Pure and Applied Mathematics Quarterly
Volume 16, Number 5, 1723–1738, 2020

Vanishing viscosity limit to the 3D Burgers equation in Gevrey class

Ridha Selmi* and Abdelkerim Chaabani

Abstract: We consider the Cauhcy problem to the 3D diffusive periodic Burgers equation. We prove that a unique solution exists on time interval independent of the viscosity and tends, as the viscosity vanishes, to the solution of the limiting equation, the inviscid periodic three-dimensional Burgers equation, in Gevrey-Sobolev spaces. Compared to Navier-Stokes equations, the main difficulties come from the lack of the divergence-free condition which is essential to handle the nonlinear term. Our alternative tool will be to use a change of functions to estimate nonlinearities. Fourier analysis and compactness methods are widely used.

Keywords: Existence and uniqueness, vanishing viscosity limit.

1. Introduction

We consider the diffusive Burgers system

$$(Bg_\nu) \qquad \begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \\ u|_{t=0} = u^{in}(x), & x\in\mathbb{T}^3 \end{cases}$$

and the inviscid Burgers system

$$(Bg_0) \qquad \begin{cases} \partial_t u + (u\cdot\nabla)u = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \\ u|_{t=0} = u^{in}(x), & x\in\mathbb{T}^3. \end{cases}$$

For all $s\geq 1$, $r\geq 0$ and $a\in(0,1)$, the homogeneous Gevrey-Sobolev space for a positive real number s is given by

$$\dot{H}^r_{a,s} = \{f\in L^2(\mathbb{T}^3);\ \ e^{a\Lambda^{1/s}}f\in \dot{H}^r(\mathbb{T}^3)\}$$

Received September 9, 2019.
2010 Mathematics Subject Classification: Primary 35A01, 35A02; secondary 35B45, 35B99.

*Corresponding author.

endowed by the norm

$$\|f\|_{\dot{H}^r_{a,s}} = \|\Lambda^r e^{a\Lambda^{1/s}} f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}^3} |k|^{2r} e^{2a|k|^{1/s}} |\hat{f}_k|^2 \right)^{1/2},$$

where $\hat{f}_k$ is the Fourier coefficient of f and $\Lambda = \sqrt{-\Delta}$. The nonhomogeneous Gevrey-Sobolev space is given by

$$H^r_{a,s} = \{ f \in L^2(\mathbb{T}^3); \quad e^{a\Lambda^{1/s}} f \in H^r(\mathbb{T}^3) \},$$

endowed by the norm

$$\|f\|_{H^r_{a,s}} = \|e^{a\Lambda^{1/s}} f\|_{L^2} + \|\Lambda^r e^{a\Lambda^{1/s}} f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}^3} (1+|k|^2)^r e^{2a|k|^{1/s}} |\hat{f}_k|^2 \right)^{1/2}.$$

Also, we will use the notation $\mathcal{F}(\cdot)$ for the Fourier transform and $\mathcal{F}^{-1}(\cdot)(x)$ to denote the inverse Fourier transform.

Burgers equation is the simplest nonlinear model equation for diffusive waves in fluid dynamics. In 1948, Burgers [4] was the first to develop the one dimensional Burgers equation to shed light on the study of turbulence described by the interaction of the two opposite effects of convection and diffusion. Later on, the three-dimensional viscous Burger equation was considered in cosmology as the Zeldovich approximation [15]. The non-viscous Burgers equation is perhaps the simplest equation that models the nonlinear phenomena in a force free mass transfer as said in [1].

Mathematically, Burgers equation is the incompressible Navier-Stokes ones, without the incompressibility condition and pressure term. This similarity leads to think about a parallel mathematical analysis to the one done for Navier-Stokes equations, as in [9] where authors proved that the periodic diffusive three-dimensional Burgers equation is globally in time well posed in Sobolev space $H^{1/2}$, and in [14], where authors proved existence and uniqueness of global in time solution to the viscous Burgers equation in critical Gevrey class. In [6], Kato proved that the solution to the three-dimensional Navier-Stokes equations exists on a time interval independent of the viscosity ν and tends to the solution of the Euler equation when $\nu \to 0$, provided that u^{in} belongs to H^m, for $m \geq 3$. Other results concerning the viscosity limit can be found in [3, 8]. It is worth mentioning that convergence results, as a small parameter (Rossby number) vanishes, were proved in the case of geophysical magnetohydrodynamic systems, see for example ([11, 12, 13]) and references therein, or also as a regularizing small parameter goes to zero [10].

In this paper, we will prove the local-in-time existence and uniqueness of solutions to (Bg_ν) and (Bg_0) on the periodic domain $\mathbb{T}^3$, with initial data $u^{in} \in H^r_{a,s}$, for $r > 5/2$. Then, we return to prove that the solution of (Bg_ν) exists on a time interval independent of the viscosity ν and tends to the solution of the inviscid Burgers equation (Bg_0) when ν vanishes. Mainly, we have the following theorems.

Theorem 1.1. *Given u^{in} in $H^r_{a,s}(\mathbb{T}^3)$ where $r > 5/2$, then there exists a unique local solution to (Bg_ν), such that*

$$u \in C([0,T); H^r_{a,s}(\mathbb{T}^3)) \cap L^2([0,T); H^{r+1}_{a,s}(\mathbb{T}^3)).$$

Given u^{in} in $H^r_{a,s}(\mathbb{T}^3)$ where $r > 5/2$, then there exists a unique local solution to (Bg_0), such that $u \in C([0,T); H^r_{a,s}(\mathbb{T}^3))$.

