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Hilbert schemes of points and quasi-modularity

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Abstract: We study further connections between Hilbert schemes of points on a smooth projective (complex) surface and quasi-modular forms. We prove that the leading terms of certain generating series (with variable q) involving intersections with the total Chern classes of the tangent bundles of these Hilbert schemes are quasi-modular forms. The main idea is to link these leading terms with those coming from the equivariant setting for the complex plane $\mathbb{C}^2$.

Keywords: Hilbert scheme, quasi-modular form, projective surface, multiple zeta value, generalized partition.

1. Introduction

The Hilbert schemes of points on a smooth complex algebraic surface are known to be smooth irreducible varieties. They parametrize 0-dimensional closed subschemes of the surface. A fundamental and beautiful relation between these Hilbert schemes and modular forms is given by Göttsche's formula [8]:

$$q^{-\frac{\chi(X)}{24}} \cdot \sum_{n=0}^{+\infty} \chi(X^{[n]}) q^n = \eta(q)^{-\chi(X)}$$

where $\chi(X)$ denotes the Euler characteristic of the surface X, $X^{[n]}$ denotes the Hilbert schemes of n-points on X, and $\eta(q) = q^{1/24} \cdot \prod_{n=1}^{+\infty}(1 - q^n)$ is the Dedekind eta function and a modular form of weight $1/2$. Using the Ext vertex operators constructed in [5, 7], Carlsson [5, 6] studied the generating series for the intersection pairings between the total Chern class of the tangent

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bundle of the Hilbert scheme $(\mathbb{C})^{[n]}$ and the Chern characters of tautological bundles over $(\mathbb{C})^{[n]}$, and proved that the reduced series

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle' \tag{1.1}$$

(see (3.18) for the definition) is a quasi-modular form. These series may be regarded as the generalization of the Nekrasov partition function and the related correlation function. For a smooth projective complex surface X, Okounkov [14] conjectured that these reduced series are multiple q-zeta values. Okounkov's conjecture was investigated in [16] via the leading term of the reduced series

$$\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = (q;q)_\infty^{\chi(X)} \cdot \sum_n q^n \int_{X^{[n]}} \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_{X^{[n]}})$$

where $k_1, \ldots, k_N \geq 0$ are integers, $\alpha_i \in H^*(X)$, $G_{k_i}(\alpha_i, n) \in H^*(X^{[n]})$ is from Definition 5.1, $(q;q)_\infty = \prod_{n=1}^{+\infty}(1-q^n)$, and $c(T_{X^{[n]}})$ is the total Chern class of the tangent bundle $T_{X^{[n]}}$. The cohomology classes $G_k(\alpha, n) \in H^*(X^{[n]})$ play pivotal roles in studying the geometry of $X^{[n]}$ [10, 11, 15].

In this paper, we will study the quasi-modularity of the series $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$. Except its leading term and a universal expression obtained in [16], little is known about $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$. So we will concentrate on its leading term. To state the main result, for an integer $k \geq 0$ and a class $\alpha \in H^*(X)$, let $\Theta^\alpha_k(q)$ denote

$$- \sum_{\substack{\ell(\lambda)=k+2 \\ |\lambda|=0}} \langle (1_X - K_X)^{\sum_{n\geq 1} i_n}, \alpha \rangle \cdot \prod_{n \geq 1} \left(\frac{(-1)^{i_n}}{i_n!} \frac{q^{n i_n}}{(1-q^n)^{i_n}} \frac{1}{\tilde{i}_n!} \frac{1}{(1-q^n)^{\tilde{i}_n}} \right)$$

where $\lambda = (\cdots (-n)^{\tilde{i}_n} \cdots (-1)^{\tilde{i}_1} 1^{i_1} \cdots n^{i_n} \cdots)$ denotes a generalized partition with size $|\lambda|$ and length $\ell(\lambda)$ (see Definition 2.1 (i)), and 1_X and K_X denote the fundamental class and canonical class of X respectively.

Theorem 1.1. *Let $k_1, \ldots, k_N \geq 0$ be integers. Let X be a smooth projective complex surface, and $\alpha_1, \ldots, \alpha_N \in H^*(X;\mathbb{Q})$.*

(i) *If $\langle K_X^2, \alpha_i \rangle = 0$ and $2|k_i$ for every i, then the leading term $\prod_{i=1}^N \Theta^{\alpha_i}_{k_i}(q)$ of $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ is either 0 or a quasi-modular form of weight $\sum_{i=1}^N (k_i + 2)$.*

(ii) *Let $|\alpha_i| = 4$ for every i. If $2 \nmid k_i$ for some i, then $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = 0$. If $2|k_i$ for every i, then $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ is a quasi-modular form of weight $\sum_{i=1}^N (k_i + 2)$.*

To prove Theorem 1.1, we relate the leading term of the series $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ for the surface X to the leading term of $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ for $\mathbb{C}^2$ mentioned in (1.1). This is done by writing down the equivariant Chern character operators explicitly and by applying the method in [16, Section 4] to $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ which has been proved to be a quasi-modular form in [5, 6] (also see [20] for related work).

The paper is organized as follows. In Section 2, we review multiple zeta values and quasi-modular forms. Moreover, the important function $\Theta_k(q)$ will be introduced. In Section 3, the equivariant Chern character operators are expressed in terms of the Heisenberg operators. In Section 4, the leading term of the reduced series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is computed. In Section 5, we prove Theorem 1.1.

2. Multiple zeta values, quasi-modular forms and the function $\Theta_k(q)$

This section is devoted to some generalities concerning multiple zeta values and quasi-modular forms. In addition, we will define the functions $\Theta_k(q)$ and $\Theta_k(q, z)$.

First of all, multiple zeta values are series of the form

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

where $s_1, \ldots, s_k$ are positive integers with $s_1 > 1$. Their linear span over $\mathbb{Q}$ is denoted by **MZV**. It is known that **MZV** is a graded subring of $\mathbb{R}$ with the degree (or weight) of $\zeta(s_1, \ldots, s_k)$ being equal to $s_1 + \ldots + s_k$. The ring **MZV** contains the subring generated by the even Riemann zeta values

$$\zeta(2k) = \frac{1}{2} \cdot (-1)^{k+1} \cdot \frac{B_{2k}}{(2k)!} \cdot (2\pi)^{2k}$$

where $B_i \in \mathbb{Q}, i \geq 2$ are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{i=2}^{+\infty} B_i \cdot \frac{t^i}{i!}.$$

The graded ring **QM** of quasi-modular forms (of level 1 on the full modular group $\mathrm{PSL}(2;\mathbb{Z})$) over $\mathbb{Q}$ is the polynomial ring over $\mathbb{Q}$ generated by the Eisenstein series $G_2(q), G_4(q)$ and $G_6(q)$:

$$\mathbf{QM} = \mathbb{Q}[G_2, G_4, G_6] = \mathbf{M}[G_2]$$

where $\mathbf{M} = \mathbb{Q}[G_4, G_6]$ is the graded ring of modular forms (of level 1) over $\mathbb{Q}$, and

$$G_{2k} = G_{2k}(q) = \frac{1}{(2k-1)!} \cdot \left(-\frac{B_{2k}}{4k} + \sum_{n\ge 1} \Big(\sum_{d|n} d^{2k-1} \Big) q^n \right). \tag{2.1}$$

The grading is to assign G_2, G_4, G_6 weights $2, 4, 6$ respectively. By [1, p.6],

$$\begin{aligned} G_2 &= -\frac{1}{24} + Z(2), \\ G_4 &= \frac{1}{1440} + Z(2) + \frac{1}{6} Z(4), \\ G_6 &= -\frac{1}{60480} + \frac{1}{120} Z(2) + \frac{1}{4} Z(4) + Z(6) \end{aligned} \tag{2.2}$$

where $Z(2k)$ is from [14] and defined by

$$Z(2k) = \sum_{n\ge 1} \frac{(q^n)^k}{(1-q^n)^{2k}}. \tag{2.3}$$

It follows that

$$\mathbf{QM} = \mathbb{Q}[Z(2), Z(4), Z(6)]. \tag{2.4}$$

Moreover, there exists a homomorphism

$$\mathbf{QM} \to \mathbb{Q}[\zeta(2), \zeta(4), \zeta(6)] = \mathbb{Q}[\pi^2] \subset \mathbf{MZV}$$

sending a pure weight-w element $f(q) \in \mathbf{QM}$ to

$$\lim_{q\to 1} \big((1-q)^w \cdot f(q) \big).$$

Next, we will define and study the function $\Theta_k(q)$ which plays a significant role in this paper. We begin with the definition of generalized partitions.

Definition 2.1. (i) Let $\lambda = \big(\cdots (-2)^{m_{-2}} (-1)^{m_{-1}} 1^{m_1} 2^{m_2} \cdots \big)$ be a *generalized partition* of the integer $n = \sum_i i m_i$ whose part $i \in \mathbb{Z}$ has multiplicity m_i. Define $\ell(\lambda) = \sum_i m_i$, $|\lambda| = \sum_i i m_i = n$, $\lambda^! = \prod_i m_i!$, and

$$-\lambda = \big(\cdots (-2)^{m_2} (-1)^{m_1} 1^{m_{-1}} 2^{m_{-2}} \cdots \big).$$

(ii) The set of all generalized partitions is denoted by $\widetilde{\mathcal{P}}$. A generalized partition becomes a *partition* in the usual sense if the multiplicity $m_i = 0$ for all $i < 0$. The set of all partitions is denoted by $\widetilde{\mathcal{P}}_+ = \mathcal{P}$. The set of the generalized partitions with multiplicity $m_i = 0$ for all $i > 0$ is denoted by $\widetilde{\mathcal{P}}_-$.

(iii) For $\lambda^{(j)} = (\cdots(-2)^{m_{-2}^{(j)}}(-1)^{m_{-1}^{(j)}}1^{m_1^{(j)}}2^{m_2^{(j)}}\cdots)$ with $j = 1$ and 2, define

$$\lambda^{(1)} - \lambda^{(2)} = (\cdots(-2)^{m_{-2}^{(1)}-m_{-2}^{(2)}}(-1)^{m_{-1}^{(1)}-m_{-1}^{(2)}}1^{m_1^{(1)}-m_1^{(2)}}2^{m_2^{(1)}-m_2^{(2)}}\cdots)$$

with the convention that $\lambda^{(1)} - \lambda^{(2)} = \emptyset$ if $m_i^{(1)} < m_i^{(2)}$ for some i.

Definition 2.2. Fix a non-negative integer k. Define

$$\Theta_k(q) = \sum_{\ell(\lambda)=k+2,|\lambda|=0} \prod_{n\geq 1} \left(\frac{1}{i_n!} \frac{q^{ni_n}}{(1-q^n)^{i_n}} \cdot \frac{(-1)^{\tilde{i}_n}}{\tilde{i}_n!} \frac{1}{(1-q^n)^{\tilde{i}_n}} \right) \tag{2.5}$$

where $\lambda = (\cdots(-n)^{\tilde{i}_n}\cdots(-1)^{\tilde{i}_1}1^{i_1}\cdots n^{i_n}\cdots) \in \widetilde{\mathcal{P}}$. Define $\Theta_k(q,z)$ to be

$$\sum_{\substack{a,b\geq 0 \\ s_1,\ldots,s_a,t_1,\ldots,t_b\geq 1 \\ \sum_{i=1}^a s_i + \sum_{i=1}^b t_i = k+2}} \prod_{i=1}^a \frac{1}{s_i!} \cdot \prod_{i=1}^b \frac{(-1)^{t_i}}{t_i!}$$

$$\cdot \sum_{n_1>\cdots>n_a} \prod_{i=1}^a \frac{(qz)^{n_i s_i}}{(1-q^{n_i})^{s_i}} \cdot \sum_{n_1>\cdots>n_b} \prod_{i=1}^b \frac{z^{-n_i t_i}}{(1-q^{n_i})^{t_i}}.$$

Let $\mathrm{Coeff}_{z^i} g(z)$ denote the coefficient of z^i in a formal power series $g(z)$.

