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The Formation of Singularities in the Ricci Flow

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1 The Equation. We have many cases now where some geometrical object can be improved by evolving it with a parabolic partial differential equation. In the Ricci Flow we try to improve a Riemannian metric $g(x, y)$ by evolving it by its Ricci curvature $Rc(x, y)$ under the equation

$$\frac{\partial}{\partial t} g(X, Y) = -2Rc(X, Y).$$

In local geodesic coordinates $\{x^i\}$ at a point P where the metric is

$$ds^2 = g_{ij} dx^i dx^j$$

we find that the ordinary Laplacian of the metric is

$$\text{``}\Delta\text{''} g_{ij} = g^{pq} \frac{\partial^2}{\partial x^p \partial x^g} g_{ij} = -2Rc(X, Y)$$

so the Ricci flow is really the heat equation for a Riemannian metric

$$\frac{\partial}{\partial t} g = \text{``}\Delta\text{''} g.$$

In this paper we will survey some of the basic geometrical properties of the Ricci Flow with a view to considering what kind of singularities might form. This has proven to be a useful technique even where we want to prove convergence; sometimes if we know enough about the singularities we can see there aren't any. It is also the first step toward continuing the flow through essential singularities where the topology of the manifold may change, and hopefully simplify.

2 Exact Solutions. In order to get a feel for the equation we present some examples of specific solutions.

(a) Einstein Metrics

If the initial metric is Ricci flat, so that $Rc = 0$, then clearly the metric remains stationary. This happens, for example, on a flat torus $T^m = S^1 \times \cdots \times S^1$, or on a K^3 Kähler surface with a Calabi-Yau metric.

If the initial metric is Einstein with positive scalar curvature, the metric will shrink under the flow by a time-dependent factor. For example, on a sphere S^n_r

of radius r and dimension n, the sectional curvatures are all $1/r^2$ and the Ricci curvatures are all $(n-1)/r^2$. This gives the ordinary differential equation

$$\frac{dr}{dt} = -\frac{n-1}{r}$$

with the solution

$$r^2 = 1 - 2(n-1)t$$

which starts as a unit sphere $r = 1$ at $t = 0$ and shrinks to a point as

$$t \to T = 1/2(n-1).$$

Any Einstein metric of positive scalar curvature behaves the same way, and shrinks to a point homothetically as t approaches some finite time T, while the curvature becomes infinite like $1/(T-t)$.

By contrast, if we start with an Einstein metric of negative scalar curvature, the metric will expand homothetically for all time, and the curvature will fall back to zero like $-1/t$. For example, on a hyperbolic manifold of constant curvature $-1/r^2$ we get the ordinary differential equation

$$\frac{dr}{dt} = -\frac{n-1}{r}$$

which has the solution

$$r = 1 + 2(n-1)t,$$

with $K = -1$ at $t = 0$. Note that now the solution only goes back in time to $T = -\frac{1}{2}(n-1)$, when the metric explodes out of a single point in a big bang.

(b) Product Metrics

If we take a product metric on a product manifold $M \times N$ to start, the metric will remain a product metric under the Ricci Flow, and the metric on each factor evolves by the Ricci Flow there independently of the other factor. Thus on $S^2 \times S^1$ the S^2 shrinks to a point in a finite time while the S^1 stays fixed; hence the manifold collapses to a circle. On a product $S^2 \times S^2$ with different radii, the sphere of smaller radius collapses faster, and shrinks to a point while the other metric is still non-degenerate, and the limit manifold is S^2. If the radii start the same, they remain the same, and the whole product shrinks to a point in finite time.

(c) Quotient Metrics

If the Riemannian manifold $N = M/\Gamma$ is a quotient of a Riemannian manifold M by a group of isometries Γ at the start, it will remain so under the Ricci Flow. This is because the Ricci Flow on M preserves the isometry group. For example, a projective space $RP^n = S^n/Z_2$ of constant curvature shrinks to a point the same as its cover S^n. The S^2 bundle over S^1 where the gluing map reverses orientation can be written as a quotient $W^2 \tilde{\times} S^1 = S^2 \times S^1/Z_2$ where Z_2 flips S^2 antipodally and rotates S^1 by 180°. The product metric on $S^2 \times S^1$ induces a quotient metric on $S^2 \tilde{\times} S^1$ which evolves under the Ricci Flow to collapse to S^1.

(d) Homogeneous Metrics

Since the Ricci Flow is invariant under the full diffeomorphism group, any isometries in the initial metric will persist as isometries in each subsequent metric. A metric is homogeneous when the isometry group is preserved; hence if we start with a homogeneous metric the metric will stay homogeneous. For a given isometry group there is only a finite dimensional space of homogeneous metrics, and the Ricci Flow can be written for these metrics as a system of a finite number of ordinary differential equations. In three dimensions there are eight distinct homogeneous geometries; in [**8**] the Ricci Flow has been worked out on each. We give two examples typical of the phenomena that occur.

The Berger spheres are homogeneous metrics on S^3 which respect the Hopf fibration over S^2 with fibre S^1. Under the Ricci Flow the metrics on S^2 and on S^1 shrink to points in a finite time, but in such a way that the ratio of their radii goes to 1.

There is a torus bundle over the circle which is made with a Dehn twist in the fibre. This manifold admits a nilpotent homogeneous metric. It evolves under the Ricci Flow by stretching some ways and shrinking others, but so as to reduce the total twisting. As $t \to \infty$ the curvature falls off to zero like $1/t$.

(e) Solitons

A solution to an evolution equation which moves under a one-parameter subgroup of the symmetry group of the equation is called a soliton. The symmetry group of the Ricci Flow contains the full diffeomorphism group. A solution to the Ricci Flow which moves by a one-parameter group of diffeomorphisms is called a Ricci soliton. The equation for a metric to move by a diffeomorphism in the direction of a vector field V is that the Ricci term Rc is the Lie derivative $\mathcal{L}_V g$ of the metric g in the direction of the vector field V; thus

$$Rc = \mathcal{L}_V g \quad \text{or} \quad R_{ij} = g_{ik} D_i V^k + g_{jk} D_i V^k$$

is the Ricci soliton equation. If the vector field V is the gradient of a function f we say the soliton is a gradient Ricci soliton; thus

$$Rc = D^2 f \quad \text{or} \quad R_{ij} = D_i D_j f$$

is the gradient Ricci soliton equation. In two dimensions [**22**] the complete metric on the xy plane given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

is a gradient Ricci soliton of positive curvature with the metric flowing in along the conformal vector field

$$V = \partial/\partial r = x\partial/\partial x + y\,\partial/\partial y.$$

This metric is asymptotic to a cylinder of finite circumference 2π at ∞, while R falls off like e^{-s}. Robert Bryant [**3**] has found a complete gradient Ricci soliton metric on R^3 with positive curvature operator by solving an ordinary differential equation up to quadrature. The metric now opens like a paraboloid so that the sphere at radius s has diameter like $\sqrt{s}$, while R falls off only like $1/s$. (Presumably the same is true for $n > 3$.)

On a Kähler manifold the equation for a gradient Ricci soliton splits into two parts:

$$R_{\alpha\overline{\beta}} = D_\alpha D_{\overline{\beta}} f \quad \text{and} \quad D_\alpha D_\beta f = 0.$$

The first equation says f is a potential function for the Chern class; the second says that the gradient of f is a holomorphic vector field, so that the flow along the vector field preserves the complex structure. The gradient Ricci soliton on $R^2 = \mathbb{C}^1$ given above is a gradient Ricci-Kähler soliton in the usual complex structure, and the conformal vector field is of course holomorphic. Cao [**5**] has found similar gradient Ricci-Kähler solitons on $\mathbb{C}^n$ with positive holomorphic bisectional curvature. The sphere S^{2n-1} at radius s looks like an S^1 bundle over $\mathbb{C}P^{n-1}$ where the $\mathbb{C}P^{n-1}$ has diameter on the order of $\sqrt{s}$ while the S^1 fibre has diameter on the order of 1 (it remains finite as $s \to \infty$). He has also found a gradient Ricci-Kähler soliton on the tangent bundle TS^2 to the sphere $S^2 = \mathbb{C}P^1$ where the metrics on the $R^2 = \mathbb{C}^1$ fibres also are asymptotic to a cylinder of finite circumference. Again these are found by quadrature of an ODE.

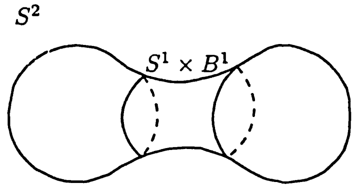
More generally, we can look for a solution to the Ricci Flow which moves by a diffeomorphism and also shrinks or expands by a factor at the same time. Such a solution is called a homothetic Ricci soliton. The equation for a homothetic Ricci soliton is

$$Rc = \rho g + \mathcal{L}vg \quad \text{or} \quad R_{ij} = \rho g_{ij} + g_{ik} D_j V^k + g_{jk} D_i V^k$$

where ρ is the homothetic constant. For $\rho > 0$ the soliton is shrinking, for $\rho < 0$ it is expanding. The case $\rho = 0$ is a steady soliton discussed before; the case $V = 0$ is an Einstein metric discussed before. We only have a few examples, but there should be more. Koiso [**33**] has found a shrinking gradient Ricci-Kähler soliton on a compact Kähler surface. If we enlarge the category of solutions from manifolds to orbitfolds, we can find shrinking gradient Ricci-Kähler solitons on the teardrop and football surface orbitfolds (see [**22**] and [**45**]), which are quotients of S^3 by and S^1 action with one or two exceptional orbits.

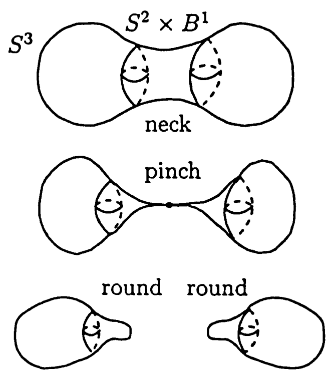
3 Intuitive Solutions. It is always good to keep in mind what we expect, as well as what we know (provided we keep the distinction clear). In this section we will show the sort of behavior which is likely for the Ricci Flow in some general settings where exact solutions are not available, based on drawing pictures, using computer models, and making analogies with other equations (particularly the Mean Curvature Flow). Beware that these results here are conjectures, not theorems.

First consider a metric on the two-sphere S^2 shaped like a dumbell. (We draw it in R^3, but the Ricci Flow is for the intrinsic metric and has no relation to the embedding.)

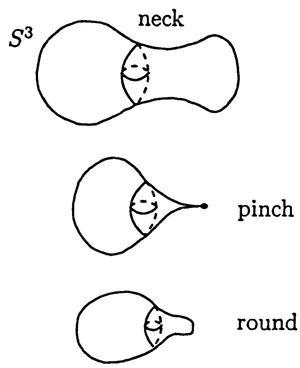


At the ends of the dumbell the curvature is positive and the metric will contract, while in the neck in the middle, which looks like $S^1 \times B^1$ and has slightly negative curvature, the metric will expand slightly. Thus we expect the sphere S^2 to round itself out. (Note that in the Mean Curvature Flow the neck would shrink because S^1 has extrinsic curvature, but in the Ricci Flow it doesn't because S^1 has no intrinsic curvature.)

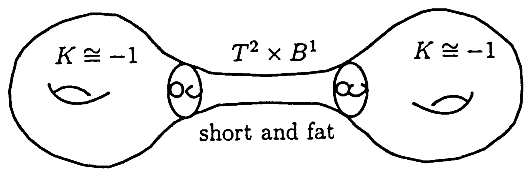
By contrast, if we take a dumbell metric on S^3 with a neck like $S^2 \times B^1$, we expect the neck will shrink because the positive curvature in the S^2 direction will dominate the slightly negative curvature in the B^1 direction. In some finite time we expect the neck will pinch off. There may be a weak solution extending past the pinching moment when the sphere splits into two spheres. (Weak solutions are known to exist for the Mean Curvature Flow, but have not even been defined for the Ricci Flow.) The movie would look like this.

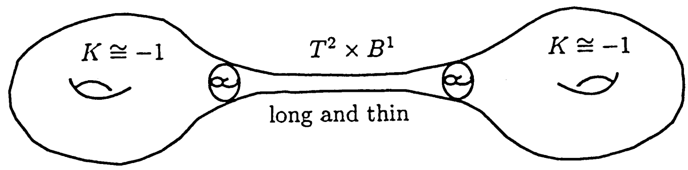


The picture above is symmetric; we could however pinch off a little sphere from a big one. If we let the size of the little sphere go to zero, we expect to get a degenerate singularity where there is nothing on the other side. The movie would now look like this.



We could also imagine a three-manifold with a toroidal neck $T^2 \times B^1$ formed by joining two complete hyperbolic manifolds of finite volume where each has a single toroidal end. Since T^2 has no intrinsic curvature the neck is flat or has slightly negative curvature and should expand slowly, while each hyperbolic piece should expand more rapidly. The solution should exist as $t \to \infty$ with the negative curvature falling back to zero like $-1/t$. Thus no collapse should happen, unless we rescale the solution to see the geometry better. If we rescale to keep the volume constant and the curvature about -1 in each hyperbolic piece, then the toroidal neck should become very long and thin as in this movie.





We can summarize these observations with the remark that a neck $N^p \times B^q$ in a manifold M^m (with $m = p + q$) will only pinch if the B^p has some positive intrinsic curvature to shrink it. Thus in two dimensions we can do surgery

$$S^2 \times B^1 \to B^3 \times S^0$$

because S^2 has intrinsic curvature, but not the surgery

$$S^1 \times B^2 \to B^2 \times S^1$$

because S^1 is intrinsically flat. When surgeries only occur in one direction the topology of the manifold must get simpler each time.

We can ask about similar neck pinches in higher dimensions. In dimension 4 we expect the Ricci Flow could perform surgeries

$$S^3 \times B^1 \to B^4 \times S^0 \quad \text{and} \quad S^2 \times B^2 \to B^3 \times S^1$$

but not the reverse; this gives hope the Ricci Flow may provide topological information on 4-manifolds also. But already in dimension 5 we expect the Ricci Flow to perform the surgery

$$S^2 \times B^3 \to B^3 \times S^2$$

which is its own inverse; this destroys any hope of getting purely topological results. Now it is exceedingly fortunate that this is just the dimension where the h-cobordism theory kicks in, so the Ricci Flow can only work where the topology doesn't!

4 Evolution of Curvature. Whenever a Riemannian metric evolves so does its curvature. It is best to study the evolution of the representative of the tensor in an orthonormal frame F. Since the metric evolves, we must evolve the frame also to keep it orthonormal. If the frame F consisting of an orthonormal basis of vectors

$$F = \{F_1, \ldots, F_a, \ldots, F_n\}$$

given in local coordinates by

$$F_a = F_a^i \frac{\partial}{\partial x^i}$$

evolves by the formula

$$\frac{\partial}{\partial t} F_a^i = g^{ij} R_{jk} F_a^k$$

it will remain orthonormal in the Ricci Flow. We will use indices $a, b, \ldots$ on a tensor to denote its components in an evolving frame, and D_t to denote the change of the components with respect to time in the evolving frame. The Riemannian curvature tensor has components

$$Rm = \{R_{abcd}\} \quad \text{where} \quad R_{abcd} = R_{ijkl} F_a^i F_b^j F_c^k F_d^l$$

in a frame which evolve by the formula ([**24**])

$$D_t R_{abcd} = \Delta R_{abcd} + 2(B_{abcd} + B_{acbd} - B_{abdc} - B_{adbc})$$

where

$$B_{abcd} = R_{aebf} R_{cedf}.$$

This is a diffusion-reaction equation. The problem of singularity formation is related to the competition between the diffusion, which tries to spread the curvature evenly over the manifold, and the reaction, which concentrates the curvature causing it to blow up in finite time.

We can understand the geometry of this equation better if we think of the curvature tensor R_{abcd} as a symmetric bilinear form on the two-forms Λ^2 given by the formula

$$Rm(\varphi, \phi) = R_{abcd}\varphi_{ab}\psi_{cd}.$$

A form in Λ^2 can be regarded as an element of the Lie algebra so(n), in which case it is an infinitesimal rotation; or as an infinitesimal loop, in which case it is a sum of primitive two-forms each of which is a little loop in a place where enclosed area is the coefficient of the primitive two-form, and the sum of the primitive two-form is the composition of the loops, modulo an obvious equivalence. Then the curvature tensor is the infinitesimal generator of the local holonomy group; going around an infinitesimal loop represented by $\varphi \in \Lambda^2$ gives rise under parallel translation to an infinitesimal rotation $Rm(\varphi) \in \Lambda^2$ where

$$Rm(\varphi, \phi) = \langle Rm(\varphi), \phi \rangle$$

turns the bilinear form into a symmetric operator. In order to treat the curvature tensor as a bilinear form on Λ^2, we choose an orthonormal basis

$$\Phi = \left\{ \varphi^1, \ldots, \varphi^\alpha, \ldots, \varphi^{n(n-1)/2} \right\}$$

where in the frame F we have

$$\varphi^\alpha (F_a, F_b) = \varphi^\alpha_{ab}$$

and write the matrix $M_{\alpha\beta}$ of the curvature operator in this basis, so that

$$R_{abcd} = M_{\alpha\beta}\varphi^\alpha_{ab}\varphi^\beta_{cd}.$$

Let $c_{\alpha\beta\gamma}$ be the structure constants of the Lie algebra $so(n)$ in this basis, so that

$$c_{\alpha\beta\gamma} = \langle \left[\varphi^\alpha, \varphi^\beta \right], \varphi^\gamma \rangle.$$

Then the evolution of the curvature operator M is given by

$$D_t M_{\alpha\beta} = \Delta M_{\alpha\beta} + M^2_{\alpha\beta} + M^\#_{\alpha\beta}$$

where $M^2_{\alpha\beta}$ is the operator square

$$M^2_{\alpha\beta} = M_{\alpha\gamma}M_{\beta\gamma}$$

and $M^\#_{\alpha\beta}$ is the Lie algebra square

$$M^\#_{\alpha\beta} = c_{\alpha\gamma\zeta}c_{\beta\delta\eta}M_{\gamma\delta}M_{\zeta\eta}.$$

As an example, on a surface the curvature is all given by the scalar curvature R, which evolves by

$$\frac{\partial R}{\partial t} = \Delta R + R^2.$$

On a three-manifold the sectional curvatures are given by a 3×3 matrix $M_{\alpha\beta}$. Since the Lie structure constants are always given by $c_{123} = 1$, the matrix $M^{\#}$ is just the adjoint matrix of determinants of 2×2 cofactors.

We can now use this representation of the curvature to derive the following result.

THEOREM 4.1. *If the initial metric has its local holonomy group restricted to a subgroup G of $SO(n)$, it remains so under the Ricci Flow.*

Proof. We refer the reader to [**21**] for details. The idea is that the local holonomy is restricted to G if and only if the image of the curvature operator M at each point is restricted to the Lie algebra $\mathcal{G}$ of G. In this case since M is self-adjoint, the orthogonal complement $\mathcal{G}^{\perp}$ is contained in the kernel of M. Then the same properties hold for both M^2 and $M^{\#}$, and hence are preserved by the Ricci Flow by the maximum principle. □

As an example, the local holonomy of a Riemannian manifold of even dimension reduces from $SO(2n)$ to $U(n)$ if and only if there is a complex structure with respect to which the metric is Kähler. Also, the local holonomy reduces from $SO(n)$ to $SO(p) \times SO(q)$ with $p + q = n$ if and only if the metric is locally a product. (It need not be a product globally, as we see from $S^2 \tilde{\times} S^1$.)

5 Preserving Curvature Conditions. A number of curvature pinching inequalities, mostly representing some form of positive curvature, are preserved by the Ricci Flow. It always happens that if we start with a metric satisfying a weak inequality, either for all $t > 0$ it immediately becomes a strict inequality or else the curvature is restricted everywhere; this is a consequence of the strong maximum principle. (The reader will find the details in [**21**].)

The proof that a weak inequality is preserved is always by the maximum principle, usually for a system. If a tensor F evolves by a diffusion-reaction equation

$$\frac{\partial F}{\partial t} = \Delta F + \Phi(F)$$

and if Z is a closed subset of the tensor bundle which is invariant under parallel translation and such that its intersection with each fibre is convex, and if Z is preserved by the system of ordinary differential equations in each fibre given by the reaction.

$$\frac{dF}{dt} = \Phi(F)$$

then Z is also preserved by the diffusion-reaction equation, in the sense that if the tensor lies in Z at each point at the start, then it continues to lie in Z subsequently. For preserving curvature inequalities in the Ricci Flow we take Z to be a subset of the bundle of curvature operators M which is convex in each fibre, and check that Z is preserved by the curvature reaction

$$\frac{dM}{dt} = M^2 + M^\#$$

(a) Positive Scalar Curvature

The Ricci Flow preserves positive scalar curvature $R \geq 0$ on a manifold in any dimension. This follows from the evolution of the scalar curvature

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2$$

and the observation that $|Rc|^2 \geq 0$. Note that the scalar curvature immediately becomes strictly positive $R > 0$ everywhere unless the manifold is Ricci flat everywhere.

(b) Negative Scalar Curvature on a Surface

In dimensions $n > 2$ negative scalar curvature is not preserved; however on a surface $n = 2$ it is, since

$$Rc(X,Y) = \frac{1}{2} Rg(X,Y)$$

gives

$$\frac{\partial R}{\partial t} = \Delta R + R^2.$$

In this case the scalar curvature immediately becomes strictly negative unless $R \equiv 0$ and the metric is flat. This is the only case we know where negative curvature is preserved by the Ricci Flow.

(c) Positive Sectional Curvature on a Three-Manifold

In dimension $n = 3$ (but no higher) positive sectional curvature is preserved. Indeed since every two-form is primitive in this dimension, positive sectional curvature is the same as positive curvature operator. In an orthonormal frame where M is diagonal

$$M = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}$$

the square M^2 and the adjoint $M^\#$ are also both diagonal

$$M^2 = \begin{pmatrix} \alpha^2 & & \\ & \beta^2 & \\ & & \gamma^2 \end{pmatrix} \quad \text{and} \quad M^\# = \begin{pmatrix} \beta\gamma & & \\ & \alpha\gamma & \\ & & \alpha\beta \end{pmatrix}$$

and the reaction equation for M (in the space of 3×3 matrices) descends to the reaction on the diagonal terms $(\alpha, \beta, \gamma) \in R^3$ given by

$$\begin{cases} \dfrac{d\alpha}{dt} = \alpha^2 + \beta\gamma \\ \dfrac{d\beta}{dt} = \beta^2 + \alpha\gamma \\ \dfrac{d\gamma}{dt} = \gamma^2 + \alpha\beta \end{cases} .$$

Clearly the set of positive matrices $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$ is preserved by this reaction. If the sectional curvature starts weakly positive, it immediately becomes strictly positive unless the manifold is flat, or locally a product of a surface of positive curvature with a line.

(d) Positive Ricci Curvature on a Three-Manifold

In dimension $n = 3$ positive Ricci curvature is equivalent to 2-positive curvature operator; in terms of the eigenvalues α, β, γ of the curvature operator this gives the inequalities

$$\alpha + \beta \geq 0, \quad \alpha + \gamma \geq 0, \quad \beta + \gamma \geq 0$$

which are clearly preserved by the reaction. Again if the Ricci curvature starts weakly positive, it immediately becomes strictly positive unless the manifold is flat, or locally a product of a surface of positive curvature with a line.

(e) Positive Curvature Operator

In every dimension positive curvature operator $M \geq 0$ is preserved by the Ricci Flow. To see this we must check the reaction

$$\frac{dM}{dt} = M^2 + M^{\#}.$$

Choose an orthonormal frame where $M_{\alpha\beta}$ is diagonal with

$$M_{\alpha\alpha} = \lambda_\alpha \quad \text{and} \quad M_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta$$

with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n(n-1)/2}$. Now λ_1 is Lipschitz-continuous as a function of M, but may not be differentiable; however we have an inequality

$$\frac{d\lambda_1}{dt} \geq \frac{d}{dt} M_{11}$$

in the sense of the lim sup of forward difference quotients (as explained in [**21**]). Now

$$\frac{d}{dt} M_{11} = M_{11}^2 + M_{11}^{\#} = \lambda_1^2 + \sum_{\beta\gamma} c_{1\beta\gamma}^2 \lambda_\beta \lambda_\gamma$$

so if $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ then $d\lambda_1/dt \geq 0$ and the result is true.

If the curvature operator is weakly positive to start, it becomes strictly positive immediately unless the holonomy group reduces to a proper subgroup (again the details are in [**21**]).

(f) Two-Positive Curvature Operator

A symmetric bilinear form is called 2-positive if the sum of its two smallest eigenvalues is positive. Chen ([**12**]) has observed that two-positive curvature operator is also preserved by the Ricci Flow. To see this, we must show the reaction preserves

$$\lambda_1 + \lambda_2 \geq 0.$$

Now as before

$$\frac{d}{dt}(\lambda_1 + \lambda_2) \geq \frac{d}{dt}(M_{11} + M_{22})$$

and

$$\frac{d}{dt}(M_{11} + M_{22}) = \lambda_1^2 + \lambda_2^2 + \sum_{pq}(c_{1pq}^2 + c_{2pq}^2)\lambda_p\lambda_q.$$

Now we do not know if λ_1 is positive, but surely $\lambda_2, \ldots, \lambda_n$ are. Hence we only need to worry about terms $\lambda_p\lambda_q$ with p or q equal to 1, and then $c_{1pq} = 0$ so we only have to worry about the terms

$$c_{21q}^2\lambda_1\lambda_q$$

where $p = 1$ (or actually twice this because we could switch p and q). Then $q \geq 3$ and we also have a positive term when $p = 1$ coming from c_{1pq} of the form

$$c_{12q}^2\lambda_2\lambda_q.$$

Recall that for the Lie structure constants $c_{21q} = -c_{12q}$. Grouping these we get

$$\frac{d}{dt}(M_{11} + M_{22}) = \lambda_1^2 + \lambda_2^2 + 2\sum_{q\geq 3} c_{12q}^2(\lambda_1 + \lambda_2)\lambda_q + \sum_{p,q\geq 3}(c_{1pq}^2 + c_{2pq}^2)\lambda_p\lambda_q$$

and since $\lambda_1 + \lambda_2 \geq 0$ we see $d(\lambda_1 + \lambda_2)/dt \geq 0$ which proves the result.

(g) Positive Holomorphic Bisectional Curvature

A Kähler metric has positive holomorphic bisectional curvature if

$$R(Z, \overline{Z}, \overline{W}, W) \geq 0$$

for all complex vectors Z and W. Mok [**38**] has shown that positive holomorphic bisectional curvature is preserved by the Ricci Flow. To check this result it is only necessary to check that

$$\frac{d}{dt}R(Z, \overline{Z}, \overline{W}, W) \geq 0$$

when $R(Z, \overline{Z}, \overline{W}, W) = 0$. Now for all vectors U and V

$$R(Z + U, \overline{Z} + \overline{U}, \overline{W} + \overline{V}, W + V) \geq 0$$

and it follows that the part quadratic in U and V

$$R(Z, \overline{Z}, \overline{V}, V) + R(Z, \overline{U}, \overline{W}, V) + R(Z, \overline{U}, \overline{V}, W) + R(U, \overline{Z}, \overline{W}, V)$$
$$+ R(U, \overline{Z}, \overline{V}, W) + R(U, \overline{V}, \overline{W}, W) \geq 0$$

for all U and V. Replace U by iU and V by $-iV$ and average; then

$$R(Z,\overline{Z},\overline{V},V)+R(Z,\overline{U},\overline{V},W)+R(U,\overline{Z},\overline{W},V)+R(U,\overline{V},\overline{W},W)\geq 0$$

for all U and V. Let us write

$$\begin{aligned} L(X,\overline{Y}) &= R(X,\overline{Y},\overline{Z},Z)\\ M(X,Y) &= R(X,\overline{Z},\overline{W},Y)\\ N(X,\overline{Y}) &= R(X,\overline{Y},\overline{W},W). \end{aligned}$$

Note that $L={}^t\,\overline{L}$ and $N={}^t\,\overline{N}$ are Hermitian. Then the above says

$$L(V,\overline{V})+M(U,V)+\overline{M}(\overline{U},\overline{V})+N(U,\overline{U})\geq 0$$

or in matrix form

$$\begin{pmatrix} L & \overline{M}\\ {}^tM & \overline{N}\end{pmatrix}\geq 0$$

as we see by applying the matrix to the vector of V and $\overline{V}$.

Conjugate the above matrix by

$$\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}$$

to see that we also have

$$\begin{pmatrix} \overline{N} & -{}^tM\\ -\overline{M} & L\end{pmatrix}\leq 0$$

and taking conjugates

$$\begin{pmatrix} N & -{}^t\overline{M}\\ -M & \overline{L}\end{pmatrix}\geq 0.$$

Now if two Hermitian matrices are positive, the trace of their product is also positive; so

$$tr\begin{pmatrix} L & \overline{M}\\ {}^tM & \overline{N}\end{pmatrix}\begin{pmatrix} N & -{}^t\overline{M}\\ -M & \overline{L}\end{pmatrix}\geq 0$$

and the trace has two equal parts because the trace of a matrix equals the trace of its transpose, making

$$tr(LN-\overline{M}M)\geq 0.$$

This makes

$$L(U,\overline{V})N(V,\overline{U})-\overline{M}(U,\overline{V})M(V,\overline{U})\geq 0$$

where we adopt the summation convention that whenever a complex vector and its conjugate appear together in an expression we sum over the vector in a Hermitian basis.

Writing this in terms of the curvature tensor gives

$$R(U,\overline{V},\overline{Z},Z)R(V,\overline{U},\overline{W},W)-R(Z,\overline{U},\overline{V},W)R(V,\overline{Z},\overline{W},U)\geq 0.$$

We also have

$$R(U,\overline{V},\overline{W},Z)R(V,\overline{U},\overline{Z},W) \geq 0$$

since it is a sum of products of numbers with their conjugates. Now the reaction equation for the curvature tensor in the Kähler case simplifies using the Kähler identities to

$$\frac{d}{dt}R(Z,\overline{Z},\overline{W},W) = 2[R(U,\overline{V},\overline{Z},Z)R(V,\overline{U},\overline{W},W) - R(Z,\overline{U},\overline{V},W)R(V,\overline{Z},\overline{W},U) + R(U,\overline{V},\overline{W},Z)R(V,\overline{U},\overline{Z},W)]$$

and we have seen this is a sum of two positive terms. This completes the proof.

6 Short-Tine Existence and Uniqueness. Short-time existence for solutions to the Ricci Flow on a compact manifold was first shown in [**20**] using the Nash-Moser Theorem. This sophisticated machinery was employed because the Ricci Flow itself is only weakly parabolic, since it is invariant under the whole diffeomorphism group. Shortly thereafter De Turck [**16**] showed that by modifying the flow by a reparametrization using a fixed background metric to break the symmetry the equation could be replaced by an equivalent one which is strictly parabolic, and the classical inverse function theorem suffices. Here we present a version of De Turck's Trick by combining the Ricci Flow with the Harmonic Map Flow.

Eells and Sampson [**17**] evolve a map $F : M \to N$ from a Riemannian manifold M of dimension m with coordinates $\{x^i\}$, $1 \leq i \leq m$, and metric g_{ij} to a Riemannian manifold N of dimension n with coordinates $\{y^\alpha\}, 1 \leq \alpha \leq n$, and metric $h_{\alpha\beta}$ by the formula

$$\frac{\partial F}{\partial t} = \Delta F$$

where ΔF is the harmonic map Laplacian, defined as follows. The tangent bundle TM of M has the Levi-Civita connection Γ^k_{ij} of the metric g_{ij}, the tangent bundle TN of N has the Levi-Civita connection $\Delta^\gamma_{\alpha\beta}$ of the metric $h_{\alpha\beta}$ and the pull-back bundle F^*TN of TN by F is a bundle over M with the pull-back connection

$$F^*\Delta^\alpha_{\beta\ell} = \Delta^\alpha_{\beta\gamma} \frac{\partial y^\gamma}{\partial x^\ell}.$$

The derivative DF given locally by

$$D_i F^\alpha = \frac{\partial y^\alpha}{\partial x^i}$$

is a section of the bundle $L(TM, F^*TN)$ of linear maps of TM into F^*TN. The second derivative D^2F is the covariant derivative of DF using the induced connection in the bundle $L(TM, F^*TN)$ coming from the connections on M and F^*TN, and is given locally by

$$D^2_{ij}F^\alpha = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial y^\alpha}{\partial x^k} + F^*\Delta^\alpha_{\beta\gamma} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j}.$$

The harmonic map Laplacian ΔF is the trace if D^2F; locally

$$\Delta F^\alpha = g^{ij} D^2_{ij} F^\alpha.$$

The Harmonic Map Flow is strictly parabolic, so solutions with any initial data exist for a short time. When the target manifold N has weakly negative sectional curvature Eells and Sampon prove the solution exists for all time and converges to a harmonic map, one with $\Delta F = 0$, homotopic to the initial map.

Now we want to combine the Ricci Flow on M with the Harmonic Map Flow for the map F from M to N, keeping the metric on the target N fixed. This gives the system of equations

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2Rc_g \\ \\ \dfrac{\partial}{\partial t} F = \Delta_{g,h} F \end{cases}$$

where we write Rc_g to denote the Ricci curvature of g, and $\Delta_{g,h}$ to denote the Laplacian using the metrics g on M and h on N. The first equation is independent of the second. There is at least one interesting advantage; now if we look at the evolution of the energy density

$$e = g^{ij} h_{\alpha\beta} D_i F^\alpha D_j F^\beta$$

we find that the usual term involving the Ricci curvature of g is cancelled by the Ricci Flow, and we just get

$$\frac{\partial e}{\partial t} = \Delta e - 2|D^2F|^2 + 2Rm(DF, DF, DF, DF)$$

where

$$|D^2F|^2 = g^{ik} g^{j\ell} h_{\alpha\beta} D^2_{ij} F^\alpha D^2_{k\ell} F^\beta$$

and

$$Rm(DF, DF, DF, DF) = g^{ik} g^{j\ell} R_{\alpha\beta\gamma\delta} D_i F^\alpha D_j F^\beta D_k F^\gamma D_\ell F^\delta.$$

Consequently if N has weakly negative sectional curvature the maximum of the energy density e is weakly monotone decreasing in time regardless of the sign of the Ricci curvature on M. (In the classical case where the metric on M is fixed we would need weakly positive Ricci curvature on M for this to hold.)

Consider now the case where M and N have the same dimension and the initial map F is a diffeomorphism. Then F will stay a diffeomorphism at least for a short time. Let us write $\hat{g} = F_* g$ for the push-forward of the metric g from M to N; then locally $\hat{g} = \{\hat{g}_{\alpha\beta}\}$ where

$$g_{ij} = \hat{g}_{\alpha\beta} \, \frac{\partial y^\alpha}{\partial x^i} \, \frac{\partial y^\beta}{\partial x^j}.$$

We can now ask how $\hat{g}$ evolves under the dual Ricci-Harmonic Map Flow. The answer is

$$\frac{\partial}{\partial t} \, \hat{g} = \mathcal{L}_V \hat{g} - 2\widehat{Rc}$$

where V is the vector field $V = \widehat{tr}(\widehat{\Gamma} - \Delta)$

$$V^\gamma = \hat{g}^{\alpha\beta}\left(\widehat{\Gamma}^\gamma_{\alpha\beta} - \Delta^\gamma_{\alpha\beta}\right)$$

formed by training the tensor which is the difference between the Levi-Civita connections $\widehat{\Gamma}$ of $\hat{g}$ and Δ of h with the inverse $\hat{g}^{\alpha\beta}$ of the metric $\hat{g}_{\alpha\beta}$, $\mathcal{L}_V\hat{g}$ is the Lie derivative of $\hat{g}$ in the direction V and where $\widehat{Rc}$ is the Ricci tensor of $\hat{g}$. In local coordinates

$$\frac{\partial}{\partial t}\,\hat{g}_{\alpha\beta} = \hat{g}_{\alpha\gamma}\widehat{D}_\beta V^\gamma + \hat{g}_{\beta\gamma}\widehat{D}_\alpha V^\gamma - 2\widehat{R}_{\alpha\beta}$$

where $\widehat{D}$ is the Levi-Civita connection of $\hat{g}$. Note this flow now only talks of the manifold N, not M, and only uses the metric $\hat{g}$, not g, and the background connection Δ, which need not have come from the Levi-Civita connection of a metric h. However since it does use Δ, it is no longer invariant under the diffeomorphism group.

Now a straightforward computation in local coordinates shows we can write the Ricci-De Turck Flow as

$$\frac{\partial}{\partial t}\,\hat{g} = tr\ \widehat{D}D\hat{g}$$

where $\widehat{D}$ is the covariant derivative using the connection $\widehat{\Gamma}$ of $\hat{g}$ and D is the covariant derivative in the background connection Δ and $\widehat{tr}$ is the trace using the metric $\hat{g}$; locally

$$\frac{\partial}{\partial t}\,\hat{g}_{\alpha\beta} = \hat{g}^{\gamma\delta}\widehat{D}_\gamma D_\delta \hat{g}_{\alpha\beta}.$$

This is a quasilinear equation because D is independent of $\hat{g}$ and $\widehat{D}$ only involves first derivatives of $\hat{g}$. Its symbol $\sigma(\xi)$ in the direction of a covector ξ is

$$\sigma(\xi) = \hat{g}^{\alpha\beta}\xi_\alpha\xi_\beta \cdot I$$

where I is the identity on tensors $\hat{g}$. It follows that the Ricci-De Turck flow is strictly parabolic. If the initial metric is smooth, then there exists a unique smooth solution for at least a short time.

We can recover the solution g for the original Ricci Flow on M from the solution $\hat{g}$ for the Ricci-De Turck flow on N as follows. The vector field V on N pulls back to the vector field of motion $\partial F/\partial t$ in F^*TN; thus

$$\partial F/\partial t = V \circ F$$

or locally

$$\frac{\partial y^\alpha}{\partial t} = V^\alpha(y,t).$$

Now once we know V^α on N, this is just a system of ordinary differential equations on the domain M. Hence there is no problem with the existence of a solution. If the initial metric for g is C^∞ smooth, the initial metric for $\hat{g}$

will be C^∞ smooth; then the solution for $\hat{g}$ will be smooth, and the map F constructed by solving the ODE system will be smooth. We can then recover g as the pull-back $g = F^*\hat{g}$; locally

$$g_{ij} = \hat{g}_{\alpha\beta}\,\frac{\partial y^\alpha}{\partial x^i}\,\frac{\partial y^\beta}{\partial x^j}$$

and g will be a smooth solution of the Ricci Flow as desired.