Theorem 1.2. *Let $u^{in} \in H^r_{a,s}$ where $r > 5/2$ and $T > 0$, then*

i) there exists T_0 depending on $\|u^{in}\|_{H^r_{a,s}}$ but not on ν, such that (Bg_ν) has a unique solution

$$u_\nu \in C([0,T_0]; H^r_{a,s}) \cap L^1([0,T_0); H^{r+1}_{a,s}).$$

Furthermore, u_ν is bounded in $C([0,T_0]; H^r_{a,s})$ for all $\nu > 0$.

ii) For each $t \in [0,T_0]$, $u_0(t) = \lim_{\nu\to 0} u_\nu(t)$ exists strongly in $H^{r-1}_{a,s}$ and weakly in $H^r_{a,s}$ uniformly in t:

$$\lim_{\nu\to 0} \sup_{t\in[0,T_0]} \|u_\nu - u_0\|_{H^{r-1}_{a,s}} = 0$$

and

$$\lim_{\nu\to 0} \sup_{t\in[0,T_0]} \langle u_\nu - u_0, f\rangle_{H^r_{a,s}} = 0$$

where $f \in H^{r+1}_{a,s}$, u_0 is a unique solution to (Bg) satisfying

$$u_0 \in C([0,T_0]; H^r_{a,s}).$$

Unlike prior efforts, the main difficulty in proving our results stems from the nonlinear term. Particularly, the lack of divergence-free condition which usually played the key role in Euler and Navier-Stokes theory prevents us from applying the usual estimates as in [7, 6, 5]. Also, we note that as the viscosity ν is destined to vanish when taking the limit, any estimates that depend singularly on ν will fail to control the non-linearity as $\nu \to 0$. However, we arrived in the framework of Gevrey-Sobolev space to beat the odds via several applications of change of functions and Plancherel's identity.

The following section is assigned to prove the technical lemmas used later on to control the nonlinear term. In the third section, we prove unique solution to (Bg_ν) and to (Bg_0). In the last section, we investigate the convergence result.

2. Estimates of the nonlinear term

We denote by c a generic constant that may change from line to another.

Lemma 2.1. *Let u, v and w be three-dimensional vector valued functions, such that u and v belong to $H^r_{a,s}$ and $w \in H^{r+1}_{a,s}$ for $r > 3/2$, then there exists a positive constant C, such that*

$$|\langle (u\cdot\nabla)v, w\rangle_{H^r_{a,s}}| \leq C\|u\|_{H^r_{a,s}}\|v\|_{H^r_{a,s}}\|w\|_{H^{r+1}_{a,s}}. \tag{1}$$

If v and w belong to $H^r_{a,s}$ and $u \in H^{r+1}_{a,s}$, $r > 3/2$, then there exists a positive constant C, such that

$$|\langle (u\cdot\nabla)v, w\rangle_{H^r_{a,s}}| \leq C\|u\|_{H^{r+1}_{a,s}}\|v\|_{H^r_{a,s}}\|w\|_{H^r_{a,s}}. \tag{2}$$

Proof. By Parseval's identity, the following holds

$$\begin{aligned}|\langle (u\cdot\nabla)v, w\rangle_{H^r_{a,s}}| &= |\sum_k e^{a|k|^{1/s}}\mathcal{F}((u\cdot\nabla)v)_k e^{a|k|^{1/s}}\langle k\rangle^{2r}\overline{\mathcal{F}(w)(k)}| \\ &\leq \sum_k e^{a|k|^{1/s}}|\mathcal{F}((u\cdot\nabla)v)_k| e^{a|k|^{1/s}}\langle k\rangle^{2r}|\overline{\mathcal{F}(w)(k)}|,\end{aligned}$$

where $\langle k\rangle := (1+|k|^2)^{1/2}$. We estimate the Fourier transform of the nonlinear term $\mathcal{F}((u\cdot\nabla)v)_k$ as follows

$$\begin{aligned}e^{a|k|^{1/s}}|\mathcal{F}((u\cdot\nabla)v)_k| &= e^{a|k|^{1/s}}|\mathcal{F}(u)*\mathcal{F}(\nabla v)| \\ &= e^{a|k|^{1/s}}|\sum_p \mathcal{F}(u)(p)\mathcal{F}(\nabla v)(k-p)| \\ &\leq \sum_p e^{a|p|^{1/s}}|\hat{u}_p|e^{a|k-p|^{1/s}}|k-p||\hat{v}(k-p)| \\ &\leq \sum_p e^{a|p|^{1/s}}|\hat{u}(p)|(|k|+|p|)e^{a|k-p|^{1/s}}|\hat{v}(k-p)| \\ &\leq |k|\sum_p e^{a|p|^{1/s}}|\hat{u}(p)|e^{a|k-p|^{1/s}}|\hat{v}(k-p)|\end{aligned}$$

