

# Categorical representation of superschemes

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**Abstract:** In the present paper, we prove that a locally noetherian superscheme  $X^\otimes$  may be reconstructed (up to certain equivalence) category-theoretically from the category of noetherian superschemes over  $X^\otimes$ . This result is a supergeometric generalization of the result proved by Shinichi Mochizuki concerning categorical reconstruction of schemes.

**Keywords:** Superscheme, supersymmetry.

## Introduction

Superschemes (or, supermanifolds) were introduced and discussed in various works, especially in connection with the important physical applications, which stem from superstring theory. Besides having such physical applications, the theory of superschemes will be interesting on its own from the purely mathematical viewpoint. In the present paper, we are interested in understanding the richness of algebraic supergeometry from category-theoretic aspects.

The main result of our study provides a supergeometric generalization of the result proved by S. Mochizuki (cf. [5], Theorem A) concerning categorical reconstructibility of locally noetherian schemes, as described below. (Note that Mochizuki also proved a generalization of his own result to arithmetic log schemes, see [6].) Let  $X^\otimes = (X_b, \mathcal{O}_{X^\otimes})$  be a superscheme (cf. Definition 1.1 (i)), i.e., a scheme  $X_b$  together with a certain quasi-coherent sheaf of superalgebras  $\mathcal{O}_{X^\otimes}$  on  $X_b$ . Suppose that  $X^\otimes$  is locally noetherian in the sense of Definition 1.2. For each such  $X^\otimes$ , one obtains the category

$$(1) \quad \mathfrak{Sch}_{/X^\otimes}^\otimes$$

consisting of noetherian superschemes over  $X^\otimes$  (cf. (3) for the precise definition of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ ). The problem that we consider in the present paper is to

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know to what extent one can reconstruct the superscheme-theoretic structure of  $X$  from the categorical structure of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . Our main result is the following assertion.

**Theorem A.** *Let  $X^\otimes$  and  $X'^\otimes$  be two locally noetherian superschemes. Then,*

$$(2) \quad X^\otimes \overset{f}{\sim} X'^\otimes \text{ if and only if } \mathfrak{Sch}_{/X^\otimes}^\otimes \cong \mathfrak{Sch}_{/X'^\otimes}^\otimes.$$

(Here,  $\overset{f}{\sim}$  denotes the equivalence relation defined in (24).)

Theorem A implies, unlike the result of [5], that isomorphism classes of locally noetherian superschemes may not be determined uniquely from the categorical structure of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . Indeed, suppose that  $X^\otimes \overset{f}{\sim} X'^\otimes$ , that is to say,  $X'^\otimes$  is isomorphic to a fermionic twist of  $X^\otimes$  (cf. Definition 1.7). By definition,  $X'^\otimes$  may be obtained by twisting the fermionic portion of  $X^\otimes$  by means of some element  $a$  in the first étale cohomology group  $H_{\text{ét}}^1(X_b, \mu_2)$ . By twisting various superschemes over  $X^\otimes$  by means of  $a$  in the same manner, we obtain the assignment from each object in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  to an object in  $\mathfrak{Sch}_{/X'^\otimes}^\otimes$ ; this assignment gives an equivalence of categories  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes$ , and hence, shows one direction of the equivalence in Theorem A (cf. Proposition 1.10 and the discussion in its proof).

On the other hand, the proof of the reverse direction (i.e.,  $\mathfrak{Sch}_{/X^\otimes}^\otimes \cong \mathfrak{Sch}_{/X'^\otimes}^\otimes$  implies  $X^\otimes \overset{f}{\sim} X'^\otimes$ ) is technically much more difficult. To complete the proof, we reconstruct step-by-step various partial information of (the equivalence class of)  $X^\otimes$  from the categorical structure of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ , as discussed in §2. If  $X^\otimes$  is a scheme in the classical sense, then any fermionic twists of  $X'^\otimes$  are in fact isomorphic to  $X^\otimes$  (in particular,  $X'^\otimes$  is a scheme); in this case, Theorem A is exactly the same as the result by S. Mochizuki.

In the last section of the present paper, we shall prove further rigidity properties concerning the category  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  (cf. Propositions 3.1 and 3.2).

Finally, we want to remark that, as a different type of reconstruction of a superscheme, one may find a result by U. V. Dubey and V. M. Mallik (cf. [3]); according to this result, Balmer’s construction of Spec of a tensor triangulated category can be used to reconstruct superschemes from its category of perfect complexes.

### 1. Superschemes

In this section, we recall first the definition of a superscheme (cf. Definition 1.1). Basic references for the notion of a superscheme are, e.g., [1], [2]. Then, we introduce the notion of a *fermionic twist* (cf. Definition 1.6), and the equivalence relation “ $\overset{f}{\sim}$ ” (cf. (24)) appearing in the statement of Theorem A. One direction of the equivalence in Theorem A (which is much easier to prove than the reverse direction) will be proved in § 1.4 (cf. Proposition 1.9).

Throughout the present paper, we denote, for any category  $\mathcal{C}$ , by  $\text{Ob}(\mathcal{C})$  the set of objects of  $\mathcal{C}$ . Also, if both  $A$  and  $B$  are objects of  $\mathcal{C}$  (i.e.,  $A, B \in \text{Ob}(\mathcal{C})$ ), then we shall denote by  $\text{Map}_{\mathcal{C}}(A, B)$  the set of morphisms (in  $\mathcal{C}$ ) from  $A$  to  $B$ . By a *superalgebra*, we mean a supercommutative  $\mathbb{Z}/2\mathbb{Z}$ -graded algebra  $R = R_b \oplus R_f$  (where  $R_b$  and  $R_f$  denote the even and odd parts respectively) such that it includes  $R_b$  as a central subalgebra, and  $x^2 = 0$  for all  $x \in R_f$ .