Now we claim the solution with given smooth initial conditions on a compact manifold is unique. For suppose g_1 and g_2 are two solutions which agree at $t = 0$. We can solve the Ricci-Harmonic Map Flow for maps F_1 and F_2 with the metrics g_1 and g_2 on M into the same target N with the same fixed h, and starting at the same map, as there is no problem with existence for the Harmonic Map Flow even with a time-varying metric on M as long as this metric is known. This gives two solutions $\hat{g}_1$ and $\hat{g}_2$ to the Ricci-De Turck flow on N with the same initial metric. By the standard uniqueness result for strictly parabolic equations $\hat{g}_1 = \hat{g}_2$. Then the corresponding vector fields $V_1 = V_2$. Thus the two ODE systems

$$\frac{\partial F_1}{\partial t} = V_1 \circ F_1 \quad \text{and} \quad \frac{\partial F_2}{\partial t} = V_2 \circ F_2$$

with the same initial values must have the same solutions, and hence the induced metrics

$$g_1 = F_1^*\hat{g}_1 \quad \text{and} \quad g_2 = F_2^*\hat{g}_2$$

must agree also.

There is one case where it would clearly be desirable to have weak solutions to the Ricci Flow. It is possible to construct geometrically metrics on triangulated manifolds which are constant on each simplex, continuous on the interfaces, and satisfy certain curvature conditions in terms of angle defects. It would be nice to smooth out these metrics to smooth metrics by running the Ricci Flow for a short time, and convert the angle defect curvature condition to some Riemannian curvature condition. Of course the initial curvature is now concentrated as δ-functions on subvarieties and zero almost everywhere, so the initial curvature is not even in L^p for any p.

7 Derivative Estimates. Whenever we have a bound on the curvature, the smoothing property of the parabolic equation gives us a bound on the derivatives of the curvature at any time $t > 0$.

THEOREM 7.1. *There exist constants C_k for $R \geq 1$ such that if the curvature is bounded*

$$|Rm| \leq M$$

up to time t with $0 < t \leq 1/M$ then the covariant derivative of the curvature is bounded

$$|DRm| \leq C_1 M/t^{1/2}$$

and the k^{th} covariant derivative of the curvature is bounded

$$|D^k Rm| \leq C_k M/T^{k/2}.$$

Proof. We need to apply the maximum principle to the right quantity. We denote by $A * B$ any tensor product of two tensors A and B when we do not need the precise expression. We have a formula

$$D_t Rm = \Delta Rm + Rm * Rm$$

which gives a formula

$$D_t|Rm|^2 \leq \Delta|Rm|^2 - 2|DRm|^2 + C|Rm|^3$$

for some constant C. Taking the covariant space derivative of the first formula yields

$$D_t DRm = \Delta DRm + Rm * DRm$$

which leads to a formula

$$D_t|DRm|^2 \leq \Delta|DRm|^2 - 2|D^2Rm|^2 + C|Rm||DRm|^2.$$

Now let F be the function

$$F = t|DRm|^2 + A|Rm|^2$$

where A is a constant we shall choose in a minute. Then discarding $|D^2Rm|^2 \geq 0$ we find that

$$D_t F \leq \Delta F + (Ct|Rm| - 2A)|DRm|^2 + CA|Rm|^3$$

for some constant C. Now we assume $|Rm| \leq M$ and $tM \leq 1$; then if we take $A \geq C$ we get

$$D_t F \leq \Delta F + CM^3$$

for some constant C. Also

$$F \leq CM^2$$

at $t = 0$ (since $t|DRm|^2 = 0!$) and hence by the maximum principle

$$F \leq CM^2 + CtM^3.$$

Now as long as $tM \leq 1$ this gives $F \leq CM^2$ for some constant C, and

$$t|DRm|^2 \leq F \leq CM^2$$

yields

$$|DRm| \leq C_1 M/t^{1/2}$$

for some constant C_1.

The general case follows in the same way. Differentiating k times gives a formula

$$D_t D^k Rm = \Delta D^k Rm + \sum D^{j_1} Rm * D^{j_2} Rm * \cdots * D^{j_p} Rm$$

where the sum extends over $p \geq 2$ with

$$0 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq k$$

with

$$j_1 + j_2 + \cdots + j_p = k + 4 - 2p.$$

If we have bounds

$$|D^k Rm| \leq C_k M / t^{k/2}$$

we get an estimate (using $tM \leq 1$)

$$D_t |D^k Rm|^2 \leq \Delta |D^k Rm|^2 - 2|D^{k-1} Rm|^2 + CM^3/t^k$$

and another estimate (using $tM \leq 1$)

$$\begin{aligned} D_t |D^{k+1} Rm|^2 \leq \Delta |D^{k+1} Rm|^2 &- 2|D^{k+2} Rm|^2 + CM|D^{k+1} Rm|^2 \\ &+ CM^2 |D^{k+1} Rm| / t^{(k+1)/2} + CM^3/t^{k+1}. \end{aligned}$$

Now we can bound

$$M^2 |D^{k+1} Rm| / t^{(k+1)/2} \leq M|D^{k+1} Rm|^2 + CM^3/t^{k+1}$$

and discard $|D^{k+2} Rm|^2 \geq 0$ to get

$$D_t |D^{k+1} Rm|^2 \leq \Delta |D^{k+1} Rm|^2 + CM|D^{k+1} Rm|^2 + CM^3/t^{k+1}.$$

Now we let

$$F_k = t|D^{k+1} Rm|^2 + A_k |D^k Rm|^2$$

where A_k is a constant we shall choose soon. Then

$$D_t F_k \leq \Delta F_k + (CtM - 2A_k)|D^{k+1} Rm|^2 + CA_k M^3/t^k$$

and if we take $tM \leq 1$ and $A_k \geq C$ then

$$D_t F_k \leq \Delta F_k + CM/t^k$$

for some constant C. Also at $t = 0$

$$F_k \leq CM^2/t^k$$

so by the maximum principle for $t \geq 0$

$$F_k \leq CM^2/t^k + CM^3/t^{k-1}.$$

Now since $tM \leq 1$ we just get $F_k \leq CM^2/t^k$, and then

$$t|D^{k+1}Rm|^2 \leq F_k \leq CM^2/t^k$$

gives

$$|D^{k+1}Rm| \leq C_R M/t^{(k+1)/2}$$

which is the induction step we need. This completes the proof of the Theorem. □

COROLLARY 7.2. *There exist constants $C_{j,k}$ such that if the curvature is bounded $|Rm| \leq M$ then the space-time derivatives are bounded*

$$|D_t^j D^k Rm| \leq C_{j,k} M/t^{j+(k/2)}.$$

Proof. We can express $D_t Rm$ in terms of $\Delta Rm = trD^2 Rm$ and $Rm * Rm$. Likewise we can differentiate this equation to express any space-time derivative $D_t^j D^k Rm$ just in terms of space derivatives, and recover the bound above. □

8 Long Time Existence. We now get the following result on the maximal existence time for a solution.

THEOREM 8.1. *For any smooth initial metric on a compact manifold there exists a maximal time T on which there is a unique smooth solution to the Ricci flow for $0 \leq t < T$. Either $T = \infty$ or else the curvature is unbounded as $t \to T$.*

Proof. Any two smooth solutions agree, so there is a unique smooth solution on a maximal time interval $0 \leq t < T$ for $T \leq \infty$. Suppose $T < \infty$ and $|Rm|$ remains bounded as $t \to T$. Then so do all the space-time derivatives $D_t D^k Rm$. We claim that the metric g and all its derivatives, i.e. ordinary derivatives in a local coordinate chart, also remain bounded, and g remains bounded away from zero below. Then the metric g_t at time t converges to a smooth limit metric g_T as $t \to T$. Once we know this, we can continue the solution past T, and so T was not maximal after all.

To see that g remains bounded above and below, consider the evolution of the length of a vector from

$$|V|^2 = g(V, V).$$

By the equation

$$\frac{\partial}{\partial t} g(V, V) = -2Rc(V, V)$$

and if $|Rm| \leq M$ then

$$|Rc(V,V)| \leq CMg(V,V)$$

for some constant C depending on the dimension only. Thus

$$\left|\frac{\partial}{\partial t}g(V,V)\right| \leq CMg(V,V)$$

and it follows that

$$e^{-CMt}g_0(V,V) \leq g_t(V,V) \leq e^{CMt}g_0(V,V)$$

and the metrics g_t are uniformly bounded above and below for $0 \leq t < T$. As a result, it does not matter what metric we use to measure the length of a vector or tensor from now on in the argument.

Fix a background connection $\overline{\Gamma}$ (i.e. the zero connection in a local chart) and let $\overline{D}$ be the covariant derivative in $\overline{\Gamma}$ (i.e., an ordinary derivative). Then the difference $\Gamma - \overline{\Gamma}$ is a tensor, and in fact

$$(\Gamma - \overline{\Gamma})^{\ell}_{ij} = \frac{1}{2}g^{k\ell}(\overline{D}_i g_{jk} + \overline{D}_j g_{ik} - \overline{D}_k g_{ij}).$$

We can then compute its evolution

$$\frac{\partial}{\partial t}\left(\Gamma - \overline{\Gamma}\right)^{\ell}_{ij} = g^{k\ell}\left(\overline{D}_k R_{ij} = -\overline{D}_i R_{jk} - \overline{D}_j R_{ik}\right).$$

Since there is a formula

$$\overline{D}Rc = DRc + (\Gamma - \overline{\Gamma}) * Rc$$

we get a formula

$$\left|\frac{\partial}{\partial t}\left(\Gamma - \overline{\Gamma}\right)\right| \leq C|DRc| + C|Rc|\ |\Gamma - \overline{\Gamma}|.$$

Bounds on $|Rc|$ and $|DRc|$ give

$$\left|\frac{\partial}{\partial t}\left(\Gamma - \overline{\Gamma}\right)\right| \leq C + C|\Gamma - \overline{\Gamma}|$$

from which we get at most exponential growth in $\Gamma - \overline{\Gamma}$. In finite time we bound $\Gamma - \overline{\Gamma}$. Hence from now on in the argument all covariant derivatives are equivalent. Bounds on $\overline{D}g$ can be recovered from

$$\overline{D}_i g_{jk} = g_{k\ell}(\Gamma - \overline{\Gamma})^{\ell}_{ij} + g_{j\ell}(\Gamma - \overline{\Gamma})^{\ell}_{ik}.$$

We can now recover the second derivatives $\overline{D}^2 g$ from a formula for their time evolution

$$\frac{\partial}{\partial t}\overline{D}^2 g = -2\overline{D}^2 Rc$$

and a formula

$$\overline{D}^2 Rc = D^2 Rc + \overline{D}^2 g * Rc + \overline{D} g * \overline{D} g * Rc$$

contracting tensors with g, to see that

$$\frac{\partial}{\partial t} \left| \overline{D}^2 g \right| \leq C + C \left| \overline{D}^2 g \right|$$

and again we get at worst exponential growth, giving bounds on $\overline{D}^2 g$. Higher derivatives of g are the same. This proves the Theorem. □

9 Convergence. In a number of cases the solution to the Ricci Flow converges, often after rescaling, to an Einstein metric. This is the most important application of the Ricci Flow to geometry. Here we discuss the known results and likely conjectures.

(a) Dimension Two

If a compact surface has Euler class $\chi = 0$, then with any initial metric the solution to the Ricci Flow exists for all time, and converges (without rescaling) to a flat metric. This applies on the torus or the Klein bottle.

If a compact surface has Euler class $\chi > 0$, then with any initial metric the solution to the Ricci Flow exists up to a finite time T when the metric shrinks to a point, and the metrics can be rescaled to converge to a metric of constant positive curvature. This applies on the sphere or the projective plane.

If a compact surface has Euler class $\chi < 0$, then with any initial metric the solution to the Ricci Flow exists for all time. As $t \to \infty$ the diameter goes to ∞ and the curvature R falls off like $1/t$, and the metrics can be rescaled to converge to a metric of constant negative scalar curvature. This applies on surfaces of higher genus.

In each case above the limiting constant curvature metric is conformal to the initial metric, so this reproves the classification of surfaces and the Uniformization Theorem. The results for $\chi \leq 0$ and $\chi > 0$ with $R > 0$ are in [**22**], and the final case of $\chi > 0$ with any R is due to Chow [**14**].

(b) Dimension Three

If the initial metric on a compact three-manifold has strictly positive Ricci curvature, then the solution to the Ricci Flow exists up to a finite time T when the metric shrinks to a point, and the metrics can be rescaled to converge to a metric of constant positive curvature. It follows that the manifold is diffeomorphic to the sphere S^3 or a quotient by a finite linear group S^3/Γ. This result is in [**20**]].

(c) Dimension Four

If the initial metric on a compact four-manifold has positive curvature operator ([**21**]) or even 2-positive curvature operator ([**12**]), the solution to the Ricci Flow exists up to a finite time T when the metric shrinks to a point, and the metrics can be rescaled to converge to a metric of constant positive curvature.

It follows that the manifold is diffeomorphic to the sphere or the projective space of dimension four.

(d) Positive Curvature Operator

We know that positive curvature operator is preserved in all dimensions. It is reasonable to conjecture that the solution shrinks to a point, and can be rescaled to converge to constant positive curvature. A proof similar to the three and four dimensional cases may suffice. This would involve showing that any given compact subset of the set of positive curvature operators M is contained in a convex subset Z of positive curvature operators such that Z is invariant under the reaction

$$\frac{dM}{dt} = M^2 + M^{\#}$$

and such that if a matrix M in Z is large enough then it is pinched as close to constant positive curvature as we wish. It would be necessary to find the right invariants of M using the Lie algebra structure of $so(n)$.

It might even be true that the Ricci Flow converges, after rescaling, for 2-positive curvature operator?

(e) Positive δ-Pinched Curvature

Huisken [**29**] has shown that if the initial metric has positive sectional curvatures which are sufficiently pinched pointwise, in the sense that for any two planes P_1 and P_2 at the same point X the sectional curvatures satisfy

$$1 - \delta \leq K(x, P_1)/K(x, P_2) \leq 1 + \delta$$

for δ sufficiently small depending only on the dimension, then the solution to the Ricci Flow exists for a finite time T when the metric shrinks to a point, and the metrics can be rescaled to converge to a metric of constant positive curvature.

Hence the manifold is diffeomorphic to a sphere S^n or to a quotient S^n/Γ by a finite linear group. This improves on the δ-pinching theorem from classical geometry (see [**9**]) because the pinching hypothesis is only pointwise, and does not compare sectional curvatures at different points.

10 Kähler Metrics. H.-D. Cao[**4**] has studied the Ricci Flow on Kähler manifolds. He introduces the hypothesis that the Chern cohomology class is a multiple of the Kähler cohomology class in $H^{1,1}$, so that

$$[Rc] = \rho[g].$$

This condition is preserved by the Ricci Flow, and must hold if the flow can be rescaled to converge to an Einstein metric. Hence for studying convergence it is appropriate to assume it holds for the initial metric.

We can understand the importance of this condition by considering product metrics $g = g_1 \times g_2$ on $S^2 \times S^2$ where each factor is a sphere. Then the cohomology splits as a direct sum

$$H^{1,1}(S^2 \times S^2) = H^{1,1}(S^2) \oplus H^{1,1}(S^2)$$

and so do the Kähler and Chern classes $[g] = [g_1] \oplus [g_2]$ and $[Rc] = [Rc_1] \oplus [Rc_2]$. On S^2 the Kähler class $[g]$ is just the area, while the Chern class $[Rc]$ is a fixed element $2\pi[1]$ by Gauss-Bonnet. Hence the condition on $S^2 \times S^2$ that

$$[Rc_1] \oplus [Rc_2] = \rho([g_1] \oplus [g_2])$$

is just that g_1 and g_2 have the same areas. Now the Ricci Flow on the product $S^2 \times S^2$ is just the Ricci Flow on each factor, as we observed before, and the area of each sphere shrinks at a fixed rate

$$\frac{dA}{dt} = -\int R\, da = -4\pi.$$

So the spheres have the same area if and only if they shrink to points at the same time. Now if each S^2 is round, the product metric is Einstein if and only if the radii are the same. Even though $S^2 \times S^2$ has an Einstein metric, the Ricci Flow even after rescaling will not approach it for a product metric unless the two spheres start with the same area.

Under the condition $[Rc] = \rho[g]$, Cao has proven the following results. If $\rho = 0$ the solution to the Ricci Flow exists for all time and converges to a Ricci flat metric (for example on a $K2$ surface). If $\rho < 0$ the solution to the Ricci flow exists for all time, the diameter goes to ∞ and the curvature Rm falls off like $1/t$. We can rescale the metrics to converge to a limit metric which is Kähler-Einstein. (The existence of these Kähler-Einstein metrics in the case $\rho \leq 0$ was known from previous work of Yau on the complex Monge-Ampere equation.) If $\rho > 0$, the solution to the Ricci Flow exists up to some finite time T. As $t \to T$ the volume goes to zero. (This is much stronger than the usual assertion that the curvature is unbounded.)

Not much else is known in the case $\rho > 0$. The Koiso soliton **[33]** shows that it may be impossible to rescale the metrics to converge to an Einstein metric; indeed Koiso's manifold has $\rho > 0$ but no Einstein metric exists. We hope that in many cases the rescaled metrics will converge to a compact Ricci soliton.

There is a useful normalization of the Ricci Flow to study convergence on Kähler manifolds. If $[Rc] = \rho[g]$ we consider the normalized Ricci-Kähler flow

$$\frac{\partial}{\partial t} g(X, Y) = 2\rho g(X, Y) - Rc(X, Y).$$

Now the volume remains constant and the scaling factor ρ remains constant also. The solution to the normalized flow differs from the usual one only by a change in the space and time scales. Whenever $Rc = \rho[g]$ Cao's result shows that the normalized flow has a solution for all time, and if $\rho \leq 0$ it converges to a Kähler-Einstein metric.

There is a further modification that is useful for studying approach to solitons other than Kähler-Einstein metrics. We can choose a potential function f so that

$$D^2_{\alpha\overline{\beta}} f = R_{\alpha\overline{\beta}} - = \rho g_{\alpha\overline{\beta}}$$

by the cohomology condition $[Rc - \rho g] = 0$, and f is unique up to a constant at each time. If we choose the constant right, the potential f varies by the equation

$$\frac{\partial f}{\partial t} = \Delta f + \rho f.$$

If the metric is a Ricci-Kähler soliton then it moves along the holomorphic vector field which is the gradient of f. Since f is determined up to a constant, its gradient ∇f is determined uniquely. The way to best see approach to a soliton metric is to modify the Ricci-Kähler flow by also flowing by the diffeomorphism generated by the gradient vector field ∇f, as in De Turck's trick. In real coordinates this gives the modified Ricci Flow

$$\frac{\partial}{\partial t} g(X,Y) = 2\rho g(X,Y) - 2Rc(X,Y) - 2D^2 f(X,Y).$$

However, unless we are on a soliton already, the gradient vector field ∇f will not be holomorphic, so the complex structure will change, although only by a diffeomorphism. In complex coordinates the components of the metric tensor and the Ricci tensor

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) \quad \text{and} \quad R_{\alpha\beta} = Rc\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right)$$

satisfy $g_{\alpha\beta} \equiv 0$ for a Kähler metric and $R_{\alpha\beta} \equiv 0$ also, so the normalized Ricci Flow takes the form

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -2D^2_{\alpha\bar{\beta}} f \quad \text{and} \quad \frac{\partial}{\partial t} g_{\alpha\beta} = 0.$$

Now for the modified Ricci Flow we get

$$\frac{\partial}{\partial t} g_{\alpha\beta} = 2D^2_{\alpha\beta} f \quad \text{and} \quad \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = 0.$$

Thus for the normalized flow the complex structure is preserved and the symplectic structure changes, while for the modified flow the symplectic structure is preserved and the complex structure changes.

It is well known that if we give the Teichmüller space of equivalence classes of complex structures (under conjugation by diffeomorphism) its quotient topology may not be Hansdorff, particularly at a complex structure which has a nontrivial holomorphic vector field. Thus if the modified Ricci-Kähler flow does converge to a soliton, it may be one with a complex structure not equivalent to the original one by any diffeomorphism.

The only case where we know the modified Ricci-Kähler flow converges is in one complex dimension, not on a smooth surface but on the teardrop and football orbifolds, by work of Lang-Fang Wu[**45**].

When $\rho > 0$, the only case where we always expect to have the rescaled flow converge to a Kähler-Einstein metric is when we start with positive holomorphic bisectional curvature. Mok [**38**] showed this is preserved by the Ricci Flow as we mentioned earlier, and we already know from the Frankel conjecture, proved by Siu and Yau, that the manifold is biholomorphic to $\mathbb{C}P^n$. There is however a problem with trying to prove this in the usual way. There is a solution to the reaction ODE

$$\frac{dM}{dt} = M^2 + M^\#$$

which emerges unstably at $t = -\infty$ from the curvature operator matrix of $\mathbb{C}P^2$ and approaches the curvature operator matrix of $S^2 \times R^2$ as $t \to +\infty$. To see this, consider the three-parameter family of curvature operator matrices in dimension four, decomposed by splitting Λ^2 into self-dual and anti-self-dual forms $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, in the form

$$M = \left(\begin{array}{ccc|ccc} 0 & & & 0 & & \\ & 0 & & & 0 & \\ & & 2x+y & & & u \\ \hline & & & x & & \\ & & & & x & \\ & & & & & y \end{array}\right).$$

These matrices have image in $su(2)$ and so are compatible with a Kähler structure, and satisfy the first Bianchi identity. We get $\mathbb{C}P^2$ with $x = 1, y = 1, u = 0$ and $S^2 \times R^2$ with $x = 0, y = 1, u = 1$. The reaction ODE system shows the matrix remains in this form and reduces to the system

$$\begin{cases} \dfrac{dx}{dt} = x^2 + 2xy \\ \dfrac{dy}{dt} = 2x^2 + y^2 + u^2 \\ \dfrac{du}{dt} = 2xu + 2yu \end{cases}$$

as we can easily compute from the formulas in [**21**]. This is a 3×3 system homogeneous of degree 2. The way to study the solution curves of a homogeneous system

$$\frac{dV}{dt} = \Phi(V)$$

is to consider an associated system

$$\frac{dV}{dt} = \Phi(V) - \lambda(V)V$$

where $\lambda(V)$ is a scalar function of V; the solution curves of the original system and the associated system are projectively equivalent (i.e., define the same curve in projective space). This is enough if we only wish to study the ratios of the components of V. If we take $\lambda = 2x + 2y$ the associates system keeps u constant; if we then take $u = 1$ we get the system

$$\begin{cases} \dfrac{dx}{dt} = x^2 \\ \dfrac{dy}{dt} = 2x^2 - 2xy - y^2 + 1 \\ u = 1 \end{cases}$$

whose solutions are projectively equivalent to those of the original system. Starting with $x \approx y$ near 1 and u small but positive is equivalent to starting with $x \approx y$ large and $u = 1$. The associated system clearly has solutions where $x \to 0$ from the first equation and then $y \to 1$ from the second. This implies that the original system has solutions with $x/u \to 0$ and $y/u \to 1$, so we emerge from $\mathbb{C}P^2$ and approach $S^2 \times R^2$ in the reaction system.

By no means does this imply the same for the Ricci Flow, but we must hope to have $\mathbb{C}P^2$ become attractive under the effect of the diffusion on the curvature because the reaction above is unstable. Notice that Cao's hypothesis that $[Rc] = \rho(g)$ prevents the solution from forming a singularity looking like $S^2 \times R^2$, because the S^2 carries a non-zero element in the Chern class $[Rc]$ and hence an analytic S^2 in the manifold can only shrink proportional to the total volume of the manifold. However the reaction ODE just happens pointwise and knows nothing about this cohomology condition.

11 Metrics with Symmetry. Any symmetries present in the initial metric will be preserved by the Ricci Flow. This fact can sometimes be used to simplify the equations and prove convergence in the special class of metrics with a given symmetry. We will give a very simple example to illustrate the idea, but there are many potential applications to finding new Einstein metrics (or Ricci soliton metrics), particularly on manifolds where the orbit space of some group action is one dimensional. Even though the Einstein equations reduce in this case to a system of non-linear ordinary differential equations, a parabolic flow can be a useful approach to prove the existence of a solution. This is the case, for example, in the Kervaire spheres studied by W.-Y. Hsiang and A. Back [**2**].

For our simple example, consider a 3-manifold M^3 where the torus group $T^2 = S^1 \times S^1$ acts freely. Then M^3 is a T^2 bundle over the circle S^1. There is a larger group G which is the isometry group of the square flat torus R^2/Z^2, containing T^2 as a subgroup. For any point P in the square torus the stabilizer G_P is a copy of the group D_4 of isometries of the square. Consider metrics on M^3 which have G as their isometry group with the subgroup T^2 acting freely. We call these metrics square torus bundles over the circle. For any point P in the bundle, the stabilizer G_P is again a copy of D_4, and the fixed point set of G_P defines a global section of the bundle M^3 which must be totally geodesic and hence horizontal (because G_P contains an element which acts as -I on the normal bundle to the fixed point set). Therefore the bundle is trivial, and the connection on the bundle is trivial. Topologically M^3 is T^3, whose universal cover is R^3. Choosing coordinates (x, y, z) on R^3 so that x is a coordinate on the orbit space S^1 and, for each fixed x, y and z are coordinates on the fibre so that each section where y and z are constant is horizontal, and translation in y and z is an isometry, we get coordinates which are unique up to a diffeomorphism in x and a translation in y and z

$$(x, y, z) \to (a(x), y + b(x), z + c(x)).$$

In such a coordinate system the metric on a square torus bundle takes the form

$$ds^2 = f(x)^2 dx^2 + g(x)^2[dy^2 + dz^2].$$

Note that $ds = f(x)dx$ is the arc length for the quotient metric on the orbit space S^1, and $g(x)$ is the length of the side of the square fibre over x. If the initial metric has this form, it must continue to have this form under the Ricci Flow because the symmetry group G is preserved. We can see this directly by computing the Ricci tensor. Just as the metric g defines a quadratic form

$$ds^2 = g_{ij} dx^i dx^j$$

the Ricci tensor Rc defines a quadratic form

$$d\sigma^2 = R_{ij} dx^i dx^j.$$

For a square torus bundle over the circle we compute

$$d\sigma^2 = p(x)dx^2 + q(x)[dy^2 + dz^2]$$

where

$$\begin{cases} p = -\dfrac{2}{g}\dfrac{\partial^2 g}{\partial x^2} + \dfrac{2}{fg}\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} \\ q = -\dfrac{g}{f^2}\dfrac{\partial^2 g}{\partial x^2} + \dfrac{g}{f^3}\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} - \dfrac{1}{f^2}\left(\dfrac{\partial g}{\partial x}\right)^2. \end{cases}$$

It follows that the Ricci Flow on M^3 reduces to the system of evolution equations

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{2}{fg}\dfrac{\partial^2 g}{\partial x^2} - \dfrac{2}{f^2 g}\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} \\ \dfrac{\partial g}{\partial t} = \dfrac{1}{f^2}\dfrac{\partial^2 g}{\partial x^2} - \dfrac{1}{f^3}\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} + \dfrac{1}{f^2 g}\left(\dfrac{\partial g}{\partial x}\right)^2. \end{cases}$$

for two functions $f(x,t)$ and $g(x,t)$ periodic in x with initial conditions at $t = 0$. Note the equation for g is parabolic, but the second derivative of f does not even enter the equations because f is just the arc length on the orbit space and has no intrinsic geometric meaning up to diffeomorphism, while g is the size of an orbit so g does.

We can simplify these equations by introducing the unit vector field on the orbit space

$$\frac{\partial}{\partial s} = \frac{1}{f}\frac{\partial}{\partial u}$$

whose evolution is given by the commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -\frac{1}{f}\frac{\partial f}{\partial t}\frac{\partial}{\partial s}.$$

Then the Ricci Flow takes the form of the parabolic equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial s^2} + \frac{1}{g}\left(\frac{\partial g}{\partial s}\right)^2$$

on a circle whose unit vector field $\partial/\partial s$ varies by the commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -\frac{2}{g}\frac{\partial^2 g}{\partial s^2}\frac{\partial}{\partial s}.$$

Now we make some interesting geometrical observations before proving convergence.

LEMMA 11.1. *The length L of the orbit circle always increases.*

Proof. The arc length ds on the orbit circle varies by

$$\frac{\partial}{\partial t}\, ds = \frac{2}{g}\,\frac{\partial^2 g}{\partial s^2} ds$$

and the length

$$L = \int 1 ds$$

varies by

$$\frac{dL}{dt} = \int \frac{2}{g}\,\frac{\partial^2 g}{\partial s^2}\, ds = 2\int \frac{1}{g^2}\left(\frac{\partial g}{\partial s}\right)^2 ds \geq 0.$$

□

LEMMA 11.2. *The total volume V of the bundle always decreases.*

Proof. Since

$$V = \int g^2 ds$$

we compute

$$\frac{dV}{dt} = -2\int \left(\frac{\partial g}{\partial s}\right)^2 ds \leq 0.$$

□

LEMMA 11.3. *The size of the largest square torus fibre decreases, and the size of the smallest one increases.*

Proof. At the maximum of g

$$\frac{\partial g}{\partial s} = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial s^2} \leq 0$$

so $\partial g/\partial t \leq 0$ and the maximum decreases. Likewise at the minimum

$$\frac{\partial g}{\partial s} = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial s^2} \geq 0$$

so $\partial g/\partial t \geq 0$ and the minimum increases. □

COROLLARY 11.4. *The length of the orbit circle remains bounded above.*

Proof. Since V is bounded above and g is bounded below, L must be bounded above. □

THEOREM 11.5. *The Ricci Flow on a square torus bundle over a circle has a solution which exists for all time and converges as $t \to \infty$ to a flat metric.*

Proof. Using the commutator relation

$$\frac{\partial}{\partial t}\frac{\partial g}{\partial s} = \frac{\partial^2}{\partial s^2}\frac{\partial g}{\partial s} - \frac{2}{g^2}\left(\frac{\partial g}{\partial s}\right)^3$$

which shows that the maximum of $\partial g/\partial s$ decreases. Since g is bounded above, the maximum principle shows that the maximum value

$$w = \max\left|\frac{\partial g}{\partial s}\right|$$

satisfies an ordinary differential equation

$$\frac{dw}{dt} \leq -cw^3$$

for some constant $c > 0$, and hence satisfies an estimate

$$\left|\frac{\partial g}{\partial s}\right| \leq C/\sqrt{t}$$

for some constant $C < \infty$. □

Differentiating the equation once more gives

$$\frac{\partial}{\partial t}\frac{\partial^2 g}{\partial s^2} = \frac{\partial^2}{\partial s^2}\frac{\partial^2 g}{\partial s^2} - \frac{2}{g}\left(\frac{\partial^2 g}{\partial s^2}\right)^2 - \frac{6}{g^2}\left(\frac{\partial g}{\partial s}\right)^2\frac{\partial^2 g}{\partial s^2} + \frac{4}{g^3}\left(\frac{\partial g}{\partial s}\right)^4.$$

Since g is bounded above by a constant C and $\partial g/\partial s$ is bounded above by $C/\sqrt{t}$ we find that

$$\frac{\partial}{\partial t}\frac{\partial^2 g}{\partial s^2} \leq \frac{\partial^2}{\partial s^2}\frac{\partial^2 g}{\partial s^2} - c\left(\frac{\partial^2 g}{\partial s^2}\right) + \frac{C}{t^2}$$

for some constants $c > 0$ and $C < \infty$. Again the maximum principle shows that the maximum value

$$z = \max\left|\frac{\partial^2 g}{\partial s^2}\right|$$

satisfies an ordinary differential equation

$$\frac{dz}{dt} \le \frac{C}{t^2} - cz^2$$

from which we get an estimate

$$\left| \frac{\partial^2 g}{\partial s^2} \right| \le \frac{C}{t}$$

for some constant C.

In terms of the arc length the sectional curvature has components

$$K_V = -\frac{1}{g^2} \left(\frac{\partial g}{\partial s} \right)^2 \quad \text{and} \quad K_H = -\frac{1}{g} \frac{\partial^2 g}{\partial s^2}$$

where K_H is the sectional curvature of a horizontal plane and K_V that of a vertical one. Note that K_V must be negative in general but zero somewhere while K_H must have both signs. Now the estimate above shows that the curvature remains bounded, so the solution exists for all time $t < \infty$.

Moreover since the maximum value g_{MAX} of g decreases while the minimum value g_{MIN} of g increases, and since

$$g_{MAX} - g_{MIN} \le \int \left| \frac{\partial g}{\partial s} \right| ds \le C/\sqrt{t}$$

we see that g must converge to some constant value $\overline{g}$ as $t \to \infty$. Moreover

$$|K_V| \le C/t \quad \text{and} \quad |K_H| \le C/t$$

so the curvature goes to zero. We can do even better. We can compute

$$\frac{d}{dt} \int \left(\frac{\partial g}{\partial s} \right)^2 ds = -\int \left\{ 2 \left(\frac{\partial^2 g}{\partial s} \right)^2 + \frac{5}{g^2} \left(\frac{\partial g}{\partial s} \right)^4 \right\} ds$$

and use Wirtenger's inequality

$$\int \left(\frac{\partial^2 g}{\partial s^2} \right)^2 ds \ge \left(\frac{2\pi}{L} \right)^2 \int \left(\frac{\partial g}{\partial s} \right)^2 ds$$

and even throw away the term with $(\partial g/\partial s)^4$ and get

$$\frac{d}{dt} \int \left(\frac{\partial g}{\partial s} \right)^2 ds \le -\frac{8\pi^2}{L^2} \int \left(\frac{\partial g}{\partial s} \right)^2 ds.$$

Since L is bounded above, it follows that

$$\int \left(\frac{\partial g}{\partial s} \right)^2 ds \le Ce^{-ct}$$

for some constants $C < \infty$ and $c > 0$. Then

$$g_{MAX} - g_{MIN} \leq \int \left| \frac{\partial g}{\partial s} \right| ds \leq L^{1/2} \left\{ \int \left(\frac{\partial g}{\partial s} \right)^2 ds \right\}^{1/2}$$

shows that g approaches the constant $\overline{g}$ exponentially. A little more work along these lines would show all the derivatives of g, and hence the curvature and its derivatives, go to zero exponentially as well.

12 Geodesic Loops and Minimal Surfaces. Consider a loop γ of length L in a manifold. If T is the unit tangent vector to the loop and ds is the arc length along the loop, the length L evolves by the formula

$$\frac{\partial L}{\partial t} = -\int_\gamma Rc(T,T)ds$$

under the Ricci Flow if we keep the loop fixed. If the loop γ varies in space with a velocity V and if the loop has curvature k in the unit normal direction N then the length L varies at a rate

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} - \int_\gamma KN \cdot V \; ds.$$

When the loop varies so as to remain a geodesic loop the curvature $k = 0$ and the last term drops out, so $dL/dt = \partial L/\partial t$.

Now fix the time and consider a one-parameter family of loops with parameter r starting at the given loop γ at $t = 0$ with a point P on the loop moving with velocity

$$\frac{\partial P}{\partial r} = V.$$

We can always parametrize the loops so V is normal. If γ is a geodesic loop the first variation $\partial L/\partial r = 0$ and the second variation is given by the standard formula

$$\frac{\partial^2 L}{\partial r^2} = \int \left\{ \left(\frac{\partial V}{\partial s} \right)^2 - Rm(T,V,T,V) \right\} ds.$$

Consider first the case where the geodesic loop lies on a surface. If the loop is orientation preserving, we can choose V to be the unit normal vector N. Then

$$\frac{\partial N}{\partial s} = 0 \quad \text{and} \quad Rm(T,N,T,N) = K$$

where K is the Gauss curvature. This gives

$$\frac{\partial L}{\partial t} = -\int K \; ds = \frac{\partial^2 L}{\partial r^2}.$$

Thus L satisfies a kind of heat equation! If γ is weakly stable then $\partial^2 L/\partial r^2 \geq 0$ and $\partial L/\partial t \geq 0$. If we vary the loop γ to keep it a stable geodesic then $dL/dt \geq 0$ also. If the maximum curvature on the manifold is M, then any loop with $L \leq 2\pi/\sqrt{M}$ is stable. This gives the following result.

THEOREM 12.1. *On a surface evolving by the Ricci Flow a weakly stable geodesic loop which preserves orientation has it length increase. (Of course if the loop is not strictly stable it may disappear.)*

COROLLARY 12.2. *For a solution of the Ricci Flow on a compact surface we can find a constant ρ_0 depending only on the initial metric such that if the solution subsequently has sectional curvatures bounded above by M then the injectivity radius ρ is bounded below by*

$$\rho \geq \min\{\rho_0, \pi/2\sqrt{M}\}.$$

Proof. Any loop of length $L < 2\pi/\sqrt{M}$ is strictly stable, so there is a smooth 1-parameter family of loops varying over time that contains it. Their length L is not decreasing if they preserve orientation. Hence it was never longer at an earlier time, and there must have been a stable geodesic loop that short in the initial metric. But we can bound its length by $2\rho_0$ for ρ_0 small. If the loop does not preserve orientation, at least its double cover does, and if $L < \pi/\sqrt{M}$ the previous argument applies. Now the injectivity radius can be bounded by the smaller of $\pi/\sqrt{M}$ and half the length of the shortest geodesic loop. Note our result is precise on P^2. □

The argument extends to three dimensions but the result is not as nice. We can choose an orthonormal frame V_1 and V_2 for the normal bundle to an orientation-preserving loop γ, and consider two one-parameter families of loops with parameters r_1 and r_2 where

$$\frac{dP}{dr_1} = V_1 \quad \text{and} \quad \frac{dP}{dr_2} = V_2.$$

For the best result, choose the frame $\{V_1, V_2\}$ to rotate at a constant rate τ so that

$$\frac{dV_1}{ds} = \tau V_2 \quad \text{and} \quad \frac{dV_2}{ds} = -\tau V_1.$$

The rotation rate τ is related to the holonomy angle of rotation around the loop η by $\eta = \tau L$. Then we can compute

$$\Delta L = \frac{\partial^2 L}{\partial r_1^2} + \frac{\partial^2 L}{\partial r_2^2} = 2\frac{\eta^2}{L} - \int_\gamma Rc(T,T)ds$$

and get the formula

$$\frac{\partial L}{\partial T} = \Delta L - 2\frac{\eta^2}{L}.$$

If the geodesic loop γ is weakly stable then $\Delta L \geq 0$. This gives the following result.

THEOREM 12.3. *A weakly stable orientation-preserving geodesic loop in a three-manifold has its length L shrink at a rate*

$$\frac{dL}{dt} \geq -2\frac{\eta^2}{L}$$

where η is the holonomy angle of rotation around the loop. (Of course if the loop is not strictly stable it may disappear.)

Finally consider a surface Σ^2 in a three-manifold M^3.