$$+ \quad \sum_p e^{a|p|^{1/s}}|p||\hat{u}(p)|e^{a|k-p|^{1/s}}|\hat{v}(k-p)|,$$

where we used the inequality $e^{a|k|^{1/s}} \leq e^{a|p|^{1/s}}e^{a|k-p|^{1/s}}$, for all p, k in $\mathbb{Z}^3$. Let $f_1 = \mathcal{F}^{-1}(e^{a|k|^{1/s}}|\hat{u}_k|)$, $f_2 = \mathcal{F}^{-1}(e^{a|k|^{1/s}}|\hat{v}_k|)$ and $f_3 = \mathcal{F}^{-1}(|k|e^{a|k|^{1/s}}|\hat{u}_k|)$ for all k in $\mathbb{Z}^3$, it follows that

$$\begin{aligned} e^{a|k|^{1/s}}|\mathcal{F}((u\cdot\nabla)v)| &\leq |k|\mathcal{F}(f_1)*\mathcal{F}(f_2)+\mathcal{F}(f_3)*\mathcal{F}(f_2) \\ &= |k|\mathcal{F}(f_1\cdot f_2)+\mathcal{F}(f_3\cdot f_2). \end{aligned}$$

Therefore, it turns out that

$$\begin{aligned} |\langle(u\cdot\nabla)v,w\rangle_{H^r_{a,s}}| &\leq \sum_k \mathcal{F}(f_1\cdot f_2)|k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)| \\ &+ \sum_k \mathcal{F}(f_3\cdot f_2)e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)|. \end{aligned}$$

Let $f_4 = \mathcal{F}^{-1}(\overline{e^{a|k|^{1/s}}\langle k\rangle^{2r}|\hat{w}_k|})$, the fact that by definition, the Fourier coefficients of f_3, f_2 and f_4 are all non-negative real valued functions for all $k\in\mathbb{Z}^3$ yields

$$\sum_k \mathcal{F}(f_3\cdot f_2)\overline{e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)_k|} = \left|\sum_k \mathcal{F}(f_3\cdot f_2)\overline{e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)|}\right|.$$

Then, by applying Parseval's identity once again, we obtain

$$\begin{aligned} \sum_k \mathcal{F}(f_3\cdot f_2)e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)| &= \sum_k \mathcal{F}(f_3\cdot f_2)\overline{e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)|} \\ &= c|\langle f_3\cdot f_2, f_4\rangle_{L^2}| \\ &\leq c\|f_2\|_{L^\infty}|\langle f_3, f_4\rangle_{L^2}|. \end{aligned}$$

By definition of f_3 and f_4, we have

$$|\langle f_3, f_4\rangle_{L^2}| = c\sum_k e^{a|k|^{1/s}}|\hat{u}_k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|k||\hat{w}_k|$$

$$\leq c\left(\sum_k(1+|k|^2)^r e^{2a|k|^{1/s}}|\hat{u}_k|^2\right)^{1/2}\left(\sum_k(1+|k|^2)^r e^{2a|k|^{1/s}}|\mathcal{F}(\nabla w)|^2\right)^{1/2}$$

$$\leq c\|u\|_{H^r_{a,s}}\|w\|_{H^{r+1}_{a,s}},$$

where we achieved the last step by using the Cauchy-Schwarz inequality. It remains to estimate $\sum_k \mathcal{F}(f_1\cdot f_2)|k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)|$, to do so, we proceed as follows

$$\sum_k \mathcal{F}(f_1 \cdot f_2)|k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)| = \sum_k \mathcal{F}(f_1 \cdot f_2)\overline{|k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)|}.$$

Let $f_5 = \mathcal{F}^{-1}(|k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|\hat{w}_k|)$. Then, we infer that

$$\begin{aligned}\sum_k \mathcal{F}(f_1 \cdot f_2)|k|e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}w_k| &= |\langle f_1 \cdot f_2, f_5\rangle_{L^2}| \\ &\leq \|f_2\|_{L^\infty}|\langle f_1, f_5\rangle_{L^2}|.\end{aligned}$$

By definition of f_1 and f_5, we have

$$\begin{aligned}|\langle f_1, f_5\rangle_{L^2}| &= \sum_k \hat{f}_1(k)\overline{\hat{f}_5(k)} \\ &= \sum_k e^{a|k|^{1/s}}|\hat{u}_k|(1+|k|^2)^r|k|e^{a|k|^{1/s}}|\hat{w}_k| \\ &\leq \left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}}|\hat{u}_k|^2\right)^{1/2}\left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}}|\mathcal{F}(\nabla w)|^2\right)^{1/2} \\ &\leq \|u\|_{H^r_{a,s}}\|w\|_{H^{r+1}_{a,s}},\end{aligned}$$

where we achieved the last step by using the Cauchy-Schwarz inequality. The Fourier expansion of f_2 is given by

$$f_2 = \sum_k \mathcal{F}(f_2)(k)e^{ikx} = \sum_k e^{a|k|^{1/s}}|\hat{v}_k|e^{ikx}.$$

It follows that

$$\begin{aligned}\|f_2\|_{L^\infty} &\leq \sum_k e^{a|k|^{1/s}}|\hat{v}_k| \\ &= \sum_k (1+|k|^2)^{-r/2}(1+|k|^2)^{r/2}e^{a|k|^{1/s}}|\hat{v}_k| \\ &\leq \left(\sum_k \frac{1}{(1+|k|^2)^r}\right)^{1/2}\|v\|_{H^r_{a,s}}.\end{aligned}$$

