilde{M}}_\rho(z)| - \Theta_{\widetilde{M}}(z) \le C\epsilon, \quad z \in S_+(w_1,\sigma_1), \quad \rho \in (0, \tfrac{1}{4}], \tag{7.5}$$

where

$$(7.6)\qquad 0 \in S_+(w_1, \sigma_1) \equiv \{z \in \overline{B}_1(0) : \Theta_{\widetilde{M}}(z) \geq \theta_0\} = \overline{B}_1(0) \cap \eta_{w_1,\sigma_1} S_+.$$

Notice that $S_+(w_1, \sigma_1)$ corresponds exactly to the S_+ of §§5, 6 with $\widetilde{M}$ in place of M. Also, recall that by Lemma 2.16, we can, and we shall, assume that $\sigma_0 = \sigma_0(M, w_0, \epsilon)$ is chosen small enough so that S_+ has the ϵ-approximation property of 2.16 and hence $S_+(w_1, \sigma_1)$ does also. Thus (Cf. 5.4)

$$(7.7)\qquad S_+(w_1, \sigma_1) \cap B_\sigma(z) \subset \textit{the } (\epsilon\sigma)\textit{-neighbourhood of } L_{z,\sigma},$$

for each $z \in S_+(w_1, \sigma_1)$ and each $\sigma \in (0, 1]$, where $L_{z,\sigma}$ is an m-dimensional affine space containing z. We fix these affine spaces in the sequel. Without loss of generality we assume

$$(7.8)\qquad L_{0,1} = \{0\} \times \mathbf{R}^m.$$

We emphasize that 7.5 and 7.7 hold automatically if $\sigma_0 = \sigma_0(\epsilon, u, w_0)$ is chosen sufficiently small. We henceforth assume $\sigma_0(\epsilon, u, w_0)$ has been so chosen, and we continue to take $\widetilde{M}$ as in 7.4. Notice also that by 7.4 (choosing a new ϵ if necessary) 5.1, 5.3, 5.5 all hold with $S_+(w_1, \sigma_1)$ in place of S_+ and with $\theta_0 = \Theta_M(w_0)$. Thus we can apply the results of §5, §6 with $\widetilde{M}$ in place of M, and with $S_+ = S_+(w_1, \sigma_1)$, $\theta_0 = \Theta_M(w_0)$.

Before we begin, we need to establish the following lemma, which is a simple inequality for real numbers:

7.9 Lemma. *If* $0 < a < b \leq 1$, $\alpha \in (0, 1)$, $\beta > 0$ *and* $a^{2-\alpha} \leq \beta(b - a)$, *then*

$$a^{-1+\alpha/2} - b^{-1+\alpha/2} \geq Ca^{-\alpha/2}, \quad C = C(\beta, \alpha) > 0.$$

Proof. In case $b/a > 2$ we have trivially that

$$a^{-1+\alpha/2} - b^{-1+\alpha/2} \geq Ca^{-1+\alpha/2} \geq Ca^{-\alpha/2},$$

so the required inequality holds in this case. In case $b/a \leq 2$ we have

$$\begin{aligned} a^{-1+\alpha/2} - b^{-1+\alpha/2} &= (1 - \alpha/2)c^{-2+\alpha/2}(b - a) && \text{for some } c \in (a, b) \\ &\geq \frac{1 - \alpha/2}{4} a^{-\alpha/2} \frac{b - a}{a^{2-\alpha}} && \text{since } a \geq b/2 \\ &\geq \frac{\beta(1 - \alpha/2)}{4} a^{-\alpha/2} && \text{since } a^{2-\alpha} \leq \beta(b - a), \end{aligned}$$

so again the required inequality is satisfied, and the lemma is proved.

Proof of Theorem 4.

Let T_ρ^+, μ^+ (corresponding to given δ with $\epsilon < \delta/8$, and with $\widetilde{M}$ as in 7.4 in place of M) be as in §6. $\delta \leq \delta_0(n, k, \beta)$ and $\epsilon < \delta/8$ will be chosen later.

Now, with $\widetilde{M}$ as in 7.4, by virtue of 7.3, 7.5, we can apply all the results of §6 to $\widetilde{M}$, and hence

$$(1)\qquad \int_{T_{\theta\rho}^+} \psi \leq C \left(\int_{T_\rho^+ \setminus T_{\theta\rho}^+} \psi \right)^{1/(2-\alpha)}$$

with ψ the deviation function of §6 with $\widetilde{M}$ in place of M, where $\theta = \theta(n, k, \beta) > 0$, and $\alpha = \alpha(n, k, \beta) \in (0, 1)$.

In view of Lemma 7.9 we can use (1) to get

$$\left(\int_{T^+_{\theta\rho}} \psi\right)^{-1+\alpha/2} - \left(\int_{T^+_{\rho}} \psi\right)^{-1+\alpha/2} \geq C I_0^{-\alpha/2}, \tag{2}$$

where $I_0 = \int_{T^+_{\frac{1}{4}}} \psi$. Then starting with $\rho = \frac{1}{4}$ we can iterate the inequality (2) in order to obtain

$$\left(\int_{T^+_{\theta^j/4}} \psi\right)^{-1+\alpha/2} \geq C j I_0^{-\alpha/2}, \quad j = 1, 2, \dots,$$

and hence

$$\int_{T^+_{\theta^j/4}} \psi \leq C j^{-1-2\gamma} I_0^{2\gamma}, \quad j = 1, 2, \dots, \tag{3}$$

where $2\gamma = \alpha/(2-\alpha) > 0$. Since $(j+1)^{1+\gamma} - j^{1+\gamma} \geq C j^{\gamma}$, this implies

$$\sum_{j=0}^{\infty} ((j+1)^{1+\gamma} - j^{1+\gamma}) \int_{T^+_{\theta^j/4}} \psi \leq C I_0^{2\gamma} \sum_{j=1}^{\infty} j^{-1-\gamma} \leq C I_0^{2\gamma}.$$