#### 1.1. Superschemes

**Definition 1.1.** (i) A **superscheme** is a pair  $X^{\otimes} := (X_b, \mathcal{O}_X)$  consisting of a scheme  $X_b$  and a quasi-coherent sheaf of superalgebras  $\mathcal{O}_{X^{\otimes}}$  over  $\mathcal{O}_{X_b}$  such that the natural morphism  $\mathcal{O}_{X_b} \rightarrow \mathcal{O}_{X^{\otimes}}$  is injective and its image coincides with the bosonic (i.e., even) part of  $\mathcal{O}_{X^{\otimes}}$ . We shall write  $\mathcal{O}_{X_f}$  for the fermionic (i.e., odd) part of  $\mathcal{O}_{X^{\otimes}}$  and identify  $\mathcal{O}_{X_b}$  with the bosonic part via the injection  $\mathcal{O}_{X_b} \rightarrow \mathcal{O}_{X^{\otimes}}$  (hence,  $\mathcal{O}_{X^{\otimes}} = \mathcal{O}_{X_b} \oplus \mathcal{O}_{X_f}$ ).

(ii) Let  $X^{\otimes} := (X_b, \mathcal{O}_{X^{\otimes}})$  and  $Y^{\otimes} := (Y_b, \mathcal{O}_{Y^{\otimes}})$  be two superschemes. A **morphism of superschemes** from  $Y^{\otimes}$  to  $X^{\otimes}$  is a pair  $f^{\otimes} := (f_b, f^b)$  consisting of a morphism  $f_b : Y_b \rightarrow X_b$  of schemes and a morphism of superalgebras  $f^b : f_b^*(\mathcal{O}_{X^{\otimes}}) \left( := \mathcal{O}_{Y_b} \otimes_{f_b^{-1}(\mathcal{O}_{X_b})} f_b^{-1}(\mathcal{O}_{X^{\otimes}}) \right) \rightarrow \mathcal{O}_{Y^{\otimes}}$  over  $\mathcal{O}_{Y_b}$ .

In the following, let us fix a superscheme  $X^{\otimes} := (X_b, \mathcal{O}_{X^{\otimes}})$ .

**Definition 1.2.** We shall say that  $X^{\otimes}$  is **locally noetherian** (resp., **noetherian**) if  $X_b$  is locally noetherian (resp., noetherian) and the  $\mathcal{O}_{X_b}$ -module  $\mathcal{O}_{X_f}$  is coherent.

We shall denote by

$$(3) \qquad \text{Sch}_{/X^{\otimes}}^{\otimes}$$

the category defined as follows:

- the *objects* are morphisms of superschemes  $Y^\otimes (= (Y_b, \mathcal{O}_{Y^\otimes})) \rightarrow X^\otimes$  to  $X^\otimes$  such that  $Y^\otimes$  is noetherian and the underlying morphism  $Y_b \rightarrow X_b$  of schemes is *of finite type*;
- the *morphisms* (from an object  $Y_1^\otimes \rightarrow X^\otimes$  to an object  $Y_2^\otimes \rightarrow X^\otimes$ ) are morphisms of superschemes  $Y_1^\otimes \rightarrow Y_2^\otimes$  lying over  $X^\otimes$ .

Note that the fiber products and finite coproducts exist in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  (cf. [2], Corollary 10.3.9).

**Remark 1.3.** Let  $X$  be a scheme (in the usual sense). Then,  $X$  carries a superschemes of the form  $X_{\text{triv}}^\otimes := (X, \mathcal{O}_{X_{\text{triv}}^\otimes} (= \mathcal{O}_X \oplus \mathcal{O}_{X_f}))$  with  $\mathcal{O}_{X_f} = 0$ . (Conversely, any superscheme with vanishing fermionic part arises uniquely from a scheme in this manner.) In the rest of the present paper, we shall not distinguish between  $X$  and  $X_{\text{triv}}^\otimes$ .

### 1.2. Superschemes arising from a bilinear map

Let  $X^\otimes := (X_b, \mathcal{O}_{X^\otimes})$  be a superscheme. The multiplication morphism  $\mathcal{O}_{X^\otimes} \otimes \mathcal{O}_{X^\otimes} \rightarrow \mathcal{O}_{X^\otimes}$  restricts to a skew-symmetric  $\mathcal{O}_{X_b}$ -bilinear map

$$(4) \quad m_{X^\otimes} : \mathcal{O}_{X_f}^{\otimes 2} \left( := \mathcal{O}_{X_f} \otimes_{\mathcal{O}_{X_b}} \mathcal{O}_{X_f} \right) \rightarrow \mathcal{O}_{X_b}.$$

The associative property of the multiplication gives rise to the equality

$$(5) \quad m_{X^\otimes} \otimes \text{id}_{\mathcal{O}_{X_f}} = \text{id}_{\mathcal{O}_{X_f}} \otimes m_{X^\otimes} : \mathcal{O}_{X_f}^{\otimes 3} \rightarrow \mathcal{O}_{X_f}.$$

One verifies that the superscheme  $X^\otimes$  is uniquely determined (up to isomorphism) by the triple

$$(6) \quad \mathcal{A}_{X^\otimes} := (X_b, \mathcal{O}_{X_f}, m_{X^\otimes}).$$