Under the Ricci Flow the area A of the fixed surface Σ^2 changes at a rate

$$\frac{\partial A}{\partial t} = -\int_{\Sigma} \{2Rm(T) + Rc(N)\} da$$

where $Rm(T)$ is the sectional curvature of the tangent plane T and $Rc(N)$ is the Ricci curvature in the normal direction N. If we move the surface over time with a velocity V, the area A changes at a rate

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} - \int_{\Sigma} = HN \cdot V\, da$$

where H is the mean curvature. If Σ is a minimal surface, $H = 0$ and the latter term drops out. So if we move Σ so as to keep it a minimal surface then $dA/dt = \partial A/\partial t$.

Assume Σ has an orientable normal bundle, and consider at a fixed time a one-parameter family of surfaces with parameter r starting at the given surface Σ at $r = 0$ and moving in the normal direction N (choose one side) with velocity 1. If Σ is minimal then $\partial A/\partial r = 0$, and the second variation is given by the standard formula

$$\frac{\partial^2 A}{\partial r^2} = \iint_{\Sigma} \{2 \det B - Rc(N)\}\, da$$

where B is the second fundamental form of Σ. The Gauss curvature K of Σ in the induced metric is given by

$$K = \det B + Rm(P)$$

and by the Gauss-Bonnet theorem

$$\int_{\Sigma} K\, da = 2\pi\chi$$

where χ is the Euler class of $\sum$. This gives the formula

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial r^2} - 4\pi\chi$$

which is also a heat equation! If Σ is a weakly stable minimal surface then $\partial^2 A/\partial r^2 \geq 0$. This gives the following result.

THEOREM 12.4. *On a three-manifold a weakly stable minimal surface with orientable normal bundle has its area A vary by*

$$\frac{dA}{dt} \geq -4\pi\chi$$

where χ is the Euler class of the surface. If $\chi \leq 0$ the area of the surface increases. (Of course if the surface is only weakly stable it may disappear.)

Suppose for example that the three-manifold contains an incompressible torus (so that its fundamental group injects). Then there will always be a minimal surface of least area A representing the incompressible torus, and it will always have Euler class $\chi = 0$. The surface may not be unique or vary continuously, but its area must. It now follows that the least area A must increase. This shows that a toroidal neck cannot pinch off (except by rescaling). A spherical neck can pinch since $\chi > 0$, but only at a controlled rate.

13 Local Derivative Estimates. It is often useful to be able to estimate the derivatives of curvature just from a local bound M on the curvature, without requiring the curvature to be bounded by M everywhere. Such estimates were given by W.–X. Shi ([**43**]). We give the estimate for the first derivative, higher derivative are similar.

THEOREM 13.1. *There exists a constant $C < \infty$, depending only on the dimension, with the following property. Suppose we have a smooth solution to the Ricci Flow in an open neighborhood U of a point P in a manifold for times $0 \leq t \leq T$. Assume that the curvature is bounded*

$$|Rm| \leq M$$

with some constant M everywhere on $U \times [0, T]$, and assume that the closed ball of some radius r at time $t = 0$ is a compact set continued in U. Then at the point P at time T we can estimate the covariant derivatives of the curvature by

$$|DRm(P,T)|^2 \leq CM^2\left(\frac{1}{r^2} + \frac{1}{T} + M\right).$$

Proof. Without losing any generality, for any constant $c > 0$ depending only on the dimension we can assume $r \leq c/\sqrt{M}$ by reducing the radius r, and $T \leq c/M$ by starting the argument later and translating in time; in each case we would only increase the constant C in the Theorem by a fixed amount depending on c. Moreover we can assume that the exponential map at P at time $t = 0$ is injective on the ball of radius r, by passing to a local cover if necessary, pulling back the local solution of the Ricci Flow to the ball of radius r in the tangent space at P at time $t = 0$. □

LEMMA 13.1. *We can choose constants $b > 0$ and $B < \infty$, depending only on the dimension, such that the function*

$$F = b(BM^2 + |Rm|^2)|DRm|^2/M^4$$

satisfies the estimate

$$\frac{\partial F}{\partial t} \leq \Delta F - F^2 + M^2$$

on the set $U \times [0, T]$ where $|Rm| \leq M$.

Proof. We have equations

$$\begin{aligned} D_t Rm &= \Delta Rm + Rm * Rm \\ D_t DRm &= \Delta DRm + Rm * DRm \end{aligned}$$

where $*$ denotes some tensor product. From this we find that the function

$$S = (BM^2 + |Rm|^2)|DRm|^2$$

satisfies an inequality

$$\begin{aligned} D_t S \leq \Delta S &- 2BM^2|D^2Rm|^2 - 2|DRm|^4 \\ &+ CM|DRm|^2|D^2Rm| + CBM^3|DRm|^2 \end{aligned}$$

for some constants C depending only on the dimension. Using the inequality

$$2xy \leq x^2 + y^2$$

if we choose B large enough compared to $C, B \geq C^2/4$ to be exact, we can bound first the term

$$CM|DRm|^2|D^2Rm| \leq 2BM^2|D^2Rm|^2 + \frac{1}{2}|DRm|^4$$

and then the term

$$CBM^2|DRm|^2 \leq \frac{1}{2}|D^2Rm|^4 + \frac{1}{2}C^2B^2M^6.$$

This gives

$$D_t S \leq \Delta S - |DRm|^4 + CB^2M^6$$

for the appropriate B. Now

$$S \leq (B+1)M^2|DRm|^2$$

and this yields

$$D_t S \leq \Delta S - \frac{S^2}{(B+1)^2M^4} + CB^2M^6.$$

If we take $F = bS/M^4$ we get

$$D_t F \le \Delta F - \frac{F^2}{b(B+1)^2} + CbB^2M^2.$$

Taking $b \le 1/(B+1)^2$ and $b \le 1/cB^2$ leads to

$$D_t F \le \Delta F - F^2 + M^2$$

as desired. □

LEMMA 13.2. *There exists a constant $A < \infty$ depending only on the dimension such that we can construct a smooth function φ with compact support in the ball of radius r around P at time $t = 0$ such that*

$$\varphi(P) = r$$

and

$$0 \le \varphi \le Ar,\ |D\varphi| \le A,\ |D^2\varphi| \le A/r.$$

Proof. Introduce harmonic coordinates (see [**32**]) and take φ to be a suitable function of the radius in these coordinates. A bound on the curvature in C^1 gives a bound on the curvature in L^p for $p < \infty$. In harmonic coordinates the Euclidean Laplacian of the metric is minus twice the Ricci curvature, so the metric has two derivatives bounded in L^p for $p < \infty$ from the C^0 bound M on Rm. This gives C^1 bounds on the metric, and hence C^0 bounds on the connection, and the second covariant derivatives of the harmonic coordinate functions are given by the components of the connection. This yields bounds on the second covariant derivatives of φ in terms of M and r. For $r \le c/\sqrt{M}$ the precise form follows from the case $r = 1, M \le r$ by a scaling argument. □

Now extend φ to $U \times [0,T]$ by letting φ be independent of time. Choose a constant $\lambda = 12 + 4\sqrt{n}$ and introduce the barrier function

$$H = \frac{\lambda A^2}{\varphi^2} + \frac{1}{t} + M$$

which is defined and smooth on the set where $\varphi > 0$ and $t > 0$. Let V denote the open set in U where $\varphi > 0$. Then V is contained in the ball of radius r around P at $t = 0$, and H is defined and smooth on $V \times (0,T]$. As the metric evolves, we will still have $0 \le \varphi \le A_r$; but $|D\varphi|^2$ and $\varphi|D^2\varphi|$ may increase. By continuity it will be a while before they double.

LEMMA 13.3. *As long as*

$$|D\varphi|^2 \le 2A^2 \quad \textit{and} \quad \varphi|D^2\varphi|^2 \le 2A^2$$

we have the reverse strict inequality

$$\frac{\partial H}{\partial t} > \Delta H - H^2 + M^2.$$

Proof. Since none of the terms is zero

$$H^2 > \frac{\lambda^2 A^4}{\varphi^4} + \frac{1}{t^2} + M^2.$$

Now

$$\Delta\left(\frac{1}{\varphi^2}\right) = \frac{6|D\varphi|^2 - 2\varphi\Delta\varphi}{\varphi^4}$$

and so the hypothesized bounds on $|D\varphi|$ and $|D^2\varphi|$ give

$$6|D\varphi|^2 - 2\varphi\Delta\varphi \leq (12 + 4\sqrt{n})A^2 = \lambda A^2.$$

Then

$$\Delta H = \lambda A^2 \Delta\left(\frac{1}{\varphi^2}\right) \leq \frac{\lambda^2 A^4}{\varphi^4}$$

and

$$-\frac{\partial H}{\partial t} = \frac{1}{t^2}$$

so

$$H^2 > \Delta H - \frac{\partial H}{\partial t} + M^2$$

which is equivalent to the conclusion of the Lemma. □

Since $H \to \infty$ as $t \to 0$ or as $\varphi \to 0$, it is clear that $F \leq H$ at least for $0 < t < \delta$ for some positive time δ.

LEMMA 13.4. *If the constant $c > 0$ is small enough compared to b, B, A, λ and the dimension n, it will have the following property. As long as $r \leq c/\sqrt{M}$ and $t \leq c/M$ and $F \leq H$ we will have*

$$|D\varphi|^2 \leq 2A^2 \quad \textit{and} \quad \varphi|D^2\varphi| \leq 2A^2.$$

Proof. In the frame $\{F_a\}$ which evolves so as to stay orthonormal under the Ricci Flow we have

$$D_t D_a \varphi = D_a D_t \varphi + R_{ab} D_b \varphi$$

for any function φ where $D_a\varphi$ are the components of its derivative in the frame $\{F_a\}$ and D_t is the time derivative in the moving frame with the term in Rc from the motion of the frame. Since $D_t\varphi = 0$,

$$\frac{\partial}{\partial t}|D\varphi|^2 \leq CM|D\varphi|^2$$

and for $|D\varphi|^{\leq}A^2$ at $t = 0$ we get

$$|D\varphi|^2 \leq A^2 e^{CMt}$$

for $t \geq 0$. Now if $t \leq c/M$ with $c \leq (\ln 2)/C$ then $|D\varphi|^2 \leq 2A^2$.

We also have the formula

$$\begin{aligned} D_t D_a D_b \varphi = D_a D_b D_t \varphi + R_{ac} D_b D_c \varphi + R_{bc} D_a D_c \varphi \\ + (D_c R_{ab} - D_a R_{bc} - D_b R_{ac}) D_c \varphi \end{aligned}$$

with the terms in Rc coming from the motion of the frame and the terms in DRc coming from the motion of the connection. This formula gives a bound

$$\frac{\partial}{\partial t}|D^2\varphi| \leq C|Rm||D^2\varphi| + C|DRm||D\varphi|.$$

We can use the bound $|Rm| \leq M$ as before, but we get a bound on $|DRm|$ from $F \leq H$. In particular

$$b(BM^2 + |Rm|^2)|DRm|^2/M^4 \leq \frac{\lambda A^2}{\phi^2} + \frac{1}{t} + M$$

gives a bound (for $t \leq 1/M$ at least)

$$|DRm| \leq CM\left(\frac{1}{\phi} + \frac{1}{\sqrt{t}}\right)$$

with a constant C depending on b, B, A and λ. This yields the estimate

$$\frac{\partial}{\partial t}|D^2\varphi| \leq CM|D^2\varphi| + CM\left(\frac{1}{\varphi} + \frac{1}{\sqrt{t}}\right)|D\varphi|.$$

We can estimate $|D\varphi|^2 \leq 2A^2$ from before. Since φ is fixed in time and $\varphi \leq Ar$,

$$\frac{\partial}{\partial t}\varphi|D^2\varphi| \leq CM\left[\varphi|D^2\varphi| + 1 + \frac{r}{\sqrt{t}}\right].$$

Now viewing this as an ordinary differential inequality at a fixed point, we see that $\varphi|D^2\varphi| \leq 2A^2$ if $\varphi|D^2\varphi| \leq A^2$ at $t = 0$ and $t \leq c/M$ for a suitably small constant $c > 0$.

To see this, consider the ordinary differential equation

$$\frac{du}{dt} \leq CM\left(u + 1 + \frac{r}{\sqrt{t}}\right)$$

for $u = \varphi|D^2\varphi|$ at a fixed point. Then

$$\frac{d}{dt}e^{-CMt}u \leq CMe^{-CMt}\left(1 + \frac{r}{\sqrt{t}}\right).$$

Since $e^{-CMt} \leq 1$ we get

$$\frac{d}{dt} e^{-CMt} u \leq CM \left(1 + \frac{r}{\sqrt{t}}\right).$$

Since $u \leq A^2$ at $t = 0$, we have

$$u \leq e^{CMt} \left[A^2 + CM \int_{\theta=0}^{t} \left(1 + \frac{r}{\sqrt{t}}\right) dt\right].$$

The latter improper integral is finite and gives

$$u \leq e^{CMt} \left[A^2 + CM \left(t + 2r\sqrt{t}\right)\right]$$

and if $r \leq c/\sqrt{M}$ and $t \leq c/M$ for a suitably small c then $u \leq 2A^2$ as desired. □

LEMMA 13.5. *We have $F \leq H$ for $0 < t \leq T$ on the set V where $\varphi > 0$.*

Proof. Since $H \to \infty$ for $t \to 0$ or $\varphi \to 0$, the set where $F \geq H$ is a compact subset of $V \times (0, T]$. Unless it is empty, the continuous function t assumes its minimum $t^* > 0$ on this set at some point P^*. Then $F \leq H$ on all of V for $t \leq t^*$, while $F = H$ at P^* at time t. This forces

$$\frac{\partial F}{\partial t} \geq \frac{\partial H}{\partial t} \quad \text{and} \quad \Delta F \leq \Delta H$$

at P^* at time t^*. But since

$$\frac{\partial F}{\partial t} \leq \Delta F - F^2 + M^2$$

and

$$\frac{\partial H}{\partial t} > \Delta H - H^2 + M^2$$

we have a contradiction. Thus the set $F \geq H$ is empty, and $F < H$ everywhere. □

We conclude that

$$|DRm|^2 \leq CM^2 \left(\frac{1}{\varphi^2} + \frac{1}{t} + M\right)$$

on $V \times (0, T]$ for some constant C depending only on the dimension. Since $\varphi = r$ at P we are done.

14 The Harnack Inequality. There is an interesting differential Harnack inequality for the Ricci Flow (see [**24**]). In addition to the curvature tensor R_{abcd} we consider the tensors

$$P_{abc} = D_a R_{bc} - D_b R_{ac}$$

and

$$M_{ab} = \Delta R_{ab} - \frac{1}{2} D_a D_b R + 2R_{acbd} R_{cd} - R_{ac} R_{bc}.$$

For any two-form U_{ab} and one-form W_a we form the quadratic

$$Z = \left(M_{ab} + \frac{1}{2t} R_{ab} \right) W_a W_b + 2P_{abc} U_{ab} W_c + R_{abcd} U_{ab} U_{cd}.$$

THEOREM 14.1. *Suppose we have a solution to the Ricci Flow for $t > 0$ which is either compact or complete with bounded curvature, and suppose the curvature operator is weakly positive. Then the Harnack quadratic Z is also weakly positive for all two-forms U_{ab} and one-forms W_a for all $t > 0$.*

The proof is given in the reference quoted, and uses the maximum principle. The Harnack quadratic is found by the fact that it vanishes identically on a homothetically expanding soliton, which shows it is a delicate and precise estimate. Now there probably exists a homothetically expanding soliton which is rotationally symmetric and can be found by solving an ordinary differential equation, but no one has bothered to do this yet as far as we know. It would represent a solution emerging from a cone. There may also be non-rotationally symmetric ones, which would be more interesting.

In the proof, assume for simplicity the manifold is compact. Then there will be a first time the quadratic is zero, and a point where this happens, and a choice of U and W giving the null eigenvector. We can extend U and W any way we like in space and time and still have $Z \geq 0$, up to the critical time and we can profit by extending them with

$$D_a U_{bc} = \frac{1}{2} (R_{ab} W_c - R_{ac} W_b) = +\frac{1}{4t} (g_{ab} W_c - g_{ac} W_b)$$

and

$$D_a W_b = 0$$

at the critical point where $Z = 0$. This is an optimal choice for the following. We also take

$$(D_t - \Delta) W_a = \frac{1}{t} W_a \quad \text{and} \quad (D_t - \Delta) U_{ab} = 0$$

at the critical point. We then compute

$$\begin{aligned}(D_t - \Delta) Z =& (P_{abc} W_c + R_{abcd} U_{cd}) \ (P_{abe} W_e + R_{abef} U_{ef}) \\ &+ 2R_{acbd} M_{cd} W_a W_b - 2P_{acd} P_{bdc} W_a W_b \\ &+ 8R_{adce} P_{abe} U_{ab} W_c + 4R_{aecf} R_{bedf} U_{ab} U_{cd}\end{aligned}$$

and indeed this computation is most of the work in the proof. We then check algebraically that if $Z \geq 0$ then $(D_t - \Delta)Z \geq 0$ and apply the maximum principle. Because of the factor $1/t$ in Z we have Z positive for small t, and then it must stay positive.

There is an interesting interpretation of this formula which follows from a remark by Nolan Wallach. Suppose we have a Lie algebra $\mathcal{G}$ with Lie bracket $[,]$ and with an inner product $<,>$. We can then define a system of ordinary differential equations for an element M in the symmetric tensor product $\mathcal{G} \otimes_s \mathcal{G}$ as follows. Choose any basis $\{\phi^\alpha\}$ for $\mathcal{G}$. The Lie bracket is given by the Lie structure constants

$$[\phi^\alpha, \phi^\beta] = c_\gamma^{\alpha\beta} \phi^\gamma$$

and the inner product is given by a matrix

$$\langle \phi^\alpha, \phi^\beta \rangle = g^{\alpha\beta}$$

while the element $M \in \mathcal{G} \otimes_s \mathcal{G}$ is given by

$$M = M_{\alpha\beta} \phi^\alpha \otimes \phi^\beta$$

for some matrix $M_{\alpha\beta}$. Then the ODE is given (independently of the choice of a basis) by

$$\frac{d}{dt} M_{\alpha\beta} = g^{\gamma\delta} M_{\alpha\gamma} M_{\beta\delta} + c_\alpha^{\gamma\zeta} c_\beta^{\delta\eta} M_{\gamma\delta} M_{\zeta\eta}.$$

This is the reaction system in the Ricci Flow for the evolution of the curvature operator M when we compute $(D_t - \Delta)M$ and drop the Laplacian and replace $D_t - \Delta$ by d/dt. Here the Lie algebra is the two-forms Λ^2 which can be identified as the Lie algebra of the rigid rotations $So(n)$, regarding $M = R_{abcd}$ as an element of $\Lambda^2 \otimes_s \Lambda^2$. The inner product used on Λ^2 is the standard one.

Now the Harnack quadratic can be regarded as an element of

$$\left(\Lambda^2 \oplus \Lambda^1\right) \otimes_s \left(\Lambda^2 \oplus \Lambda^1\right)$$

and $\Lambda^2 \oplus \Lambda^1$, the space of pairs of a two-form and a one-form, is the Lie algebra of the group of rigid motions, which is a natural extension of the group of rigid rotations, with the group of translations as kernel. The Lie bracket on $\Lambda^2 = \oplus \Lambda^1$ is given by

$$[U \oplus W, V \oplus X] = [U, V] \oplus (U \rfloor X - V \rfloor W).$$

We can also introduce a degenerate inner product on $\Lambda^2 \oplus \Lambda^1$ by letting

$$\langle U \oplus W, V \oplus X \rangle = \langle U, V \rangle$$

ignoring the Λ^1 factor. Now if we form the ODE on the Lie algebra $\Lambda^2 \oplus \Lambda^1$ according to the rules above for a quadratic Z, we get exactly the reaction system for $(D_t - \Delta)Z$ as given above! The geometry would seem to suggest that the Harnack inequality is some sort of jet extension of positive curvature operator on some bundle including translation as well as rotation, and this is somehow all related to solitons where the solution moves by translation. It would be very helpful to have a proper understanding of this suggestion.

At any rate, we can see why the Harnack expression stays positive.

Write the Harnack quadratic Z as a sum of squares of linear functions (eigenvalues) weighed by constants (eigenvectors)

$$Z = \sum_M \lambda_M \left(\langle V_M, U \rangle + \langle X_M, W \rangle \right)^2 .$$

Then the previous formula yields

$$\begin{aligned}(D_t - \Delta) Z = & \left| \sum \lambda_M \left(\langle V_M, U \rangle + \langle X_M, W \rangle \right) V_M \right|^2 \\ & + \sum_{MN} \lambda_M \lambda_N \left(\langle [V_M, V_N], U \rangle \right. \\ & \left. + \langle V_M \rfloor X_N - V_N \rfloor X_M, W \rangle \right)^2 .\end{aligned}$$

This gives the identification of $(D_t - \Delta)Z$ in terms of the Lie algebra. Now if all $\lambda_M \geq 0$ then clearly $(D_t - \Delta)Z \geq 0$, which is all we need to prove the Theorem.

H.D. Cao([**6**]) has shown that the same conclusion holds if instead of a Riemannian metric with weakly positive curvature operator we have a Kähler metric with weakly positive holomorphic bisectional curvature (a weaker hypothesis in the Kähler case). This suggests trying to prove a Harnack inequality with other curvature hypotheses. For example, does there exist a Harnack inequality on three-manifolds with positive Ricci curvature?

In two dimensions we can rewrite the Harnack inequality using the identification of two-forms with scalars and the rotation by 90° on the tangent space using a local orientation.

THEOREM 14.2. *If we have a solution for $t > 0$ to the Ricci Flow on a surface which is compact, or complete with bounded curvature, and if the curvature R is weakly positive, and if we let*

$$N_{ab} = D_a D_b R + \frac{1}{2} R^2 g_{ab}$$

and define the quadratic

$$Z = \left(N_{ab} + \frac{1}{2t} R g_{ab} \right) X_a X_b + 2 D_a R \cdot X_a \cdot V + R V^2$$

then $Z \geq 0$ for all vectors X_a and all scalars V for all $t > 0$.

Proof. We substitute

$$W_a = \sqrt{2} \mu_{ab} X_b \quad \text{and} \quad U_{ab} = -\frac{1}{\sqrt{2}} \mu_{ab} V$$

in the original formula where μ_{ab} is the volume 2-form in a local orientation. Note the choice of orientation disappears when we square. □

In all cases we can trace the Harnack inequality by writing $U = V \wedge W$ and summing over an orthonormal basis of W to get the trace Harnack inequality

$$\frac{\partial R}{\partial t} + \frac{1}{t}R + 2D_a R \cdot V_a + R_{ab}V_a V_b \geq 0$$

for all vectors V_a for all $t > 0$. This has the consequence, letting $V = 0$, that

$$\frac{\partial R}{\partial t} + \frac{1}{t}R \geq 0$$

or

$$\frac{\partial}{\partial t}(tR) \geq 0$$

which implies that tR is increasing at each point! This is very useful if we combine it with the local derivative estimate of Shi.

COROLLARY 14.3. *Suppose we have a solution to the Ricci Flow for $t > 0$ which is compact or complete with bounded curvature, and has weakly positive curvature operator or is Kähler with weakly positive holomorphic bisectional curvature. Suppose moreover that at some time $t > 0$ we have the scalar curvature $R \leq M$ for some constant M in the ball of radius r around some point P. Then the derivatives of the curvature at P at time t satisfy a bound*

$$|DRm(P,t)|^2 \leq CM^2\left(\frac{1}{r^2} + \frac{1}{t} + M\right)$$

for some constant C depending only on the dimension.

Proof. Since tR increases, we get a bound $R \leq 2M$ in the given region for times between $t/2$ and t. The positive curvature hypotheses each imply a bound on all the curvatures $|Rm|$ from a bound on the trace R. The result now follows from the standard estimate. Likewise we get bounds on higher derivatives.

Such instantaneous derivative estimates are more like what we expect for solutions of elliptic equations. We will use them subsequently in a variety of ways. □

15 The Little Loop Lemma. The following result gives a bound on the injectivity radius at a point in terms of a local bound on the curvature.

LITTLE LOOP LEMMA 15.1. *There exists a constant $\beta > 0$ such that for any initial metric g_0 on a compact manifold which either has positive curvature operator or is Kähler with positive holomorphic bisectional curvature, we can find a constant $\gamma > 0$ depending on g_0 with the following property. If g_t is the* *subsequent solution of the Ricci Flow with initial value g_0 and if P is a point where*

$$R \leq \beta/W^2$$

in the ball around P of radius W at time t, then the injectivity radius of the metric g_t at P at time t satisfies

$$inj(P,t) \geq \gamma/W.$$

Proof. Since the injectivity radius at P can be estimated in terms of the maximum curvature in a ball around P and the length of the shortest closed geodesic loop starting and ending at P, it suffices to get a lower bound on the length of the loop. The Lemma then follows from the following statement, which is what we actually prove. □

THEOREM 15.2. *There exists a constant $\beta > 0$ such that for any initial metric g_0 on a compact manifold which either has positive curvature operator or is Kähler with positive biholomorphic sectional curvature, we can find a constant $B < \infty$ with the following property. If g_t is the subsequent solution of the Ricci flow and if P is a point where*

$$R \leq \beta/(W-s)^2$$

in the ball of radius W around P at time t where s is the distance of a point in the ball from P, then any geodesic loop starting and ending at P at time t has length L with

$$W/L \leq B.$$

Proof. If g_0 has either positive curvature operator or is Kähler with positive holomorphic bisectional curvature, then the subsequent solution g_t does also, and hence g_t has positive Ricci curvature. Moreover from [**24**] or from [**6**] we know that g satisfies a trace Harnack inequality

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2DR(V) + Rc(V,V) \geq 0$$

for all vectors V at any time $t > 0$. Any solution of the Ricci flow satisfying the trace Harnack inequality will also satisfy the Little Loop Lemma, as this is all we use in the proof.

Since the Ricci curvature is always positive, distances always shrink as time increases. This makes it easier to control the geometry. Moreover since all the Ricci curvatures are positive, we have all the Ricci curvatures bounded by the scalar curvature, so

$$0 \leq Rc(V,V) \leq Rg(V,V),$$

and we can control the rate at which any distance shrinks by controlling R from above. □

The first step is to check that we can find a constant B_1 which works in the Theorem up to some time $\tau > 0$. The reason is that the control on R from the Harnack estimate is not so good for small t.

LEMMA 15.3. *For any $\beta > 0$ and any initial metric g_0 as above, we can find $\tau > 0$ and a constant B_1 with the following property. If at some subsequent time t with $0 \leq t \leq \tau$ we have*

$$R \leq \beta/(W - s)^2$$

in the ball of radius W around some point P, then any geodesic loop at P at time t has length L with

$$W/L \leq B_1.$$

Proof. Let M_0 be the maximum curvature at $t = 0$, and let M_t be the maximum curvature up to time t. Since

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2$$

it follows from the maximum principle that

$$\frac{dM}{dt} \leq CM^2$$

for some constant C (in the sense of the lim sup of forward difference quotients) and hence if we take $\tau = c/M_0$ for some small constant $c > 0$ then

$$M_t \leq 2M_0 \quad \text{for} \quad 0 \leq t \leq \tau.$$

Since $R > 0$, we can let $m_0 > 0$ be the minimum value of R at $t = 0$; then by the maximum principle

$$R \geq m_0$$

everywhere for all $t \geq 0$. Since at the center point P at time t

$$R_P W^2 \leq \beta$$

we see that

$$W \leq \sqrt{\beta/M_0}$$

gives an upper bound on W. □

Suppose now that there is a short loop at P of length L with

$$W/L \geq B_1.$$

Then

$$L \leq \sqrt{\beta/M_0} \Big/ B_1$$

and if B_1 is large enough we can make

$$L \leq \varepsilon \Big/ \sqrt{M_0}$$

for any $\varepsilon > 0$ we like. Now as long as

$$L \leq c/\sqrt{M_0}$$

for an appropriately small constant $c > 0$, the standard existence theory for geodesics tells us that in any nearby metric there will exist a geodesic loop starting and ending at P close to the original one; this is just an application of the inverse function theorem together with the observation that for $L^2 M_0 \leq c$ there are no nonvanishing Jacobi fields on the loop which vanish at the end points. Thus we get a family of geodesic loops parametrized by t and varying smoothly, at least for some time backward.

Under the Ricci flow the length of the loop varies by

$$\frac{dL}{dt} = -\int Rc(V, V)ds$$

where we integrate the Ricci curvature in the tangent direction V with respect to the arc length over the loop. (Since the loop is kept geodesic, the first variation in L from the motion of the loop is zero, and we only get the contribution from the change in the metric.) This gives an estimate

$$\frac{dL}{dt} \geq -CM_0 L$$

showing the loop does not shrink too fast. In fact the length L_t at t is related to the length L_θ at θ for $\theta \leq t$ by

$$L_\theta \leq L_t e^{CM_0(t-\theta)}$$

and hence in time $0 \leq t \leq \tau$ with $\tau = c/M_0$ for a suitably small c, if the loop ends with length $L \leq \varepsilon/\sqrt{M_0}$ it is never more than twice as large for as far back in time as we can continue it as a perturbation. But then we can do this all the way to $t = 0$ taking $\varepsilon > 0$ small. Hence then must have been a geodesic loop at $t = 0$ of length at most $2\varepsilon/\sqrt{M_0}$. Now for any g_0 we can take ε so small this is false. Then making B_1 large compared to ε gives us a contradiction if $W/L \geq B_1$. Thus $W/L \leq B_1$, and we have established the Lemma.

This Lemma has one very useful consequence. It is a Corollary of the trace Harnack inequality that for a solution of the Ricci Flow for $t \geq 0$ the quantity tR is pointwise increasing in t. Now we only have to worry for $t \geq \tau$ with $\tau > 0$. Moreover we can find a time T depending on g_0 (in fact $T = C/m_0$ for some constant C, since by the maximum principle the minimum m_t of R at time t grows by a rate

$$\frac{d}{dt}m_t \geq cm_t^2$$

for some constant $c > 0$) such that the solution cannot exist longer than time T. Then for any time t_1 and t_2 with

$$0 < \tau \leq t_1 \leq t_2 \leq T$$

and any point X we have

$$R(X, t_1) \leq CR(X, t_2)$$

for the constant $C = T/\tau$.

The next step is to find a constant B_2 which works if W is not too small.

THEOREM 15.4. *For any $\beta > 0$ and any initial metric g_0 as above and any $W_0 > 0$ we can find a constant B_2 with the following property. If at some subsequent time $t \geq 0$ we have*

$$R \leq \beta/(W - s)^2$$

in the ball of radius W around some point P with $W \geq W_0$, then any geodesic loop at P at time t has length L with $W/L \leq B_2$.

Proof. If we take $B_2 \geq B_1$, we can assume $t \geq \tau$. Suppose $W/L \geq B_2$; then if B_2 is large we can make

$$L \leq \varepsilon W$$

for any $\varepsilon > 0$ we like. Since distances shrink, if a point X has distance s at most $W/2$ from P at some earlier time $\theta \leq t$, it also has distance s at most $W/2$ from P at the later time t. By assumption

$$R \leq \beta/(W - s)^2$$

and hence

$$R(X, t) \leq 4\beta/W^2.$$

Now for $\tau \leq \theta \leq t \leq T$ we have

$$R(X, \theta) \leq 4\beta T/\tau W^2$$

Putting $C = 4\beta T/\tau$ we get

$$R(X, \theta) \leq C/W^2$$

on the ball of radius $W/2$ around P at times θ in $\tau \leq \theta \leq t$. □

Now from the existence of a short loop at P at time t we can deduce the existence of a short loop at earlier times θ, just as before. As long as the loop at P has length $L \leq W$, it must stay in the ball of radius $W/2$ around P where we have a curvature estimate $R \leq C/W^2$. Then again the loop shrinks at a rate

$$\frac{dL}{dt} = -\int Rc(V, V)ds \geq -CL/W^2$$

and for $\tau \leq \theta \leq t$ the length L_t of the loop at time t is related to the length of the loop $L\theta$ at time θ by

$$L_\theta \leq AL_t$$

for the constant

$$A = e^{C(T-\tau)/W_0^2}$$

since $t - \theta \leq T - \tau$ and $W \geq W_0$. If $\varepsilon \leq 1/A$ and $L_t \leq \varepsilon W$ then $L_\theta \leq W$ and we can continue backward all the way to time τ.

Now at time τ we have $L_\tau \leq \varepsilon AW$ and we still have $R \leq C/W^2$ in the ball around P of radius $W/2$. Letting $\widetilde{W} = \delta W$ for an appropriate $\delta > 0$ gives

$$R \leq \beta/(\widetilde{W} - s)^2$$

in the ball of radius $\widetilde{W}$ around P at time θ.

Let $\tilde{L} = L_\tau$ be the length of the loop at P at time τ we constructed by continuation.. Then

$$\widetilde{W}/\tilde{L} \geq \delta W/\varepsilon AW = \delta/\varepsilon A > B_1$$

if $\varepsilon < \delta/AB_1$. This contradicts our first estimate, which proves $W/L \leq B_2$ if ε is chosen small compared to B_2.

COROLLARY 15.5. *For any $\beta > 0$ and for any initial metric g_0 as above and any $\alpha > 0$ there exists a constant B_3 with the following property. If $R \leq \beta/(W - s)^2$ in a ball of radius W around some point P at some time t with $W^2 \geq \alpha t$, then any geodesic loop at P at time t has length L with $W/L \leq B_3$.*

Proof. Choose $\tau > 0$ from Lemma 15.3 and let $W_0^2 = \alpha\tau$ in Theorem 15.4. Then take B_3 to be the larger of B_1 or B_2. If $t \leq \tau$ then 15.3 gives the result; while if $t \geq \tau$ and $W^2 \geq \alpha t$ then $W \geq W_0$ and 15.4 gives the result. □

Now we come to the important case where $W^2 \leq \alpha t$.

LEMMA 15.6. *For any constant $B \geq B_3$, if there exists a loop of length L at the center of a ball of radius W with $R \leq \beta/(W - s)^2$ and $W/L \geq B$, then there exists a first such time $t_* > 0$, and at t_* there is a point P_* with a loop of length L_* and a ball at P_* of radius S_* as above with $W_*/L_* = B$. Moreover $W_*^2 \leq \alpha t$.*

Proof. Pick a decreasing sequence of times t_j, and points P_j with loops of length L_j at P_j and balls of radius W_j with $R \leq \beta/(W_j - s_j)^2$ on the ball, where s_j is the distance to P_j at time t_j, such that t_j converges to the greatest lower bound t_* of all such t. For a subsequence, $P_j \to P_*$ and $s_j \to s_*$, the distance from P_* at time t_*. Since $B \geq B_3$, we know $t_j \geq \tau > 0$ so $t_* \geq \tau > 0$. Also $W_j \leq W_0$ so a subsequence $W_j \to W_*$ with $W_* \leq W_0$. Now for $t \leq t_1$

there is some $\delta > 0$ such that every geodesic loop has length $L \geq \delta$; so $L_j \geq \delta$. This makes $W_j \geq \delta B > 0$, so $W_* > 0$.

If $S_j/L_j = B_j$ with $B_j \geq B$, we have $B_j \leq w/\delta$, and a subsequence $B_j \to B_*$ with $B_* \geq B$. Thus $L_j \to L_*$ where $L_* = W_*/B_*$. □

Choose a subsequence so that the initial unit velocity vectors V_j of the loop at X_j at time t_j of length L_j converge to a vector V_* ; then V_* is the initial unit velocity vector of a loop at X_* at time t_* of length L_*. Moreover in the ball of radius W_* at X_* at time t_* we have $R \leq \beta/(W_* - s_*)^2$ by continuity. This gives a loop of length L_* in a ball of radius W_* at time t_* with $W_*/L_* = B_*$. Since B_* is large enough, there is still a loop of almost the same length at X_* at a slightly earlier time in a ball of radius almost as large where $R \leq \beta/(W - s)$. This would contradict the minimality of t_* unless $B_* = B$. Finally $W_*^2 \leq \alpha t_*$ follows from Corollary 15.5.

Now in reality we always have $W/L < B$ above. To see this, we suppose not, pick the first time t_* when $W_*/L_* = B$, and get a contradiction. The contradiction will come from demonstrating a loop and a ball as above at P_* just a little before t_* with $W/L \geq B$.

First we show there will be a loop L at P_* at earlier times which is not much longer. Since $R \leq \beta/(W_* - s_*)^2$ in the ball of radius W_* around P_* at time t_* where s_* is the distance to P_* at time t_*, we can bound R near P_* at earlier times $t \leq t_*$ using the Harnack inequality. Recall that tR is pointwise increasing, so that

$$tR(X,t) \leq t_* R(X, t_*)$$

for $t \leq t_*$. Now if $s(X,Y,t)$ denotes the distance from X to Y at time t, since lengths shrink we have

$$s = s(X, P_*, t) \geq s(X, P_*, t_*) = s_*$$

and $W_* - s \leq W_* - s_*$ and

$$\beta/(W_* - s_*)^2 \leq \beta/(W_* - s)^2.$$

This makes

$$R \leq \frac{t^*}{t} \cdot \frac{\beta}{(W_* - s)^2}$$

in the ball of radius W_* around X_* at times $t \leq t_*$.

Thus R stays small compared to L_*^2 in a ball of radius large compared to L_* at times a little earlier than t_*, so by the theory of geodesics there will exist unique loop at X_* near the original one for times t a little less than t_*. Moreover the length L of this loop varies by

$$\frac{dL}{dt} = -\int Rc(V,V)ds$$

integrating over the loop. As long as $L \leq W_*$ the loop will stay in the ball of radius $W_*/2$, and as long as $t \geq t_*/2$ we have $t_*/t \leq 2$. Then on the loop

$$Rc(V,V) \leq R \leq 8\beta/W_*^2$$

and

$$dL/dt \geq -8\beta L/W_*^2$$

from estimating the integral. If L does not shrink fast and ends at L_*, it was not much larger than L_* a little before t_*. In fact

$$\frac{d}{dt}\log L \geq -8\beta/W_*^2$$

and

$$L \leq L_* e^{8\beta(t_*-t)/W_*^2}$$

for t a little earlier than t_*.

Finally, we want to show that at a time t a little before t_* the curvature is small enough in a ball of radius W around P_* with W appreciably larger than W_*, so that $W/L > B$. This will finish the proof.

LEMMA 15.7. *At each time $t \leq t_*$ there is a largest W such that if s is the distance to P_* then*

$$R(W-s)^2 \leq \beta$$

on the ball of radius W around P_* at time t. Moreover there is at least one point X where the equality is attained with $0 =< s < W$.