Lemma 2.3. *Fix a non-negative integer k. Then,*

(i) $\Theta_k(q,z) = \dfrac{1}{(k+2)!} \cdot \left(\displaystyle\sum_{m>0} \frac{(qz)^m}{1-q^m} - \sum_{m>0} \frac{z^{-m}}{1-q^m} \right)^{k+2}$;

(ii) $\Theta_k(q) = \mathrm{Coeff}_{z^0}\Theta_k(q,z)$;

(iii) *If k is an odd positive integer, then $\Theta_k(q) = 0$.*

Proof. (i) For a fixed positive integer s, we have

$$\begin{aligned} &\sum_{\sum_{i=1}^a s_i = s} \prod_{i=1}^a \frac{1}{s_i!} \cdot \sum_{n_1>\cdots>n_a} \prod_{i=1}^a \frac{(qz)^{n_i s_i}}{(1-q^{n_i})^{s_i}} \\ = \; &\frac{1}{s!} \sum_{m_1,\ldots,m_s>0} \prod_{i=1}^s \frac{(qz)^{m_i}}{1-q^{m_i}} \end{aligned} \tag{2.6}$$

$$= \frac{1}{s!}\left(\sum_{m>0}\frac{(qz)^m}{1-q^m}\right)^s.$$

Therefore, by the definition of $\Theta_k(q,z)$, we obtain

$$\begin{aligned} \Theta_k(q,z) &= \sum_{\substack{s,t\geq 0\\ s+t=k+2}} \frac{1}{s!}\left(\sum_{m>0}\frac{(qz)^m}{1-q^m}\right)^s \cdot \frac{1}{t!}\left(\sum_{m>0}\frac{-z^{-m}}{1-q^m}\right)^t \\ &= \frac{1}{(k+2)!}\cdot\left(\sum_{m>0}\frac{(qz)^m}{1-q^m} - \sum_{m>0}\frac{z^{-m}}{1-q^m}\right)^{k+2}. \end{aligned} \tag{2.7}$$

(ii) Denote the *positive* integers in the *ordered* list $\{i_1,\ldots,i_n,\ldots\}$ by $s_a,\ldots,s_1$ respectively (e.g., if the ordered list $\{i_1,\ldots,i_n,\ldots\}$ is equal to $\{2,0,5,4,0,\ldots\}$, then $a=3$ with $s_3=2, s_2=5, s_1=4$). Similarly, denote the *positive* integers in the *ordered* list $\{\tilde{i}_1,\ldots,\tilde{i}_n,\ldots\}$ by $t_b,\ldots,t_1$ respectively. Rewriting the right-hand-side of (2.5) in terms of $s_a,\ldots,s_1$ and $t_b,\ldots,t_1$, we see that $\Theta_k(q) = \mathrm{Coeff}_{z^0}\Theta_k(q,z)$.

(iii) Replacing λ in (2.5) by $-\lambda = (\cdots(-n)^{i_n}\cdots(-1)^{i_1}1^{\tilde{i}_1}\cdots n^{\tilde{i}_n}\cdots)$, we conclude that $\Theta_k(q) = (-1)^{k+2}\cdot\Theta_k(q)$. So $\Theta_k(q)=0$ if k is odd. □

Definition 2.4. Fix two positive integers s and t. Define

$$U_{s,t}(q,z) = \frac{1}{s!}\left(\sum_{m>0}\frac{(qz)^m}{1-q^m}\right)^s \cdot \frac{1}{t!}\left(\sum_{m>0}\frac{z^{-m}}{1-q^m}\right)^t.$$

Notice that the weight of $\mathrm{Coeff}_{z^0}U_{s,t}(q,z)$ is equal to $s+t$.

Lemma 2.5. $\lim\limits_{q\to 1}\left((1-q)^{s+t}\cdot \mathrm{Coeff}_{z^0}U_{s,t}(q,z)\right) \in \mathbf{MZV}$.

Proof. First of all, let $\mathcal{U}(s,t)$ denote

$$\sum_{\substack{n_1,\cdots,n_s,m_1,\cdots,m_t>0\\ n_1+\cdots+n_s=m_1+\cdots+m_t}} \prod_{i=1}^{s}\frac{1}{n_i}\cdot\prod_{j=1}^{t}\frac{1}{m_j}. \tag{2.8}$$

The function $\mathcal{U}(s,t)$ is from [2], and is a special case of the extended Mordell-Tornheim-Witten zeta function values [3, 4, 17, 18]. By the Theorem 2 in [2], if $s\geq t\geq 1$, then $\mathcal{U}(s,t)$ is equal to

$$s!\cdot t!\cdot\sum_{n=1}^{t}\frac{1}{n!}\sum_{\substack{j_1,\ldots,j_n,k_1,\ldots,k_n\geq 1\\ j_1+\ldots+j_n=s,k_1+\ldots+k_n=t}}\prod_{i=1}^{n}\left(\frac{(j_i+k_i-1)!}{j_i!\cdot k_i!}\zeta(j_i+k_i)\right) \tag{2.9}$$

where $\zeta(\cdot)$ is the Riemann zeta function.

Next, we have

$$\begin{aligned}\lim_{q\to 1}(1-q)^{s+t}U_{s,t}(q,z) &= \frac{1}{s!}\left(\sum_{m>0}\frac{z^m}{m}\right)^s\cdot\frac{1}{t!}\left(\sum_{m>0}\frac{z^{-m}}{m}\right)^t \\ &= \frac{1}{s!}\cdot\frac{1}{t!}\cdot\sum_{n_1,\cdots,n_s>0}\prod_{i=1}^{s}\frac{z^{n_i}}{n_i}\cdot\sum_{m_1,\cdots,m_t>0}\prod_{j=1}^{t}\frac{z^{-m_j}}{m_j}.\end{aligned}$$

Combining with (2.8) and (2.9), we conclude that

$$\begin{aligned}\lim_{q\to 1}\left((1-q)^{s+t}\cdot\mathrm{Coeff}_{z^0}U_{s,t}(q,z)\right) &= \frac{1}{s!}\cdot\frac{1}{t!}\cdot\mathcal{U}(s,t) \\ &\in \mathbb{Q}[\zeta(2),\zeta(3),\zeta(4),\zeta(5),\ldots] \\ &\subset \mathbf{MZV}.\end{aligned}\tag{2.10}$$

This completes the proof of the lemma. □

Let $k\geq 0$ be an even integer. By Lemma 2.3 (ii), (2.7) and Definition 2.4,

$$\Theta_k(q)=\sum_{\substack{s,t>0\\ s+t=k+2}}(-1)^t\cdot\mathrm{Coeff}_{z^0}U_{s,t}(q,z).\tag{2.11}$$

So we conclude from Lemma 2.5 that

$$\lim_{q\to 1}\left((1-q)^{k+2}\Theta_k(q)\right)\in\mathbf{MZV}.\tag{2.12}$$

In fact, by Corollary 4.9 (i) below, $\Theta_k(q)$ is a quasi-modular form.

3. The equivariant Chern character operators

In this section, X denotes the complex affine plane $\mathbb{C}^2$. We will define the equivariant Chern character operators, and express them in terms of Heisenberg operators. Moreover, we will recall the series $\langle \mathrm{ch}_{k_1}\cdots\mathrm{ch}_{k_N}\rangle$ and the reduced series $\langle \mathrm{ch}_{k_1}\cdots\mathrm{ch}_{k_N}\rangle'$, which have been studied in [5, 6]. Unless otherwise specified, all cohomology groups in this paper are in $\mathbb{C}$-coefficients.

We begin with the equivariant setting. Let $X=\mathbb{C}^2$, and let u,v be the standard coordinate functions on $\mathbb{C}^2$. Define the action of $\mathbb{T}=\mathbb{C}^*$ on X by

$$s\cdot(u,v)=(su,s^{-1}v),\quad s\in\mathbb{T}.\tag{3.1}$$

The origin of X is the only fixed point. This action of $\mathbb{T}$ on X induces natural actions of $\mathbb{T}$ on the Hilbert schemes $X^{[n]}$. Define a bilinear form

$$\langle \cdot, \cdot \rangle : \ (H^*_{\mathbb{T}}(X^{[n]}) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)) \times (H^*_{\mathbb{T}}(X^{[n]}) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)) \to \mathbb{C}(t)$$

by putting

$$\langle A, B \rangle = (-1)^n p_! \iota_!^{-1}(A \cup B) \tag{3.2}$$

where p is the projection $(X^{[n]})^{\mathbb{T}} \to \mathrm{pt}$, and $\iota : (X^{[n]})^{\mathbb{T}} \to X^{[n]}$ is the inclusion map.

A Heisenberg algebra action on the middle cohomology groups

$$\mathbb{H}_X = \bigoplus_{n \geq 0} H^{2n}_{\mathbb{T}}(X^{[n]})$$

was constructed in [19] (see also [9, 12, 13]). Equivalently, the space

$$\mathbb{H}'_X = \bigoplus_{n \geq 0} H^*_{\mathbb{T}}(X^{[n]}) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \tag{3.3}$$

is an irreducible module over the Heisenberg algebra which is generated by the Nakajima operators $\mathfrak{a}_m(\alpha), m \in \mathbb{Z}$ and $\alpha \in H^*_{\mathbb{T}}(X) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) = \mathbb{C}(t) \cdot 1_X$ with

$$[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = m \cdot \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \tag{3.4}$$

where $\langle \alpha, \beta \rangle$ denotes the equivariant pairing (3.2). For every integer m, put

$$\mathfrak{a}_m = \mathfrak{a}_m(1_X),$$

and $|0\rangle = 1 \in H^*_{\mathbb{T}}(X^{[0]}) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) = \mathbb{C}(t)$. Since $H^*_{\mathbb{T}}(X) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) = \mathbb{C}(t) \cdot 1_X$,

$$\mathbb{H}'_X = \mathbb{C}(t)[\mathfrak{a}_{-1}, \mathfrak{a}_{-2}, \ldots] \cdot |0\rangle \tag{3.5}$$

together with the Heisenberg commutation relation

$$[\mathfrak{a}_m, \mathfrak{a}_n] = m \cdot \delta_{m,-n} \cdot t^{-2}. \tag{3.6}$$

The following operators play an essential role in this paper:

$$\Gamma_{\pm}(z) = \exp\left(\sum_{n>0} \frac{z^{\mp n}}{n} \mathfrak{a}_{\pm n} \right). \tag{3.7}$$

Using (3.6), we obtain

$$\Gamma_+(x)\Gamma_-(y) = \left(1 - \frac{y}{x}\right)^{-t^{-2}} \cdot \Gamma_-(y)\Gamma_+(x) \tag{3.8}$$

(see also the Lemma 5 in [5]).

In the rest of the paper, we set $t = 1$. We see from (3.5) and (3.6) that

$$\mathbb{H}'_X = \mathbb{C}[\mathfrak{a}_{-1}, \mathfrak{a}_{-2}, \ldots] \cdot |0\rangle \tag{3.9}$$

together with the following Heisenberg commutation relation

$$[\mathfrak{a}_m, \mathfrak{a}_n] = m \cdot \delta_{m,-n}. \tag{3.10}$$

Remark 3.1. In our setup here, we have implicitly defined

$$\mathfrak{a}_n = (-1)^n \cdot \mathfrak{a}^*_{-n} \tag{3.11}$$

for $n > 0$, which is consistent with the setup in [11]. The commutation relation (3.10) is the same as the commutation relation in [6, (27)].

Let p_1, p_2 be the projections of $X^{[n]} \times X$ to $X^{[n]}, X$ respectively. Let $\mathcal{Z}_n$ be the universal codimension-2 closed subscheme of $X^{[n]} \times X$, i.e.,

$$\mathcal{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \mathrm{Supp}(\xi)\} \subset X^{[n]} \times X.$$

The tautological rank-n vector bundle $\mathcal{O}_X^{[n]}$ over $X^{[n]}$ is defined to be $p_{1*}\mathcal{O}_{\mathcal{Z}_n}$, and is $\mathbb{T}$-equivariant. Let $\mathrm{ch}_{k,\mathbb{T}}(\mathcal{O}_X^{[n]})$ be its k-th $\mathbb{T}$-equivariant Chern character.

Definition 3.2. For $k \geq 0$, define $\mathfrak{G}_k$ to be *the k-th equivariant Chern character operator* which acts on $\mathbb{H}'_X$ by cup product with $\bigoplus_n \mathrm{ch}_{k,\mathbb{T}}(\mathcal{O}_X^{[n]})$.

For a generalized partition $\lambda = (\cdots(-2)^{m_{-2}}(-1)^{m_{-1}}1^{m_1}2^{m_2}\cdots)$, put

$$\mathfrak{a}_\lambda = \prod_i \mathfrak{a}_i^{m_i} = (\cdots \mathfrak{a}_{-2}^{m_{-2}}\mathfrak{a}_{-1}^{m_{-1}}\mathfrak{a}_1^{m_1}\mathfrak{a}_2^{m_2}\cdots).$$

Proposition 3.3. *The equivariant Chern character operator $\mathfrak{G}_k$ is equal to*

$$\sum_{\substack{\ell(\lambda)\le k+2\\|\lambda|=0}} \frac{\mathfrak{a}_\lambda}{\lambda^!} \cdot \mathrm{Coeff}_{z^k} \frac{1}{(x-1)(1-x^{-1})} \prod_{n>0}\left(\frac{x^n-1}{n}\right)^{m_{-n}} \cdot \prod_{n>0}\left(\frac{1-x^{-n}}{n}\right)^{m_n}$$

where $|x^{-1}| < 1$, $x = e^z$, and $\lambda = (\cdots(-2)^{m_{-2}}(-1)^{m_{-1}}1^{m_1}2^{m_2}\cdots)$.