To make the discussion precise, let us define

$$(7) \quad \mathfrak{A}$$

to be the category, where

- the *objects* are triples  $(Y, \mathcal{F}, \omega)$  consisting of a locally noetherian scheme  $Y$ , a coherent  $\mathcal{O}_Y$ -module  $\mathcal{F}$ , and a skew-symmetric  $\mathcal{O}_Y$ -bilinear map  $\omega : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{O}_Y$  on  $\mathcal{F}$  satisfying the equality  $\omega \otimes \text{id}_{\mathcal{F}} = \text{id}_{\mathcal{F}} \otimes \omega : \mathcal{F}^{\otimes 3} \rightarrow \mathcal{F}$ ;

- the *morphisms* from  $(Y, \mathcal{F}, \omega)$  to  $(Y', \mathcal{F}', \omega')$  (where both  $(Y, \mathcal{F}, \omega)$  and  $(Y', \mathcal{F}', \omega')$  are objects of  $\mathfrak{A}$ ) are pairs  $(f, f^b)$  consisting of a morphism  $f : Y \rightarrow Y'$  of schemes and an  $\mathcal{O}_Y$ -linear morphism  $f^b : f^*(\mathcal{F}') \rightarrow \mathcal{F}$  satisfying the equality

$$(8) \quad \omega \circ (f^b \otimes f^b) = f^*(\omega') : f^*(\mathcal{F}') \otimes f^*(\mathcal{F}') (= f^*(\mathcal{F}' \otimes \mathcal{F}')) \rightarrow \mathcal{O}_{Y'}.$$

Then, the following proposition may be verified.

**Proposition 1.4.** *The assignment  $X^\otimes \mapsto \mathcal{A}_{X^\otimes}$  defined above is functorial, and the resulting functor from the category of locally noetherian superschemes to  $\mathfrak{A}$  is an equivalence of categories.*

*Proof.* Let us take an object  $(Y, \mathcal{F}, \omega)$  of  $\mathfrak{A}$ . Then, the direct sum  $\mathcal{O}_Y \oplus \mathcal{F}$  admits a structure of  $\mathcal{O}_Y$ -superalgebra (where the first and second factors are the bosonic and fermionic parts respectively) with multiplication given by

$$(9) \quad (\mathcal{O}_Y \oplus \mathcal{F}) \otimes (\mathcal{O}_Y \oplus \mathcal{F}) \rightarrow \mathcal{O}_Y \oplus \mathcal{F} \\ (a, \epsilon_a) \otimes (b, \epsilon_b) \mapsto (ab + \omega(\epsilon_a, \epsilon_b), a\epsilon_b + b\epsilon_a).$$

The pair  $Y_{\mathcal{F}, \omega}^\otimes := (Y, \mathcal{O}_Y \oplus \mathcal{F})$  forms a superscheme and the resulting assignment  $(Y, \mathcal{F}, \omega) \mapsto Y_{\mathcal{F}, \omega}^\otimes$  is functorial in  $\mathfrak{A}$ . This assignment defines a functor forming the inverse to the functor  $X^\otimes \mapsto \mathcal{A}_{X^\otimes}$ . This completes the proof of Proposition 1.4. □

### 1.3. From superschemes to schemes

In the following, we shall fix a superscheme  $X^\otimes := (X_b, \mathcal{O}_{X^\otimes} (= \mathcal{O}_{X_b} \oplus \mathcal{O}_{X_f}))$ . By considering the morphism

$$(10) \quad \beta_{X^\otimes} : X^\otimes \rightarrow X_b$$

corresponding to the inclusion  $\mathcal{O}_{X_b} \rightarrow \mathcal{O}_{X^\otimes}$ , we may regard  $X^\otimes$  as a superscheme over the scheme  $X_b$ . The construction of  $\beta_{X^\otimes}$  is evidently functorial in  $X^\otimes$ , that is to say,  $\beta_{X^\otimes} \circ f^\otimes = f_b \circ \beta_{Y^\otimes}$  for any superscheme  $Y^\otimes$  and any morphism  $f^\otimes (:= (f_b, f^b)) : Y^\otimes \rightarrow X^\otimes$  of superschemes.

Denote by

$$(11) \quad \mathcal{N}_{X^\otimes}$$

the superideal of  $\mathcal{O}_{X^\otimes}$  generated by  $\mathcal{O}_{X_f}$ . The quotient of  $\mathcal{O}_{X^\otimes}$  by  $\mathcal{N}_{X^\otimes}$  determines a scheme  $X_t$  equipped with a morphism

$$(12) \quad \tau_{X^\otimes} : X_t \rightarrow X^\otimes$$

of superschemes. The composite

$$(13) \quad \gamma_X := \beta_{X^\otimes} \circ \tau_{X^\otimes} : X_t \rightarrow X_b$$

is a closed immersion of schemes corresponding to the quotient  $\mathcal{O}_{X_b} \twoheadrightarrow \mathcal{O}_{X_b}/\mathcal{O}_{X_f}^2 (= \mathcal{O}_{X^\otimes}/\mathcal{N}_{X^\otimes})$  by the nilpotent ideal  $\mathcal{O}_{X_f}^2 \subseteq \mathcal{O}_{X_b}$ .

If  $f^\otimes : Y^\otimes \rightarrow X^\otimes$  is a morphism of superschemes, then it induces a morphism

$$(14) \quad f_t : Y_t \rightarrow X_t$$

of schemes satisfying that  $\tau_{X^\otimes} \circ f_t = f^\otimes \circ \tau_{Y^\otimes}$ .

