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Spaces of Algebraic Cycles

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Introduction

In this article [1] we shall review a recent body of work which is concerned with the structure of the spaces of algebraic cycles on an algebraic variety. Before embarking on this survey we should offer some general motivation for such a study.

The fundamental objects of interest in algebraic geometry are the sets of solutions of polynomial equations in affine or projective space. Any profound understanding of such sets must be at least in part geometric. However, if the field in question is, say finite, in what sense can one speak of geometry ? This geometry comes from the network of algebraic subsets. An algebraic variety has not only points, but also a family of "algebraic curves", algebraic subsets of dimension 1. As in most geometries the distinguished curves give structure to the space. Of course here there are also distinguished algebraic surfaces, 3-folds, etc. It is the interlocking web of these subvarieties which endows an algebraic variety with a rich geometric structure.

For an affine variety $X \subset \mathbb{C}^N$ this picture translates faithfully into algebraic terms. Irreducible subvarieties of X correspond to prime ideals in the ring $\mathcal{O}(X)$ of polynomials restricted to X. The inclusion of subvarieties corresponds to the (reverse) inclusion of ideals.

For general X Grothendieck took all this a step further. He taught us to consider the irreducible subvarieties to be "points" of the space. On this enhanced set of points he introduced a topology and a sheaf of rings – classical stuctures of geometry.

In this spirit of purely elementary considerations, there is a related construction which also uses subvarieties and in fact predates Grothendieck. Fix a variety X, and for $p \geq 0$ let $X(p)$ denote the set of "p-dimensional points" of X, i.e., the set of irreducible p-dimensional subvarieties. Then one defines the **Chow monoid of** X to be simply the free abelian monoid

$$\mathcal{C}_p(X) \;=\; \mathbb{Z}^+ \cdot X(p)$$

generated by this set. The points $c \in \mathcal{C}_p(X)$, which are expressed uniquely as finite formal sums $c = \sum n_i V_i$ with $n_i \in \mathbb{Z}^+$ and $V_i \in X(p)$, are called **effective algebraic** p **cycles** on X.

Now the surprizing fact — established by Chow and Van der Waerden in 1937 — is that when $X \subset \mathbb{P}^N$ is projective, *this monoid itself is an algebraic variety.* Specifically, it can be written as a countable disjoint union

$$\mathcal{C}_p(X) \;=\; \coprod_{\alpha \in H_{2p}(X;\mathbb{Z})} \mathcal{C}_{p,\alpha}(X) \tag{0.1}$$

[1] Partially supported by the NSF and I.H.E.S.

where each $\mathcal{C}_{p,\alpha}(X)$ canonically carries the structure of a projective algebraic set. This gives us a constellation of geometric objects naturally associated to X. They can be thought of as compactifications of the moduli spaces of p-dimensional algebraic subsets of X, and have considerable independent interest, particularly when $X = \mathbb{P}^n$. (For example, for varieties over $\mathbb{C}$, $\mathcal{C}_{p,\alpha}(X)$ has been shown to represent the space of solutions to the Plateau problem in the homology class α.) These $\mathcal{C}_{p,\alpha}(X)$ fit together to form a monoid whose addition when restricted to the algebraic pieces is a morphism of varieties.

There is of course also the free abelian group

$$\mathcal{Z}_p(X) = \mathbb{Z} \cdot X(p)$$

of *all algebraic p-cycles on X*. It is functorially related to X, but appears at first to be just a huge, infinitely generated group. However, it carries a very interesting structure which comes from the Chow monoid

$$\mathcal{C}_p(X) \subset \mathcal{Z}_p(X)$$

as follows. Note that $\mathcal{Z}_p(X)$ can be written as a quotient

$$\mathcal{Z}_p(X) = \mathcal{C}_p(X) \times \mathcal{C}_p(X)/\sim \tag{0.2}$$

where $(a, b) \sim (a', b') \iff a + b' = a' + b$. By (0.1) $\mathcal{C}_p(X)$ can be written as a monotone union $V_1 \subset V_2 \subset \ldots$ of projective algebraic sets. Therefore, $\mathcal{Z}_p(X)$ carries an intrinsic filtration

$$K_1 \subset K_2 \subset K_3 \subset \ldots \subset \mathcal{Z}_p(X) \tag{0.3}$$

where

$$K_\ell = \bigcup_{i+j\leq\ell} V_i \times V_j/\sim .$$

Each K_ℓ is the quotient of an algebraic set by a proper algebraic equivalence relation.

Note that when X is defined over $\mathbb{C}$, each K_ℓ is a compact Hausdorff space. This induces a topology on $\mathcal{Z}_p(X)$ in the standard way (by defining $C \subset \mathcal{Z}_p(X)$ to be closed iff $C \cap K_\ell$ is closed for all ℓ), making $\mathcal{Z}_p(X)$ a topological abelian group. Its homotopy groups, as we shall see, constitute an interesting set of invariants. They characterize $\mathcal{Z}_p(X)$ up to homotopy equivalence and reflect the algebraic structure of X.

I have gotten somewhat ahead of myself. Let's return to elementary considerations. As mentioned above, spaces of cycles have considerable geometric interest, particularly when $X = \mathbb{P}^n$. Consider for example the set $\mathcal{C}_{p,1}(\mathbb{P}^n)$ of effective p-cycles of homology degree 1. This is exactly the Grassmannian of $(p+1)$-planes in $\mathbb{C}^{n+1}$, a space of fundamental importance in geometry. One reason for its importance is that, as n goes to infinity $\mathcal{C}_{p-1,1}(\mathbb{P}^n)$ approximates the classifying space BU_p for p-dimensional vector bundles; and as both n and p go to infinity, one obtains the classifying space BU for reduced K-theory.

Despite the beauty and importance of the Grassmannians, until seven years ago surprizingly little was known about spaces of cycle of higher degree. In fact, the work surveyed here was motivated by a desire to understand these other components of

$$\mathcal{C}_p(\mathbb{P}^n) = \coprod_{d=0}^{\infty} \mathcal{C}_{p,d}(\mathbb{P}^n).$$

(Here $\mathcal{C}_{p,d}(\mathbb{P}^n)$ denotes the effective p-cycles of homology degree d.) One could see straightforwardly that $\mathcal{C}_{p,d}(\mathbb{P}^n)$ is always connected and simply-connected, and it seemed plausible to conjecture that $\pi_2\mathcal{C}_{p,d}(\mathbb{P}^n) \cong \mathbb{Z}$. This and much more turned out to be true.

The first interesting discovery was that as $d \to \infty$ the sets $\mathcal{C}_{p,d}(\mathbb{P}^n)$ "stabilize" to become classifying spaces for integral cohomology in even degrees. This says much about their structure. It also means that the Chow varieties are in fact fundamental objects in topology.

This stablization result can be rigorously expressed by the assertion that there exists a homotopy equivalence

$$\mathcal{Z}_p(\mathbb{P}^n) \cong K(\mathbb{Z},0)\times K(\mathbb{Z},2)\times K(\mathbb{Z},4)\times\cdots\times K(\mathbb{Z},2(n-p)) \tag{0.4}$$

for all $0 \le p \le n$, where $K(\mathbb{Z},2k)$ denotes the Eilenberg-MacLane space (See I.3 below). Since $\mathcal{Z}_p(\mathbb{P}^n)$ is a group, this is equivalent to the assertion that

$$\pi_i\mathcal{Z}_p(\mathbb{P}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0,2,4,...,2(n-p) \\ 0 & \text{otherwise.} \end{cases}.$$

Note the simplicity of these homotopy groups. By contrast the homology groups of $\mathcal{Z}_p(\mathbb{P}^n)$ are quite complicated. Note also that

$$\mathcal{Z}_p(\mathbb{P}^n) \cong \mathcal{Z}_{p+1}(\mathbb{P}^{n+1}). \tag{0.5}$$

This equivalence is induced by an *algebraic suspension* mapping which is described in Chapter II.

Now from the introduction of cycle groups into topology something new emerges. The first surprizing fact is that the simple inclusion $\mathcal{C}_{p,1}(\mathbb{P}^n) \subset \mathcal{Z}_p(\mathbb{P}^n)$ canonically represents the total Chern class of the tautological $(n-p)$-plane bundle over the Grassmannian $\mathcal{C}_{p,1}(\mathbb{P}^n)$. Furthermore, on projective algebraic cycles there exists an elementary binary operation, called the *algebraic join*. It is a direct generalization of the direct sum of linear spaces, which gives a pairing on Grassmannians and corresponds to addition in K-theory. It turns out to canonically represent the cup product in cohomology. Using this join structure one has been able to answer some old questions in homotopy theory (cf. Chapter III).

Now the homotopy groups of $\mathcal{Z}_p(\mathbb{P}^n)$ turn out to be simple and to play a central role in certain universal constructions in topology. It seems reasonable to think therefore that the groups $\pi_i\mathcal{Z}_p(X)$ might be important for any projective variety X.[2] They are functorial. Furthermore there is an Algebraic Suspension Theorem:

$$\mathcal{Z}_p(X) \cong \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X)$$

generalizing (0.5) above, which gives $\pi_i\mathcal{Z}_p(X)$ an unexpected and useful structure. Consider some basic examples.

Example 1.

$$\pi_0\mathcal{Z}_p(X) \;=\; \mathcal{A}_p(X)$$

$$= \text{ algebraic } p\text{–cycles on } X \text{ modulo algebraic equivalence,}$$

Example 2. By a classical theorem of Dold and Thom, one has that for all $k \geq 0$

$$\pi_k\mathcal{Z}_0(X) \;=\; H_k(X;\mathbb{Z}).$$

This shows that the functor $\pi_*\mathcal{Z}_*(X)$ not only contains the *integral* homology of X but it also contains the groups $\mathcal{A}_*(X)$ which are purely algebraic invariants. So this functor represents something new which should be of interest to algebraic geometers.

On the other hand the groups $\pi_*\mathcal{Z}_*(X)$ have definite geometric interest since they tell us about the global structure of the Chow varieties $\mathcal{C}_{p,d}(X)$. (See Chapter I.8.)

For these reasons the groups $\pi_*\mathcal{Z}_*(X)$ have been systematically studied over the past few years. They turn out to have a rich internal structure and to be related to many of the standard invariants of algebraic geometry. For example P. Lima-Filho has shown that these groups can be defined for quasi-projective varieties, and they fit into localization exact sequences. This allows complete computations in many cases. He has also extended the definition from quasi-projective to general algebraic varieties. It was Eric Friedlander who laid the foundations for the study of these invariants. He realized the importance of Example 1 above and introduced methods of formal group completion into the theory. He made sense of the groups $\pi_*\mathcal{Z}_*(X)$ for varieties defined over any algebraically closed field and proved the suspension theorem in this context. In his fundamental paper he introduced the notation

$$L_pH_k(X) \overset{def.}{=} \pi_{k-2p}\mathcal{Z}_p(X)$$

[2]It may seem at first that homotopy groups, which involve continuous mappings of spheres, are particularly non-algebraic in their construction. However, the homotopy of an abelian topological group Z has a beautiful realization as the homology of the simplicial group Sing.(Z).

where L_p indicates that the algebraic level is p, i.e., there are p algebraic parameters, and where H_k indicates that the homology degree is k. Friedlander and Mazur have shown that the algebraic join of cycles leads to a natural transformation

$$s : L_pH_k(X) \longrightarrow L_{p-1}H_k(X)$$

which in turn induces filtrations on the groups $H_*(X;\mathbb{Z})$ and $\mathcal{A}_*(X)$. These filtrations have alternative, purely algebraic interpretations and are subordinate to the filtrations of Grothendieck and of Bloch-Ogus. Grothendieck's standard Conjecture B actually implies that the filtrations coincide. The suspension theorem has been extended by Friedlander and Gabber to a general intersection product in $\mathcal{Z}_*(X)$ which gives a graded ring structure to $L_*H_*(X)$. There exists a local-to-global spectral sequence with an identifiable E_2-term as in Bloch-Ogus theory. There are relations to algebraic K-theory and to Bloch's higher Chow groups. All this is discussed in Chapter IV.

Now the groups $L_*H_*(X)$ behave like a homology theory on the category of quasi-projective varieties and proper morphisms, and it is natural to ask whether there is an associated "cohomology" theory. In $[FL_{1,2}]$ such a theory was introduced, based on a new concept of an *effective algebraic cocycle* on a variety X. Such a cocycle is defined as an algebraic family of affine subvarieties parameterized by X. The set of all such cocycles in degree-q is roughly speaking the monoid

$$\mathcal{C}^q(X;\mathbb{C}^n) = \mathrm{Mor}(X, \mathcal{C}^q(\mathbb{C}^n))$$

with a natural topology. Taking homotopy groups of the group completion gives a contravariant functor $L^*H^*(X)$ which enjoys a rich structure. There is a "cup product" induced by taking the pointwise join of cocycles, there is a natural transformation of *rings*

$$L^qH^k(X) \longrightarrow H^k(X;\mathbb{Z}),$$

there are s-maps and filtrations, Chern classes, etc. All this is discussed in Chapter V.

Although the functors L_*H_* and L^*H^* are quite differently defined, they are surprizingly related. There is for example a Kronecker pairing between them. However, much more interesting is the recently established fact that on smooth projective varieties they satisfy **Poincaré duality**. In fact for any projective variety X of dimension n there is a naturally defined homomorphism

$$L^qH^k(X) \xrightarrow{\tilde{\mathcal{D}}} L_{n-q}H_{2n-k}(X)$$

for all q, k which under the natural transformation to singular theory becomes the Poincaré duality map, i.e., there is a commutative diagram

$$\begin{array}{ccc} L^qH^k(X) & \xrightarrow{\tilde{\mathcal{D}}} & L_{n-q}H_{2n-k}(X) \\ \downarrow & & \downarrow \\ H^k(X;\mathbb{Z}) & \xrightarrow{\mathcal{D}} & H_{2n-k}(X;\mathbb{Z}) \end{array}$$

where $\mathcal{D}(\alpha) = \alpha \cap [X]$. **When X is smooth and projective, the map $\tilde{\mathcal{D}}$ is an isomorphism.** This result, discussed in Chapter VI, has a broad range of consequences.

I would like to express my sincere thanks to Eric Friedlander, Paulo Lima-Filho and Pawel Gajer for having made a number of suggestions which greatly improved the original version of this manuscript.

Chapter I - Algebraic Cycles

§1. Algebraic subsets. Let $\mathbb{P}^n$ denote complex projective n-space, the space of all lines through the origin in $\mathbb{C}^{n+1}$. Then there is a natural map

$$\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$$

which assigns to v the 1-dimensional subspace it generates.

Definition 1.1. A subset $V \subset \mathbb{P}^n$ is said to be **algebraic** if there exists a finite collection of homogeneous polynomials $p_1, ..., p_N \in \mathbb{C}[z_0, z_1, ..., z_n]$ such that

$$\pi^{-1}(V) \cup \{0\} = \{z \in \mathbb{C}^{n+1} : p_1(z) = \cdots = p_N(z) = 0\}.$$

An algebraic subset V is said to be **irreducible** if it cannot be written as a union $V = V_1 \cup V_2$ of two algebraic subsets where $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$.

Basic results in algebra tell us that every algebraic subset $V \subset \mathbb{P}^n$ can be written uniquely as a finite union of irreducible ones, and each irreducible one has a well defined dimension (cf. [**4**], [**43**]).

From a differential geometric point of view, irreducibility is nicely characterized as follows. For $V \subset \mathbb{P}^n$, let $\mathrm{Reg}(V)$ denote the set of manifold points of V, i.e., the set of points $x \in V$ for which there is an open neighborhood U and local holomorphic coordinates $(\zeta_1, \cdots, \zeta_n)$ on U such that

$$V \cap U = \{p \in U : \zeta_1(p) = \cdots = \zeta_n(p) = 0\}.$$

From the Weierstrass Preparation Theorem one proves the following. If V is an algebraic subset, then so is $\mathrm{Sing}(V) \stackrel{\mathrm{def}}{=} V - \mathrm{Reg}(V)$. Furthermore,

$$V \text{ is irreducible } \iff \mathrm{Reg}(V) \text{ is connected},$$

and the algebraic dimension of an irreducible V equals the complex dimension of $\mathrm{Reg}(V)$. For a general algebraic subset V, $\mathrm{Reg}(V)$ can be written as a finite disjoint union $\mathrm{Reg}(V) = R_1 \amalg \cdots \amalg R_N$ of submanifolds, and the unique decomposition $V = V_1 \cup \cdots \cup V_N$ is given by $V_j = \overline{R}_j$.

We now introduce some terminology. An irreducible algebraic subset $V \subset \mathbb{P}^n$ is called a **projective subvariety.** The set theoretic difference $V = V_1 - V_2$ of two projective subvarieties is called a **quasi-projective subvariety.** For any such V, let $\mathcal{R}(V)$ denote the **field of rational functions on** V (the restrictions of rational functions on $\mathbb{P}^n$ whose polar divisor does not contain V). Then a **morphism** between quasi-projective subvarieties is a map $f : V_1 \to V_2$ such that $f^*\mathcal{R}(V_2) \subseteq \mathcal{R}(V_1)$. By a **projective** or **quasi-projective variety** we mean an isomorphism class of such subvarieties.

Of fundamental importance to us here is the fact that projective subvarieties $V \subset \mathbb{P}^n$ determine "topological cycles". This can be seen, for example, from the following. Let $\Sigma_1 = V - \mathrm{Reg}(V), \Sigma_2 = \Sigma_1 - \mathrm{Reg}(\Sigma_1)$, etc. denote the **singular strata** of V. Then there exists a semi-algebraic triangulation of V for which the singular strata are subcomplexes. This triangulation is *unique* up to PL homeomorphism (see [**85**]). If $W \subset V$ is also a subvariety, then this triangulation of V can be chosen so that W is a subcomplex and the induced triangulation on W is as above. See [**44**] for an elementary proof. Now fix V with such a triangulation, and suppose $p = \dim(V)$. Let $[V]$ be the chain consisting of all $2p$-dimensional simplices oriented by the canonical orientation of $\mathrm{Reg}(V)$. Then $\partial[V]$ lies in the $2p-2$ skeleton (since it is supported in $V - \mathrm{Reg}(V)$), and so $\partial[V] = 0$. This is the *fundamental cycle* of V. It can be seen to generate $H_{2p}(V;\mathbb{Z}) \cong \mathbb{Z}$.

The cycle $[V]$ also determines a class in $H_{2p}(\mathbb{P}^n;\mathbb{Z}) \cong \mathbb{Z}[\mathbb{P}^p]$ where $\mathbb{P}^p$ is a p-dimensional linear subspace. The integer d such that $[V]$ is homologous to $d[\mathbb{P}^p]$ is called the **degree** of V. One has that $\#(V \cap \mathbb{P}^{n-p}) = d$ for almost all linear subspaces $\mathbb{P}^{n-p}$ of codimension p. Furthermore for almost all $\mathbb{P}^{n-p-1}$ we have $\mathbb{P}^{n-p-1} \cap V = \emptyset$, and the linear projection $\pi : \mathbb{P}^n - \mathbb{P}^{n-p-1} \to \mathbb{P}^p$ restricts to a map

$$\pi : V \to \mathbb{P}^p$$

of degree d.

There is another more intrinsic definition of the cycle $[V]$ in terms of deRham-Federer Theory. Denote by $\mathcal{E}^k(M)$ the space of smooth differential k-forms on a manifold M equipped with the C^∞ topology (uniform convergence of derivatives on compacta). The topological dual space $\mathcal{E}_k(M) \overset{\text{def}}{=} \mathcal{E}^k(M)'$ is called the space of **deRham currents of dimension** k on M. Taking the adjoint of exterior differentation gives a complex $(\mathcal{E}_*(M), d)$ whose homology is isomorphic to $H_*(M;\mathbb{R})$ (cf. [**16**]).

Let now $V \subset \mathbb{P}^n$ be a projective subvariety of dimension p. Then the Hausdorff $2p$-measure of V is finite, and so V defines an element $[V] \in \mathcal{E}_{2p}(\mathbb{P}^n)$ by

$$[V](\varphi) = \int_{\mathrm{Reg}(V)} \varphi \tag{1.1}$$

for all $\varphi \in \mathcal{E}^{2p}(\mathbb{P}^n)$. As a current we have that $d[V] = 0$ i.e.,

$$d[V](\psi) \overset{\text{def}}{=} [V](d\psi) = 0$$

for all $\psi \in \mathcal{E}^{2p-1}(\mathbb{P}^n)$. (For proofs of these and of subsequent assertions about currents, see [**42**]). Of course we have $[V] \cong d[\mathbb{P}^p]$ also in deRham cohomology.

§2. Algebraic cycles. Let $X \subset \mathbb{P}^N$ be an n-dimensional projective subvariety and for each p, $0 \leq p \leq n$ consider the set $X(p)$ of all p-dimensional subvarieties contained in X. In Grothendieck's picture these are the p-dimensional points of X - the p^{th} level of the web of points, curves surfaces etc, which encode the rigid algebro-geometric structure of X. It is natural to consider the following.

Definition 2.1. The group of **p-cycles on** X is the free abelian group $\mathcal{Z}_p(X)$ generated by $X(p)$. **The positive** (or **effective**) **p-cycles on** X are the elements of the free abelian monoid $\mathcal{C}_p(X) \subset \mathcal{Z}_p(X)$ generated by $X(p)$. We will call $\mathcal{C}_p(X)$ the **Chow monoid of** X.

In other words $\mathcal{Z}_p(X)$ consists of all finite formal sums

$$c = \sum n_i V_i$$

where $V_i \in X(p)$ and $n_i \in \mathbb{Z}$ for each i ; and we have $c \in \mathcal{C}_p(X)$ if each $n_i \geq 0$. There is a group homomorphism

$$\deg : \mathcal{Z}_p(X) \to \mathbb{Z}$$

given by $\deg(c) = \sum n_i \text{ degree}(V_i)$.

Letting $\mathcal{C}_{p,d}(X) \subset \mathcal{Z}_{p,d}(X)$ denote the subset of cycles of degree d gives us a graded group and a graded submonoid :

$$\mathcal{C}_p(X) = \coprod_{d=0}^{\infty} \mathcal{C}_{p,d}(X) \subset \mathcal{Z}_p(X) = \bigoplus_{d=-\infty}^{\infty} \mathcal{Z}_{p,d}(X). \tag{2.1}$$

Now comes the magic. In 1937 Chow and van der Waerden discovered the following fundamental result (cf. [**14**], [**70**], [**81**]).

Theorem 2.2. (Chow [**13**]) *Each of the sets $\mathcal{C}_{p,d}(X)$ for $d \geq 0$ carries the structure of a projective algebraic subset.*

When $X = \mathbb{P}^n$ Chow's construction goes as follows. Let $G_{n-p-1}(\mathbb{P}^n) \stackrel{\text{def}}{=} G$ denote the Grassmannian of linear subspaces of codimension $p+1$ on $\mathbb{P}^n$. Holomorphic line bundles on G are in one-to-one correspondence with $\mathbb{Z}$ via the first Chern class c_1. Suppose $V \subset \mathbb{P}^n$ is a projective variety of dimension p and degree d. Set $D_V = \{L \in G : L \cap V \neq 0\}$. Then D_V can be seen to be an algebraic subset of codimension one in G. Any such set is the divisor of a holomorphic section σ_V of a holomorphic line bundle ℓ_d of Chern classs d on G. The section σ_V is unique up to scalar multiples. Thus V determines a point

$[\sigma_V]$ in $\mathbb{P}\left(H^0\left(G; \mathcal{O}\left(\ell_d\right)\right)\right)$. To a general positive cycle $c = \sum n_i V_i$ we associate the section $\sigma_c = \sigma_{V_1}^{n_1} \otimes \cdots \otimes \sigma_{V_k}^{n_k}$. This gives an embedding

$$\mathcal{C}_{p,d}\left(\mathbb{P}^n\right) \hookrightarrow \mathbb{P}\left(H^0\left(G; \mathcal{O}\left(\ell_d\right)\right)\right).$$

A careful analysis involving resultants shows the image to be an algebraic subset. Furthermore it is proven that if $X \subset \mathbb{P}^n$ is an algebraic subset, then $\mathcal{C}_{p,d}(X) \subset \mathcal{C}_{p,d}\left(\mathbb{P}^n\right)$ is also an algebraic subset.

Notice what this gives us. Our monoid $\mathcal{C}_p(X)$ is now equipped with a topology so that each piece $\mathcal{C}_{p,d}(X)$ is a compact Hausdorff space, in fact an *algebraic set*. The addition map $\mathcal{C}_p(X) \times \mathcal{C}_p(X) \to \mathcal{C}_p(X)$ is easily seen to be an algebraic map on these components. Hence, $\mathcal{C}_p(X)$ is an algebraic abelian monoid – quite a nice object !

It is natural to wonder about the uniqueness of this canonical algebraic structure on $\mathcal{C}_p(X)$. For this we need the following.