The series $\left(\sum_k \frac{1}{(1+|k|^2)^r}\right)^{1/2}$ is convergent since $r > 3/2$, and (1) follows. Estimate (2) follows the same way, only the application of Cauchy-Schwarz on $|\langle f_3, f_4\rangle_{L^2}|$ and $|\langle f_1, f_5\rangle_{L^2}|$ differs. In fact,

$$\begin{aligned}|\langle f_3, f_4\rangle_{L^2}| &= \sum_k \hat{f}_3(k)\overline{\hat{f}_4(k)} \\ &= \sum_k |k| e^{a|k|^{1/s}} |\hat{u}_k| (1+|k|^2)^r e^{a|k|^{1/s}} |\hat{w}_k| \\ &\leq \left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}} |\mathcal{F}(\nabla u)|^2\right)^{1/2} \\ &\times \left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}} |\mathcal{F}(w)(k)|^2\right)^{1/2} \\ &\leq \|u\|_{H^{r+1}_{a,s}} \|w\|_{H^r_{a,s}},\end{aligned}$$

and

$$\begin{aligned}|\langle f_1, f_5\rangle_{L^2}| &= \sum_k |k| e^{a|k|^{1/s}} |\mathcal{F}(u)(k)| (1+|k|^2)^r e^{a|k|^{1/s}} |\mathcal{F}(w)(k)| \\ &\leq \left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}} |\mathcal{F}(\nabla u)|^2\right)^{1/2} \\ &\times \left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}} |\mathcal{F}(w)(k)|^2\right)^{1/2} \\ &\leq \|u\|_{H^{r+1}_{a,s}} \|w\|_{H^r_{a,s}}.\end{aligned}$$

□

Remark 2.2. Let u, v and w be three-dimensional vector valued functions such that u, v and w are all in $H^r_{a,s}$ where $r > 5/2$, then there exists a constant C, such that

$$|\langle (u\cdot\nabla)v, w\rangle_{H^r_{a,s}}| \leq C\|u\|_{H^r_{a,s}}\|v\|_{H^r_{a,s}}\|w\|_{H^r_{a,s}}. \tag{3}$$

Proof. By Parseval's identity, the following holds

$$|\langle (u\cdot\nabla)v, w\rangle_{H^r_{a,s}}| \leq \sum_k e^{a|k|^{1/s}} |\mathcal{F}((u\cdot\nabla)v)_k| e^{a|k|^{1/s}} \langle k\rangle^{2r} |\overline{\mathcal{F}(w)_k}|.$$

Where $\langle k\rangle^{2r} := (1+|k|^2)^{1/2}$. The non-linear term can be estimated, by using the following steps:

$$e^{a|k|^{1/s}} |\mathcal{F}((u\cdot\nabla)v)| \leq \sum_p e^{a|p|^{1/s}} |\hat{u}(p)||k-p| e^{a|k-p|^{1/s}} |\hat{v}(k-p)|,$$

let

$$\begin{aligned} g_1 &= \mathcal{F}^{-1}(e^{a|k|^{1/s}}|\hat{u}_k|), \\ g_2 &= \mathcal{F}^{-1}(|k|e^{a|k|^{1/s}}|\hat{v}_k|) \end{aligned}$$

and $g_3 = \mathcal{F}^{-1}(e^{a|k|^{1/s}}\langle k\rangle^{2r}|\hat{w}_k|)$. It turns out that

$$\begin{aligned} e^{a|k|^{1/s}}|\mathcal{F}((u\cdot\nabla)v)| &\leq \mathcal{F}(g_1)*\mathcal{F}(g_2) \\ &= \mathcal{F}(g_1\cdot g_2), \end{aligned}$$

and hence,

$$|\langle (u\cdot\nabla)v, w\rangle_{H^r}| \leq c\sum_k \mathcal{F}(g_1\cdot g_2)_k e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)_k|,$$

Thus, by applying Plancherel's identity once again, we obtain

$$\begin{aligned} \sum_k \mathcal{F}(g_1\cdot g_2)e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)| &= \sum_k \mathcal{F}(g_1\cdot g_2)\overline{e^{a|k|^{1/s}}\langle k\rangle^{2r}|\mathcal{F}(w)(k)|} \\ &= |\langle g_1\cdot g_2, g_3\rangle_{L^2}| \\ &\leq \|f_2\|_{L^\infty}|\langle g_1, g_3\rangle_{L^2}|. \end{aligned}$$

By definition of g_1 and g_3, we have

$$\begin{aligned} |\langle g_1, g_3\rangle_{L^2}| &= \sum_k e^{a|k|^{1/s}}|\hat{u}_k|(1+|k|^2)^r e^{a|k|^{1/s}}|\hat{w}_k| \\ &\leq \left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}}|\hat{u}_k|^2\right)^{1/2}\left(\sum_k (1+|k|^2)^r e^{2a|k|^{1/s}}|\hat{w}_k|^2\right)^{1/2} \\ &\leq \|u\|_{H^r_{a,s}}\|w\|_{H^r_{a,s}}, \end{aligned}$$

where we achieved the last step by using the Cauchy-Schwarz inequality. The Fourier expansion of g_2 is given by

$$g_2 = \sum_k \mathcal{F}(g_2)(k)e^{ikx} = \sum_k |k|e^{a|k|^{1/s}}|\hat{v}_k|e^{ikx}.$$