Using summation by parts we obtain that

$$\sum_{j=1}^{\infty} j^{1+\gamma} \int_{T^+_{\theta^{j-1}/4} \setminus T^+_{\theta^j/4}} \psi \leq C I_0^{2\gamma},$$

so that

$$\int_{T^+_{\frac{1}{4}}} |\log d|^{1+\gamma} \psi \leq C I_0^{2\gamma}, \tag{4}$$

where d is defined on $T^+_{\frac{1}{4}}$ by

$$d(x, y) = \begin{cases} 2^{-k} & \text{if } (x, y) \in T^+_{2^{-k}} \setminus T^+_{2^{-k-1}}, \ k \geq 2, \\ 0 & \text{if } (x, y) \in T^+_0. \end{cases} \tag{5}$$

Now for $z \in S_+ \cap T^+_{\frac{1}{4}}$ and $(x, y) \in T^+_{\frac{1}{4}}$ we claim that

$$d(x, y) \leq 4 R_z(x, y), \quad (x, y) \in T^+_{\frac{1}{4}} \setminus B_{\frac{d(z)}{2}}(z), \tag{6}$$

where $R_z(x, y) = |(x, y) - z|$. Here we include $z \in T^+_0$, in which case $d(z) = 0$ so (6) says $d(x, y) \leq 4R_z(x, y)$, $\forall\, (x, y) \in T^+_{\frac{1}{4}}$. To prove this we can of course assume $d(x,y) > 0$, so take any $w = (x,y) \in T^+_{2^{-k}} \backslash T^+_{2^{-k-1}}$ for some $k \geq 2$, and consider cases as follows:

Case (a): $z \in T^+_{2^{-q}}$ for some $q \geq k+2$. (If $z \in T^+_0$, then this case will be applicable $\forall q \geq k+2$.) Then by Remark 3.4(2)(d) of §3 we have $|w - z| \geq 2^{-k-2} = 2^{-k}/4 = d(w)/4$.

Case (b): $z \in T^+_{2^{-q}} \backslash T^+_{2^{-q-1}}$ with $q \leq k+1$. In this case, if we assume that $w \notin B_{\frac{d(z)}{2}}(z)$, then (keeping in mind that $z \in S_+$ and $d(z) = 2^{-q}$ in case $z \in T^+_{2^{-q}} \backslash T^+_{2^{-q-1}}$), we have $|w - z| \geq 2^{-q-1} \geq 2^{-k-2} = d(w)/4$.

Thus (6) is always satisfied as claimed. Now inequality (4) states that

$$\int_{T^+_{\frac{1}{4}}} |\log d|^{1+\gamma} \int_{S_+} \frac{|((x,y) - z)^\perp|^2}{R_z^{n+2}} \, d\mu^+(z) \, dxdy \leq C I_0^{2\gamma}, \tag{7}$$

so that by interchanging the order of integration we deduce that

$$\int_{T^+_{\frac{1}{4}}} |\log d|^{1+\gamma} \frac{|((x,y) - z)^\perp|^2}{R_z^{n+2}} \, dxdy \leq I_0^\gamma, \tag{8}$$

for all $z \in S_+$ with the exception of a set of μ^+-measure $\leq C I_0^\gamma$. (We must keep in mind here that there will in general be lots of points $z \in S_+$ which are not in the support of μ^+, and these have μ^+-measure zero, so in particular (8) need not hold for them.)

In view of (6), (8) implies

$$\int_{T^+_{\frac{1}{4}} \backslash B_{\frac{d(z)}{2}}(z)} |\log R_z|^{1+\gamma} \frac{|((x,y) - z)^\perp|^2}{R_z^{n+2}} \, dxdy \lesssim I_0^\gamma, \tag{9}$$

for all $z \in S_+$ with the exception of a set of μ^+-measure $\leq C I_0^\gamma$.

Next note that according to Lemma 3.7 we have a countable set $\mathcal{S} = \{z_{j,k} : j = 1, \ldots, Q_k, \ k \geq 2\} \subset S_+ \cap T^+_{\frac{1}{4}}$ such that

$$z_{j,k} \in T^+_{2^{-k}} \backslash T^+_{2^{-k-1}}, \text{ so } d(z_{j,k}) = 2^{-k}, \quad j = 1, \ldots, Q_k, \ k \geq 2, \tag{10}$$

$$\mu = C_1 \delta^{m/2} \sum_{k=2}^{\infty} 2^{-mk} \sum_{j=1}^{Q_k} [\![z_{k,j}]\!] + C_2 \mathcal{H}^m \llcorner T^+_0, \quad C_i = C_i(m), \ i = 1, 2, \tag{11}$$

and

$$S \cap T^+_{2^{-k}} \backslash T^+_{2^{-k-1}} \subset \cup_{\ell=\max(k-2,2)}^{k+1} \cup_{j=1}^{Q_\ell} B_{\delta^{1/2} 2^{-k}}(z_{\ell,j}) \quad \forall k \geq 2. \tag{12}$$

Now let $\mathcal{E}_0 \subset \mathcal{S}$ be the collection of all $z_{j,k} \in \mathcal{S}$ such that

$$\int_{T^+_{\frac{1}{4}} \backslash B_{\frac{d(z_{j,k})}{2}}(z_{j,k})} |\log R_{z_{j,k}}|^{1+\gamma} \frac{|((x,y) - z_{j,k})^\perp|^2}{R_{z_{j,k}}^{n+2}} \, dxdy \geq I_0^\gamma, \tag{13}$$

and let $\mathcal{E}_1 \subset T_0^+$ be the collection of all $z \in T_0^+$ such that

$$(14) \qquad \int_{T^+_{\frac{1}{4}} \setminus B_{\frac{d(z)}{2}}(z)} |\log R_z|^{1+\gamma} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\, dxdy \geq I_0^\gamma.$$