Next, we denote by

$$(15) \quad \mathfrak{Sch}/_{X^\otimes}$$

the full subcategory of  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  consisting of objects of the form  $Y \rightarrow X^\otimes$ , where  $Y$  is a scheme. The assignment  $Y^\otimes \mapsto Y_t$  (where  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}/_{X^\otimes}^\otimes)$ ) defines a functor

$$(16) \quad \tau : \mathfrak{Sch}/_{X^\otimes}^\otimes \rightarrow \mathfrak{Sch}/_{X_t}$$

which turns out to be a right adjoint functor of the functor

$$(17) \quad \begin{aligned} \mathfrak{Sch}/_{X_t} &\rightarrow \mathfrak{Sch}/_{X^\otimes}^\otimes \\ "Z \rightarrow X_t" &\mapsto "Z \rightarrow X_t \xrightarrow{\tau_{X^\otimes}} X^\otimes". \end{aligned}$$

That is to say, the functorial map of sets

$$(18) \quad \text{Map}_{\mathfrak{Sch}/_{X^\otimes}^\otimes}(Z, Y^\otimes) \rightarrow \text{Map}_{\mathfrak{Sch}/_{X_t}}(Z, Y_t)$$

is bijective, where  $Y \in \text{Ob}(\mathfrak{Sch}/_{X_t})$  and  $Z^\otimes \in \text{Ob}(\mathfrak{Sch}/_{X^\otimes}^\otimes)$ . In particular, we obtain an equivalence of categories  $\mathfrak{Sch}/_{X_t} \xrightarrow{\sim} \mathfrak{Sch}/_{X^\otimes}$  (given as in (17)).

**Definition 1.5.** Let  $X^\otimes$  be a superscheme and  $U$  an fppf scheme over  $X_b$ . Then, we shall write

$$(19) \quad X^\otimes|_U := X^\otimes \times_{\beta_{X^\otimes}, X_b} U.$$

By an **open subsuperscheme** (resp., a **quasi-compact open subsuperscheme**) of  $X^\otimes$ , we mean a superscheme of the form  $X^\otimes|_U$  for some open subscheme (resp., quasi-compact subscheme)  $U$  of  $X_b$ .

### 1.4. Fermionic twists

Let us define the notion of a fermionic twist of a given superscheme. In the following, let us fix a locally noetherian superscheme  $X^\otimes := (X_b, \mathcal{O}_{X^\otimes})$ .

Denote by  $(\mu_2)_{X_b}$  the fppf sheaf on  $X_b$  represented by the group of square roots of unity  $\mu_2 := \text{Spec}(\mathbb{Z}[T]/(T^2-1))$ . For each  $\zeta \in \mu_2(X_b) (= (\mu_2)_{X_b}(X_b))$ , we shall define  $\langle \zeta \rangle_{X^\otimes}$  to be the automorphism

$$(20) \quad \langle \zeta \rangle_{X^\otimes} := (\text{id}_{X_b}, \langle \zeta \rangle_{X^\otimes}^b) : X^\otimes \xrightarrow{\sim} X^\otimes$$

of  $X^\otimes$ , where  $\langle \zeta \rangle_{X^\otimes}^b$  denotes the automorphism of  $\mathcal{O}_{X^\otimes} = \mathcal{O}_{X_b} \oplus \mathcal{O}_{X_f}$  given by assigning  $(a, \epsilon_a) \mapsto (a, \zeta \cdot \epsilon_a)$ . In particular,  $\langle \zeta \rangle_{X^\otimes} \circ \langle \zeta \rangle_{X^\otimes} = \text{id}_{X^\otimes}$ , and if  $X^\otimes$  is a scheme (i.e.,  $\mathcal{O}_{X_f} = 0$ ), then the equality  $\langle \zeta \rangle_{X^\otimes} = \text{id}_{X^\otimes}$  holds. If, moreover,  $Y^\otimes$  is a locally noetherian superscheme and  $f^\otimes (= (f_b, f^b)) : Y^\otimes \rightarrow X^\otimes$  is a morphism of superschemes, then (by regarding  $\zeta$  as an element of  $\mu_2(Y_b)$  via  $f_b$ ) we have the equality of morphisms  $f^\otimes \circ \langle \zeta \rangle_{Y^\otimes} = \langle \zeta \rangle_{X^\otimes} \circ f^\otimes$ . Hence, the collection of automorphisms  $\{\langle \zeta \rangle_{Y^\otimes}\}_{Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)}$  defines a center of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  (i.e., an automorphism of the identity functor  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X^\otimes}^\otimes$ ).

**Definition 1.6.** We shall refer to  $\langle \zeta \rangle_{X^\otimes}$  as the **fermionic involution of  $X^\otimes$  associated with  $\zeta$** .

Write  $\mathcal{A}ut_{X_b}(X^\otimes)$  for the fppf sheaf on  $X_b$  consisting of locally defined automorphisms of  $X^\otimes$  over  $X_b$  (i.e., the sheaf which, to any fppf scheme  $U$  over  $X_b$ , assigns the group of automorphisms of  $X^\otimes|_U$  over  $U$ ). Then, we have a homomorphism

$$(21) \quad \eta_{X^\otimes} : (\mu_2)_{X_b} \rightarrow \mathcal{A}ut_{X_b}(X^\otimes)$$

determined by  $\eta_{X^\otimes}(\zeta) = \langle \zeta \rangle_{X^\otimes}$ . By applying the functor  $H_{\text{fppf}}^1(X_b, -)$ , we have a homomorphism

$$(22) \quad H_{\text{fppf}}^1(\eta_{X^\otimes}) : H_{\text{fppf}}^1(X_b, \mu_2) \rightarrow H_{\text{fppf}}^1(X_b, \mathcal{A}ut_{X_b}(X^\otimes))$$