Definition 2.3. A **continuous algebraic map** is a map $\varphi : V \to W$ between projective algebraic subsets whose graph is an algebraic subset of the product $V \times W$.

If V is normal (in particular if V is smooth) every such map is a morphism. Note however that the inverse of the map $\mathbb{C} \to Y = \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$ given by $t \mapsto (t^3, t^2)$, is continuous algebraic but not a morphism. We now have the following.

Proposition 2.4. *The canonical algebraic structure on $\mathcal{C}_p(X)$ is uniquely determined up to algebraic homeomorphism by any projective embedding of X.*

Proof. (Sketch.) Let $j : X \hookrightarrow \mathbb{P}^n$ be the given embedding and suppose $j' : X \hookrightarrow \mathbb{P}^{n'}$ is another. Using the Veronese embedding (i.e., the tensor product of homogeneous coordinates) we get an embedding $j \times j' : X \times X \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{n'} \subset \mathbb{P}^{nn'+n+n'}$. Define $\Delta : X \hookrightarrow \mathbb{P}^{nn'+n+n'}$ via the diagonal in $X \times X$. The Veronese is linear on each factor, so our original embeddings are recaptured by restriction. Now we see above that if $A \subset B \subset \mathbb{P}^N$ are projective varieties, then $\mathcal{C}_{p,d}(A)$ is an algebraic subset of $\mathcal{C}_{p,d}(B)$ for all d. Thus we have three algebraic embeddings

$$\mathcal{C}_p(X) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathcal{C}_p(X) \times \mathcal{C}_p(X) \subset \mathcal{C}_p(X \times X)$$

corresponding to j, j' and Δ. Since $\Delta\mathcal{C}_p(X)$ is the graph of the identity map on $\mathcal{C}_p(X)$, and it is also algebraic, we are done. □

Note. In the above proof it is better to use the intrinsic grading of $\mathcal{C}_p(X)$ given by the map

$$\mathcal{C}_p(X) \to H_{2p}(X; \mathbb{Z}). \tag{2.2}$$

From 2.4 we conclude that the topology on $\mathcal{C}_p(X)$ is intrinsically defined, i.e., it depends only on the isomorphism class of X. This makes it natural to pass the topology on to the group completion $\mathcal{Z}_p(X)$. Note that

$$\mathcal{Z}_p(X) = \mathcal{C}_p(X) \times \mathcal{C}_p(X)/\sim \tag{2.3}$$

where $(c_1, c_2) \sim (c_1', c_2') \iff c_1 + c_2' = c_2 + c_1'$. Therefore, taking equivalence classes gives a surjective map

$$\mathcal{C}_p(X) \times \mathcal{C}_p(X) \xrightarrow{\pi} \mathcal{Z}_p(X). \tag{2.4}$$

Now $\mathcal{C}_p(X) \times \mathcal{C}_p(X)$ is a monotone union of compact sets

$$\widehat{K_i} = \coprod_{d+d' \leq i} \mathcal{C}_{p,d}(X) \times \mathcal{C}_{p,d'}(X) \tag{2.5}$$

for $i \geq 0$. The equivalence relation is closed and so the quotients

$$K_i = \pi \widehat{K_i}$$

are compact Hausdorff spaces topologically embedded in one another:

$$K_0 \subset K_1 \subset K_2 \subset K_3 \subset \cdots \subset \mathcal{Z}_p(X) \tag{2.6}$$

with $\bigcup K_i = \mathcal{Z}_p(X)$. Whenever one is in this situation, there is a natural topology induced on the space, called **the topology associated to the family** $\{K_i\}$. It is defined by declaring subset C to be closed if and only if $C \cap K_i$ is closed for all i. With this topology $\mathcal{Z}_p(X)$ *is a topological group.*

This group is characterized by the universal property that any continuous homomorphism $h : \mathcal{C}_p(X) \to G$ into an abelian topological group G determines a continuous homomorphism $\widetilde{h} : \mathcal{Z}_p(X) \to G$ so that

$$\begin{array}{ccc} \mathcal{C}_p(X) & & \\ & \searrow & \\ \downarrow & & G \\ & \nearrow & \\ \mathcal{Z}_p(X) & & \end{array}$$

commutes.

Remark 2.5. In a beautiful paper [**54**] P. Lima-Filho recently established several equivalent formulations of the topology on $\mathcal{Z}_p(X)$. One is engendered by considering flat families of cycles over smooth base spaces and is related to work of Rojtman. With this definition many properties, such as functoriality, are clear. Another definition involves "Chow envelopes" and is useful for establishing the existence of fibration sequences, etc. Lima-Filho shows that these definitions with all their properties extend to *arbitrary algebraic varieties* (not just quasi-projective ones), and that on this general category they *coincide.*

At this point it could be useful to examine a number of examples.

§3. Symmetric products. Note that for any projective variety X

$$\begin{aligned} \mathcal{C}_{0,d}(X) &= \{\Sigma n_i x_i : x_i \in X \text{ and } n_i \in \mathbb{Z}^+ \text{ with } \Sigma n_i = d\} \\ &= X \times \cdots \times X / S_d \\ &\stackrel{\text{def}}{=} SP^d(X) \text{ (the } d\text{–}fold\ symmetric\ product) \end{aligned}$$

where S_d is the symmetric group acting by permutation of the factors. Hence,

$$\mathcal{C}_0(X) = \coprod_{d \geq 0} SP^d(X)$$

is the free abelian topological monoid generated by the space: its components are evidently varieties. A particularly nice case is that where $X = \mathbb{P}^1$.

Lemma 3.1. *As projective varieties we have that $SP^d(\mathbb{P}^1) = \mathbb{P}^d$. Hence,*

$$\mathcal{C}_0(\mathbb{P}^1) = \coprod_{d \geq 0} \mathbb{P}^d.$$

Proof. Define the map $\mathbb{P}^d \to SP^d(\mathbb{P}^1)$ by assigning to the point with homogeneous coordinates $[a_0 : a_1 : \cdots : a_d]$ the zeros of the homogeneous polynomial equation

$$\sum_{k=0}^{d} a_k z_0^k z_1^{d-k} = 0.$$

The inverse to this map is given by expressing the coefficients of a polynomial as elementary symmetric functions of its roots; namely, if $[\xi_1 : \eta_1], \cdots, [\xi_d : \eta_d]$ are d-points in $\mathbb{P}^1$, then $[a_0 : \cdots : a_d]$ are the coefficients of the polynomial

$$p(z_0, z_1) = \prod_{i=1}^{d} (\xi_i z_1 - \eta_i z_0). \qquad \square$$

Note that the additive structure in this monoid is given by the maps

$$\mathbb{P}^d \times \mathbb{P}^{d'} \to \mathbb{P}^{d+d'}$$

$$([a], [b]) \mapsto [c]$$

where $c_k = \sum_{i+j=k} a_i b_j \quad \text{for} \quad k = 0, \cdots, d + d'$.

The case where X is a non-singular curve of higher genus is even more interesting. Here we must use more sophisticated geometry. By an elementary construction (cf. **[64]** and §4 below) one associates to every positive 0-cycle $\sum n_i x_i$ on the curve X, a holomorphic line bundle ℓ of degree $d = \sum n_i$, and a holomorphic section σ of ℓ such that $\sum n_i x_i$ is the zero divisor of σ. The pair (ℓ, σ) is unique up to scalar multiples of σ. Now holomorphic line bundles on X correspond to elements in $H^1(X, \mathcal{O}^\times)$ where $\mathcal{O}^\times$ is the sheaf of non-vanishing holomorphic functions on X. It sits in an exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$ which gives an exact sequence

$$0 \to H^1(X;\mathcal{O})/H^1(X;\mathbb{Z}) \to H^1(X;\mathcal{O}^\times) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to 0 \tag{3.1}$$

where c_1 is the degree or first Chern class of the bundle. Resolving $\mathcal{O}$ and using harmonic theory gives an isomorphism $H^1(X;\mathcal{O}) \cong H^1(X, \mathbb{R})$, and (3.1) is of the form

$$0 \to \mathbb{R}^{2g}/\mathbb{Z}^{2g} \to H^1(X;\mathcal{O}^\times) \to \mathbb{Z} \to 0.$$

In particular the components of $H^1(X;\mathcal{O}^\times)$ are all tori of dimension $2g$ where g is the genus of X. The map above gives us a monoid homomorphism

$$\mathcal{C}_0(X) \to H^1(X;\mathcal{O}^\times) \tag{3.2}$$

The preimage of each point ℓ is the projective space $\mathbb{P}\left(H^0\left(X;\mathcal{O}(\ell)\right)\right)$ of all global holomorphic sections of ℓ. Hence, component by component we get maps

$$SP^d(X) \to \mathbb{R}^{2g}/\mathbb{Z}^{2g}.$$

For d sufficiently large, this is surjective. In fact it is a fibre bundle whose fibre is $\mathbb{P}^{d-g}$ (a non-obvious result even topologically).

We now observe that for any topological space Y, the symmetric products $SP^d(Y) = Y \times \cdots \times Y/S_d$ and therefore the topological monoid $\mathcal{C}_0(Y)$ are well defined. When Y is compact and Hausdorff we can also define the topological group $\mathcal{Z}_0(Y)$ exactly as in §2 above. The spaces $SP^d(Y)$ are beautiful and of fundamental importance in topology. This is due to the following classical result originally conjectured by Serre. It was proved by Dold and Thom and, independently and simultaneously, by Ioan James.

Theorem 3.2. (Dold and Thom [**17**], [**18**]). *Let Y be a connected finite complex with base point y_0. Then under the embeddings $SP^d(Y) \hookrightarrow SP^{d+1}(Y)$ given by $c \mapsto c + y_0$, there is an isomorphism*

$$\varinjlim_d \pi_*\left(SP^d(Y)\right) \cong \tilde{H}_*(Y;\mathbb{Z}). \tag{3.3}$$

Furthermore for any finite complex Y, there is an isomorphism

$$\pi_* \mathcal{Z}_0(Y) \cong H_*(Y;\mathbb{Z}). \tag{3.4}$$

The first statement can be rephrased by considering the limiting space

$$SP(Y) = \varinjlim_d SP^d(Y)$$

with topology induced by the family of compact sets $K_i = SP^i(Y)$ as in (2.6)-forward. This space, called **the infinite symmetric product of** Y, has the property that

$$\pi_* SP(Y) \cong \tilde{H}_*(Y;\mathbb{Z}). \tag{3.5}$$

One nice consequence of 3.2 is that it gives beautiful models of Eilenberg-MacLane spaces. Recall that for any finitely generated abelian group Γ, the Eilenberg-MacLane space $K(\Gamma, n)$ is the space, unique up to homotopy equivalence in the category of countable CW-complexes, such that

$$\pi_k K(\Gamma, n) = \begin{cases} \Gamma & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

(See [**80**] for this and what follows). These spaces are classifying spaces for the functor $H^n(\bullet;\Gamma)$ in the sense that for any finite complex Y, there is a natural isomorphism

$$H^n(Y;\Gamma) \cong [Y, K(\Gamma, n)] \tag{3.6}$$

where $[Y, K(\Gamma, n)] = \pi_0 \text{ Map } (Y, K(\Gamma, n))$ denotes homotopy classes of maps from Y to $K(\Gamma, n)$.

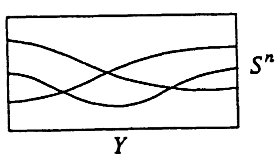
Theorem 3.2 gives homotopy equivalences

$$K(\mathbb{Z}, n) \cong SP(S^n) \cong \mathcal{Z}_0(S^n) \tag{3.7}$$

for all n. Hence, from (3.6) we see that for a connected finite complex Y,

$$\begin{aligned} H^n(Y;\mathbb{Z}) &\cong [Y, SP(S^n)] \cong \varinjlim_d \, [Y, SP^d(S^n)] \\ &= \varinjlim_d \, \pi_0 \text{ Map}\left(Y, SP^d(S^n)\right). \end{aligned} \tag{3.8}$$

This is interesting since maps from Y to $SP^d(S^n)$ are simply d-valued maps from Y to S^n. They correspond, under graphing, to topological cycles in the product $Y \times S^n$ which project d-to-1 onto Y. We will return to this point later when we discuss algebraic cocycles.



Equation (3.8) generalizes to higher homotopy groups. One has that

$$H^{n-k}(Y;\mathbb{Z}) \cong \pi_k \{\text{Map}(Y, SP(S^n))\} \tag{3.9}$$

for all $k \geq 0$.

There is a relative version of Theorem 3.2 which was important in the original proof.

Theorem 3.3. (Dold and Thom [**17**], [**18**]). *Let $A \subset Y$ be a pair of finite complexes (i.e., A is a subcomplex of Y). Then there is a natural isomorphism*

$$\pi_* \left\{\mathcal{Z}_0(Y)/\mathcal{Z}_0(A)\right\} \cong H_*(X, A; \mathbb{Z}). \tag{3.10}$$

Furthermore, given any integer $m > 0$, there is a natural isomorphism

$$\pi_* \left\{\mathcal{Z}_0(Y)/m\mathcal{Z}_0(Y)\right\} \cong H_*(Y; \mathbb{Z}_m). \tag{3.11}$$

Proving (3.10) involves proving that the homomorphism $Z_0(X) \to Z_0(X)/Z_0(Y)$ is a principal bundle. The long exact sequence for π_* then results in the long exact sequence in homology for the pair.

Note that algebraically we have that

$$\mathcal{Z}_0(X)/m\mathcal{Z}_0(X) = \mathcal{Z}_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}_m$$

is just the free $\mathbb{Z}_m$-module generated by the points of X. The topology on this is interesting to contemplate.

It is an important fact that the results of Dold-Thom completely determine the homotopy type of these spaces. This is due to the following result.

Theorem 3.4 (John Moore [**66**]). *Let $\mathcal{A}$ be a connected topological abelian monoid or a topological abelian group. Then $\mathcal{A}$ is homotopy equivalent to a product of Eilenberg-MacLane spaces.*

In other words the Postnikov k-invariants all vanish, and so Y is completely determined by its homotopy groups. Thus Theorem 3.2 implies that for any connected finite complex Y there are homotopy equivalences

$$\mathbb{Z} \times SP(Y) \cong \mathcal{Z}_0(Y) \cong \prod_{p=0}^{\infty} K\left(H_p(Y; \mathbb{Z}), p\right). \tag{3.12}$$

There are analogous statements for $\mathcal{Z}_0(Y)/\mathcal{Z}_0(A)$ and $\mathcal{Z}_0(Y)/m\mathcal{Z}_0(Y)$ corresponding to (3.10) and (3.11).

Now for a projective variety X we see that $\mathcal{C}_p(X)$ and $\mathcal{Z}_p(X)$ are natural generalizations of $\mathcal{C}_0(X)$ and $\mathcal{Z}_0(X)$ to the p-dimensional points of the space. It is certainly intriguing to speculate about the extent to which these gorgeous results can be generalized.

§**4. Divisors.** Let us now examine cycles of codimension one. Suppose X is a non-singular projective variety of dimension n. The fundamental result here is that locally every effective cycle of codimension one on X is the divisor of a holomorphic function, in fact a rational function which is regular in the neighborhood and is unique up to multiplication by non-vanishing functions. Thus given $c \in \mathcal{C}_{n-1}(X)$, there is a family $\{(U_i, f_i)\}_{i=1}^N$, where $\{U_i\}_{i=1}^N$ is an open covering of X, and $f_i \in \mathcal{O}(U_i)$ has the property that

$$c_{|U_i} = \mathrm{Div}(f_i).$$

The quotients $g_{ij} = f_i/f_j : U_i \cap U_j \to \mathbb{C} - \{0\}$ define transition functions for a line bundle ℓ_c on X for which the f_i determine a global holomorphic cross-section. (See **[64]** for more details).

When $X = \mathbb{P}^n$ there is exactly one holomorphic line bundle for each integer d, which is denoted $\mathcal{O}(d)$. The sections of $\mathcal{O}(d)$ for $d > 0$ are in natural one-to-one correspondence with homogeneous polynomials of degree d in $(n+1)$-variables. This gives the following generalization of Lemma 3.1 above.

Lemma 4.1.

$$\mathcal{C}_{n-1}(\mathbb{P}^n) = \coprod_{d \geq 0} \mathbb{P}^{\binom{n+d}{d}-1}.$$

For a general X, the construction above gives a homomorphism

$$\mathcal{C}_{n-1}(X) \to H^1(X; \mathcal{O}^\times) \tag{4.1}$$

generalizing (3.2). The preimage of $\ell \in H^1(X;\mathcal{O}^\times)$ under (4.1) is the projective space $\mathbb{P}\left(H^0(X, \mathcal{O}(\ell))\right)$ of holomorphic sections of ℓ. One can prove that there is an exact sequence

$$0 \to H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) \to H^1(X;\mathcal{O}^\times) \to NS(X) \to 0$$

where $NS(X) = H^{1,1}(X;\mathbb{Z}) \subseteq H^2(X;\mathbb{Z})$ is the set of classes whose image in $H^2(X;\mathbb{C})$ is represented by a $(1,1)$-form. $NS(X)$ is called the **Neron-Severi group** of X. Thus $H^1(X;\mathcal{O}^\times)$ is a discrete group extended by a torus of dimension $b_1(X) = \mathrm{rank}\,(H_1(X;\mathbb{R}))$.

By universality, the homomorphism (4.1) extends to a continuous homomorphism

$$\mathcal{Z}_{n-1}(X) \to H^1(X;\mathcal{O}^\times). \tag{4.2}$$

This extension is explicitly given by extending the construction given at the beginning of this section to general (non-positive) cycles. Every line bundle admits a meromorphic (i.e., rational) section, and the quotient of two sections of the same bundle is a rational function. Hence, (4.2) expands to an exact sequence

$$0 \to \mathbb{P}\left(\mathcal{K}^\times(X)\right) \to \mathcal{Z}_{n-1}(X) \to H^1(X;\mathcal{O}^\times) \to 0 \tag{4.3}$$

where $\mathbb{P}(\mathcal{K}^\times(X))$ is the projective group of the non-zero rational functions on X under multiplication.

In more classical terms there is a tautologically defined exact sequence of groups

$$0 \to \mathbb{P}\left(\mathcal{K}^\times(X)\right) \to \mathcal{Z}_{n-1}(X) \to \mathrm{Pic}(X) \to 0$$

where $\mathrm{Pic}(X)$ is the Picard group of divisors on X modulo rational equivalence. The above remarks constitute a computation of $\mathrm{Pic}(X)$ in terms of sheaf cohomology.

What is of interest here is that with our given topology on $\mathcal{Z}_{n-1}(X)$ the homomorphism (4.2) is continuous, and in fact (4.3) is a fibration. Setting $\mathbb{P}^\infty = \varinjlim_d \mathbb{P}^d$ we have the following result of E. Friedlander.

Theorem 4.2 [22] *Let X be any non-singular projective variety of dimension n. Then there is a homotopy equivalence*

$$\mathcal{Z}_{n-1}(X) \cong NS(X) \times \left(\frac{H^1(X;\mathbb{R})}{H^1(X;\mathbb{Z})}\right) \times \mathbb{P}^\infty.$$

From this we see that certain classical invariants occur as homotopy groups of $\mathcal{Z}_{n-1}(X)$ namely

$$\pi_0\mathcal{Z}_{n-1}(X) \cong NS(X)$$
$$\pi_1\mathcal{Z}_{n-1}(X) \cong H^1(X;\mathbb{Z}) \cong H_{2n-1}(X,\mathbb{Z})$$

and also $\pi_2\mathcal{Z}_{n-1}(X) \cong \mathbb{Z}$.

§5. Curves on a 3-fold. The next interesting case to examine is when $p = 1$ and $n = 3$. Here life can be quite complicated. Consider for example $X = \mathbb{P}^3$. Every cycle of degree 1 is linear, so

$$\mathcal{C}_{1,1}\left(\mathbb{P}^3\right) = \mathcal{G}_2\left(\mathbb{C}^4\right) \equiv \mathcal{G}$$

where $\mathcal{G}_2\left(\mathbb{C}^4\right)$ denotes the Grassmannian of 2-planes in $\mathbb{C}^4$. It is not difficult to see that

$$\mathcal{C}_{1,2}\left(\mathbb{P}^3\right) = SP^2(\mathcal{G}) \cup Q$$

where $SP^2(\mathcal{G})$ corresponds to pairs of lines in $\mathbb{P}^3$ and Q consists of plane quadrics, i.e., all quadratic curves lying in hyperplanes in $\mathbb{P}^3$. These two subsets of $\mathcal{C}_{1,2}\left(\mathbb{P}^3\right)$ are algebraically irreducible and of dimension 8. The set $SP^2(\mathcal{G}) \cap Q$ consists of degenerate plane quadrics, i.e., pairs of lines which meet in $\mathbb{P}^3$. It has dimension 7.

In degree 3 we have the decomposition

$$\mathcal{C}_{1,3}\left(\mathbb{P}^3\right) = SP^3(\mathcal{G}) \cup (\mathcal{G} + Q) \cup C \cup N$$

where $SP^3(\mathcal{G})$ consists of triples of lines, $\mathcal{G} + Q = \{\ell + q : \ell \in \mathcal{G} \text{ and } q \in Q\}$, C consists of plane cubics, and N consists of "twisted cubics" and their limits. Each of these is irreducible and of dimension 12. The elements in N which do not belong to other components are exactly the images of maps $\mathbb{P}^1 \to \mathbb{P}^3$ given by a full basis of homogeneous polynomials of degree 3, the so-called rational normal curves in $\mathbb{P}^3$. One example is the map

$$[z_0 : z_1] \mapsto \left[z_0^3 : \sqrt{3}z_0^2 z_1 : \sqrt{3}z_0 z_1^2 : z_1^3\right]$$

(which has constant curvature 1/3 in the standard metric). All others are obtained from this one by a change of basis in $\mathbb{C}^4$, i.e., N is the closure of an orbit of $PGL_4(\mathbb{C})$ acting on $\mathcal{C}_{1,3}\left(\mathbb{P}^3\right)$.

It is interesting to examine the intersections of the various components of $\mathcal{C}_{1,3}\left(\mathbb{P}^3\right)$. For example $C \cap N$ consists of those plane cubics which are rational. It has dimension 11 and generically fibres over $\left(\mathbb{P}^3\right)^*$.

Clearly as d increases the geometry of $\mathcal{C}_{1,d}\left(\mathbb{P}^3\right)$ becomes tremendously complicated. Each map

$$\mathcal{C}_{1,d'}\left(\mathbb{P}^3\right) \times \mathcal{C}_{1,d''} \to \mathcal{C}_{1,d}\left(\mathbb{P}^3\right)$$

given by addition, where $d' + d'' = d$, contributes a large number of irreducible components to $\mathcal{C}_{1,d}\left(\mathbb{P}^3\right)$. In addition to this there will always be new ones, for example plane curves of degree d. For further discussion see [**74**] for example.

§6. The Euler-Poincaré series of the Chow monoid. Despite their complicated structure, it is possible to compute the Euler characteristics of the Chow sets $\mathcal{C}_{p,d}(\mathbb{P}^n)$. An intriguing consequence of the calculation is that the generating function associated to these numbers is rational. For fixed $p \le n$ consider the formal power series in one variable

$$\Psi_{p,n}(t) = \sum_{d=0}^{\infty} \chi\left(\mathcal{C}_{p,d}(\mathbb{P}^n)\right) t^d \tag{6.1}$$

where $\chi(Y)$ denotes the Euler characteristic of Y. Then in collaboration with Steven Yau the following was proved.

Theorem 6.1. ([**60**]) *For all* $0 \le p < n$

$$\Psi_{p,n}(t) = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}} \tag{6.2}$$

When $p = n - 1$, i.e., in the case of divisors, this is a classical Hilbert polynomial calculation. When $p = 0$, it is a special case of the general MacDonald formula [**61**]

$$\sum_{d=0}^{\infty} \chi\left(SP^d(Y)\right) t^d = \left(\frac{1}{1-t}\right)^{\chi(Y)}$$

which holds for any connected finite complex Y. Nevertheless, the rationality of $\Psi_{p,n}(t)$ for general p is somewhat surprising. Note that the exponent in (6.2) is just the Euler characteristic of the Grassmannian $\mathcal{C}_{p,1}(\mathbb{P}^n)$ of p-planes in $\mathbb{P}^n$.