Since $\mu^+(\mathcal{E}_0 \cup \mathcal{E}_1) \leq C I_0^\gamma$ by (9), by (11) we have that

$$(15) \qquad \sum_{w \in \mathcal{E}_0} d(w)^m + \mathcal{H}^m(\mathcal{E}_1) \leq C I_0^\gamma, \quad C = C(n,k,\delta).$$

Now take any $z \in T^+_{\frac{1}{4}} \cap S_+ \setminus T_0^+$. From (12) it follows that $z \in B_{d(z_{j,k})/4}(z_{j,k})$ for some $z_{j,k} \in \mathcal{S}$; if this $z_{j,k} \notin \mathcal{E}_0$ then by (9)

$$(16) \qquad \int_{T^+_{\frac{1}{4}} \cap B_{\frac{d(z_{j,k})}{2}}(z_{j,k})} |\log R_{z_{j,k}}|^{1+\gamma} \frac{|((x,y)-z_{j,k})^\perp|^2}{R_{z_{j,k}}^{n+2}}\, dxdy \leq I_0^\gamma.$$

Regardless of whether $z_{j,k} \in \mathcal{E}_0$ or not, by (10) and Remark 3.4(2)(d) (with $k+2$, $k+1$ in place of ℓ, k) we have that $z \in B_{d(z_{j,k})/4}(z_{j,k}) \subset \mathbf{R}^n \setminus T^+_{2^{-k-2}}$ so that

$$(17) \qquad d(z) \geq 2^{-k-1} = \frac{1}{2} d(z_{j,k}).$$

Thus by (16), (17), for any $z \in S_+ \cap T^+_{\frac{1}{4}} \setminus T_0^+$,

$$(18) \qquad \begin{aligned} &\textit{either } z \in S_+ \cap (\cup_{z_{j,k} \in \mathcal{E}_0} B_{d(z_{j,k})/4}(z_{j,k})) \\ &\textit{or } \exists\, \tilde{z}\ (= \textit{some } z_{j,k} \in S_+ \cap T^+_{\frac{1}{4}} \setminus (T_0^+ \cup \mathcal{E}_0)) \\ &\quad \textit{with } d(z) \geq \tfrac{1}{2} d(\tilde{z}),\ z \in B_{d(\tilde{z})/4}(\tilde{z}), \quad \textit{and} \\ &\quad \int_{T^+_{\frac{1}{4}} \setminus B_{\frac{d(\tilde{z})}{2}}(\tilde{z})} |\log R_{\tilde{z}}|^{1+\gamma} \frac{|((x,y)-\tilde{z})^\perp|^2}{R_{\tilde{z}}^{n+2}}\, dxdy \leq I_0^\gamma. \end{aligned}$$

On the other hand if $z \in T_0^+ \setminus \mathcal{E}_1$, then by definition $d(z) = 0$, and (9) implies

$$(19) \qquad \int_{T^+_{\frac{1}{4}} \setminus B_{\frac{d(z)}{2}}(z)} |\log R_z|^{1+\gamma} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\, dxdy \leq I_0^\gamma.$$

(18) and (19) are the main estimates. Using them we now want to check that we have all the hypotheses needed to apply the rectifiability lemma of §2 (in case $\rho = 1$ and $S = S_+$). For this purpose, we first assume

(20) *no ball* $B_\rho(z)$ *with* $z \in \overline{B}_{5/8}(0) \cap S_+$ *and* $\rho \in [\frac{1}{8}, \frac{1}{4}]$ *has a* δ*-gap.*

By the definition of δ-gap in §3, from (20) it follows that

(20)′ S_+ *has no* 4δ*-gaps in the ball* $B_1(0)$.

Also by Definition 3.1 and the definition of $d(\tilde{z})$ we have

$$(20)'' \qquad d(z) \le \frac{1}{32} \qquad z \in \overline{B}_{5/8}(0) \cap S_+,$$

provided ϵ is sufficiently small (depending on ϵ, n, k, β), which we subsequently assume. Using (20), 7.7, 7.8, and the fact that $\epsilon < \delta/8$ we obtain

$$(21) \qquad B_{5/8}(0) \cap \{(x,y) : |x| \le \tfrac{1}{8}\} \subset T^+_{\frac{1}{4}}.$$

In this case (19) implies

$$(22) \qquad \int_{Q \setminus B_{\frac{d(z)}{2}}(z)} |\log R_z|^{1+\gamma} \frac{|((x,y)-z)^\perp|^2}{R_z^{n+2}}\, dxdy \le I_0^\gamma,$$

for any $z \in T_0^+ \setminus \mathcal{E}_1$, where $Q = B_{5/8}(0) \cap \{(x,y) : |x| \le \frac{1}{8}\}$, and (18) implies that for any $z \in (S_+ \cap \overline{B}_{1/2}(0) \setminus T_0^+) \setminus (\cup_{z_{j,k} \in \mathcal{E}_0} B_{d(z_{j,k})/4}(z_{j,k}))$ there is always a point $\tilde{z} \in S_+ \cap T^+_{\frac{1}{4}} \setminus T_0^+$ such that

$$(23) \qquad \begin{gathered} \int_{Q \setminus B_{\frac{d(\tilde{z})}{2}}(\tilde{z})} |\log R_{\tilde{z}}|^{1+\gamma} \frac{|((x,y)-\tilde{z})^\perp|^2}{R_{\tilde{z}}^{n+2}}\, dxdy \le I_0^\gamma, \\ z \in B_{d(\tilde{z})/4}(\tilde{z}), \quad d(z) \ge \tfrac{1}{2} d(\tilde{z}). \end{gathered}$$

Now take an arbitrary point $z \in (S_+ \cap \overline{B}_{1/2}(0) \setminus T_0^+) \setminus (\cup_{z_{j,k} \in \mathcal{E}_0} B_{d(z_{j,k})/4}(z_{j,k}))$ and let $\tilde{z}$ be as in (23).