Proof. By [6, Lemma 2 (b)] (see also the proof of [5, Lemma 6]), $\mathfrak{G}_k$ is equal to

$$\text{Coeff}_{z^k y^0} \frac{1}{1-x} \left(\frac{1}{1-x^{-1}} - \Gamma_-(xy)\Gamma_+(xy)^{-1}\, \Gamma_-(y)^{-1}\Gamma_+(y) \right) \tag{3.12}$$

where $|x^{-1}| < 1$ and $x = e^z$. By (3.8), we have

$$\Gamma_+(xy)^{-1}\Gamma_-(y)^{-1} = \frac{1}{1-x^{-1}}\, \Gamma_-(y)^{-1}\Gamma_+(xy)^{-1}.$$

Combining this with (3.12), we conclude that $\mathfrak{G}_k$ is equal to

$$\text{Coeff}_{z^k y^0} \frac{1}{(1-x)(1-x^{-1})} \Big(1 - \Gamma_-(xy)\Gamma_-(y)^{-1}\Gamma_+(xy)^{-1}\Gamma_+(y) \Big). \tag{3.13}$$

Next, by (3.7) and using generalized partitions, we obtain

$$\begin{aligned}
&\Gamma_-(xy)\Gamma_-(y)^{-1}\Gamma_+(xy)^{-1}\Gamma_+(y) \\
= \;& \exp\left(\sum_{n>0} \frac{(x^n-1)y^n}{n} \mathfrak{a}_{-n} \right) \cdot \exp\left(\sum_{n>0} \frac{(1-x^{-n})y^{-n}}{n} \mathfrak{a}_n \right) \\
= \;& 1 + \sum_{\lambda \in \widetilde{\mathcal{P}}} \prod_{n>0} \left(\frac{(x^n-1)y^n}{n} \right)^{m_{-n}} \cdot \prod_{n>0} \left(\frac{(1-x^{-n})y^{-n}}{n} \right)^{m_n} \cdot \frac{\mathfrak{a}_\lambda}{\lambda!} \\
= \;& 1 + \sum_{\lambda \in \widetilde{\mathcal{P}}} \prod_{n>0} \left(\frac{x^n-1}{n} \right)^{m_{-n}} \cdot \prod_{n>0} \left(\frac{1-x^{-n}}{n} \right)^{m_n} \cdot y^{-|\lambda|} \cdot \frac{\mathfrak{a}_\lambda}{\lambda!}
\end{aligned}$$

where $\lambda = (\cdots (-2)^{m_{-2}} (-1)^{m_{-1}} 1^{m_1} 2^{m_2} \cdots)$. By (3.13), $\mathfrak{G}_k$ is equal to

$$\sum_{|\lambda|=0} \frac{\mathfrak{a}_\lambda}{\lambda!} \cdot \text{Coeff}_{z^k} \frac{1}{(x-1)(1-x^{-1})} \prod_{n>0} \left(\frac{x^n-1}{n} \right)^{m_{-n}} \cdot \prod_{n>0} \left(\frac{1-x^{-n}}{n} \right)^{m_n}.$$

Finally, note that if $\ell(\lambda) = \sum_{n>0} (m_{-n} + m_n) > k+2$, then

$$\text{Coeff}_{z^k} \frac{1}{(x-1)(1-x^{-1})} \prod_{n>0} \left(\frac{x^n-1}{n} \right)^{m_{-n}} \cdot \prod_{n>0} \left(\frac{1-x^{-n}}{n} \right)^{m_n} = 0.$$

This completes the proof of our proposition. □

The leading term of $\mathfrak{G}_k$ is given by the following corollary.

Corollary 3.4. *The operator* $\mathfrak{G}_k$ *is equal to*

$$\sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{\mathfrak{a}_\lambda}{\lambda!} + \sum_{\ell(\lambda)\le k, |\lambda|=0} g_\lambda \cdot \frac{\mathfrak{a}_\lambda}{\lambda!}$$

where $g_{-\lambda} = g_\lambda \in \mathbb{Q}$.

Proof. For $|\lambda| = \sum_n n m_n = 0$, we define the rational number g_λ to be

$$\begin{aligned} (3.14) \qquad & \mathrm{Coeff}_{z^k} \frac{1}{(x-1)(1-x^{-1})} \prod_{n\neq 0} \left(\frac{1-x^{-n}}{n}\right)^{m_n} \\ = \ & \mathrm{Coeff}_{z^k} \frac{1}{(x-1)(1-x^{-1})} \prod_{n>0} \left(\frac{x^n-1}{n}\right)^{m_{-n}} \cdot \prod_{n>0} \left(\frac{1-x^{-n}}{n}\right)^{m_n} \end{aligned}$$

where $|x^{-1}| < 1$ and $x = e^z$. Since $|\lambda| = \sum_n n m_n = 0$, we have

$$\begin{aligned} g_\lambda &= \mathrm{Coeff}_{z^k} \frac{x^{\sum_{n>0}(n m_{-n} - n m_n)}}{(x-1)(1-x^{-1})} \prod_{n>0} \left(\frac{1-x^{-n}}{n}\right)^{m_{-n}} \cdot \prod_{n>0} \left(\frac{x^n-1}{n}\right)^{m_n} \\ &= \mathrm{Coeff}_{z^k} \frac{1}{(x-1)(1-x^{-1})} \prod_{n>0} \left(\frac{x^n-1}{n}\right)^{m_n} \cdot \prod_{n>0} \left(\frac{1-x^{-n}}{n}\right)^{m_{-n}} \\ &= g_{-\lambda}. \end{aligned}$$

Next, a straightforward computation shows that

$$g_\lambda = \begin{cases} 1 & \text{if } \ell(\lambda) = k+2, \\ 0 & \text{if } \ell(\lambda) = k+1. \end{cases}$$

So we conclude from Proposition 3.3 and (3.14) that

$$\mathfrak{G}_k = \sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{\mathfrak{a}_\lambda}{\lambda!} + \sum_{\ell(\lambda)\le k, |\lambda|=0} g_\lambda \cdot \frac{\mathfrak{a}_\lambda}{\lambda!}.$$

This completes the proof of our corollary. □

Let $k_1, \ldots, k_N \ge 0$ and $m \in \mathbb{Z}$. Following [5, 6], define $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle$ to be the series

$$(3.15) \qquad \sum_{n=0}^{+\infty} q^n \int_{X^{[n]}} \mathrm{ch}_{k_1,\mathbb{T}}(\mathcal{O}_X^{[n]}) \cdots \mathrm{ch}_{k_N,\mathbb{T}}(\mathcal{O}_X^{[n]}) \cdot c(T_{X^{[n]},m})$$

where $T_{X^{[n]},m}$ denotes the tangent bundle of $X^{[n]}$ with a scaling action of $\mathbb{C}^*$ of character m. By the setups in [5, 6],

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle = \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^{N} \mathfrak{G}_{k_i} \tag{3.16}$$

where $\mathfrak{d}$ is the number-of-points operator, i.e., $\mathfrak{d}|_{H^*_{\mathbb{T}}(X^{[n]})} = n\,\mathrm{Id}$. In particular, by the result in Subsection 4.1 of [5],

$$\langle 1 \rangle = (q;q)_\infty^{m^2-1}. \tag{3.17}$$

Following [14], we define the reduced series

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle' = \frac{\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle}{\langle 1 \rangle} = (q;q)_\infty^{-m^2+1} \cdot \langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle. \tag{3.18}$$

Lemma 3.5. *Let $k_1, \ldots, k_N \geq 0$, and $m \in \mathbb{Z}$.*

(i) *As a function of q, the reduced series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ is a quasi-modular form of weight at most $\sum_{i=1}^N (k_i + 2)$. As a function of m, it is a polynomial in m^2 of degree at most $\sum_{i=1}^N \left(\lfloor \frac{k_i}{2} \rfloor + 1\right)$.*

(ii) *If $\sum_{i=1}^N k_i$ is odd, then $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle' = 0$.*

Proof. (i) By the formula (36) in [5], we have

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \left(\prod_{\ell=1}^{N} \sum_{\square \in D_\lambda} \frac{c(\square)^{k_\ell}}{k_\ell!} \right) \prod_{\square \in D_\lambda} \frac{h(\square)^2 - m^2}{h(\square)^2} \tag{3.19}$$

where $c(\square)$ and $h(\square)$ denote the content and the hook length, respectively, of a cell $\square$ in the Young diagram D_λ associated to the partition λ. So the reduced series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ can be regarded as a function of m^2. Now the statements in (i) are the Theorem 2 in [5] (see also the Theorem 2 in [6]).

(ii) For a partition $\lambda \in \mathcal{P}$, let λ^* be its conjugate partition. Then,

$$\prod_{\ell=1}^{N} \sum_{\square \in D_{\lambda^*}} \frac{c(\square)^{k_\ell}}{k_\ell!} = (-1)^{\sum_{\ell=1}^N k_\ell} \cdot \prod_{\ell=1}^{N} \sum_{\square \in D_\lambda} \frac{c(\square)^{k_\ell}}{k_\ell!} = -\prod_{\ell=1}^{N} \sum_{\square \in D_\lambda} \frac{c(\square)^{k_\ell}}{k_\ell!}$$

since $\sum_{i=1}^N k_i$ is odd. Therefore, we conclude from (3.19) that

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle = \sum_{\lambda^* \in \mathcal{P}} q^{|\lambda^*|} \left(\prod_{\ell=1}^{N} \sum_{\square \in D_{\lambda^*}} \frac{c(\square)^{k_\ell}}{k_\ell!} \right) \prod_{\square \in D_{\lambda^*}} \frac{h(\square)^2 - m^2}{h(\square)^2}$$

$$= -\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \left(\prod_{\ell=1}^{N} \sum_{\square \in D_\lambda} \frac{c(\square)^{k_\ell}}{k_\ell !} \right) \prod_{\square \in D_\lambda} \frac{h(\square)^2 - m^2}{h(\square)^2}$$
$$= -\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle.$$

Hence $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle = 0$. It follows that $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle' = 0$. □

The series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle$ and the reduced series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ have been investigated in [5, 6]. In particular, the leading coefficient of $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ (as a polynomial of m^2) would follow from [5, Proposition 2] or [6, Proposition 4]. However, we are unable to achieve this due to the complexity of [5, Proposition 2] and [6, Proposition 4]. So instead, we will determine the leading coefficient of $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ as a polynomial of m^2 by applying the methods in [16, Section 4]. This will be done in the next section.

4. The leading coefficient of $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ as a polynomial of m^2

In this section, we will continue to use the setup in Section 3. So $X = \mathbb{C}^2$. Our goal is to study the reduced series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ by going through the process in [16, Section 4], and determine its leading coefficient. The quasi-modularity of this leading coefficient is guaranteed by Lemma 3.5 (i).

By (3.16), Corollary 3.4 and (3.7), to understand $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle$, we must study

$$\begin{aligned} &\mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!} \\ = \;& \mathrm{Tr}\, q^{\mathfrak{d}}\, \exp\left(\sum_{n>0} \frac{m z^n}{n} \mathfrak{a}_{-n} \right) \exp\left(- \sum_{n>0} \frac{m z^{-n}}{n} \mathfrak{a}_{n} \right) \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!} \end{aligned}$$

where $\lambda^{(i)}$ denotes a generalized partition in $\widetilde{\mathcal{P}}$. The purpose of the next two lemmas is to remove the term $\Gamma_-(z)^m$ from the above trace.

Lemma 4.1. *Let $n \neq 0$, $m \in \mathbb{Z}$, and $\lambda \in \widetilde{\mathcal{P}}$ be a generalized partition. Then,*

$$\begin{aligned} &\frac{\mathfrak{a}_\lambda}{\lambda!} \exp\left(\frac{m z^n}{n} \mathfrak{a}_{-n} \right) \\ = \;& \exp\left(\frac{m z^n}{n} \mathfrak{a}_{-n} \right) \cdot \sum_{i \geq 0} \frac{(m z^n)^i}{i!} \frac{\mathfrak{a}_{\lambda - (n^i)}}{(\lambda - (n^i))!}, \end{aligned} \tag{4.1}$$

$$\exp\left(-\frac{mz^{-n}}{n}\mathfrak{a}_n\right)\cdot\frac{\mathfrak{a}_\lambda}{\lambda!} \tag{4.2}$$
$$= \sum_{i\geq 0}\frac{(-mz^{-n})^i}{i!}\frac{\mathfrak{a}_{\lambda-((-n)^i)}}{(\lambda-((-n)^i))!}\exp\left(-\frac{mz^{-n}}{n}\mathfrak{a}_n\right).$$

Proof. This is parallel to Lemma 4.1 in [16]. To prove (4.1), note from (3.10) that for $n \neq 0$, $s > 0$ and $t > 0$, we have

$$\frac{\mathfrak{a}_n^s}{s!}\cdot\frac{\mathfrak{a}_{-n}^t}{t!} = \sum_{i=0}^{\min(s,t)}\frac{n^i}{i!}\cdot\frac{\mathfrak{a}_{-n}^{t-i}}{(t-i)!}\cdot\frac{\mathfrak{a}_n^{s-i}}{(s-i)!}. \tag{4.3}$$

To prove (4.2), we see from (4.1) that

$$\exp\left(-\frac{mz^n}{n}\mathfrak{a}_{-n}\right)\cdot\frac{\mathfrak{a}_\lambda}{\lambda!} = \sum_{i\geq 0}\frac{(mz^n)^i}{i!}\frac{\mathfrak{a}_{\lambda-(n^i)}}{(\lambda-(n^i))!}\exp\left(-\frac{mz^n}{n}\mathfrak{a}_{-n}\right).$$

Now (4.2) follows from replacing m, n by $-m, -n$ respectively. □

Lemma 4.2. *Let $\lambda^{(1)}, \ldots, \lambda^{(N)} \in \widetilde{\mathcal{P}}$ be generalized partitions. Then, the trace $\mathrm{Tr}\, q^{\mathfrak{d}}\,\Gamma_-(z)^m\Gamma_+(z)^{-m}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}$ is equal to*

$$(q;q)_\infty^{m^2}\cdot\sum_{\substack{\mu^{(i,s)}\in\widetilde{\mathcal{P}}_+\\ 1\leq i\leq N, s\geq 1}}\prod_{\substack{1\leq i\leq N\\ s,n\geq 1}}\frac{(m(zq^s)^n)^{m_n^{(i,s)}}}{m_n^{(i,s)}!}\mathrm{Tr}\, q^{\mathfrak{d}}\Gamma_+(z)^{-m}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}-\sum_{s\geq 1}\mu^{(i,s)}}}{\left(\lambda^{(i)}-\sum_{s\geq 1}\mu^{(i,s)}\right)!}$$

where $\mu^{(i,s)} = \left(1^{m_1^{(i,s)}}\cdots n^{m_n^{(i,s)}}\cdots\right) \in \widetilde{\mathcal{P}}_+$ for $1 \leq i \leq N$ and $s \geq 1$.