The result above has been generalized by J. Elizondo to general toric varieties. Note that for any projective variety X and any $p < \dim(X)$ we can define

$$\Psi_p = \sum_{\alpha \in \Gamma_{2p}} \chi(\mathcal{C}_{p,\alpha}(X))\,\alpha$$

where $\Gamma_{2p} = H_{2p}(X;\mathbb{Z})_{mod\ torsion}$, by convention $\chi(\emptyset) = 0$. Given a basis $e_1, \cdots, e_N$ of Γ_{2p} we let $t_1, \cdots, t_N$ denote the linear coordinates on $H^{2p}(X;\mathbb{R})$ with respect to the dual basis. We then "rewrite" Ψ_p as a formal sum

$$\Psi_p = \sum_{n \in \mathbb{Z}^N} \chi(\mathcal{C}_{p,n}(X))\,t^n. \tag{6.3}$$

Theorem 6.2. (J. Elizondo [**19**], [**20**]). *If X is a non-singular toric variety, then Ψ_p is an intrinsically defined rational function on $H^{2p}(X;\mathbb{R})$ which can be explicitly and canonically computed from the combinatorial data (the "fan") of the variety.*

This result has an elegant formulation in terms of equivariant cohomology suggested by E. Bifet (See [**19**], [**20**]).

§**7. Functoriality.** Despite their complicated nature, the Chow monoid and its group completion do behave nicely under algebraic maps. Let

$$f : X \to Y$$

be a morphism of projective varieties and suppose $V \subset X$ is a irreducible subvariety of codimension p. Then $f(V) \subset Y$ is a subvariety of Y, and we define

$$f_*V = \begin{cases} 0 & \text{if } \dim(f(V)) < p \\ kf(V) & \text{if } \dim(f(V)) = p \end{cases}$$

where k is the degree of the map $f : V \to f(V)$. This determines group homomorphisms

$$f_* : \mathcal{C}_p(X) \to \mathcal{C}_p(Y) \tag{7.1}$$

and

$$f_* : \mathcal{Z}_p(X) \to \mathcal{Z}_p(Y). \tag{7.2}$$

Proposition 7.1. (Friedlander [**22**]) *The homomorphisms* (7.1) *and* (7.2) *are continuous.*

These homomorphisms clearly have the property that

$$(g \circ f)_* = g_* \circ f_*$$

for morphisms $f : X \to Y$ and $g : Y \to Z$.

For nice morphisms there are maps in the opposite direction.

Proposition 7.2. (cf. [**34**], [**22**]) *Suppose $f : X \to Y$ is a flat morphism of projective varieties with fibre dimension k. Then the flat pull-back of cycles gives continuous homomorphisms*

$$f^* : \mathcal{C}_p(Y) \to \mathcal{C}_{p+k}(X)$$
$$f^* : \mathcal{Z}_p(Y) \to \mathcal{Z}_{p+k}(X).$$

Examples of flat morphisms are proper submersions and branched coverings. For a rigorous definition of flatness see [**43**]

§8. The homotopy relationship between $\mathcal{C}_p(X)$ and $\mathcal{Z}_p(X)$. In our discussion there are two objects of interest. The first is the Chow monoid $\mathcal{C}_p(X)$, a geometric object. The components of $\mathcal{C}_p(X)$ are algebraic spaces whose structure we would like to understand and relate to X.

The second object is $\mathcal{Z}_p(X)$, the topological group of all p-cycles on X. This is a fundamental algebraic object attached to X. It would also be nice to understand the topological structure of $\mathcal{Z}_p(X)$. In fact by Theorem 3.4 above we know that the homotopy type of $\mathcal{Z}_p(X)$ is completely determined by the homotopy groups $\pi_*\mathcal{Z}_p(X)$. Each group $\pi_*\mathcal{Z}_p(\bullet)$ is an interesting functor on the category of projective varieties.

Now $\mathcal{Z}_p(X)$ is simply the naïve topological group completion of $\mathcal{C}_p(X)$ and one might hope that the homotopy of $\mathcal{Z}_p(X)$ is somehow a "completion" of that of $\mathcal{C}_p(X)$. In fact we can be quite specific. Let $\mathcal{M}$ be an abelian topological monoid, and set $\Gamma = \pi_0\mathcal{M}$. Let

$$\mathcal{M} = \coprod_{\alpha\in\Gamma} \mathcal{M}_\alpha$$

denote the decomposition into connected components, and choose an element $x_a \in \mathcal{M}_\alpha$ for each α. Then we can define continuous maps

$$f_\alpha : \mathcal{M} \times \Gamma \to \mathcal{M} \times \Gamma$$

by setting $f_\alpha(x, \beta) = (x + x_\alpha, \beta + \alpha)$. The homotopy class of f_α depends only on $\alpha \in \Gamma$. Now Γ is a directed system (where $\beta > \alpha \iff \beta = \alpha + \gamma$ for $\gamma \in \Gamma$), and $f_{\alpha+\beta}$ is homotopic to $f_\alpha + f_\beta$ for all α, β. Hence, for any covariant homotopy functor h we can define the direct limit

$$\varinjlim_{\alpha\in\Gamma} h(\mathcal{M} \times \{\alpha\}).$$

Suppose now that $\widetilde{\mathcal{M}} = \mathcal{M}\times\mathcal{M}/\sim$ is the naïve topological group completion, where $(x,y) \sim (x',y') \iff \exists z \in \mathcal{M}$ with $x + y' + z = x' + y + z$, and where one takes on $\widetilde{\mathcal{M}}$ the quotient topology in the compactly generated category (cf. [**80**]). Then one might naïvely hope that

$$h(\widetilde{\mathcal{M}}) = \varinjlim_{\alpha\in\Gamma} h(\mathcal{M} \times \{\alpha\}), \tag{8.1}$$

and in particular that

$$\pi_k(\widetilde{\mathcal{M}}) = \varinjlim_{\alpha\in\Gamma} \pi_k(\mathcal{M}_\alpha) \tag{8.2}$$

for $k > 0$. In fact for general $\mathcal{M}$ this is almost certainly not true. There are nice cases, such as

$$\mathcal{M} = \mathcal{C}_0(X) \qquad \text{and} \qquad \widetilde{\mathcal{M}} = \mathcal{Z}_0(X)$$

where it does hold (cf. Theorem 3.2). However the standard proofs are difficult and quite indirect.

In general homotopy theorists ignore this question because there exists a substitute for $\widetilde{\mathcal{M}}$ which has the good property (8.1). It is the homotopy-theoretic group completion

$$\mathcal{M}^+ \stackrel{\text{def}}{=} \Omega B\mathcal{M} \tag{8.3}$$

where $B\mathcal{M}$ is the classifying space of the monoid obtained from the standard bar construction, and ΩY denotes the loop space of Y. (See [**22**] for a detailed discussion.) There is a canonical homotopy class of maps $\mathcal{M} \to \mathcal{M}^+$ which is an equivalence when $\mathcal{M}$ is a group, and there is a model [**52**] for $\mathcal{M}^+$ which admits a map

$$\mathcal{M}^+ \xrightarrow{\psi} \widetilde{\mathcal{M}} \tag{8.4}$$

so that

$$\begin{array}{ccc} & & \mathcal{M}^+ \\ & \nearrow & \\ \mathcal{M} & & \downarrow\psi \\ & \searrow & \\ & & \widetilde{\mathcal{M}} \end{array}$$

commutes. The desired relationship (8.1) (and (8.2)) will hold if ψ is a homotopy equivalence.

The author conjectured several years ago that this should be true when $\mathcal{M} = \mathcal{C}_p(X)$ for a projective variety X. The conjecture proved to be useful but quite hard. Friedlander made important progress on it. The first complete proof was given by P. Lima-Filho in a beautiful paper [**52**] in which several other basic results are established. Following this a somewhat stronger result was proved by E. Friedlander and O. Gabber [**28**]. (This stronger result can also be obtained from methods in [**52**].)

Theorem 8.1. (Lima-Filho [**52**], Friedlander, Gabber [**28**]) *Let $\mathcal{M} = \mathcal{C}_p(X)$ be the Chow monoid in dimension p of a projective variety X. Then the map (8.4) between the homotopy-theoretic and naïve group completions of $\mathcal{M}$ is a homotopy equivalence.*

Corollary 8.2. *Let $\mathcal{C}_p(X)$ be as above. Set $\Gamma = \pi_0\, \mathcal{C}_p(X)$ and let*

$$\mathcal{C}_p(X) = \coprod_{\alpha\in\Gamma} \mathcal{C}_{p,\alpha}(X)$$

be the decomposition into connected components. Then

$$\pi_0 \mathcal{Z}_p(X) \cong \widetilde{\Gamma}$$

(the algebraic group completion of the monoid Γ), and

$$\pi_k \mathcal{Z}_p(X) \cong \varinjlim_{\alpha}\ \pi_k\, \mathcal{C}_{p,\alpha}(X)$$

for all $k > 0$. Furthermore,

$$H_* \mathcal{Z}_p(X) \cong H_*\left(\mathcal{C}_p(X)\right) \oplus_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\widetilde{\Gamma}].$$

The Friedlander-Gabber proof uses the fact that $\mathcal{Z}_p$ can built out of quotients of varieties by algebraic equivalence relations. (They work in a nice category which contains varieties and is closed under push-out's). Lima-Filho's arguments are couched in more topological terms, and give also the Dold-Thom result that $\mathcal{Z}_0(X) \cong \mathbb{Z} \times SP(X)$ for any connected finite complex X.

The result above shows that the functors $\pi_* \mathcal{Z}_*(X)$ and $H_* \mathcal{Z}_*(X)$ are related to the homotopy and homology of the Chow varieties of X.

In the absence of this theorem one could replace $\mathcal{Z}_p(X)$ by $\Omega B\mathcal{C}_p(X)$ (or equivalently just $B\mathcal{C}_p(X)$) and obtain interesting functors. In fact this is important for extending the theory to varieties defined in characteristic $p > 0$ (See IV. 13.). However, in doing this one loses direct contact with the variety and such theorems as localization, discussed in Chapter IV, are difficult to establish.

§**9. Cycles and the Plateau problem.** For varieties defined over $\mathbb{C}$ the components of $\mathcal{C}_p(X)$ have a beautiful geometric interpretation. Fix $\gamma \in H_{2p}(X;\mathbb{Z})$ and let $\mathcal{C}_{p,\gamma}(X)$ denote the set of all $c \in \mathcal{C}_p(X)$ whose homology class is γ. This is a finite union of connected components of $\mathcal{C}_p(X)$ and is very possibly empty. However, whenever there exists a cycle $c \in \mathcal{C}_{p,\gamma}(X)$, H. Federer [**26**] proved the following. *For any Kähler metric on X, c is a current of least mass (i.e., weighted volume) in its homology class γ. That is,*

$$\text{Mass}(c) \le \text{Mass}(c') \qquad (9.1)$$

for all rectifiable cycles c' on X which represent γ (cf. [**27**]*). Furthermore, equality occurs in* (9.1) *if and only if* $c' \in \mathcal{C}_{p,\gamma}(X)$. (Note: When X is not smooth, a "Kähler metric on X" means a Kähler metric defined on some neighborhood of X in $\mathbb{P}^N$.) Hence whenever it is non-empty, $\mathcal{C}_{p,\gamma}(X)$ is precisely the moduli space of all solutions to the Plateau problem for the homology class γ on X.

This says that when it is not empty, $\mathcal{C}_{p,\gamma}(X)$ embeds into the space $\mathfrak{Z}_{2p,\gamma}(X)$ of rectifiable $2p$-cycles on X with homology class γ, as the set of minimum points of the continuous function

$$\text{Mass} : \mathfrak{Z}_{2p,\gamma}(X) \longrightarrow \mathbb{R}^+.$$

One may wonder whether $\mathcal{C}_{p,\gamma}(X)$ is connected. This turns out not to hold in general. However, for certain basic spaces, such as projective spaces, Grassmannians, general flag manifolds, etc, this and much more are true. For such spaces the inclusion

$$\mathcal{C}_{p,\gamma}(X) \hookrightarrow \mathfrak{Z}_{2p,\gamma}(X)$$

becomes a **homotopy equivalence** as γ tends to infinity in the partially ordered monoid $\pi_0\mathcal{C}_p(X)$. These results are discussed in Chapter IV.

In recent years there have been examples of geometric variational problems where "as the degree goes to infinity" the set of absolute minima gives a homotopy approximation to the space. This was seen in the work of G. Segal and others [**72**], [**12**], [**65**] where the space of rational maps of $\mathbb{P}^1$ into a good variety (as above) approximates the space of all continuous maps. It also appears in the theory of SU_2 gauge fields over S^4 where as the degree of the bundle increases, the finite-dimensional space of self-dual connections approximates the space of all connections modulo gauge equivalence (cf. [**3**], [**77**], [**11**]). Algebraic cycles provide another example of this phenomenon.

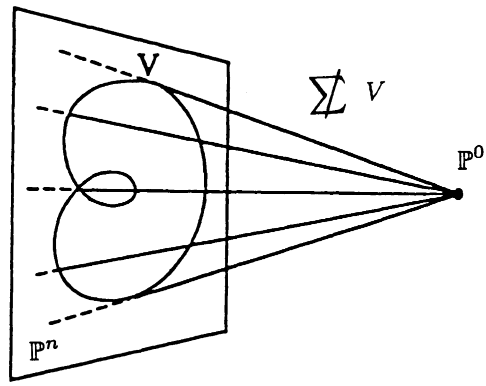
Chapter II - Suspension and Join

In this section we introduce two elementary constructions on projective subvarieties and discuss the Algebraic Suspension Theorem which is the key to much of the subsequent material.

§1. Algebraic suspension. Fix a hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ and a point $\mathbb{P}^0 \in \mathbb{P}^{n+1} - \mathbb{P}^n$.

Definition 1.1. Let $V \subset \mathbb{P}^n$ be a closed set. By the **algebraic suspension of** V (or **complex cone** on V) with vertex $\mathbb{P}^0$ we mean the set

$$\Sigma\!\!\!/ V = \bigcup\{\ell : \ell \text{ is a projective line through } \mathbb{P}^0 \text{ which meets } V\}.$$



The projection $\mathbb{P}^{n+1} - \mathbb{P}^0 \to \mathbb{P}^n$ is a holomorphic line bundle. It is the normal bundle to $\mathbb{P}^n$ and is equivalent to $\mathcal{O}(1)$. Its fibres are the lines through $\mathbb{P}^0$ (with $\mathbb{P}^0$ removed). Thus *$\not\Sigma V$ is homeomorphic to the Thom space of $\mathcal{O}(1)|_V$.*

The construction $\not\Sigma V$ is particularly simple in terms of homogeneous coordinates. Suppose $\mathbb{C}^{n+2}$ is a choice of homogeneous coordinates for $\mathbb{P}^{n+1}$ with projection

$$\pi : \mathbb{C}^{n+2} - \{0\} \to \mathbb{P}^{n+1}.$$

Given any subset $S \subset \mathbb{P}^{n+1}$, let

$$C(S) \stackrel{\text{def}}{=} \pi^{-1}(S) \cup \{0\}, \tag{1.1}$$

and suppose the coordinates are chosen so that $C(\mathbb{P}^n) = \mathbb{C}^{n+1} \times \{0\}$ and $C(\mathbb{P}^0) = \{0\} \times \mathbb{C}$. Then for any closed set $V \subset \mathbb{P}^n$,

$$C(\not\Sigma V) = C(V) \times C(\mathbb{P}^0) = C(V) \times \mathbb{C}. \tag{1.2}$$

From this we see that if $\mathbb{P}^p \subset \mathbb{P}^n$ is a linear subspace, then

$$\not\Sigma \mathbb{P}^p = \mathbb{P}^{p+1} \tag{1.3}$$

is also a linear subspace. Furthermore, we see that if V is a projective subvariety of $\mathbb{P}^n$, then $\not\Sigma V$ is a projective subvariety of $\mathbb{P}^{n+1}$. In fact if V is defined by homogeneous polynomial equations $p_1(z_0, \dots, z_n) = \cdots = p_N(z_0, \dots, z_n) = 0$, then $\not\Sigma V$ is defined by exactly the same polynomials, now considered to be functions of an additional "hidden" variable z_{n+1}. Hence by linearity the algebraic suspension gives a homomorphism

$$\not\Sigma : \mathcal{C}_p(\mathbb{P}^n) \longrightarrow \mathcal{C}_{p+1}(\mathbb{P}^{n+1}) \tag{1.4}$$

which is easily seen to be continuous. Consequently, we have

Proposition 1.2. *For any algebraic subset $X \subset \mathbb{P}^n$, the algebraic suspension gives a continuous monoid homomorphism*

$$\Sigma\!\!\!/ : \mathcal{C}_p(X) \longrightarrow \mathcal{C}_{p+1}(\Sigma\!\!\!/ X) \tag{1.5}$$

which extends to a continuous group homomorphism

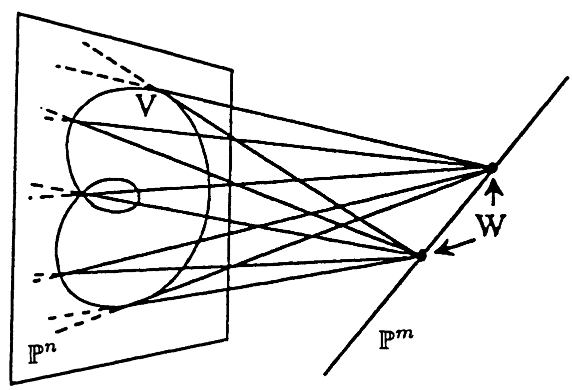
$$\Sigma\!\!\!/ : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X) \tag{1.6}$$

for all p, $0 \le p \le \dim(X)$.

§2. Algebraic join. Fix disjoint linear subspaces $\mathbb{P}^n \amalg \mathbb{P}^m \subset \mathbb{P}^{n+m+1}$.

Definition 2.1. Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be closed subsets. By the **algebraic join of** V and W we mean the set

$$V\#W = \bigcup \{\ell : \ell \text{ is a projective line which meets both } V \text{ and } W\}.$$



Suppose $\mathbb{C}^{n+m+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$ is a choice of homogeneous coordinates such that $C(\mathbb{P}^n) = \mathbb{C}^{n+1} \times \{0\}$ and $C(\mathbb{P}^m) = \{0\} \times \mathbb{C}^{m+1}$. Then we have that

$$C(V\#W) = C(V) \times C(W). \tag{2.1}$$

From this it is clear that the join takes linear subspaces to linear subspaces, i.e.,

$$\mathbb{P}^p \# \mathbb{P}^q = \mathbb{P}^{p+q+1} \tag{2.2}$$

for $0 \le p \le n$ and $0 \le q \le m$. Furthermore, one has

$$(2.3) \qquad \Sigma\!\!\!\!/\, V = V\#\mathbb{P}^0 \quad \text{and}$$

$$\Sigma\!\!\!\!/^{m+1} V \stackrel{\text{def}}{=} \underbrace{\Sigma\!\!\!\!/\Big(\Sigma\!\!\!\!/(\cdots(\Sigma\!\!\!\!/ V)\cdots)\Big)}_{m+1-\text{times}} = V\#\mathbb{P}^m.$$

Of course, one has symmetrically that $\mathbb{P}^n\#W \cong \Sigma\!\!\!\!/^{n+1}W$, and this gives the basic relation

$$(2.4) \qquad V\#W = (\Sigma\!\!\!\!/^{m+1}V) \cap (\Sigma\!\!\!\!/^{n+1}W)$$

that the join pairing is obtained by suspending and then intersecting. Note that this suspension always puts cycles in good position, i.e., so they intersect properly. Note from (2.1) that if V and W are projective subvarieties then so is $V\#W$. In fact if V is defined by homogeneous polynomials $p_1(z) = \cdots = p_N(z) = 0$ and W is defined by $q_1(\zeta) = \cdots = q_M(\zeta) = 0$, then $V\#W$ is defined by the vanishing of all p_i's and q_j's simultaneously. The join extends to algebraic cycles by bilinearity. Suspension is continuous, and the proper intersection of cycles in $\mathbb{P}^k$ is continuous on the subset of pairs which meet properly. (See Fulton [**34**] or Barlet [**6**].) Hence, we have the following.

Proposition 2.2. *Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be algebraic subsets. Then the algebraic join defines a continuous biadditive pairing*

$$(2.5) \qquad \mathcal{C}_p(X) \times \mathcal{C}_r(Y) \xrightarrow{\#} \mathcal{C}_{p+r+1}(X\#Y)$$

which extends to a continuous biadditive pairing

$$(2.6) \qquad \mathcal{Z}_p(X) \times \mathcal{Z}_r(Y) \xrightarrow{\#} \mathcal{Z}_{p+r+1}(X\#Y)$$

for all $0 \le p \le \dim(X)$ *and* $0 \le r \le \dim(Y)$.

In particular, if we choose the notation

$$(2.7) \qquad \mathcal{C}^q(X) \stackrel{\text{def}}{=} \mathcal{C}_{n-q}(X) \quad \text{and} \quad \mathcal{Z}^q(X) \stackrel{\text{def}}{=} \mathcal{Z}_{n-q}(X)$$

where $n = \dim(X)$, then (2.5) and (2.6) give basic pairings:

$$(2.8) \qquad \mathcal{C}^q(\mathbb{P}^n) \times \mathcal{C}^{q'}(\mathbb{P}^{n'}) \longrightarrow \mathcal{C}^{q+q'}(\mathbb{P}^{n+n'+1}),$$

$$(2.9) \qquad \mathcal{Z}^q(\mathbb{P}^n) \times \mathcal{Z}^{q'}(\mathbb{P}^{n'}) \longrightarrow \mathcal{Z}^{q+q'}(\mathbb{P}^{n+n'+1}).$$

§3. The Algebraic Suspension Theorem. The importance of algebraic suspension comes from the following result.

Theorem 3.1. ([**47**]) *For any algebraic subset $X \subset \mathbb{P}^n$ and any p, $0 \leq p \leq \dim(X)$, the algebraic suspension homomorphism*

$$\Sigma\!\!\!/ : \mathcal{Z}_p(X) \hookrightarrow \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X)$$

is a homotopy equivalence.

Idea of Proof. Suppose $X = \mathbb{P}^n$. (The general case will follow by restricting to cycles in X.) For simplicity set $\mathcal{C}_{p+1} = \mathcal{C}_{p+1}(\mathbb{P}^{n+1})$ and $\mathcal{Z}_{p+1} = \mathcal{Z}_{p+1}(\mathbb{P}^{n+1})$. Consider the subset

$$\mathcal{J}_{p+1} \stackrel{\text{def}}{=} \left\{\Sigma n_i V_i \in \mathcal{C}_{p+1} : \dim(V_i \cap \mathbb{P}^n) = p \ \forall i\right\}$$

of cycles which meet the hyperplane $\mathbb{P}^n$ in proper dimension. Let $\widetilde{\mathcal{J}}_{p+1} \subset \mathcal{Z}_{p+1}$ be the subgroup generated by $\mathcal{J}_{p+1}$. The proof breaks into two steps.

Assertion 1. The subset $\Sigma\!\!\!/\big(\mathcal{Z}_p(\mathbb{P}^n)\big) \subset \widetilde{\mathcal{J}}_{p+1}$ is a deformation retract.

Assertion 2. The inclusion $\widetilde{\mathcal{J}}_{p+1} \subset \mathcal{Z}_{p+1}$ is a homotopy equivalence.

For the first step we recall that $\mathbb{P}^{n+1} - \mathbb{P}^0 \to \mathbb{P}^n$ is a line bundle. Scalar multiplication by $t > 0$ in this bundle defines a one-parameter family of automorphisms $\varphi_t : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ which fixes $\mathbb{P}^n \amalg \mathbb{P}^0$. It induces a 1-parameter family of automorphisms

$$\Phi_t : \mathcal{C}_{p+1} \longrightarrow \mathcal{C}_{p+1}$$

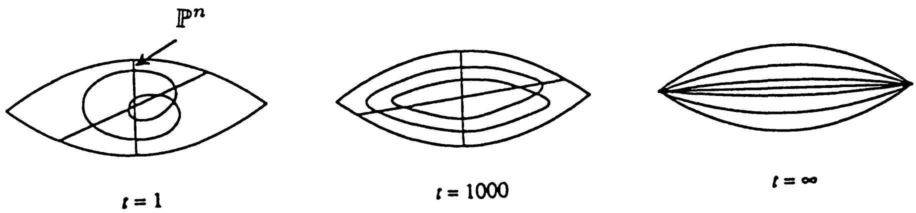
which leaves invariant the submonoid $\mathcal{J}_{p+1}$ and fixes the submonoid $\Sigma\!\!\!/\big(\mathcal{C}_p(\mathbb{P}^n)\big)$. The main point here is that on the subset $\mathcal{J}_{p+1}$ the map Φ_t extends continuously to $t = \infty$ where

$$\Phi_\infty : \mathcal{J}_{p+1} \longrightarrow \Sigma\!\!\!/\big(\mathcal{C}_p(\mathbb{P}^n)\big)$$

is the retraction defined by

$$\Phi_\infty(c) = \Sigma\!\!\!/(c \cdot \mathbb{P}^n)$$

$c \cdot \mathbb{P}^n$ denoting the intersection of $c \in \mathcal{J}_{p+1}$ with the hyperplane $\mathbb{P}^n$. The continuity of this process, called "pulling to the normal cone" is established in the book of Fulton [**34**].