Proof. Since the manifold is compact, the function

$$s + \sqrt{\beta/R}$$

attains its infinimum W at some point X. (Even if it were not compact but complete, this would hold since $s \to \infty$ as $X \to \infty$.) Clearly $W > 0$ and $s < W$. Since s is conelike at P_* but R is smooth, the minimum is not at P_*, so $s > 0$. Now W is a function of t. □

Choose a minimal geodesic γ from P_* to X at time t, and let Y be its unit tangent vector at X pointing away from P_*. The distance function s along the geodesic γ is realized by the arc length, so

$$Ds(Y) = 1.$$

Now on γ

$$R(W-s)^2 \leq \beta$$

and equality is attained at the end X, so

$$DR(Y) \geq \frac{2R}{W-s}.$$

The Harnack Estimate [**24**] in section 14 tells us that for all V

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2DR(V) + Rc(V,V) \geq 0$$

and since $Rc(V, V) \leq R|V|^2$ we have

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2DR(V) + R|V|^2 \geq 0.$$

Choose $V = \lambda Y$ where Y is the unit tangent vector at the end of the geodesic above. Then

$$\frac{\partial R}{\partial t} + \frac{R}{t} + \lambda \frac{4R}{W - s} + \lambda^2 R \geq 0$$

for all λ. Choose $\lambda = -2/(W - s)$; then

$$\frac{\partial R}{\partial t} + \frac{R}{t} \geq \frac{4R}{(W - s)^2}.$$

Now $(W - s)^2 \leq W^2$, and we can choose α so that if $W_*^2 \leq \alpha t_*$ and t is near t_* and W near W_* then $W^2 \leq 2t$ (as long as $\alpha < 2$). This gives

$$\frac{\partial R}{\partial t} \geq \frac{2R}{(W - s)^2.}$$

This inequality holds at any time t a little before t_* at any point X where $R(W - s)^2 = \beta$, and there is at least one such point.

Now the distance s from X_* must decrease as t increases.

Then W must decrease fast enough to keep $R(W - s)^2 \leq \beta$ at the point X above. The function W may not be differentiable,

so we proceed carefully. We know

$$W \leq s + \sqrt{\beta/R}$$

at each point and time with equality at X at time t, and W depends only on t while s decreases. Then at X at time t

$$\liminf_{h \downarrow 0} \frac{W(t + h) - W(t)}{h} \leq -\frac{1}{W}$$

holds for all t a little before t_*.

Since we end up with W_* at t_*, the usual argument gives us that

$$W^2 \geq W_*^2 + 2(t_* - t)$$

for all t a little before t_*. Combining this with our previous estimate

$$L \leq L_* e^{8\beta(t_* - t)/W_*^2}$$

shows that for small β we get $W/L > W_*/L_*$. To see this, expand in power series to get

$$W \geq W_* + \frac{t_* - t}{W_*} + 0(t_* - t)^2$$

and

$$L \leq L_* + \frac{8\beta(t_* - t)L_*}{W_*^2} + 0(t_* - t)^2$$

and

$$W/L \geq W_*/L_* + (1 - 8\beta)\frac{t_* - t}{L_* W_*} + 0(t_* - t)^2$$

showing we need $\beta < 1/8$. This completes the proof.

16 Limits of Solutions to the Ricci Flow. Given a sequence of manifolds $\mathcal{X}_j$ with origin O_j, frames $\mathcal{F}_j$ at O_j and Riemannian metrics g_j, we say that the sequence $(\mathcal{X}_j, O_j, \mathcal{F}_j, g_j)$ converges to the limit $(\mathcal{X}, O, \mathcal{F}, g)$ if there exists a sequence of compact set K_j exhausting $\mathcal{X}$ and a sequence of diffeomorphisms φ_j of K_j in $\mathcal{X}$ to $\mathcal{X}_j$ such that φ_j takes O to O_j and $\mathcal{F}$ to $\mathcal{F}_j$, and the pull-back metrics $\varphi_j^* g_j$ converge to g uniformly on compact sets together with all their derivatives. This is the topology of C^∞ convergence on compact sets. If the limit exists, it is unique up to a unique isometry preserving the origin and frame.

If $(\mathcal{X}_j, O_j, \mathcal{F}_j, g_j)$ converges to $(\mathcal{X}, O, \mathcal{F}, g)$, then we clearly have the following properties:

(a) for every radius s and every integer k there exists a constant $B(s,k)$ independent of j such that the k^{th} covariant derivative of the curvature Rm_j of the metric g_j satisfies a bound

$$|D^k Rm_j| \leq B(s,k)$$

on the ball of radius s around O_j in $\mathcal{X}_j$ in the metric g_j; and

(b) there exists a constant $b > 0$ independent of j such that the injectivity radii ρ_j of $\mathcal{X}_j$ at O_j in the metric g_j satisfy the bound

$$\rho_j \geq b.$$

Conversely we have the following existence result.

THEOREM 16.1. *Given any sequence of manifolds $(\mathcal{X}_j, O_j, \mathcal{F}_j, g_j)$ satisfying the bounds that $|D^k Rm_j| \leq B(s,k)$ on balls of radius s and $\rho_j \geq b > 0$, there exists a subsequence which converges in the C^∞ topology on compact sets to a manifold $(\mathcal{X}, O, \mathcal{F}, g)$.*

Proof. This is slightly more general even than what we did in [**26**], but follows again from an easy modification of the argument in Greene and Wu[**19**]. The only essential new feature is to bound the injectivity radius below at points at a large distance s from O_j in terms of the bounds on the curvature in a slightly larger ball. A lot of the subtlety of getting convergence using only bounds on curvature Rm and not its derivatives DRm is entirely unnecessary for solutions to parabolic equations which are automatically smoothing, such as the Ricci Flow. We have already seen how estimates on Rm give estimates on DRm. □

Now if we have a sequence of solutions to the Ricci Flow on some time interval, we can take a limit (if we have the appropriate bounds) and get another solution to the Ricci Flow. At each time t the metric in the limit solution is the limit of the metrics at the same time in each solution in the sequence. To extract the limit we only need bounds on the curvature at each point at each time, and bounds on the injectivity radius at the origins at time 0 (see [**26**]).

Consider a maximal solution g to the Ricci Flow on a manifold $\mathcal{X}$ for $0 \le t < T$, where either $\mathcal{X}$ is compact or at each time t the metric g is complete with bounded curvature, and either $T = \infty$ or $|Rm|$ is unbounded as $t \to T$. We let $M(t)$ denote the maximum curvature at time t, i.e.,

$$M(t) = \sup\{|Rm(P,t)|\}.$$

We need to assume a bound on the injectivity radius in terms of the maximum curvature. Let $\rho(t)$ denote the infimum of the injectivity radii at all points at time t.

DEFINITION 16.2. *The solution satisfies an injectivity radius bound if there exists a constant $c > 0$ such that*

$$\rho(t) \ge c/\sqrt{M(t)}$$

at every time t.

We classify maximal solutions into three types; every maximal solution is clearly of one and only one of the following three types:

Type I: $T < \infty$ and $\sup(T-t)M(t) < \infty$.
Type II(a): $T < \infty$ but $\sup(T-t)M(t) = \infty$.
Type II(b): $T = \infty$ but $\sup tM(t) = \infty$.
Type III: $T = \infty$ and $\sup tM(t) < \infty$.

For each type of solution we get a different type of limit singularity model.

DEFINITION 16.3. A solution to the Ricci Flow, where either the manifold is compact or at each time t the metric g is complete with bounded curvature, is called a singularity model if it is not flat and of one of the following three types:

Type I: The solution exists for $-\infty < t < \Omega$ for some Ω with $0 < \Omega < +\infty$ and

$$|Rm| \le \Omega/(\Omega - t)$$

everywhere with equality somewhere at $t = 0$.

Type II: The solution exists for $-\infty < t < +\infty$ and $|Rm| \le 1$ everywhere with equality somewhere at $t = 0$.

Type III: The solution exists for $-A < t < \infty$ for some constant A with $0 < A < \infty$ and

$$|Rm| \le A/(A + t)$$

with equality somewhere at $t = 0$.

We always take the equality to hold at some origin 0 at time 0.

THEOREM 16.4. *For any maximal solution to the Ricci Flow which satisfies an injectivity radius estimate of the type above, of Type I, II, or III, there exists a* *sequence of dilations of the solution which converges in the limit to a singularity model of the corresponding type.*

Proof. For Type I, let

$$\Omega = \limsup (T-t)M(t) < \infty.$$

There is some $\varepsilon > 0$ such that we always have $\Omega \geq \varepsilon$; for $M(t)$ satisfies an ODE

$$\frac{dM}{dt} \leq CM^2$$

and hence $M(t)$ couldn't go to ∞ at time T if $(T-t)M(t) < \varepsilon$ at any $t < T$ when ε is small compared to this constant C. Pick a sequence of points P_j and times t_j with $t_j \to T$ and

$$\lim (T-t_j)R(P_j, t_j) = \Omega.$$

For each of these solutions, let P_j be the origin 0, translate in time so that t_j becomes 0, dilate in space by a factor λ so that $R(P_j, t_j)$ becomes 1 at the origin at $t = 0$, and dilate in time by λ^2 so it is still a solution to the Ricci Flow. The dilated solutions exist on a time interval

$$-A_j \leq t < \Omega_j$$

where

$$\Omega_j = (T-t_j)M(t_j) \to \Omega$$

and

$$A_j = t_j \Omega_j / (T-t_j) \to \infty.$$

Moreover, they satisfy curvature bounds. For any $\varepsilon > 0$ we can find a time $\tau < T$ such that for $\tau \leq t < T$ we have

$$|Rm| \leq (\Omega + \varepsilon)/(T-t)$$

by assumption, before dilation. After dilation this becomes a curvature bound

$$|Rm| \leq (\Omega + \varepsilon)/(\Omega_j - t)$$

for times $-\theta_j \leq t < \Omega_j$ where

$$\theta_j = (t_j - \tau)\Omega_j / (T - t_j) \to \infty.$$

Consequently the limit exists on the time interval $-\infty < t < \Omega$ and satisfies

$$|Rm| \leq \Omega/(\Omega - t)$$

everywhere, while $|Rm(0,0)| = 1$.

For Type II(a), we have to be a little more subtle. We start by picking a sequence $T_j < T < \infty$ with $T_j \to T$. If the manifold is compact we can pick points P_j and t_j where

$$(T_j - t_j)|Rm(P_j, t_j)| = \sup_{P, t < T_j} (T_j - t)|Rm(P,t)|$$

as the latter goes to zero as $t \nearrow T_j$. If the manifold is not compact, we can take $\gamma_j \nearrow 1$ and find P_j and t_j so that at least

$$(T_j - t_j)|Rm(P_j, t_j)| \geq \gamma_j \sup_{P, t \leq T_j} (T_j - t)|Rm(P, t)|.$$

Now pick P_j to be the origin O_j, translate in time so t_j becomes zero, dilate in space by a factor λ so $R(P_j, t_j)$ becomes 1 at the origin at $t = 0$, and dilate time by λ^2 so that we still have a solution to the Ricci Flow. The dilated solution exists for $A_j \leq t < \Omega_j$ where

$$\Omega_j = (T_j - t_j)|Rm(P_j, t_j)| \to \infty$$

and

$$A_j = \frac{t_j}{T_j - t_j}\Omega_j = t_j|Rm(P_j, t_j)| \to \infty$$

also. To see $\Omega_j \to \infty$ for Type II(a) where $T < \infty$, note that $T_j \nearrow T, P_j$ and t_j are chosen maximally and

$$\limsup (T - t)|Rm(P, t)| = \infty.$$

To see $A_j \to \infty$ for Type II(a), use the fact that $\Omega_j \to \infty$ forces $t_j \to T$ and $|Rm(P_j, t_j)| \to \infty$. We also get a bound on curvature. We have for $0 \leq t \leq T_j$

$$(T_j - t)|Rm(P, t)| \leq \Gamma_j (T_j - t_j)|Rm(P_j, t_j)|$$

where $\Gamma_j = 1/\gamma_j \to 1$ also, before dilation. After dilation this becomes for $-A_j \leq t \leq \Omega_j$

$$(\Omega_j - t)|Rm(P, t)| \leq \Gamma_j \Omega_j.$$

Write this as

$$|Rm(P, t)| \leq \Gamma_j \Omega_j / (\Omega_j - t).$$

As $j \to \infty, \Gamma_j \to 1$ and $\Omega_j/(\Omega_j - t) \to 1$ for any fixed t. Hence the limit exists and satisfies

$$|Rm(P, t)| \leq 1$$

everywhere for $-\infty < t < +\infty$, while $|Rm(0, 0)| = 1$.

For type II(b), we again choose a sequence $T_j \nearrow T = \infty$, but now we pick P_j and t_j so that

$$t_j(T_j - t_j)|Rm(P_j, t_j)| \geq \gamma_j \sup_{P, t \leq T_j} t(T_j - t)|Rm(P, t)|$$

where again $\gamma_j \nearrow 1$. Pick P_j to be the origin O_j, translate in time so t_j becomes zero, dilate in space by a factor λ so that $R(P_j, t_j)$ becomes 1 at the origin at $t = 0$, and dilate time by λ^2 so it is still a solution of the Ricci Flow. Suppose T_j dilates to Ω_j and 0 dilates to $-A_j$. The solution now exists after dilation on a time interval $-A_j < t < \Omega_j$ where by dilation invariance

$$\frac{A_j \Omega_j}{A_j + \Omega_j} = \frac{t_j(T_j - t_j)}{T_j} |Rm(P_j, t_j)| \to \infty$$

since by assumption

$$\limsup t|Rm(P,t)| = \infty.$$

This forces $A_j \to \infty$ and $\Omega_j \to \infty$ as well since

$$\frac{xy}{x+y} = \frac{1}{1/x + 1/y}.$$

Before dilation we have an estimate for $0 \le t \le T_j$

$$t(T_j - t)|Rm(P,t)| \le \Gamma_j t_j (T_j - t_j)|Rm(P_j, t_j)|$$

where again $\Gamma_j = 1/\gamma_j \to 1$. After dilation this becomes for $-A_j \le t \le \Omega_j$

$$(t + A_j)(\Omega_j - t)|Rm(P,t)| \le \Gamma_j A_j \Omega_j.$$

Write this as

$$|Rm(P,t)| \le \frac{\Gamma_j A_j \Omega_j}{(t+A_j)(\Omega_j - t)}.$$

As $j \to \infty, \Gamma_j \to 1$ and $A_j \Omega_j/(t + A_j)(\Omega_j - t) \to 1$ also for any fixed t. Hence the limit exists and satisfies

$$|Rm(P,t)| \le 1$$

everywhere for $-\infty < t < \infty$ while $|Rm(0,0)| = 1$.

Finally we come to Type III, where $T = \infty$ and

$$A = \limsup tM(t) < \infty.$$

First we claim $A > 0$. Indeed if $t|Rm(P,t)| \le \varepsilon$ for large t then the diameter L satisfies an estimate

$$\frac{dL}{dt} \le C\varepsilon L$$

for a constant C independent of ε and L. This makes L grow at most like $t^{C\varepsilon}$ while $|Rm|$ falls off at least like $1/t$. If $C\varepsilon < 1/2$ we see that $L^2 M \to 0$, which means that after rescaling the curvature collapses with bounded diameter. By a well-known result of Gromov the manifold is nilpotent; more to the point the injectivity radius bound we assumed would fail. Thus $A > 0$.

Now pick a sequence of points P_j and times t_j so that $t_j \to \infty$ and

$$\lim t_j |Rm(P_j, t_j)| = A.$$

Choose P_j to be the origin O_j, translate in time so t_j becomes O_j, dilate in space by a factor λ so that $|Rm(P_j, t_j)|$ becomes 1 at the origin at time $t = 0$, and dilate in time by a factor λ^2 so we still have a solution of the Ricci Flow. After dilation the solution will exist for times $-A_j \le t < \infty$ where time 0 dilates to

$$A_j = t_j |Rm(P_j, t_j)| \to A.$$

Moreover for any $\varepsilon > 0$ we can find a time $\tau < \infty$ such that for $t \geq \tau$

$$t|Rm(P,t)| \leq A + \varepsilon$$

by hypothesis, before dilation. After dilation this becomes a bound

$$(t + A)|Rm(P,t)| \leq A + \varepsilon$$

for time $t \geq -\theta_j$ where

$$\theta_j = (t_j - \tau)A_j/t_j \to A$$

since $t_j \to \infty$ and $A_j \to A > 0$ while τ is fixed. Hence we get a limit which satisfies

$$|Rm(P,t)| \leq A/(t + A)$$

on $-A < t < \infty$ while $|Rm(0,0)| = 1$. This completes the proof of the Theorem. □

In the case of manifolds with positive curvature operator, or Kähler metrics with positive holomorphic bisectional curvature, there is a small modification which is quite useful for Type II and Type III. Because we have positive curvature, we can bound the Riemannian curvature tensor just by the scalar curvature, with a bound

$$|Rm| \leq CR$$

for a constant C depending only on the dimension. Then if we repeat the previous argument we get the following result. Note that we do not need to assume an injectivity radius bound this time; if the solution is compact the injectivity radius bound follows from the Little Loop Lemma in section 15, while if the manifold is complete but not compact and has positive sectional curvature the injectivity radius bound follows from the argument of Gronmoll + Meyer (see [**9**]) in the real case.

THEOREM 16.5. *For any maximal solution to the Ricci Flow with strictly positive sectional curvature on a compact manifold, or with a metric which is complete with bounded curvature at each time with strictly positive sectional curvature, or on a compact Kähler manifold with strictly positive holomorphic bisectional curvature, there exists a sequence of dilations which converges to a singularity model. For Type I solutions the limit exists for* $-\infty < t < \Omega$ *and has* $R(P,t) \leq \Omega/(\Omega - t)$ *with* $R(0,0) = 1$, *for Type II the limit exists for* $-\infty < t < \infty$ *with* $R \leq 1$ *and* $R(0,0) = 1$, *and for Type III the limit exists for* $-A < t < \infty$ *with* $R \leq A/(t + A)$ *and* $R(0,0) = 1$. *These limits will have weakly positive curvature operator, or weakly positive holomorphic bisectional curvature.*

COROLLARY 16.6. *In the real case such a Type II limit must be a Ricci soliton with*

$$Rc = D^2 f.$$

Proof. This follows from the result in [**25**] on eternal solutions have weakly positive curvature operator and where the scalar curvature assumes its maximum, which happens by our construction at the origin at time zero. The proof is by applying the strong maximum principle to the Harnack inequality. □

CONJECTURE 16.7 . *In the Kähler case such a Type II limit must be a Ricci-Kähler soliton with* $Rc = \partial\bar{\partial} f$ *and* $\partial\partial f = 0$.

Proof. Try to use the strong maximum principle on Cao's Harnack inequality. □

CONJECTURE 16.8 . *In the real case such a Type III singularity must be a homothetically expanding Ricci soliton with* $Rc = D^2 f + \rho g$ *for some constant* $\rho > 0$. *In the Kähler case such a Type III singularity must also be an expanding Ricci-Kähler soliton with* $Rc = \partial\bar{\partial} f + \rho g$ *and* $\partial\partial f = 0$.

Proof. Apply the strong maximum principle to the Harnack inequality for solutions on $t > 0$ with the extra term $(1/2t)Rc$. We haven't checked the details, but it must work. □

Unfortunately we don't have injectivity radius bounds available in many cases; in fact in many cases we expect them to fail, particularly as $t \to \infty$ for example on a nilmanifold or two hyperbolic manifolds of finite volume joined along their cusps. However, recent work of Cheeger, Gromov and Futake ([**11**]) suggests that we should be able to get some kind of limit anyway. The manifolds will collapse to a lower dimensional manifold (or orbifold). However the solution to the Ricci Flow on the limiting manifolds may not converge to a solution to the Ricci Flow on the lower dimensional limit manifold (or orbifold). Rather there will be some extra information in the fibres that collapse, which should be represented by some information in a bundle over the lower dimensional limit manifold (or orbifold), and there should be a system for the joint evolution of the metric on the base and the information in the fibre reflecting the Ricci Flow in the limiting manifolds.

17 Bounds on Changing Distances. It is useful to see how the actual geometry changes under the Ricci Flow. For this purpose we need to control the change in the distance $d(P, Q, t)$ between two points P and Q at time t when P and Q are fixed but t increases. The basic obvious estimate is the following.

THEOREM 17.1. *There exists a constant* C *depending only on the dimension, such that if the curvature* Rm *is bounded by a constant* M

$$|Rm| \le M$$

then

$$e^{-CM(t_2-t_1)} d(P, Q, t_1) \le d(P, Q, t_2) \le e^{CM(t_2-t_1)} d(P, Q, t_1)$$

for any points P *and* Q *and any times* t_1 *and* t_2.

There is a more subtle bound on how fast distances can shrink which is much better when the distance is large compared to the curvature.

THEOREM 17.2. *There exists a constant C depending only on the dimension such that if*

$$|Rm| \leq M$$

then

$$d(P, Q, t_2) \geq d(P, Q, t_1) - C\sqrt{M}(t_2 - t_1)$$

for any points P and Q and any times $t_1 \leq t_2$.

The second estimate says that the rate at which a distance shrinks can be bounded independently of how large it is. It is due to the fact that on a long minimal geodesic there cannot be too much positive curvature along its middle or it would be unstable.

Both theorems are proved by the following observation. For any path γ its length L changes at a rate

$$\frac{dL}{dt} = -\int_\gamma Rc(T, T)ds$$

where T is the unit tangent vector to the path γ and we integrate along the path with respect to the arc length s. The function $d(P, Q, t)$ is the least length L of all paths. In general it will not be smooth in t for fixed P and Q, but at least it will be Lipschitz continuous. Hence we can estimate its derivative above and below, in the sense of giving an upper bound on the lim sup of all forward difference quotients and a lower bound on the lim inf of all forward difference quotients.

LEMMA 17.3. *The distance $d(P, Q, t)$ satisfies the estimate*

$$-\sup_{\gamma\in\Gamma}\int_\gamma Rc(T, T)ds \leq \frac{d}{dt}d(P, Q, t) \leq 0 \inf_{\gamma\in\Gamma}\int_\gamma Rc(T, T)ds$$

where the sup and inf are taken over the compact set Γ of all geodesics γ from P to Q realizing the distance as a minimal length.

Proof. We can restrict our attention to the compact set of geodesics of some large but finite length and apply the argument in [**21**]. □

For the first theorem we apply the bound

$$-CMd(P, Q, t) \leq \int_\gamma Rc(T, T)ds \leq CMd(P, Q, t)$$

to conclude

$$-CM \leq \frac{d}{dt}\log d(P, Q, t) \leq CM$$

and integrate and exponentiate to get the result. For the second theorem we apply the following result, which is an integral version of Meyer's Theorem.

THEOREM 17.4. *On a Riemannian manifold suppose we have a geodesic from P to Q of length L with arc length s and unit tangent vector T. For $0 < \sigma \leq L/2$*

(a) if $Rc(T,T) \geq 0$ along γ then

$$\int_\sigma^{L-\sigma} Rc(T,T)ds \leq \frac{2(n-1)}{\sigma} \quad ;$$

(b) if $Rc(T,T) \geq (n-1)\rho^2$ then

$$\int_\sigma^{L-\sigma} Rc(T,T)ds \leq \frac{2(n-1)\rho}{\tan \rho\sigma} \quad ;$$

(c) if $Rc(T,T) \geq -(n-1)\rho^2$ then

$$\int_\sigma^{L-\sigma} Rc(T,T)ds \leq \frac{2(n-1)\rho}{\tan h\rho\sigma} \quad .$$

We give the proof shortly for convenience, but first we finish the proof of Theorem 17.2. We can bound the integral over the whole path γ in three pieces

$$\int_0^L Rc(T,T)ds \leq \int_0^\sigma Rc(T,T)ds + \int_\sigma^{L-\sigma} Rc(T,T)ds + \int_{L-\sigma}^L Rc(T,T)ds.$$

We bound the first and third piece using the maximum of the curvature

$$\int_0^\sigma Rc(T,T)ds \leq CM\sigma \quad \text{and} \quad \int_{L-\sigma}^L Rc(T,T)ds \leq CM\sigma.$$

We bound the middle piece using Theorem 17.4

$$\int_{L-\sigma}^L Rc(T,T)ds \leq \frac{C\sqrt{M}}{\tan \mathrm{h}(\sqrt{M}\sigma)} \quad .$$

If we take $\sigma = 1/\sqrt{M}$ both bounds are the same and we get

$$\int_0^L Rc(T,T)ds \leq C\sqrt{M}.$$

Using this bound in Lemma 17.3 gives the result in Theorem 17.2. Now we prove Theorem 17.4.

Consider a geodesic from P to Q of length L with arc length s and unit tangent vector T. Choose an orthonormal frame $F_0, F_1, \ldots, F_{n-1}$ at P with $F_0 = T$, and extend it along the geodesic by parallel translation so that

$$\frac{d}{ds}F_a = 0 \quad \text{for} \quad 0 \leq a \leq n-1.$$

Then F_0 continues to be T and the frame continues to be orthonormal.

Jacobi's equation for a normal vector field V to the geodesic representing an infinitesimal geodesic perturbation is

$$\left\langle \frac{d^2}{ds^2}V, W \right\rangle + R(T, V, T, W) = 0$$

for all normal vectors W. Choose a basis $V_1, \ldots, V_{n-1}$ for the Jacobi's fields vanishing at P by choosing

$$V_\alpha = 0 \quad \text{and} \quad \frac{d}{ds}V_\alpha = F_\alpha \quad \text{at} \quad P \quad \text{for} \quad 1 \leq \alpha \leq n-1.$$

In terms of the parallel frame we can write

$$V_\alpha = V_\alpha^\beta F_\beta \quad \text{for} \quad 1 \leq \alpha, \beta \leq n-1.$$

Then Jacobi's equation becomes

$$I_{\alpha\gamma}\frac{d^2}{ds^2}V_\beta^\gamma + R_{0\alpha 0\gamma}V_\beta^\gamma = 0$$

for the functions $V_\beta^\gamma(s)$ for $0 \leq s \leq L$, with initial conditions

$$V_\beta^\gamma = 0 \quad \text{and} \quad \frac{d}{ds}V_\beta^\gamma = I_\beta^\gamma \quad \text{at} \quad s = 0$$

where

$$R_{0\alpha 0\gamma} = R(F_0, F_\alpha, F_0, F_\gamma)$$

so that $R_{0\alpha 0\alpha}$ is the sectional curvature of the plane spanned by the tangent to the geodesic and the α^{th} normal frame vector. Of course $R_{0\alpha 0\gamma}$ is symmetric in α and γ.

LEMMA. *The matrix*

$$S_{\alpha\beta} = I_{\gamma\delta}\frac{d}{ds}V_\alpha^\gamma \cdot V_\beta^\delta$$

is symmetric.

Proof. We compute

$$\frac{d}{ds}S_{\alpha\beta} = I_{\gamma\delta}\,\frac{d}{ds}\,V_\alpha^\gamma \cdot \frac{d}{ds}V_\beta^\delta - R_{0\gamma 0\delta}V_\alpha^\gamma V_\beta^\delta$$

using Jacobi's equation. This shows the derivative of $S_{\alpha\beta}$ is symmetric. But $S_{\alpha\beta} = 0$ at $s = 0$, so $S_{\alpha\beta}$ is always symmetric. □

Now by definition if the geodesic has no conjugate points to P before Q, then any Jacobi field vanishing at P does not vanish again before Q. Consequently the matrix V_α^γ is invertible on $0 < s < L$ with an inverse we call W_γ^α. Define the matrix

$$Z_{\alpha\beta} = I_{\alpha\gamma} W_\beta^\delta \frac{d}{ds} V_\delta^\gamma.$$

Since

$$Z_{\alpha\beta} = W_\alpha^\gamma W_\beta^\delta S_{\gamma\delta}$$

we see that $Z_{\alpha\beta}$ is symmetric also. The formula for the derivative of the inverse of a matrix is

$$\frac{d}{ds} W_\beta^\delta = -W_\beta^\eta W_\theta^\delta \frac{d}{ds} V_\eta^\theta$$

and we can easily compute

$$\frac{d}{ds} Z_{\alpha\beta} + I^{\gamma\delta} Z_{\alpha\gamma} Z_{\beta\delta} + R_{0\alpha 0\beta} = 0$$

using Jacobi's equation. The trace

$$Z = I^{\alpha\beta} Z_{\alpha\beta} = \frac{d}{ds} \log\det V_\alpha^\beta$$

represents the rate of growth of the transversal area along the geodesic. The function Z is defined and smooth on the interior $0 < s < L$, while $Z \to +\infty$ as $s \to 0$, and $Z \to -\infty$ as $s \to L$ also if and only if Q is a conjugate point to P. The usual inequality gives

$$I^{\alpha\beta} I^{\gamma\delta} Z_{\alpha\gamma} Z_{\beta\delta} \geq \frac{1}{n-1} Z^2$$

with equality when $Z_{\alpha\beta}$ is a multiple of the identity. Taking the trace of the equation above gives the inequality

$$\frac{d}{ds} Z + \frac{1}{n-1} Z^2 + Rc(T,T) \leq 0$$

where $I^{\alpha\beta} R_{0\alpha 0\beta} = Rc(T,T)$ is the Ricci curvature in the direction T tangent to the geodesic. The only fact we use for the following estimate is that there is some smooth function Z finite on $0 < s < L$ for which this inequality holds.

Since $Z^2 \geq 0$ we always have

$$\frac{dZ}{ds} + Rc(T,T) \leq 0$$

and hence

$$\int_\sigma^{L-\sigma} Rc(T,T) ds \leq Z(\sigma) - Z(L-\sigma)$$

for any σ in $0 < \sigma \leq L/2$. If $Rc(T,T) \geq 0$ along the geodesic then

$$\frac{d}{ds} Z + \frac{1}{n-1} Z^2 \leq 0$$

and we find that

$$Z(\sigma) \leq \frac{n-1}{\sigma} \quad \text{and} \quad Z(L-\sigma) \geq -\frac{n-1}{\sigma}$$

so we get

$$\int_{\sigma}^{L-\sigma} Rc(T,T)ds \leq \frac{2(n-1)}{\sigma}.$$

If $Rc(T,T) \geq (n-1)\rho^2$ then

$$\frac{d}{ds}Z + \frac{1}{n-1}Z^2 + (n-1)\rho^2 \leq 0$$

and we find that

$$Z(\sigma) \leq \frac{(n-1)\rho}{\tan \rho\sigma} \quad \text{and} \quad Z(L-\sigma) \geq -\frac{(n-1)\rho}{\tan \rho\sigma}$$

so we get

$$\int_{\sigma}^{L-\sigma} Rc(T,T)ds \leq \frac{2(n-1)\rho}{\tan \rho\sigma}.$$

Finally if $Rc(T,T) \geq -(n-1)\rho^2$ then

$$\frac{d}{ds}Z + \frac{1}{n-1}Z^2 - (n-1)\rho^2 \leq 0$$

and we find that

$$Z(\sigma) \leq \frac{(n-1)\rho}{\tanh \rho\sigma} \quad \text{and} \quad Z(L-\sigma) \geq -\frac{(n-1)\rho}{\tanh \rho\sigma}$$

so we get

$$\int_{\sigma}^{L-\sigma} Rc(T,T)ds \leq \frac{2(n-1)\rho}{\tanh \rho\sigma}.$$

This completes the proof.

18 Geometry of Complete Manifolds at Infinity. Given a complete Riemannian manifold, we define its aperture in the following way. Pick an origin 0, and let $S(s)$ be the sphere if radius s around the origin, the set of points whose distance to the origin 0 is exactly s. Its diameter diam S_s is the maximum distance between two points in the sphere. The aperture α of the manifold is defined as

$$\alpha = \limsup_{s\to\infty} \ \text{diam } S_s/2s.$$

Clearly α is invariant under dilation. We note that the aperture is independent of the choice of the origin. To see this, suppose 0 and 0′ are two origins. Choose points P and Q on the sphere S_s around 0 with s very large compared to the distance r between 0 and 0′, and so that $d(P,Q)$ is nearly αs where α is the aperture at 0. Then P and Q are nearly at distance αs from $0'$, and by making one shorter we can make the distances equal, and at least $s - r$. For s large we can make

$$\alpha s/(s-r)$$

as close to α as we like. Then the aperture α' at $0'$ is at least the aperture α at 0. By symmetry $\alpha = \alpha'$. Note the aperture of the paraboloid is 0, the aperture of a convex cone is between 0 and 1, the aperture of Euclidean space is 1, and the aperture of hyperbolic space is ∞.

In much the same way we can prove the following result. For a solution to the Ricci Flow the aperture $\alpha = \alpha(t)$ is defined for each t.

THEOREM 18.1. *For a complete solution to the Ricci flow with bounded curvature and weakly positive Ricci curvature the aperture α is constant.*

Proof. Suppose $Rc \geq 0$ and $|Rm| \leq M$. In time $\Delta t \geq 0$ the distance between two points shrinks but not by more than $C\sqrt{M}\Delta t$. Let α be the aperture at time t. For any $\tilde{\alpha} < \alpha$ and any $\sigma < \infty$ we can find $s \geq \sigma$ and two points P and Q such that

$$d(0,P,t) = s \qquad d(0,Q,t) = s \qquad \text{and} \qquad d(P,Q,t) \geq \tilde{\alpha}s.$$

Then

$$s - C\sqrt{M}\Delta t \leq d(0,P,t \pm \Delta t) \leq s + C\sqrt{M}\Delta t$$
$$s - C\sqrt{M}\Delta t \leq d(0,Q,t \pm \Delta t) \leq s + C\sqrt{M}\Delta t$$

and

$$\tilde{\alpha}s - C\sqrt{M}\Delta t \leq d(P,Q,t+\Delta t) \leq \tilde{\alpha}s + C\sqrt{M}\Delta t.$$

Now depending on which is further from 0, we can more P or Q back toward 0 by no more than $C\sqrt{M}\Delta t$ and make the distances of P and Q from 0 equal again, without reducing the distance between P and Q by more than $C\sqrt{M}\Delta t$. Since

$$\frac{\tilde{\alpha}s - C\sqrt{M}\Delta t}{s + C\sqrt{M}\Delta t} \longrightarrow \tilde{\alpha} \quad \text{as} \quad s \to \infty$$

we see that the aperture at time $t \pm \Delta t$ is at least α also. Hence the aperture is constant. □

THEOREM 18.2. *Suppose we have a solution to the Ricci Flow on a complete manifold with bounded curvature. If $|Rm| \to 0$ as $s \to \infty$ at $t = 0$, this remains true for $t \geq 0$.*

Proof. Suppose $|Rm| \leq M$ for some constant M. For every $\varepsilon > 0$ we can find $\sigma < \infty$ such that $|Rm| \leq \varepsilon$ for $s \geq \sigma$. The curvature tensor evolves by a formula

$$D_t Rm = \Delta Rm + Rm * Rm$$

which gives a formula

$$\frac{\partial}{\partial t}|Rm|^2 = \Delta|Rm|^2 - 2|DRm|^2 + Rm * Rm * Rm$$

and an estimate

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 + C|Rm|^3$$

for some constant C depending only on the dimension. For any $\delta > 0$ choose

$$\rho = \sigma + (M^2 - \varepsilon^2)/\delta$$

and choose the continuous function

$$\psi = \begin{cases} M^2 & \text{if} \quad s \leq \sigma, \\ M^2 - \delta(s-\sigma) = \varepsilon^2 + \delta(\rho - s) & \text{if} \quad \sigma \leq s \leq \rho, \\ \varepsilon^2 & \text{if} \quad s \geq \rho. \end{cases}$$

where s is the distance from some origin at $t = 0$. Then ψ is Lipschitz continuous since s is, and since $|Ds| \leq 1$ almost everywhere we also have $|D\psi| \leq \delta$ almost everywhere.

Now we can smooth ψ locally and patch together with a partition of unity to get a function $\tilde{\psi}$ which is smooth and has

$$-\varepsilon^2 \leq \tilde{\psi} \leq M^2 + \varepsilon^2 \quad \text{and} \quad |D\tilde{\psi}| \leq 2\delta \quad \text{everywhere}$$

and

$$\tilde{\psi} \geq M^2 - \varepsilon^2 \quad \text{for} \quad s \leq \sigma \quad \text{and} \quad \tilde{\psi} \leq \varepsilon^2 \quad \text{if} \quad s \geq \rho.$$

Lastly take $\varphi = \psi + 2\varepsilon^2$. Then

$$\varepsilon^2 \leq \varphi \leq M^2 + 3\varepsilon^2 \quad \text{and} \quad |D\varphi| \leq 2\delta \quad \text{everywhere}$$

and

$$\varphi \geq M^2 \quad \text{if} \quad s \leq \sigma \quad \text{and} \quad \varphi \leq 3\varepsilon^2 \quad \text{if} \quad s \geq \rho.$$

Now define φ for $t \geq 0$ by solving the scalar heat equation

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi$$

in the Laplacian of the metric evolving by the Ricci Flow. By the maximum principle we still have $\varepsilon^2 \leq \varphi \leq M^2$ everywhere for $t \geq 0$. The derivative $D_a\varphi$ evolves in an evolving orthonormal frame by the formula

$$D_t D_a \varphi = \Delta D_a \varphi$$

and hence

$$\frac{\partial}{\partial t}|D\varphi|^2 = \Delta|D\varphi|^2 - 2|D^2\varphi|^2.$$

Note this formula does not involve the curvature. Hence $|D\varphi| \leq 2\delta$ everywhere for $t \geq 0$ by the maximum principle.

The second derivative $D_a D_b \varphi$ evolves by the formula

$$D_t D_a D_b \varphi = \Delta D_a D_b \varphi + 2R_{acbd} D_c D_d \varphi$$

and hence

$$\frac{\partial}{\partial t}|D^2\varphi|^2 = \Delta|D^2\varphi|^2 - 2|D^3\varphi|^2 + 4R_{acbd} D_a D_b \varphi D_c D_d \varphi$$

which gives an estimate

$$\frac{\partial}{\partial t}|D^2\varphi|^2 \leq \Delta|D^2\varphi|^2 + CM|D^2\varphi|^2$$

for some constant C depending only on the dimension. Let us put

$$F = t|D^2\varphi|^2 + |D\varphi|^2$$

and compute

$$\frac{\partial F}{\partial t} \leq \Delta F - (1 - CMt)|D^2\varphi|^2.$$

Then if $t \leq c/M$ where $c = 1/C$ depends only on the dimension, we have

$$\frac{\partial f}{\partial t} \leq \Delta F$$

and the maximum of F decreases. But

$$F \leq 4\delta^2$$

at $t = 0$, and hence for $t \geq 0$ also. Thus

$$|D^2\varphi| \leq 2\delta/\sqrt{t} \quad \text{for} \quad 0 < t \leq c/M.$$

Since $|\Delta\varphi|^2 \leq n|D^2\varphi|^2$ and φ solves the heat equation,

$$\left|\frac{\partial \varphi}{\partial t}\right| \leq C\delta\sqrt{t} \quad \text{for} \quad 0 < t \leq c/M$$

where this constant $C = 2\sqrt{n}$ depends only on the dimension n. Now $1/\sqrt{t}$ has an improper integral which is $2\sqrt{t}$ which is finite, so for all P

$$|\varphi(P,t) - \varphi(P,0)| \leq 2C\delta\sqrt{t} \quad \text{for} \quad 0 < t \leq c/M.$$

Since $\delta > 0$ is arbitrarily small, we can take

$$\delta \leq \varepsilon^2\sqrt{M}/2C\sqrt{c}$$

so that $2C\delta\sqrt{t} \leq \varepsilon^2$ for $t \leq c/M$. Then $\varphi \leq 4\varepsilon^2$ at times $t \leq c/M$ on the set where $s \geq \rho$ at $t = 0$. Now distances can expand, but only at an exponential rate governed by M. In particular if $s = s(P, 0, t)$ is the distance between a point P and the origin 0 at time t, we have

$$\frac{\partial s}{\partial t} \leq CMs$$

and

$$s(t) \leq s(0)e^{CMt}.$$

This gives us a constant C depending only on the dimension such that if $s \geq C\rho$ at P at time $t \leq c/M$ then $s \geq \rho$ at P at $t = 0$, and $\varphi \leq 4\epsilon^2$ at P at time t.