Proof. For simplicity, put $Q_1 = \mathrm{Tr}\, q^{\mathfrak{d}}\,\Gamma_-(z)^m\Gamma_+(z)^{-m}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}$. We have

$$Q_1 = \mathrm{Tr}\,\Gamma_-(zq)^m\, q^{\mathfrak{d}}\,\Gamma_+(z)^{-m}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!} = \mathrm{Tr}\, q^{\mathfrak{d}}\,\Gamma_+(z)^{-m}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}\cdot\Gamma_-(zq)^m.$$

Applying (4.1) repeatedly, we conclude that

$$\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}\cdot\Gamma_-(zq)^m$$

$$= \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!} \cdot \exp\left(\sum_{n>0} \frac{m(zq)^n}{n} \mathfrak{a}_{-n}\right)$$

$$= \Gamma_-(zq)^m \sum_{\substack{\mu^{(i,1)} \in \widetilde{\mathcal{P}}_+ \\ 1 \le i \le N}} \prod_{\substack{1 \le i \le N \\ n \ge 1}} \frac{(m(zq)^n)^{m_n^{(i,1)}}}{m_n^{(i,1)}!} \cdot \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)} - \mu^{(i,1)}}}{\left(\lambda^{(i)} - \mu^{(i,1)}\right)!}$$

where $\mu^{(i,1)} = \left(1^{m_1^{(i,1)}} \cdots n^{m_n^{(i,1)}} \cdots\right)$. Therefore, Q_1 is equal to

$$\begin{aligned} & \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_+(z)^{-m} \Gamma_-(zq)^m \\ & \quad \cdot \sum_{\substack{\mu^{(i,1)} \in \widetilde{\mathcal{P}}_+ \\ 1 \le i \le N}} \prod_{\substack{1 \le i \le N \\ n \ge 1}} \frac{(m(zq)^n)^{m_n^{(i,1)}}}{m_n^{(i,1)}!} \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)} - \mu^{(i,1)}}}{\left(\lambda^{(i)} - \mu^{(i,1)}\right)!} \\ = \; & (1-q)^{m^2} \cdot \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(zq)^m \Gamma_+(z)^{-m} \\ & \quad \cdot \sum_{\substack{\mu^{(i,1)} \in \widetilde{\mathcal{P}}_+ \\ 1 \le i \le N}} \prod_{\substack{1 \le i \le N \\ n \ge 1}} \frac{(m(zq)^n)^{m_n^{(i,1)}}}{m_n^{(i,1)}!} \cdot \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)} - \mu^{(i,1)}}}{\left(\lambda^{(i)} - \mu^{(i,1)}\right)!} \end{aligned}$$

where we have used (3.8) with $t = 1$.

Repeat the above process s times. Then, Q_1 is equal to

$$\begin{aligned} & (q;q)_{s-1}^{m^2} \cdot \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(zq^s)^m \Gamma_+(z)^{-m} \\ & \cdot \sum_{\substack{\mu^{(i,r)} \in \widetilde{\mathcal{P}}_+ \\ 1 \le i \le N \\ 1 \le r \le s}} \prod_{\substack{1 \le i \le N \\ n \ge 1 \\ 1 \le r \le s}} \frac{(m(zq^r)^n)^{m_n^{(i,r)}}}{m_n^{(i,r)}!} \cdot \prod_{i=1}^{N} \frac{\mathfrak{a}_{\lambda^{(i)} - \sum_{r=1}^{s} \mu^{(i,r)}}}{\left(\lambda^{(i)} - \sum_{r=1}^{s} \mu^{(i,r)}\right)!} \end{aligned}$$

where $\mu^{(i,r)} = \left(1^{m_1^{(i,r)}} \cdots n^{m_n^{(i,r)}} \cdots\right)$. Letting $s \to +\infty$ proves our lemma. □

Lemma 4.2 contains traces of the following form

$$Q_2 := \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_+(z)^{-m} \prod_{i=1}^{N} \frac{\mathfrak{a}_{\tilde{\lambda}^{(i)}}}{\tilde{\lambda}^{(i)}!} \tag{4.4}$$

where $\tilde{\lambda}^{(1)}, \ldots, \tilde{\lambda}^{(N)} \in \widetilde{\mathcal{P}}$ are generalized partitions. Our next two lemmas deal with Q_2. The first one eliminates the term $\Gamma_+(z)^{-m}$ from the trace Q_2.

Lemma 4.3. *Let* $\tilde\lambda^{(1)}, \ldots, \tilde\lambda^{(N)} \in \widetilde{\mathcal{P}}$. *Then, the trace* Q_2 *in* (4.4) *is equal to*

$$\sum_{\sum_{i=1}^N (|\tilde\lambda^{(i)}| - \sum_{t\ge 1} |\tilde\mu^{(i,t)}|)=0} \prod_{\substack{1\le i\le N\\ t,n\ge 1}} \frac{(-mz^{-n}q^{(t-1)n})^{\tilde m_n^{(i,t)}}}{\tilde m_n^{(i,t)}!} \cdot \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)} - \sum_{t\ge 1}\tilde\mu^{(i,t)}}}{\big(\tilde\lambda^{(i)} - \sum_{t\ge 1}\tilde\mu^{(i,t)}\big)^!}$$

where $\tilde\mu^{(i,t)} = \big(\cdots(-n)^{\tilde m_n^{(i,t)}}\cdots(-1)^{\tilde m_1^{(i,t)}}\big) \in \widetilde{\mathcal{P}}_-$ *for* $1\le i\le N$ *and* $t\ge 1$.

Proof. By (3.7),

$$Q_2 = \mathrm{Tr}\, q^{\mathfrak{d}} \, \exp\left(-\sum_{n>0} \frac{mz^{-n}}{n}\mathfrak{a}_n\right) \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}}}{\tilde\lambda^{(i)!}}.$$

Applying (4.2) repeatedly, we see that Q_2 is equal to

$$\begin{aligned}
&\sum_{\substack{\tilde\mu^{(i,1)}\in\widetilde{\mathcal{P}}_-\\ 1\le i\le N}} \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,1)}}}{\tilde m_n^{(i,1)}!} \cdot \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}}}{\big(\tilde\lambda^{(i)} - \tilde\mu^{(i,1)}\big)^!} \cdot \Gamma_+(z)^{-m}\\
=\; &\sum_{\substack{\tilde\mu^{(i,1)}\in\widetilde{\mathcal{P}}_-\\ 1\le i\le N}} \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,1)}}}{\tilde m_n^{(i,1)}!} \cdot q^{\sum_{i=1}^N (|\tilde\mu^{(i,1)}| - |\tilde\lambda^{(i)}|)}\\
&\cdot \mathrm{Tr} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}}}{\big(\tilde\lambda^{(i)} - \tilde\mu^{(i,1)}\big)^!} \cdot q^{\mathfrak{d}}\Gamma_+(z)^{-m}\\
=\; &\sum_{\substack{\tilde\mu^{(i,1)}\in\widetilde{\mathcal{P}}_-\\ 1\le i\le N}} \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,1)}}}{\tilde m_n^{(i,1)}!} \cdot q^{\sum_{i=1}^N (|\tilde\mu^{(i,1)}| - |\tilde\lambda^{(i)}|)}\\
&\cdot \mathrm{Tr}\, q^{\mathfrak{d}}\Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}}}{\big(\tilde\lambda^{(i)} - \tilde\mu^{(i,1)}\big)^!}
\end{aligned}$$

where $\tilde\mu^{(i,1)} = \big(\cdots(-n)^{\tilde m_n^{(i,1)}}\cdots(-1)^{\tilde m_1^{(i,1)}}\big) \in \widetilde{\mathcal{P}}_-$. Note that

$$\mathrm{Tr}\, q^{\mathfrak{d}}\Gamma_+(z)^{-m}\mathfrak{a}_\mu = 0$$

if $|\mu| > 0$. If $|\mu| = 0$, then $\mathrm{Tr}\, q^{\mathfrak{d}}\Gamma_+(z)^{-m}\mathfrak{a}_\mu = \mathrm{Tr}\, q^{\mathfrak{d}}\mathfrak{a}_\mu$. So Q_2 is equal to

$$\sum_{\sum_{i=1}^N(|\tilde\lambda^{(i)}|-|\tilde\mu^{(i,1)}|)<0} \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,1)}}}{\tilde m_n^{(i,1)}!} \cdot q^{\sum_{i=1}^N(|\tilde\mu^{(i,1)}|-|\tilde\lambda^{(i)}|)}$$
$$\cdot \mathrm{Tr}\, q^{\mathfrak{d}}\Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}}}{\big(\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}\big)^!}$$
$$+ \sum_{\sum_{i=1}^N(|\tilde\lambda^{(i)}|-|\tilde\mu^{(i,1)}|)=0} \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,1)}}}{\tilde m_n^{(i,1)}!} \cdot \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}}}{\big(\tilde\lambda^{(i)}-\tilde\mu^{(i,1)}\big)^!}.$$

Repeating the process in the previous paragraph t times, we see that

$$Q_2 = U(t) + V(t)$$

where $U(t)$ is given by

$$\sum_{\sum\limits_{i=1}^N(|\tilde\lambda^{(i)}|-\sum\limits_{r=1}^t |\tilde\mu^{(i,r)}|)<0} \prod_{r=1}^t \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,r)}}}{\tilde m_n^{(i,r)}!} q^{\sum\limits_{i=1}^N(\sum\limits_{\ell=1}^r |\tilde\mu^{(i,\ell)}|-|\tilde\lambda^{(i)}|)} \tag{4.5}$$

$$\cdot \mathrm{Tr}\, q^{\mathfrak{d}}\Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\sum_{r=1}^t \tilde\mu^{(i,r)}}}{\big(\tilde\lambda^{(i)}-\sum_{r=1}^t \tilde\mu^{(i,r)}\big)^!} \tag{4.6}$$

with $\tilde\mu^{(i,r)} = \big(\cdots(-n)^{\tilde m_n^{(i,r)}} \cdots (-1)^{\tilde m_1^{(i,r)}}\big) \in \widetilde{\mathcal{P}}_-$, and $V(t)$ is given by

$$\sum_{\sum_{i=1}^N(|\tilde\lambda^{(i)}|-\sum_{r=1}^t |\tilde\mu^{(i,r)}|)=0} \prod_{r=1}^t \prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,r)}}}{\tilde m_n^{(i,r)}!} q^{\sum_{i=1}^N(\sum_{\ell=1}^r |\tilde\mu^{(i,\ell)}|-|\tilde\lambda^{(i)}|)}$$
$$\cdot \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\sum_{r=1}^t \tilde\mu^{(i,r)}}}{\big(\tilde\lambda^{(i)}-\sum_{r=1}^t \tilde\mu^{(i,r)}\big)^!}.$$

Denote line (4.5) by $\widetilde{U}(t)$. Since $\sum_{i=1}^N(|\tilde\lambda^{(i)}| - \sum_{r=1}^t |\tilde\mu^{(i,r)}|) < 0$ and $|\tilde\mu^{(i,r)}| < 0$, $\widetilde{U}(t)$ is a polynomial in q with coefficients being bounded in terms of $-\sum_{i=1}^N |\tilde\lambda^{(i)}|$. Moreover, $q^t|\widetilde{U}(t)$. Line (4.6) is contained in a finite set of traces, which depends only on the generalized partitions $\tilde\lambda^{(i)}$ and is independent of t. Since $0 < q < 1$, $U(t) \to 0$ as $t \to +\infty$. Letting $t \to +\infty$, we see that Q_2

equals

$$\sum_{\sum_{i=1}^N(|\tilde\lambda^{(i)}|-\sum_{t\ge 1}|\tilde\mu^{(i,t)}|)=0} \prod_{t\ge 1}\prod_{\substack{1\le i\le N\\ n\ge 1}} \frac{(-mz^{-n})^{\tilde m_n^{(i,t)}}}{\tilde m_n^{(i,t)}!} q^{\sum_{i=1}^N(\sum_{\ell=1}^t|\tilde\mu^{(i,\ell)}|-|\tilde\lambda^{(i)}|)}$$
$$\cdot \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde\lambda^{(i)}-\sum_{t\ge 1}\tilde\mu^{(i,t)}}}{\big(\tilde\lambda^{(i)}-\sum_{t\ge 1}\tilde\mu^{(i,t)}\big)^!}.$$

Replacing $q^{\sum_{i=1}^N(\sum_{\ell=1}^t|\tilde\mu^{(i,\ell)}|-|\tilde\lambda^{(i)}|)}$ by $q^{-\sum_{i=1}^N\sum_{\ell\ge t+1}|\tilde\mu^{(i,\ell)}|}$ completes the proof of our lemma. ☐

Lemma 4.4. *Let $\lambda^{(1)},\ldots,\lambda^{(N)}\in\widetilde{\mathcal{P}}$ be generalized partitions. Put*

$$A_{\lambda^{(1)},\ldots,\lambda^{(N)}} = \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)!}}.$$

For $n\ne 0$, let $m_n^{(i)}$ denote the multiplicity of part n in $\lambda^{(i)}$.