$S_+ = S_+(w_1, \sigma_1)$ has no δ-gaps in $B_\rho(\tilde{z})$ for $\rho \ge d(\tilde{z})$, and hence for all $\rho \in [\frac{d(\tilde{z})}{4}, \frac{1}{4}]$ we can select $z_1, \ldots, z_m \in S_+ \cap B_\rho(\tilde{z})$ such that 5.6 holds with $\tilde{z}$ in place of z_0 and with γ depending only on n, k, β. Let $\eta \in (0, \delta^3]$ be given and let L be as in 5.6 (with $\tilde{z}$ in place of z_0). For ϵ small enough (depending on η, n, k, β) and for $\rho \in [\frac{d(\tilde{z})}{4}, \frac{1}{8}]$ we have all the hypotheses needed to apply Lemma 5.9 with 2ρ in place of ρ and with η in place of ζ. Hence there is $\mathbf{C}^{(\rho)} \in \mathcal{T}_{C\beta}$ with $\operatorname{sing} \mathbf{C}^{(\rho)} = L$ and $u^{(\rho)} \in C^3((\tilde{z} + \mathbf{C}^{(\rho)}) \cap B_{4\rho/3}(\tilde{z}) \setminus \{(x,y) : \operatorname{dist}((x,y), \tilde{z} + L) \le \rho/8\}; (\mathbf{C}^{(\rho)})^\perp)$ such that

$$(24) \qquad \begin{aligned} &M \cap B_{4\rho/3}(\tilde{z}) \setminus \{(x,y) : \operatorname{dist}((x,y), \tilde{z}+L) \le \rho/8\} \\ &\qquad = \operatorname{graph} u^{(\rho)} \cap B_{4\rho/3}(\tilde{z}) \setminus \{(x,y) : \operatorname{dist}((x,y), \tilde{z}+L) \le \rho/8\} \end{aligned}$$

and

$$(25) \qquad \sum_{j=0}^{3} \sup \rho^{j-1} |D^j u^{(\rho)}| \le \eta,$$

provided that ϵ is small enough depending only on η, n, k, β. Since for $\rho \ge d(\tilde{z})/4$, by 3.1, 3.5 and 7.8 we have automatically

$$(26) \qquad \|L - \{0\} \times \mathbf{R}^{n-3}\| \le 4\delta, \qquad \|L - L_{\tilde{z},\rho}\| \le C\epsilon.$$

Using (24), (25), (26) with $\rho = \frac{1}{8}$ we then can select a maximum interval $[\rho_0, \frac{1}{8}] \subset [\frac{d(\tilde{z})}{4}, \frac{1}{8}]$ such that there is $w \in C^3(\mathbf{C} \cap B_{1/8}(\tilde{z}) \backslash (B_{\rho_0}(\tilde{z}) \cup K(\tilde{z})); \mathbf{C}^{\perp})$ such that

$$\widetilde{M} \cap B_{1/8}(\tilde{z}) \backslash (B_{\rho_0}(\tilde{z}) \cup (\tilde{z} + K)) = \operatorname{graph}(w) \cap B_{1/8}(\tilde{z}) \backslash (B_{\rho_0}(\tilde{z}) \cup (\tilde{z} + K)),$$

and

$$\sum_{j=0}^{3} \rho^{j-1} |D^j w| \leq \eta^{2/3}, \tag{27}$$

where $\mathbf{C} \in \mathcal{T}_{C\beta}$ with $\operatorname{sing} \mathbf{C} = \{0\} \times \mathbf{R}^m$ (we can take $\mathbf{C} = q(\mathbf{C}^{(1/8)})$ with q orthogonal such that $q(\operatorname{sing} \mathbf{C}^{(1/8)}) = \{0\} \times \mathbf{R}^m$ and $\|q - 1_{\mathbf{R}^{n+k}}\| \leq C\epsilon$), and where

$$K = \{(x, y) : |x| \leq \frac{1}{4} |(x, y)|\}.$$

By (20), 3.1, (24), (25) it is clear that

$$\rho_0 \leq \tfrac{1}{32}$$

so long as ϵ is small enough. Further, from (27) and (24), (25), (26) with $\rho \in [\rho_0, \frac{1}{8}]$, it follows that w can be extended to give $\tilde{w} \in C^3(\mathbf{C} \cap (B_{1/8}(\tilde{z}) \backslash (B_{\rho_0/2}(\tilde{z}) \cup (\tilde{z} + \widetilde{K})); \mathbf{C}^{\perp})$ with

$$\begin{aligned} \widetilde{M} \cap (B_{1/8}(\tilde{z}) \backslash (B_{\rho_0/2}(\tilde{z}) \cup (\tilde{z} + \widetilde{K})) \\ = \operatorname{graph}(w) \cap (B_{1/8}(\tilde{z}) \backslash (B_{\rho_0/2}(\tilde{z}) \cup (\tilde{z} + \widetilde{K})) \end{aligned} \tag{28}$$

and

$$\sum_{j=0}^{3} \rho^{j-1} \sup |D^j \tilde{w}| \leq C\eta^{2/3}, \tag{29}$$

where

$$\widetilde{K} = \{(x, y) : |x| \leq \frac{9}{40} |(x, y)|\},$$

On the other hand by the identity (8) in the proof of Lemma 4.4, applied with $\tilde{w}$ in place of u and with $C\eta^{2/3}$ in place of η, C as in (29), we have

$$\tfrac{1}{2} |((x, y) + \tilde{w}(x, y) - \tilde{z})^{\perp}| \leq |R_{\tilde{z}}^2 (u/R_{\tilde{z}})_{R_{\tilde{z}}}|, \tag{30}$$

provided η is small enough depending on n, k, β.