Now at $t = 0$ we have

$$|Rm|^2 \leq M^2 \leq \varphi \quad \text{if} \quad s \leq \sigma$$

and

$$|Rm|^2 \leq \varepsilon^2 \leq \varphi \quad \text{if} \quad s \geq \sigma$$

so $|Rm|^2 \leq \varphi$ everywhere at $t = 0$. Since

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 + C|Rm|^3$$

we have

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 + CM|Rm|^2$$

while

$$\frac{\partial}{\partial t}\left(e^{CMt}\varphi\right) = \Delta\left(e^{CMt}\varphi\right) + CM\left(e^{CMt}\varphi\right)$$

so $|Rm|^2 \leq e^{CMt}\varphi$ by the maximum principle. For $t \leq c/M$ this gives $|Rm|^2 \leq C\varphi$ for some other constant C depending only on the dimension. Hence at time t we have

$$|Rm|^2 \leq C\varepsilon^2 \quad \text{for} \quad s \geq C\rho$$

where these constants C depend only on the dimension and are independent of ε. Thus $|Rm| \to 0$ for $t \leq c/M$ also as $s \to \infty$. Since the time interval can always be advanced by c/M as long as $|Rm| \leq M$, we get the result until $|Rm|$ becomes unbounded or $t \to \infty$. □

Next we define the asymptotic volume ratio. Again let s denote the distance to an origin 0 in a complete manifold of dimension n, let B_s denote the ball of radius s around the origin, and let $V(B_s)$ be its volume. If the manifold has weakly positive Ricci curvature, then the standard volume comparison theorem tells us that $V(B_s)/s^n$ is monotone decreasing in s. We define the asymptotic volume ratio

$$\nu = \lim_{s\to\infty} V(B_s)/s^n.$$

In Euclidean space ν is the volume $\overline{\nu}$ of the unit ball, otherwise $\nu \leq \overline{\nu}$. For all s, $V(B_s) \geq \nu s^n$. In the same way as for α before, the value of ν is independent of the choice of the origin. (We omit the details.) Hence the lower bound holds on any ball around any point P

$$V(B_s(P)) \geq \nu s^n.$$

Often a volume bound can substitute for an injectivity radius bound. Of course we also have

$$V(B_s(P)) \leq \overline{\nu} s^n.$$

THEOREM 18.3. *Suppose we have a complete solution to the Ricci Flow with bounded curvature and weakly positive Ricci curvature, where $|Rm| \to 0$ as $s \to \infty$ (a condition preserved by the flow). Then the asymptotic volume ratio ν is constant.*

Proof. Let γ be a small constant we shall choose soon, and consider the annulus

$$N_\sigma = \{\gamma\sigma \leq s \leq \sigma\}.$$

Since

$$N_\sigma = B_\sigma - B_{\gamma\sigma}$$

we have

$$V(N_\sigma) = V(B_\sigma) - V(B_{\gamma\sigma}).$$

If the asymptotic curvature ratio is at least ν, then

$$V(N_\sigma) \geq (\nu - \gamma^n \overline{\nu})\sigma^n.$$

When γ is small, $\nu - \gamma^n\overline{\nu}$ is nearly ν and most of the volume of the ball is in the annulus. □

The volume of the annulus changes at a rate

$$\frac{d}{dt}V(N_\sigma) = -\int_{N_\sigma} R \, dv.$$

For every ε and every γ we can find σ_0 so that if $\sigma \geq \sigma_0$ then $|Rm| \leq \varepsilon$ on N_σ. This makes

$$\left|\frac{d}{dt}V(N_\sigma)\right| \leq \varepsilon V(N_\sigma).$$

If $V_1(N_\sigma)$ is the volume at time t_1 and $V_2(N_\sigma)$ is the volume at time t_2 we have

$$V_2(N_\sigma) \geq e^{-\varepsilon|t_2-t_1|}V_1(N_\sigma).$$

Let ν_1 be the asymptotic volume ratio at time t_1 and ν_2 the ratio at time t_2. Then

$$V_1(N_\sigma) \geq (\nu_1 - \gamma^n\overline{\nu})\sigma^n$$

for all σ and all $\gamma > 0$. If $V_2(B_\sigma)$ is the volume of B_σ at time t_2 then

$$V_2(B_\sigma) \geq V_2(N_\sigma).$$

Together these make

$$V_2(B_\sigma) \geq e^{-\varepsilon|t_2-t_1|}(V_1 - \gamma^n \overline{\nu})\sigma^n.$$

Fix $\gamma > 0$ and let $\sigma \to 0$. Then $\varepsilon \to 0$ and

$$\nu_2 = \lim_{\sigma \to \infty} V_2(B_\sigma)/\sigma^2 \geq \nu_1 - \gamma^n \overline{\nu}.$$

Since this is true for all $\gamma > 0, \nu_2 \geq \nu_1$. But we can switch t_1 and t_2, so $\nu_1 = \nu_2$ and ν is constant.

19 Ancient Solutions. There is one other geometric invariant we shall consider. Let 0 be an origin, s the distance to the origin, and R the scalar curvature. We define the asymptotic scalar curvature ratio

$$A = \limsup_{s \to \infty} Rs^2.$$

Again the definition is independent of the choice of an origin and invariant under dilation. This is particularly useful on manifolds of positive curvature where R bounds $|Rm|$. On Euclidean space $A = 0$, on a manifold which opens like a cone $0 < A < \infty$, and on a manifold which opens like a paraboloid $A = \infty$. Eschenberg, Shrader and Strake (**[18]**) have shown that on a complete odd-dimensional manifold of strictly positive sectional curvature $A > 0$; it is unknown whether this is true in even dimensions.

THEOREM 19.1. *For a complete solution to the Ricci Flow with bounded curvature which is ancient (defined for $-\infty < t < T$), and either with weakly positive curvature operator or Kähler with weakly positive holomorphic bisectional curvature, the asymptotic scalar curvature ratio A is constant.*

Proof. In either positive curvature case the Harnack estimate holds, and we conclude that the scalar curvature R is pointwise increasing. If the asymptotic curvature ratio is A at time t then for any finite $\widetilde{A} < A$ and any $\tilde{s}$ we can find a point P at distance $s \geq \tilde{s}$ from 0 at time t where $Rs^2 \geq \widetilde{A}$. At a later time $t + \Delta t$ with $\Delta t \geq 0$ the scalar curvature R at P is at least as big, while if M is a bound on the curvature everywhere the distance s of P from 0 will not have shrunk by more than $C\sqrt{M}\Delta t$. Since

$$\frac{s - C\sqrt{M}\Delta t}{s} \to 1 \quad \text{as} \quad s \to \infty$$

we see that the asymptotic scalar curvature ratio is at least A still at time $t+\Delta t$. Hence A does not decrease.

To see A does not increase either, first suppose at some time t that A is finite. Then for any $\widehat{A} > A$ we can find $\tilde{s} \geq 1/\sqrt{M}$ so that $Rs^2 \leq \widehat{A}$ for $s \geq \tilde{s}$

at time t. Moreover for any $\tilde{s}$ and any $\widetilde{A} < A$ we can again pick a point P at time t with $Rs^2 \geq \widetilde{A}$ and $s \geq 2\tilde{s}$. Consider any point Q at distance

$$d(P, Q, \tau) \leq s/2$$

for any $\tau \leq t$. Since $Rc \geq 0$, distances shrink and

$$d(P, Q, t) \leq s/2$$

also. Then

$$d(Q, 0, t) \geq s/2 \geq \tilde{s}$$

and by our choice of $\tilde{s}$

$$R(Q, t) \leq 4\widehat{A}/s^2$$

and since R increases pointwise

$$R(Q, \tau) \leq 4\widehat{A}/s^2$$

also. Our interior derivative estimates allow us to bound DR and also D^2R, and hence $\partial R/\partial t$. Recall from section 13 that if $|Rm| \leq M$ at all points at distance at most r from P for all times between $\tau - r^2$ and τ with $Mr^2 \leq 1$ then

$$\left|\frac{\partial}{\partial t} Rm\right| \leq CM/r^2$$

with a constant C depending only on the dimension. We can bound $|Rm|$ by R and take

$$M = C\widehat{A}/s^2.$$

When $\widehat{A} \leq 1/C$ we can take $r = s/2$; when $\widehat{A} \geq 1/C$ we can take $r = 1/2\sqrt{M} \leq s/2$. In the first case we find that

$$\frac{\partial R}{\partial t}(P, \tau) \leq C\widehat{A}/s^4$$

and in the second case we find

$$\frac{\partial R}{\partial t}(P, \tau) \leq C\widehat{A^2}/s^4$$

for some constant C depending only on n, at all $\tau \leq t$. Use $\widehat{A} + \widehat{A}^2$ for either case.

Pick $\Delta t \geq 0$. Then

$$R(P, t - \Delta t) \geq R(P, t) - C(\widehat{A} + \widehat{A^2})\Delta t/s^4.$$

Also

$$d(P, 0, t - \Delta t) \geq d(P, 0, t) = s.$$

Taking s very big compared to Δt and $\widetilde{A}$ and $\widehat{A}$ so that

$$R(P, t) \geq \widetilde{A}/s^2 \geq C(\widehat{A} + \widehat{A^2})\Delta t/s^4$$

we have

$$R(P, t - \Delta t)s(P, 0, t - \Delta t)^2 \geq s^2 \left[\frac{\widetilde{A}}{s^2} - \frac{C(\widehat{A} + \widehat{A}^2)\Delta t}{s^4}\right] \to \widetilde{A}$$

as $s \to \infty$. Hence lim sup $Rs^2 \geq A$ at time $t - \Delta t$ as well.

In the case where $A = \infty$ at time t, so that

$$\limsup_{s\to\infty} R(Q, t)d(Q, 0, t)^2 = \infty$$

we have to be more careful. For any $\widetilde{A} < \infty$ choose the largest $\tilde{s}$ so that

$$\sup\{R(Q, t)d(Q, 0, t)^2 : d(Q, 0, t) \leq \tilde{s}\} \leq \widetilde{A}.$$

That a largest $\tilde{s}$ exists is clear since if Q is any point at distance $\tilde{s}$ we can find Q_j at distance $\tilde{s}_j$ with $\tilde{s}_j \nearrow \tilde{s}$ and $Q_j \to Q$. Moreover since the sphere of radius $\tilde{s}$ is compact, there must exist a $\widetilde{Q}$ with

$$d(\widetilde{Q}, 0, t) = \tilde{s}$$

and

$$R(\widetilde{Q}, t)d(\widetilde{Q}, 0, t)^2 = \widetilde{A}$$

or else $\tilde{s}$ would not be maximal. Now choose P so that

$$d(P, 0, t) \geq \tilde{s}$$

and

$$R(P, t) \geq \frac{1}{2} \sup \{R(Q, t) : d(Q, 0, t) \geq \tilde{s}\}$$

which is possible since R is bounded. Since $\widetilde{Q}$ is a possible choice

$$R(P, t) \geq \frac{1}{2} R(\widetilde{Q}, t)$$

and then

$$R(P, t)\ d(P, 0, t)^2 \geq \frac{1}{2}\widetilde{A}.$$

If Q is any point with

$$d(P, Q, \tau) \leq \frac{1}{2} d(P, 0, t)$$

at some time $\tau \leq t$, then since distances shrink

$$d(P, Q, t) \leq \frac{1}{2} d(P, 0, t)$$

as well, and

$$\frac{1}{2} d(P, 0, t) \leq d(Q, 0, t) \leq \frac{3}{2} d(P, 0, t).$$

Either

$$d(Q, 0, t) \leq \tilde{s}$$

in which case

$$R(Q, t) d(Q, 0, t)^2 \leq \widetilde{A}$$

by our choice of $\tilde{s}$, and

$$R(Q, t) \leq 2\widetilde{A}/d(P, 0, t)^2 \leq 4R(P, t);$$

or else

$$d(Q, 0, t) \geq \tilde{s}$$

in which case

$$R(Q, t) \leq 2R(P, t)$$

by our choice of P; and so in either case

$$R(Q, t) \leq 4R(P, t).$$

Since R increases pointwise,

$$R(Q, \tau) \leq 4R(P, t)$$

for $\tau \leq t$ whenever

$$d(P, Q, \tau) \leq \frac{1}{2} d(P, 0, t).$$

Now we can use the interior derivative estimate again, for $\widetilde{A} \geq 1$ we get

$$\frac{\partial R}{\partial t}(P, \tau) \leq C\widetilde{A}^2 / d(P, 0, t)^4$$

and as before

$$R(P, 0, t - \Delta t) d(P, 0, t - \Delta t)^2 \geq \frac{1}{2}\widetilde{A} - C\widetilde{A}^2 \Delta t / d(P, 0, t)^2$$

where $d(P, 0, t)$ is large compared to Δt and $\widetilde{A}$. As $d(P, 0, t) \to \infty$ we see that

$$\limsup Rs^2 = \infty$$

a time $t - \Delta t$ as well. This finishes the proof of the Theorem. □

Now we prove several results that show an ancient solution with positive curvature operator whose scalar curvature R falls off rapidly in space and time behaves like a cone at infinity.

THEOREM 19.2. *Suppose we have a solution to the Ricci Flow on an ancient time interval $-\infty < t < T$, complete with bounded curvature and strictly positive curvature operator. Assume*

$$\limsup_{t \to -\infty} (T - t) R < \infty$$

(as happens in Type I) and assume the asymptotic scalar curvature ratio (which we saw is constant in time) is finite

$$A = \limsup_{s\to\infty} Rs^2 < \infty.$$

Then we get the following results:

(a) The asymptotic volume ratio (which we saw is constant in time) is strictly positive

$$\nu = \lim_{s\to\infty} V(B_s)/s^n > 0; \textit{ and}$$

(b) for any origin 0 and any time t there exists a constant $\phi(0,t) > 0$ such that at all points at the time t

$$Rs^2 \geq \phi(0,t).$$

Proof. We begin with a good estimate giving an upper bound on the curvature at all pairs of points and all time. □

LEMMA. *There exists a constant $\overline{C}$ such that for all points P and Q at all times $t \leq 0$ we have*

$$\min[R(P,t), R(Q,t)]d(P,Q,t)^2 \leq \overline{C}$$

where $d(P,Q,t)$ is the distance from P to Q at time t.

Proof. Since $A < \infty$, some constant C_0 works at $t = 0$, so

$$\min[R(P,0), R(Q,0)]d(P,Q,0)^2 \leq C_0$$

for all P and Q. Since R increases pointwise,

$$R(P,t) \leq R(P,0) \quad \text{and} \quad R(Q,t) \leq R(Q,0)$$

for $t \leq 0$. Since $R \leq C/(T-t)$, we can use Theorem 1.72 to get

$$d(P,Q,t) \leq d(P,Q,0) + C\sqrt{T-t}\ .$$

This makes

$$d(P,Q,t)^2 \leq 2d(P,Q,0)^2 + C(T-t).$$

Thus

$$\begin{aligned} &\min[R(P,t), R(Q,t)]d(P,Q,t)^2 \\ &\leq 2\min[R(P,0), R(Q,0)]d(P,Q,0)^2 \\ &\quad + C\min[R(P,t), R(Q,t)](T-t) \leq \overline{C} \end{aligned}$$

for some constant $\overline{C}$ using the bound on the first term at $t = 0$ and the bound $R \leq C/(T-t)$ everywhere. □

LEMMA. *There exists a constant $c > 0$ such that for every $t \leq 0$ we can find a point P_t where*

$$R(P_t, t) \geq c/(T-t).$$

Proof. The maximum $R_{\max}$ of R satisfies the ordinary differential inequality

$$\frac{d}{dt} R_{\max} \leq C R_{\max}^2$$

for some constant C, by applying the maximum principle to the evolution of R. If $R_{\max}(t)$ were even smaller than $c/(T-t)$ for c small, it could not make it up to $R_{\max}(0)$ in time. □

Now fix an origin 0 and let $s = d(P,t) = d(P,0,t)$ be the distance of P to the origin at time t.

LEMMA. *There exists a constant C^* so that $Rs^2 \leq C^*$ for all $t \leq 0$.*

Proof. Since

$$R(P_t, 0) \geq R(P_t, t) \geq c/(T-t)$$

while

$$\min[R(P_t, 0), R(0,0)] d(P_t, 0, 0)^2 \leq C_0$$

we get an estimate

$$d(P_t, 0, 0) \leq C\sqrt{T-t}$$

(where the case $R(P_t, 0) \geq R(0,0)$ can be handled separately because $R(0,0) \leq C/T$ anyway while $T - t \geq T$). Then using our distance shrinking bound

$$d(P_t, 0, t) \leq C\sqrt{T-t}$$

for a larger constant C. For any P

$$d(P, 0, t) \leq d(P, P_t, t) + d(P_t, 0, t)$$

by the triangle inequality. We already have

$$\min[R(P,t), R(P_t, t)] d(P, P_t, t)^2 \leq \overline{C}$$

for some constant $\overline{C}$ independent of t. If

$$R(P,t) \geq R(P_t, t) \geq c/(T-t)$$

then the same argument that worked for P_t proves that

$$d(P, 0, t) \leq C\sqrt{T-t}$$

and since $R \leq C/(T-t), Rs^2 \leq C^*$ for some C^*. The other case when

$$R(P,t) \leq R(P_t,t)$$

gives

$$R(P,t)d(P,P_t,t)^2 \leq \overline{C}$$

in the estimate above, and since

$$d(P,0,t) \leq d(P,P_t,t) + c\sqrt{T-t}$$

and $R(P,t) \leq C/\sqrt{T-t}$, we get

$$R(P,t)d(P,0,t)^2 \leq C^*$$

also for some C^*. This proves this Lemma. □

Now we turn to the volume estimate. It is useful first to look at annuli.

LEMMA. *There exists a constant $c > 0$ such that the annulus at $t = 0$*

$$N_\sigma = \{\sigma \leq s \leq 3\sigma\}$$

has volume

$$V(N_\sigma) \geq c\sigma^n.$$

Proof. Let $\varepsilon > 0$ be a small constant we can choose later. Look at time

$$\tau = -\varepsilon\sigma^2$$

at the annulus

$$\widehat{N}_\sigma = \{2\sigma \leq s \leq 3\sigma\}.$$

Since distances shrink as t increases from τ to 0, the outer sphere of $\widehat{N}_\sigma$ surely lies inside the outer sphere of N_σ. But we have seen

$$d(P,0,\tau) \geq d(P,0,0) - C\sqrt{T-\tau}$$

and so if σ is large (which is our only concern), in particular $\sigma \geq \sqrt{T/\varepsilon}$, then $T - \tau \leq 2|\tau| \leq 2\varepsilon\sigma^2$ and

$$d(P,0,\tau) \geq d(P,0,0) - C\sqrt{2\varepsilon}\,\sigma.$$

Choose ε so small that $C\sqrt{2\varepsilon} \leq 1$. Then

$$d(P,0,\tau) \geq d(P,0,0) - \sigma$$

so no distance from the origin shrinks by more than σ. Hence the inner sphere of $\widehat{N}_\sigma$ lies outside the inner sphere of N_σ, and $\widehat{N}_\sigma \subseteq N_\sigma$. (Of course we don't need these to be topological annuli, we only estimate distances.) □

Next we claim we can find $\delta > 0$ (depending on the ε we choose) so that $\widehat{N}_\sigma$ has volume

$$V(\widehat{N}_\sigma) \geq \delta\sigma^n$$

at time $\tau = -\varepsilon\sigma^2$. Since the curvature (for $\sigma \geq \sqrt{T/\varepsilon}$ again) satisfies a bound

$$R \leq C/|\tau| \leq C/\varepsilon\sigma^2,$$

this remark follows from the following result by dilation, with $\delta = \zeta(\varepsilon/C)^{n/2}$.

LEMMA. *For every $\rho > 0$ there exists a $\zeta > 0$ so that if a complete manifold with positive sectional curvature has $0 < R \leq 1$, then the annulus*

$$\widehat{N}_\rho = \{2\rho \leq s \leq 3\rho\}$$

has volume

$$V(\widehat{N}_\rho) \geq \zeta.$$

Proof. Since the manifold is complete with positive curvature but not compact, we can bound the injectivity radius by some apriori constant $c > 0$ below. The annulus contains a minimal geodesic of length ρ, as we see by intersecting it with a ray to infinity. If $\rho \leq c/2$ the result is easy using geodesic coordinates at the origin, while if $\rho > c/2$ we can put a ball of radius $c/2$ inside the annulus. (In fact for large ρ we see the area is at least a constant times ρ. This is the best we can do if the manifold opens like a cylinder.) □

Now we want to see that $V(\widehat{N}_\sigma)$ still has a large area at $t = 0$. At each time $\tau \leq t \leq 0$ we still have all of $\widehat{N}_\sigma$ outside the ball of radius σ, where $R \leq C^*/\sigma^2$. Therefore we can estimate the rate at which the volume shrinks by

$$\frac{d}{dt}V(\widehat{N}_\sigma) = -\int_{\widehat{N}_\sigma} R\,da \geq -\frac{C^*}{\sigma^2}\,V(N_\sigma).$$

This makes

$$V(\widehat{N}_\sigma)\Big|_{t=0} \geq e^{-C^*\tau/\sigma^2}V(\widehat{N}_\sigma)\Big|_{t=\tau}.$$

Since $\tau = \varepsilon\sigma^2$ we get

$$V(\widehat{N}_\sigma)\Big|_{t=0} \geq cV(\widehat{N}_\sigma)\Big|_{t=\tau} \geq c\delta\sigma^n.$$

But at $t = 0, V(B_\sigma) \geq V(N_\sigma) \geq V(\widehat{N}_\sigma)$ so 19.2(a) is done. Next we look at 19.2(b). Given a point P at distance $\sigma = d(P,0,0)$ from the origin at time $t = 0$, we let $\tau = -\varepsilon\sigma^2$ as before and find P_τ where

$$R(P_\tau,\tau) \geq c/(T-\tau) \quad \text{and} \quad d(P_\tau,0,\tau) \leq c\sqrt{T-\tau}.$$

The Harnack inequality on a manifold with positive curvature operator in its integrated form (see [**29**]) gives

$$R(P,0) \geq R(P_\tau,\tau)e^{-Cd(P_\tau,P,\tau)^2/|\tau|}$$

for some constant C. The triangle inequality gives

$$d(P_\tau, P, \tau) \leq d(P_\tau, 0, \tau) + d(P, 0, \tau)$$

and

$$d(P, 0, \tau) \leq d(P, 0, 0) = \sigma.$$

Then

$$d(P_\tau, P, \tau) \leq \sigma + C\sqrt{T - \tau}.$$

Again if $\sigma \geq \sqrt{T/\varepsilon}$ we have $T - \tau \leq 2|\tau|$ and

$$d(P_\tau, P, \tau) \leq C\sigma$$

for some constant C, making

$$d(P_\tau, P, \tau)/|\tau| \leq C$$

for some other constant C depending on ε. This yields $R(P, 0) \geq c/\sigma^2$ as desired. For $\sigma \leq \sqrt{T/\varepsilon}$ some constant $c > 0$ works because $R > 0$. Hence the Lemma is proved. A similar bound can be derived at any time.

20 Ricci Solitons. We will now examine the structure of a steady Ricci soliton of the sort we frequently get as a limit.

THEOREM 20.1. *Suppose we have a complete Ricci soliton with bounded curvature, so that*

$$D^2 f = Rc$$

for some function f. Assume the Ricci curvature is weakly positive

$$Rc \geq 0$$

and assume the scalar curvature attains its maximum M at an origin. Then the function f is weakly convex and attains its minimum at the origin, and furthermore

$$|Df|^2 + R = M$$

everywhere on the soliton. The soliton is not compact unless $Rc = 0$.

Proof. We show the equality first. Since

$$D_i D_j f = R_{ij}$$

we have

$$D_i D_j D_k f = D_i R_{jk}$$

and

$$D_i D_j D_k f - D_j D_i D_k f = D_i R_{jk} - D_j R_{ik}$$

and

$$D_i D_j D_k f - D_j D_i D_k f = R_{ijk\ell} D_\ell f$$

so

$$D_i R_{jk} - D_j R_{ik} = R_{ijk\ell} D_\ell f.$$

Taking a trace on j and k, and using the contracted second Bianchi identity

$$D_j R_{ij} = \frac{1}{2} D_i R$$

we get that

$$D_i R + 2R_{ij} D_j f = 0.$$

Then

$$D_i(|Df|^2 + R) = 2D_j f(D_i D_j f - R_{ij}) = 0$$

so $|Df|^2 + R$ is constant. Call it M^*.

If $M^* = M$, then $Df = 0$ at the origin. Since $D_i D_j f = R_{ij} \geq 0$, along any geodesic through the origin $x^i = x^i(s)$ parameterized by arc length s we have

$$\frac{df}{ds} = D_i f \cdot \frac{dx^i}{ds}$$

and

$$\frac{d^2 f}{ds^2} = D_i D_j f \cdot \frac{dx^i}{ds} \frac{dx^j}{ds} \geq 0$$

so f is convex and hence least at the origin. Since any point can be joined to the origin by a geodesic, we are done in this case.

If $M^* > M$, consider a gradient path of f through the origin $x^i = x^i(n)$ parametrized by the parameter u with x^i at the origin at $u = 0$ and

$$\frac{dx^i}{du} = g^{ij} D_j f.$$

Now $|Df|^2 = M^* - R$ so $|Df|^2 \geq M^* - M > 0$ everywhere, while $|Df|^2$ is smallest at the origin. But we compute

$$\frac{d}{du}|Df|^2 = 2g^{ik} g^{j\ell} R_{ij} D_k f D_\ell f \geq 0$$

since $R_{ij} \geq 0$ and $|Df|^2 \geq 0$. Then $|Df|^2$ isn't smaller at the origin, and we have a contradiction.

If the solution is compact then

$$\Delta f = R \geq 0$$

implies f is constant, so $Rc = D^2 f = 0$. □

THEOREM 20.2. *For a complete Ricci soliton with bounded curvature and strictly positive sectional curvature of dimension $n \geq 3$ where the scalar curvature assumes its maximum at an origin, the asymtotic scalar curvature ratio is infinite;*

$$A = \limsup_{s \to \infty} R s^2 = \infty$$

where s is the distance to the origin.

Proof. Suppose $Rs^2 \le C$. The solution to the Ricci Flow corresponding to the soliton exists for $-\infty < t < \infty$ and is obtained by flowing along the gradient of f. We will show that the limit

$$\bar{g}_{ij}(x) = \lim_{t \to -\infty} g_{ij}(x,t)$$

exists for $x \ne 0$ on the manifold $\mathcal{X}$ and is a flat metric on $\mathcal{X} - \{0\}$ which is complete. Since $\mathcal{X}$ has positive curvature operator it is diffeomorphic to R^n, and $\mathcal{X} - \{0\}$ to $S^{n-1} \times R^1$. For $n \ge 3$ there is no flat metric on this space, and this will finish the proof. □

To see the limit metric exists, note that unless $Rs^2 \to \infty$ as $s \to \infty$, surely $R \to 0$ as $X \to \infty$ so $|Df|^2 \to M$ as $X \to \infty$, at least at $t = 0$. The function f itself can be taken to evolve with time, using the definition

$$\frac{\partial f}{\partial t} = -|Df|^2 = \Delta f - M$$

which pulls f back by the flow along the gradient of f. Then we continue to have $D_i D_j f = R_{ij}$ for all time, and $|Df|^2 \to M$ as $s \to \infty$ for each time.

When we go backwards in time, this is equivalent to flowing outwards along the gradient of f, and our speed approaches $\sqrt{M}$. If s is the distance from 0, then $s/|t| \to \sqrt{M}$. Since $Rs^2 \le C$ for some constant C, $R \le C/s^2$, and starting outside of any neighborhood of 0 we have $R \le C/M|t|^2$ and hence

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \ge -2Rg_{ij}$$

gives

$$0 \ge \frac{\partial}{\partial t} g_{ij} \ge -\frac{2C}{M|t|^2}\, g_{ij}.$$

If V is a tangent vector and $|V|_t$ denotes its length at time t, so

$$|V|_t^2 = g_{ij}(t) V^i V^j$$

then

$$0 \ge \frac{d}{dt}|V|_t^2 \ge -\frac{2C}{M|t|^2}|V|_t^2$$

so

$$0 \le \frac{d}{d|t|} \log |V|_t^2 \le \frac{2C}{M|t|^2}$$

with $t < 0$ decreasing and $|t|$ increasing. This makes $|V|_t^2$ increasing in $|t|$ with

$$\frac{d}{d|t|}\left(\log |V|_t^2 + \frac{2C}{M|t|}\right) \le 0$$

so that

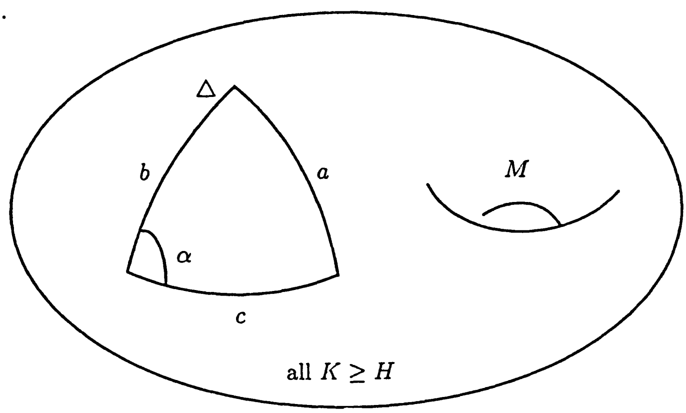
$$\log |V|_t^2 + \frac{2C}{M|t|}$$

is actually decreasing. This shows $|V|_t$ has a limit as $t \to -\infty$.

Since the metrics are all essentially the same, it always takes an infinite length to get out to ∞. On the other hand, any point X other than 0 will eventually be arbitrarily far from 0, so the metric in the limit is also complete away from 0 in $\mathcal{X} - \{0\}$. Using the derivative estimates of W.-X. Shi [**43**] on the curvature it is straightforward to see that the $g_{ij}(X,t)$ converge in C^∞ to a smooth limit metric $\overline{g}_{ij}(X)$ as $t \to -\infty$. Since $R \leq C/s^2$ and $s \approx \sqrt{M}t$ we have the result that the limit metric is flat. This proves the theorem.

21 Bumps of Curvature. We shall show an interesting fact in this section about the influence of a bump of strictly positive curvature in a complete manifold of weakly positive curvature. Namely, minimal geodesic paths that go past the bump have to avoid it. As a consequence we get a bound on the number of bumps of curvature. This principle will be important for studying the behavior of singularity models at infinity when we do a dimension reduction argument.

We begin by reviewing Toponogov's Theorem as given in Cheeger and Ebin [**9**]. Let M be a complete Riemannian manifold with all sectional curvatures K bounded below by a constant H. Suppose we have a geodesic triangle $\triangle$ in M with sides of lengths a, b, and c, and let α be the angle opposite the side of length a.



We make the following assumptions

(1) the geodesics of lengths a and b are minimal

(2) $c \leq a + b$ (surely true if the geodesic of length c is also minimal) and

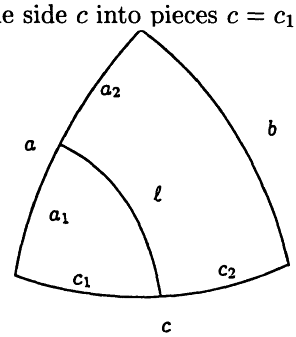
(3) $c \leq \pi/\sqrt{H}$ if $H > 0$.

THEOREM 21.1. *There exists a traingle $\overline{\triangle}$ in the space $\overline{M}$ with constant curvature H whose sides have length a, b and c, such that the angle $\overline{\alpha}$ in $\overline{\triangle}$ opposite the side of length a satisfies $\overline{\alpha} \leq \alpha$.*

THEOREM 21.2. *There exists a unique triangle $\overline{\triangle}$ in $\overline{M}$ with sides b and c and angle α, such that the length $\overline{a}$ of the side opposite α satisfies $\overline{a} \geq a$.*

REMARK. . It is not necessary to have sectional curvatures $\kappa \geq H$ in all of

M; it suffices to have this hold in the ball of radius $a + b$ around any point in the triangle; because the construction only uses κ on minimal geodesics joining two points on Δ, and these all lie in such a ball. To see this, consider a geodesic triangle with sides a, b, and $c \leq a + b$. If we join a point on the side a to a point on the side b with a minimal geodesic of length ℓ, clearly $\ell \leq a + b$. If we join a point on the side a to a point on the side c with a minimal geodesic of length ℓ, and if the first point divides the side a into pieces $a = a_1 a_2$, and likewise the second point divides the side c into pieces $c = c_1 + c_2$, then



$\ell \leq a_1 + c_1$ and $\ell \leq a_2 + b + c_2$ and by averaging

$$\ell \leq \frac{1}{2}(a + b + c) \leq a + b$$

as claimed.

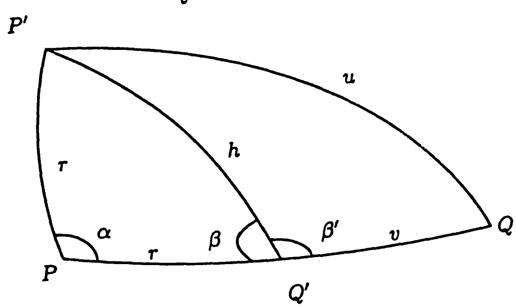
LEMMA 21.3. *For every $\varepsilon > 0$ there exist $\lambda < \infty$ and $\delta > 0$ such that if M is complete with $K \geq 0, P$ is a point in M and $K \geq \varepsilon / r^2$ everywhere in $B_{2r}(P)$, if $d(P, P') = r$ and if $d(P, Q) \geq \lambda r$, if PP', PQ and $P'Q$ are minimal geodesics and if*

$$\angle P'PQ \leq \frac{\pi}{2} + \delta$$

then

$$d(P', Q) < d(P, Q).$$

Proof. Pick a point Q' on the geodesic PQ at distance r from P, and choose a minimal geodesic from P' to Q'. □



Let $h = |P'Q'|, u = |P'Q|$ and $v = |QQ'|$ and let $\alpha = \angle P'PQ$ and $\beta = \angle PQ'P'$ and $\beta' = \pi - \beta$. We make three applications of Toponogov's Theorems.

(1) First note for every $\varepsilon > 0$ we can find $\delta > 0$ and $\eta > 0$ such that if $\alpha \leq \frac{\pi}{2} + \delta$ then $h \leq (\sqrt{2} - \eta)\, r$. This is because $K \geq \varepsilon/r^2$ in $B_{2r}(P)$ and we can compare the triangle $P'PQ'$ to the triangle with two sides equal to r and angle α in the sphere of curvature $H = \varepsilon/r^2$ using $T2$. All the sides are minimal, and we only need to check that

$$h \leq 2r \leq \pi/\sqrt{H}$$

if $\varepsilon < 1 < (\pi/2)^2$. Hence the comparison can be made.

Now on the sphere of radius 1, take an isosceles triangle of equal sides $\ell \leq 1$ with angle $\alpha \leq \frac{\pi}{2} + \delta$ between them and call the length of the third side k. In an isosceles right triangle k is strictly less than the Euclidean value of $\sqrt{2}\,\ell$, and hence depending on ℓ we can find $\delta > 0$ and $\eta > 0$ such that if $\alpha \leq \frac{\pi}{2} + \delta$ then still $k \leq (\sqrt{2} - \eta)\ell$. If we scale the result to a sphere of radius $r/\sqrt{\varepsilon}$ with curvature $H = \varepsilon/r^2$, then taking $\ell = \sqrt{\varepsilon}$ gives the desired result.

(2) Now we just use $K \geq 0$. We compare the triangle $P'Q'P$ with two sides equal to r and one equal to $h \leq (\sqrt{2} - \eta)r$ to the Euclidean triangle with the same three sides using $T1$. again all the sides are minimal, and we can do the comparison. We find that there exists a $\theta > 0$ depending on η only so that $\beta \geq \frac{\pi}{4} + \theta$. By scaling it suffices to observe that an isosceles Euclidean triangle with two equal sides 1 and the third side less than $\sqrt{2} - \eta$ has the equal angles at least $\frac{\pi}{4} + \theta$.

(3) Finally we use $T2$ again to compare the triangle $P'Q'Q$ to the Euclidean triangle with sides h and v and angle $\beta' \leq \frac{3\pi}{4} - \theta$. Again all the sides are minimal, and we find

$$u^2 \leq h^2 + v^2 - 2hv \cos\beta'.$$

Now $h \leq \sqrt{2}r$ while

$$\cos\beta' \geq -\frac{1}{\sqrt{2}} + \zeta$$

for some $\zeta > 0$ depending only on $\theta > 0$.

Therefore

$$u^2 \leq (v + r)^2 + r[r - 2\sqrt{2}\zeta v]$$

and for every $\zeta > 0$ we can choose $\lambda < \infty$ so that if

$$v + r = |PQ| \geq \lambda r$$

then $v \geq (\lambda - 1)r$ and $2\sqrt{2}\zeta v > r$. Thus $|P'Q| = u < v + r = |PQ|$ as desired.

Now we prove an important repulsion principle.