It follows that

$$\begin{aligned} \|g_2\|_{L^\infty} &\leq \sum_k e^{a|k|^{1/s}}|k||\hat{v}_k| \\ &= \sum_k (1+|k|^2)^{\frac{1-r}{2}}(1+|k|^2)^{\frac{r-1}{2}}|k|e^{a|k|^{1/s}}|\hat{v}_k| \\ &\leq \left(\sum_k \frac{1}{(1+|k|^2)^{r-1}}\right)^{1/2}\|v\|_{H^r_{a,s}}. \end{aligned}$$

The series $\left(\sum_k \frac{1}{(1+|k|^2)^{r-1}}\right)^{1/2}$ is convergent since $r > 5/2$, and (3) follows. □

3. Existence and uniqueness results

3.1. Existence and uniqueness result to ($\boldsymbol{Bg_\nu}$)

We use Galerkin approximation. For $n \in \mathbb{N}$, let P_n be the projection onto the Fourier modes of order up to n, that is $P_n\left(\sum_{k\in\mathbb{Z}^3} \hat{u}_k e^{ixk}\right) = \sum_{|k|\le n} \hat{u}_k e^{ixk}$. Let $u_n = P_n u$ be the solution to

$$\partial_t u_n + P_n[(u_n \cdot \nabla)u_n] - \nu\Delta u_n = 0 \tag{4}$$

$$u_n(0) = P_n u_0. \tag{5}$$

This is a finite-dimensional locally-Lipschitz system of ODEs. So that, for some T_n, there exists a solution $u_n \in C^\infty([0,T_n) \times \mathbb{T}^3)$. By the $H^r_{a,s}$-inner product, we have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^r_{a,s}} + \nu\|\nabla u_n\|^2_{H^r_{a,s}} + \langle (u_n\cdot\nabla)u_n, u_n\rangle_{H^r_{a,s}} = 0. \tag{6}$$

The fact that $H^r_{a,s}$ is a Banach algebra, as $r > 5/2$, yields

$$|\langle (u_n\cdot\nabla)u_n, u_n\rangle_{H^r_{a,s}}| \quad \le \quad c\|u_n\|^2_{H^r_{a,s}}\|\nabla u_n\|_{H^r_{a,s}}. \tag{7}$$

Hence, applying the Young inequality yields

$$\frac{d}{dt}\|u_n\|^2_{H^r_{a,s}} + \nu\|\nabla u_n\|^2_{H^r_{a,s}} \le C^*\|u_n\|^4_{H^r_{a,s}}, \tag{8}$$

where C^* is a positive constant that depends on ν. Thus,

$$\|u_n\|^2_{H^r_{a,s}} \le \frac{\|u^{in}|^2_{H^r_{a,s}}}{1 - C^*t\|u^{in}|^2_{H^r_{a,s}}}, \tag{9}$$

as long as $t < T^* := \dfrac{1}{C^*\|u^{in}|^2_{H^r_{a,s}}}$. So, there exists $T > 0$, saying $T = T^*/2$, such that $T_n \geq T$ for all n. We use (8) to close the energy estimate

$$\sup_{t\in[0,T]} \|u_n\|^2_{H^r_{a,s}} \leq \frac{\|u^{in}|^2_{H^r_{a,s}}}{1 - C^*T\|u^{in}|^2_{H^r_{a,s}}} := C_T$$

and

$$\int_0^T \|\nabla u_n(t)\|^2_{H^r_{a,s}} dt \leq C^*_T = C^*TC^2_T/\nu.$$

These are uniform bounds on the approximate solution u_n in $L^\infty(0,T;H^r_{a,s})$ and $L^2(0,T;H^{r+1}_{a,s})$. Then, Aubin lemma [2] allows to extract a subsequence of u_n that converges strongly to u. To prove uniqueness, we suppose that u_1 and u_2 are two solutions to (Bg_ν) that have the same initial data $u_0 \in H^r_{a,s}(\mathbb{T}^3)$ (without loss of generality we suppose that $\nu = 1$). Let $U = u_1 - u_2$, we have

$$\langle \partial_t u_1, U\rangle_{H^r_{a,s}} - \langle \Delta u_1, U\rangle_{H^r_{a,s}} + \langle (u_1\cdot\nabla)u_1, U\rangle_{H^r_{a,s}} = 0,$$

and so on for u_2. Since

$$\begin{aligned}(u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2 &= (u_1\cdot\nabla)u_1 - (u_1\cdot\nabla)u_2\\ &+ (u_1\cdot\nabla)u_2 - (u_2\cdot\nabla)u_2\\ &= (u_1\cdot\nabla)U + (U\cdot\nabla)u_2,\end{aligned}$$

one takes the difference to obtain

$$\frac{d}{dt}\|U(t)\|^2_{H^r_{a,s}} + 2\|\nabla U(t)\|^2_{H^r_{a,s}} \leq \underbrace{|\langle (u_1\cdot\nabla)U, U\rangle_{H^r_{a,s}}|}_{M_1} + \underbrace{|\langle (U\cdot\nabla)u_2, U\rangle_{H^r_{a,s}}|}_{M_2}.$$

Using estimate (3), one controls M_1 and M_2 and infers

$$\frac{d}{dt}\|U(t)\|^2_{H^r_{a,s}} \leq C(\|u_1\|_{H^r_{a,s}} + \|u_2\|_{H^r_{a,s}})\|U(t)\|^2_{H^r_{a,s}}.$$

The Grönwall inequality yields

$$\|U(t)\|^2_{H^r_{a,s}} \leq \|U(0)\|^2_{H^r_{a,s}} e^{Ct\sup_{\tau\in[0,t]}(\|u_1(\tau)\|_{H^r_{a,s}} + \|u_2(\tau)\|_{H^r_{a,s}})}.$$

As u_1 and u_2 are in $C([0,T);H^r_{a,s})$ and $U(0) = 0$, uniqueness follows.