Extending Φ_t, $0 < t \leq \infty$, to the group completions proves Assertion I.

To prove Assertion 2 it suffices to prove that the homomorphism

$$\pi_k(\widetilde{\mathcal{J}}_{p+1}) \longrightarrow \pi_k(\mathcal{Z}_{p+1}) \tag{3.1}$$

induced by the inclusion $\widetilde{\mathcal{J}}_{p+1} \subset \mathcal{Z}_{p+1}$ is an isomorphism for all $k \geq 0$. Note that the inclusion map on positive cycles $\mathcal{J}_{p+1} \subset \mathcal{C}_{p+1}$ is very far from being a homotopy equivalence. It does induce a bijection of connected components, but the corresponding components have very different dimensions in general. It is in this step that we must use the group completion strongly.

For this we erect a superstructure. Fix a linear embedding $\mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$ and two points $x_0, x_1 \in \mathbb{P}^{n+2} - \mathbb{P}^{n+1}$. The projections

$$\pi_k : \mathbb{P}^{n+2} - \{x_k\} \longrightarrow \mathbb{P}^{n+1},\ k = 0, 1 \tag{3.2}$$

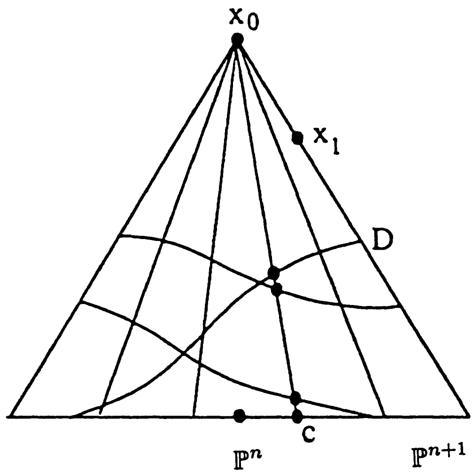
give each set $\mathbb{P}^{n+2} - \{x_k\}$ the structure of a holomorphic line bundle over $\mathbb{P}^{n+1}$.

Consider now a positive divisor D on $\mathbb{P}^{n+2}$ of degree d with $x_0 \notin D$ and $x_1 \notin D$. One can think of D as a d-valued section of the bundle $\pi_0 : \mathbb{P}^{n+2} - \{x_0\} \to \mathbb{P}^{n+1}$. The key observation is that any positive cycle $c \in \mathcal{C}_{p+1}(\mathbb{P}^{n+1})$ can be "lifted" to a cycle with support in D. This lifting is defined to be the intersection

$$\Psi_D(c) \equiv \not\Sigma_{x_0}(c) \cdot D$$

of the divisor D with the suspension of c to the point x_0. This gives us a continuous map

$$\Psi_D : \mathcal{C}_{p+1}(\mathbb{P}^{n+1}) \longrightarrow \mathcal{C}_{p+1}\left(\mathbb{P}^{n+2} - \{x_0, x_1\}\right).$$



Note that $(\pi_0)_* \circ \Psi_D \equiv d$ (multiplication by the integer d in the monoid). However, the composition $(\pi_1)_* \circ \Psi_D$ is very interesting. It gives us a transformation of cycles in $\mathbb{P}^{n+1}$ which makes most of them "transversal" to $\mathbb{P}^n$, i.e., which moves most of them into $\mathcal{J}_{p+1}$.

Consider now the family of divisors tD, $0 \le t \le 1$, given by scalar multiplication by t in the bundle $\pi_0 : \mathbb{P}^{n+2} - \{x_0\} \to \mathbb{P}^{n+1}$. We assume $x_1 \notin tD$ for all such t. (This will be true for all divisors in a neighborhood of $d \cdot \mathbb{P}^{n+1}$.) The above construction then gives us a family of transformations

$$F_{D,t} \stackrel{\text{def}}{=} (\pi_1)_* \circ \Psi_{tD} : \mathcal{C}_{p+1}(\mathbb{P}^{n+1}) \longrightarrow \mathcal{C}_{p+1}(\mathbb{P}^{n+1})$$

for $0 \le t \le 1$ such that $F_0 \equiv d$ (multiplication by d).

Fix $c \in \mathcal{C}_{p+1}(\mathbb{P}^{n+1})$ and ask which divisors D of degree d have the property that

$$F_{D,t}(c) \in \mathcal{J}_{p+1}$$

for all $t > 0$. Let $B_c \subset \mathcal{C}_{n+1,d}(\mathbb{P}^{n+2}) \cong \mathbb{P}^{\binom{n+2+d}{d}-1}$ be the subset of divisors for which this fails, i.e., for which there is some $t > 0$ such that $F_{D,t}(c) \notin \mathcal{J}_{p+1}$. Then the main algebro-geometric calculation is that

$$\text{(3.3)} \qquad \operatorname{codim}_{\mathbb{C}} B_c \ge \binom{p+d+1}{d} - 1.$$

We can now apply these transformations with $d = 1$ to prove that $\mathcal{J}_{p+1} \hookrightarrow \mathcal{C}_{p+1}$ induces a bijection on connected components. Hence, (3.1) is an isomorphism for $k = 0$.

Suppose now that $f : S^k \to \mathcal{C}_{p+1}$ is a continuous map for $k > 0$. We may assume f to be PL up to homotopy. Then for all d sufficiently large, we see that the map $d \cdot f$ is homotopic to a map $S^k \to \mathcal{J}_{p+1}$. Indeed just consider the family

$$F_{t,D} \circ f : S^k \longrightarrow \mathcal{C}_{p+1}$$

$0 \le t \le 1$, where D lies outside the union

$$\bigcup_{x \in S^k} B_{f(x)}$$

which is a set of real codimension $\ge 2\binom{p+d+1}{d} - (k+2)$.

Similarly, suppose we are given a map of pairs $f : (D^{k+1}, S^n) \to (\mathcal{C}_{p+1}, \mathcal{J}_{p+1})$. Then for all sufficiently large integers d, the map $d \cdot f$ can be deformed through a map of pairs to one with image in $\mathcal{J}_{p+1}$.

From this we deduce that the map

$$\Sigma\!\!\!/_* : \varinjlim_{\alpha} \pi_k(\mathcal{C}_{p,\alpha}) \longrightarrow \varinjlim_{\alpha} \pi_k(\mathcal{C}_{p+1,\alpha})$$

is an isomorphism for all $k > 0$. Hence the induced map on homotopy group completions is a homotopy equivalence. One then applies Theorem I.8.2 for the statement concerning naïve group completions. □

Note 3.2. With a little more care the arguments above can be applied to prove *directly* that $\Sigma\!\!\!/ : \mathcal{Z}_p \to \mathcal{Z}_{p+1}$ is a homotopy equivalence (without using Theorem I.8.1). See [**48**] for example.

The Algebraic Suspension Theorem can be thought of as a "stability result". If we choose notation

$$\mathcal{Z}^q(X) \equiv \mathcal{Z}_{n-q}(X) \tag{3.4}$$

where $n = \dim(X)$, then Theorem 3.1 can be restated by saying that

$$\Sigma\!\!\!/ : \mathcal{Z}^q(X) \xrightarrow{\cong} \mathcal{Z}^q(\Sigma\!\!\!/ X) \tag{3.5}$$

is a homotopy equivalence for all $q \le \dim(X)$.

§4. Some immediate applications. For cycles in projective space one can make a construction which strictly generalizes the Dold-Thom construction of SP to the "p-dimensional points". Fix a linear subspace ℓ_0 of dimension p in $\mathbb{P}^n$, and consider the sequence of embeddings

$$\cdots \mathcal{C}_{p,d-1}(\mathbb{P}^n) \hookrightarrow \mathcal{C}_{p,d}(\mathbb{P}^n) \hookrightarrow \mathcal{C}_{p,d+1}(\mathbb{P}^n) \hookrightarrow \cdots$$

given by $c \mapsto c + \ell_0$. Define

$$\widehat{\mathcal{C}}_p(\mathbb{P}^n) \equiv \varinjlim_{d} \mathcal{C}_{p,d}(\mathbb{P}^n)$$

to be the limiting space with topology generated by this family of compact sets. (A set C is closed iff its intersection with each $C_{p,d}$ is closed.) Note that $\widehat{C}_0(\mathbb{P}^n) = SP(\mathbb{P}^n)$. As in (3.4) we write $\widehat{C}^q(\mathbb{P}^n) \equiv \widehat{C}_{n-q}(\mathbb{P}^n)$ as the connected monoid of codimension-q cycles.

Theorem 4.1. ([**47**]) *There are homotopy equivalences*

$$\widehat{C}^q(\mathbb{P}^n) \cong K(\mathbb{Z},2)\times K(\mathbb{Z},4)\times\cdots\times K(\mathbb{Z},2q) \tag{4.1}$$

$$\mathcal{Z}^q(\mathbb{P}^n) \cong K(\mathbb{Z},0)\times K(\mathbb{Z},2)\times K(\mathbb{Z},4)\times\cdots\times K(\mathbb{Z},2q) \tag{4.2}$$

for all $n \geq q \geq 0$.

Proof. Apply Theorem 3.1 to see that $\mathcal{Z}^q(\mathbb{P}^n) \cong \mathcal{Z}^q(\mathbb{P}^q) = \mathcal{Z}_0(\mathbb{P}^q)$ and then apply the Dold-Thom Theorem (cf. (I.3.12)). The space $\widehat{C}^q$ similarly reduces down to $\widehat{C}_0(\mathbb{P}^q) = SP(\mathbb{P}^q)$. □

Theorem 4.2. ([**47**]) *Let* $\mathbb{P}^{n-1}\subset\mathbb{P}^n$ *be a hyperplane, then there are homotopy equivalences*

$$\mathcal{Z}^q(\mathbb{P}^n)/\mathcal{Z}^{q-1}(\mathbb{P}^{n-1}) \cong K(\mathbb{Z},2q) \tag{4.3}$$

for all $n \geq q \geq 0$.

Theorem 4.3 ([**47**]). *Let* $m > 0$ *be any positive integer, and let* $\mathcal{Z}^q(\mathbb{P}^n)\otimes\mathbb{Z}_m = \mathcal{Z}^q(\mathbb{P}^n)/m\mathcal{Z}^q(\mathbb{P}^n)$ *be the topological group of codimension-*q *cycles with coefficients in* $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. *Then there are homotopy equivalences*

$$\mathcal{Z}^q(\mathbb{P}^n)\otimes\mathbb{Z}_m \cong K(\mathbb{Z}_m,0)\times K(\mathbb{Z}_m,2)\times\ldots\times K(\mathbb{Z}_m,2q) \tag{4.4}$$

and

$$\mathcal{Z}^q(\mathbb{P}^n)\otimes\mathbb{Z}_m/\mathcal{Z}^{q-1}(\mathbb{P}^{n-1})\otimes\mathbb{Z}_m \cong K(\mathbb{Z}_m,2q) \tag{4.5}$$

for all $n \geq q \geq 0$.

Theorem 4.1 can be applied to give results about the structure of the Chow sets $C_{p,d}(\mathbb{P}^n)$. We say that a map $f: A\to B$ between spaces **has a right homotopy inverse through dimension** k, if there is a finite complex C and a map $i: C\to A$ so that the composition $f\circ i$ is k-connected.

Theorem 4.4 ([**47**]). The inclusion

$$C_{p,d}(\mathbb{P}^n) \hookrightarrow \widehat{C}_p(\mathbb{P}^n)$$

has a right homotopy inverse through dimension 2*d*. *In particular the induced maps*

$$H_k\big(C_{p,d}(\mathbb{P}^n);\Lambda\big) \longrightarrow H_k\big(\widehat{C}_p(\mathbb{P}^n);\Lambda\big)$$

are surjective, and the maps

$$H^k\big(\widehat{\mathcal{C}}_p(\mathbb{P}^n);\Lambda\big) \longrightarrow H^k\big(\mathcal{C}_{p,d}(\mathbb{P}^n);\Lambda\big)$$

are injective for $k \le 2d$ *and for any coefficient ring* Λ.

This establishes the existence of a lot of "stable" cohomology in the classical Chow varieties. If $\Lambda = \mathbb{Z}_2$, we pick up much of the Steenrod algebra as $d, n \to \infty$.

Note. In this context it is natural to ask how much of the "stable" homology of $\mathcal{C}_{p,d}$ is represented by algebraic cycles. Recently Michelsohn has found such representatives for essentially all possible classes [**94**].

§5. The relation to topological cycles. Let X be a projective variety of dimension n. Fix a triangulation of X compatible with the smooth stratification and let $X \hookrightarrow \mathbb{R}^N$ be a linear embedding of this simplicial complex. Then for any $k \le 2n$, the Lipschitz singular k-chains in X can be completed to a group of **rectifiable** k**-currents**

$$\mathcal{R}_k(X) \subset \mathcal{E}_k(X)'$$

using the Federer mass norm. (See [**26**] and [**27**].) These currents retain certain manifold-like properties, and the spaces have nice compactness properties. The restriction of the de Rham differential d makes $\big(\mathcal{R}_*(X), d\big)$ a complex whose homology is $H_*(X;\mathbb{Z})$ ([**27**]). In particular in each dimension $k \le 2n$ we have the topological group

$$\mathfrak{Z}_k(X) = \{T \in \mathcal{R}_k(X) : dT = 0\}$$

of rectifiable k**-cycles on** X. The group depends only on the PL-structure of X. The topology is the restriction of the standard weak topology of the space of de Rham currents on $\mathbb{R}^N$ with support in X.

Now there is a beautiful theorem of Fred Almgren which generalizes the Dold-Thom result.

Theorem 5.1 (Almgren [**1**]). *For each pair of non-negative integers* k, ℓ, *there is an isomorphism*

$$\pi_\ell \mathfrak{Z}_k(X) \cong H_{\ell+k}(X;\mathbb{Z}).$$

There is a natural continuous homomorphism

$$\mathcal{Z}_p(X) \hookrightarrow \mathfrak{Z}_{2p}(X) \tag{5.1}$$

for each p, $0 \le p \le n$. Theorem 5.1 above can be restated in the following way.

Theorem 5.2. *When* $X = \mathbb{P}^n$, *the inclusion* (5.1) *is a homotopy equivalence for all* p.

Thus in projective space the algebraic p-cycles carry the full homotopy-type of the space of all rectifiable $2p$-cycles. This strongly generalizes the basic fact that every homology class is represented by an algebraic cycle. We will see that Theorem 5.2 remains true for a large family of varieties including Grassmannians and in fact all generalized flag manifolds. However, such a result is necessarily false (even at the level of connected components) for any variety X for which $H_{2p}(X;\mathbb{Q}) \not\subset H_{p,p}(X)$. In this case the bigrading of $\pi_\ell \mathcal{Z}_p(X)$; $\ell, p \geq 0$ becomes more interesting than it is in the topological case.

§6. The ring $\pi_* Z(\mathbb{P}^0)$. Let $X \subset \mathbb{P}^N$ be a projective subvariety. Then the algebraic join gives pairings

$$\mathcal{Z}^q(X) \times \mathcal{Z}^{q'}(\mathbb{P}^n) \longrightarrow \mathcal{Z}^{q+q'}(\Sigma\!\!\!/^{n+1} X)$$

which, since $0\#C = 0$ and $C'\#0 = 0$, descend to the smash product

$$\mathcal{Z}^q(X) \wedge \mathcal{Z}^{q'}(\mathbb{P}^n) \longrightarrow \mathcal{Z}^{q+q'}(\Sigma\!\!\!/^{n+1} X). \tag{6.1}$$

Now the Suspension Theorem 3.1 gives a canonical homotopy equivalence $\mathcal{Z}^{q+q'}(X) \cong \mathcal{Z}^{q+q'}(\Sigma\!\!\!/^{n+1} X)$, provided that $q + q' \leq \dim X$. Hence taking homotopy groups in (6.1) gives a pairing

$$\pi_k \mathcal{Z}^q(X) \otimes \pi_{k'} \mathcal{Z}^{q'}(\mathbb{P}^n) \longrightarrow \pi_{k+k'} \mathcal{Z}^{q+q'}(X) \tag{6.2}$$

introduced by E. Friedlander and B. Mazur [**29**]. This pairing can be extended somewhat as follows. For any $q \geq 0$ we have canonical homotopy equivalences $\mathcal{Z}^q(\Sigma^k X) \cong \mathcal{Z}^q(\Sigma\!\!\!/^{n+1} X) \cong \cdots$ for all n such that $n + \dim(X) \geq q$. Let us define

$$\mathbf{Z}^q(X) = \varinjlim_n \ \mathcal{Z}^q(\Sigma\!\!\!/^n X) \tag{6.3}$$

to be this well-defined homotopy type. For example

$$\mathbf{Z}^q(\mathbb{P}^0) \cong \mathbf{Z}^q(\mathbb{P}^n)$$

for any $n \geq q$. The pairing (5.2) now extends to a pairing

$$\pi_k \ \mathbf{Z}^q(X) \otimes \pi_{k'} \ \mathbf{Z}^{q'}(\mathbb{P}^0) \longrightarrow \pi_{k+k'} \ \mathbf{Z}^{q+q'}(X) \tag{6.4}$$

defined for all $k, k', q, q' \geq 0$.

Theorem 6.1. (Friedlander and Mazur [**29**]). *When $X = \mathbb{P}^0$, the pairing* (6.4) *gives $\mathcal{FM} \stackrel{def}{=} \pi_* \ \mathbf{Z}^*(\mathbb{P}^0)$ the structure of a commutative bigraded ring. In fact this ring is isomorphic to a polynomial ring $\mathbb{Z}[\mathbf{s},\mathbf{h}]$ on two generators where*

$$\mathbf{s} \in \pi_2 \mathbf{Z}^1(\mathbb{P}^0) \ \textit{and}\ \mathbf{h} \in \pi_1 \mathbf{Z}^1(\mathbb{P}^0).$$

For any projective subvariety $X \subset \mathbb{P}^N$, the pairing (6.4) *gives $\pi_* \mathbf{Z}^*(X)$ the structure of a bigraded $\mathcal{FM}$-module.*

The operation $\mathbf{h}$ is related to the Lefschetz map in homology coming from intersection with a hyperplane. The operation $\mathbf{s}$ is more subtle and interesting. We will return to this in Chapter IV.

§7. **Suspension and symmetry.** In all of the discussion above it is natural to ask what happens in the presence of symmetries. Consider, for example, a projective variety X and a finite group $G \subset \text{Aut}(X)$ of automorphisms. There is a naturally induced action of G on $\mathcal{Z}_p(X)$ for each p, and in analogy with §3 above, one can consider the topological groups

$$\mathcal{Z}_{G,p}(X) \stackrel{\text{def}}{=} \mathcal{Z}_p(X)^G / G \cdot \mathcal{Z}_p(X)$$

introduced in [**58**], where $\mathcal{Z}_p(X)^G = \{c \in \mathcal{Z}_p(X) : g_* c = c \ \forall g \in G\}$ denotes the fixed-point set and $G\mathcal{Z}_p(X) = \{\sum_{g\in G} g_* c : c \in \mathcal{Z}_p(X)\}$ is the subgroup of averaged cycles. These are functors on the category of G-varieties and G-equivariant morphisms, and therefore are the homotopy groups $\pi_k \mathcal{Z}_{G,p}(X)$. If we are given a representation $G \to GL_{N+1}(\mathbb{C})$ and a G-equivariant embedding, then algebraic suspension defines homomorphisms

$$\text{(7.1)} \qquad \Sigma\!\!\!/ : \mathcal{Z}_p(X)^G \to \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X)^G \quad \text{and} \quad \Sigma\!\!\!/ : G\mathcal{Z}_p(X) \to G\mathcal{Z}_{p+1}(\Sigma\!\!\!/ X).$$

Together with M.-L. Michelsohn it was proved that these maps are homotopy equivalences if one localizes away from the order of the group. In particular we have

Theorem 7.1. ([**58**]) *The suspension homomorphisms* (7.1) *and the induced homomorphism*

$$\text{(7.2)} \qquad \Sigma\!\!\!/ : \mathcal{Z}_{G,p}(X) \to \mathcal{Z}_{G,p+1}(\Sigma\!\!\!/ X)$$

induce isomorphisms on the homotopy group $\pi_k(\bullet) \otimes \Lambda$ for all $k \geq 0$ and any ring Λ in which the order of the group G is invertible.

By "ring" we mean a commutative ring with unit. Examples of such rings are $\Lambda = \mathbb{Z}_{(q)}$ (the integers localized at the prime q) and $\Lambda = \mathbb{Z}/q\mathbb{Z}$ where q is any prime which does not divide the integer $|G|$.

This condition that $|G|$ be invertible is strictly necessary. It is shown by example in [**58**] that the suspension homomorphism fails to be a homotopy equivalence at primes dividing the order of the group. However, with Lima-Filho and Michelsohn it has been proved that the *m-fold suspension map is a G-homotopy equivalence, when m exceeds the codimension of the cycles.* One can also suspend to a general representation. Here one obtains the delicate result that *suspension to the regular representation of G is stably a G-homotopy equivalence* [**93**].

Chapter III - Cycles on $\mathbb{P}^n$ and Classifying Spaces

It is an interesting fact that the algebraic cycles in projective space can be used to construct models of certain universal spaces in topology – spaces that represent such everyday functors as K-theory and cohomology. Elementary constructions with cycles lead to Chern classes, Stiefel-Whitney classes and the cup product at the universal level. Families of cycles correspond to Steenrod operations.

§1. The total Chern class. It is a basic fact presented in most books on characteristic classes and K-theory that the space of linear cycles

$$\mathcal{C}_{n-q,1}(\mathbb{P}^n) \stackrel{\text{def}}{=} \mathcal{G}^q(\mathbb{P}^n) \tag{1.1}$$

i.e., the Grassmannian of linear subspaces of codimension-q in $\mathbb{P}^n$ is a classifying space for vector bundles. Specifically, for any finite complex Y of dimension $\leq 2(n-q)$, there is an equivalence of functors

$$\mathrm{Vect}^q(Y) \cong [Y, \mathcal{G}^q(\mathbb{P}^n)] \tag{1.2}$$

where $\mathrm{Vect}^q(Y)$ denotes the set of equivalence classes of rank-q complex vector bundles on Y. The equivalence is given by associating to $f : Y \to \mathcal{G}^q(\mathbb{P}^n)$, the pull-back $f^*\xi_q$ of the **tautological q-plane bundle** ξ_q over $\mathcal{G}^q(\mathbb{P}^n)$.

Analogously we see from II.4.1 and (I.3.6) that the space $\mathcal{Z}^q(\mathbb{P}^n)$ has the property that for any finite complex Y there is an equivalence of functors

$$\bigoplus_{k=0}^{q} H^{2k}(Y;\mathbb{Z}) \cong [Y, \mathcal{Z}^q(\mathbb{P}^n)]\,. \tag{1.3}$$

Observe now that we have a very natural map

$$\mathcal{G}^q(\mathbb{P}^n) \hookrightarrow \mathcal{Z}^q(\mathbb{P}^n) \tag{1.4}$$

given by considering linear subspaces as cycles of degree 1. By (1.3) this corresponds to a cohomology class c on $\mathcal{G}^q(\mathbb{P}^n)$. In collaboration with M.-L. Michelsohn the following was proved.

Theorem 1.1 ([**57**]). *The cohomology class $c = 1 + c_1 + \cdots + c_q$ determined by the cycle inclusion* (1.4) *is the total Chern class of the tautological bundle ξ_q over $\mathcal{G}^q(\mathbb{P}^n)$*

Under algebraic suspension the maps (1.4) sit in a grid of inclusions :

$$
(1.5)\qquad
\begin{array}{cccccc}
 & \vdots & & \vdots & \\
\Sigma\!\!\!/ & \cap & & \cap & \Sigma\!\!\!/ \\
 & \mathcal{G}^q(\mathbb{P}^n) & \subset & \mathcal{Z}^q(\mathbb{P}^n) & \\
\Sigma\!\!\!/ & \cap & & \cap & \Sigma\!\!\!/ \\
 & \mathcal{G}^q(\mathbb{P}^{n+1}) & \subset & \mathcal{Z}^q(\mathbb{P}^{n+1}) & \\
\Sigma\!\!\!/ & \cap & & \cap & \Sigma\!\!\!/ \\
 & \vdots & & \vdots &
\end{array}
$$

where the vertical maps on the right are all homotopy equivalences. Hence we may pass to the limit

$$
(1.6)\qquad BU_q \stackrel{\text{def}}{=} \varinjlim_n \mathcal{G}^q(\mathbb{P}^n).
$$

This space has the classifying property (1.2) for all finite complexes.

Corollary 1.2 ([**57**]) *Passing to the limit in (1.5) gives a map*

$$
(1.7)\qquad BU_q \longrightarrow \mathcal{Z}^q(\mathbb{P}^\infty) \cong K(\mathbb{Z},0) \times K(\mathbb{Z},2) \times \cdots \times K(\mathbb{Z},2q)
$$

which represents the total Chern class of the universal q-plane bundle over BU_q.