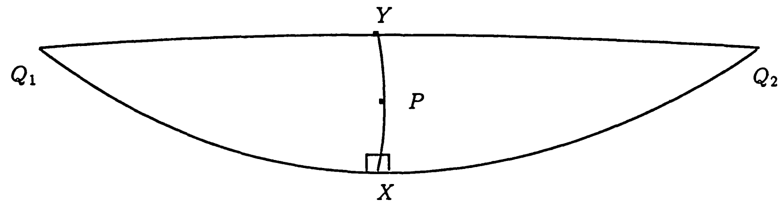
THEOREM 21.4. *For every $\varepsilon > 0$ we can find $\lambda < \infty$ such that if M is a complete Riemannian manifold with $K \geq 0$, if P is a point in M such that $K \geq \varepsilon/r^2$ everywhere in $B_{3r}(P)$, if $s \geq r$ and Q_1 and Q_2 lie outside $B_{\lambda s}(P)$ and γ is a minimal geodesic from Q_1 to Q_2, then γ stays outside $B_s(P)$.*

Proof. Let X be the closest point on Q_1Q_2 to P. Draw a minimal geodesic from X to P and let its length be σ. Extend the geodesic XP an equal length

σ beyond P, ending at a point Y. Draw minimal geodesics Q_1Y and Q_2Y. We claim

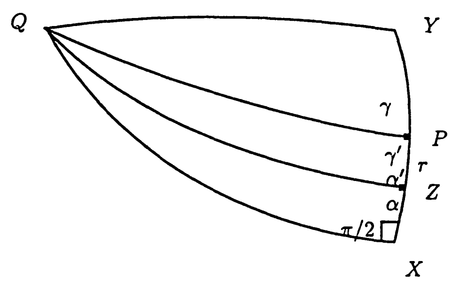
$$|Q_1Y| < |Q_1X| \quad \text{and} \quad |Q_2Y| < |Q_2X|$$

which will show Q_1Q_2 is not minimal, provided $\sigma \le s$.



Since both halves of the argument are the same, we drop the subscripts 1 and 2.

Consider the geodesic triangle QXY with P the midpoint of XY, where QX and QY and PX are minimal and $\angle QXP = \pi/2$.



Choose the point Z at distance r from P towards X, and draw minimal geodesics QP and QZ. Let $\alpha = \angle QZX$ and $\alpha' = \pi - \alpha$, while $\gamma = \angle QPY$ and $\gamma' = \pi - \gamma$. Again we make several applications of Toponogov's Theorems.

First note that

$$|QP| \ge \lambda s \quad \text{and} \quad |PX| \le s$$

so

$$|QZ| \ge |QP| - |PZ| \ge (\lambda - 1)s$$

and

$$|QX| \le |QP| + |PX| \le (\lambda + 1)s.$$

Therefore comparing the triangle QZX to the Euclidean one with the same three sides, we find by $T2$ that for every $\delta > 0$ there exists a $\lambda < \infty$ such that $\alpha \ge \frac{\pi}{2} - \delta$, as is easily seen by first comparing the Euclidean triangle to one of sides proportional to $\lambda + 1, \lambda - 1$, and 1 with a more extreme angle α, and observing $\alpha \to \pi/2$ as $\lambda \to \infty$. Consequently $\alpha' \le \frac{\pi}{2} + \delta$.

Now choosing δ small and λ large compared to ε, and noting that if $K \ge \varepsilon/r^2$ in $B_{3r}(P)$ then

$$B_{3r}(P) \supseteq B_{2r}(Z),$$

we see that Lemma 1 implies $|QP| < |QZ|$. Now if we also had $\gamma' \le \frac{\pi}{2} + \delta$ we would also have $QZ < QP$ by Lemma 3, and we cannot have both. Hence $\gamma' \ge \frac{\pi}{2} + \delta$ and this gives $\gamma' \le \frac{\pi}{2} - \delta$.

Now we apply Toponogov's Theorem 21.2 to the triangle QPY to compare it to the Euclidean triangle of sides $|QP|$ and $|PY|$ and angle γ. We do not know if PY is minimal, but QP and QY are by construction, and

$$|PY| = \sigma \leq s \quad \text{while} \quad |QP| \geq \lambda s$$

and hence $|PY| \leq |QP| + |QY|$, which is all we need. Then by the law of cosines

$$|QY|^2 \leq |QP|^2 + |PY|^2 - 2|QP| \cdot |PY| \cdot \cos\gamma.$$

But we also have

$$|QP|^2 \leq |QX|^2 + |PX|^2$$

by $T1$ on the triangle of sides QX and PX and angle $\pi/2$. Then

$$|QY|^2 \leq |QX|^2 + |PX|^2 + |PY|^2 - 2|QP| \cdot |PY| \cdot \cos\gamma.$$

Use $|PX| = |PY| = \sigma \leq s$ and $|QP| \geq \lambda s$ and $\gamma \leq \frac{\pi}{2} - \delta$ to get

$$|QY|^2 \leq |QX|^2 + 2\sigma^2 \left[1 - \lambda \cos\left(\frac{\pi}{2} - \delta\right)\right].$$

Picking λ large compared to δ, we get

$$\lambda \cos\left(\frac{\pi}{2} - \delta\right) = \lambda \sin\delta > 1$$

and $|QY| < |QX|$ as desired. This proves the theorem. □

We apply the previous repulsion theorem to prove a result on remote curvature bumps in complete manifolds of positive curvature.

DEFINITION 21.5 . *A ball $B_r(P)$ of radius r around P is a curvature β-bump if $K \geq \beta/r^2$ at all points in the ball. The ball is λ-remote from the origin 0 if $d(P, 0) \geq \lambda r$.*

THEOREM 21.6. *For every $\beta > 0$ there exists $\lambda < \infty$ such that in any complete manifold of positive curvature there are at most a finite number of disjoint balls which are λ-remote curvature β-bumps.*

Proof. If the ball $B_r(P)$ is a λ-remote curvature β-bump, and if Q is any point such that

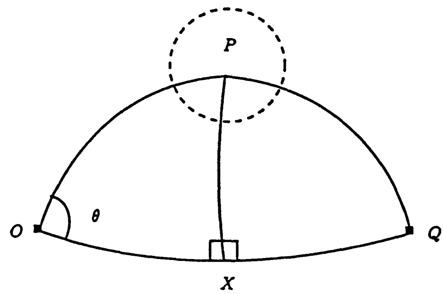
$$d(0, Q) \geq 2d(0, P)$$

then if we take minimal geodesics $0P$ and OQ, we claim that for any $\beta > 0$ we can find $\lambda < \infty$ and $\theta > 0$ such that

$$\angle POQ \geq \theta.$$

To see this, let X be the point on OQ with $OX = OP$. Since $K \geq 0$ everywhere and ΔOPQ has minimal sides, if the angle $\angle POQ$ is $< \theta$,then for every $\lambda < \infty$

we can find $\theta > 0$ such that $PX < OP/\lambda$. But this contradicts the repulsion theorem.



Note there is a curvature β-bump at P, $OP > \lambda PX$ and

$$PQ \geq OQ - OP \geq OP$$

so the theorem applies (with $\varepsilon = \beta/9$ to get $K \geq \varepsilon/\rho^2$ on the ball of radius $\rho = r/3$).

Now pick any sequence P_j of curvature β-bumps with

$$d(O, P_{j+1}) \geq 2d(O, P_j)$$

and we find for $j < k$ the angle

$$\angle P_j O P_k \geq \theta$$

for a fixed $\theta > 0$. This is impossible. Hence there cannot be an infinite sequence of λ-remote disjoint curvature β-bumps; for since K is bounded on any compact set and $r \geq \sqrt{\varepsilon/K}$ on each bump, we can only get a finite number of disjoint bumps into any compact set, and this lets us find P_{j+1} with $d(P_{j+1}, 0) \geq 2d(P_j, 0)$. Thus the theorem is proved. □

22 Dimension Reduction. There is a general principle of dimension reduction which has proved useful in minimal surface theory and also the theory of Harmonic maps. The idea is that having first taken a limit of a sequence of dilations to model a singularity, we should study this limit by next taking a sequence of origins going out to infinity and shrinking back down to get a new limit of lower dimension. On a complete manifold the idea is that in dimension at least three, as we go out to infinity the radial curvatures will fall off faster than the meridian curvature, so the new limit of the contractions will be flat in the radial direction. We will illustrate this idea by proving a result on solutions with positive curvature operator, where the Little Loop Lemma gives injectivity radius control; but the same idea will work in any other case where we can control the injectivity radius.

THEOREM 22.1. *Suppose we have a solution to the Ricci Flow on a compact manifold M^m of dimension m with weakly positive curvature operator for a* *maximal time interval* $0 \leq t < T$. *Then we can find a sequence of dilations which converge to a complete solution of the Ricci Flow with curvature bounded at each time on an ancient time interval* $-\infty < t < \Omega$ *with scalar curvature* R *bounded by*

$$R \leq \Omega/(\Omega - t)$$

everywhere and $R = 1$ *at some origin* O *at time* $t = 0$, *which again has weakly positive curvature operator. Moreover the limit splits as a quotient of a product* $N^n \times R^k$ *with* $m = n + k$ *flat in the directions* R^k *with* $k \geq 0$, *and where the interesting factor* N^n *either is compact or has finite asymptotic curvature ratio*

$$\lim_{s \to \infty} Rs^2 = A < \infty.$$

Moreover the limit factor N^n *will still satisfy a local injectivity radius estimate.*

Of course we conjecture the only possible limit is the round sphere S^n or a quotient of it shrinking to a point. In dimension 3 or 4 we have pinching estimates that keep the curvature operator strictly positive if it starts strictly positive, that prevent limits $N^n \times R^k$ with $k > 0$. We do not know any examples of complete non compact ancient solutions of positive curvature operator with $Rs^2 < \infty$ and $R|t| < \infty$, and we conjecture none exist, since the curvature has had plenty of space and time to dissipate.

Proof. The Little Loop Lemma gives us a bound on the injectivity radius in terms of the local maximum of the curvature; if $R \leq 1/r^2$ in the ball of radius r around a point P, then the injectivity radius at P is at least δr for some $\delta > 0$. This allows us to take limits by dilating to make the maximum curvature 1. From the results in section 16 we get a limit solution of Type I or Type II . Any such limit will split as a product $N^n \times R^k$ with $k \geq 0$ as large as possible, and where N^n has strictly positive sectional curvature; for any zero sectional curvature is a zero eigenvector of the curvature operator, producing a reduction of the holonomy to the nilgroup $\mathcal{O}(n) \subseteq \mathcal{O}(m)$. Among all possible Type I or II limits choose one where k is maximal. We shall then get a contradiction unless N^n has finite asymptotic scalar curvature ratio $A < \infty$. We have seen in Corollary 16.6 that a Type II limit with weakly positive curvature operator must be a Ricci soliton, and in Theorem 19.2 we have seen that in dimension $n \geq 3$ such a Ricci soliton must have $A = \infty$. In dimension $n = 2$ the only Ricci soliton is the cigar $\sum^2$ (see [**22**]) which does not satisfy the local injectivity radius bound, since R goes to zero exponentially in the distance s from the origin, while the circumference of the circle at distance s approaches 1 as it opens like a cylinder. Thus if we prove N^n is compact or has $A < \infty$, it must be Type I. Suppose therefore that N^n is not compact and $A = \infty$, and we shall contradict k maximal.

We shall pick a sequence of dilations of N^n which converges to a limit with a flat factor. We need the following result.

LEMMA 22.2. *Given a complete noncompact solution to the Ricci Flow on an ancient time interval* $-\infty < t < T$ *with* $T > 0$ *with curvature bounded at* *each time and with asymptotic scalar curvature ratio*

$$A = \limsup_{s\to\infty} Rs^2 = \infty$$

we can find a sequence of points $P_j \to \infty$ at time $t = 0$, a sequence of radii r_j and a sequence of numbers $\delta_j \to 0$ such that

(a) $R(P, 0) \leq (1 + \delta_j)R(P_j, 0)$ for all P in the ball $B_r(P_j, 0)$ of radius r_j around P_j at time $t = 0$

(b) $r_j^2 R(P_j, 0) \to \infty$

(c) if $s_j = d(P_j, O, 0)$ is the distance of P_j from some origin O at time $t = 0$, then $\lambda_j = s_j/r_j \to \infty$

(d) the balls $Br_j(P_j, 0)$ are disjoint.

Proof. Pick a sequence $\varepsilon_j \to 0$, then choose $A_j \to \infty$ so that $A_j\varepsilon_j^2 \to \infty$ as well. As in Theorem 18.2, let σ_j be the largest number such that

$$\sup\{R(Q, 0)d(Q, O, 0)^2 : d(Q, O, 0) \leq \sigma_j\} \leq A_j.$$

Then

$$R(P, 0)d(Q, O, 0)^2 \leq A_j \quad \text{if} \quad d(P, O, 0) \leq \sigma_j$$

while there exists some Q_j with

$$R(Q_j, 0)d(Q, O, 0)^2 = A_j \quad \text{and} \quad d(Q, O, 0) = \sigma_j$$

(or else σ_j would not be maximal). Now pick P_j so that $d(P_j, O, 0) \geq \sigma_j$ and

$$R(P_j, 0) \geq \frac{1}{1 + \varepsilon_j} \sup\{R(Q, 0) : d(Q, O, 0) \geq \sigma_j\}$$

which is possible since even on a noncompact set we can come as close to the sup as we wish. Finally pick $r_j = \varepsilon_j\sigma_j$.

First we check (a). If P is in the ball of radius r_j around P_j at time $t = 0$, either $d(P, O, 0) \geq \sigma_j$ or $d(P, O, 0) \leq \sigma_j$. In the first case we have from the choice of P_j

$$R(P, 0) \leq (1 + \varepsilon_j)R(P_j, 0)$$

which satisfies condition (a) with $\delta_j = \varepsilon_j$. In the second case, we have from the choice of σ_j

$$R(P, 0) \leq A_j/d(P, O, 0)^2$$

and

$$d(P, O, 0) \geq d(P_j, O, 0) - d(P, P_j, 0) \geq \sigma_j - r_j = (1 - \varepsilon_j)\sigma_j$$

so

$$R(P, 0) \leq \frac{1}{(1 - \varepsilon_j)^2} \cdot \frac{A_j}{\sigma_j^2}.$$

On the other hand, from the choice of Q_j

$$R(Q_j, 0) = \frac{A_j}{\sigma_j^2}$$

and from the choice of P_j

$$R(P_j, 0) \geq \frac{1}{1+\varepsilon_j} R(Q_j, 0)$$

since Q_j is a possible choice of Q, then

$$R(P_j, 0) \geq \frac{1}{1+\varepsilon_j} \cdot \frac{A_j}{\sigma_j^2}$$

and

$$R(P, 0) \leq \frac{1+\varepsilon_j}{(1-\varepsilon_j)^2} R(P_j, 0)$$

which satisfies condition (a) with

$$\delta_j = \frac{1+\varepsilon_j}{(1-\varepsilon_j)^2} - 1$$

and in either case $\delta_j \to 0$ as $\varepsilon_j \to 0$.

Next we check condition (b). We have from our previous estimate

$$r_j^2 R(P_j, 0) \geq \frac{\varepsilon_j^2}{1+\varepsilon_j} A_j \to \infty$$

by our choice of A_j. To check condition (c) note $s_j \geq \sigma_j$ so that $\lambda_j \geq 1/\varepsilon_j \to \infty$. Finally note that (a), (b) and (c) continue to hold if we pass to a subsequence. Any point P in $B_{r_j}(P_j, 0)$ has distance from the origin at time 0

$$d(P, 0, 0) \geq d(P_j, 0, 0) - d(P, P_j, 0) \geq (1-\varepsilon_j)\sigma_j$$

and since $A_j \to \infty$ we must have $\sigma_j \to \infty$. Thus any fixed compact set does not meet the balls $Br_j(P_j, 0)$ for large enough j. If we pass to a subsequence, the balls will all avoid each other. This proves the Lemma. □

The next step is to take a sequence of dilations of the limit factor N^n around a sequence of points P_j which we take as our new origins O_j, only now we shrink down instead of expanding to make $R(P_j, 0)$ dilate to $R(O_j, 0) = 1$. The points P_j are chosen at time $t = 0$ according to the previous Lemma. The balls $B_{r_j}(P_j, 0)$ dilate to balls of radius $\tilde{r}_j \to \infty$ by condition 4(b).

Condition (a) gives good bounds on the curvature in these balls at time $t = 0$, while the same bounds for $t \leq 0$ follow from the Harnack inequality, which has as a Corollary that R is pointwise increasing on an ancient solution with weakly positive curvature operator. The Little Loop Lemma provides a bound on the injectivity radius at a point in terms of the maximum curvature in a ball around the point, in a form invariant under dilation. Hence this local injectivity radius estimate survives into the limit N^n, and gives an injectivity radius estimate at each P_j from the estimate on R in the ball of radius r_j. We now have everything we need to take a limit of the dilations of the Ricci Flow around the $(P_j, 0)$, dilating time like distance squared and keeping $t = 0$ in N^n as $t = 0$ in the new limit, which we call $\overline{N}^n$. This new limit will be a complete solution to the Ricci Flow on an ancient time interval $-\infty < t \leq 0$ with bounded curvature and weakly positive curvature operator. (Note our bounds on R do not hold for $t > 0$. Once we have $\overline{N}^n$ we could extend it for $t > 0$ by Shi's existence result [**42**].) Moreover $\overline{N}^n$ has an origin O and $R(O, 0) = 1$, while $R \leq 1$ everywhere for $t \leq 0$ since $\delta_j \to \infty$.

We claim a cover of $\overline{N}^n$ splits as a product with a flat factor. To show this, it suffices to show that $\overline{N}^n$ has a zero sectional curvature at $(O, 0)$. Suppose it does not. Then we have some lower bound $\gamma > 0$ on the sectional curvatures at $(O, 0)$. This means that there will be a uniform lower bound γ' (say $\gamma' = \gamma/2$) so that we have a lower bound $K \geq \gamma' R(P_j, 0)$ on the sectional curvatures at the $(P_j, 0)$ for all large enough j. The bounds on R in the balls $Br_j(P_j, 0)$ give bounds on R backwards in time by the Harnack inequality (as we mentioned), and now since R bounds $|Rm|$ the interior derivative estimates give bounds on the first derivatives $|DRm|$ in smaller balls. Since these bounds are dilation invariant, we find that the sectional curvatures all have a uniform lower bound γ'' (say $\gamma'/2$) so that we have a lower bound $K \geq \gamma'' R(P_j, 0)$ in balls around the P_j at time $t = 0$ of radii

$$\rho_j = c/\sqrt{R(P_j, 0)}$$

for some constant $c > 0$ depending only on the dimension. Thus there exists a $\beta > 0$ such that for large j every P_j at $t = 0$ is the center of a β-bump, and these bumps are all disjoint. Moreover since

$$\rho_j^2 R(P_j, 0) = c^2 \quad \text{and} \quad r_j^2 R(P_j, 0) \to \infty$$

we see $r_j/\rho_j \to \infty$; and also $s_j/r_j \to \infty$ where s_j is the distance of P_j from the origin O in N^n at time $t = 0$, so for any $\lambda < \infty$ the β-bumps at P_j are λ-remote for large j. But this contradicts Theorem 21.6. Hence a cover of $\overline{N}^n$ splits as a product and

$$\overline{N}^n = \widehat{N}^p \times R^q/\Gamma$$

with $p + q = n$ and $q > 0$, and Γ is a group of isometries. (Is $\Gamma = 0$?)

The limit factor $\widehat{N}^p$ may not be yet of Type I or II because we did not choose it in the usual way. What we can do is to take a further limit of dilations of $\widehat{N}^p$, also by shrinking, to get yet another limit $\overset{\vee}{\to} N^p$ which will be of Type I or II. We get a Type I limit when the backwards limit is

$$\Omega = \limsup_{t \to -\infty} |t| \sup_P R(P, t) < \infty$$

and Type II when this limit is infinite.

To extract the Type I limit we choose a sequence of points P_j= and times $t_j \to -\infty$ so that the lim sup is attained

$$|t_j| R(P_j, t_j) \to \Omega$$

and then make P_j the new origin O_j, translate in time so t_j becomes 0, dilate in space so $R(P_j, t_j)$ becomes 1 and dilate time like distance squared. To extract the Type II limit we choose a sequence $\Omega_j \to \infty$, pick τ_j with $|\tau_j|$ as large as possible so that

$$\sup\{|t|R(P,t) : \tau_j \leq t \leq 0\} \leq \Omega_j$$

and pick P_j and $t_j \leq \tau_j$ so that

$$R(P_j, t_j) \geq \frac{1}{1+\varepsilon_j} \sup\{R(P,t) : t \leq \tau_j\}$$

where $\varepsilon_j \to 0$, and dilate the same way. In both cases we have an injectivity radius estimate coming originally from the Little Loop Lemma on M^m and surviving all the dilating and limiting procedures. The rest of this argument proceeds as before.

Now a sequence of dilations of M^m converges to $\overline{N}^n \times R^k$, and a sequence of dilations of $\overline{N}^n$ converges to $\widehat{N}^p \times R^q$, and a sequence of dilations of $\widehat{N}^p$ converges to $\overset{\vee}{\to} N^p$ which is Type I or II.

Thus a sequence of dilations of $\overline{N}^n \times R^k$ converges to $\widehat{N}^p \times R^{q+k}$, and a sequence of dilations of $\widehat{N}^p \times R^{q+k}$ converges to $\overset{\vee}{\to} N^p \times R^{q+k}$. Now a dilation of a dilation is a dilation, and a limit of limits is a limit by picking an appropriate subsequence. Thus a limit of dilations of M^m converges to $\overset{\vee}{\to} N^p \times R^{q+k}$ where $q+k > k$. This contradicts the hypothesis that k is maximal, which proves the Theorem. □

There is another case where the blow-down argument can be used.

THEOREM 22.3. *Suppose we have a complete Ricci soliton solution*

$$D_i D_j f = R_{ij}$$

in odd dimension $2n+1$ with bounded curvature and strictly positive curvature operator. Then there exists a sequence of dilations around origins P_j at time 0 which converges to a limit which splits as a product of R^1 with a solution of even dimension $2n$ which is ancient and complete with bounded curvature and weakly positive curvature operator.

Proof. In section 19 we say that $|Df|^2$ approaches the maximum curvature M as $s \to \infty$ where s is the distance from some origin. Thus for every $\delta > 0$ we can find $\sigma < \infty$ so that for $s \geq \sigma$

$$(\sqrt{M} - \delta)s \leq f \leq (\sqrt{M} + \delta)s$$

which makes f comparable to s. Hence on the level set

$$S_\varphi = \{f = \varphi\}$$

the distance s is nearly $\varphi/\sqrt{M}$ for large r, in particular. Hence on the level set

$$S_\mu = \{f = \mu\}$$

the distance s is nearly $\mu/\sqrt{M}$, in particular

$$\mu/(\sqrt{M}+\delta) \leq s \leq \mu/(\sqrt{M}-\delta)$$

for large μ. □

Now choose the point P_j and radii r_j as before and let $R_j = R(P_j, 0)$ and $\mu_j = f(P_j, 0)$. Then the curvature R at any point P on any sphere S_μ at time $= 0$ with

$$|\mu - \mu_j| \leq r_j/\sqrt{M}$$

satisfies an estimate

$$R \leq (1+\varepsilon_j)R_j$$

for large j, where again

$$R_j r_j^2 \to \infty \quad \text{and} \quad \varepsilon_j \to 0.$$

We can argue as before if we can control the injectivity radii ρ_j at $(P_j, 0)$ with an estimate

$$\rho_j \geq c/\sqrt{R_j}.$$

We get this estimate in odd dimensions as follows.

Each level set S_μ for large μ is a smooth submanifold which is strictly convex since f is convex. The second fundamental form II of S_μ is given by

$$II(X,Y) = D^2 f(X,Y)/|Df|$$

on vectors X and Y where

$$Df(X) = Df(Y) = 0$$

makes them tangent to S_μ. Since $|Df| \to \sqrt{M}$ and $D^2 f = Rc$, we can control the second fundamental form on S_μ by the maximum of Rc on S_μ, hence by R_j. Thus

$$|II| \leq CR_j/\sqrt{M}$$

on all S_μ with $|\mu - \mu_j| \leq r_j/\sqrt{M}$. Each S_μ has positive sectional curvature in the induced metric by the Gauss curvature equation, and each S_μ is orientable since the whole soliton is diffeomorphic to R^{2n+1} and the normal bundle is oriented by $Df > 0$. If the dimension $2n+1$ of the soliton is odd, the dimension $2n$ of S_μ is even. Then by a theorem in **[9]** the injectivity radius of S_μ in the induced metric can be bounded $\geq c/\sqrt{R_j}$.

This gives a similar bound on the injectivity radius of the soliton at P_j in the following way. Since the curvature is positive it is bounded below, and it suffice to show that a ball around P_j in the soliton of radius $\alpha/\sqrt{R_j}$ has volume $\geq c/\sqrt{R_j^{2n+1}}$ for some $\alpha > 0$ and $c > 0$ independent of j. We do this by taking a coordinate chart inside the ball and estimating its volume. First go a distance $\alpha/\sqrt{R_j}$ from P_j in the direction of $\pm Df$. This moves us out and back some comparable distance. Then take the exponential map of radius $\alpha/\sqrt{R_j}$

out from each point on this curve in the spheres S_μ in their induced metric. Start with a frame on the tangent space at P_j and parallelly translate it along the curve in the direction Df to get a frame at each point on this curve, and use it to refer the exponential map on a standard ball in R^{2n} into S_μ for each μ. Then this gives a coordinate chart in a neighborhood of P_j on the soliton. Since each curvature in the soliton and each second fundamental form on the hypersurfaces S_μ can be controlled by R_j, for a suitable small α the coordinate chart will inject with derivative close to an isometry. This shows the image has volume $\geq c/\sqrt{R_j^{2n+1}}$. The rest of the proof proceeds just as before, up to taking the first limit. Unfortunately we cannot do the backward limit in time without more injectivity radius control.

23 An Isoperimetric Ratio Bound in Dimension Three. In this section we shall prove an isoperimetric ratio bound for solutions to the Ricci Flow in dimension three in the special case of a Type I singularity where we have a solution for $0 \leq t < T < \infty$ with

$$|Rm|(T-t) \leq \Omega$$

for some constant $\Omega < \infty$, and where we also assume a bound below on the total volume $\mathcal{V}(t)$ of the form

$$\mathcal{V} \geq \alpha(T-t)^{3/2}$$

for some constant $\alpha > 0$. The first assumption is special; but the second is not so important, since if $|Rm|(T-t) \leq \Omega < \infty$ but $\mathcal{V}/(T-t)^{3/2} \to 0$ (at least for a subsequence of times) then $|Rm|\mathcal{V}^{2/3} \to 0$, and since $|Rm|$ controls all the curvatures, the curvature collapes with bounded volume; and it follows from the work of Cheeger and Gromov **[10]** the manifold has an F-structure, and hence its topology is understood already.

THEOREM 23.1. *For every $\beta > 0, \rho < \infty, T < \infty, \Omega < \infty$ and $\alpha > 0$ we can find a constant $\gamma = \gamma(\beta, \rho, T, \Omega, \alpha)$ with the following property. If an initial metric g_0 has the property that every surface which bounds a volume at least V on each side has area $A \geq \beta V^{2/3}$, and the initial metric has scalar curvature $R \geq -\rho$, and if the subsequent solution if the Ricci Flow exists for $0 \leq t < T$ with*

$$|Rm|(T-t) \leq \Omega \quad \textit{and} \quad \mathcal{V} \geq \alpha(T-t)^{3/2},$$

then at any time t any surface which bounds a volume at least V on each side has area

$$A \geq \gamma \min(T-t, V^{2/3}).$$

Proof. Let $G(V,t)$ be the function defined for $0 \leq t < T$ and $0 \leq V \leq \mathcal{V}(t)$ which for $0 < V < \mathcal{V}(t)$ is the infimum of the areas of surfaces of any type which divide the manifold into regions of volumes V and $\mathcal{V} - V$, with $G = 0$ if $V = 0$ or $V = \mathcal{V}$. Then so much is known about the theory of minimal surfaces (see Almgren [**1**]) that we know G is continuous in V and t, and for any t in $0 \le t < T$ and any V in $0 < V < \mathcal{V}$ the infimum is attained on a smooth surface of constant mean curvature H. Moreover if $\beta < \beta_E$ where β_E is its Euclidean value

$$\beta_E = (36\pi)^{1/3}$$

then for any metric on a compact manifold we can find $\delta > 0$ depending on the metric so that any surface bounding a volume $V \le \delta$ has area $A \ge \beta V^{2/3}$. This means we do not need to concern ourselves with very small volumes. □

We shall prove a lower bound of the form $G > F$ for $0 < V < \mathcal{V}$ where the function $F(V, t)$ is chosen of the form

$$\frac{1}{F} = e^{\frac{2}{3}\rho t}\left\{\frac{Q}{T-t} + \frac{B}{V^{2/3}} + \frac{B}{[\mathcal{V}(t) - V]^{2/3}}\right\}$$

for some suitably large constants Q and B which we are free to choose later. Since $e^{\rho t} \le e^{\rho T}$ we can find $\gamma > 0$ in terms of $e^{\rho T}$, A and B, which will prove the Theorem. If B is large enough, then by the previous remark we do not have to worry when V or $\mathcal{V} - V$ is very small.

If this estimate fails, there will be a first time t^* and a volume V^* with $0 < V^* < [\alpha(T-t)]^{3/2}$ when $G = F$, and $G(V^*, t^*)$ will be attained by the area of a smooth surface Σ^* of constant mean curvature H. Consider a one-parameter family of smooth surfaces $\Sigma(r)$ for r near 0 by taking the parallel surface to Σ^* at distance r, with $\Sigma(r)$ inside the part with volume V^* for $r < 0$ and outside for $r > 0$. Note that $\Sigma(0) = \Sigma^*$. Define the smooth functions $A(r,t)$ and $V(r,t)$ for r near 0 and t near t^* by letting $A(r,t)$ be the area of $\Sigma(r)$ at time t, and letting $V(r,t)$ be the volume enclosed by $\Sigma(r)$ at time t on the side of the part with volume V^*. Note that $A(0,t^*)$ is the area of Σ^* which is $G(V^*, t^*)$, while $V(0, t^*) = V^*$.

It is clear we have the inequality

$$A(r, t^*) \ge G(V(r,t^*), t^*)$$

since G is the least area among all surfaces enclosing the given volume at the given time. But $G \ge F$ up to time t^*, so

$$A(r, t^*) \ge F(V(r,t^*), t^*)$$

for all r near 0, and equality is attained at $r = 0$ where $G = F$ at time t^*. Since A and F are both smooth, at $r = 0$ and $t = t^*$ we get

$$\frac{\partial A}{\partial r} = \frac{\partial F}{\partial V}\frac{\partial V}{\partial r}$$

and

$$\frac{\partial^2 A}{\partial r^2} \ge \frac{\partial^2 F}{\partial V^2}\left(\frac{\partial V}{\partial r}\right)^2 + \frac{\partial F}{\partial V}\frac{\partial^2 V}{\partial r^2}.$$

In addition, it is also clear that we have the inequality

$$A(0,t) \geq G(V(0,t),t)$$

for $t \leq t^*$, and since $G \geq F$ up to time t^* we get

$$A(0,t) \geq F(V(0,t),t)$$

for $t \leq t^*$, with equality at $t = t^*$. Thus at $r = 0$ and $t = t^*$,

$$\frac{\partial A}{\partial t} \leq \frac{\partial F}{\partial t} + \frac{\partial F}{\partial V}\frac{\partial V}{\partial t}.$$

Now at $r = 0$ and $t = t^*$,

$$\frac{\partial V}{\partial r} = A = F$$

and

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial A}{\partial r} = HA = HF$$

where H is the constant mean curvature. Then the equality

$$\frac{\partial A}{\partial r} = \frac{\partial F}{\partial V}\frac{\partial V}{\partial r}$$

makes

$$\frac{\partial F}{\partial V} = H.$$

From this we get

$$\frac{\partial^2 V}{\partial r^2} = F\frac{\partial F}{\partial V}.$$

Now our inequality on $\partial^2 A/\partial r^2$ becomes

$$\frac{\partial^2 A}{\partial r^2} \geq F^2\frac{\partial^2 F}{\partial V^2} + F\left(\frac{\partial F}{\partial V}\right)^2.$$

The volume V shrinks at a rate

$$\frac{\partial V}{\partial t} = -\int R\,dv \leq \rho V$$

(since the inequality $R \geq -\rho$, which we assume at $t = 0$, is preserved by the Ricci Flow). Since both F and G are symmetric in $V \to \mathcal{V}(t) - V$, it is no loss to assume $V^* \leq \mathcal{V}(t)/2$; and this makes $\partial F/\partial V = H \geq 0$ at $r = 0$ and $t = t^*$. Then our inequality on $\partial A/\partial t$ becomes

$$\frac{\partial A}{\partial t} \leq \frac{\partial F}{\partial t} + \rho V\frac{\partial F}{\partial V}.$$

These are the inequalities we need.

The remaining fact we use comes from section 12, where we showed that for a family of parallel surfaces $\Sigma(r)$ we have

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial r^2} - 4\pi\chi$$

where χ is the Euler class of Σ^*. We claim that for a suitably large constant Q in the definition of F, which makes

$$F \leq (T-t)/Q,$$

we can make $\chi \leq 0$, so that Σ^* is not a sphere or a projective plane. What is required for this? We have

$$2\pi\chi = \int_{\Sigma^*} [Rm(P) + K] da$$

where $Rm(P)$ is the ambient curvature of the tangent plane P to Σ^*, and K is the determinant of the second fundamental form. We have

$$Rm(P) \leq C\Omega/(T-t)$$

for some constant C. Then

$$\int_{\Sigma^*} Rm(P) da \leq C\Omega/Q$$

which is as small as we like for Q large. Also $K \leq H^2/4$ and $H = \partial F/\partial V$ so

$$\int_{\Sigma^*} K\, da \leq \frac{1}{4} F \left(\frac{\partial F}{\partial V}\right)^2 .$$

Now we claim that by making B large we can make $F(\partial F/\partial V)^2$ as small as we like. Recall

$$\frac{1}{F} = e^{\frac{2}{3}\rho t} \left\{ \frac{Q}{T-t} + \frac{B}{V^{2/3}} + \frac{B}{[\mathcal{V}(t) - V]^{2/3}} \right\}.$$

Differentiate implicitly to get

$$\frac{\partial F}{\partial V} = \frac{2}{3} B e^{\frac{2}{3}\rho t} F^2 \left\{ \frac{1}{V^{5/3}} - \frac{1}{[\mathcal{V}(t) - V]^{5/3}} \right\}.$$

Since we have assumed $V \leq \mathcal{V}(t)/2$,

$$0 \leq \frac{\partial F}{\partial V} \leq \frac{4}{3} B e^{\frac{2}{3}\rho t} F^2 / V^{5/3}$$

and

$$F \leq V^{2/3} / B e^{\frac{2}{3}\rho t}.$$

This makes

$$F \left(\frac{\partial F}{\partial V}\right)^2 \leq 16 / (9 B^3 e^{2\rho t})$$

which is indeed as small as we like when B is large. Thus $\chi < 1$, and since it is an integer we must have $\chi \leq 0$. This makes

$$\frac{\partial A}{\partial t} \geq \frac{\partial^2 A}{\partial r^2}.$$

Now we can combine our inequalities to include that

$$\frac{\partial F}{\partial t} + \rho V \frac{\partial F}{\partial V} \geq F^2 \frac{\partial^2 F}{\partial V^2} + F\left(\frac{\partial F}{\partial V}\right)^2$$

at $V = V^*$ and $t = t^*$. However, we claim if Q and B are large enough the opposite inequality holds everywhere. This contradiction implies $G > F$ for all $t < T$, which will prove the Theorem. □

LEMMA 23.2. *If Q and B are large enough then the function F defined by*

$$\frac{1}{F} = e^{\frac{2}{3}\rho t}\left\{\frac{Q}{T-t} + \frac{B}{V^{2/3}} + \frac{B}{[\mathcal{V}(t) - V]^{2/3}}\right\}$$

satisfies

$$\frac{\partial F}{\partial t} + \rho V \frac{\partial F}{\partial V} \leq F^2 \frac{\partial^2 F}{\partial V^2} + F\left(\frac{\partial F}{\partial V}\right)^2$$

for $0 \leq t < T$ and $0 < V \leq \mathcal{V}(t)/2$.

Proof. We look for a function F in the form $F = 1/H$. Then we need

$$H^3\left(\frac{\partial H}{\partial t} + \rho V \frac{\partial H}{\partial V}\right) + 3\left(\frac{\partial H}{\partial V}\right)^2 \geq H \frac{\partial^2 H}{\partial V^2}.$$

If H takes the form

$$H = e^{\frac{2}{3}\rho t} K$$

then K must satisfy

$$e^{\frac{4}{3}\rho t} K^3\left[\frac{\partial K}{\partial t} + \rho\left(\frac{2}{3}K + V\frac{\partial K}{\partial V}\right)\right] + 3\left(\frac{\partial K}{\partial V}\right)^2 \geq K\frac{\partial^2 K}{\partial V^2}.$$

Our K has the form

$$K = \frac{Q}{T-t} + B\left[\frac{1}{V^{2/3}} + \frac{1}{(\mathcal{V} - V)^{2/3}}\right].$$

We compute

$$\frac{\partial K}{\partial t} + \rho\left(\frac{2}{3}K + V\frac{\partial K}{\partial V}\right) = \frac{Q}{(T-t)^2} + \frac{2}{3}\frac{\rho Q}{T-t} + \frac{2}{3}B \cdot \frac{1}{(\mathcal{V} - V)^{5/3}}\left(\rho\mathcal{V} - \frac{\partial \mathcal{V}}{\partial t}\right).$$

Since $R \geq -\rho$ makes
$$\frac{\partial \mathcal{V}}{\partial t} \leq \rho \mathcal{V}$$
we have
$$\frac{\partial K}{\partial t} + \rho\left(\frac{2}{3}K + V\frac{\partial K}{\partial V}\right) \geq \frac{Q}{(T-t)^2}.$$
Since $e^{\frac{4}{3}\rho t} \geq 1$ it is sufficient to verify
$$K^3 \frac{Q}{(T-t)^2} + 3\left(\frac{\partial K}{\partial V}\right)^2 \geq K\frac{\partial^2 K}{\partial V^2}$$
and now we can forget about ρ.