Now let

$$\Gamma = \mathbf{C} \cap S^{n+k-1} \backslash (\tilde{z} + \widetilde{K}),$$

and let $\hat{w}(\sigma)$ denote the $L^2(\Gamma)$ function given by $\hat{w}(s)(\omega) = \tilde{w}(\tilde{z} + s\omega)$, $\omega \in \Gamma$. Then by direct integration, the Cauchy-Schwarz inequality, and (30) we obtain (31)

$$\begin{aligned}\|\hat{w}(\sigma) - \hat{w}(\tau)\|_{L^2(\Gamma)} &\leq \int_\sigma^\tau \|\frac{\partial \hat{w}(s)}{\partial s}\|_{L^2(\Gamma)}\, ds \\ &\leq \Big(\int_\sigma^\tau |\log s|^{1+\gamma} s \|\frac{\partial \hat{u}(s)}{\partial s}\|^2_{L^2(S^{\ell-1})}\, ds\Big)^{1/2} \Big(\int_\sigma^\tau s^{-1}|\log s|^{-1-\gamma}\, ds\Big)^{1/2} \\ &\leq \gamma^{-1/2}\left(\int_{Q\setminus B_{d(\tilde{z})/2}(\tilde{z})} |\log R_{\tilde{z}}|^{1+\gamma} \frac{\|R_{\tilde{z}}\tilde{u}_{R_{\tilde{z}}}\|^2}{R_{\tilde{z}}^{n+2}}\right)^{1/2} |\log\tau|^{-\gamma/2} \\ &\leq C I_0^{\gamma/2} |\log\tau|^{-\gamma/2}\end{aligned}$$

for any $\rho_0/2 < \sigma < \tau \leq 1/8$. Taking $\tau = \frac{1}{8}$ and using (24)–(26) with $\rho = 1/8$, and also (27) again, we then deduce that

$$\|\hat{w}(\sigma)\|_{L^2(\Gamma)} \leq C\eta$$

for $\sigma \in [\frac{\rho_0}{2}, \frac{1}{8}]$. Thus

$$\sup_{\rho\in[\rho_0,\frac{1}{8}]} \rho^{-n-2} \int_{B^M_\rho(\tilde{z})\setminus(B_{\rho_0/2}(\tilde{z})\cup(\tilde{z}+\widetilde{K}))} |\tilde{w}|^2 \leq C\eta, \tag{32}$$

and hence by PDE estimates we can improve the estimate (29) to

$$\sum_{j=0}^{3} \rho^{j-1} \sup |D^j \tilde{w}| \leq C\eta < \eta^{2/3}, \tag{33}$$

provided η is small enough depending on n, k, β. However this contradicts the maximality of the interval $[\rho_0, \frac{1}{8}]$ unless $\rho_0 = d(\tilde{z})/4$. Thus $\rho_0 = d(\tilde{z})/4$ and, in consequence of (32),

$$\sup_{[\frac{d(\tilde{z})}{4},\frac{1}{8}]} \rho^{-n-2} \int_{B^M_\rho(\tilde{z})(\setminus\{(x,y)\,:\,|x-\tilde{x}|<9\rho/40\}\cup B_{d(\tilde{z})/8}(\tilde{z}))} \operatorname{dist}^2((x,y),\mathbf{C}) \leq C\eta. \tag{34}$$

Since $z \in B_{d(\tilde{z})/4}(\tilde{z})$ and $\frac{1}{2}d(\tilde{z}) \leq d(z) \leq \frac{1}{32}$, from (34) it follows that

$$\sup{}_{[2d(z),\frac{1}{8}]} \rho^{-n-2} \int_{B^M_{\rho/2}(z)\setminus\{(x,y)\,:\,|x-\xi|\leq\rho/5\}} \operatorname{dist}^2((x,y),\mathbf{C}) \leq C\eta,$$

where ξ is the projection of z onto its first $\ell + k$ coordinates; in other words, by writing ρ in place of $\rho/2$,

$$\sup{}_{[d(z),\frac{1}{8}]} \rho^{-n-2} \int_{B^M_\rho(z)\setminus\{(x,y)\,:\,|x-\xi|\leq 2\rho/5\}} \operatorname{dist}^2((x,y),\mathbf{C}) \leq C\eta. \tag{35}$$

If $\zeta \in (0, \delta^2]$ is given and ϵ is sufficiently small, depending on δ, n, k, β, then (35) combined with Corollary 4.6 gives

$$(36) \quad S_+ \cap B_\rho(z) \subset \text{ the } (\zeta\rho)\text{-neighbourhood of } z + \{0\} \times \mathbf{R}^m, \quad \rho \in [\tfrac{d(z)}{2}, \tfrac{1}{8}].$$

So, using the definition 3.3, we would have $z \in T^+_{d(z)/2}$ unless one of the balls $B_\rho(z)$, $\rho \in [\frac{1}{2}d(z), \frac{1}{8}]$ has a δ-gap.

But of course $z \in T^+_{\frac{d(z)}{2}}$ contradicts the definition of $d(z)$ for $d(z) > 0$, so we conclude finally, keeping in mind (20),

$$(37) \quad \begin{aligned} &\forall\, z \in (S_+ \backslash T_0^+) \cap \overline{B}_{1/2}(0) \backslash (\cup_{z_{j,k} \in \mathcal{E}_0} B_{d(z_{j,k})/4}(z_{j,k})), \\ &\qquad \exists\, \sigma_z \in [\tfrac{d(z)}{2}, \tfrac{1}{8}] \textit{ such that } S_+ \textit{ has a } \delta\textit{-gap in } B_{\sigma_z}(z). \end{aligned}$$

Next notice that since T_0^+ is a subset of the graph of a Lipschitz function over $\{0\} \times \mathbf{R}^m$ with Lipschitz constant $\leq C\delta$, in view of (15) we can select a $\{B_{\sigma_k}(z_k)\}$ such that

$$(38) \qquad \sigma_k \in (0, \tfrac{1}{8}), \quad \mathcal{E}_1 \subset \cup_k B_{\sigma_k}(z_k), \quad \sum_k \sigma_k^m \leq C I_0^\gamma.$$