(i) *If $\sum_{i=1}^N m_n^{(i)} \ne \sum_{i=1}^N m_{-n}^{(i)}$ for some $n\ne 0$, then $A_{\lambda^{(1)},\ldots,\lambda^{(N)}}=0$.*

(ii) *Assume that $\sum_{i=1}^N m_n^{(i)} = \sum_{i=1}^N m_{-n}^{(i)}$ for every $n\ne 0$. Then, $A_{\lambda^{(1)},\ldots,\lambda^{(N)}}$ is a linear combination of expressions of the form:*

$$(q;q)_\infty^{-1}\cdot\prod_{i=1}^{\ell}\frac{n_i^{\sum_{j=1}^N m_{n_i}^{(j)}} q^{n_i w_i}}{(1-q^{n_i})^{w_i}}$$

where $0<n_1<\cdots<n_\ell$ are the positive parts in $\lambda^{(1)}+\ldots+\lambda^{(N)}$, and $0\le w_i\le\sum_{j=1}^N m_{n_i}^{(j)}$. Moreover, the coefficients of this linear combination are independent of q and the integers n_i.

Proof. (i) is clear. For (ii), using (4.3) to move the creation operators in $\prod_{i=1}^N \mathfrak{a}_{\lambda^{(i)}}$ to the left, we see that $A_{\lambda^{(1)},\ldots,\lambda^{(N)}}$ is a linear combination of the expressions

$$\prod_{n>0} n^{\sum_{i=1}^N m_n^{(i)}-m_n}\cdot \mathrm{Tr}\, q^{\mathfrak{d}}\frac{\mathfrak{a}_\lambda}{\lambda^!} = \prod_{n>0} n^{\sum_{i=1}^N m_n^{(i)}-m_n}\cdot A_\lambda \tag{4.7}$$

where $m_n^{(i)}$ and m_n denote the multiplicities of part $n\in\mathbb{Z}-\{0\}$ in $\lambda^{(i)}$ and λ respectively, $m_n\le\sum_{i=1}^N m_n^{(i)}$ for every $n\in\mathbb{Z}-\{0\}$, and $|\lambda|=0$. Moreover, the coefficients of this linear combination are independent of q and the integers

$n > 0$ with $\sum_{i=1}^N m_n^{(i)} > 0$. If $\lambda = \emptyset$, then $A_\lambda = \mathrm{Tr}\, q^{\mathfrak{d}} = (q;q)_\infty^{-1}$. Assume that $\lambda \neq \emptyset$. By (i), we further assume that $m_{-n} = m_n$ for every $n > 0$. Since $\lambda \neq \emptyset$, there exists some $n > 0$ such that $m_{-n} = m_n > 0$. Then, we conclude that

$$\begin{aligned} A_\lambda &= q^n \cdot \mathrm{Tr}\, \frac{\mathfrak{a}_{-n}}{m_n} q^{\mathfrak{d}} \frac{\mathfrak{a}_{\lambda-(-n)}}{(\lambda-(-n))^!} \\ &= q^n \cdot \mathrm{Tr}\, q^{\mathfrak{d}} \frac{\mathfrak{a}_{\lambda-(-n)}}{(\lambda-(-n))^!} \frac{\mathfrak{a}_{-n}}{m_n} \\ &= q^n A_\lambda + q^n \cdot \frac{n}{m_n} \mathrm{Tr}\, q^{\mathfrak{d}} \frac{\mathfrak{a}_{\lambda-((-n)n)}}{(\lambda-((-n)n))^!} \end{aligned}$$

where we have applied (4.3) in the last step. Hence we obtain

$$\begin{aligned} A_\lambda &= \frac{1}{m_n} \cdot \frac{nq^n}{1-q^n} \cdot \mathrm{Tr}\, q^{\mathfrak{d}} \frac{\mathfrak{a}_{\lambda-((-n)n)}}{(\lambda-((-n)n))^!} \\ &= (q;q)_\infty^{-1} \cdot \prod_{n>0} \frac{1}{m_n!} \left(\frac{nq^n}{1-q^n} \right)^{m_n} \\ &= (q;q)_\infty^{-1} \cdot \prod_{n>0} \frac{1}{m_n!} \frac{n^{m_n} q^{n m_n}}{(1-q^n)^{m_n}}. \end{aligned}$$

By (4.7), we see that $A_{\lambda^{(1)},\ldots,\lambda^{(N)}}$ is a linear combination of the expressions

$$(q;q)_\infty^{-1} \cdot \prod_{n>0} \frac{1}{m_n!} \frac{n^{\sum_{i=1}^N m_n^{(i)}} q^{n m_n}}{(1-q^n)^{m_n}}$$

where $0 \le m_n \le \sum_{i=1}^N m_n^{(i)}$ for every $n > 0$. Deleting the factors (in the above product) with $\sum_{i=1}^N m_n^{(i)} = 0$ completes the proof of our lemma. □

Remark 4.5. Let N be even. In the special case that $\lambda^{(i)} = (n_i)$ for $1 \le i \le N$ and $n_i \neq 0$, an argument similar to the proof of Lemma 4.4 shows that

$$\mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \mathfrak{a}_{n_i} = \frac{(-n_1) q^{-n_1}}{1-q^{-n_1}} \cdot \sum_{\substack{2 \le j \le N \\ n_j = -n_1}} \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{\substack{2 \le i \le N \\ i \neq j}} \mathfrak{a}_{n_i}.$$

By induction, we conclude that

$$\sum_{\sum_{i=1}^N n_i = 0, n_i \neq 0} \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N z_i^{n_i} \mathfrak{a}_{n_i} \tag{4.8}$$

$$= \quad (q;q)_\infty^{-1} \cdot \sum_{\substack{n_1,\ldots,n_{N/2}\neq 0\\ \mathbf{P}_N}} \prod_{s=1}^{N/2} \frac{(-n_s)q^{-n_s}}{1-q^{-n_s}} (z_{i_s} z_{j_s}^{-1})^{n_s}$$

where $\mathbf{P}_N$ runs over all the partitions of the set $\{1, \ldots, N\}$ into pairs

$$\{i_1, j_1\}, \ldots, \{i_{N/2}, j_{N/2}\}$$

with $i_1 < j_1, \ldots, i_{N/2} < j_{N/2}$ and $i_1 < \ldots < i_{N/2}$ (so necessarily, $i_1 = 1$).

Proposition 4.6. *Let* $\lambda^{(i)} = (\cdots(-n)^{\tilde m_n^{(i)}} \cdots (-1)^{\tilde m_1^{(i)}} 1^{m_1^{(i)}} \cdots n^{m_n^{(i)}} \cdots) \in \widetilde{\mathcal{P}}$ *for* $1 \le i \le N$, *and let* $m \in \mathbb{Z}$. *Then,* $\operatorname{Tr} q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}$ *is equal to*

$$(q;q)_\infty^{m^2-1} \cdot z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot m^{\sum_{i=1}^N \ell(\lambda^{(i)})} \tag{4.9}$$

$$\cdot \prod_{1\le i\le N, n\ge 1} \frac{1}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1-q^n)^{m_n^{(i)}}} \frac{(-1)^{\tilde m_n^{(i)}}}{\tilde m_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde m_n^{(i)}}} + (q;q)_\infty^{m^2-1} \cdot \widetilde{W}$$

where $\widetilde{W}$ *is a polynomial of* m *with degree* $< \sum_{i=1}^N \ell(\lambda^{(i)})$.

Proof. For simplicity, we put $\operatorname{Tr}_\lambda = \operatorname{Tr} q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}$. Combining Lemma 4.2 and Lemma 4.3, we conclude that $\operatorname{Tr}_\lambda$ is equal to

$$(q;q)_\infty^{m^2} \cdot \sum_{\substack{\sum_{i=1}^N (|\lambda^{(i)}| - \sum_{s\ge1} |\mu^{(i,s)}| - \sum_{t\ge 1} |\tilde\mu^{(i,t)}|)=0\\ \mu^{(i,s)}\in\widetilde{\mathcal{P}}_+,\, \tilde\mu^{(i,t)}\in\widetilde{\mathcal{P}}_-}} \prod_{\substack{1\le i\le N\\ s,n\ge 1}} \frac{(mz^n q^{sn})^{m_n^{(i,s)}}}{m_n^{(i,s)}!}$$

$$\cdot \prod_{\substack{1\le i\le N\\ t,n\ge 1}} \frac{(-mz^{-n} q^{(t-1)n})^{\tilde m_n^{(i,t)}}}{\tilde m_n^{(i,t)}!} \cdot \operatorname{Tr} q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)} - \sum_{s\ge1}\mu^{(i,s)} - \sum_{t\ge1}\tilde\mu^{(i,t)}}}{\left(\lambda^{(i)} - \sum_{s\ge1}\mu^{(i,s)} - \sum_{t\ge1}\tilde\mu^{(i,t)}\right)!}$$

where $\mu^{(i,s)} = \left(1^{m_1^{(i,s)}} \cdots n^{m_n^{(i,s)}} \cdots\right)$ and $\tilde\mu^{(i,t)} = \left(\cdots(-n)^{\tilde m_n^{(i,t)}} \cdots (-1)^{\tilde m_1^{(i,t)}}\right)$. The sum of all the exponents of z is $\sum_{i=1}^N |\lambda^{(i)}|$. So $\operatorname{Tr}_\lambda$ is equal to

$$(q;q)_\infty^{m^2} \cdot z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot \sum_{\substack{\sum_{i=1}^N (|\lambda^{(i)}| - \sum_{s\ge1} |\mu^{(i,s)}| - \sum_{t\ge 1} |\tilde\mu^{(i,t)}|)=0\\ \mu^{(i,s)}\in\widetilde{\mathcal{P}}_+,\, \tilde\mu^{(i,t)}\in\widetilde{\mathcal{P}}_-}} \prod_{\substack{1\le i\le N\\ s,n\ge 1}} \frac{(mq^{sn})^{m_n^{(i,s)}}}{m_n^{(i,s)}!}$$

$$\cdot \prod_{\substack{1\le i\le N\\ t,n\ge 1}} \frac{(-mq^{(t-1)n})^{\tilde m_n^{(i,t)}}}{\tilde m_n^{(i,t)}!} \cdot \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}-\sum_{s\ge1}\mu^{(i,s)}-\sum_{t\ge1}\tilde\mu^{(i,t)}}}{\big(\lambda^{(i)}-\sum_{s\ge1}\mu^{(i,s)}-\sum_{t\ge1}\tilde\mu^{(i,t)}\big)^!}.$$

By our convention, $\sum_{s\ge1}\mu^{(i,s)}+\sum_{t\ge1}\tilde\mu^{(i,t)}\le\lambda^{(i)}$ for every $1\le i\le N$. Put

$$\begin{aligned}\tilde\lambda^{(i)} &= \sum_{s\ge1}\mu^{(i,s)}+\sum_{t\ge1}\tilde\mu^{(i,t)} \\ &= \big(\cdots(-n)^{\tilde p_n^{(i)}}\cdots(-1)^{\tilde p_1^{(i)}}1^{p_1^{(i)}}\cdots n^{p_n^{(i)}}\cdots\big).\end{aligned} \tag{4.10}$$