**Definition 1.7.** A **fermionic twist of  $X^\otimes$**  is a superscheme defined to be the twisted form of  $X^\otimes$  (over the fppf topology on  $X_b$ ) corresponding to  $H_{\text{fppf}}^1(\eta_{X^\otimes})(a) \in H_{\text{fppf}}^1(X_b, \text{Aut}_{X_b}(X^\otimes))$  for some  $a \in H_{\text{fppf}}^1(X_b, \mu_2)$ . We shall refer to this superscheme as the **fermionic twist of  $X^\otimes$  associated with  $a$**  and denote it by

$$(23) \quad {}^a X^\otimes.$$

**Remark 1.8.** By the definition of a fermionic twist, the set of isomorphism classes of fermionic twists of  $X^\otimes$  corresponds bijectively to the set  $\text{Im}(H_{\text{fppf}}^1(\eta_{X^\otimes}))$ . For example, if  $k$  is a separably closed field in which 2 is invertible and  $X_b$  is proper over  $k$ , then  $H_{\text{fppf}}^1(\eta_{X^\otimes})$  is finite (cf. [4], Chap. III, §3, Theorem 3.9, and Chap. VI, §2, Corollary 2.8). This implies that there are only a finite number of isomorphism classes of fermionic twists of  $X^\otimes$ . Also, if  $H_{\text{fppf}}^1(X_b, \mu_2) = 0$  (e.g.,  $X_b$ , as well as  $X_t$ , is simply connected and 2 is invertible in  $\Gamma(X_b, \mathcal{O}_{X_b})$ ) or  $X^\otimes$  is a scheme (i.e.,  $\mathcal{O}_{X_f} = 0$ ), then all fermionic twists of  $X^\otimes$  are isomorphic.

Consider a relation “ $\overset{f}{\sim}$ ” in the set of locally noetherian superschemes defined as follows:

$$(24) \quad Y^\otimes \overset{f}{\sim} Z^\otimes \stackrel{\text{def}}{\iff} Y^\otimes \text{ is isomorphic to a fermionic twist of } Z^\otimes.$$

One verifies immediately that this relation forms an equivalence relation. The following proposition is one direction of the equivalence in Theorem A.

**Proposition 1.9.** *Let  $X^\otimes$  and  $Y^\otimes$  be two locally noetherian superschemes and suppose that  $X^\otimes \overset{f}{\sim} Y^\otimes$ . Then, there exists an equivalence of categories  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/Y^\otimes}^\otimes$ .*

*Proof.* Let  $a \in H_{\text{fppf}}^1(X_b, \mu_2)$ , and suppose that  $Y^\otimes = {}^a X^\otimes$  (i.e.,  $Y^\otimes$  is a fermionic twist of  $X^\otimes$  associated with  $a$ ). If we are given a morphism  $f^\otimes : Z^\otimes \rightarrow X^\otimes$  in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ , then the homomorphism  $H_{\text{fppf}}^1(X_b, \mu_2) \rightarrow H_{\text{fppf}}^1(Z_b, \mu_2)$  induced by  $f_b$  sends  $a$  to an element of  $H_{\text{fppf}}^1(Z_b, \mu_2)$ ; we write, by abuse of notation, for  ${}^a Z^\otimes$  the fermionic twist of  $Z^\otimes$  associated with this element. It follows from the functoriality (with respect to  $X^\otimes$ ) of  $\langle \zeta \rangle_{X^\otimes}$  for each  $\zeta \in \mu_2(X_b)$  that  $f^\otimes$  induces a morphism  ${}^a f^\otimes : {}^a Z^\otimes \rightarrow Y^\otimes$  ( $= {}^a X^\otimes$ ) in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$ . The assignment  $Z^\otimes \mapsto {}^a Z^\otimes$  is functorial, and hence, defines a functor

$$(25) \quad \mathfrak{Sch}_{/X^\otimes}^\otimes \rightarrow \mathfrak{Sch}_{/Y^\otimes}^\otimes.$$



Since  $X^\otimes$  is fermionic twist of  $Y^\otimes$  associated with  $-a$  (under the natural identification  $H_{\text{fppf}}^1(X_b, \mu_2) = H_{\text{fppf}}^1(Y_b, \mu_2)$ ), the same argument as above gives rise to a functor  $\mathfrak{Sch}_{/Y^\otimes}^\otimes \rightarrow \mathfrak{Sch}_{/X^\otimes}^\otimes$ , which turns out to be the inverse to the functor (25). This completes the proof of Proposition 1.9.  $\square$

### 1.5. Fermionic twists in the Zariski topology

Denote by  $(\mathbb{G}_m)_{X_b}$  the fppf sheaf on  $X_b$  represented by the multiplicative group  $\mathbb{G}_m$ . The Kummer sequence

$$(26) \quad 0 \rightarrow (\mu_2)_{X_b} \rightarrow (\mathbb{G}_m)_{X_b} \rightarrow (\mathbb{G}_m)_{X_b} \rightarrow 0$$

$$a \quad \mapsto \quad a^2$$

induces an exact sequence

$$(27) \quad 0 \rightarrow \mu_2(X_b) \rightarrow \Gamma(X_b, \mathcal{O}_{X_b}^\times) \rightarrow \Gamma(X_b, \mathcal{O}_{X_b}^\times) \xrightarrow{\delta} H_{\text{fppf}}^1(X_b, \mu_2) \xrightarrow{\sigma} \text{Pic}(X_b) \rightarrow \text{Pic}(X_b)$$

$$a \quad \mapsto \quad a^2 \quad \quad \quad [\mathcal{L}] \quad \mapsto \quad [\mathcal{L}^{\otimes 2}].$$