We consider two cases. The first is where $V \leq \varepsilon \mathcal{V}$ for some small absolute constant ε we shall choose shortly. We have
$$\frac{\partial K}{\partial V} = -\frac{2}{3}B\left[\frac{1}{V^{5/3}} - \frac{1}{(\mathcal{V}-V)^{5/3}}\right]$$
and
$$\frac{\partial^2 K}{\partial V^2} = \frac{10}{9}B\left[\frac{1}{V^{8/3}} + \frac{1}{(\mathcal{V}-V)^{8/3}}\right].$$
Then
$$3\left(\frac{\partial K}{\partial V}\right)^2 \geq \frac{12}{9}B^2\left[1-\varepsilon^{5/3}\right]^2 \cdot \frac{1}{V^{10/3}}$$
and
$$\frac{\partial^2 K}{\partial V^2} \leq \frac{10}{9}B\left[1+\varepsilon^{8/3}\right] \cdot \frac{1}{V^{8/3}}.$$
Also
$$B \cdot \frac{1}{V^{2/3}} \leq K \leq \frac{Q}{T-t} + B\left[1+\varepsilon^{2/3}\right] \cdot \frac{1}{V^{2/3}}.$$
Therefore our inequality will hold if we choose $\varepsilon > 0$ so small that
$$\frac{12}{9}\left[1-\varepsilon^{5/3}\right]^2 \geq \frac{10}{9}\left[1+\varepsilon^{8/3}\right] + \frac{1}{9}$$
and if in addition we have
$$B^3 Q \cdot \frac{1}{(T-t)^2 V^{6/3}} + \frac{1}{9}B^2 \cdot \frac{1}{V^{10/3}} \geq \frac{10}{9}BQ\left[1+\varepsilon^{8/3}\right] \cdot \frac{1}{T-t} \cdot \frac{1}{V^{8/3}}.$$
Since ε is small we can take
$$\frac{10}{9}\left[1+\varepsilon^{8/3}\right] \leq 2$$
and then this inequality holds if $B^3 \geq 3Q$. Thus we have the estimate for $V \leq \varepsilon(\mathcal{V}-V)$ by making B large compared to Q.

Consider the other case where $V \geq \varepsilon(\mathcal{V} - V)$. This is easier because we can compare everything to $\mathcal{V}$ and

$$\mathcal{V}^{2/3} \geq \alpha(T-t)$$

for some $\alpha > 0$ by our hypothesis. With various constants $C < \infty$ and $c > 0$ independent of B, Q, and α (but depending on $\varepsilon > 0$ which is fixed) we have

$$\frac{\partial^2 K}{\partial V^2} \leq CB \cdot \frac{1}{\mathcal{V}^{8/3}}$$

and

$$K \geq cB \cdot \frac{1}{\mathcal{V}^{2/3}}.$$

Our estimate holds if

$$\frac{\partial^2 K}{\partial V^2} \leq K^2 \frac{Q}{(T-t)^2}$$

which holds if $\mathcal{V} \geq \alpha(T-t)^{3/2}$ and

$$BQ \geq C/\alpha^{4/3}$$

for some constant C as above. This is easily arranged also, and the Theorem is established. □

COROLLARY 23.3. *If $|Rm| \leq \Omega(T-t)$ and $\mathcal{V} \geq \alpha(T-t)^{2/3}$ as before then the injectivity radius r satisfies an estimate*

$$r \geq \theta\sqrt{T-t}$$

for some $\theta > 0$ depending on $\beta, \rho, T, \Omega, \alpha$ as before.

Proof. If the injectivity radius is very small compared to the maximum curvature then the isoperimetric ratio $A/V^{2/3}$ will also be very small for a torus of area A enclosing a volume V very small compared to the maximum curvature. □

24 Curvature Pinching in Three Dimensions. In three dimensions we can extract more information from the explicit form of the curvature reaction. Recall from 5(c) that when the curvature operator matrix M is diagonal

$$M = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}$$

where

$$M(X,Y) = R\, g(X,Y) - 2Rc(X,Y)$$

and the trace of M is the scalar curvature R the reaction ODE system becomes

$$\begin{cases} \frac{d\lambda}{dt} = \lambda^2 + \mu\nu \\ \frac{d\mu}{dt} = \mu^2 + \lambda\nu \\ \frac{d\nu}{dt} = \nu^2 + \lambda\mu \end{cases}$$

Any closed convex set of curvature operator matrices M which is $SO(3)$ invariant (and hence invariant under parallel translation) and preserved by the reaction ODE is also preserved by the Ricci Flow.

Since the system of ODEs is homogeneous, it is natural to first study the radial motion, and then examine the solution curves projectivly. The radius ρ is given by

$$\rho^2 = \lambda^2 + \mu^2 + \nu^2$$

and we compute

$$\frac{d}{dt}\left(\lambda^2 + \mu^2 + \nu^2\right) = (\lambda + \mu + \nu)\left[(\lambda + \mu)^2 + (\lambda + \nu)^2 + (\mu + \nu)^2\right].$$

which shows the radius ρ increases for positive scalar curvature $R = \lambda+\mu+\nu > 0$, and decreases for negative scalar curvature $R = \lambda+\mu+\nu < 0$. Next note that if a vector $V \in R^n$ evolves by a system of ODE s

$$\frac{dV}{dt} = F(V)$$

then this system and the associated system

$$\frac{dV}{dt} = a(V)F(V) - b(V)V$$

have the same oriented family of solution curves in the projective sphere $S^{n-1} = R^n - \{0\}/R_+$, for any scalar valued functions $a(V)$ and $b(V)$. We take $V = (\lambda, \mu, \nu)$ and

$$a = \left(\lambda^2 + \mu^2 + \nu^2\right)$$
$$b = \lambda^3 + \mu^3 + \nu^3 + 3\lambda\mu\nu\ .$$

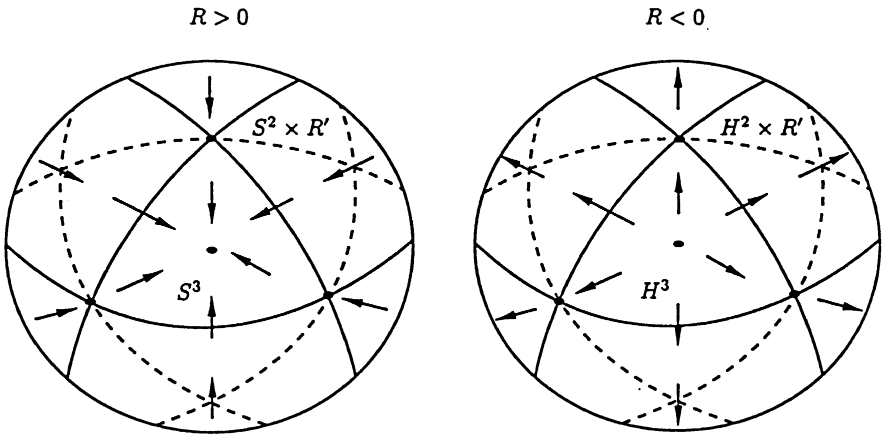
Then the associated system keeps $\lambda^2 + \mu^2\nu^2$ constant, so we can restrict our attention to the unit sphere $\rho = 1$. It has the explicit form

$$\frac{d\lambda}{dt} = \lambda^2(\mu - \nu)^2 - \mu^3(\lambda - \nu) - \nu^3(\lambda - \mu).$$

Clearly it has fixed points $\lambda = \mu = \nu$ and $\lambda = \mu = 0, \lambda = \nu = 0, \mu = \nu = 0$. This gives eight fixed points on the sphere $\rho = 1$.

It is easiest to display the flow on the front of the sphere $R > 0$ and on the back $R < 0$. We denote the circles $\lambda = 0, \mu = 0, \nu = 0$ with solid lines, and the circles $\lambda + \mu = 0, \lambda + \nu = 0, \mu + \nu = 0$ with dotted lines. In the hemisphere

$R > 0$ the region of positive sectional curvature lies inside the solid triangle, the region of positive Ricci curvature inside the dotted one; similarly for negative sectional and Ricci curvature on the other.



The center point $\lambda = \mu = \nu > 0$ represents the sphere S^3, and the center point $\lambda = \mu = \nu < 0$ represents the hyperbolic space H^3. Note S^3 is attractive while H^3 is repulsive. The three vertices $\lambda > 0, \mu = \nu = 0$ and $\mu > 0, \lambda = \nu = 0$ and $\nu > 0, \lambda = \mu = 0$ represent the cylinder $S^2 \times R^1$, while the three vertices $\lambda < 0, \mu = \nu = 0$ and $\mu < 0, \lambda = \nu = 0$ and $\nu < 0, \lambda = \mu = 0$ represent $H^2 \times R^1$. These are degenerate fixed points which all attract in one direction from one side, and repel in the opposite direction on the other side. Of course the picture on the back $R < 0$ is the reverse of the picture on the front.

We can examine the degenerate fixed point at the cylinder $S^2 \times R^1$ where $\lambda = \mu = 0$ more precisely by taking instead the associated system with

$$a = \nu \quad \text{and} \quad b = \nu^2 + \lambda\mu$$

which preserves the planes where ν is constant. Restricting to $\nu = 1$ gives the associated system

$$\begin{cases} \dfrac{d\lambda}{dt} = \mu - \lambda + \lambda^2 - \lambda^2\mu \\ \dfrac{d\mu}{dt} = \lambda - \mu + \mu^2 - \lambda\mu^2 \end{cases}$$

with a degenerate fixed point at $\lambda = \mu = 0$. If we substitute

$$\lambda = x + y \qquad \mu = x - y$$

we get the system

$$\begin{cases} \dfrac{dx}{dt} = x^2 + y^2 - x(x^2 - y^2) \\ \dfrac{dy}{dt} = -\left[2(1-x) + x^2 - y^2\right] y. \end{cases}$$

When we are close to the origin x increases and $|y|$ decreases. On the parabola

$$y^2 + 3x = 0$$

we have

$$\frac{d}{dt}\left(y^2+3x\right)=3x\left(1+x^2\right)\leq 0$$

so if we start inside this parabola we must stay inside, and if we start close to the origin we must appraoch the origin. But on the parabola

$$y^2+4x-\varepsilon=0$$

we have

$$\frac{d}{dt}\left(y^2+4x-\varepsilon\right)=4x^2(1+x)+2\varepsilon\left(\varepsilon-4x-x^2\right)\geq 0$$

when $-1\leq x\leq 0$, so if we start outside this parabola but close to the origin with $x<0$ we must stay outside until $x>0$, after which x becomes large before y reaches 0. The envelope of all the solution curves attracted to the origin will again be a solution curve between the parabolas $y^2+3x=0$ and $y^2+4x=0$, so this separatrix has a vertical tangent near the orgin.

On the other hand, near the origin

$$\frac{dx}{dt}\approx x^2+y^2 \quad\text{and}\quad \frac{dy}{dt}\approx -2y$$

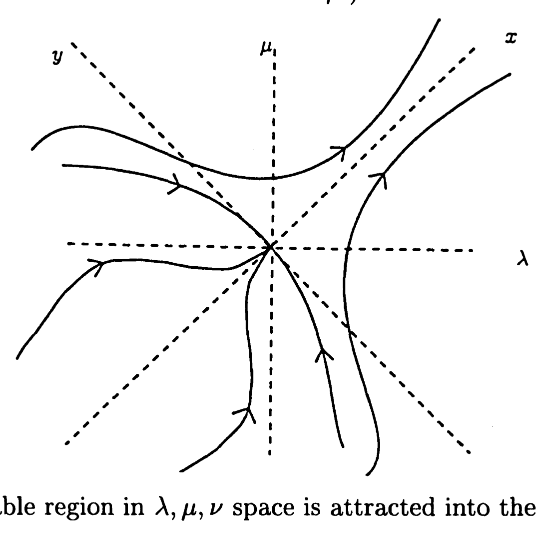
to a good approximation. If $x\leq 0$ and $0\leq y\ll |x|$ then

$$\frac{dx}{dt}\approx x^2 \quad\text{and}\quad \frac{dy}{dt}\approx -2y$$

which gives solution curves

$$y\approx Ce^{2/x}$$

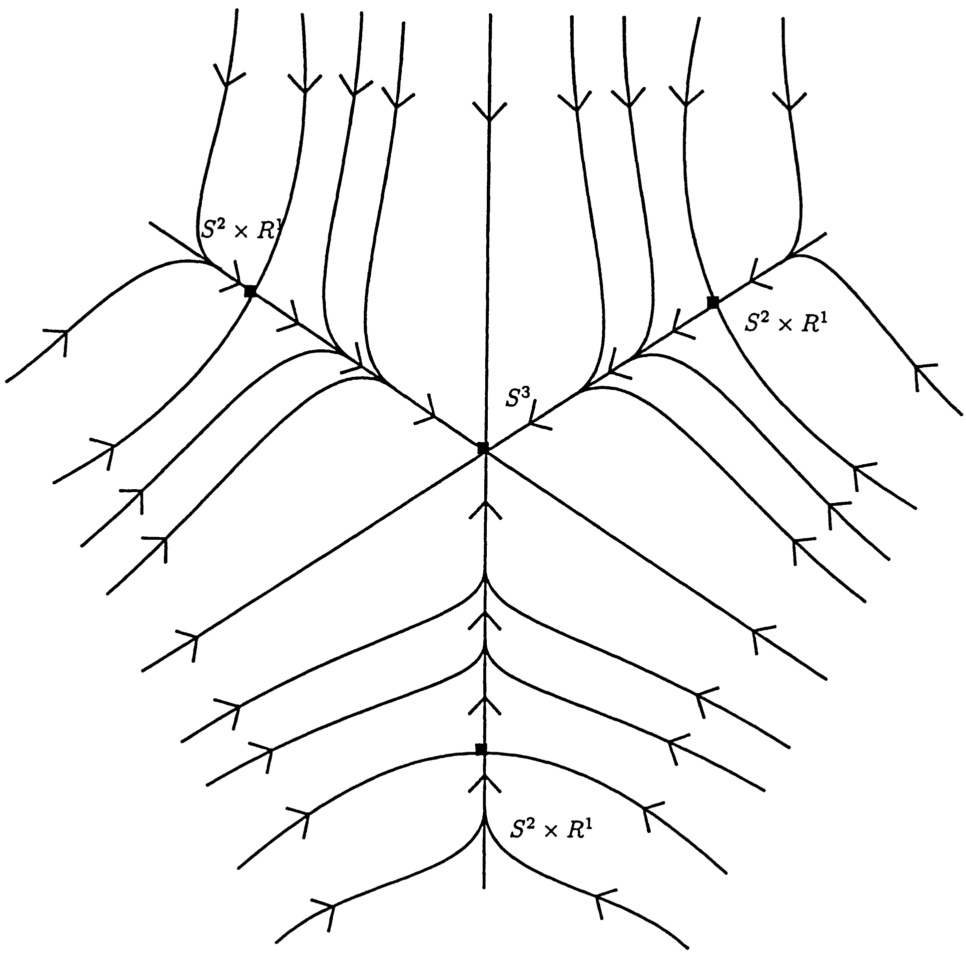
which keep $y\ll|x|$ and approach the x-axis very fast. We expect the solution curves inside the separatrix to look like these. In fact we expect the solution curves of the original system and the simple approximation are conjugate by a diffeomorphism. This gives the following picture for the solution curves near $S^2\times R$ where $\lambda=0,\mu=0,\nu=1$, projected radially onto the plane $\nu=1$. (Recall x and y are rotated 90° from λ and μ.)



Note that a sizable region in λ,μ,ν space is attracted into the fixed radial line

$\lambda = \mu = 0$ while the rest flows past it, on towards S^3 along the fixed radial line $\lambda = \mu = \nu$.

Tom Ivey has used a computer to produce a picture of the solution curves for the associated system obtained by projecting radially on the plane $\lambda+\mu+\nu = 1$. The picture looks like this



Hopefully some geometric insight into the following pinching results.

THEOREM 24.1. *For any ε in $0 \le \varepsilon \le 1/3$, the pinching condition $R \ge 0$ and*

$$Rc(x,y) \ge \varepsilon R\, g(x,y)$$

is preserved by the Ricci Flow in dimension three.

Proof. If the curvature operator M has eigenvalues $\lambda \ge \mu \ge \nu$, the pinching conditions become $\mu + \nu \ge 0$ and

$$\mu + \nu \ge \delta\lambda$$

with $\delta = 2\varepsilon/(1-2\varepsilon)$. Since λ is a convex function of M while $\mu+\nu$ is a concave function, the inequalities define a convex set of matrices, so we only have to check that this set is preserved by the ODE system. So we must check

$$\frac{d}{dt}(\mu+\nu) \ge \delta \frac{d}{dt}\lambda$$

or

$$\mu^2 + \lambda\nu + \nu^2 + \lambda\mu \ge \delta\left(\lambda^2 + \mu\nu\right)$$

on the boundary where

$$\mu + \nu = \delta\lambda \ge 0.$$

This is equivalent (solving for δ) to

$$\lambda\left(\mu^2 + \lambda\nu + \nu^2 + \lambda\mu\right) \ge (\mu+\nu)\left(\lambda^2 + \mu\nu\right)$$

which reduces to

$$\lambda^2(\mu+\nu) \ge \mu\nu(\mu+\nu)$$

which clearly holds if $\mu + \nu \ge 0$ and $\lambda \ge \mu \ge \nu$. ◻

THEOREM 24.2. *For any $\beta > 0, B < \infty$, and $\gamma > 0$ we can find a constant $C < \infty$ depending on β, B and γ with the following property. If a solution to the Ricci Flow in dimension three has*

$$\beta g(x,y) \le Rc(x,y) \le B g(x,y)$$

at the beginning $t = 0$, then for all subsequent times $t \ge 0$ we have

$$\left| Rc - \frac{1}{3} R\, g \right| \le \gamma R + C$$

as a bound on the trace-free part of the Ricci tensor.

Proof. Depending on β and B we can choose $\delta > 0$ so that

$$\delta\lambda \le \mu + \nu$$

at $t = 0$, and hence for $t \geq 0$ by the proof of the previous theorem. Choose the constant A so that the inequality

$$\lambda - \nu \leq A(\mu + \nu)^{1-\delta}$$

holds at $t = 0$, which is possible since

$$\lambda - \nu \leq \lambda + \mu \leq B \quad \text{and} \quad \mu + \nu \geq \beta$$

at $t = 0$. We claim this inequality is also preserved by the Ricci Flow. Clearly it defines a convex set of matrices M with eigenvalues $\lambda \geq \mu \geq \nu$ and $\mu + \nu \geq 0$. So we only must check that the inequality is preserved by the ODE system. Now

$$\frac{d}{dt}(\lambda - \nu) = \lambda^2 + \mu\nu - \nu^2 - \lambda\mu$$

so

$$\frac{d}{dt}\ln(\lambda - \nu) = \lambda - \mu + \nu$$

while

$$\frac{d}{dt}(\mu + \nu) = \mu^2 + \lambda\nu + \nu^2 + \lambda\mu \geq \lambda(\mu + \nu)$$

so

$$\frac{d}{dt}\ln(\mu + \nu) \geq \lambda.$$

Then

$$\frac{d}{dt}\ln\left[(\lambda - \nu)/(\mu + \nu)^{1-\delta}\right] \leq \delta\lambda - (\mu + \nu) \leq 0$$

so the ratio $(\lambda - \nu)/(\mu + \nu)^{1-\delta}$ decreases. If it is less than A to start, it remains so.

We can estimate

$$\left|Rc - \frac{1}{3}Rg\right| \leq C(\lambda - \nu)$$

for some constant C, and

$$\mu + \nu \leq CR$$

for some the constant C. For any $\zeta > 0$ we can find yet another constant $C(\zeta)$ with

$$R^{1-\delta} \leq \zeta R + C(\zeta)$$

for all $R \geq 0$. Then we get

$$\left|Rc - \frac{1}{3}Rg\right| \leq C\zeta R + C(\zeta)$$

and we only need to take $\zeta \leq \gamma/C$ to finish the proof. □

COROLLARY 24.3. *For any $\beta > 0, B < \infty$ and $\theta > 0$ we can find a constant $C < \infty$ with the following property. If a solution to the Ricci Flow in dimension three has*

$$\beta g(x, y) \leq Rc(x, y) \leq Bg(x, y)$$

at the beginning $t = 0$, then for any subsequent $\tau \geq 0$

$$\max_{t \leq \tau} \max_{P} |DRm(P,t)| \leq \theta \max_{t \leq \tau} \max_{P} |Rm(P,t)|^{3/2} + C.$$

Proof. We can recover this result by a limiting procedure; an explicit estimate using the maximum principle is given in [**20**]. Suppose the estimate fails for all C. Pick a sequence $C_j \to \infty$, and pick points P_j and times τ_j such that

$$|DRm(P_j, \tau_j)| \geq \theta \max_{t \leq \tau_j} \max_{P} |Rm(P,t)|^{3/2} + C_j.$$

Choose the P_j to be the origin, and pull the metric back to a small ball of radius r_j proportional to the reciprocal of the square root of the maximum curvature up to time τ. Clearly these go to infinity by our derivative bounds. Dilate the metrics so

$$\max_{t \leq \tau_j} \max_{P} |Rm(P,t)|$$

becomes 1 and translate so time τ_j becomes time 0. Then C_j dilates to zero, but in the limit metric

$$|DRm(0,0)| \geq \theta.$$

However the limit metric has

$$Rc - \frac{1}{3} R\, g = 0$$

by the previous theorem. But then it has constant curvature, which is a contradiction. This proves the corollary. □

We can now see that the solution to the Ricci Flow on a compact three-manifold with positive Ricci curvature becomes round. Since $R_{\text{MIN}} > 0, R_{\text{MAX}}$ goes to infinity in a finite time. Pick a sequence of points P_j and times τ_j where the curvature at P_j is as large as it has been anywhere for $0 \leq t \leq \tau_j$. Since $|DRm|$ is very small conpared to $R(P_j, t_j)$ and $|Rc - \frac{1}{3} R\, g|$ is also, the curvature is nearly constant and positive in a large ball around P_j. But then Myer's Theorem tells us this is the whole manifold.

Our next result is even more interesting, because it applies to any three-manifold regardless of the sign of the curvature tensor. It was also observed independently by Ivey [**30**]. Consider the function

$$y = f(x) = x \log x - x$$

for $1 \leq x < \infty$, where it is increasing and convex with range $-1 \leq y < \infty$. We let $f^{-1}(y) = x$ be the inverse function, which is also increasing but concave and satisfies

$$\lim_{y \to \infty} f^{-1}(y)/y = 0.$$

THEOREM 24.4. *Suppose we have a solution to the Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on a compact three-manifold which satisfies the inequalities $R \geq -1$ *and*

$$M_{ij} + f^{-1}(R) g_{ij} \geq 0$$

at $t = 0$. *Then it will continue to satisfy them for* $t \geq 0$.

Note that since $f^{-1}(y) \geq +1$ always, any matrix with eigenvalues at least -1 and trace at least -1 satisfies the inequalities. For any metric we can achieve this by dilation. Then the inequalities will continue to hold under the Ricci flow. Then if the curvatures go to infinity, the most negative will be small compared to the most positive.

LEMMA. *The set* P *of matrices* M_{ab} *defined by the inequalities*

$$P : \begin{cases} \lambda + \mu + \nu \leq -1 \\ \nu + f^{-1}(\lambda + \mu + \nu) \geq 0 \end{cases}$$

is closed, convex and preserved by the ODE.

Proof. P is closed because f^{-1} is continuous. The function $\lambda + \mu + \nu$ is just the trace, which is a linear function. Therefore the first inequality defines a linear half-space, which is convex. The function ν is concave, and f^{-1} is concave and increasing, so the second inequality defines a convex set as well. □

Under the ODE

$$\frac{d}{dt}(\lambda + \mu + \nu) = \lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \lambda\nu + \mu\nu$$

and this quadratic can be written as

$$\frac{1}{2}\left[(\lambda + \mu)^2 + (\lambda + \nu)^2 + (\mu + \nu)^2\right]$$

so it is clearly non-negative. Thus the first inequality is preserved. The second inequality can be written as

$$\lambda + \mu + \nu \geq f(-\nu)$$

which becomes

$$\lambda + \mu \geq (-\nu) \log(-\nu).$$

It is easier to keep track of the signs if we let $n = -\nu$, and write it as

$$\lambda + \mu \geq n \log n.$$

To show the inequality is preserved we only need to look at points on the boundary of the set. If $\nu + f^{-1}(\lambda + \mu + \nu) = 0$ then $\nu = -f^{-1}(\lambda + \mu + \nu) \leq -1$ since $f^{-1}(y) \geq 1$ for all y. This makes $n \geq 1$, so $n \log n \geq 0$ and $\lambda + \mu \geq 0$. Since $\lambda \geq \mu$ we must at least have $\lambda \geq 0$. But μ may have either sign.

We deal first with the case where $\mu \geq 0$. Then we need to verify

$$\frac{d\lambda}{dt} + \frac{d\mu}{dt} \geq (\log n + 1)\frac{dn}{dt}$$

when $\lambda + \mu = n \log n$. Solving for

$$\log n = \frac{\lambda + \mu}{n}$$

and substituting above, we must show

$$\lambda^2 - \mu n + \mu^2 - \lambda n \geq \left(\frac{\lambda + \mu}{n} + 1\right)\left(-n^2 - \lambda\mu\right)$$

which reduces to

$$\left(\lambda^2 + \mu^2\right) n + \lambda\mu(\lambda + \mu + n) + n^3 \geq 0$$

and since λ, μ and n are all positive or zero we are done here.

In the other case where $\mu \leq 0$ we again change the sign by letting $\mu = -m$. Then the inequality becomes

$$\lambda \geq m + n \log n.$$

To show the inequality is preserved we must verify that

$$\frac{d\lambda}{dt} \geq \frac{dm}{dt} + (\log n + 1)\frac{dn}{dt}$$

when $\lambda = m + n \log n$. Solving for

$$\log n = \frac{\lambda - m}{n}$$

and substituting above, we must show

$$\lambda^2 + mn \geq \lambda n - m^2 + \left(\frac{\lambda - m}{n} + 1\right)\left(\lambda m - n^2\right)$$

when $\lambda \geq 0$ and $0 \leq m \leq n$ (and $n \geq 1$). This simplifies algebraically to showing

$$\lambda^2 n + \lambda m^2 + m^2 n + n^3 \geq \lambda^2 m + \lambda m n$$

which is equivalent to

$$\left(\lambda^2 - \lambda m + m^2\right)(n - m) + m^3 + n^3 \geq 0$$

which must hold because

$$\lambda^2 - \lambda m + m^2 \geq 0 \quad \text{and} \quad n - m \geq 0.$$

Hence the proof is complete.

COROLLARY 24.5. *For any constants $B < \infty$ and $\delta > 0$ there exists a constant $C < \infty$ with the following property. If any solution to the Ricci Flow on a complete three-manifold with bounded curvature satisfies $|Rm| \leq B$ at $t = 0$, then for $t \geq 0$ it satisfies the estimate*

$$M(X,Y) \geq -(\delta R + C)g(X,Y)$$

on the curvature operator M. Hence when the curvature R is big, any negative curvature is very small in comparison.

The following refinement of these techniques gives a curvature pinching result useful for classifying Type I singularities on a three-manifold.

THEOREM 24.6. *Suppose we have a solution to the Ricci Flow on a compact three-manifold on a maximal time interval $0 \leq t < T$ which is Type I, so*

$$\limsup_{t \to T} (T-t)|M| < \infty$$

and suppose the manifold never acquires positive sectional curvature everywhere. Then there exists a $\theta > 0$ such that for every $\tau < T$ and every $\delta > 0$ we can find a time t in $\tau \leq t < T$ and a point P where $(T-t)|M| \geq \theta$ and a frame at P in which

$$|M - RE| \leq \delta |M|$$

where the scalar curvature $R = trM$ is the trace of the curvature operator M and E is the curvature operator matrix of a round cylinder $S^2 \times R^1$ given by

$$E = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} .$$

COROLLARY 24.7. *The limit of dilations of the solution around these points and times gives an ancient solution with bounded non-negative sectional curvature whose holonomy reduces. Consequently it splits as a product of a surface with R^1.*

Proof. Since the minimum of R increases, we can choose a constant $\rho \geq 0$ so that

$$R + \rho > 0$$

for all $t \geq 0$. The pinching estimates imply that for large $|M|$ any negative eigenvalues M may have are not nearly as great in absolute value as some positive one; and hence there is some constant $A < \infty$ so that

$$|M| \leq A(R + \rho)$$

for all $t \geq 0$. □

We shall prove the converse of the Theorem. Suppose that for every $\theta > 0$ there exist $\tau < T$ and $\delta > 0$ such that at every point and in every frame at any time t with $\tau \le t < T$ we always have

$$(T-t)|M| \le \theta \quad \text{or else} \quad |M - RE| \le \delta |M|.$$

We shall then show the manifold shrinks to a point and becomes round. We shall let $C < \infty$ and $c > 0$ denote various constants which may depend on A and ρ (as well as the dimension $n = 3$) but which for now are independent of the parameters $\theta, \tau, \delta, \eta, \varepsilon$ which we will choose as follows. We pick θ small enough to start, choose τ and δ depending on θ from the new hypothesis, pick η depending on δ, and finally choose ε depending on ζ. The exact choices of $\theta, \tau, \delta, \eta, \varepsilon$ will be explained as the proof evolves.

Using $R + \rho > 0$, consider the function

$$F = (T-t)^{\varepsilon} |\overset{\circ}{\to} M|^2/(R+\rho)^{2-\varepsilon}$$

where

$$\overset{\circ}{\to} M = M - \frac{1}{3} R I$$

is the trace-free part of M when I is the identity matrix in an orthonormal frame. The matrix M evolves by

$$D_t M = \Delta M + M'$$

where if

$$M = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}$$

in an appropriate frame then

$$M' = \begin{pmatrix} \lambda^2 + \mu\nu & & \\ & \mu^2 + \lambda\nu & \\ & & \nu^2 + \lambda\mu \end{pmatrix}.$$

The trace R evolves by

$$D_t R = \Delta R + R'$$

where R' is the trace of M', and the trace-free part $\overset{\circ}{\to} M$ evolves by

$$D_t \overset{\circ}{\to} M = \Delta \overset{\circ}{\to} M + \overset{\circ}{\to} M'$$

where $\overset{\circ}{\to} M'$ is the trace-free part of M'.

Using the identity

$$\begin{aligned} &\Delta \left[|\overset{\circ}{\to} M|^2/(R+\rho)^{2-\varepsilon} \right] \\ &\quad + (2-\varepsilon)\left[DR/(R+\rho)\right] D \left[|\overset{\circ}{\to} M|^2/(R+\rho)^{2-\varepsilon} \right] \\ &= 2 \overset{\circ}{\to} M \cdot \Delta \overset{\circ}{\to} M/(R+\rho)^{2-\varepsilon} - (2-\varepsilon)|\overset{\circ}{\to} M|^2 \Delta R/(R+\rho)^{3-\varepsilon} \\ &\quad + \left\{ \varepsilon |D \overset{\circ}{\to} M|^2 + (2-\varepsilon)\left| D \overset{\circ}{\to} M - \overset{\circ}{\to} \frac{MDR}{R+\rho} \right|^2 \right\} (R+\rho)^{\varepsilon-2} \end{aligned}$$

and discarding the last term in braces which is clearly positive, we can compute the evolution of F as

$$D_t F = \Delta F + (2-\varepsilon)[DR/(R+\rho)] \cdot DF + F'$$

where F' is computed from the ODE's as

$$\begin{aligned} F' = 2(T-t)^\varepsilon \overset{\circ}{\rightarrow} M \cdot \overset{\circ}{\rightarrow} \quad & M' / (R+\rho)^{2-\varepsilon} - \varepsilon (T-t)^{\varepsilon-1} \\ & |\overset{\circ}{\rightarrow} M|^2 / (R+\rho)^{2-\varepsilon} - (2-\varepsilon)(T-t)^\varepsilon | \\ & \overset{\circ}{\rightarrow} M|^2 R' / (R+\rho)^{3-\varepsilon}. \end{aligned}$$

We can regroup this as

$$F' = (T-t)^\varepsilon [X - 2Y]/(R+\rho)^{3-\varepsilon}$$

where

$$X = 2\rho \overset{\circ}{\rightarrow} M \cdot \overset{\circ}{\rightarrow} M' + \varepsilon |\overset{\circ}{\rightarrow} M|^2 R' - \varepsilon (R+\rho) |\overset{\circ}{\rightarrow} M|^2 / (T-t)$$

and

$$Y = |\overset{\circ}{\rightarrow} M|^2 R' - R \overset{\circ}{\rightarrow} M \cdot \overset{\circ}{\rightarrow} M'.$$

(Note Y is the only term we would have if $\varepsilon = 0$ and $\rho = 0$.) Using the ODE's we compute explicitly

$$Y = \lambda^2(\mu-\nu)^2 + \mu^2(\lambda-\nu)^2 + \nu^2(\lambda-\mu)^2$$

and note $Y = 0$ on the symmetric spaces $S^3, S^2 \times R^1, R^3, H^2 \times R^1$, and H^3 where $\lambda = \mu = \nu$ or $\lambda = \mu = 0$ or $\lambda = \nu = 0$ or $\mu = \nu = 0$, while $Y > 0$ elsewhere.We can estimate X from above as follows. The matrix $\overset{\circ}{\rightarrow} M$ has diagonal entries like

$$\frac{1}{3}[(\lambda-\mu) + (\lambda-\nu)]$$

so $|\overset{\circ}{\rightarrow} M|$ is comparable to

$$(\lambda-\mu) + (\lambda-\nu) + (\mu-\nu)$$

up to a constant factor above and below. The matrix $\overset{\circ}{\rightarrow} M'$ has diagonal entries like

$$\frac{1}{3}[(\lambda-\mu)(\lambda+\mu-\nu) + (\lambda-\nu)(\lambda+\nu-\mu)]$$

so $|\overset{\circ}{\rightarrow} M'| \le C|M||\overset{\circ}{\rightarrow} M|$ for some constant C. This gives a bound on the first term in X

$$2\rho \overset{\circ}{\rightarrow} M \cdot \overset{\circ}{\rightarrow} M' \le C\rho |M||\overset{\circ}{\rightarrow} M|^2.$$

We also have a bound

$$|M'| \le C|M|^2$$

and R' is the trace of M', so we get a bound on the second term in X

$$\varepsilon |\overset{\circ}{\rightarrow} M|^2 R' \le C\varepsilon |M|^2 |\overset{\circ}{\rightarrow} M|^2.$$

Finally $|M| \leq A(R+\rho)$ so we get a bound on the third term

$$\varepsilon(R+\rho)| \overset{\circ}{\rightarrow} M|^2/(T-t) \geq c\varepsilon|M|| \overset{\circ}{\rightarrow} M|^2/(T-t).$$

This gives a bound

$$X \leq C\rho|M|| \overset{\circ}{\rightarrow} M|^2 + C\varepsilon|M|^2| \overset{\circ}{\rightarrow} M|^2 - c\varepsilon|M|| \overset{\circ}{\rightarrow} M|^2/(T-t).$$

on the quantity X.

We can also estimate Y from below.

LEMMA 24.8. *For every $\delta > 0$ there exists an $\zeta > 0$ such that if the matrix M satisfies*

$$|M - RE| \geq \delta|M|$$

in every frame then

$$Y \geq \zeta|M|^2| \overset{\circ}{\rightarrow} M|^2.$$

Proof. We saw $Y > 0$ if we avoid the lines where $M = RE$ or $M = \frac{1}{3}RI$. Hence by homogeneity $Y \geq \zeta|M|^4$ for some $\zeta > 0$ if

$$|M - RE| \geq \delta|M| \quad \text{and} \quad \left|M - \frac{1}{3}RI\right| \geq \delta|M|.$$

If $\left|M - \frac{1}{3}RI\right| \leq \delta|M|$ for δ small, we surely have all the eigenvalues of the same sign with comparable magnitudes, and

$$\begin{aligned}\lambda^2(\mu-\nu)^2 &+ \mu^2(\lambda-\nu)^2 + \nu^2(\lambda-\mu)^2 \\ &\geq \zeta(\lambda^2+\mu^2+\nu^2)[(\lambda-\mu)^2 + (\lambda-\nu)^2 + (\mu-\nu)^2]\end{aligned}$$

for some $\zeta > 0$. Hence in either case we are done.

Given ζ as above, choose $\varepsilon > 0$ so small that $C\varepsilon \leq \zeta$ for the constant C in the bound on X. If $|M - RE| \geq \delta|M|$ then

$$X - 2Y \leq C\rho|M|| \overset{\circ}{\rightarrow} M|^2 - \zeta|M|^2| \overset{\circ}{\rightarrow} M|^2.$$

On the other hand, if $(T-t)|M| \leq \theta$ then neglecting $Y \geq 0$ we have

$$X - 2Y \leq C\rho|M|| \overset{\circ}{\rightarrow} M|^2 - \left(\frac{c}{\theta} - C\right)|M|^2| \overset{\circ}{\rightarrow} M|^2$$

and if we pick $\theta > 0$ at the beginning with $\theta \leq c/(C+1)$ then $c/\theta - C \geq 1$. Since ε is small compared to ζ, we have

$$X - 2Y \leq C\rho|M|| \overset{\circ}{\rightarrow} M|^2 - \varepsilon|M|^2| \overset{\circ}{\rightarrow} M|^2$$

in either case. As a consequence

$$X - 2Y \leq \left(\frac{C^2\rho}{2\varepsilon} - \frac{\varepsilon}{2}|M|^2\right)| \overset{\circ}{\rightarrow} M|^2.$$

Having come this far, since ε is now chosen we loose nothing to let our constants C and c depend on ε from now on . Then we can write this as

$$X - 2Y \leq (C\rho - c|M|^2)| \overset{\circ}{\rightarrow} M|^2.$$

□

We summarize our argument so far.

LEMMA 24.9. *There exist constants $\rho \geq 0, A < \infty, C < \infty, c > 0$ and $\varepsilon > 0$ such that $R + \rho > 0$ and $|M| \leq A(R + \rho)$, and if*

$$F = (T - t)^{\varepsilon} | \overset{\circ}{\rightarrow} M|^2/(R + \rho)^{2-\varepsilon}$$

then

$$D_t F = \Delta F + V \cdot DF + F'$$

where

$$V = (2 - \varepsilon) DR/(R + \rho)$$

and

$$F' \leq (T - t)^{\varepsilon}[C\rho^2 - c|M|^2]| \overset{\circ}{\rightarrow} M|^2/(R + \rho)^{3-\varepsilon}.$$

COROLLARY 24.10. *We have $F \to 0$ as $t \to T$.*

Proof. Choose any $\lambda > 0$. When

$$(T - t)|M| \leq \lambda$$

since $| \overset{\circ}{\rightarrow} M| \leq |M|$ and $|M| \leq A(R + \rho)$ we have

$$F \leq A^{2-\varepsilon} \lambda^{\varepsilon}$$

which is as small as we like if λ is small enough. But when

$$(T - t)|M| \geq \lambda$$

we have $|M|$ quite large for t near T, so

$$C - c|M|^2 \leq -\frac{1}{2} c|M|^2$$

and

$$F' \leq -\frac{1}{2} c|M|^2 F/(R + \rho).$$

On the other hand, now that $|M|$ is large

$$R + \rho \leq \sqrt{3}\, |M| + \rho \leq 2|M|$$

so

$$F' \leq -\frac{1}{4}c|M|F.$$

Using $|M| \geq \lambda/(T-t)$ we get

$$F' \leq -\frac{1}{4}c\lambda F/(T-t)$$

for t near T. Thus when the maximum F_{MAX} of F exceeds $A^{2-\varepsilon}\lambda^{\varepsilon}$ it must decrease at a rate

$$\frac{d}{dt}F_{\text{MAX}} \leq -pF_{\text{MAX}}/(T-t)$$

where $p = \frac{1}{4}c\lambda > 0$. This implies

$$\frac{d}{dt}(T-t))^{-p}F_{\text{MAX}} \leq 0$$

so if $(T-t)^{-p}F_{\text{MAX}} = B$ at some time τ close enough to T for the above estimates to hold then subsequently

$$F_{\text{MAX}} \leq B(T-t)^p \quad \text{or} \quad F_{\text{MAX}} \leq A^{2-\varepsilon}\lambda^{\varepsilon}$$

and so when t is even closer to T the second holds. But $\lambda > 0$ is arbitrary, so $F \to 0$.