For $z \in S_+ \cap T_0^+ \backslash \cup_k B_{\sigma_k}(z_k)$ we have again

$$(31)' \qquad \|\hat{w}(\sigma) - \hat{w}(\tau)\|_{L^2(\Gamma)} \leq C I_0^{\gamma/2} |\log \tau|^{-\gamma/2}$$

($\hat{w}$ as in (31) with z in place of $\tilde{z}$), and

$$(36)' \quad S_+ \cap B_\rho(z) \subset \text{ the } (\zeta\rho)\text{-neighbourhood of } z + \{0\} \times \mathbf{R}^m, \quad \rho \in (0, \tfrac{1}{8}],$$

by the same argument used to derive (31) and (36), except that we use (22) in place of (23) and z in place of $\tilde{z}$ everywhere. In view of (15), (36), (37), (38), and (36)$'$ it is now evident that, provided (20) holds, we can take the collection $\{B_{d(z_{j,k})/4}(z_{j,k})\}_{z_{j,k} \in \mathcal{E}_0} \cup \{B_{\sigma_k}(z_k)\}$ to be the collection corresponding to $\mathcal{T}_{x_0,\rho_0}$ in the rectifiability lemma of §2 in case we use ζ in place of ϵ, and then hypothesis (b) of that lemma is satisfied in case $x_1 = 0$ and $\rho_1 = 1$.

On the other hand if (20) fails, then some ball $B_{\frac{1}{4}}(y)$ with $y \in B_{\frac{1}{2}}(0) \cap \{0\} \times \mathbf{R}^m$ must have a $\frac{\delta}{2}$-gap, and so the first alternative hypothesis of the rectifiability lemma holds in case $x_1 = 0$ and $\rho_1 = 1$.

Thus, provided ϵ is sufficiently small, depending on δ, n, k, β, we have shown that $S_+(w_1, \sigma_1)$ satisfies the hypotheses of the rectifiability lemma 2.2 for $x_1 = 0$, $\rho_1 = 1$. But then trivially any closed subset of $S_+(w_1, \sigma_1)$, including $\eta_{w_1,\sigma_1}(\overline{B}_{\epsilon\sigma_0/4}(w_0) \cap S_+)$, also satisfies such hypotheses. That is, in view of the arbitrariness of w_1, σ_1, we have shown that $S = \overline{B}_{\epsilon\sigma_0/4}(w_0) \cap S_+$ satisfy the hypotheses of 2.2 for any $x_1 \in S$, $\rho_1 \in (0, \rho_0]$, where $\rho_0 = \epsilon\sigma_0/4$.

Thus the rectifiability lemma 2.2 implies that $\overline{B}_{\epsilon\sigma_0/4}(w_0) \cap S_+$ is m-rectifiable.

Finally, let B be any closed ball contained in U_M. Then by monotonicity 1.7 there is a fixed $\beta > 0$ such that $\Theta_M(y) \leq \beta$ for each $y \in B$. In particular

$\Theta_{\mathbf{C}}(0) \le \beta$ for any tangent cone of M at any point $y \in B$, and by Lemma 4.3 we know that $\{\Theta_M(y) : y \in \operatorname{sing}_* M \cap B\}$ is a finite set $\alpha_1 < \cdots < \alpha_N$ of positive numbers, where $\operatorname{sing}_* M$ is as in 2.17. Let

$$S_j = \{z \in \operatorname{sing} M : \Theta_M(z) = \alpha_j\},$$
$$S_j^+ = \{z \in \operatorname{sing} M : \Theta_M(z) \ge \alpha_j\}.$$

Notice that S_j^+ is closed in Ω by the upper semi-continuity 1.13 of Θ_M. For any $j \in \{1, \dots, N\}$ and any $y \in S_j$, according to the above discussion, there is $\rho > 0$ such that $B_\rho(y) \cap S_j^+$ is m-rectifiable. Thus, in view of the arbitrariness of y, the set S_j has an open neighbourhood U_j such that

$$S_j^+ \cap U_j \text{ is locally } m\text{-rectifiable.} \tag{41}$$

Of course the $S_j^+ \cap U_j$ are also locally compact, because S_j^+ is closed and U_j is open. Now let

$$V_j = \{z \in \operatorname{sing} M : \Theta_M(z) < \alpha_{j+1}\}, \quad j = 0, \dots, N-1, \quad V_N = \Omega.$$

Then the V_j are open in Ω by the upper semi-continuity 1.13 of Θ_M, and with $\alpha_0 = 0$, $\alpha_{N+1} = \infty$, $S_0^+ = \operatorname{sing} M$, and $U_0 = \emptyset$, we can write

$$\begin{aligned} B \cap \operatorname{sing} M &= \cup_{j=0}^N \{z \in B \cap \operatorname{sing} M : \alpha_j \le \Theta_M(z) < \alpha_{j+1}\} \\ &= \cup_{j=0}^N B \cap S_j^+ \cap V_j \\ &= (\cup_{j=0}^N (B \cap S_j^+ \cap U_j \cap V_j)) \cup (\cup_{j=0}^N (B \cap S_j^+ \backslash U_j) \cap V_j). \end{aligned}$$

This is evidently a decomposition of $B \cap \operatorname{sing} M$ into a finite union of pairwise disjoint locally compact sets, each of which is locally m-rectifiable; in fact for each j the set $(B \cap S_j^+ \backslash U_j) \cap V_j \subset \operatorname{sing} M \backslash \operatorname{sing}_* M$, and hence has Hausdorff dimension $\le m-1$ by 2.17, and the set $B \cap S_j^+ \cap U_j \cap V_j$ is locally m-rectifiable by (30). This completes the proof of Theorem 2.

Proof of Remark 1.14. We have to show that for $\mathcal{H}^m$-a.e. $z \in \operatorname{sing} M$ there is a unique tangent space for $\operatorname{sing} M$ at z in the Hausdorff distance sense, and also that M has a unique tangent cone at z.