Recall (cf. [4], Chap. III, §4, the discussion following Lemma 4.10) that any element of  $H_{\text{fppf}}^1(X_b, \mu_2)$  may be represented by a collection of data

$$(28) \quad s := (\{U_\alpha\}_{\alpha \in I}, \{s_\alpha\}_{\alpha \in I}, \{t_{\alpha, \beta}\}_{(\alpha, \beta) \in I_2}),$$

where

- $I$  is an index set;
- $\{U_\alpha\}_{\alpha \in I}$  is a Zariski open covering of  $X_b$ ;
- each  $s_\alpha$  ( $\alpha \in I$ ) is an element of  $\Gamma(U_\alpha, \mathcal{O}_{U_\alpha}^\times)$ ;
- $I_2 := \{(\alpha, \beta) \in I \times I \mid U_{\alpha, \beta} := U_\alpha \cap U_\beta \neq \emptyset\}$ ;
- $\{t_{\alpha, \beta}\}_{(\alpha, \beta) \in I_2}$  is a 1-cocycle of  $\{U_\alpha\}_{\alpha \in I}$  with coefficients in  $\mathcal{O}_{X_b}^\times$  satisfying the equality  $s_\beta|_{U_{\alpha, \beta}} \cdot t_{\alpha, \beta}^2 = s_\alpha|_{U_{\alpha, \beta}}$  for any  $(\alpha, \beta) \in I_2$ .

The homomorphism  $\delta$  (resp.,  $\sigma$ ) is given by assigning  $u \mapsto \overline{(\{X_b\}, \{u\}, \{1\})}$  for any  $u \in \Gamma(X_b, \mathcal{O}_{X_b}^\times)$  (resp.,  $\bar{s} \mapsto \overline{(\{U_\alpha\}_\alpha, \{t_{\alpha, \beta}\}_{\alpha, \beta})}$  for any  $s$  as in (28)).

Now, let  $u \in \Gamma(X_b, \mathcal{O}_{X_b}^\times)$ . We shall write

$$(29) \quad {}^u X^\otimes := \delta(u) X^\otimes$$

by abuse of notation. One verifies that it is a unique (up to isomorphism) superscheme such that the triple  $\mathcal{A}_{{}^u X^\otimes}$  associated with it (cf. Proposition 1.4)

coincides with  $(X_b, \mathcal{O}_{X_f}, u \cdot m_{X^\otimes})$ . (In particular,  $\mathcal{O}_{uX^\otimes} = \mathcal{O}_{X^\otimes}$  as an  $\mathcal{O}_{X_b}$ -module.) Indeed, let us write  $Y^\otimes$  for the superscheme corresponding to  $(X_b, \mathcal{O}_{X_f}, u \cdot m_{X^\otimes})$  (hence,  $Y_b = X_b$ ). Also, let us take an étale covering  $U \rightarrow X_b$  such that there exists  $v \in \Gamma(U, \mathcal{O}_U^\times)$  with  $v^2 = u$ . The automorphism of the  $\mathcal{O}_U$ -module  $\mathcal{O}_U \oplus \mathcal{O}_{X_f}|_U$  given by assigning  $(a, \epsilon_a) \mapsto (a, v \cdot \epsilon_a)$  determines an isomorphism  $X^\otimes|_U \xrightarrow{\sim} Y^\otimes|_U$  that induces the identity morphism of  $X_b$ . This implies that  $Y^\otimes$  is the fermionic twist of  $X^\otimes$  associated with  $\delta(u)$ , as desired.

Conversely, any fermionic twist of  $X^\otimes$  is, Zariski locally on  $X_b$ , isomorphic to  ${}^u X^\otimes$  (for some local section  $u \in \mathcal{O}_{X_b}^\times$ ), as described in the following proposition.

**Proposition 1.10.** *Let  $a$  be an element of  $H_{\text{fppf}}^1(X_b, \mu_2)$  (hence, we have the fermionic twist  ${}^a X^\otimes$  of  $X^\otimes$  associated with  $a$ ). Also, let  $(\{U_\alpha \in I\}_{\alpha \in I}, \{s_\alpha\}_{\alpha \in I}, \{t_{\alpha, \beta}\}_{(\alpha, \beta) \in I_2})$  be a representative of  $a$  as in (28). Then, there exists a collection of isomorphisms*

$$(30) \quad \{\xi_\alpha^\otimes : {}^a X^\otimes|_{U_\alpha} \xrightarrow{\sim} {}^{s_\alpha} X^\otimes|_{U_\alpha}\}_{\alpha \in I}$$

satisfying the following two conditions:

- For each  $\alpha \in I$ , the morphism  $(\xi_\alpha^\otimes)_b$  of schemes underlying  $\xi_\alpha^\otimes$  coincides with the identity morphism of  $U_\alpha$ ;
- For each  $(\alpha, \beta) \in I_2$ , the automorphism

$$(31) \quad \xi_\beta^\otimes \circ (\xi_\alpha^\otimes)^{-1} : {}^{s_\alpha} X^\otimes|_{U_{\alpha, \beta}} \xrightarrow{\sim} {}^{s_\beta} X^\otimes|_{U_{\alpha, \beta}}$$

corresponds to the automorphism of the  $\mathcal{O}_{U_{\alpha, \beta}}$ -module  $\mathcal{O}_{U_{\alpha, \beta}} \oplus \mathcal{O}_{X_f}|_{U_{\alpha, \beta}}$  given by assigning  $(a, \epsilon_a) \mapsto (a, t_{\alpha, \beta} \cdot \epsilon_a)$ .