Taking the limit as $q \to \infty$ in (1.7) gives us map

$$
(1.8)\qquad c : BU \longrightarrow K(\mathbb{Z},2*)
$$

where BU and $K(\mathbb{Z},2*)$ have the property that

$$
\widetilde{K}(Y) = [Y, BU] \qquad \text{and} \qquad H^{\text{even}}(Y;\mathbb{Z}) = [Y, K(\mathbb{Z},2*)]
$$

for all finite complexes Y. That is BU and $K(\mathbb{Z},2*)$ are the classifying spaces for reduced K-theory and even cohomology respectively. From 1.2 we immediately have

Corollary 1.3. *The map* (1.8) *represents the universal total Chern class from K-theory to cohomology.*

Note that by Bott Periodicity the homotopy groups of BU and $K(\mathbb{Z},2*)$ are the same in positive dimensions. In fact Bott's fundamental results show that the homomorphism

$$
\begin{array}{ccc}
\pi_{2k}BU & \xrightarrow{c_*} & \pi_{2k}K(\mathbb{Z},2*) \\
\wr\| & & \wr\| \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
$$

is multiplication by $(k-1)!$ (See [**57**]).

These results lead to some interesting questions. One can consider the spaces

$$\mathcal{D}(d) = \lim_{\substack{p\to\infty \\ n\to\infty}} C_{p,d}(\mathbb{P}^n) \tag{1.9}$$

(with the compactly generated topology) for all $d = 1, 2, \cdots, \infty$. There is a natural sequence of inclusion mappings

$$BU = \mathcal{D}(1) \subset \mathcal{D}(2) \subset \cdots \subset \mathcal{D}(\infty) = K(\mathbb{Z}, 2*). \tag{1.10}$$

Note that $\mathcal{D}(d)$ is the space of all positive projective cycles of degree d. There are a number of interesting questions concerning this filtration. A simple one is whether the maps $\pi_*\mathcal{D}(d) \to \pi_*\mathcal{D}(\infty)$ are injective, and if so, at which levels d do the factors in the homomorphism $\pi_{2k}\mathcal{D}(1) \to \pi_{2k}\mathcal{D}(\infty)$, which is multiplication by $(k-1)!$, appear ?

§2. **Algebraic join and the cup product.** From II.2 we know that the algebraic join induces a continuous biadditive pairing

$$\mathcal{Z}^q(\mathbb{P}^n) \times \mathcal{Z}^{q'}\left(\mathbb{P}^{n'}\right) \xrightarrow{\#} \mathcal{Z}^{q+q'}\left(\mathbb{P}^{p+n'+1}\right). \tag{2.1}$$

From Theorem II.4.1. and its proof we obtain a canonical homotopy equivalence

$$\mathcal{Z}^q(\mathbb{P}^n) \longrightarrow \prod^q \stackrel{\text{def}}{=} \prod_{k=0}^q K(\mathbb{Z}, 2k) \tag{2.2}$$

for all $n \geq q$. It is natural to ask what the join map becomes when interpreted as a map of Eilenberg-MacLane spaces. Certainly the most basic pairing of such spaces comes from the cup-product in cohomology. This product

$$H^a(Y;\mathbb{Z}) \otimes H^b(Y;\mathbb{Z}) \longrightarrow H^{a+b}(Y;\mathbb{Z})$$

on spaces Y can be represented universally, via (I.3.6), by a map

$$K(\mathbb{Z}, a) \times K(\mathbb{Z}, b) \xrightarrow{\cup} K(\mathbb{Z}, a+b), \tag{2.3}$$

which is determined up to homotopy by the fact that it classifies the cup product $\iota_a \otimes \iota_b$ of the fundamental cohomology classes, where $\iota_k \in H^k(K(\mathbb{Z}, k);\mathbb{Z}) \cong \mathbb{Z}$ is the generator. The map (2.3) can be constructed explicitly by extending the smash product map $S^a \times S^b \to S^a \wedge S^b = S^{a+b}$ bilinearly to $\mathcal{Z}_0(S^a) \times \mathcal{Z}_0(S^b) \to \mathcal{Z}_0(S^{a+b})$ and using (I.3.7).

These basic cup product maps (2.3) assemble naturally to give a mapping

$$\prod^q \times \prod^{q'} \xrightarrow{\cup} \prod^{q+q'} \tag{2.4}$$

which classifies the cup-product mapping in even-cohomology

$$\left(\bigoplus_{k=0}^{q} H^{2k}(Y;\mathbb{Z})\right) \otimes \left(\bigoplus_{k=0}^{q'} H^{2k}(Y;\mathbb{Z})\right) \longrightarrow \left(\bigoplus_{k=0}^{q+q'} H^{2k}(Y;\mathbb{Z})\right)$$

In collaboration with M.-L. Michelsohn the following was proved.

Theorem 2.1. ([**57**]) *Under the canonical homotopy equivalence* (2.2) *the algebraic join pairing # is (homotopic to) the cup product mapping* (2.4)

Observe now that the inclusion of degree-1 cycles into $\mathcal{Z}^q(\mathbb{P}^n)$ gives a commutative diagram

$$\begin{array}{ccc} \mathcal{G}^q(\mathbb{P}^n) \times \mathcal{G}^{q'}\left(\mathbb{P}^{n'}\right) & \xrightarrow{\oplus} & \mathcal{G}^{q+q'}\left(\mathbb{P}^{n+n'+1}\right) \\ \downarrow & & \downarrow \\ \mathcal{Z}^q(\mathbb{P}^n) \times \mathcal{Z}^{q'}\left(\mathbb{P}^{n'}\right) & \xrightarrow{\#} & \mathcal{Z}^{q+q'}\left(\mathbb{P}^{n+n'+1}\right). \end{array} \tag{2.5}$$

The restriction of the join to linear subspaces corresponds to taking the direct sum (cf. (II.2.1)). Passing to the limit as $q, q' \to \infty$ and applying Theorem 2.1 gives a commutative diagram

$$\begin{array}{ccc} BUq \times BU_{q'} & \xrightarrow{\oplus} & BU_{q+q'} \\ {\scriptstyle c\times c}\downarrow & & \downarrow{\scriptstyle c} \\ \prod^{q} \times \prod^{q'} & \xrightarrow{\cup} & \prod^{q+q'} \end{array} \tag{2.6}$$

where the map $\oplus$ classifies the Whitney sum of vector bundles. The commutativity of this diagram corresponds to the Fundmanental Whitney Duality Formula

$$c(E \oplus E') = c(E)c(E')$$

for the total Chern class of complex vector bundles E, E' over a space Y.

The importance of (2.5) was realized early by E. Friedlander. He pointed out that in conjunction with Theorem 1.1, it can be used to prove Theorem 2.1 over the rationals.

§3. Real cycles and the total Stiefel-Whitney class. It was suggested by Deligne that if one worked with cycles modulo 2, some of the results above might carry over to real algebraic geometry. Indeed, with the correct formulation of "reality" this turns out to be true, and the results are surprisingly nice. Both the formulation of the theory and the proofs of the results are due to T.-K. Lam.

Following Atiyah [**2**] we define a **Real projective variety** to be a pair (X, C) where X is a projective variety and $C : X \to X$ is an antiholomorphic map with $C^2 = Id$. A basic example is that of projective space $(\mathbb{P}^n, C)$ where C is defined by complex conjugation in homogeneous coordinates. The fixed-point set of C is real projective n-space

$$\mathbb{P}^n(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{C}) = \mathbb{P}^n.$$

The choice of this real form corresponds to the choice of Real structure on $\mathbb{P}^n$.

Observe that if $V \subset X$ is an algebraic subvariety of a Real variety X, then its conjugate $C(V)$ is also a subvariety. Thus C induces an involution C_* on the set of subvarieties of X which extends by linearity to cycles.

Definition 3.1. Let (X, C) be a Real projective variety. An algebraic p-cycle on X is said to be **Real** if it is fixed by C_*. Let

$$\mathcal{Z}_p^{\mathbb{R}}(X) = \{c \in \mathcal{Z}_p(X) : C_*(c) = c\}$$

denote the topological group of all Real p-cycles on X.

Note that any Real cycle can be uniquely written as $\sum V_i + \sum m_i\,(W_i + C_* W_i)$ where the V_i's are C_*-invariant subvarieties. It is enticing (and naturally suggested by Galois theory) to divide by the subgroup

$$(1 + C_*)\,\mathcal{Z}_p(X) = \{c + C_*(c) : c \in \mathcal{Z}_p(X)\}$$

of "averaged" cycles. Therefore following [**48**] we introduce the topological quotient group

$$R\mathcal{Z}_p(X) \stackrel{\text{def}}{=} \mathcal{Z}_p^{\mathbb{R}}(X) / \,(1 + C_*)\,\mathcal{Z}_p(X) \tag{3.1}$$

of **reduced Real p-cycles on X.**

Algebraically $R\mathcal{Z}_p(X)$ is just the $\mathbb{Z}_2$-vector space generated by the Real (irreducible) subvarieties of X. However, this group is also furnished with a natural topology, and T.-K. Lam proves the following theorems.

Theorem 3.2 ([**48**]). *Let $(X, C) \subseteq (\mathbb{P}^n, C)$ be a Real algebraic subvariety. Then C-equivariant algebraic suspension gives a homotopy equivalence*

$$\Sigma\!\!\!/ : R\mathcal{Z}_p(X) \xrightarrow{\approx} R\mathcal{Z}_{p+1}(\Sigma\!\!\!/ X)$$

for all p, $0 \le p \le \dim(X)$.

As above we set $R\mathcal{Z}^q(X) = R\mathcal{Z}_{n-q}(X)$ where $n = \dim(X)$.

Theorem 3.3 ([**48**]). *There are homotopy equivalences*

$$R\mathcal{Z}^q\,(\mathbb{P}^n) \cong K\,(\mathbb{Z}_2, 0) \times K\,(\mathbb{Z}_2, 1) \times K\,(\mathbb{Z}_2, 2) \times \cdots \times K\,(\mathbb{Z}_2, q) \tag{3.2}$$

for all $q \le n$.

A given Real structure C on $\mathbb{P}^n$ induces a real structure on $\mathcal{G}^q(\mathbb{P}^n)$ whose fixed-point set is the real Grassmannian

$$\mathcal{G}^q(\mathbb{P}^n(\mathbb{R})) \simeq O_{n+1}/O_q \times O_{n+1-q}$$

of $\mathbb{R}$-linear subspaces of codimension-q in $\mathbb{R}^{n+1}$.

Theorem 3.4 ([48]). *The natural inclusion*

$$\mathcal{G}^q(\mathbb{P}^n(\mathbb{R})) \hookrightarrow R\mathcal{Z}^q(\mathbb{P}^n) \cong K(\mathbb{Z}_2, 0) \times \cdots \times K(\mathbb{Z}_2, q) \tag{3.3}$$

represents the total Stiefel Whitney class of the tautological q-plane bundle over $\mathcal{G}^q(\mathbb{P}^n(\mathbb{R}))$. *Therefore passing the limit as* $q \to \infty$ *in* (3.3) *gives a map*

$$BO_q \to \prod_{k=0}^{q} K(\mathbb{Z}_2, k) \tag{3.4}$$

which represents the total Stiefel-Whitney class of the universal q-plane bundle over $BO_q = \varinjlim_n \mathcal{G}^q(\mathbb{P}^n(\mathbb{R}))$.

Taking the limit as $q \to \infty$ in (3.4) gives a map

$$BO \to K(\mathbb{Z}_2, *) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} K(\mathbb{Z}_2, k).$$

Theorem 3.5. ([48]) *The algebraic join*

$$R\mathcal{Z}^q\left(\mathbb{P}^n\right) \times R\mathcal{Z}^{q'}\left(\mathbb{P}^{n'}\right) \longrightarrow R\mathcal{Z}^{p+q'}\left(\mathbb{P}^{n+n'+1}\right)$$

correspond via (3.2) *to the map which classifies the cup-product in* $\mathbb{Z}_2$*-cohomology.*

From 3.4 and 3.5 one retrives the classical Whitney Duality Formula

$$w(E \otimes E') = w(E)w(E')$$

for the total Stiefel-Whitney class.

§4. A conjecture of G. Segal. We have seen that the elementary inclusions (1.4) and (3.3) determine maps

$$BU \xrightarrow{c} K(\mathbb{Z}, 2*) \quad \text{and} \quad BO \xrightarrow{w} K(\mathbb{Z}_2, *) \tag{4.1}$$

which correspond to the universal total Chern and Stiefel-Whitney classes respectively. The question naturally arises whether these maps extend to transformations of generalized (connective) cohomology theories. In other words, can we extend these to maps of spectra where the mappings (4.1) occur at the 0^{th}-level ? Finding such an extension amounts to finding infinite loop space structures on these spaces such that c and w are infinite loop maps.

Now the spaces BU and BO have natural infinite loop space structures coming from Bott periodicity (e.g., $BU = \Omega^2 BU$). Each $K(G,n)$ is also an infinite loop space since it is an abelian topological group. However these structures are not even compatible at the 1-loop level. If they were, then c and w would preserve the loop-product ($\Omega Y \times \Omega Y \to \Omega Y$). However loop product in BU (and BO) is equivalent to Whitney sum, and in $K(G,n)$ it is equivalent to addition. The Whitney Duality Formulas show that c and w are not additive homomorphisms.

In fact if we fix Bott's loop space structures on BU, then a compatible loop structure on $K(\mathbb{Z}, 2*)$ will yield a quite different "addition" on even cohomology. This different additive structure $\widehat{+}$ was pointed out and used by Grothendieck in 1958 [**37**]. It is given on $H^{2*}(Y;\mathbb{Z}) = H^0(Y;\mathbb{Z}) \oplus H^{>0}(Y;\mathbb{Z})$ by setting $(a_0, a') \,\widehat{+}\, (b_0, b') = (c_0, c')$ where

$$c_0 = a_0 + b_0 \qquad \text{and} \qquad (1 + c') = (1 + a') \cup (1 + b').$$

This is precisely the addition given by the cup product pairing on $K(\mathbb{Z}, 2*)$ discussed in §2.

In 1975 G. Segal [**71**] asked the following question :

(4.2) *Do the cup product pairings on $K(\mathbb{Z}, 2*)$ and $K(\mathbb{Z}_2, *)$ enhance to infinite loop space structures such that c and w become infinite loop maps ?*

Several such structures were proposed and shown not to work (cf.[**73**], [**75**], [**82**]-[**84**], [**86**]. See [**7**] for a history).

Question (4.2) is very complicated in nature. For any proposed structure one must check compatibility on an infinite pyramid of higher associativity relations. Fortunately topologists have found simpler sets of compatibility hypotheses which yield infinite loop space structures and infinite loop maps. One such machine, due to Peter May, uses the linear isometries operator $\mathcal{L}$. I will spare you the definition and say only that any $\mathcal{L}$-space (a topological space with an action of $\mathcal{L}$) is canonically an infinite loop space, and any $\mathcal{L}$-map between $\mathcal{L}$-spaces is an infinite loop map.

Happily for us there is an elementary method for constructing $\mathcal{L}$-spaces and $\mathcal{L}$-maps. It involves the category $\mathcal{I}_*$ whose objects are finite dimensional inner product spaces and whose morphisms are linear isometric embeddings. Let $\mathcal{T}$ denote the category of compactly generated, Hausdorff topological spaces with base point. The sets of morphisms in $\mathcal{T}$ are given the compact-open topology.

Definition 4.1. An **$\mathcal{I}_*$-functor** (T, ω) is a continuous functor $T : \mathcal{I}_* \to \mathcal{T}$ together with a commutative, associative, and continuous natural transformation $\omega : T \times T \to T \circ \oplus$ such that

a) If $X \in TV$, and $1 \in T\{0\}$ is the basepoint, then

$$\omega(x, 1) = x \in T(V \oplus \{0\}) = TV.$$

b) If $V = V' \oplus V''$, then the map $TV' \to TV$ given by $x \mapsto \omega(x, 1)$ is a homeomorphism onto a closed subset.

Theorem 4.2. ([**63**]) *If T is an $\mathcal{I}_*$-functor, then*

$$T(\mathbb{C}^\infty) = \lim_{V \subset \mathbb{C}^\infty} T(V)$$

where the limit is taken over finite dimensional subspaces of $\mathbb{C}^\infty$, is an $\mathcal{L}$-space. Any natural transformation $\Phi : T \to T'$ of $\mathcal{I}_*$-functors induces a mapping $\Phi : T(\mathbb{C}^\infty) \to T'(\mathbb{C}^\infty)$ of $\mathcal{L}$-spaces.

An illuminating example is given by the "Bott functor" T_B which associates to each Hermitian V of dimension n the Grassmannian

$$T_B(V) = \mathcal{G}^n(V \oplus V)$$

of n-planes in $V \oplus V$, with distinguished point $1 = V \oplus \{0\}$. Given an isometry $f : V \to W$ define $T_B(f) : T_B(V) \to T_B(W)$ on a plane U by $(T_B f)(U) = \big((fV)^\perp \oplus \{0\}\big) \oplus (f \oplus f)(U)$. The natural transformation ω_B is given by

$$\omega_B(U, U') = \tau(U \oplus U')$$

where $\tau : V \oplus V \oplus V' \oplus V' \to V \oplus V' \oplus V \oplus V'$ is the obvious shuffle. This is an $\mathcal{I}_*$-functor, and clearly

$$T_B(\mathbb{C}^\infty) = BU.$$

It is shown in [**63**, p.16] that the induced infinite loop structure is the standard one of Bott.

Now in parallel fashion one may define the **Chow monoid** functor $T_{\mathcal{C}}$ by setting

$$T_{\mathcal{C}}(V) = \mathcal{C}^n(\mathbb{P}(V \oplus V)) \quad where \quad n = \dim(V)$$

with distinguished point $1 = \mathbb{P}(V \oplus \{0\})$. In dimension 0 we set $T\{0\} = \mathbb{N}$ with distinguished element 1. For an isometric embedding $f : V \to W$, we define $T_{\mathcal{C}}(f) : T_{\mathcal{C}}(V) \to T_{\mathcal{C}}(W)$ on a cycle c by

$$T_{\mathcal{C}}(f)c = \mathbb{P}\left(f(V)^\perp \oplus \{0\}\right) \#(f \oplus f)_*(c).$$

The natural transformation $\omega_{\mathcal{C}}$ is given on cycles c, c' by

$$\omega_{\mathcal{C}}(c, c') = \tau_*(c\#c')$$

with τ as above. One sees that

$$T_{\mathcal{C}}(\mathbb{C}^\infty) = \mathcal{D} \stackrel{\text{def}}{=} \coprod_{d=0}^{\infty} \mathcal{D}(d)$$

where $\mathcal{D}(d)$ is the space of all degree d cycles (See (1.9)).

One verifies that $T_{\mathcal{C}}$ is an $\mathcal{I}_*$-functor and concludes the following (due to Boyer, Mann, Lima-Filho, Michelsohn and the author).

Theorem 4.3. ([**7**]). *The stabilized cycle space $\mathcal{D}$ is an $\mathcal{L}$-space where the structure maps are defined via the algebraic join pairing $\# : \mathcal{D}(d) \times \mathcal{D}(d') \to \mathcal{D}(dd')$. Furthermore, the infinite loop space structure induced on $\mathcal{D}(1) = BU$ agrees with the standard one of Bott.*

Of course $T_{\mathcal{C}}$ has values in abelian topological monoids. From this one can deduce that $\mathcal{D}$ is an E_∞-ring space in the sense of P. May [**63**]. Associated to $\mathcal{D}$ is an E_∞-ring spectrum. This quickly leads to a positive answer to (4.2). However in the spirit of the exposition here one can proceed as follows.

Again in parallel with the above, we define an $\mathcal{I}_*$-functor $T_{\mathcal{Z}}$ by setting

$$T_{\mathcal{Z}}(v) = \mathcal{Z}^n(\mathbb{P}(V \oplus V))$$

where $n = \dim(V)$, and continuing as in the definition of $T_{\mathcal{C}}$. (Here $T_{\mathcal{C}}\{0\} = \mathbb{Z}$ with distinguished element 1). Note that $T_{\mathcal{Z}}(V)$ is the naïve group completion of $T_{\mathcal{C}}(V)$ and the limit

$$T_{\mathcal{Z}}(\mathbb{C}^\infty) = \mathcal{Z} = \coprod_{d=-\infty}^{\infty} \mathcal{Z}(d)$$

is the additive group completion of $\mathcal{D}$.

Theorem 4.4. ([**7**]). *The natural map $\mathcal{D} \to \mathcal{Z}$ of $\mathcal{D}$ into its additive group completion is a map of $\mathcal{L}$-spaces. In particular, the infinite loop structure induced by the complex join on $\mathcal{Z}(1)$ is such that the total Chern class map $\mathcal{D}(1) \hookrightarrow \mathcal{Z}(1)$ is an infinite loop map.*

This also carries through for Real cycles and we have the following.

Theorem 4.5. ([**7**]). *The multiplication on $K(\mathbb{Z}, 2*)$ and $K(\mathbb{Z}_2, *)$ induced by the algebraic join enhances to an infinite loop structure with respect to which the maps* (4.1) *are infinite loop maps.*

Let $M^0(\bullet)$ and $MO^0(\bullet)$ denote the functors $H^{2*}(\bullet; \mathbb{Z})$ and $H^*(\bullet; \mathbb{Z}_2)$ with the Grothendieck addition defined above.

Corollary 4.6. ([**7**]). *The functors M^0 and MO^0 enhance to generalized cohomology theories $\mathbf{M}^*$ and $\mathbf{MO}^*$ such that the maps* (4.1) *extend to natural transformations*

$$c : \mathbf{k}^* \to \mathbf{M}^* \quad \text{and} \quad w : \mathbf{ko}^* \to \mathbf{MO}^*.$$

This implies for example the existence of transfer maps in cohomology which commute with c and w.

It has been shown by Totaro [**79**] that the maps c and w cannot extend to natural transformations of multiplicative theories.

§5. **Equivariant Theories.** It is natural to ask what happens in the constructions above if one introduces the action of a finite group. The very pleasant answer is that one finds a new equivariant cohomology theory with some very nice properties.

To be more specific let G be a finite group, and to each finite-dimensional complex representation space V of G associate the cycle group

$$T_{\mathbb{C}}(V) = \mathcal{Z}^n((\mathbb{P}(V \otimes V)) \quad where \quad n = \dim(V)$$

as in section 4 above. This space has a natural action of G which respects the algebraic join pairing. It thereby gives us a "G-equivariant $\mathcal{I}_*$-functor". Now the theory of May has been carried through in this case [**59**], and we find the following. Let $\mathcal{U} = V_0 \oplus V_0 \oplus \cdots$ be the direct sum of infinitely many copies of the regular representation V_0 of G, and consider the limiting G-space

$$\mathcal{Z}_G \stackrel{\text{def}}{=} \varinjlim_{V \subset \mathcal{U}} \mathcal{Z}^n(\mathbb{P}(V \oplus V)). \tag{5.1}$$

Theorem 5.1. ([**55**], [**56**]). *The cycle space $\mathcal{Z}_G$ is a G-equivariant E_∞-ring space (in the sense of* [**59**]) *and gives rise to an E_∞-ring G-spectrum. In particular it determines an equivariant cohomology theory $\mathcal{H}^*_G(\bullet)$ which is ring-valued (and indexed by $R(G)$). This theory admits a natural transformation to a canonically associated theory of Borel type*

$$\mathcal{H}^*_G(\bullet) \to \mathcal{H}^*_G(\bullet)_{Borel} \tag{5.2}$$

where

$$\mathcal{H}^0_G(\bullet)_{Borel} = \prod_{i \geq 0} H^{2i}_G(\bullet; \mathbb{Z})$$

and where H^{2i}_G denotes the usual Borel equivariant cohomology.

The component $\mathcal{Z}_G(1)$, which is closed under the join $\#$, determines a related equivariant cohomology theory $\mathcal{M}^*_G(\bullet)$ which is only group-valued. It admits a natural transformation to its canonically associated "Borel counterpart"

$$\mathcal{M}^*_G(\bullet) \to \mathcal{M}^*_G(\bullet)_{Borel} \tag{5.3}$$

where

$$\mathcal{M}^0_G(\bullet)_{Borel} = 1 + \prod_{i \geq 1} H^{2i}_G(\bullet; \mathbb{Z})$$

is the group of units in $\mathcal{H}^0_G(\bullet)_{Borel}$.

Remark 5.2. Given a G-spectrum S, we define the associated Borel G-spectrum to be $S_{Borel} = F(EG_+, S)$, where $F(X,Y)$ denotes the space of pointed G-maps from X to Y. This gives the "associated Borel theories" referred to in 5.1.