Then, $\sum_{s\ge1}m_n^{(i,s)}=p_n^{(i)}$ and $\sum_{t\ge1}\tilde m_n^{(i,t)}=\tilde p_n^{(i)}$. So Tr_λ is equal to

$$\begin{aligned}&(q;q)_\infty^{m^2}\cdot z^{\sum_{i=1}^N|\lambda^{(i)}|}\cdot \sum_{\sum_{i=1}^N(|\lambda^{(i)}|-|\tilde\lambda^{(i)}|)=0} (-1)^{\sum_{i=1}^N\sum_{n\ge1}\tilde p_n^{(i)}} m^{\sum_{i=1}^N\ell(\tilde\lambda^{(i)})}\\ &\qquad\cdot\mathrm{Tr}\, q^{\mathfrak{d}}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}-\tilde\lambda^{(i)}}}{(\lambda^{(i)}-\tilde\lambda^{(i)})^!}\\ &\cdot\sum_{\substack{\sum_{s\ge1}m_n^{(i,s)}=p_n^{(i)}\\ 1\le i\le N,n\ge1}}\prod_{\substack{1\le i\le N\\ s,n\ge1}}\frac{q^{snm_n^{(i,s)}}}{m_n^{(i,s)}!}\cdot\sum_{\substack{\sum_{t\ge1}\tilde m_n^{(i,t)}=\tilde p_n^{(i)}\\ 1\le i\le N,n\ge1}}\prod_{\substack{1\le i\le N\\ t,n\ge1}}\frac{q^{(t-1)n\tilde m_n^{(i,t)}}}{\tilde m_n^{(i,t)}!}.\end{aligned}$$

Since $\displaystyle\sum_{\sum_{s\ge1}i_{s,n}=i_n,\,n\ge1}\prod_{s,n\ge1}\frac{(q^{(s-1)n})^{i_{s,n}}}{i_{s,n}!}=\prod_{n\ge1}\left(\frac{1}{i_n!}\frac{1}{(1-q^n)^{i_n}}\right)$, Tr_λ is equal to

$$\begin{aligned}&(q;q)_\infty^{m^2}\cdot z^{\sum_{i=1}^N|\lambda^{(i)}|}\cdot\sum_{\sum_{i=1}^N(|\lambda^{(i)}|-|\tilde\lambda^{(i)}|)=0} m^{\sum_{i=1}^N\ell(\tilde\lambda^{(i)})}\\ &\qquad\cdot\mathrm{Tr}\, q^{\mathfrak{d}}\prod_{i=1}^N\frac{\mathfrak{a}_{\lambda^{(i)}-\tilde\lambda^{(i)}}}{(\lambda^{(i)}-\tilde\lambda^{(i)})^!}\\ &\cdot\prod_{1\le i\le N,n\ge1}\left(\frac{1}{p_n^{(i)}!}\frac{q^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}}\cdot\frac{(-1)^{\tilde p_n^{(i)}}}{\tilde p_n^{(i)}!}\frac{1}{(1-q^n)^{\tilde p_n^{(i)}}}\right).\end{aligned} \tag{4.11}$$

By Lemma 4.4, Tr_λ is a linear combination of expressions of the form:

$$(q;q)_\infty^{m^2-1}\cdot z^{\sum_{i=1}^N|\lambda^{(i)}|}\cdot\sum_{\sum_{i=1}^N(|\lambda^{(i)}|-|\tilde\lambda^{(i)}|)=0} m^{\sum_{i=1}^N\ell(\tilde\lambda^{(i)})} \tag{4.12}$$

$$\cdot \prod_{i=1}^{\ell} \frac{n_i^{\sum_{j=1}^N (m_{n_i}^{(j)} - p_{n_i}^{(j)})} q^{n_i w_i}}{(1-q^{n_i})^{w_i}}$$

$$\cdot \prod_{1 \le i \le N, n \ge 1} \left(\frac{1}{p_n^{(i)}!} \frac{q^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}} \cdot \frac{(-1)^{\tilde{p}_n^{(i)}}}{\tilde{p}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{p}_n^{(i)}}} \right)$$

where $0 < n_1 < \cdots < n_\ell$ are the positive parts in $\sum_{i=1}^N (\lambda^{(i)} - \tilde{\lambda}^{(i)})$, the multiplicities of parts n and $-n$ in $\sum_{i=1}^N (\lambda^{(i)} - \tilde{\lambda}^{(i)})$ are equal for every $n \neq 0$, and $0 \le w_i \le \sum_{j=1}^N (m_{n_i}^{(j)} - p_{n_i}^{(j)})$. Moreover, the coefficients of this linear combination are independent of q and the integers $n_1, \dots, n_\ell$.

As a polynomial of m, the degree of $(q;q)_\infty^{-m^2+1} \cdot \mathrm{Tr}_\lambda$ is the largest possible

$$\sum_{i=1}^N \ell(\tilde{\lambda}^{(i)}) \le \sum_{i=1}^N \ell(\lambda^{(i)}).$$

So the degree of $(q;q)_\infty^{-m^2+1} \cdot \mathrm{Tr}_\lambda$ is equal to $\sum_{i=1}^N \ell(\lambda^{(i)})$, and its leading term corresponds to the unique term in (4.11) with $\tilde{\lambda}^{(i)} = \lambda^{(i)}$ for every $1 \le i \le N$:

$$(q;q)_\infty^{m^2-1} \cdot z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot m^{\sum_{i=1}^N \ell(\lambda^{(i)})}$$

$$\cdot \prod_{1 \le i \le N, n \ge 1} \left(\frac{1}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1-q^n)^{m_n^{(i)}}} \cdot \frac{(-1)^{\tilde{m}_n^{(i)}}}{\tilde{m}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{m}_n^{(i)}}} \right).$$

Combining this with (4.12), we conclude that Tr_λ is of the form (4.6). □

By Lemma 3.5 (ii), $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle' = 0$ if $\sum_{i=1}^N k_i$ is odd. So we are only interested in even $\sum_{i=1}^N k_i$ which is equivalent to the condition that up to a re-ordering of $k_1, \dots, k_N$, there exists an even integer M such that $0 \le M \le N$, $k_1, \dots, k_M$ are odd positive integers, and $k_{M+1}, \dots, k_N$ are even non-negative integers. Our next lemma deals with the leading coefficent in this situation.

Lemma 4.7. *Let $0 \le M \le N$, M be even, and $k_1, \dots, k_M$ be odd positive integers. For $M < i \le N$, fix $\lambda^{(i)} = (\cdots (-n)^{\tilde{m}_n^{(i)}} \cdots (-1)^{\tilde{m}_1^{(i)}} 1^{m_1^{(i)}} \cdots n^{m_n^{(i)}} \cdots)$. Define $O_{k_1,\dots,k_M}$ to be*

$$(4.13) \quad (q;q)_\infty^{-m^2+1} \cdot \sum_{\substack{\ell(\lambda^{(i)})=k_i+2, |\lambda^{(i)}|=0 \\ 1 \le i \le M}} \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}.$$

Then, $O_{k_1,\ldots,k_M}$ is equal to

$$(q;q)_\infty \cdot z^{\sum_{i=M+1}^N |\lambda^{(i)}|} \cdot m^{\sum_{i=1}^M (k_i+1)+\sum_{i=M+1}^N \ell(\lambda^{(i)})}$$
$$\cdot \prod_{M+1\le i\le N, n\ge 1} \left(\frac{1}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1-q^n)^{m_n^{(i)}}} \cdot \frac{(-1)^{\tilde{m}_n^{(i)}}}{\tilde{m}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{m}_n^{(i)}}} \right)$$
$$\cdot \sum_{\substack{\sum_{i=1}^M n_i=0 \\ n_i\neq 0}} \mathrm{Tr} q^{\mathfrak{d}} \prod_{i=1}^M \mathfrak{a}_{n_i} \sum_{\substack{\ell(\tilde{\lambda}^{(i)})=k_i+1 \\ |\tilde{\lambda}^{(i)}|=-n_i \\ 1\le i\le M}} \prod_{\substack{1\le i\le M \\ n\ge 1}} \frac{q^{np_n^{(i)}}/p_n^{(i)}!}{(1-q^n)^{p_n^{(i)}}} \frac{(-1)^{\tilde{p}_n^{(i)}}/\tilde{p}_n^{(i)}!}{(1-q^n)^{\tilde{p}_n^{(i)}}} + \widetilde{W}_1$$

where $\tilde{\lambda}^{(i)} = (\cdots(-n)^{\tilde{p}_n^{(i)}} \cdots (-1)^{\tilde{p}_1^{(i)}} 1^{p_1^{(i)}} \cdots n^{p_n^{(i)}} \cdots)$ for $1\le i\le M$, and $\widetilde{W}_1$ is a polynomial of m with degree $< \sum_{i=1}^M (k_i+1) + \sum_{i=M+1}^N \ell(\lambda^{(i)})$.

Proof. We apply Proposition 4.6 and its proof to the term

$$\mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)}!}$$

in (4.13). By (4.11), the contributions to $O_{k_1,\ldots,k_M}$ of the tuples $\big(\tilde{\lambda}^{(1)},\ldots,\tilde{\lambda}^{(N)}\big)$ defined in (4.10) satisfying $\tilde{\lambda}^{(1)} = \lambda^{(1)}$ is a multiple of

$$\sum_{\ell(\lambda^{(1)})=k_1+2, |\lambda^{(1)}|=0} \prod_{n\ge 1} \left(\frac{1}{m_n^{(1)}!} \frac{q^{nm_n^{(1)}}}{(1-q^n)^{m_n^{(1)}}} \cdot \frac{(-1)^{\tilde{m}_n^{(1)}}}{\tilde{m}_n^{(1)}!} \frac{1}{(1-q^n)^{\tilde{m}_n^{(1)}}} \right) = \Theta_{k_1}(q)$$

where $\lambda^{(1)} = (\cdots(-n)^{\tilde{m}_n^{(1)}} \cdots (-1)^{\tilde{m}_1^{(1)}} 1^{m_1^{(1)}} \cdots n^{m_n^{(1)}} \cdots)$. By Lemma 2.3 (iii), the contributions to $O_{k_1,\ldots,k_M}$ of the tuples $\big(\tilde{\lambda}^{(1)},\ldots,\tilde{\lambda}^{(N)}\big)$ satisfying $\tilde{\lambda}^{(1)} = \lambda^{(1)}$ is equal to 0. By the symmetry of the odd positive integers $k_1,\ldots,k_M$, we see that $O_{k_1,\ldots,k_M}$ is the sum of the contributions of the tuples $\big(\tilde{\lambda}^{(1)},\ldots,\tilde{\lambda}^{(N)}\big)$ satisfying $\tilde{\lambda}^{(1)} < \lambda^{(1)},\ldots,\tilde{\lambda}^{(M)} < \lambda^{(M)}$. Since $\tilde{\lambda}^{(i)} \le \lambda^{(i)}$ for $M+1\le i\le N$, we conclude from (4.12) that as a polynomial of m, the degree of $O_{k_1,\ldots,k_M}$ is at most

$$\sum_{i=1}^N \ell(\tilde{\lambda}^{(i)}) \le \sum_{i=1}^M (k_i+1) + \sum_{i=M+1}^N \ell(\lambda^{(i)}).$$

Moreover, the leading term in $O_{k_1,\ldots,k_M}$ comes from the contributions of the tuples $\big(\tilde{\lambda}^{(1)},\ldots,\tilde{\lambda}^{(M)},\lambda^{(M+1)},\ldots,\lambda^{(N)}\big)$ satisfying

$$\ell(\tilde{\lambda}^{(1)}) = \ell(\lambda^{(1)}) - 1, \ldots, \ell(\tilde{\lambda}^{(M)}) = \ell(\lambda^{(M)}) - 1.$$

Put $\lambda^{(i)} - \tilde{\lambda}^{(i)} = (n_i)$ for $1 \le i \le M$. By (4.11), the leading term in $O_{k_1,\ldots,k_M}$ equals

$$(q;q)_\infty \cdot z^{\sum_{i=M+1}^N |\lambda^{(i)}|} \cdot m^{\sum_{i=1}^M (k_i+1) + \sum_{i=M+1}^N \ell(\lambda^{(i)})}$$
$$\cdot \prod_{M+1\le i\le N, n\ge 1} \left(\frac{1}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1-q^n)^{m_n^{(i)}}} \cdot \frac{(-1)^{\tilde{m}_n^{(i)}}}{\tilde{m}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{m}_n^{(i)}}} \right)$$
$$\cdot \sum_{\substack{\sum_{i=1}^M n_i = 0 \\ n_i \ne 0}} \operatorname{Tr} q^{\mathfrak{d}} \prod_{i=1}^M \mathfrak{a}_{n_i} \sum_{\substack{\ell(\tilde{\lambda}^{(i)}) = k_i+1 \\ |\tilde{\lambda}^{(i)}| = -n_i \\ 1\le i\le M}} \prod_{\substack{1\le i\le M \\ n\ge 1}} \frac{q^{np_n^{(i)}}/p_n^{(i)}!}{(1-q^n)^{p_n^{(i)}}} \frac{(-1)^{\tilde{p}_n^{(i)}}/\tilde{p}_n^{(i)}!}{(1-q^n)^{\tilde{p}_n^{(i)}}}$$

where $\tilde{\lambda}^{(i)} = \left(\cdots (-n)^{\tilde{p}_n^{(i)}} \cdots (-1)^{\tilde{p}_1^{(i)}} 1^{p_1^{(i)}} \cdots n^{p_n^{(i)}} \cdots \right)$ for $1 \le i \le M$. Finally, let $\widetilde{W}_1$ consist of all the terms in $O_{k_1,\ldots,k_M}$ with degrees $< \sum_{i=1}^M (k_i+1) + \sum_{i=M+1}^N \ell(\lambda^{(i)})$. □

The following is the main result in this section. It presents the leading term in the reduced series $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ in terms of the function $\Theta_k(q,z)$ from Definition 2.2.