*Proof.* The assertion follows immediately from the definition of a fermionic twist and the above discussion. □

### 1.6. $\mathbb{A}^{0|1}$ -twists

For each pair  $(n, m)$  of nonnegative integers, we shall denote by

$$(32) \quad \mathbb{A}^{n|m}$$

the  $(n|m)$ -dimensional affine superspace over  $\mathbb{Z}$ , i.e., the superspectrum of the superalgebra  $\mathbb{Z}[t_1, \dots, t_n, \psi_1, \dots, \psi_m]$ , where the  $t_1, \dots, t_n$  are ordinary indeterminates and  $\psi_1, \dots, \psi_m$  are odd indeterminates. Also, let us write

$$(33) \quad \mathbb{A}_{X^\otimes}^{n|m} := X^\otimes \times \mathbb{A}^{n|m}.$$

For any  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$  and any nonnegative integers  $n, m$ , the superscheme  $\mathbb{A}_{Y^\otimes}^{n|m}$  belongs to  $\text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ . Also, we have a sequence of functorial (in  $Y^\otimes$ ) bijections of sets:

$$\begin{aligned}
 (34) \quad & \text{Map}_{\mathfrak{Sch}_{/Y^\otimes}^\otimes}(Y^\otimes, \mathbb{A}_{Y^\otimes}^{1|1}) \\
 & \xrightarrow{\sim} \text{Map}_{\mathfrak{Sch}_{/Y^\otimes}^\otimes}(Y^\otimes, \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{0|1}) \\
 & \xrightarrow{\sim} \text{Map}_{\mathfrak{Sch}_{/Y^\otimes}^\otimes}(Y^\otimes, \mathbb{A}_{Y^\otimes}^{1|0}) \times \text{Map}_{\mathfrak{Sch}_{/Y^\otimes}^\otimes}(Y^\otimes, \mathbb{A}_{Y^\otimes}^{0|1}) \\
 & \xrightarrow{\sim} \Gamma(Y_b, \mathcal{O}_{Y_b}) \times \Gamma(Y_b, \mathcal{O}_{Y_f}) \\
 & \xrightarrow{\sim} \Gamma(Y_b, \mathcal{O}_{Y^\otimes}),
 \end{aligned}$$

where the third bijection is given by  $(h_1^\otimes, h_2^\otimes) \mapsto (h_1^b(t), h_2^b(\psi))$ . The multiplication and addition in  $\Gamma(Y_b, \mathcal{O}_{Y^\otimes})$  correspond, via (34), to morphisms

$$(35) \quad \mu_{Y^\otimes} : \mathbb{A}_{Y^\otimes}^{1|1} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|1} \rightarrow \mathbb{A}_{Y^\otimes}^{1|1} \quad \text{and} \quad \alpha_{Y^\otimes} : \mathbb{A}_{Y^\otimes}^{1|1} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|1} \rightarrow \mathbb{A}_{Y^\otimes}^{1|1}$$

respectively. That is to say, the set  $\text{Map}_{\mathfrak{Sch}_{/Y^\otimes}^\otimes}(Y^\otimes, \mathbb{A}_{Y^\otimes}^{1|1})$  admits a structure of superalgebra by means of  $\mu_{Y^\otimes}$  and  $\alpha_{Y^\otimes}$  (together with the decomposition  $\mathbb{A}_{Y^\otimes}^{1|1} \xrightarrow{\sim} \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{0|1}$ , which induces the parity), and the composite bijection (34) becomes an isomorphism of superalgebras. In particular, each element  $a$  of  $\Gamma(Y_b, \mathcal{O}_{Y_b})$  corresponds to a morphism

$$(36) \quad \sigma_{Y^\otimes}^{[a]} : Y^\otimes \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}.$$

Let  $\mathcal{A}ut_{Y^\otimes}(\mathbb{A}_{Y^\otimes}^{1|0}, \sigma_{Y^\otimes}^{[0]})$  denote the Zariski sheaf on  $Y_b$  such that to every open subsuperscheme  $U$  of  $Y_b$ , assigned is the group of automorphisms of  $\mathbb{A}_{Y^\otimes|U}^{1|0}$  over  $Y^\otimes|_U$  which are compatible with  $\sigma_{Y^\otimes}^{[0]}|_{Y^\otimes|U}$ . The homomorphism

$$(37) \quad \mathcal{O}_{Y_b}^\times \xrightarrow{\sim} \mathcal{A}ut_{Y^\otimes}(\mathbb{A}_{Y^\otimes}^{1|0}, \sigma_{Y^\otimes}^{[0]})$$

which, to any local section  $a \in \mathcal{O}_{Y_b}^\times$ , assigns the automorphism of  $\mathbb{A}_{Y^\otimes}^{1|0}$  over  $Y^\otimes$  determined by  $t \mapsto a \cdot t$  turns out to be bijective. By applying the functor  $H_{\text{Zar}}^1(Y_b, -)$ . we have an isomorphism

$$(38) \quad \text{Pic}(Y_b) \xrightarrow{\sim} H_{\text{Zar}}^1(Y_b, \mathcal{A}ut_{Y^\otimes}(\mathbb{A}_{Y^\otimes}^{1|0}, \sigma_{Y^\otimes}^{[0]})).$$

**Definition 1.11.** (i) An  $\mathbb{A}^{0|1}$ -twist over  $Y^\otimes$  is a twisted form of  $(\mathbb{A}_{Y^\otimes}^{1|0}, \sigma_{Y^\otimes}^{[0]})$  (over the Zariski topology on  $Y_b$ ) determined, via (38), by some  $a \in \text{Pic}(Y_b)$ ; it may be described as a pair

$$(39) \quad (Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$$

consisting of a twisted form  $Z^\otimes$  of  $\mathbb{A}_{Y^\otimes}^{1|0}$  over  $Y^\otimes$  and a section  $\sigma_{Z^\otimes/Y^\otimes} : Y^\otimes \rightarrow Z^\otimes$  of the structure morphism of  $Z^\otimes$ . We shall refer to the pair  $(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  as the  $\mathbb{A}^{0|1}$ -twist over  $Y^\otimes$  associated with  $a$ .