We may now restrict our attention to the subspaces

$$\mathcal{G}^n(\mathbb{P}(V\oplus V)) \subset \mathcal{Z}^n(\mathbb{P}(V\oplus V)), \qquad \textit{where } n = \dim(V), \tag{5.4}$$

of linear cycles, (i.e., the Grassmannians, which are contained in the degree-one component), together with the pairing given by the join which is simply the direct sum of subspaces. This is also a G-equivariant $\mathcal{I}_*$-functor. Hence the limiting space

$$BU_G = \varinjlim_{V\subset\mathcal{U}} \mathcal{G}^n(\mathbb{P}(V\oplus V)) \tag{5.5}$$

is the 0^{th} space of a canonically determined G-spectrum. This spectrum classifies reduced equivariant K-theory $K^*_G(\bullet)$.

The following theorem establishes, among other things, a solution of the equivariant Segal Problem (cf. (4.2)) for Borel cohomology.

Theorem 5.3. ([**55**], [**56**]). *The inclusion* (5.4) *determines a natural transformation of equivariant cohomology theories*

$$K^*_G(\bullet) \xrightarrow{c} \mathcal{M}^*_G(\bullet).$$

Composing with (5.3) gives a natural transformation to the associated Borel theory, which on the 0^{th}*-level is the usual total Chern class map*

$$K_G(X) \xrightarrow{c} 1 + \prod_{i\geq 1} H^{2i}_G(X;\mathbb{Z})$$

into Borel cohomology.

This analogue of the results in §4 should have applications, for example, to the computation of Chern classes of induced representations.

The ring functor $\mathcal{H}^*_G(\bullet)$ is a new equivariant cohomology theory which arises quite naturally and may have some interesting uses. The coefficients of the theory have been computed in basic cases using techniques of degeneration via $\mathbb{C}^x$-actions. (cf. [**55**], [**56**]).

For example if G is abelian, then $\mathcal{H}^0_G(\text{ pt}) = \mathbb{H}^*(G)_S$ where $\mathbb{H}^*(G)$ is a ring functor on finite abelian groups such that

(i) $\mathbb{H}^0(G) = \mathbb{Z}$ and $\mathbb{H}^1(G) \cong G$
(ii) $\mathbb{H}^*(G_1\oplus G_2) = \mathbb{H}^*(G_1)\otimes \mathbb{H}^*(G_2)$.
(iii) If G is cyclic, then $\mathbb{H}^*(G) = H^{2*}(G;\mathbb{Z})$.

and where $\mathcal{S}$ is the multiplicative system generated by the total Chern classes of the irreducible representations of G.

Note. The discussion above provokes the interesting question of what functor is defined by taking the equivariant homotopy groups of $Z_0(X)$ for a G-space X. Is there an equivariant Dold-Thom Theorem? The very nice answer, due to P.Lima-Filho, is that for G-spaces one gets Bredon cohomology with $\mathbb{Z}$-coefficients, and for G-spectra one gets the RO_G-graded homology with coefficients in the Burnside ring Mackey functor [**92**].

Chapter IV - The Functor L_*H_*

The groups $\pi_*\mathcal{Z}_*(X)$ constitute a set of interesting invariants attached to any projective variety X. Some work has been done recently in trying to systematically understand these invariants. At least part of this is presented below.

Before embarking let me offer some general motivation. As we have seen, for any projective variety X, the p-dimensional subvarieties generate an interesting topological group $\mathcal{Z}_p(X)$. This group is functorially related to X. Its geometry is a limit of the Chow sets of X. Specifically, we know from I.8.2 that

$$\pi_k\mathcal{Z}_p(X) = \varinjlim_{\alpha} \pi_k\mathcal{C}_{p,\alpha}(X) \qquad \text{and} \qquad H_k\mathcal{Z}_p(X) = \varinjlim_{\alpha} H_k\mathcal{C}_{p,\alpha}(X)$$

for all $k > 0$. In fact all "stable" topological invariants of the Chow sets of X are carried in this fashion by $\mathcal{Z}_p(X)$.

Now the homotopy type of $\mathcal{Z}_p(X)$ is completely determined by the groups $\pi_k\mathcal{Z}_p(X)$. Such a statement is false for general spaces. However for an abelian topological group Z, the invariants π_*Z are special. For example π_*Z appears as primitive elements in the Hopf algebra $H_*(Z;\mathbb{Z})$. It can also be computed as the homology of the simplicial group $Sing.(Z)$. So the groups π_*Z are simpler than other invariants, like $H_*(Z)$, but nevertheless they determine Z up to homotopy equivalence. This makes $\pi_k\mathcal{Z}_p(X)$ natural to consider in studying $\mathcal{Z}_p(X)$.

It is useful to think of $\mathcal{Z}_p(X)$ as a generalized torus associated to X, much like the intermediate Jacobians. In fact there is a homotopy equivalence

$$\mathcal{Z}_p \cong \mathcal{A}_p \times \left(S^1\right)^{b_1} \times \left(BS^1\right)^{b_2} \times \left(B^2S^1\right)^{b_3} \times \cdots \times Tor_p$$

where $\mathcal{A}_p$ is the group of p-cycles on X modulo algebraic equivalence, $b_k = \operatorname{rank}(\pi_k\mathcal{Z}_p)$, and Tor_p is a connected space with $\pi_k Tor_p$ finite for all k. The various tori which are delooped in this picture can in fact be directly related to intermediate jacobians and their generalizations.

From another perspective $\pi_*\mathcal{Z}_*$ is the direct analogue of $\pi_*\mathfrak{Z}_*$ where $\mathfrak{Z}_*(X)$ denotes the rectifiable cycles on X, (cf. II.5). Now by Almgren's theorem II.5.1

there is an equivalence:

$$\pi_* \mathfrak{Z}_*(\bullet) = H_{*+*}(\bullet; \mathbb{Z}).$$

This indicates that the functor $\pi_* \mathcal{Z}_*$ might behave like a homology theory on the category of projective varieties. In fact the map $\pi_* \mathcal{Z}_* \to \pi_* \mathfrak{Z}_{2*}$ constitutes a natural transformation to standard integral homology. Thinking of $\pi_* \mathcal{Z}_*$ in this way gives a systematic approach to the study of these invariants. Note however that $\pi_* \mathcal{Z}_*$ is far from being a simple topological theory. For example, $\pi_0 \mathcal{Z}_p$ is the group of algebraic p-cycles modulo algebraic equivalence. This already shows these groups to be non-trivially related to the *algebraic* structure of X. We shall see below that the theory in fact emcompasses many new algebraic invariants.

§1. **Definitions and basic properties.** With the motivation above E. Friedlander introduced in [**22**] the groups

$$L_p H_k(X) \stackrel{\text{def}}{=} \pi_{k-2p} \mathcal{Z}_p(X) \tag{1.1}$$

for $k \geq 2p \geq 0$. Here k denotes the **homology dimension**, and p the **algebraic level** (the number of algebraic parameters). From the Algebràic Suspension Theorem II.3.1 we have canonical isomorphisms

$$L_p H_k(X) = L_{p+1} H_{k+2}(\Sigma\!\!\!/ X) \tag{1.2}$$

for all $k \geq 2p \geq 0$.

From I.7.1 we see that $L_* H_*$ *is a functor* on the category of projective varieties, i.e., if $f : X \to Y$ is a morphism, then there are induced homomorphisms

$$f_* : L_p H_k(X) \to L_p H_k(Y) \tag{1.3}$$

for all $k \geq 2p \geq 0$, and if $g : Y \to Z$ is another morphism, then

$$(g \circ f)_* = g_* \circ f_*. \tag{1.4}$$

From I.7.2 we have *Gysin maps.* If $f : X \to Y$ is a flat morphism, then there are induced homomorphisms

$$f^* : L_p H_k(Y) \to L_{p+r} H_{k+r}(X) \tag{1.6}$$

where $r = \dim X - \dim Y$. If $g : Y \to Z$ is also flat, then

$$(g \circ f)^* = f^* \circ g^*. \tag{1.7}$$

Note. For those who are simplicially minded we note that the definition in (1.1) could be replaced by $H_{k-2p} \overline{\mathcal{Z}}_p(X)$ where $\overline{\mathcal{Z}}_p(X) = N\text{Sing.}(\mathcal{Z}_p(X))$ is the normalized chain complex of the simplicial group Sing. $(\mathcal{Z}_p(X))$.

§2. The natural transformation to $H_*(\bullet;\mathbb{Z})$. The continuous homomorphism

$$\mathcal{Z}_*(X) \hookrightarrow \mathfrak{Z}_{2*}(X) \tag{2.1}$$

defined in II.5 induces a map on homotopy groups which is independent of the choice of smooth triangulation. Applying Almgren's theorem II.5.1 gives the following.

Theorem 2.1. *There is a natural transformation of functors*

$$\Phi : L_pH_k(X) \to H_k(X;\mathbb{Z}) \tag{2.2}$$

for all $0 \le 2p \le k$.

Note that it is *integral* (not rational or real) homology that appears here.

§3. Coefficients in $\mathbb{Z}_m$. In the preceeding two sections, one could replace $\mathfrak{Z}_*(X)$ with the quotient group $\mathfrak{Z}_*(X)/m\mathfrak{Z}_*(X)$ for a fixed integer $m > 0$. This yields a functor

$$L_pH_k(X;\mathbb{Z}_m) = \pi_{k-2p}\left\{\mathcal{Z}_p(X)/m\mathcal{Z}_p(X)\right\} \tag{3.1}$$

with a natural transformation

$$\Phi : L_pH_k(X;\mathbb{Z}_m) \to H_k(X;\mathbb{Z}_m). \tag{3.2}$$

Most results discussed below will carry through in this case.

§4. Relative groups. Let $Y \subset X$ be an algebraic subset of a projective variety X. For each p, $\mathcal{Z}_p(Y)$ is a closed subgroup of $\mathcal{Z}_p(X)$ and we can consider the quotient group $\mathcal{Z}_p(X)/\mathcal{Z}_p(Y)$ with the quotient topology. We set

$$L_pH_k(X,Y) \stackrel{\text{def}}{=} \pi_{k-2p}\mathcal{Z}_p(X)/\mathcal{Z}_p(Y) \tag{4.1}$$

for $k \ge 2p \ge 0$. Then we have the following.

Theorem 4.1 ([**47**], [**50**], [**51**]). *There is a long exact sequence*

$$\cdots \to L_pH_k(Y) \to L_pH_k(X) \to L_pH_k(X,Y) \to L_pH_{k-1}(Y) \to \cdots$$

which is functorial for morphisms of pairs. This sequence terminates with

$$\cdots \to L_pH_{2p+1}(X,Y) \to \mathcal{A}_p(Y) \to \mathcal{A}_p(X) \to \mathcal{A}_p(X)/\mathcal{A}(Y) \to 0$$

where $\mathcal{A}_p$ denotes the group of p-cycles modulo algebraic equivalence.

§5. **Localization**. Fundamental results in the theory are the localization theorems of P. Lima-Filho.

Theorem 5.1 (Lima-Filho [**50**], [**51**]). *Let X, X' be projective varieties with algebraic subsets $Y \subset X$ and $Y' \subset X'$, and suppose*

$$f : X - Y \xrightarrow{\cong} X' - Y'$$

is an isomorphism of quasi-projective varieties. Then there is a naturally induced isomorphism of groups

$$f_* : \mathcal{Z}_p(X)/\mathcal{Z}_p(Y) \xrightarrow{\cong} \mathcal{Z}_p(X')/\mathcal{Z}_p(Y')$$

which is a homeomorphism. In particular there is a naturally induced isomorphism

$$f_* : L_*H_*(X,Y) \xrightarrow{\cong} L_*H_*(X',Y')$$

This theorem enables us to extend the theory to quasi-projective varieties.

Definition 5.2. Let $U \subset \mathbb{P}^N$ be a quasi-projective variety with closure $\overline{U}$. Then we define the **topological group of p-cycles on** U to be the quotient

$$\mathcal{Z}_p(U) \stackrel{\text{def}}{=} \mathcal{Z}_p(\overline{U})/\mathcal{Z}_p(\overline{U} - U),$$

and we set

$$L_pH_k(U) \stackrel{\text{def}}{=} \pi_{k-2p}\mathcal{Z}_p(U)$$

for all $k \geq 2p \geq 0$.

By 5.1, $\mathcal{Z}_p(U)$ and $L_pH_k(U)$ are independent of the projective embedding of U. They are, in fact, functors on the category of quasi-projective varieties and proper morphisms. Furthermore, the following holds.

Theorem 5.3. (Lima-Filho [**50**], [**51**]). *Let $V \subset U$ be a Zariski open subset of a quasi-projective variety U. Then there is a long exact "localization" sequence:*

$$\cdots \to L_pH_k(U - V) \to L_pH_k(U) \to L_pH_k(V) \to L_pH_{k-1}(U - V) \to \cdots$$

From this one can inductively build a Zariski open covering and do computations.

The proof of Theorem 5.1 uses strongly that one can work with the naïve group completion. The idea is as follows. Suppose $\varphi : X - Y \to X' - Y'$ is an isomorphism. By replacing X with the closure of the graph of φ in $X \times X'$ we can assume φ extends to a morphism on X. One then has a well-defined map $\varphi_* : \mathcal{Z}_p(X)/\mathcal{Z}_p(Y) \to \mathcal{Z}_p(X')/\mathcal{Z}_p(Y')$, which a direct technical argument shows to be a homeomorphism.

The proof of Theorem 5.3 amounts to proving that a short exact sequence of groups is a principal fibration.

Theorem 5.3 is quite useful. One can inductively build a space from a suitable open covering and apply the localization sequence step by step. In this way for example one can "untwist" the Suspension Theorem to get the following pretty result.

Theorem 5.4. (Friedlander-Gabber [**28**]). *Let U be a quasi-projective variety. Then algebraic suspension induces isomorphisms*

$$L_pH_k(U) \xrightarrow{\approx} L_{p+1}H_{k+2}(U \times \mathbb{C}) \tag{5.1}$$

for all $p \geq 2p \geq 0$. More generally if $\pi : E \to U$ is an algebraic vector bundle of rank r over U, then the flat pull-back of cycles induces isomorphisms

$$\pi^* : L_pH_k(U) \xrightarrow{\approx} L_{p+r}H_{k+2r}(E) \tag{5.2}$$

for all $k \geq 2p \geq 0$.

There is a related "projective bundle theorem" which we will discuss soon.

§**6. Computations.** With the results discussed thus far one can compute the groups $L_*H_*(X)$ in a number of interesting cases. We begin with the following.

A projective variety X is said to **admit a cell-decomposition** if there exists a nested family

$$X_0 \subset X_1 \subset \cdots \subset X_N = X$$

of algebraic subsets with the property that $X_k - X_{k-1}$ is isomorphic to $\mathbb{C}^{n_k}$ for all k (where $0 = n_0 \leq n_1 \leq n_2 \leq \cdots$). Spaces of this type include : Grassmannians and in fact all generalized flag manifolds, hermitian symmetric spaces, and varieties on which a reductive group acts with isolated fixed points.

Theorem 6.1. (Lima-Filho [**50**], [**53**]). *Let X be a projective variety which admits a cell decomposition. Then the inclusion*

$$\mathcal{Z}_*(X) \hookrightarrow \mathfrak{Z}_{2*}(X)$$

is a homotopy equivalence and the natural transformation

$$\Phi : L_pH_k(X) \xrightarrow{\approx} H_k(X;\mathbb{Z})$$

is an isomorphism for all $p \geq 2p \geq 0$.

This represents a vast generalization of the fact that on such spaces every homology class is represented by an algebraic cycle unique up to algebraic equivalence. (This fact corresponds to the isomorphism $\pi_0\mathcal{Z}_*(X) \cong \pi_0\mathfrak{Z}_{2*}(X)$).

Of course such a result does not hold for general projective manifolds. It is precisely for this reason that the groups L_*H_* are interesting. A good example where it fails is a product of elliptic curves, or more generally any abelian variety. This follows directly from Hodge theory, since the homology class of an algebraic cycle is always of type (p,p). Other examples can be constructed from the following result (cf. (I.4.2)).

Theorem 6.2 (Friedlander [**22**], [**28**]). *Let X be a non-singular projective variety of dimension n. Then there are isomorphisms*

$$\begin{aligned} L_{n-1}H_{2n}(X) &\cong \mathbb{Z}, \\ L_{n-1}H_{2n-1}(X) &\cong H_{2n-1}(X;\mathbb{Z}), \\ L_{n-1}H_{2n-2}(X) &\cong H_{n-1,n-1}(X;\mathbb{Z}) = NS(X) \end{aligned}$$

and $L_{n-1}H_k(X) = 0$ for $k > 2n$.

This computes the groups completely for smooth algebraic surfaces.

In [**28**] Friedlander and Gabber extend the Algebraic Suspension Theorem to a refined intersection theorem with divisors (cf. §8). This enabled them to prove the following "projective bundle theorem".

Theorem 6.3 (Friedlander-Gabber [**28**]). *Let E be an algebraic vector bundle of rank r over a quasi- projective variety U. Then for each $p \geq r-1$ there is a homotopy equivalence*

$$\mathcal{Z}_p(\mathbb{P}(E))) \cong \prod_{k=0}^{r-1} \mathcal{Z}_{p-k}(U)$$

where $\mathbb{P}(E)$ denotes the projectivization of the bundle E.

A direct consequence of localization and Theorem 6.2 is the following :

Theorem 6.4. *Let X be a smooth projective 3-fold. Then each of the groups $L_1H_k(X)$ for $k \geq 6$ is a birational invariant of X.*

It is not unreasonable to conjecture that $L_pH_k(X) = 0$ for all $p > 2\dim_{\mathbb{C}}(X)$. This would be interesting if true. If false, then in the first dimension for which it fails one finds non-trivial birational invariants.

§7. A local-to-global spectral sequence. By using the Localization Theorem of Lima-Filho (Theorem 5.3), Friedlander and Gabber are able to construct an analogue of Quillen's local-to-global spectral sequence in algebraic K-Theory [**69**]. Fix a quasi-projective variety X and, as before, let $X(p)$ denote the set of p-dimensional subvarieties of X. For each $x \in X(p)$, set

$$\widetilde{L_rH_k}(x) \stackrel{\text{def}}{=} \lim_{U\subset x} L_rH_k(U)$$

where the limit is taken over all Zariski open subsets U of x. From the localization exact sequence one constructs an exact couple which yields the following.

Propostion 7.1 ([**28**]). *Let X be a quasi-projective variety and $r \geq 0$ an integer. Then there is a spectral sequence of homological type of the form :*

$$E^1_{p,q} = \bigoplus_{x \in X(p)} \widetilde{L_r H_{p+q}}(x) \;\Rightarrow\; L_r H_{p+q}(X).$$

Following ideas of Quillen [**69**] and Bloch-Ogus [**8**], one can compute the E^2-term of this spectral sequence. Let $\mathcal{L}_r\mathcal{H}_k$ denote the Zariski sheaf on X associated to the presheaf

$$U \mapsto L_r H_k(U).$$

Theorem 7.2 ([**28**]). *Let X be a quasi-projective variety of dimension n, and fix $0 \leq 2r \leq k$. Then there is an exact sequence of sheaves on X :*

$$\begin{aligned} 0 \to \mathcal{L}_r\mathcal{H}_k \to \quad & \bigoplus\nolimits_{x \in X(n)} i_x \left(\widetilde{L_r H_k}(x)\right) \to \\ & \bigoplus\nolimits_{x \in X(n-1)} i_x \left(\widetilde{L_r H_{k-1}}(x)\right) \to \cdots \to \\ & \bigoplus\nolimits_{x \in X(n-k+2r)} i_x \left(\widetilde{L_r H_{2r}}(x)\right) \to 0 \end{aligned}$$

where $i_x\left(\widetilde{L_r H_j}(x)\right)$ denotes the constant sheaf $\widetilde{L_r H_j}(x)$ on x extended by zero to all of X, and the spectral sequence of 7.1 *has the form*

$$E^2_{p,q} = H^{n-p}(X, \mathcal{L}_r\mathcal{H}_{n+q}) \;\Rightarrow\; L_r H_{p+q}(X).$$

§**8. Intersection Theory.** In [**28**] E. Friedlander and O. Gabber succeed in extending the Algebraic Suspension Theorem to a beautiful intersection pairing defined at the level of the groups $\mathcal{Z}_*$. (Recall that intersection theory is conventionally defined in the quotient $A_* \equiv \mathcal{Z}_*/\sim$ of cycles modulo rational equivalence (cf. [**34**])). This pairing enables us to define a graded commutative ring structure on $L_*H_*(X)$ for X smooth.

To begin suppose $E_D \xrightarrow{\pi} X$ is a line bundle associated to a divisor D on X. Let $i : X \hookrightarrow E_D$ be the inclusion as the zero-section. Then composing with the homotopy inverse in (5.2) gives a map

$$\mathcal{Z}_p(X) \xrightarrow{i_*} \mathcal{Z}_p(E_D) \xrightarrow{\cong} \mathcal{Z}_{p-1}(X). \tag{8.1}$$

This represents "intersection with D". In fact if one lets $\mathcal{C}_p(X, D)$ denote the effective p-cycles which meet D in dimension $\leq p-1$, then the restriction of (8.1) to the naive group completion of $\mathcal{C}_p(X, D)$ is homotopic to the intersection product

$$c \mapsto c \cdot D \tag{8.2}$$

which is continuously defined as in [**34**].

It would be a sharper and more useful result to know that the image of the composition (8.1) consisted of cycles in the support $|D|$ of D. In [**28**] this and much more are accomplished.

We recall that an **effective Cartier divisor** on X is one which is defined by the vanishing of a regular section of a line bundle on X.

Theorem 8.1 (Friedlander-Gabber [**28**]). *Let D be an effective Cartier divisor on a quasi-projective variety X. Then for each $p \geq 1$ there is a canonical homotopy class of maps*

$$i_D^! : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p-1}(|D|)$$

which on the subgroup generated by $\mathcal{C}_p(X, D)$ is induced by the intersection map (8.2). The composition

$$c_1(L) \stackrel{def}{=} (i_D)_* \circ i_D^! : Z_p(X) \longrightarrow Z_{p-1}(|D|) \longrightarrow Z_{p-1}(X)$$

(where $i_D : |D| \hookrightarrow X$ denotes the inclusion) depends only on the isomorphism class of the line bundle $L = \mathcal{O}(D)$.

If D, D' are two such divisors then

$$i_D^! + i_{D'}^! = i_{D+D'}^! : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p-1}(|D| \cup |D'|)$$

and

$$i_D^! \circ i_{D'}^! = i_{D'}^! \circ i_D^! : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p-1}(|D| \cap |D'|).$$

Note 8.2. In [**28**] the authors work in the category of chain complexes localized with respect to quasi isomorphisms (maps of chain complexes inducing isomorphisms in homology). This makes the statements slightly neater and stronger.

Note 8.3. Let $p : \mathbb{P}(E) \to U$ be the projectivization of a bundle of rank r, and let L_E denote the standard line bundle on $\mathbb{P}(E)$. Then the equivalence in the Projective Bundle Theorem 6.3 is given by

$$\sum_{k=0}^{r-1} c_1 (L_E)^k \circ p^* : \prod_{k=0}^{r-1} \mathcal{Z}_{p-k}(U) \xrightarrow{\cong} \mathcal{Z}_p(\mathbb{P}(E)).$$

Note 8.4. The intersection pairing above leads to a general pairing $\mathcal{Z}_p(X) \wedge \mathrm{Div}(X)^+ \to \mathcal{Z}_{p-1}(X)$ where, in the case that X is smooth, $\mathrm{Div}(X)^+ \cong \mathcal{Z}_1(X)$. The induced pairing on homotopy groups yields the operators $c_1(L)$ above (corresponding to elements $L \in \pi_0 \mathrm{Div}(X)^+ \cong NS(X)$), and also yields the operator s of Theorem II.6.1 (corresponding to the generator of $\pi_2 \mathrm{Div}(X)^+ \cong \mathbb{Z}$).

Theorem 8.1 can be generalized from divisors to subvarieties of general codimension.

Theorem 8.5. (Friedlander-Gabber [**28**]). *Let X be a quasi-projective variety and $i_V : V \hookrightarrow X$ a regular (closed) embedding of codimension-q. Then for all $p \geq q$ there is a naturally defined homotopy class of mapping*

$$i_V^! : \mathcal{Z}_p(X) \to \mathcal{Z}_{p-q}(V).$$

This map has the property that on the subgroup generated by the effective cycles which meet V in proper dimension, it is homotopic to the intersection-theoretic mapping $c \mapsto c \cdot V$. These maps behave as expected with respect to composition and flat pull-back of cycles, namely

$$(i_V \circ i_{V'})^! = i_{V'}^! \circ i_V^!$$

and

$$i_{\widetilde{V}}^! \circ g^* = \widetilde{g}^* \circ i_V^!$$

if $g : Y \to X$ is flat and $\widetilde{g} : \widetilde{V} \stackrel{def}{=} Y \times_X V \to V$ is the pull-back of g via i_V.