Proposition 4.8. *Let $0 \le M \le N$. Let $k_1, \ldots, k_M$ be odd positive integers, and $k_{M+1}, \ldots, k_N$ be even non-negative integers.*

(i) *If M is an odd integer, then $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle' = 0$.*
(ii) *If M is an even integer, then $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle'$ is equal to*

$$m^{\sum_{i=1}^N (k_i+2) - M} \cdot \sum_{\substack{n_1,\ldots,n_{M/2} \ne 0 \\ \mathbf{P}_M}} \mathrm{Coeff}_{z_1^0 \cdots z_N^0} \left(\prod_{i=1}^M \Theta_{k_i-1}(q, z_i) \right.$$
$$\left. \cdot \prod_{i=M+1}^N \Theta_{k_i}(q, z_i) \cdot \prod_{s=1}^{M/2} \frac{(-n_s) q^{-n_s}}{1-q^{-n_s}} (z_{i_s} z_{j_s}^{-1})^{n_s} \right) + W$$

where $\mathbf{P}_M$ runs over all the partitions of the set $\{1, \ldots, M\}$ into pairs $\{i_1, j_1\}, \ldots, \{i_{M/2}, j_{M/2}\}$ such that $i_1 < j_1, \ldots, i_{M/2} < j_{M/2}$ and $i_1 < \ldots < i_{M/2}$, and W is a polynomial of m with degree $< \sum_{i=1}^N (k_i+2) - M$.

Proof. Part (i) is Lemma 3.5 (ii). In the following, we prove (ii). By (3.16),

$$\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N} \rangle = \operatorname{Tr} q^{\mathfrak{d}} \, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \mathfrak{G}_{k_i}.$$

Furthermore, in view of Corollary 3.4, we have

$$\mathfrak{G}_k = \sum_{\ell(\lambda)=k+2,|\lambda|=0} \frac{\mathfrak{a}_\lambda}{\lambda^!} + \sum_{\ell(\lambda)\le k,|\lambda|=0} g_\lambda \cdot \frac{\mathfrak{a}_\lambda}{\lambda^!}. \tag{4.14}$$

Therefore, $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is a $\mathbb{Q}$-linear combination of

$$(q;q)_\infty^{-m^2+1} \cdot \sum_{\substack{\ell(\lambda^{(i)})=k_i+2,|\lambda^{(i)}|=0\\ i\in I}} \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)!}} \tag{4.15}$$

where $I \subset \{1,\ldots,M\}$, $\ell(\lambda^{(i)}) \le k_i$ if $i \in \{1,\ldots,M\} - I$, and $\ell(\lambda^{(i)}) \le k_i + 2$ if $M < i \le N$. By Lemma 4.7, (4.15) is a polynomial in m of degree at most

$$\begin{aligned} &\sum_{i\in I}(k_i+1) + \sum_{i\in\{1,\ldots,M\}-I} \ell(\lambda^{(i)}) + \sum_{i=M+1}^N \ell(\lambda^{(i)}) \\ \le\ &\sum_{i\in I}(k_i+1) + \sum_{i\in\{1,\ldots,M\}-I} k_i + \sum_{i=M+1}^N (k_i+2). \end{aligned}$$

Hence $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is a polynomial in m of degree at most

$$d := \sum_{i=1}^M (k_i+1) + \sum_{i=M+1}^N (k_i+2) = \sum_{i=1}^N (k_i+2) - M.$$

In addition, the degree-d term in $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is from (4.15) with $I = \{1,\ldots,M\}$ and $\ell(\lambda^{(i)}) = k_i + 2$ for $M < i \le N$. Combining with (4.14), we see that the degree-d term of $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is the same as the degree-d term of

$$(q;q)_\infty^{-m^2+1} \cdot \sum_{\substack{\ell(\lambda^{(i)})=k_i+2,|\lambda^{(i)}|=0\\ 1\le i\le N}} \mathrm{Tr}\, q^{\mathfrak{d}}\, \Gamma_-(z)^m \Gamma_+(z)^{-m} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}}{\lambda^{(i)!}}. \tag{4.16}$$

By Lemma 4.7 again, the degree-d term of $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is equal to

$$(q;q)_\infty \cdot m^d \sum_{\substack{\ell(\lambda^{(i)})=k_i+2\\ |\lambda^{(i)}|=0\\ M<i\le N}} \prod_{\substack{M<i\le N\\ n\ge 1}} \frac{q^{nm_n^{(i)}}/m_n^{(i)}!}{(1-q^n)^{m_n^{(i)}}} \frac{(-1)^{\tilde{m}_n^{(i)}}/\tilde{m}_n^{(i)}!}{(1-q^n)^{\tilde{m}_n^{(i)}}} \tag{4.17}$$

$$(4.18) \qquad \cdot \sum_{\substack{\sum_{i=1}^M n_i=0 \\ n_i\neq 0}} \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^M \mathfrak{a}_{n_i} \sum_{\substack{\ell(\tilde\lambda^{(i)})=k_i+1 \\ |\tilde\lambda^{(i)}|=-n_i \\ 1\le i\le M}} \prod_{\substack{1\le i\le M \\ n\ge 1}} \frac{q^{np_n^{(i)}}/p_n^{(i)}!}{(1-q^n)^{p_n^{(i)}}} \frac{(-1)^{\tilde p_n^{(i)}}/\tilde p_n^{(i)}!}{(1-q^n)^{\tilde p_n^{(i)}}}$$

where $\lambda^{(i)} = \big(\cdots(-n)^{\tilde m_n^{(i)}}\cdots(-1)^{\tilde m_1^{(i)}} 1^{m_1^{(i)}}\cdots n^{m_n^{(i)}}\cdots\big)$ for $M < i \le N$, and $\tilde\lambda^{(i)} = \big(\cdots(-n)^{\tilde p_n^{(i)}}\cdots(-1)^{\tilde p_1^{(i)}} 1^{p_1^{(i)}}\cdots n^{p_n^{(i)}}\cdots\big)$ for $1 \le i \le M$. Line (4.17) is equal to

$$(q;q)_\infty \cdot m^d \cdot \prod_{i=M+1}^N \Theta_{k_i}(q) = (q;q)_\infty \cdot m^d \cdot \mathrm{Coeff}_{z_{M+1}^0\cdots z_N^0}\left(\prod_{i=M+1}^N \Theta_{k_i}(q,z_i)\right)$$

by Lemma 2.3 (ii). Similarly, we conclude that line (4.18) is equal to

$$\sum_{\substack{\sum_{i=1}^M n_i=0 \\ n_i\neq 0}} \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^M \mathfrak{a}_{n_i} \cdot \mathrm{Coeff}_{z_1^0\cdots z_M^0}\left(\prod_{i=1}^M z_i^{n_i}\Theta_{k_i-1}(q,z_i)\right).$$

It follows that the degree-d term of $\langle \mathrm{ch}_{k_1}\cdots \mathrm{ch}_{k_N}\rangle'$ is equal to

$$(q;q)_\infty \cdot m^d \cdot \mathrm{Coeff}_{z_1^0\cdots z_N^0}\left(\prod_{i=1}^M \Theta_{k_i-1}(q,z_i) \cdot \prod_{i=M+1}^N \Theta_{k_i}(q,z_i) \cdot \sum_{\sum_{i=1}^M n_i=0, n_i\neq 0} \mathrm{Tr}\, q^{\mathfrak{d}} \prod_{i=1}^M z_i^{n_i}\mathfrak{a}_{n_i}\right).$$

By (4.8), the degree-d term of $\langle \mathrm{ch}_{k_1}\cdots \mathrm{ch}_{k_N}\rangle'$ is equal to

$$m^d \cdot \sum_{\substack{n_1,\ldots,n_{M/2}\neq 0 \\ \mathbf{P}_M}} \mathrm{Coeff}_{z_1^0\cdots z_N^0}\left(\prod_{i=1}^M \Theta_{k_i-1}(q,z_i) \cdot \prod_{i=M+1}^N \Theta_{k_i}(q,z_i) \cdot \prod_{s=1}^{M/2} \frac{(-n_s)q^{-n_s}}{1-q^{-n_s}}(z_{i_s}z_{j_s}^{-1})^{n_s}\right)$$

where $\mathbf{P}_M$ runs over all the partitions of $\{1,\ldots,M\}$ into pairs

$$\{i_1, j_1\}, \ldots, \{i_{M/2}, j_{M/2}\}$$

such that $i_1 < j_1, \ldots, i_{M/2} < j_{M/2}$ and $i_1 < \ldots < i_{M/2}$. Let W consist of terms in $\langle \mathrm{ch}_{k_1}\cdots \mathrm{ch}_{k_N}\rangle'$ with degrees $< d$. This proves (ii). $\square$

Corollary 4.9. (i) *If $k \ge 0$ and $2|k$, then $\Theta_k(q)$ is a quasi-modular form.*
(ii) *Let $k_1, \ldots, k_M$ be odd positive integers with M being even. Then,*

$$\sum_{\substack{n_1,\ldots,n_{M/2}\neq 0\\ \mathbf{P}_M}} \mathrm{Coeff}_{z_1^0\cdots z_M^0}\left(\prod_{i=1}^{M}\Theta_{k_i-1}(q,z_i)\cdot\prod_{s=1}^{M/2}\frac{(-n_s)q^{-n_s}}{1-q^{-n_s}}(z_{i_s}z_{j_s}^{-1})^{n_s}\right)$$

is a quasi-modular form, where $\mathbf{P}_M$ runs over all the partitions of the set $\{1, \ldots, M\}$ into pairs $\{i_1, j_1\}, \ldots, \{i_{M/2}, j_{M/2}\}$ such that $i_1 < j_1, \ldots, i_{M/2} < j_{M/2}$ and $i_1 < \ldots < i_{M/2}$.

Proof. By Lemma 3.5 (i), as a function of q, $\langle \mathrm{ch}_{k_1} \cdots \mathrm{ch}_{k_N}\rangle'$ is a quasi-modular form. So the coefficient

$$\sum_{\substack{n_1,\ldots,n_{M/2}\neq 0\\ \mathbf{P}_M}} \mathrm{Coeff}_{z_1^0\cdots z_N^0}\left(\prod_{i=1}^{M}\Theta_{k_i-1}(q,z_i)\cdot\prod_{i=M+1}^{N}\Theta_{k_i}(q,z_i)\right.$$
$$\left.\cdot\prod_{s=1}^{M/2}\frac{(-n_s)q^{-n_s}}{1-q^{-n_s}}(z_{i_s}z_{j_s}^{-1})^{n_s}\right)$$

of $m^{\sum_{i=1}^N (k_i+2)-M}$ in Proposition 4.8 (ii) is a quasi-modular form.

(i) Let $M = 0, N = 1, k_1 = k$ and $z_1 = z$. Combining with Lemma 2.3 (ii), we see that $\Theta_k(q) = \mathrm{Coeff}_{z^0}\Theta_k(q, z)$ is a quasi-modular form.

(ii) Follows immediately by letting $N = M$. □

Example 4.10. Let $Z(2), Z(4)$ and $Z(6)$ be from (2.3). Recall from (2.4) that the ring **QM** of quasi-modular forms is generated by $Z(2), Z(4), Z(6)$ over $\mathbb{Q}$. By comparing the coefficients of q^i for $0 \le i \le 6$ and the weights, we conclude that

$$\Theta_4(q) = \frac{1}{6!}\left(-Z(2)^2 + Z(4) - 40Z(2)^3 + 24Z(2)Z(4) - 2Z(6)\right) \in \mathbf{QM}.$$

5. Hilbert schemes of points and quasi-modularity

In this section, X denotes a smooth projective complex surface. As applications of the results in previous sections, we will investigate the relation between the Hilbert schemes $X^{[n]}$ and quasi-modular forms. We will do this by studying the leading term in the reduced series $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$.

We begin with the definition of the cohomology class $G_k(\alpha, n) \in H^*(X^{[n]})$. Let $\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n})$ be the Chern character of the structure sheaf $\mathcal{O}_{\mathcal{Z}_n}$ where $\mathcal{Z}_n$ is the universal codimension-2 closed subscheme of $X^{[n]} \times X$.