Now we can show that the manifold shrinks to a point and becomes round. By assumption

$$(T-t)|M| \leq \Omega$$

for some constant Ω. On the other hand there exists a constant $\omega > 0$ such that at each t we have

$$(T-t)|M| \geq \omega$$

somewhere, or else $|M|$ could not go to infinity as $t \to T$ because

$$D_t M = \Delta M + M^2 + M^{\#}$$

would not allow such rapid growth. Hence the maximum of $|M|$ is always proportional to $1/(T-t)$. The quantity F is dilation-invariant, so when we form the Type I limit (which must exist by our injectivity radius estimate which we proved in Corollary 23.3) we have $F = 0$ on the limit. Hence the limit metric has $\overset{\circ}{\to} M = 0$, and hence has $M = \frac{1}{3}RI$. But this implies the curvature is constant, (as we have had occasion to observe before from the contracted second Bianchi identity). Since the curvature is positive, the limit is a sphere S^3 or a quotient space-form S^3/Γ. This proves the theorem. □

25 Limits with Strictly Positive Curvature Operator. Given a sequence of complete solutions to the Ricci Flow with uniformly bounded curvature on some time interval, we can extract a convergent subsequence by the result in [**26**] provided we can control the injectivity radius at the origin points. In general this may be hard, but there is one important case where we get it for free. This is based on the observation that for a complete non-compact manifold with strictly positive sectional curvature we can bound the injectivity radius by the maximum of the curvature.

The situation we consider here is not quite that simple, but with some work it is also possible to estimate the injectivity radius. We have a sequence of solutions to the Ricci flow where the sectional curvatures are bounded, where the lower bound is negative but increases to zero (as we have seen always happens after dilation if $n = 3$), and where the sectional curvatures at the origin points are uniformly bounded positive away from zero, and where the diameters go to infinity. In this case when we are far out in the sequence the curvature stays positive a long way out, and is never very negative. This is enough to produce a neighborhood of the origin which is convex and contains a ball of enough size to give a good lower bound on the injectivity radius. We now make this precise.

THEOREM 25.1. *Suppose we have a sequence of solutions to the Ricci Flow given by metrics G_j on manifolds M_j with origins O_j and frames $\mathcal{F}_j$ for times $\alpha < t < \omega$ (with $\alpha < 0 < \omega$) which are all complete, and such that for some $\rho > 0$*

(a) all the sectional curvatures of all the metrics G_j are at most $1/\rho^2$

(b) there is a sequence $\delta_j \to 0$ such that all the eigenvalues of the curvature operator Rm_j of the metric G_j are at least $-\delta_j/\rho^2$

(c) there is an $\varepsilon > 0$ such that all the eigenvalues of the curvature operator Rm_j of G_j at the origin O_j are at least ε/ρ^2

(d) the diameters d_j of the metrics G_j go to ∞.

Then there is a subsequence of the metrics such that all the injectivity radii at the origins are at least this $\rho > 0$. Hence a subsequence converges to a solution G_∞ of the Ricci Flow on $\alpha < t < \omega$.

Proof. The first step is to extract a subsequence which would want to converge if we could control the injectivity radii. To do this we introduce the notion of a geodesic tube in a manifold M with origin. Given a frame $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ at the origin O and a length L, we begin by constructing the geodesic of length L out of O in the direction F_1 and its opposite. Then we parallelly translate the frame $\mathcal{F}$ along this geodesic, and take the geodesic out of each point in the direction F_2 and its opposite of length ρ. Parallelly translate $\mathcal{F}$ along these also, and take the geodesic out of each of these points in the direction F_3 and its opposite of length ρ, and so on. Notice that only in the first direction do we go a long way L, while in the other directions we don't go farther than ρ. The curvature satisfies $|K| \leq \rho$, so this construction gives a local diffeomorphism of

$$(-L, L) \times (-\rho, \rho) \times \cdots \times (-\rho, \rho) \longrightarrow M.$$

Consider the pull-back metrics. For the Ricci Flow a bound on the curvature gives a bound on all the derivatives of the curvature. Then by ordinary differential equations we get bounds on the pull-back metric and all its derivatives with respect to the tube coordinates. (Here we omit the details.)

If we consider a fixed reference frame $\mathcal{F}_j$ at the origin in each M_j and take an element A of the orthogonal group, then $A\mathcal{F}_j$ is a frame at the origin in M_j, and we can take the pull-back metric for the geodesic tube on $A\mathcal{F}_j$. For a fixed A and a fixed L, we can always find a convergent subsequence of the pull-back metrics. By choosing a countable dense set of A's and a sequence of L's going to infinity, and by a diagonalization argument, we can find a subsequence of metrics so that the pull-back metrics to the tube on the frames $A\mathcal{F}_j$ of length L converge for every A and every L. In this case we say the metrics preconverge along geodesic tubes. (Note any convergent sequence would be preconvergent.) The advantage of preconvergence is that we do not need to control the injectivity radius to get it. Form now on we only deal with such a preconvergent sequence.

We can strengthen the notion of preconvergence to compare one tube with another. For any two vectors X and Y in R^n (which we identify with the tangent spaces at the origins 0_j in the M_j with the frames $\mathcal{F}_j$) we can consider the sequence of distances

$$d_j = d_j(\exp_j X, \exp_j Y) \leq |X| + |Y|$$

in M_j ; by picking a subsequence we can assume the d_j converge. If we do this by diagonalization for a countable dense set of pairs (X_α, Y_α) then in fact d_j will converge for every pair (X, Y). To see this take any $\varepsilon > 0$. Choose a sequence of pairs with $X_\alpha \to X$ and $Y_\alpha \to Y$. Since we have preconvergence in geodesic tubes in the directions X and Y, the metrics G_j converge to a limit G_∞^X in the tube on X, and to a limit G_∞^Y in the tube on Y. We can find a constant $c > 0$ depending only on the dimension so that for any $\zeta > 0$ small enough, if $|\widetilde{X} - X| \leq c\zeta\rho$ then in the metric G_∞^X

$$d_\infty^X \left(\exp_\infty^X \widetilde{X}, \exp_\infty^X X\right) \leq \zeta\rho$$

and likewise if $|\widetilde{Y} - Y| \leq c\zeta\rho$ then in the metric G_∞^Y

$$d_\infty^Y \left(\exp_\infty^Y \widetilde{Y}, \exp_\infty^Y Y\right) \leq \zeta\rho.$$

Given $\zeta > 0$, choose α so large that

$$|X_\alpha - X| \leq c\zeta\rho \quad \text{and} \quad |Y_\alpha - Y| \leq c\zeta\rho.$$

Then

$$d_\infty^X \left(\exp_\infty^X X_\alpha, \exp_\infty^X X\right) \leq \zeta\rho$$

and

$$d_\infty^Y \left(\exp_\infty^Y Y_\alpha, \exp_\infty^Y Y\right) \leq \zeta\rho.$$

Now choose j large enough depending on X, Y, α, η, and ρ so that

$$\left|d_j(\exp_j X_\alpha, \exp_j X) - d_\alpha^X \left(\exp_\infty^X X_\alpha, \exp_\infty^X X\right)\right| \leq \zeta\rho$$

and

$$\left|d_j(\exp_j Y_\alpha, \exp_j Y) - d^X_\alpha\left(\exp^Y_\infty Y_\alpha, \exp^Y_\infty Y\right)\right| \le \zeta\rho.$$

Finally make j large enough also depending on X, Y, α, ζ and ρ so that

$$d_j(\exp_j X_\alpha, \exp_j Y_\alpha) \le \zeta\rho$$

since X_α and Y_α are in the countable set for which the sequence is preconvergent in distances. Then

$$d_j(\exp_, X, \exp_j Y) \le 5\zeta\rho.$$

Since ζ is arbitrary, the sequence is preconvergent in distances for all X and Y as claimed.

In fact we can do a little better along the lines of [H]. Using the geodesic tube coordinates at $t = 0$, we can also consider the pull-back of the metric at earlier or later times, which we can bound using curvature bounds, since we know the metric evolves by the curvature under Ricci Flow. Then we can actually make the pull-backs of the Ricci Flow converge to a solution of the Ricci Flow in every geodesic tube. We can also keep the solutions preconvergent in the distances $d_j(\exp_m X, \exp_j Y)(t)$ for all X and Y at every time t. □

LEMMA 25.2. *For every length L we can find $\varepsilon(L) > 0$ and $J(L) > \infty$ such that all eigenvalues of the curvature operator on M_j at points within distance L of the origin have $\kappa \ge \varepsilon(L)$ when $j \ge J(L)$.*

Proof. Suppose not. Then we can find a sequence of points $X_j = \exp_0(\ell_j V_j)$ at distances $\ell_j \le L$ from the origin in some directions V_j with $|V_j| = 1$ such that some eigenvalues of the curvature operators at the X_j are not bounded away from 0 on the positive side. Since on M_j we have there eigenvalues $\ge -\delta_j/\rho^2$ with $\delta \to 0$, they in fact go to zero. □

Find a convergent subsequence $V_j \to V$ and $\ell_j \to \ell$ and pick a geodesic tube in each M_j starting in the direction V. By preconvergence we get a limit which solves the Ricci Flow in the tube and the limit will have some eigenvalue of the curvature operator equal to zero at the point ℓV with $\ell \le L$. But in the limit all the eigenvalues of the curvature operator are ≥ 0, so by the strong maximum principle (see[**29**]; the argument works locally also) there must be a zero eigenvalue of the curvature operator everywhere in the tube at every time, in particular at the origin at $t = 0$. But for the sequence we had the eigenvalues of the curvature operator at $O_j \ge \varepsilon$, so this holds in the limit also. Since this is a contradiction, the Lemma is established.

In a manifold M with origin 0, we define the function $\ell(V)$ on unit tangent vectors V at 0 with values in $[0, \infty]$ to be the distance to the cut locus in the direction V. If exp is the exponential map at the origin, then

$$\ell(V) = \max\{\ell; d(\exp \ell V, 0) = \ell\}.$$

It is well-known (see Cheeger and Ebin [**9**]) that the distance to the cut locus is a continuous function. Moreover if $\ell = \ell(V)$ then either the geodesic $\exp(sv)$

for $0 \le s \le \ell$ has a non-zero Jacobi field vanishing at the ends, or there exists another $W \neq V$ write $\exp(\ell W) = \exp(\ell V)$.

The choice of frames $\mathcal{F}_j$ at the origins O_j in M_j allows us to identify the tangent spaces at the origins with R^n. We define the set $\mathcal{D}$ of distinguished directions as those in which we can go off to infinity as $j \to \infty$. To see this is well-defined, let $\ell_j(V)$ for a unit vector V in R^n be the distance to the cut locus in M_j in the direction V relative to the frame $\mathcal{F}_j$.

LEMMA. *For any sequence $V_j \to V$, the limit $\ell_\infty(V) = \lim_{j\to\infty} \ell_j(V_j)$ exists and depends only on V and is a continuous function of V, when the sequence of manifolds is preconvergent.*

Proof. First we show the limit exists. We can always define $\ell_\infty(V) =_{j\to\infty}^{\to} \liminf \ell_j(V_j)$. Choose a subsequence of j's for which the lim inf is attained as a limit. If $\liminf = \infty$ we are done. Otherwise for each j, either there is a non-zero Jacobi field J_j or an alternate geodesic in the direction W_j. By passing to a subsequence, there is always either one or the other. □

If there is always a Jacobi field J_j, we can take its derivative dJ_j/ds at the origin to be a unit tangent vector X_j. By choosing a subsequence we can make X_j converge to some unit tangent vector X. The metrics preconverge in the geodesic tube around V, so the limit metric has a non-zero Jacobi field J vanishing at 0 with $dJ/ds = X$, and J vanishes again at $\exp_\infty(sV)$ with $s = \ell_\infty(V)$. This means that the index form

$$I(J,J) = \int \left[|DJ|^2 - R(T,J,T,J)\right] ds$$

on the geodesic $\exp_\infty(sV)$ on $0 \le s \le \ell_\infty(V)$ has a null space, and hence has a strictly negative direction on $0 \le s \le \ell_\infty(V) + \varepsilon$ for any $\varepsilon > 0$. Then it also has a negative direction on $0 \le s \le \ell_\infty(V) + \varepsilon$ in any metric G_j when j is large enough, and thus

$$\ell_j(V) \le \ell_\infty(V) + \varepsilon.$$

Therefore $\ell_j(V) \to \ell_\infty(V)$ for all $j \to \infty$, not just for the subsequence.

Otherwise we find a subsequence where $\exp_j(\ell_j W_j) = \exp_j(\ell_j V_j)$ for some sequence $W_j \neq V_j$ with $\ell_j = \ell_j(V)$. By taking a subsequence we can assume $W_j \to W$. If $W = V$, then the limit metric in a geodesic tube in the direction V again has a non-zero Jacobi field on $\exp_\infty(sV)$ vanishing at $s = \ell_\infty(V)$, and we are done. This Jacobi field J can be bound by taking $J = 0$ and $dJ/ds = X$ at the origin 0 where for some subsequence

$$X = \lim_{j\to\infty} \frac{W_j - V_j}{|W_j - V_j|} .$$

Since $\exp_j(sV)$ and $\exp_j(sW_j)$ are geodesics in the metrics G_j and $G_j \to G$ in the tube on $V = W$, we can check that

$$J(s) = \lim_{j\to\infty} \frac{\exp_j(sW_j) - \exp_j(sV_j)}{|W_j - V_j|}$$

converges for the subsequence chosen above to the desired Jacobi field, with $J = 0$ again at $s = \ell_\infty(V)$. If $W \neq V$, we take two geodesic tubes in the directions V and W. Then for our subsequence

$$d_j(\exp_j(\ell_j V_j), \exp_j(\ell_j W_j)) \to 0$$

and since $\ell_j \to \ell = \ell_\infty(V)$ and $V_j \to V$ and $W_j \to W$ we also have in the tube on V

$$d_j(\exp_j(\ell_j V_j), \exp_j(\ell V)) \to 0$$

and in the tube on W

$$d_j(\exp_j(\ell_j W_j), \exp_j(\ell W)) \to 0$$

which makes

$$d_j(\exp_j(\ell_j V_j), \exp_j(\ell W)) \to 0$$

for our subsequence. But this sequence is defined for all j, and the limit exists because we have made our metrics preconvergent in distance. Hence this sequence not just the subsequence, goes to zero for all j.

Now consider the picture in the geodesic tube in the direction V for each M_j with j large. There is the geodesic out of V from the center, and close to it is the geodesic out of V_j. At distance ℓ out the tube there is another geodesic passing through the tube which came out of W, and at a distance ℓ out of W it is close to the point at distance ℓ out of V. The metrics converge in the tube, and the geodesics out of W will converge in the tube to a limit geodesic which we call γ. Now γ passes through the point P at distance ℓ out along the geodesic $\overline{\gamma}$ down the center which came out of V. But we claim γ cannot coincide with $\overline{\gamma}$. For if it did, the corresponding γ_j out of W_j and $\overline{\gamma}_j$ out of V_j in M_j for the subsequence of j would be close, and hence both in the tube in direction V, and their starting vectors V_j and W_j would be as close as we like. But $V_j \to V$ and $W_j \to W$ with $W \neq V$. Hence γ and $\overline{\gamma}$ are distinct.

Now the argument is a little subtle, because γ is only defined in the tube around $\overline{\gamma}$. If we had a limit metric, then γ would be a geodesic out of W, and the distance to the crossing point P would be the same along γ and $\overline{\gamma}$. In this case it would be a shorter path, once we are beyond P along $\overline{\gamma}$, to go in a perpendicular from $\overline{\gamma}$ over to γ and then follow γ back to the origin. For short distances beyond P, the savings in distance is on the order of a fraction given by the sine of the angle between γ and $\overline{\gamma}$. (This would be exact for the flat metric.) Since we have a uniform curvature bound, for short distances beyond P we still save almost this much. Now if we take j large enough, since the metrics converge in the tube our savings in cutting over from the geodesic $\overline{\gamma}_j$ out of V_j to the geodesic $\overline{\gamma}_j$ out of W will still be almost this much. Thus for every $\varepsilon > 0$ we can find $J(\varepsilon)$ so that if $j \geq J(\varepsilon)$ then $\ell_j(V_j) < \ell_\infty(V) + \varepsilon$, since the geodesic out of V_j does not minimize length at distance much past $\ell = \ell_\infty(V)$. This proves the assertion that

$$\ell_\infty(V) = \lim_{j\to\infty} \ell_j(V_j)$$

always exists.

It follows easily that $\ell_\infty(V)$ is independent of the choice of the sequence $V_j \to V$. For if we have two different sequences, we can collate them to get a new sequence by odd and even j and the odd and even subsequences cannot have different limits.

It also follows that $\ell_\infty(V)$ is continuous in V. For let V_k be any sequence which converges to V. For each k choose j_k so large that in M_{jk} we have

$$|\ell_{j_k}(V_k) - \ell_\infty(V_k)| \leq 1/k$$

if $\ell_\infty(V_k) < \infty$, otherwise we make $\ell_{j_k}(V_k) \geq k$ if $\ell_\infty(V_k) = \infty$. Then for the subsequence j_k we have

$$\lim_{k\to\infty} \ell_{j_k}(V_k) = \ell_\infty(V)$$

by the previous argument. Hence

$$\lim_{k\to\infty} \ell_\infty(V_k) = \ell_\infty(V)$$

also, and we are done proving the Lemma.

Now we let $\mathcal{D}$ be the set of directions in which we can go off to infinity without hitting the cut locus in M_j as $j \to \infty$; specifically

$$\mathcal{D} = \left\{V \in S^{n-1} : \ell_\infty(V) = \infty\right\}.$$

Since the diameters of the M_j go to infinity, the set $\mathcal{D}$ is not empty. To see this, pick a sequence V_j with $\ell_j(V_j) \to \infty$ and find a subsequence with $V_{jk} \to V_j$ then $\ell_\infty(V) = \infty$ so $V \in \mathcal{D}$.

Moreover

$$\lim_{j\to\infty} \inf_{V\in\mathcal{D}} \ell_j(V) = \infty.$$

For if not, pick $V_j \in \mathcal{D}$ with $\ell_j(V_j) \leq \ell < \infty$ for some subsequence j and some $\ell < \infty$. For another subsequence $V_j \to V$. But $\ell_\infty(V) = \lim_{j\to\infty} \ell_\infty(V_j)$ since ℓ_∞ is continuous, and $\ell_\infty(V_j) = \infty$ since $V_j \in \mathcal{D}$, so $\ell_\infty(V) = \infty$ also and $V \in \mathcal{D}$. But $\ell_j(V_j) \to \ell_\infty(V)$ also, which is a contradiction. Now recall that all sectional curvatures on M_j have $\kappa_j \leq 1/\rho^2$ for some $\rho > 0$ independent of j. We define the set N_j in M_j in the following way:

$$\begin{aligned} N_j = \{\exp_j(sW) : |W| = 1 \quad &\text{and for} \quad s \leq \ell_j(W); \\ &\text{and for all} \quad V \in \mathcal{D}, s' \leq s \quad \text{and} \quad r \leq \ell_j(V) \\ &\text{we have} \quad d_j \exp_j(s'W), \exp_j(rV) \geq r - \rho\}. \end{aligned}$$

First note that N_j is closed and not empty; for N_j is defined as an intersection of closed sets, and contains the ball of radius ρ around the origin O_j in M_j.

LEMMA 25.3. *There exists an $L < \infty$ such that for all large enough j the set N_j lies in the ball of radius L around the origin O_j in M_j.*

Proof. If not, we could pass to a subsequence of j's and find a sequence $s_j \to \infty$ and W_j in the unit sphere with

$$\exp_j(s_j w_j) \in N_j.$$

For another subsequence we have $W_j h \to v$ for some V. Now $s_j \leq \ell_j(W_j)$ and $s_j \to \infty$ so $\ell_\infty(V) = \infty$ and $V \in \mathcal{D}$. However if we take any $s' > \rho$ fixed then

$$d_j(\exp_j(s'W_j), \exp_j(s'V)) \to 0$$

and we get a contradiction, since we must have

$$d_j(\exp_j(s'W_j), \exp_j(s'V)) \geq s' - \rho > 0$$

once $s_j \geq s'$. □

Among all geodesic loops starting and ending at the same point and lying entirely in the compact set N_j there will be a shortest one. Call it γ_j, and suppose j starts and ends at a point we call P_j. If γ_j has length at least ρ for all j, we are done. When γ_j is shorter than ρ we consider two cases (and rule them both out). The first case is when γ_j makes an angle π with itself at P_j, hence forming a geodesic circle. For any r no matter how large and any $V \in \mathcal{D}$ we can take j large enough to make $\ell_j(V) \geq r$. Consider the point $X_j = \exp_j(rV)$, and find the point Y_j on γ_j closest to X_j. Let $Y_j = \exp_j(sW)$ with $|W| = 1$ be an exponential representation of Y_j in N_j. Then taking $s' = s$ we get

$$d(X_j, Y_j) \geq r - \rho.$$

Now we can find $\varepsilon > 0$ depending on $L + \rho$ so that all sectional curvatures κ_j on M_j in the ball of radius $L + \rho$ around the origin O_j have $\kappa_j \geq \varepsilon/\rho^2$ independent of j, by our previous Lemma. Take a shortest geodesic ζ_j from X_j to Y_j. Then along ζ_j for a distance ρ from Y_j we have all $\kappa_j \geq \varepsilon/\rho^2 > 0$. Moreover by taking j large we can make all sectional curvatures $\kappa_j \geq -\delta_j/\rho^2$ for δ_j as small as we like, and we can make r as large as we like. In this case the standard computation shows the second variation of the arc length of the geodesic η_j fixing one endpoint at X_j and the other on γ_j is strictly negative. Indeed let Z_j be the unit tangent vector to γ_j at Y_j and extend Z_j to ζ_j by parallel translation. Choose a function φ to be identically 1 within distance ρ of Y_j along η_j and then to drop linearly to zero. The second variation of arc length in the direction φZ_j is

$$I(\varphi Z_j, \varphi Z_j) = \int \left[|D\varphi|^2 - \kappa_j \varphi^2\right] ds$$

where

$$\kappa_j = Rm(T_j, Z_j, T_j, Z_j)$$

is the sectional curvature of the plane spanned by the unit tangent vector T_j to ζ_j and by Z_j. Considering the separate contributions from the part of η_j within ρ of Y_j and the past beyond

$$I(\varphi Z_j, \varphi Z_j) \leq -\varepsilon/\rho + 1/r + \delta_j r/\rho^2.$$

First take r so large that

$$1/r \leq \varepsilon/3\rho$$

and then take j so large that

$$\delta_j r/\rho^2 \leq \varepsilon/3\rho$$

and we still have

$$I(\varphi Z_j, \varphi Z_j) \leq -\varepsilon/3\rho$$

so the second variation is strictly negative. But now we see Y_j is not the closest point on γ_j to X_j, which is a contradiction. Thus γ_j cannot be a geodesic circle.

However if γ_j makes an angle different from π at P_j , we are no better off. For now we can shorten the geodesic loop γ_j. Since its length is no more than ρ, and all sectional curvatures satisfy $\kappa_j \leq 1/\rho^2$, there will be a geodesic loop $\tilde{\gamma}_j$ close to γ_j starting and ending at any point $\widetilde{P}_j$ close to P_j. If we take $\widetilde{P}_j$ to be along γ_j itself then the loop $\tilde{\gamma}_j$ is shorter than γ_j, since for angle less than π the first variation in arc length of this motion is strictly negative. Moreover $\widetilde{P}_j$ is still in N_j. If the whole loop $\tilde{\gamma}_j$ is in N_j then γ_j wasn't the shortest. On the other hand if $\tilde{\gamma}_j$ doesn't stay in N_j there must be a point $\widetilde{Q}_j$ on $\tilde{\gamma}_j$ lying outside of N_j.

Now if $\widetilde{P}_j$ is close to P_j, then $\widetilde{Q}_j$ cannot lie far from N_j, so in particular its distance from the origin can be kept less than $L + \rho$. Let $\widetilde{Q}_j = \exp_j(\tilde{s}_j \widetilde{W}_j)$ with $|\widetilde{W}_j| = 1$ and $\tilde{s}_j \leq \ell_j(\widetilde{W}_j)$ be some exponential representation of $\widetilde{Q}_j$ and let

$$\widetilde{\theta}_j = \{\exp_j(s\widetilde{W}_j) : 0 \leq s \leq \tilde{s}_j\}$$

be the corresponding geodesic from O_j to $\widetilde{Q}_j$. Since $\widetilde{Q}_j$ is not in N_j, we can find some $\widetilde{V}_j \in \mathcal{D}$ and some $\tilde{r}_j \leq \ell_j(V_j)$ and some $\tilde{s}'_j \leq \tilde{s}_j$ such that

$$d_j(\exp_j(\tilde{s}'_j \widetilde{W}_j), \exp_j(\tilde{r}_j \widetilde{V}_j)) < \tilde{r}_j - \rho.$$

In fact we may as well take $\tilde{r}_j = \ell_j(\widetilde{V}_j)$ since the inequality gets stronger as $\tilde{r}_j$ increases. Choose $\tilde{\varepsilon} > 0$ depending on $L + 2\rho$ so that all the sectional curvatures κ_j on M_j in the ball of radius $L + 2\rho$ around the origin O_j in m_j are at least $\tilde{\varepsilon}/\rho^2$ independent of j. The previous argument shows that the closest point on $\widetilde{\theta}_j$ to $\widetilde{X}_j = \exp(\tilde{r}_j \widetilde{V}_j)$ cannot be an interior point for large j. We only need observe that $\tilde{r}_j$ is as large as we want when j is large by our previous observation.

Moreover the closest point is not the origin, since there the distance is $\tilde{r}_j$ while at $\exp_j(\tilde{s}'_j \widetilde{W}_j)$ it is less than $\tilde{r}_j - \rho$. Hence the closest point is at the end $\widetilde{Q}_j$, so

$$d_j(\widetilde{X}_j, \widetilde{Q}_j) < \tilde{r}_j - \rho$$

while surely

$$d_j(\widetilde{X}_j, \widetilde{P}_j) \geq \tilde{r}_j$$

since $\widetilde{P}_j$ lies in N_j. Thus the closest point $\widetilde{Q}'_j$ to $\widetilde{X}_j$ on $\tilde{\gamma}_j$ is not its end point $\widetilde{P}_j$. But now the second variation of arc length from $\widetilde{X}_j$ to $\widetilde{Q}'_j$ will be negative, giving a contradiction as before. Hence the only possibility is that the shortest loop γ in N_j has length at least ρ, and we have our injectivity radius estimate.

26 Singularities in Dimension Two and Three. The Ricci Flow on a compact surface cannot form any singularity except for the sphere or projective plane shrinking to a point and becoming round. One way to prove this now is to examine the possible singularities and see there are no others. We have an injectivity rdius estimate in terms of the maximum of the curvature valid for all time. So unless the solution exists for all time with curvature decaying like

$$|R| \leq C/t$$

as $t \to \infty$, we can form a singularity model of Type I or II. We examine Type I first.

THEOREM 26.1. *The only solutions to the Ricci Flow on a surface which are complete with bounded curvature on an ancient time interval $-\infty < t < T$ and where the curvature R has*

$$\limsup_{t \to -\infty} (T - t)|R| < \infty$$

are the round sphere S^2 and the flat plane R^2, and their quotients.

Proof. Since $|R| \leq C/(T-t)$ and the minimum of R increases, $R \geq 0$. Moreover by the strong maximum principle $R = 0$ everywhere and it is flat, or $R > 0$ everywhere. If the solution is compact with $R > 0$, either it is the sphere or it is the projective plane $RP^2 = S^2/Z_2$ whose double cover is the sphere. Assume it is the sphere, and we shall see it is round. Then RP^2 must be round also since its cover is.

We know from **[22]** that the sphere shrinks to a point at some future time which we can take to be T, when it becomes round. Its area A shrinks at a constant rate

$$\frac{dA}{dt} = -\int R\, da = -8\pi$$

so $A = 8\pi(T - t)$. On an even dimensional oriented manifold the injectivity radius can be bounded by the maximum curvature. Since

$$R \leq C/(T - t)$$

by hypothesis, we must have injectivity radius ρ with

$$\rho \geq c\sqrt{T-t}$$

for some c. Now the diameter L has

$$L \leq CA/\rho \leq C\sqrt{T-t}$$

as a bound also. Hence the diameter, the injectivity radius and the maximum curvature all scale proportionally to the time to blow-up.

The scaled entropy

$$E = \int R \; \ln[R(T-t)]da$$

is monotone decreasing in t. Since

$$R(T-t) \leq C \quad \text{and} \quad \int R \; da = 8\pi$$

we have an upper bound

$$E \leq 8\pi \ln C$$

so

$$E_{-\infty} = \lim_{t \to -\infty} E_t$$

exists. Pick a sequence of points $t_j \to -\infty$ and points P_j where the curvature is as big as anywhere at time t_j. Then it was never larger anywhere at any earlier time, since an ancient solution with $R > 0$ has R pointwise increasing by the Harnack inequality. Make P_j the new origin and t_j the new time 0 and T the same blow-up time by translation and dilation. We can then take a limit using the curvature and injectivity radius bounds. The backwards limit is still compact by the diameter bound. Moreover the scaled entropy is now constant at the value $E_{-\infty}$. But the only way this happens is on a shrinking soliton, and (except for orbifolds) the only one is the round sphere. Then E has its minimal value at $t = -\infty$, so it was constant all along. hence the sphere was round all along.

There remains the case where the surface is complete but not compact. Since $R > 0$, the surface is diffeomorphic to the plane. We proceed to examine such a surface until we learn enough about it to get a contradiction.

Recall first that the asymptotic scalar curvature ratio

$$A = \limsup_{s \to \infty} Rs^2$$

is constant on an ancient solution with weakly positive curvature operator by Theorem 18.3.

LEMMA 26.2. *For our solution $A < \infty$.*

Proof. Suppose $A = \infty$. Then as in the dimension reduction argument of Lemma 22.2 and the following, we can choose a sequence of points P_j at $t = 0$

and radii r_j which give λ_j remote β-bumps for a fixed $\beta > 0$ and $\lambda_j \to \infty$. This works in dimension 2 only, because the only curvature is the scalar curvature, so when it is big every curvature is big. But now this contradicts Theorem 21.6. (Once the dimension is 2 we cannot reduce it further, since everything in dimension 1 is intrinsically flat.) Thus $A < \infty$. □

From our previous results in the proof of Theorem 18.3 we know that an annulus

$$N_\sigma = \{\sigma \leq s \leq 3\sigma\}$$

has an area

$$A(N_\sigma) \geq c\sigma^2$$

for some constant c, and the scalar curvature at distance s falls off at most by

$$R \geq c/s^2$$

for some other constant c.

Now we can explain how we get a contradiction. For a complete surface with $R > 0$ we have

$$\iint R \, da \leq 4\pi$$

by the Gauss-Bonnet Theorem since the surface is exhausted by discs bounded by convex circles with geodesic curvature $k \geq 0$ and on a disc

$$\iint R \, da + 2 \int_\partial k \, ds = 4\pi.$$

However on our surface we claim

$$\iint R \, da = \infty.$$

This is because each annulus N_σ makes a contribution

$$\iint_{N_\sigma} R \, da \geq \frac{c}{\sigma^2} \cdot c\sigma^2 \geq c$$

for some constants $c > 0$, using $R \geq c/s^2$ and $A(N_\sigma) \geq c\sigma^2$. But we can take an infinite sequence of disjoint annuli, and their contributions add up to ∞. This finishes the proof of the Theorem. □

Next we examine Type II limits. Since $R > 0$ it must be a soliton which assumes it maximum at an origin.

THEOREM 26.3. *The only complete Ricci soliton on a surface with bounded curvature which assumes its maximum 1 at an origin is the "cigar" soliton Σ^2 with metric*

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

Proof. The soliton moves along a vector field $V = Df$. Since the Ricci Flow preserves the conformal structure, which gives a complex structure J, the vector field V is holomorphic. Then it turns out that JV is a Killing vector field; this trick works on any Ricci-Kähler soliton. This gives a circle action on the soliton which makes it rotationally symmetric, and the Ricci soliton equation reduces to an ordinary differential equation which we can solve. We refer the reader to **[22]** for details. □

In our paper **[28]** we prove the following isoperimetric estimate, which is similar to our study of minimal geodesic loops on a surface. Suppose a loop γ of length L divides the surface into two pieces of areas A_1 and A_2. Define the isoperimetric ratio $I(\gamma)$ of the loop γ by

$$I(\gamma) = L^2\left(\frac{1}{A_1} + \frac{1}{A_2}\right)$$

and let

$$I = \inf_{\gamma} I(\gamma)$$

be the infimum over all γ of any length or shape.

THEOREM 26.4. *On the sphere S^2 the isoperimetric ratio I is increasing.*

It follows that we cannot form the cigar as a limit on S^2, because the cigar opens like a cylinder. If the surface develops a piece like a long thin cylinder it will have a short curve in the cylinder with a comparably large area on either side, and the ratio I will be close to 0 . If we approach the cigar as a singularity forms, I must decrease to zero. But on the sphere I increases. The projective plane RP^2 can be treated by looking at its cover S^2. Other surfaces have Euler class $\chi \leq 0$ and can be treated directly (as in **[22]**) or as a special case of Kähler manifolds with $[Rc] = \rho[\omega]$ with $\rho \leq 0$ (as in **[4]**). The rescaled flow converges to a constant curvature metric.

It is very interesting to see how much we can say about the formation of singularities in dimension three.

THEOREM 26.5. *Suppose we have a solution to the Ricci Flow on a compact three- manifold, and suppose R becomes unbounded in some finite time T. Then there exists a sequence of dilations of the solution which converges to S^3 or $S^2 \times R^1$ or $\Sigma^2 \times R^1$ (where Σ^2 is the "cigar" soliton) or to a quotient of one of these solutions by a finite group of isometries acting freely (these quotients are the space forms S^3/γ, $RP^2 \times R^1$, and $RP^2 \times R^1$ and $S^2 \times S^1_r, RP^2 \times S^1_r$ and $\Sigma^2 \times S^1_r$ for circles S^1_r of any radius r), except possibly for the case of a Type I singularity where the injectivity radius times the square root of the maximum curvature goes to zero.*

Proof. When we get an injectivity radius estimate valid for finite time we can always for a singularity model of Type I or II. First consider Type I. If the sectional curvature ever becomes positive everywhere, it becomes round and our limit is S^3 or S^3/γ. Otherwise in Type I we get a limit which is an ancient solution with bounded non-negative sectional curvature which splits as a product of a surface with R^1. For the surface, if $(T-t)R \leq C$ it must be a round sphere or projective plane by Theorem 26.1. Otherwise we can take a backwards limit as $t \to -\infty$ to get a Type II limit, which must be the cigar Σ^2. Since a limit of a limit is also a limit, we get $\Sigma^2 \times R^1$ or $\Sigma^2 \times S^1$ as a limit of the three-manifold solution.

In order to take this backward limit we need an injectivity radius estimate on the surface in terms of the maximum curvature R at the current time. Since $R > 0$ this is easy. There are three cases. If the surface is compact and oriented, it is S^2 and the result follows from a theorem for positive sectional curvature in even dimensions of Klingenberg ([**9**], 5.9). If it is compact but not oriented, it is RP^2 and the double cover can be handled as before. If it is not compact, it is diffeomorphic to R^2 and we can use the estimate for complete noncompact manifolds of positive sectional curvature.

If the limit is Type II, it must be a Ricci soliton of weakly positive sectional curvature from our pinching result in Theorem 24.4. If the sectional curvature is not strictly positive, it splits as a product of a surface soliton, which must be Σ^2, with a flat factor R^2 or S^1 (of any radius). Even if it does not split, we know the asymptotic curvature ratio is infinite

$$A = \limsup_{s \to \infty} Rs^2 = \infty$$

by Theorem 20.2, and by Theorem 22.3 since the dimension $n = 3$ is odd, we can do dimension reduction to find a limit of a limit which splits as a product with R^1 of an ancient solution with bounded positive curvature on a surface. Again a limit of a limit is a limit, and we can classify the surface as a round S^2 (not RP^2 because it is oriented) or Σ^2. This finishes the proof of the Theorem. □

Of course S^3 or S^3/γ can actually occur as limits from the homothetically shrinking solutions, and we expect to get $S^2 \times R^1$ from a neck pinch (or a degenerate neck pinch after dimension reduction). We even expect $RP^2 \times R^1$ as the limit from doing a neck pinch on S^3 shaped like a dumb-bell and then quotienting by Z_2. Some of the other quotients are harder to picture. For example if $S^2 \times S^1$ has a product metric, the S^2 factor shrinks but the S^1 factor does not. Hence the limit of its dilations is $S^2 \times R^1$, not $S^2 \times S^1$. We conjecture $S^2 \times S^1$ cannot form.

More importantly, we conjecture $\Sigma^2 \times R^1$ and $\Sigma^2 \times S^1$ cannot form as limits of dilations of a compact solution. Here are the reasons for our belief. First, Σ^2 cannot form starting from a compact surface. Second, we could rule out $\Sigma^2 \times R^1$ on a three-manifold the same way we can rule out Σ^2 occuring as a factor in limits coming from compact manifolds with positive curvature operator, because Σ^2 violates the local injectivity radius estimate coming from the Little Loop Lemma. Moreover the Little Loop Lemma only depends on having some kind of backwards control on the scalar curvature R locally. This control came from the Harnack estimate, which uses positive curvature operator. But in three dimensions our pinching estimates show that we only miss positive curvature by a little bit. This gives hope that we can get an approximate Harnack estimate giving some backwards control on R as desired. Backwards control means that R does not fall off too rapidly.

This raises the following interesting problem. If we almost have a degenerate neck pinch, but at the last moment the little bubble on the end of the neck gets pulled through, leaving a little bump, how fast can the curvature of this little bump decay?

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