If $f : X \to Y$ is a morphism of varieties where Y is smooth, then the inclusion $\Gamma_f \hookrightarrow X \times Y$ of the graph of f into the product is a regular embedding. Theorem 8.5 thereby leads to the following basic result.

Theorem 8.6 (The Intersection Pairing, Friedlander-Gabber [**28**]). *Let $f : X \to Y$ be a morphism of quasi-projective varieties where Y is smooth and of dimension n. Then if $p + p' \geq n$, f determines a natural pairing*

$$\mathcal{Z}_p(X) \wedge \mathcal{Z}_{p'}(Y) \longrightarrow \mathcal{Z}_{p+p'-n}(X).$$

In particular when X is smooth and of dimension n, one obtains a pairing

$$\mathcal{Z}_p(X) \wedge \mathcal{Z}_{p'}(X) \longrightarrow \mathcal{Z}_{p+p'-n}(X) \tag{8.3}$$

which extends, up to homotopy, the usual intersection pairing on cycles which meet in proper dimension. This pairing is homotopy commutative and associative.

Corollary 8.7. *For any smooth quasi-projective variety X, the pairing*

$$L_*H_*(X) \otimes L_*H_*(X) \longrightarrow L_*H_*(X)$$

induced by (8.3) *gives $L_*H_*(X)$ the structure of a bigraded commutative ring.*

Restricted to $\bigoplus_p L_pH_{2p}(X)$ (= cycles modulo algebraic equivalence), this is the standard ring structure given by intersection product.

§9. Operations and filtrations. Using the complex join and the Suspension Theorem, E. Friedlander and B. Mazur [**29**] have introduced a ring of operators on L_*H_*. These operators lead to filtrations [**29**], [**30**], [**23**] which are compatible with, and conjecturally equal to certain standard filtrations.

Throughout this section X will denote a projective variety with a fixed embedding $X \subset \mathbb{P}^N$. The Algebraic Suspension Theorem gives us canonical isomorphisms

$$L_pH_k(X) \xrightarrow{\cong} L_{p+1}H_{k+2}(\Sigma X) \xrightarrow{\cong} L_{p+2}H_{k+4}(\Sigma^2 X) \xrightarrow{\cong} \cdots$$

Hence, we can extend our groups $L_pH_k(X)$ to negative indices by setting

$$L_pH_k(X) \stackrel{\text{def}}{=} \lim_{j\to\infty} L_{p+j}H_{k+2j}(\Sigma^j X) \tag{9.1}$$

for all $k \geq 2p$. We saw in Chapter II.5 that $L_*H_*(X)$ is a bigraded module over the ring

$$\mathcal{FM} \equiv L_*H_*\left(\mathbb{P}^0\right) \cong \mathbb{Z}[\,\mathbf{h},\ \mathbf{s}] \tag{9.2}$$

where

$$\mathbf{h} \in L_{-1}H_{-2}\left(\mathbb{P}^0\right) \qquad \text{and} \qquad \mathbf{s} \in L_{-1}H_0\left(\mathbb{P}^0\right) \tag{9.3}$$

are the additive generators. These homomorphisms are functorial, i.e., they are natural transformations of L_*H_* on the category of projective subvarieties and polorization-preserving morphisms. They constitute therefore a ring of "homology operations" which we call **Friedlander-Mazur operations.**

The first operator $\mathbf{h}$ is an operator of Lefschetz type. In fact under the natural transformation Φ of §2 we have a commutative diagram

$$\begin{array}{ccc} L_pH_k(X) & \xrightarrow{\mathbf{h}} & L_{p-1}H_{k-2}(X) \\ \Phi\downarrow & & \downarrow\Phi \\ H_k(X;\mathbb{Z}) & \xrightarrow{\lambda} & H_{k-2}(X;\mathbb{Z}) \end{array}$$

where λ is the Lefschetz map given by cap-product with the hyperplane class $c_1\left(\mathcal{O}_X(1)\right)$ (See [**52**], [**29**]). This operator $\mathbf{h}$ evidently depends on the projective embedding since λ does. We recall that if X is smooth of dimension n, then $\lambda^k : H_{n+k}(X;\mathbb{Q}) \to H_{n-k}(X;\mathbb{Q})$ is an isomorphism.

The second operator $\mathbf{s}$ is a special feature of this theory. It preserves homology degree and lowers algebraic level. In fact we have commutative diagrams (cf. [**23**])

$$\begin{array}{ccc} L_pH_k(X) & \xrightarrow{\mathbf{s}} & L_{p-1}H_k(X) \\ & \Phi\searrow \quad \swarrow\Phi & \\ & H_k(X;\mathbb{Z}) & \end{array}$$

for all p and k. Note in particular that

$$\mathbf{s}^p = \Phi : L_pH_k(X) \longrightarrow L_0H_k(X) = H_k(X;\mathbb{Z}).$$

Theorem 9.1. *The operation s of Friedlander and Mazur is a natural transformation of covariant functors. In particular it is independent of projective embedding and is also compatible with flat pull-back of cycles and localization.*

Work on the operation $\mathbf{s}$ developed as follows. It was introduced in [**29**] where it was proved that $\mathbf{s}^p : L_pH_{2p}(X) \cong \mathcal{A}_p(X) \to H_{2p}(X;\mathbb{Z})$ is the "cycle map" which associates to an algebraic cycle (modulo algebraic equivalence) its homology class. In [**52**] Lima-Filho generalized this to prove that the map $\mathbf{s}^p : L_pH_k(X) \to H_k(X;\mathbb{Z})$, for any $k \geq 2p$, agrees with Almgren's map ([**1**]) and, in particular, is independent of the projective embeddings of X. Friedlander and Gabber [**28**] then proved that every $\mathbf{s}$: $L_pH_k(X) \to L_{p-1}H_k(X)$ is independent of projective embedding. In [**23**], Friedlander established a number of interesting properties and interpretations of this operation, some of which involve the intersection theory discussed above.

Now the powers of $\mathbf{s}$ give rise to very interesting filtrations. Consider the case where $k = 2p$ is even. We have the sequence of homomorphisms

$$\begin{array}{ccccccccc} L_pH_{2p}(X) & \xrightarrow{s} & L_{p-1}H_{2p}(X) & \xrightarrow{s} & \cdots & \xrightarrow{s} & L_1H_{2p}(X) & \xrightarrow{s} & L_0H_{2p}(X) \\ \| & & & & & & & & \| \\ \mathcal{A}_p & & & & & & & & H_{2p}(X;\mathbb{Z}) \end{array} \tag{9.4}$$

where $\mathcal{A}_p$ denotes the group of algebraic p-cycles modulo algebraic equivalence.

This gives us two filtrations

$$\mathcal{A}_p \supseteq \mathcal{G}_{p,p} \supseteq \mathcal{G}_{p,p-1} \supseteq \cdots \supseteq \mathcal{G}_{p,1} \tag{9.5}$$

$$\mathcal{F}_{2p,p} \subseteq \mathcal{F}_{2p,p-1} \subseteq \cdots \subseteq \mathcal{F}_{2p,1} \subseteq H_{2p}(X;\mathbb{Z}) \tag{9.6}$$

defined by setting.

$$\mathcal{G}_{p,j} = \ker\left(\mathbf{s}^j\right) \qquad \text{and} \qquad \mathcal{F}_{2p,j} = \operatorname{Im}\left(\mathbf{s}^j\right) \tag{9.7}$$

on $\mathcal{A}_p$ and in $H_{2p}(X;\mathbb{Z})$ respectively. There are of course filtrations induced in the intermediate groups $L_jH_{2p}(X)$ as well. However the filtrations above are on classically defined groups and can be compared with well-known filtrations.

Note that $\mathcal{G}_{p,p}$ is the **Griffiths group** of p-cycles homologically equivalent to zero modulo those algebraically equivalent to zero. There is a filtration of this group due to Bloch and Ogus defined by setting

$$\mathcal{G}_{p,j}^{BO} = \{c \in \mathcal{A}_p : \ c \textit{ is homologous to zero in an algebraic subset of dimension } \leq p+1+j \ \in X\}.$$

There is an analogus **geometric filtration** on $H_{2p}(X;\mathbb{Z})$ defined by setting

$$\mathcal{F}_{2p,j}^{\mathrm{Gr}} = \operatorname{span}\{\alpha \in H_{2p}(X;\mathbb{Z}) : \ \alpha \textit{ is the image of a class supported on an algebraic subset of dimension } \leq 2p-j \ \in X\}.$$

Over $\mathbb{Q}$ this is the dual of Grothendieck's **arithmetic filtration** on $H^{2p}(X;\mathbb{Q})$. The following basic results have been proved.

Theorem 9.2 (Friedlander [**23**]). *For any projective variety X, we have*

$$\mathcal{G}_{p,j} \subseteq \mathcal{G}^{BO}_{p,j}$$

for all p, j.

Theorem 9.3 (Friedlander-Mazur [**29**], [**30**]). *For any projective variety X, we have*

$$\mathcal{F}_{2p,j} \subseteq \mathcal{F}^{Gr}_{2p,j}$$

for all p, j. Furthermore, the analogous result holds on odd-degree cohomology groups.

Theorems 9.2 and 9.3 are in fact proved in a much stronger form. It is shown that the filtrations $\mathcal{G}_{p,*}$ and $\mathcal{F}_{2p,*}$ *coincide* with certain geometrically defined filtrations. More specifically, in [**30**] one associates to a morphism $f : Y \longrightarrow \mathcal{C}_p(X)$ from any projective variety Y, an induced *Chow correspondence homomorphism* $\Phi_f : H_*(Y; \mathbb{Z}) \longrightarrow H_{2p+*}(X; \mathbb{Z})$ which generalizes very classical constructions. Let $\mathcal{F}^{C}_{2p,j} \subset H_{2p+j}(X; \mathbb{Z})$ be the subgroup generated by the images of all such maps. Then in [**30**] it is proved that

$$\mathcal{F}_{2p,j} = \mathcal{F}^{C}_{2p,j}.$$

This shows that Friedlander's functor L_*H_*, which is close to ordinary homology theory, has the property that its homologically and geometrically defined filtrations coincide.

There is an analogous story for $\mathcal{G}_{*,*}$. In [**23**] it is proved that the subgroup $\mathcal{G}_{p,j}$ is generated by cycles of the form $\Phi_f(c)$, where $f : Y \longrightarrow \mathcal{C}_{p-k}(X)$ and c is an algebraic p-cycle homologous to zero on the projective variety Y.

It is a classical result that the geometric filtration is subordinate to the Hodge filtration (as strengthened by Grothendieck in [**39**]). In particular, if X is smooth we can define

$$\mathcal{F}^{H}_{2p,j} = \rho_{\mathbb{C}}^{-1}\left(\bigoplus_{|r-s|\leq 2(p-j)} H_{r,s}(X)\right)$$

where $\rho_{\mathbb{C}} : H_{2p}(X;\mathbb{Z}) \to H_{2p}(X;\mathbb{C})$ is the coefficient homomorphism, and the decomposition $H_{2p}(X;\mathbb{C}) = \bigoplus_{r+s=2p} H_{r,s}(X)$ is dual to the standard Dolbeault decomposition of $H^{2p}(X;\mathbb{C})$. Define $\mathcal{F}^{HG}_{2p,j} \subset \mathcal{F}^{H}_{2p,j}$ to be the pull-back by $\rho_{\mathbb{C}}$ of a maximal Hodge substructure. (See [**49**] for a nice discussion.) A special consequence of the above is that

$$\mathcal{F}_{2p,j} \subseteq \mathcal{F}^{HG}_{2p,j}.$$

Question 9.4. Does one have equality in 9.2 and 9.3 when X is a smooth projective variety?

It turns out this is not so unreasonable to ask.

Theorem 9.5 (Friedlander [**23**]). *If Grothendieck's standard conjecture B holds* (cf. [**38**]) *then for a smooth projective variety X,*

$$\mathcal{F}_{2p,j} \otimes \mathbb{Q} = \mathcal{F}^{Gr}_{2p,j} \otimes \mathbb{Q}$$

for all p and j.

More modestly one might ask for an example where the filtrations $\mathcal{F}_{p,j}$ and $\mathcal{G}_{p,j}$ are at least non-trivial. For the case $\mathcal{F}_{p,j}$, consider a product of elliptic curves $X = T_1 \times \cdots \times T_n$. For each sequence $(p_1, \cdots, p_n)$ of zeros and ones with $\sum p_j = p$ we have a map

$$\mathcal{Z}_{p_1}(T_1) \wedge \cdots \wedge \mathcal{Z}_{p_n}(T_n) \longrightarrow \mathcal{Z}_p(X)$$

inducing a map

$$L_{p_1}H_{k_1}(T_n) \otimes \cdots \otimes L_{p_n}H_{k_n}(T_n) \longrightarrow L_pH_k(X)$$

where $\sum k_j = k$. This map commutes with the natural transformation Φ giving a diagram

$$\begin{array}{ccc} \bigotimes_{j=1}^{n} L_{p_j}H_{k_j}(T_j) & \longrightarrow & L_pH_k(X) \\ \downarrow & & \downarrow \\ \bigotimes_{j=1}^{n} H_{k_j}(T_j;\mathbb{Z}) & \longrightarrow & H_k(X;\mathbb{Z}) \end{array}$$

from which one can deduce that in this case the filtrations $\mathcal{F}_{*,*}$ and $\mathcal{F}^{\mathrm{Gr}}_{*,*}$ agree and coincide with the trancendental Hodge filtration.

The non-triviality of the filtration $\mathcal{G}_{*,*}$ is related to recent work of M. Nori, [**67**]. For smooth varieties X, Nori introduces a filtration $\mathcal{G}^N_{p,*}$ on $\mathcal{A}_p(X)$ which he proves to be non-trivial on certain projective hypersurfaces. It is shown in [**23**] that $\mathcal{G}^N_{p,j} \subset \mathcal{G}_{p,j}$, in fact $\mathcal{G}^N_{p,j}$ is generated in the same fashion as $\mathcal{G}_{p,j}$, by Chow correspondence homomorphisms associated to maps $f : Y \longrightarrow \mathcal{C}_{p-j}(X)$ where Y is now assumed to be smooth.

For the motivically minded, one should mention that there is an intriguing spectral sequence of homology type defined in [**23**], which incorporates both filtrations $\mathcal{F}$ and $\mathcal{G}$. Its abutment is the associated graded of $H_*(X;\mathbb{Z})$ with respect to the $\mathcal{F}$-filtration, and there are isomorphisms $E^{2k+2}_{-2p,0} \cong \mathcal{A}_p/\mathcal{G}_{p,k}$.

§10. Mixed Hodge Structures. It is a fundamental and useful fact that the groups $L_pH_k(X) \otimes \mathbb{Q}$ carry mixed Hodge structures. This fact, due to Dick Hain, was exploited for example in the work of Friedlander and Mazur mentioned in the previous section. We recall that a **Mixed Hodge Structure** over $\mathbb{Q}$ is a finite dimensional vector space W over $\mathbb{Q}$ provided with an increasing **weight filtration**

$$\cdots \subseteq W_{i-1} \subseteq W_i \subseteq W_{i+1} \subseteq$$

and a decreasing **Hodge filtration** of the space $W_{\mathbb{C}} = W \otimes_{\mathbb{Q}} \mathbb{C}$

$$\cdots F^{j-1} \supseteq F^j \supseteq F^{j-1} \supseteq \cdots$$

so that $(W_* \otimes \mathbb{C}, F_*, \overline{F}_*)$ form a triple of opposing filtrations (cf. [**15**], (1.2.7) and (1.2.13).). A **morphism** of mixed Hodge structures is a linear map of rational vector spaces which is filtration preserving. The mixed Hodge structures form an abelian category MHS which is closed under tensor products. One can expand this category to include infinite dimensional vector spaces which are direct limits in MHS, with morphisms which are direct limits of morphisms from MHS. This is again an abelian category called **limits of mixed Hodge structures** and denoted LMHS (See [**40**], [**41**]).

Deligne showed that the functor $X \mapsto H_k(X;\mathbb{Q})$ takes values in MHS.

Theorem 10.1 (Dick Hain). *The functors* $L_pH_k(X)\otimes\mathbb{Q}$ *take values in* LMHS.

In other words each group $L_pH_k(X) \otimes \mathbb{Q}$ is naturally equipped with a direct limit of mixed Hodge structures, which is respected by the maps induced by morphisms of varieties. The idea of the proof, which is given in [**29**], is that the groups

$$H_j\left(\mathcal{Z}_p(X);\mathbb{Q}\right) = \varinjlim_{\alpha} H_j\left(C_{p,\alpha}(X);\mathbb{Q}\right)$$

are naturally limits of mixed Hodge structures, and the homotopy groups $\pi_*\mathcal{Z}_p(X)\otimes\mathbb{Q}$ can be identified with the primitive elements in the Hopf algebra $H_*\left(\mathcal{Z}_p(X);\mathbb{Q}\right)$. Since LMHS is an abelian category, the subspace of primitive elements, which the kernel of the morphism

$$H_*\left(\mathcal{Z}_p(X);\mathbb{Q}\right) \to H_*\left(\mathcal{Z}_p(X);\mathbb{Q}\right) \otimes H_*\left(\mathcal{Z}_p(X);\mathbb{Q}\right),$$

given by $\alpha \mapsto \Delta_*(\alpha) - \alpha\otimes 1 - 1\otimes\alpha$, is also in LMHS.

Theorem 10.2 (Friedlander-Mazur [**29**]). *The operators* $\boldsymbol{s}$ *and* $\boldsymbol{h}$ *on* $L_*H_*(X)\otimes\mathbb{Q}$*, which are discussed in §9 above, respect the (limits of) mixed Hodge structures. In particular, the natural transformation* $\Phi : L_pH_k(X)\otimes\mathbb{Q} \to H_k(X;\mathbb{Q})$ *is a transformation of functors with values in* LMHS.

§11. Chern classes for higher algebraic K-theory. Friedlander and Gabber have defined Chern classes for the higher algebraic K-groups of a variety which have values in L_*H_*. One of the key steps is to replace a projective variety X by an "equivalent" affine variety, i.e., an affine variety with the same K-theory and LH-theory. This affine variety, whose construction is due to Jouanolou, is the total space of an affine $\mathbb{C}^m$-bundle $\pi : J_X \to X$. If $X \subset \mathbb{P}^N$, then J_X is merely the restriction of $\pi : J_{\mathbb{P}^N} \to \mathbb{P}^N$ defined as follows. Let

$$J_{\mathbb{P}^N} = \{A \in \mathcal{M}_{N+1,N+1} : A^2 = A \quad \text{and} \quad \text{rank}A = 1\}$$

where $\mathcal{M}_{N+1,N+1}$ is the space of $(N+1)\times(N+1)$ complex matrices. This is defined in $\mathbb{C}^{(N+1)^2}$ by the vanishing of the 2×2 minors and the equation

$\text{Tr}A = 1$; hence it is an affine variety. We set $\pi(A) = \text{Im}(A) \in \mathbb{P}^N$, and note that the fibres are affine subspaces.

Now since J_X is an affine variety, it is of the form Spec R where R is the ring of functions on J_X. Quillen shows that the map $K_*(X) \to K_*(\text{Spec } R) = \pi_*(BGL(R)^+)$ is an isomorphism. The homotopy property (cf. Theorem 5.4) which comes from the Suspension Theorem and Localization, give an isomorphism $L_*H_*(X) \to L_*H_*(\text{Spec } R)$ (with a shift in degrees). The Projective Bundle Theorem (cf. 8.3) and ideas of Grothendieck, lead to the following.

Theorem 11.1 (Friedlander-Gabber [**28**]). *Let X be a smooth, n-dimensional quasi-projective variety. Then for all $j > 0$ and all i with $0 \le i \le n$, there exist naturally constructed Chern classes*

$$c_{i,j} : K_j(X) \to L_{n-i}H_{2(n-i)+j}(X).$$

§12. **Relation to Bloch's higher Chow groups.** In [**5**] Spencer Bloch introduced higher Chow groups for a quasi- projective variety as follows. For each $k \ge 0$ consider the "algebraic simplex"

$$\Delta[k] = \{Z \in \mathbb{C}^{k+1} : \sum_{j=0}^{k} Z_j = 1\}$$

with combinatorial structure given as in the real case (i.e., "faces" are defined by intersections with coordinate planes). For a quasi-projective variety X, let $z^q(X,k)$ denote the free abelian group generated by irreducible subvarieties of codimension-q on $X \times \Delta[k]$ which meet $X \times F$ in proper dimension for each face F of $\Delta[k]$. Using intersection and pull-back of cycles, one defines face and degeneracy relations in the standard way, making $z^q(X,*)$ a simplicial abelian group. Let $|z^q(X,*)|$ denote its geometric realization, and let $(\tilde{z}^q(X,*), \partial)$ denote the chain complex naturally associated to $z^q(X,*)$ using the additive structure of each $z^q(X,*)$. Then by definition we have

$$CH^q(X,k) = \pi_k\left(|z^q(X,*)|\right) = H_k\left(\tilde{z}^q(X,*), \partial\right).$$

Theorem 12.1 (Friedlander-Gabber [**28**]). *Let X be a quasi-projective variety of dimension n. Then there exist natural homomorphisms*

$$CH^{n-p}(X,k) \longrightarrow L_pH_{2p+k}(X)$$

for all $0 \le p \le n$ and all k.

This map is induced by a map $\tilde{z}^{n-p} \to \tilde{\mathcal{Z}}_p$ in the derived category of chain complexes associated to simplicial abelian groups. When X is smooth and projective they show that $\tilde{z}^1(X,*) \otimes (\mathbb{Z}/m)$ and $\tilde{\mathcal{Z}}_{n-1}(X) \otimes (\mathbb{Z}/m)$ are quasi-isomorphic for any integer $m > 0$.

§13. The theory for varieties defined over fields of positive characteristic. The discussion in this article has been intentionally restricted to complex varieties. Nevertheless for many results stated above there are analogues which hold for varieties defined over arbitrary algebraically closed fields.

This highly non-trivial achievement is due to Eric Friedlander. The reader should see the announcement [**21**] and the main paper [**22**] for details. Very roughly the main ideas are these. Suppose X is defined over a field of characteristic $p \geq 0$ and ℓ is a prime $\neq p$. Then the Chow monoids $\mathcal{C}_r(X)$ are well defined, and one can construct group completions $\Omega B\mathcal{C}_r(X)_\ell$ via étale homotopy theory. Taking π_* gives ℓ-adic homology groups which we shall denote by $L_rH_k(X)_\ell$. When $2r = k$, this is the group of algebraic equivalence classes of r-cycles ; and when $r = 0$ it is isomorphic to the k^{th} étale ℓ-adic homology group of X. If X is defined over $\mathbb{C}$, this group is just the tensor product of $L_rH_k(X)$ with the ℓ-adic integers.

It is proved in [**22**] that the Algebraic Suspension Theorem is valid for $L_*H_*(X)_\ell$, and the Friedlander-Mazur operations are defined. The map $\mathbf{s}^p : L_pH_{2p}(X)_\ell \to L_0H_{2p}(X)_\ell = \lim_n H_{2p}(X_{et}, \mathbb{Z}/\ell^n)$, for $p > 0$, is just the cycle map. One has filtrations and mixed Hodge structures as in §§9-10 above.

One of the nice features of these groups is that they are Galois modules. Suppose X is defined over a field F and is provided with an embedding $X \subset \mathbb{P}^m_F$. Let K be the algebraic closure of F. Then $Gal(K/F)$ acts naturally on $L_*H_*(X_K)_\ell$, and the operations and cycle maps discused above are all $Gal(K/F)$-equivariant. So also are the maps $f_* : L_*H_*(X_K)_\ell \to L_*H_*(Y_K)_\ell$ induced by a morphism $f : X \to Y$ of varieties over F.

§14. New directions. There have been some recent enhancements of the above LH-constructions which are both algebraically and geometrically more sophisticated but, of course, less manageable than the basic theory. The first is due to Friedlander and Gabber [**28**] who construct functorial spaces where π_0 gives algebraic cycles modulo *rational* equivalence instead of the coarser algebraic equivalence. Their theory is therefore a "rational equivalence analogue" of LH-theory. The basic idea is to consider the simplicial monoid $\mathcal{F}_p(X) = Mor(\Delta[\bullet], \mathcal{C}_p(X))$ where $\Delta[k]$ is the algebraic simplex mentioned in §12 above. In the case $p = 0$, this becomes the *Suslin complex* $\mathcal{F}_0(X) = Sus.(X)$ of algebraic singular chains of the infinite symmetric product of X. It has recently been shown by Suslin and Voevodsky [**76**] that for all n,

$$H_*(Sus.(X); \mathbb{Z}/n) \cong H_*(X; \mathbb{Z}/n),$$

giving an algebraic computation of the singular homology of the variety. Furthermore the result extends to higher dimensional cycles to prove that

$$H_*(\mathcal{F}_p(X); \mathbb{Z}/n) \cong L_pH_*(X; \mathbb{Z}/n),$$

for all $p \geq 0$. (See [**24**], [**76**].) In [**24**] the groups $H_*(\mathcal{F}_p(X))$ are computed for $p = \dim(X) - 1$.