For the former of these we have to show that, for $\mathcal{H}^m$-a.e. $z \in \operatorname{sing} M$, there is an m-dimensional subspace L_z such that for each $\epsilon > 0$

$$B_1(0) \cap \eta_{z,\sigma}(\operatorname{sing} M) \subset \text{ the } \epsilon\text{-neighbourhood of } L_z \tag{1}$$

and

$$B_1(0) \cap L_z \subset \text{ the } \epsilon\text{-neighbourhood of } \eta_{z,\sigma}(\operatorname{sing} M) \tag{2}$$

for all $\sigma \in (0, \sigma_0)$ where $\sigma_0 = \sigma_0(\epsilon, M, z) \downarrow 0$ as $\epsilon \downarrow 0$. Using the notation in the last part of the proof above, let $z \in S_j$ be any point where S_j^+ has an approximate tangent space. Then there is an m-dimensional subspace L_z with

$$\lim_{\sigma \downarrow 0} \int_{\eta_{z,\sigma}(S_j^+)} f \, d\mathcal{H}^m = \int_{L_z} f \, d\mathcal{H}^m \quad \forall f \in C_c^0(\mathbf{R}^n). \tag{3}$$

(Notice such L_z exists for $\mathcal{H}^m$-a.e. $z \in S_j$ because S_j is locally m-rectifiable.) We show that (1) and (2) hold with this L_z. In fact the inclusion (2) is evidently already implied by this, so we need only to prove (1). Let $\sigma_k \downarrow 0$ be arbitrary, and let $\mathbf{C}$ be any tangent cone of M at z with $\eta_{z,\sigma_{k'}} M \to \mathbf{C}$ for some subsequence $\sigma_{k'}$. By (3) it is evident that the ϵ_k neighbourhood of $B_1(0) \cap \eta_{z,\sigma_{k'}} S_j^+$ contains all of $L_z \cap B_{1/2}(0)$ for some sequence $\epsilon_k \downarrow 0$, so that, in consequence of the upper semi-continuity 2.3,

$$\Theta_{\mathbf{C}}(y) \geq \Theta_{\mathbf{C}}(0) = \Theta_M(0) \quad \textit{everywhere on } L_z \cap B_{1/2}(0).$$

Thus by 2.5 and 2.6 we have $L_{\mathbf{C}} \supset L_z$, and since L_z has maximal dimension m, this shows that $L_{\mathbf{C}} = L_z$, so $\mathbf{C} \in \mathcal{T}$ with $L_{\mathbf{C}} = L_z$. But then by 4.6 we have

$$B_1(0) \cap \eta_{z,\sigma_{k'}}(\operatorname{sing} M) \subset \textit{ the } \epsilon_k\textit{-neighbourhood of } L_z$$

for some sequence $\epsilon_k \downarrow 0$. In view of the arbitrariness of the original sequence σ_k we thus obtain (2) as claimed.

Finally we want to show that there is a unique tangent cone of $\mathbf{C}$ at $\mathcal{H}^m$-a.e. $z \in \operatorname{sing} M$. Let $S_j = \{z \in \operatorname{sing} M : \Theta_M(z) = \alpha_j\}$ as above. For each $\epsilon > 0$, we can subdivide S_j into $\cup_{i=1}^{\infty} S_{j,i}$, where $S_{j,i}$ denotes the set of points $z \in S_j$ such that the conclusions (1) and (2) hold with $\sigma_0 = \frac{1}{i}$. Provided the original w_0, σ_1 in the definition 7.4 of $\widetilde{M}$ are selected with $w_0 \in S_{j,i}$ and $\sigma_1 = \sigma_1(\epsilon, M, w_0, i) \leq \frac{1}{i}$, by (1) and (2) we then have that *all* points of $z \in \eta_{w_0,\sigma_1} S_{j,i}$ are contained in the set T_0^+ in the proof of Theorem 2 above. Hence by (31) of the above proof we conclude that there is a unique tangent cone of M at each point $z \in S_{j,i} \cap B_{\sigma_1}(w_0)$ with the exception of a set of $\mathcal{H}^m$-measure $\leq \epsilon\sigma_1^m$. In view of the arbitrariness of ϵ, w_0 here (and keeping in mind that we have already established that $S_{j,i}$ is locally m-rectifiable) this shows that there is a unique tangent cone of M for $\mathcal{H}^m$-a.e. points $z \in S_{j,i}$. Since $\mathcal{H}^m(\operatorname{sing} M \setminus (\cup_{i,j} S_{j,i})) = 0$, the proof is complete.

7 Theorems on Countable Rectifiability. Recall that a set is countably m-rectifiable if it can be written as the countable union of m-rectifiable sets.

There are some theorems about countable rectifiability of the singular set even without the hypotheses 1.13, 1.13′ (i.e., without assuming that we are in the top dimension of singularities over the entire class of maps or surfaces under consideration). For minimizing maps such theorems are established in [**32**]. Here we want to establish such a result for $M \in \mathcal{M}$.

We are going to prove that $S^{(m)}$ is countably m-rectifiable, where, for a given $M \in \mathcal{M}$ and $m \in \{1, \ldots, n-1\}$, $S^{(m)}$ is the set of points $z \in \operatorname{sing} M$ such that all tangent cones $\mathbf{C}$ of M at z are such that $\dim \operatorname{sing} \mathbf{C} \leq m$.

In fact we shall prove the stronger result that $T^{(m)}$ is countably rectifiable, where $T^{(m)}$ is the set of points $z \in \operatorname{sing} M$ such that all tangent cones $\mathbf{C}$ of M at z have $\dim L_{\mathbf{C}} \leq m$ and $\operatorname{sing} \mathbf{C} = L_{\mathbf{C}}$ if $\dim L_{\mathbf{C}} = m$. Since trivially $S^{(m)} \subset T^{(m)}$, this will also prove the above claim about $S^{(m)}$.