Similar computation can be done for (Bg_0) and by (3) we obtain

$$\frac{d}{dt}\|u_n(t)\|^2_{H^r_{a,s}} \leq c\|u_n(t)\|^3_{H^r_{a,s}}, \tag{10}$$

which implies that

$$\|u_n(t)\|^2_{H^r_{a,s}} \quad \leq \quad \frac{\|u^{in}\|^2_{H^r_{a,s}}}{(1-2tc\|u^{in}\|_{H^r_{a,s}})^2}. \tag{11}$$

Then, compactness method applies to obtain existence of solution. Uniqueness can be proved in analogues way to the viscous equation.

4. The inviscid limit

4.1. Uniform bound in viscosity ν

Consider u_ν the solution to (Bg_ν). In equation (Bg_ν), taking the inner product $\langle e^{a\Lambda^{1/s}}\cdot, e^{a\Lambda^{1/s}}\cdot\rangle_{H^r(\mathbb{T}^3)}$, we obtain

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\|e^{a\Lambda^{1/s}}u_\nu\|^2_{H^r} \quad &+ \quad \nu\|e^{a\Lambda^{1/s}}\nabla u_\nu\|^2_{H^r} \\ &\leq \quad |\langle e^{a\Lambda^{1/s}}(u_\nu\cdot\nabla)u_\nu, e^{a\Lambda^{1/s}}u_\nu\rangle_{H^r(\mathbb{T}^3)}|. \end{aligned} \tag{12}$$

Using (3), we infer

$$\frac{1}{2}\frac{d}{dt}\|u_\nu(t)\|^2_{H^r_{a,s}} + \nu\|\nabla u_\nu(t)\|^2_{H^r_{a,s}} \leq c\|u_\nu(t)\|^3_{H^r_{a,s}}, \tag{13}$$

where c does not depend on ν. Dropping the non-negative viscous term from the left-hand side of (13), it holds

$$\frac{d}{dt}\|u_\nu(t)\|_{H^r_{a,s}} \leq 2c\|u_\nu(t)\|^2_{H^r_{a,s}}. \tag{14}$$

By Grönwall type estimate yields

$$\|u_\nu(t)\|_{H^r_{a,s}} \leq \frac{\|u^{in}\|_{H^r_{a,s}}}{1-2ct\|u^{in}\|_{H^r_{a,s}}}. \tag{15}$$

The right-hand side in (15) is continuous on a certain interval of time $[0, T_0]$, where T_0 is uniform with respect to ν. By using an iterative argument, one can prove that u_ν can be extended over $[0,T_0]$ with estimate (15) throughout. Furthermore, integrating estimate (13) over the interval of time $(0,t)$ yields

$$\nu \int_0^t \|\nabla u_\nu(\tau)\|^2_{H^r_{a,s}} d\tau \le c\Phi(t), \quad t \in [0,T_0], \tag{16}$$

where Φ is a continuous function independent of ν.

4.2. Zero viscosity limit of solutions to Burgers equation

Let u_1 and u_2 be the solution to Burgers equation respectively for $\nu=\nu_1$ and $\nu=\nu_2$, where $\nu_1<\nu_2$. Taking the difference and denoting $w := u_1-u_2$, it follows that

$$\frac{d}{dt}w - \nu_1\Delta w - (\nu_1-\nu_2)\Delta u_2 = w\cdot\nabla u_1 + u_2\cdot\nabla w. \tag{17}$$

Taking the $H^{r-1}_{a,s}$-inner product, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2_{H^{r-1}_{a,s}} + \nu_1\|\nabla w\|^2_{H^{r-1}_{a,s}} &\le (\nu_2-\nu_1)\langle -\Delta u_2, w\rangle_{H^{r-1}_{a,s}} \\ &+ |\langle (w\cdot\nabla)u_1, w\rangle_{H^{r-1}_{a,s}}| \\ &+ |\langle (u_2\cdot\nabla)w, w\rangle_{H^{r-1}_{a,s}}|.\end{aligned} \tag{18}$$

We use the same technicalities as the ones used in the previous section to estimate the exponential weight, and the fact that H^{r-1} is a Banach algebra to obtain

$$\begin{aligned}|\langle (w\cdot\nabla)u_1, w\rangle_{H^{r-1}_{a,s}}| &\le c_1\|\nabla u_1\|_{H^{r-1}_{a,s}}\|w\|^2_{H^{r-1}_{a,s}} \\ &\le c_1\|u_1\|_{H^r_{a,s}}\|w\|^2_{H^{r-1}_{a,s}}.\end{aligned}$$