The rational theory of Friedlander-Gabber actually has a bivariant formulation in analogy with the constructions of the next section (see [**90**]).

There has also recently been work of P. Gajer aimed at constructing intersection versions of $L_*H_*(X)$. He has found workable definitions and has succeeded in formulating and proving an **intersection homology version of the Dold-Thom Theorem** [**36**]. P.Gajer and C.Flannery have also established the LH-groups [**87**], [**91**].

Chapter V - The Functor L^*H^* (Morphic Cohomology)

Recently E. Friedlander and the author [**31**], [**32**] introduced the notion of an effective algebraic cocycle on a variety X as a morphism $\varphi : X \to \mathcal{C}_p(\mathbb{P}^n)$, i.e., a family of projective cycles parametrized by X. From these basic objects, bigraded rings were defined on X yielding a functor with a natural transformation to $H^*(X;\mathbb{Z})$. Although very differently defined, this functor enjoys many of the properties of L_*H_*, and for smooth projective varieties is "Poincaré dual" to L_*H_*. We present here the outlines of this theory.

§1. Effective algebraic cocycles. Recall that for any finite simplicial complex X we have the theorem of Almgren [**1**] that

$$\pi_i\mathfrak{Z}_j(X) \cong H_{i+j}(X;\mathbb{Z}) \tag{1.1}$$

for all $i, j \geq 0$, where $\mathfrak{Z}_j(X)$ is the group of integral j-cycles on X. Here the doubly indexed family of groups $\pi_i\mathfrak{Z}_j(X)$ collapse redundantly to the homology of X. However, if X is a projective variety and we replace $\mathfrak{Z}_i(X)$ by *algebraic* cycles $\mathcal{Z}_i(X)$, then the groups $\pi_i\mathcal{Z}_j(X)$ pull apart to become the distinct functors examined in the last section.

There is a parallel story for cohomology. For any finite complex X, there are natural isomorphisms

$$\pi_i\mathrm{Map}\,(X, K(\mathbb{Z}, j)) \cong H^{j-i}(X;\mathbb{Z}) \tag{1.2}$$

for all $0 \leq i \leq j$, giving redundant representations of cohomology. (The case $i = 0$ is discussed in Chapter I. (See (I.3.6)). Now the results in the section above give us algebraic models for Eilenberg-MacLane spaces, namely

$$K(\mathbb{Z}, 2j) \cong \mathcal{Z}^j\,(\mathbb{C}^n)$$

for any $n \geq j$. Thus, if X is an algebraic variety, one can replace "Map" in (1.2) with "Mor", and hope by analogy to find a doubly indexed family of groups with a natural transformation to ordinary integral cohomology.

This leads us to the following basic definition. Throughout this section we shall use the term **variety** to mean a quasi-projective variety. For reasons of exposition we shall assume our varieties X to be weakly normal. (The general case follows easily since weak normalization is a functor; cf. [**32**]).

Definition 1.1. An **effective algebraic** s**-cocycle** on a variety X with values in a projective subvariety Y is a morphism

$$\varphi : X \to C^s(Y).$$

Note that such a morphism represents a family of codimension -s cycles on Y parametrized by X. These families occur naturally and abundantly in algebraic geometry. They are in fact as abundant as cycles themselves.

The following are examples of classical synthetic constructions that naturally yield cocycles.

Example 1.2. Let $f : Y \to X$ be a flat morphism. Then the flat pull-back $\varphi(x) = f^{-1}(\{x\})$ of cycles gives a morphism

$$\varphi : X \to C^s(Y)$$

where $s = \dim(X)$. As special case considers "Noether normalization" $f : Y^n \to \mathbb{P}^n$ defined by a generic linear projection of $Y^n \subset \mathbb{P}^N$ onto a linear subspace of the same dimension. This gives an n-cocycle $\varphi : \mathbb{P}^n \to SP^d(Y)$ where $d =$ degree (Y). Composing with f yields a cocycle $f^*\varphi : Y \to SP^d(Y)$.

Example 1.3. Let $X \subset \mathbb{P}^N$ be a smooth hypersurface and suppose $Y \subset \mathbb{P}^N$ is a subvariety which does not lie in any hyperplane. Then we define

$$\varphi : X \to C^1(Y)$$

by the intersection-theoretic product

$$\varphi(x) = T_x X \cdot Y$$

of Y with the tangent hyperplanes to X. Interesting cases arise by choosing $Y = X$.

Example 1.4. Let $X, Y, Z \subset \mathbb{P}^N$ be subvarieties such that for all $x \in X$, the cone $\not{\Sigma}_x Z$ on Z with vertex x meets Y in proper dimension. Then we can define

$$\varphi(x) = (\not{\Sigma}_x Z) \cdot Y.$$

Example 1.5. Let $X \subset \mathbb{P}^N$ be any subvariety of dimension n and define an "Alexander dual" cocycle

$$\varphi_X : \mathbb{P}^N - X \longrightarrow C^{N-n-1}\left(\mathbb{P}^N\right)$$

by setting

$$\varphi_X(u) = \not{\Sigma}_u X.$$

Example 1.6. Let $A = \mathbb{C}^N/\Lambda$ be an abelian variety with Θ-divisor D. Given $Y \subset A$, define

$$\varphi : A \to C^1(Y)$$

by $\varphi(a) = (a + D) \cdot Y$.

Many similar constructions are clearly possible.

An interesting consequence of the theory we are about to describe is that *to every algebraic cocycle there is naturally associated an integral cohomology class* just as to every algebraic cycle we can associate an integral homology class.

§2. **Morphic cohomology.** Let X and Y be as in 1.1, and denote by

$$C^s(X; Y)$$

the set of effective algebraic s-cocycles on X with values in Y. We provide this mapping space with the topology of uniform convergence on compact of families of bounded degree (i.e., families mapping into compact subsets of $C^s(Y)$). This makes $C^s(X; Y)$ an abelian topological monoid.

Any cocycle $\varphi \in C^s(X; Y)$ can be "graphed" to give a cycle $\Gamma_\varphi \in C^s(X \times Y)$. We let $GC^s(X \times Y)$ denote the submonoid of cycles in $C^s(X \times Y)$ which are **equidimensional** over X, i.e., cycles c such that $\text{supp}(c) \cap (\{x\} \times Y)$ is of pure codimension s for all $x \in X$. Then we have the following.

Theorem 2.1. ([**32**]). *If X is locally irreducible (e.g., smooth), then the graphing map*

$$\Gamma : C^s(X; Y) \to GC^s(X \times Y)$$

is a homeomorphism.

Recall that the homotopy-theoretic group completion of an abelian topological monoid $\mathcal{M}$ is defined to be $\mathcal{M}^+ = \Omega B\mathcal{M}$. (See I.8 above and [**62**].)

Definition 2.2. For X and Y as above, let $\mathcal{Z}^s(X; Y) = C^s(X, Y)^+$ and define the **morphic cohomology groups** of X with values in Y by

$$L^s H^k(X; Y) = \pi_{2s-k} C^s(X; Y)$$

for all $k \leq 2s$.

Theorem 2.3. (The Algebraic Suspension Theorem for Cocycles [**32**]) *The algebraic suspension map*

$$\mathcal{Z}^s(X; Y) \xrightarrow{\Sigma\!\!\!/\,\circ} \mathcal{Z}^s(X; \Sigma\!\!\!/ Y)$$

is a homotopy equivalence.

Note that when $X = \mathbb{P}^0$, morphic cohomology reduces to $L_*H_*(Y)$, and Theorem 2.3 is just the Suspension Theorem of Chapter II. The argument outlined there essentially carries over to the more general case above.

Here we are interested principally in the case where X is non-trivial and $Y = \Sigma^n \mathbb{P}^0 = \mathbb{P}^n$.

Definition 2.4. For $n \geq s$ let $\mathcal{Z}^s(X;\mathbb{C}^n)$ be the (homotopy) quotient

$$\mathcal{Z}^s(X;\mathbb{C}^n) = \mathcal{Z}^s\left(X;\mathbb{P}^n\right)/\mathcal{Z}^{s-1}\left(X;\mathbb{P}^{n-1}\right)$$

(cf. [**32**]) and define the **morphic cohomology groups of** X by

$$L^s H^k(X) = \pi_{2s-k}\mathcal{Z}^s(X;\mathbb{C}^n)$$

for $k \leq 2s$.

Theorem 2.3 gives canonical homotopy equivalences :

$$\mathcal{Z}^s\left(X;\mathbb{C}^n\right) \cong \mathcal{Z}^s\left(X;\mathbb{C}^{n+1}\right)$$

for all $n \geq s$, and so the definition of $L^*H^*(X)$ is independent of n. Note that $\mathcal{Z}^s(X;\mathbb{C}^n)$ can be roughly thought of as *families of affine varieties of codimensions parametrized by* X.

We note that, as with cycles, it is possible to replace the homotopy-theoretic group completion above with the naïve topological group completion. Details of this equivalence appear in [**33**] and [**88**].

In the remaining sections we sketch the principal features of morphic cohomology theory established in [**32**].

§**3. Functoriality.** Morphic cohomology is a functor on the category of quasi-projective varieties and morphisms. In particular to each morphisms $f : X \to X'$, there is an associated graded group homomorphism

$$f^* : L^*H^*(X') \longrightarrow L^*H^*(X) \tag{3.1}$$

of bidegree $(0,0)$, given by the obvious pull-back of cocycles. If $g : X' \to X''$ is a morphism on X', then

$$(g \circ f)^* = f^* \circ g^*.$$

Furthermore if $f : X \to X'$ is a flat proper map of fibre dimension d, then there are induced Gysin "wrong way" maps

$$f_! : L^*H^*(X) \longrightarrow L^*H^*(X') \tag{3.2}$$

of bidegree $(d, 2d)$. These satisfy the composition law:

$$(g \circ f)_! = g_! \circ f_!$$

§**4. Ring structure.** There is a natural biadditive pairing

$$\mathcal{Z}^s(X;\mathbb{C}^n) \wedge \mathcal{Z}^{s'}(X;\mathbb{C}^{n'}) \longrightarrow \mathcal{Z}^{s+s'}(X;\mathbb{C}^{n+n'+1})$$

induced by the pointwise join

$$(\varphi \# \varphi')(x) = \varphi(x) \# \varphi'(x)$$

of effective cocycles. Taking homotopy groups gives a pairing

$$L^s H^k(X) \otimes L^{s'} H^{k'}(X) \longrightarrow L^{s+s'} H^{k+k'}(X)$$

which makes the morphic cohomology $L^* H^*(X)$ of X a *bigraded commutative ring.* With respect to this the naturally induced maps (3.1) are ring homomorphisms.

§5. The natural transformation to $H^*(\bullet; \mathbb{Z})$. Passing from morphisms to general continuous maps gives a natural transformation

$$\Phi : L^s H^k(X) \to H^k(X; \mathbb{Z})$$

of functors of all $k \leq 2s$ which carries the join-induced product to the cup product. That is, for each variety $X, \Phi : L^* H^*(X) \to H^*(X; \mathbb{Z})$ is a *homomorphism of rings.*

For any polarized projective variety Y there is also a natural transformation of functors in X:

$$\Phi : L^s H^k(X; Y) \longrightarrow \bigoplus_{i=0}^{k} H^i \left(X; H_{2m-(k-i)} Y\right) \tag{5.2}$$

where $m = \dim_{\mathbb{C}} Y$ and $H_j(Y) = H_j(Y; \mathbb{Z})$.

§6. Operations and filtrations. The algebraic join of cocycles induces an exterior product

$$L^* H^*(X; Y) \otimes L^* H^*(X'; Y') \longrightarrow L^* H^*(X \times X'; Y \# Y') \tag{6.1}$$

in morphic cohomology. The Algebraic Suspension Theorem 2.3 gives us canonical isomorphisms $L^* H^*(X; Y \# \mathbb{P}^n) \cong L^* H^*(X; Y)$. Thus when $X' = \mathbb{P}^0$ and $Y' = \mathbb{P}^n$, the product (6.1) induces an action of the algebra

$$\mathcal{FM} = L^* H^* \left(\mathbb{P}^0; \mathbb{P}^0\right) \cong \mathbb{Z}[\, \mathbf{h},\ \mathbf{s}]$$

of Friedlander-Mazur operations (cf. I.6 and IV.9), where

$$\mathbf{h} \in L^1 H^2 \left(\mathbb{P}^0; \mathbb{P}^0\right) \qquad \text{and} \qquad \mathbf{s} \in L^1 H^0 \left(\mathbb{P}^0; \mathbb{P}^0\right)$$

are the additive generators in these bidegrees. These operations are functorial. For any variety X and polarized variety Y there is a commutative diagram

$$\begin{array}{ccc} L^* H^*(X; Y) & \xrightarrow{\ \mathbf{h}\ } & L^* H^*(X; Y) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ H^*(X; H_*(Y)) & \xrightarrow{\ \lambda\ } & H^*(X; H_*(Y)) \end{array}$$

where λ denotes cap product with the hyperplane class of Y on the coefficients $H_*(Y)$. There is also a commutative diagram

$$\begin{array}{ccc} L^*H^*(X;Y) & \xrightarrow{\mathbf{s}} & L^*H^*(X;Y) \\ \Phi \searrow & & \swarrow \Phi \\ & H^*(X;H_*(Y)) & \end{array}$$

If we pass to the morphic cohomology groups $L^*H^*(X)$, the operation $\mathbf{h}$ becomes zero. However, we retain the interesting operation

$$L^*H^*(X;Y) \xrightarrow{\mathbf{s}} L^*H^*(X;Y)$$

which with respect to the natural transformation Φ gives commutatives triangles

$$\begin{array}{ccc} L^sH^k(X) & \xrightarrow{\mathbf{s}} & L^{s+1}H^k(X) \\ \Phi \searrow & & \swarrow \Phi \\ & H^k(X;\mathbb{Z}) & \end{array} \tag{6.2}$$

for all $0 \leq k \leq 2s$. Thus for any variety X, the morphic cohomology is naturally a module over $\mathcal{FM}_0 \equiv \mathbb{Z}[\mathbf{s}]$. It is shown that the product in $L^*H^*(X)$ is $\mathcal{FM}_0$-bilinear, i.e., it has the property that $\mathbf{s}(a \cdot b) = (\mathbf{s}a) \cdot b = a \cdot (\mathbf{s}b)$ for all $a, b \in L^*H^*(X)$. Thus we have

Theorem 6.1 (**[32]**). *For any variety X the morphic cohomology $L^*H^*(X)$ is a graded commutative $\mathcal{FM}_0$-algebra natural with respect to morphisms $f : X' \to X$.*

Observe now that the operator $\mathbf{s}$ gives a sequence of homomorphisms

$$\cdots \xrightarrow{\mathbf{s}} L^sH^k(X) \xrightarrow{\mathbf{s}} L^{s+1}H^k(X) \xrightarrow{\mathbf{s}} \cdots \tag{6.3}$$

which commute with the natural transformation Φ to $H^k(X;\mathbb{Z})$. Thus if we set

$$\mathcal{F}^s \stackrel{\text{def}}{=} \Phi\left(L^sH^k(X)\right)$$

we obtain from (6.3) a filtration

$$\mathcal{F}^{s_0} \subset \mathcal{F}^{s_0+1} \subset \mathcal{F}^{s_0+2} \subset \cdots \subset H^k(X;\mathbb{Z}) \tag{6.4}$$

of the integral cohomology of X, where $s_0 = [(k+1)/2]$. Set

$$\mathcal{F}^s_{\mathbb{Q}} = \mathcal{F}^s \otimes \mathbb{Q} \subset H^k(X;\mathbb{Q}).$$

Theorem 6.2 (**[32]**). *The filtration $\mathcal{F}^s_{\mathbb{Q}}$ is subordinate to the refined Hodge filtration.*

The refined Hodge filtration is defined at level s to be the maximal rational subspace of

$$H^{k-s,s}(X) \oplus H^{k-s+1,s-1}(X) \oplus \cdots \oplus H^{s,k-s}(X)$$

which is a sub-Hodge-structure.

Both exterior product and cup product in $H^*(X;\mathbb{Z})$ respect the filtration $\mathcal{F}^\bullet$.

§7. Computations at level 1. Recall that for a projective variety X, there is a classically defined Picard group Pic(X) which consists of isomorphism classes of line bundles on X under tensor product. There is a short exact sequence

$$O \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

where $\mathrm{Pic}^0(X)$ is the identity component and $NS(X)$ is the **Neron-Severi group** of algebraic equivalence classes of line bundles on X.

Theorem 7.1 ([**32**]). *For any projective variety X there is a natural homotopy equivalence*

$$\mathcal{Z}^1(X) \cong Pic(X) \times \mathbb{P}^\infty.$$

If X is smooth, then:

1) $L^1H^0(X) \cong \mathbb{Z}$,
2) $\Phi : L^1H^1(X) \stackrel{\simeq}{\to} H^1(X;\mathbb{Z})$ *is an isomorphism,*
3) $L^1H^2(X) \cong NS(X)$,
4) *with respect to* 3), *the natural transformation*

$$\Phi : L^1H^2(X) \hookrightarrow H^2(X;\mathbb{Z})$$

is the first Chern class, and

5) $L^1H^k(X) = 0$ for $k > 2$.

As a consequence of 3) above we have the naturally defined **Lefschetz operators** $L : L^sH^k(X) \to L^{s+1}H^{k+2}(X)$ given by multiplication by the class of a fixed, very ample line bundle in $L^1H^2(X)$. By 4) above, this map transforms under Φ to the standard Lefschetz opertor, given by multiplication by $c_1(L)$.

Theorem 7.1 together with the inner and other products, gives the existence of many non-trivial groups $L^*H^*(X)$. For example, $L^*H^*(\mathbb{P}^n) \to H^*(\mathbb{P}^n;\mathbb{Z})$ is surjective. This is true also for abelian varieties. Moreover, the $\mathcal{F}^\bullet$ and Hodge filtrations agree for products of elliptic curves.

§8. Chern classes. Let X be a variety and denote by $\mathrm{Vect}^q_+(X)$ the equivalence classes of rank-q algebraic vector bundles which are generated by their global cross-sections. This space can be identified with π_0 of the space

$$\varinjlim_n \mathrm{Mor}\left(X, \mathcal{G}^q\left(\mathbb{P}^n\right)\right)$$

when $\mathcal{G}^q(\mathbb{P}^n)$ is the Grassmannian of codimension -q planes in $\mathbb{P}^n$. Using results discussed in III.1, one can define Chern classes for such bundles in morphic cohomology.

Theorem 8.1 ([**32**]). *For any $q > 0$ there is a natural transformation of functors*

$$Vect^q_+(X) \xrightarrow{c} \bigoplus_{s=0}^{q} L^s H^{2s}(X)$$

with the property that

$$Vect^q_+(X) \xrightarrow{\Phi \circ c} \bigoplus_{s=0}^{q} H^{2s}(X;\mathbb{Z})$$

is the standard total Chern class.

§9. An existence theorem. Using 8.1 and results of Grothendieck one can prove the following.

Theorem 9.1 ([**32**]). *Let X be a smooth projective variety. Then every class in $H^{2*}(X;\mathbb{Q})$ which is Poincaré dual to the homology class of a (rational) algebraic cycle is represented by a rational linear combination of effective algebraic cocycles.*

In other words at the level of rational cohomology there are at least as many algebraic cocycles as there are algebraic cycles.

In the next section we shall discuss an even stronger theorem, namely Poincaré duality at the level of L^*H^*.

§10. A Kronecker pairing with L_*H_*. It is shown in [**32**] that for any projective variety X there is a pairing

$$L^s H^k(X) \otimes L_p H_k(X) \xrightarrow{\kappa} \mathbb{Z}$$

whenever

$$2p \le k \le 2s,$$

which when $p = 0$ carries over, under the natural transformation Φ, to the standard Kronecker pairing $H^k(X;\mathbb{Z}) \otimes H_k(X;\mathbb{Z}) \to \mathbb{Z}$. In the next section we examine an even more striking pairing betwen these theories.

Chapter VI - Duality

It is an striking fact the two theories L_*H_* and L^*H^* whose definitions are so completely different (one in terms of cycles and other in terms of morphsims) actually admit a Poincaré duality map which carries over under the natural transformations Φ to the standard Poincaré duality map. For smooth varieties this map is an isomorphism!

§1. **Definition.** The duality map is generated in an deceptively simple fashion. Suppose X and Y are projective varieties. Then for each s, $0 \le s \le \dim_{\mathbb{C}}(Y)$, there is a natural inclusion

$$C^s(X;Y) \hookrightarrow C^s(X \times Y) \tag{1.1}$$

as the submonoid of codimension-s cycles on $X \times Y$ which are equidimensional over X. (See V.2.1). This engenders a map

$$\mathcal{Z}^s(X;Y) \longrightarrow \mathcal{Z}^s(X \times Y) \tag{1.2}$$

of group completions.

Suppose now that $Y = \mathbb{C}^N$, i.e., consider the two cases $Y = \mathbb{P}^N$ and $Y = \mathbb{P}^{N-1}$ and pass to a quotient. Then (1.2) yields a map

$$\mathcal{Z}^s\left(X;\mathbb{C}^N\right) \longrightarrow \mathcal{Z}^s\left(X \times \mathbb{C}^N\right) \cong \mathcal{Z}_{n-s}(X) \tag{1.3}$$

where $n = \dim_{\mathbb{C}}(X)$, and the homotopy equivalence on the right comes from the Algebraic Suspension Theorem : $\mathcal{Z}_p(X) \cong \mathcal{Z}_{p+1}(X \times \mathbb{C})$. (See II.1 and IV. 5). Taking π_{2s-k} in (1.3) gives a **Duality homomorphism**

$$L^sH^k(X) \xrightarrow{\widetilde{\mathcal{D}}} L_{n-s}H_{2n-k}(X)$$

which is defined in [**33**], where the following is proved.

Theorem 1.1 ([**33**]). *For any projective variety X of dimension n, the natural transformations to singular theory give a commutative diagram*

$$\begin{array}{ccc} L^sH^k(X) & \xrightarrow{\tilde{\mathcal{D}}} & L_{n-s}H_{2n-k}(X) \\ \Phi\downarrow & & \downarrow\Phi \\ H^k(X;\mathbb{Z}) & \xrightarrow{\mathcal{D}} & H_{2n-k}(X;\mathbb{Z}) \end{array}$$

where $\mathcal{D}$ is the standard Poincaré duality map (given by cap product with the fundamental class of X.)

§2. **The duality isomorphism :** $L^*H^* \cong L_{n-*}H_{2n-*}$. The considerations above lead to the following conjecture.

Conjecture 2.1 (Friedlander-Lawson). *For X and Y smooth and projective, the map (1.2) is a homotopy equivalence.*

E. Friedlander and the author verified this in several cases, including the case $s = 1$. Ofer Gabber then suggested that a general proof could be obtained from a good version of the Chow Moving Lemma for Families. Such a Moving Lemma has now been proved by Friedlander and the author [**89**]. The result has some independent interest. It holds over arbitrary infinite fields, and applies to classical questions concerning the Chow ring. More importantly here, it leads to the following result.

Theorem 2.2 ([**33**]). *Conjecture* 2.1 *is true. In particular, for any smooth projective variety X of dimension n, the duality map*

$$L^sH^k(X) \xrightarrow{\cong} L_{n-s}H_{2n-k}(X)$$

is an isomorphism for all $k \leq 2s$.

An analogous duality result holds for quasi-projective varieties. Details of this appear in [**88**].

This result has a number of non-obvious consequences. Note for example the isomorphism

$$L^sH^{2s}(X) \xrightarrow{\cong} L_{n-s}H_{2(n-s)}(X) = \mathcal{A}_{n-s}$$

which relates families of affine varieties over X to cycles modulo algebraic equivalence inside X. Note also that this gives a complete computation of morphic cohomology for a number of spaces, including all generalized flag manifolds (projective spaces, Grassmannians, etc., c.f. IV.6.1.). In particular, for such spaces we have isomorphisms

$$L^sH^k(X) \cong H^k(X;\mathbb{Z})$$

for all k, s with $2s \geq k$, and the transformations

$$\mathcal{Z}^s(X;\mathbb{C}^n) \longrightarrow \mathrm{Map}(X, \mathcal{Z}^s(\mathbb{C}^n))$$

are homotopy equivalences for all $n \geq s$.

Another consequence of duality is that it gives rise to Gysin "wrong way" maps of L^*H^* and L_*H_* for general morphisms between smooth varieties. Such maps were constructed in [**28**]. Here however the maps have additional naturality properties which have importance for applications of the theory.

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