Definition 5.1. For $n > 0$ and a homogeneous class $\alpha \in H^*(X)$, let $|\alpha| = s$ if $\alpha \in H^s(X)$, and let $G_k(\alpha, n)$ be the homogeneous component in $H^{|\alpha|+2k}(X^{[n]})$ of

$$G(\alpha, n) = p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^*\alpha \cdot p_2^*\mathrm{td}(X)) \in H^*(X^{[n]}) \tag{5.1}$$

where $\mathrm{td}(X)$ denotes the Todd class of X. We extend the notion $G_k(\alpha, n)$ linearly to an arbitrary class $\alpha \in H^*(X)$, and set $G_k(\alpha, 0) = 0$.

It was proved in [10] that the cohomology ring of $X^{[n]}$ is generated by the classes $G_k(\alpha, n)$ where $0 \le k < n$ and α runs over a fixed linear basis of $H^*(X)$.

For $\alpha_1, \ldots, \alpha_N \in H^*(X)$ and integers $k_1, \ldots, k_N \ge 0$, define the series

$$F^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = \sum_n q^n \int_{X^{[n]}} \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_{X^{[n]}}). \tag{5.2}$$

In particular, $F(q) = (q;q)_\infty^{-\chi(X)}$. Following [14], we define the reduced series

$$\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = \frac{F^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)}{F(q)} = (q;q)_\infty^{\chi(X)} \cdot F^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q). \tag{5.3}$$

Problem 5.2. *When is $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ a quasi-modular form?*

We will study this problem for the leading term of $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$, which has been determined in [16]. To state the leading term of $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$, we recall the notations $\Theta^\alpha_k(q)$ and $\Theta^\alpha_k(q,z)$ from [16].

Definition 5.3. For a non-negative integer k and a class $\alpha \in H^*(X)$, we define $\Theta^\alpha_k(q)$ to be

$$-\sum_{\substack{\ell(\lambda)=k+2\\|\lambda|=0}} \langle (1_X - K_X)^{\sum_{n\ge1} i_n}, \alpha\rangle \cdot \prod_{n\ge1} \left(\frac{(-1)^{i_n}}{i_n!} \frac{q^{ni_n}}{(1-q^n)^{i_n}} \frac{1}{\tilde{i}_n!} \frac{1}{(1-q^n)^{\tilde{i}_n}} \right)$$

where $\lambda = (\cdots(-n)^{\tilde{i}_n} \cdots (-1)^{\tilde{i}_1} 1^{i_1} \cdots n^{i_n} \cdots)$. Define $\Theta^\alpha_k(q,z)$ to be

$$-\sum_{\substack{a,b\ge0\\ s_1,\ldots,s_a,t_1,\ldots,t_b\ge1\\ \sum_{i=1}^a s_i + \sum_{j=1}^b t_j = k+2}} \langle (1_X - K_X)^{\sum_{i=1}^a s_i}, \alpha\rangle \prod_{i=1}^a \frac{(-1)^{s_i}}{s_i!} \cdot \prod_{j=1}^b \frac{1}{t_j!}$$

$$\cdot \sum_{n_1 > \cdots > n_a} \prod_{i=1}^{a} \frac{(qz)^{n_i s_i}}{(1-q^{n_i})^{s_i}} \cdot \sum_{m_1 > \cdots > m_b} \prod_{j=1}^{b} \frac{z^{-m_j t_j}}{(1-q^{m_j})^{t_j}}.$$

We see from Definition 2.2 that if $\alpha \in H^4(X)$, then

$$\Theta_k^\alpha(q) = -\langle 1_X, \alpha \rangle \cdot \Theta_k(q), \tag{5.4}$$
$$\Theta_k^\alpha(q,z) = -\langle 1_X, \alpha \rangle \cdot \Theta_k(q,z). \tag{5.5}$$

Also, note that the weight of $\Theta_k^\alpha(q)$ is equal to $(k+2)$.

Proposition 5.4. *Let X be a smooth projective complex surface, and k be a non-negative integer. Fix $\alpha \in H^*(X;\mathbb{Q})$. Then,*

$$\lim_{q\to 1} \left((1-q)^{k+2} \Theta_k^\alpha(q) \right) \in \mathbf{MZV}. \tag{5.6}$$

Proof. By the Lemma 4.7 in [16], we have

$$\Theta_k^\alpha(q) = \mathrm{Coeff}_{z^0} \Theta_k^\alpha(q,z). \tag{5.7}$$

In view of (2.6), $\Theta_k^\alpha(q,z)$ is equal to

$$- \sum_{\substack{s,t\ge 0\\ s+t=k+2}} \langle (1_X - K_X)^s, \alpha \rangle \cdot \frac{(-1)^s}{s!} \left(\sum_{m>0} \frac{(qz)^m}{1-q^m} \right)^s \cdot \frac{1}{t!} \left(\sum_{m>0} \frac{z^{-m}}{1-q^m} \right)^t. \tag{5.8}$$

Combining with (5.7) and Definition 2.4, we see that

$$\Theta_k^\alpha(q) = - \sum_{\substack{s,t>0\\ s+t=k+2}} \langle (1_X - K_X)^s, \alpha \rangle \cdot (-1)^s \cdot \mathrm{Coeff}_{z^0} U_{s,t}(q,z).$$

By Lemma 2.5, we obtain (5.6). □

In the next lemma, we deal with the leading term of the reduced series $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$. Note that the weight of $\prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q)$ is equal to $\sum_{i=1}^N (k_i+2)$.

Lemma 5.5. *Let $\alpha_1, \ldots, \alpha_N \in H^*(X)$. Then,*

$$\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = \prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q) + W$$

where W consists of terms with weights $< \sum_{i=1}^{N}(k_i+2)$. Moreover, $W = 0$ when $\alpha_1, \ldots, \alpha_N \in H^4(X)$.

Proof. The first statement is equivalent to [16, Theorem 4.8]. When we have $\alpha_1, \ldots, \alpha_N \in H^4(X)$, the proof of [16, Theorem 4.8] shows that by degree reasons, the lower weight term W is equal to 0. $\square$

A necessary condition for the leading term $\prod_{i=1}^{N}\Theta_{k_i}^{\alpha_i}(q)$ to be a quasi-modular form is that the weight $\sum_{i=1}^{N}(k_i+2)$ must be even, i.e., the number of odd integers among $k_1, \ldots, k_N$ is even. The next example indicates that we may have to further assume that every k_i is even.

Example 5.6. Let $\alpha \in H^2(X;\mathbb{Q})$ satisfy $\langle K_X, \alpha\rangle \neq 0$. By (5.7) and (5.8),

$$\Theta_1^{\alpha}(q) = \frac{\langle K_X, \alpha\rangle}{2} \cdot \sum_{m_1, m_2 > 0} \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} \frac{1}{1-q^{m_1+m_2}}.$$

A tedious computation shows that $\Theta_1^{\alpha}(q) \cdot \Theta_1^{\alpha}(q)$ is not a quasi-modular form.

Lemma 5.7. *Let $\alpha \in H^*(X;\mathbb{Q})$ satisfy $\langle K_X^2, \alpha\rangle = 0$, and $k \geq 0$ be an even integer. Then, $\Theta_k^{\alpha}(q)$ is either 0 or a quasi-modular form of weight $(k+2)$.*

Proof. We may assume that α is a homogeneous cohomology class. If the degree $|\alpha|$ is odd, then $\Theta_k^{\alpha}(q) = 0$ by definition. If $|\alpha| = 0$, then $K_X^2 = 0$ since $\langle K_X^2, \alpha\rangle = 0$; again, it follows from the definition that $\Theta_k^{\alpha}(q) = 0$. If $|\alpha| = 4$, then $\Theta_k^{\alpha}(q)$ is a quasi-modular form by (5.4) and Corollary 4.9 (i).

In the rest of the proof, we assume that $|\alpha| = 2$. By (5.7) and (5.8),

$$\Theta_k^{\alpha}(q) = \langle K_X, \alpha\rangle \cdot \sum_{s=1}^{k+1} \frac{(-1)^s}{(s-1)! \cdot (k+2-s)!} \cdot A_s(q)$$

where

$$A_s(q) = \sum_{\substack{m_1, \ldots, m_s, n_1, \ldots, n_{k+2-s} > 0 \\ m_1 + \ldots + m_s = n_1 + \ldots + n_{k+2-s}}} \prod_{i=1}^{s} \frac{q^{m_i}}{1-q^{m_i}} \cdot \prod_{i=1}^{k+2-s} \frac{1}{1-q^{n_i}}.$$

Note that for $1 \leq s \leq k/2$, we have $A_s(q) = A_{k+2-s}(q)$ and

$$\frac{(-1)^s}{(s-1)! \cdot (k+2-s)!} + \frac{(-1)^{k+2-s}}{(k+1-s)! \cdot s!} = \frac{(k+2) \cdot (-1)^s}{s! \cdot (k+2-s)!}.$$

Thus, $\Theta_k^\alpha(q)$ is equal to

$$\langle K_X, \alpha\rangle \cdot \left(\sum_{s=1}^{k/2} \frac{(k+2)\cdot(-1)^s}{s!\cdot(k+2-s)!} A_s(q) + \frac{(k/2+1)\cdot(-1)^{k/2+1}}{\big((k/2+1)!\big)^2} A_{k/2+1}(q) \right). \tag{5.9}$$

Similarly, by Lemma 2.3 (i) and (ii), we obtain

$$\begin{aligned} \Theta_k(q) &= \frac{1}{(k+2)!} \cdot \mathrm{Coeff}_{z^0} \left(\sum_{m>0} \frac{(qz)^m}{1-q^m} - \sum_{m>0} \frac{z^{-m}}{1-q^m} \right)^{k+2} \\ &= \sum_{s=1}^{k/2} \frac{2\cdot(-1)^s}{s!\cdot(k+2-s)!} \cdot A_s(q) + \frac{(-1)^{k/2+1}}{\big((k/2+1)!\big)^2} \cdot A_{k/2+1}(q). \end{aligned}$$

Combining this with (5.9), we see that

$$\Theta_k^\alpha(q) = \langle K_X, \alpha\rangle \cdot \frac{k+2}{2} \cdot \Theta_k(q).$$

Therefore, $\Theta_k^\alpha(q)$ is a quasi-modular form when $|\alpha| = 2$. □

The following example illustrates that the assumption $\langle K_X^2, \alpha\rangle = 0$ in Lemma 5.7 is necessary.

Example 5.8. Let X be a smooth projective complex surface satisfying $\langle K_X, K_X\rangle = \langle K_X^2, 1_X\rangle \neq 0$. By (5.7) and (5.8), $\Theta_2^{1_X}(q)$ is equal to

$$\begin{aligned} \frac{\langle K_X, K_X\rangle}{4} \cdot \Bigg(2 \cdot \sum_{m_1,m_2,m_3>0} \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} \frac{q^{m_3}}{1-q^{m_3}} \frac{1}{1-q^{m_1+m_2+m_3}} \\ - \sum_{m_1,m_2,n_1,n_2>0, m_1+m_2=n_1+n_2} \frac{q^{m_1}}{1-q^{m_1}} \frac{q^{m_2}}{1-q^{m_2}} \frac{1}{1-q^{n_1}} \frac{1}{1-q^{n_2}} \Bigg). \end{aligned}$$

A straightforward computation shows that $\Theta_2^{1_X}(q)$ is not a quasi-modular form.

Theorem 5.9. *Let $k_1, \ldots, k_N \geq 0$ be integers. Let X be a smooth projective complex surface, and $\alpha_1, \ldots, \alpha_N \in H^*(X;\mathbb{Q})$.*

(i) *If* $\langle K_X^2, \alpha_i \rangle = 0$ *and* $2|k_i$ *for every* i*, then the leading term* $\prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q)$ *of* $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ *is either* 0 *or a quasi-modular form of weight* $\sum_{i=1}^N (k_i + 2)$*.*

(ii) *Let* $|\alpha_i| = 4$ *for every* i*. If* $2 \nmid k_i$ *for some* i*, then* $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = 0$*. If* $2|k_i$ *for every* i*, then* $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ *is a quasi-modular form of weight* $\sum_{i=1}^N (k_i + 2)$*.*

Proof. The first statement follows from Lemma 5.5 and Lemma 5.7. For (ii), we see from Lemma 5.5 and (5.4) that

$$\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = \prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q) = (-1)^N \cdot \prod_{i=1}^N \langle 1_X, \alpha_i \rangle \cdot \prod_{i=1}^N \Theta_{k_i}(q). \tag{5.10}$$

So by Lemma 2.3 (iii), if k_i is odd for some i, then $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q) = 0$. By Corollary 4.9 (i), if k_i is even for every $1 \le i \le N$, then $\overline{F}^{\alpha_1,\ldots,\alpha_N}_{k_1,\ldots,k_N}(q)$ is a quasi-modular form of weight $\sum_{i=1}^N (k_i + 2)$. □

Note from (5.10) that if x denotes the cohomology class corresponding to a point in X, then the reduced series $\overline{F}^{x,\ldots,x}_{k_1,\ldots,k_N}(q)$ is independent of the surface X.

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