Using estimate (2) yields

$$|\langle (u_2\cdot\nabla)w, w\rangle_{H^{r-1}_{a,s}}| \le c_2\|u_2\|_{H^r_{a,s}}\|w\|^2_{H^{r-1}_{a,s}}.$$

Dropping the second non-negative term in the left-hand side of (18) to obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|w\|^2_{H^{r-1}_{a,s}} &\le (\nu_2-\nu_1)\langle -\Delta u_2, w\rangle_{H^{r-1}_{a,s}} \\ &+ (c_1\|u_1\|_{H^r_{a,s}} + c_2\|u_2\|_{H^r_{a,s}})\|w\|^2_{H^{r-1}_{a,s}}.\end{aligned} \tag{19}$$

Since $\|u_i\|_{H^r_{a,s}}$, $1 \le i \le 2$ is uniformly bounded with respect to ν, there exists C, such that

$$\frac{1}{2}\frac{d}{dt}\|w\|_{H^{r-1}_{a,s}} \le \nu_2\|\nabla u_2\|_{H^r_{a,s}} + C\|w\|_{H^{r-1}_{a,s}}. \tag{20}$$

Using Grönwall inequality, we obtain

$$\|w\|_{H^{r-1}_{a,s}} \le (\nu_2 t)^{1/2} e^{Ct}\left(\nu_2\int_0^t \|\nabla u_2\|^2_{H^r_{a,s}}\right)^{1/2}. \tag{21}$$

By estimate (16), the right-hand side of (21) tends to zero, when ν_2 goes to zero. Consequently, there exists $u_0(t)$, such that $u_\nu(t) \longrightarrow u_0(t)$ strongly in $C([0,T_0];H^{r-1}_{a,s})$ and $u_\nu(t) \longrightarrow u_0(t)$ strongly in $L^2([0,T_0];H^{r-1}_{a,s})$. Estimate (15) ensures that $u_0(t)$ belongs also to $H^r_{a,s}$. Furthermore $u_\nu(t) \rightharpoonup u_0(t)$ weakly in $C([0,T_0];H^r_{a,s})$, and

$$\|u_0\|_{H^r_{a,s}} \le \frac{\|u^{in}\|_{H^r_{a,s}}}{1-2c't\|u^{in}\|_{H^r_{a,s}}}.$$

Let $f \in H^{r+1}_{a,s}$. Then, for all $t_0,\ t \in [0,T_0]$, the following holds

$$\begin{aligned} \langle u_\nu(t), f\rangle_{H^{r-1}_{a,s}} &+ \int_{t_0}^t \nu\langle u_\nu(t), -\Delta f\rangle_{H^{r-1}_{a,s}} ds \\ &+ \int_{t_0}^t \langle (u_\nu\cdot\nabla)u_\nu, f\rangle_{H^{r-1}_{a,s}} ds \\ &= \langle u_\nu(t_0), f\rangle_{H^{r-1}_{a,s}}. \end{aligned} \tag{22}$$

By Cauchy-Schwarz inequality, $\langle u_\nu(t), -\Delta f\rangle_{H^{r-1}_{a,s}} \le \|u_\nu(t)\|_{H^{r-1}_{a,s}}\|f\|_{H^{r+1}_{a,s}}$ and

$$\lim_{\nu\to 0}\int_{t_0}^t \nu\langle u_\nu(t), -\Delta f\rangle_{H^{r-1}_{a,s}} ds = 0. \tag{23}$$

Also, we have

$$\begin{aligned} \lim_{\nu\to 0}\langle u_\nu(t), f\rangle_{H^{r-1}_{a,s}} &= \langle u_0(t), f\rangle_{H^{r-1}_{a,s}} \\ \lim_{\nu\to 0}\langle u_\nu(t_0), f\rangle_{H^{r-1}_{a,s}} &= \langle u_0(t_0), f\rangle_{H^{r-1}_{a,s}}. \end{aligned}$$

It remains to take the limit in the non-linear term. To do so, we have

$$\begin{aligned} \langle (u_\nu\cdot\nabla)u_\nu, f\rangle_{H^{r-1}_{a,s}} - \langle (u_0\cdot\nabla)u_0, f\rangle_{H^{r-1}_{a,s}} &= \langle [(u_\nu - u_0)\cdot\nabla]u_\nu, f\rangle_{H^{r-1}_{a,s}} \\ &+ \langle (u_\nu\cdot\nabla)(u_\nu - u_0), f\rangle_{H^{r-1}_{a,s}}. \end{aligned}$$

We integrate over (t_0, t) and use Hölder's inequality, to obtain

$$|\int_{t_0}^{t} \langle [(u_\nu - u_0)\cdot\nabla]u_\nu, f\rangle_{H^{r-1}_{a,s}} ds| \quad \leq \quad \|u_\nu - u_0\|_{H^{r-1}_{a,s}} \|u_\nu\|_{L^\infty(0,T_0;H^r_{a,s})} C_f t,$$

where C_f is a positive constant that depends on f. As u_ν is bounded uniformly in ν in $L^\infty(0,T_0;H^r_{a,s})$ and u_ν converges strongly to u_0 in $H^{r-1}_{a,s}$, then

$$\lim_{\nu\to 0} |\int_{t_0}^{t} \langle [(u_\nu - u_0)\cdot\nabla]u_\nu, f\rangle_{H^{r-1}_{a,s}} ds| = 0.$$