For each $\delta > 0$, let $T_\delta^{(m)}$ denote the set of points $z \in \operatorname{sing} M$ such that, whenever $\mathbf{C} \in \mathcal{C}$ with $\inf_{\sigma \in (0,\delta]} \int_{B_1(0) \cap \eta_{z,\sigma} M} \operatorname{dist}^2((x,y), \mathbf{C}) < \delta$, then we have

$\dim L_{\mathbf{C}} \le m$ and $\operatorname{sing}\mathbf{C} = L_{\mathbf{C}}$ if $\dim L_{\mathbf{C}} = m$. We claim that

$$T^{(m)} \subset \cup_{j=1}^{\infty} T_{1/j}^{(m)}. \tag{1}$$

Indeed if $z \notin \cup_{j=1}^{\infty} T_{1/j}^{(m)}$, then for $j = 1, 2, \ldots$ we can find $\mathbf{C}_j \in \mathcal{C}$ and $\sigma_j \in (0, 1/j]$ with

$$\int_{B_1(0)\cap\eta_{z,\sigma_j}M} \operatorname{dist}^2((x,y), \mathbf{C}_j) \le 1/j \tag{2}$$

and

$$\textit{either } \dim L_{\mathbf{C}_j} > m \ \textit{ or both } \dim L_{\mathbf{C}_j} = m \ \textit{ and } \operatorname{sing}\mathbf{C}_j \ne L_{\mathbf{C}_j}. \tag{3}$$

Notice that the latter alternative here implies that

$$\operatorname{sing}\mathbf{C}_j \supset H_j, \tag{4}$$

where H_j is an $(m+1)$-dimensional half-space. Now by (2) some subsequence of $\mathbf{C}_j$ (still denoted $\mathbf{C}_j$) converges to a tangent cone $\mathbf{C}$ of M at z, and by (3), (4), 2.1, and 2.3 we have that

$$\dim L_{\mathbf{C}} > m \ \textit{ or both } \dim L_{\mathbf{C}} = m \ \textit{ and } \operatorname{sing}\mathbf{C} \ne L_{\mathbf{C}}.$$

Hence $z \notin T^{(m)}$ by definition, and (1) is proved.

Next we define $T_{\delta,\beta}^{(m)}$, for $\beta > 0$, to be the set of all $z \in T_{\delta}^{(m)}$ such that

$$\sup_{\mathbf{C}_0\cap S^{n-m-1}} \sum_{j=0}^{3} |D^j A_{\mathbf{C}_0}| \le \beta \tag{5}$$

for all $\mathbf{C} \in \mathcal{C}$ such that $\inf_{\sigma\in(0,\delta]} \|\mathbf{C} - \eta_{z,\sigma}M\|_{L^2(B_1(0))} < \delta$ and $\dim L_{\mathbf{C}} = m$, where $\mathbf{C}_0$ is such that $\operatorname{sing}\mathbf{C}_0 = \{0\}$ and $q(\mathbf{C}) = \mathbf{C}_0 \times \mathbf{R}^m$ for some orthogonal q, and $A_{\mathbf{C}_0}$ is the second fundamental form of $\mathbf{C}_0$. (Notice that by definition of $T_{\delta}^{(m)}$ there is such a $\mathbf{C}_0$ corresponding to each such $\mathbf{C}$.)

Now

$$T_{\delta,\beta}^{(m)} \ \textit{is a closed subset of } \operatorname{sing} M \tag{6}$$

for each $\delta, \beta > 0$, because if $z_j \to z \in \operatorname{sing} M$, with $z_j \in T_{\delta,\beta}^{(m)}\ \forall j$, and if $\mathbf{C} \in \mathcal{C}$ with $\int_{\eta_{z,\sigma}M\cap B_1(0)} \operatorname{dist}^2((x,y), \mathbf{C}\cap B_1(0)) < \delta$ for some $\sigma \in (0,\delta]$, then, with this σ, $\int_{\eta_{z_j,\sigma}M\cap B_1(0)} \operatorname{dist}^2((x,y), \mathbf{C}\cap B_1(0)) < \delta$ for all sufficiently large j. Since $z_j \in T_{\delta,\beta}^{(m)}$, we have $\dim L_{\mathbf{C}} \le m$, and also $\operatorname{sing}\mathbf{C} = L_{\mathbf{C}}$ and $\sup_{\Sigma\cap S^{n-m-1}} \sum_{j=0}^{3} |D^j A_{\mathbf{C}_0}| \le \beta$ in case $\dim L_{\mathbf{C}} = m$. That is, $z \in T_{\delta,\beta}^{(m)}$ and hence (6) is proved.

All the arguments used in the proof of Theorem 4 now carry over to the present setting essentially without change provided we use $T_{\delta,\beta}^{(m)} \cap S_+$ in place of S_+. (Whenever we needed 2.12 before, we can now use instead (5) above.)

Thus we conclude that for each given δ, $\beta > 0$ and for each $z \in T^{(m)}_{\delta,\beta}$ with $\Theta_M(z) = \Theta_{\mathbf{C}}(0)$ for some $\mathbf{C} \in \mathcal{C}$ with $\operatorname{sing}\mathbf{C} = L_{\mathbf{C}}$ of dimension m, there is $\rho > 0$ such that $B_\rho(z) \cap \{w \in T^{(m)}_{\delta,\beta} : \Theta_M(w) \geq \Theta_M(z)\}$ is m-rectifiable, and then the argument in the last part of the proof of Theorem 2 shows that $T^{(m)}_{\delta,\beta}$ locally decomposes into a finite union of locally m-rectifiable subsets. In view of (1) and the fact that $T^{(m)}_{\delta} = \cup_{j=1}^{\infty} T^{(m)}_{\delta,j}$, which proves that $T^{(m)}$ is countably m-rectifiable as claimed.

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