We use estimate (1) and Hölder's inequality, to obtain

$$|\langle (u_\nu\cdot\nabla)(u_\nu - u_0), f\rangle_{H^{r-1}_{a,s}}| \quad \leq \quad C\|u_\nu\|_{H^{r-1}_{a,s}} \|u_\nu - u_0\|_{H^{r-1}_{a,s}} \|f\|_{H^r_{a,s}}$$

and

$$|\int_{t_0}^{t} \langle (u_\nu\cdot\nabla)(u_\nu - u_0), f\rangle_{H^{r-1}_{a,s}} d\tau| \leq \|u_\nu - u_0\|_{H^{r-1}_{a,s}} \|u_\nu(\tau)\|_{L^\infty(0,T_0;H^r_{a,s})} C^*_f t,$$

where C^*_f is a positive constant that depends on f. As u_ν is bounded uniformly in ν in $L^\infty(0,T_0;H^r_{a,s})$ and u_ν converges strongly to u_0 in $H^{r-1}_{a,s}$, then

$$\lim_{\nu\to 0} |\int_{t_0}^{t} \langle (u_\nu\cdot\nabla)(u_\nu - u_0), f\rangle_{H^{r-1}_{a,s}} ds| = 0.$$

At this point, we infer that u_0 is a solution of (Bg_0), in the following sense

$$\langle u_0(t), f\rangle_{H^{r-1}_{a,s}} + \int_{t_0}^{t} \langle (u_0\cdot\nabla)u_0, f\rangle_{H^{r-1}_{a,s}} ds = \langle u_0(t_0), f\rangle_{H^{r-1}_{a,s}}. \tag{24}$$

Taking the limit as t_0 tends to 0, we infer that u_0 exists within the same class as u_ν with maximal time T_0.

Acknowledgements

The authors gratefully acknowledge the approval and the support of this research study by the grant number SCI-2017-1-7-F-6901 from the Deanship of Scientific Research at Northern Border University, Arar, K. S. A.

References

[1] S. Alverio, O. Rozanova, *The non-viscous Burgers equation associated with random position in coordinate space: a threshold for blow up behavior*, Math. Models Methods Appl. Sci. **19** (2009), 314–330. MR2531038

[2] J.P. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris **256** (1963), 5042–5044. MR0152860

[3] C. Bardos, E.S. Titi, E. Wiedemann, *Vanishing viscosity as a selection principle for the Euler equations: The case of 3D shear flow*, C. R. Acad. Sci. Paris **350** (2012), 757–760. MR2981348

[4] J.M. Burgers, *A mathematical model illustrating the theory of turbulence*, Academic Press, New York, 1948. MR0027195

[5] C.L. Fefferman, D.S. McCormick, J.C. Robinson, J.L. Rodrigo, *Higher order commutator estimates and local existence for the non-resistive MHD equations and related models*, J. Funct. Anal. **267** (2014), 1035–1056. MR3217057

[6] T. Kato, *Non-stationary flows of viscous and ideal fluids in* $\mathbb{R}^3$, J. Funct. Anal. **9** (1972), 296–309. MR0481652

[7] T. Kato, G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), 891–907. MR0951744

[8] N. Masmoudi, *Remarks about the inviscid limit of the Navier-Stokes system*, Commun. Math. Phys. **270** (2007), 777–788. MR2276465

[9] B.C. Pooley, J.C. Robinson, *Well-posedness for the diffusive 3D Burgers equations with initial data in* $H^{1/2}$, London Math. Soc. Lecture Note Ser. **430** (2016), 137–153. MR3497691

[10] R. Selmi, *Global well-posedness and convergence results for the 3D-regularized Boussinesq system*, Canad. J. Math. **64** (2012), 1415–1435. MR2994672

[11] R. Selmi, *Asymptotic study of mixed rotating MHD system*, Bull. Korean Math. Soc. **47** (2010), 231–249. MR2650694

[12] R. Selmi, *Asymptotic study of anisotropic periodic rotating MHD system*, Further Progress in Analysis, World Sci. (2009), 368–377. MR2581638

[13] R. Selmi, *Convergence results for MHD system*, Int. J. Math. Sci. 28704, (2006), 19 pages. MR2251633

[14] R. Selmi, A. Chaabani, *Well-posedness to 3D Burgers equation in critical Gevrey Sobolev spaces*, to appear in Arch. Math. (2018). MR3951656

[15] Y.B. Zeldovich, *Gravitational instability: an approximate theory for large density perturbations*, Astronom. Astrophys. Lib. **5** (1970), 84–89.

Ridha Selmi
Department of Mathematics, Colleage of Sciences
Northern Border University
P.O. Box 1631
Arar, 91431
Kingdom of Saudi Arabia
Department of Mathematics, Faculty of Sciences of Gabes
University of Gabes
Gabes, 6072
Tunisia
Department of Mathematics, Faculty of Sciences of Tunis
PDEs Lab (LR03ES04)
University of Tunis El Manar
Tunis, 2092
Tunisia
E-mail: ridha.selmi@nbu.edu.sa; ridhaselmiridhaselmi@gmail.com

Abdelkerim Chaabani
Department of Mathematics, Faculty of Sciences of Tunis
PDEs Lab (LR03ES04)
University of Tunis El Manar
Tunis, 2092
Tunisia
E-mail: abdelkerim.chaabani@fst.utm.tn