(ii) Let  $(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  and  $(Z'^\otimes, \sigma_{Z'^\otimes/Y^\otimes})$  be two  $\mathbb{A}^{0|1}$ -twists over  $Y^\otimes$ . An **isomorphism of  $\mathbb{A}^{0|1}$ -twists** from  $(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  to  $(Z'^\otimes, \sigma_{Z'^\otimes/Y^\otimes})$  is an isomorphism  $h^\otimes : Z^\otimes \xrightarrow{\sim} Z'^\otimes$  of superschemes over  $Y^\otimes$  with  $h^\otimes \circ \sigma_{Z^\otimes/Y^\otimes} = \sigma_{Z'^\otimes/Y^\otimes}$ .

By (38), there exists canonically a bijective correspondence between  $\text{Pic}(Y_b)$  and the set of isomorphism classes of  $\mathbb{A}^{0|1}$ -twists over  $Y^\otimes$ .

### 1.7. The multiplication morphisms of fermionic twists

Let  $u \in \Gamma(Y_b, \mathcal{O}_{Y_b}^\times)$ . Since  $\mathcal{O}_{uY^\otimes} = \mathcal{O}_{Y^\otimes}$  as  $\mathcal{O}_{Y_b}$ -modules, the multiplication in  $\mathcal{O}_{uY^\otimes}$  gives rise to a morphism

$$(40) \quad \mu_{Y^\otimes \rightsquigarrow uY^\otimes} : \mathbb{A}_{Y^\otimes}^{1|1} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|1} \rightarrow \mathbb{A}_{Y^\otimes}^{1|1}$$

over  $Y^\otimes$  under the bijection (34). The morphism  $\mu_{Y^\otimes \rightsquigarrow uY^\otimes}$  corresponds to the homomorphism of superalgebras over  $\mathcal{O}_{Y^\otimes}$  described as follows:

$$(41) \quad \begin{aligned} \mathcal{O}_{Y^\otimes}[t, \psi] &\rightarrow \mathcal{O}_{Y^\otimes}[t, \psi] \otimes_{\mathcal{O}_{Y^\otimes}} \mathcal{O}_{Y^\otimes}[t, \psi] \\ t &\mapsto t \otimes t + s \cdot \psi \otimes \psi \\ \psi &\mapsto \psi \otimes t + t \otimes \psi. \end{aligned}$$

Next, let  $a$  be an element of  $H_{\text{ppf}}^1(Y_b, \mu_2)$  and let  $Z^\otimes := {}^a Y^\otimes$ . We shall choose a representative  $(\{U_\alpha\}_{\alpha \in I}, \{s_\alpha\}_{\alpha \in I}, \{t_{\alpha, \beta}\}_{(\alpha, \beta) \in I_2})$  of  $a$  as in (28) (of the case where the superscheme  $X^\otimes$  is replaced with  $Y^\otimes$ ). Write

$$(42) \quad (\mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{0|1}, \sigma_{\mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{0|1}})$$

for the  $\mathbb{A}^{0|1}$ -twist over  $Y^\otimes$  determined by  $\sigma(a) \in \text{Pic}(Y_b)$ , and write

$$(43) \quad \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1} := \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{0|1} \times \mathbb{A}^{1|0}.$$

The multiplication morphisms  $\mu_{Y^\otimes|U_\alpha \rightsquigarrow s_\alpha Y^\otimes|U_\alpha}$  ( $\alpha \in I$ ) may be glued together to obtain a morphism

$$(44) \quad \mu_{Y^\otimes \rightsquigarrow Z^\otimes} : \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1} \rightarrow \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1}$$

over  $Y^\otimes$ . This morphism does not depend on the choice of a representative of  $a$ . Also, the morphisms  $\alpha_{s_\alpha Y^\otimes|U_\alpha}$  may be glued together to obtain a morphism

$$(45) \quad \alpha_{Y^\otimes \rightsquigarrow Z^\otimes} : \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1} \rightarrow \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1}$$

over  $Y^\otimes$ . This morphism depends only on the  $\mathbb{A}^{0|1}$ -twist  $\mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{0|1}$  (i.e., the class  $\sigma(a) \in \text{Pic}(Y_b)$ ). Owing to the morphisms  $\alpha_{Y^\otimes \rightsquigarrow Z^\otimes}$  and  $\mu_{Y^\otimes \rightsquigarrow Z^\otimes}$ , we have an isomorphism of superalgebras

$$(46) \quad \Gamma(Z_b, \mathcal{O}_{Z^\otimes}) \xrightarrow{\sim} \text{Map}_{\mathfrak{Sch}_{/Y^\otimes}^\otimes} (Y^\otimes, \mathbb{A}_{Y^\otimes \rightsquigarrow Z^\otimes}^{1|1})$$

which is functorial with respect to  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ .

## 2. Proof of Theorem A

This section is devoted to proving the remaining portion of Theorem A, i.e., that the equivalence class defined by “ $\overset{f}{\sim}$ ” of a locally noetherian superscheme  $X$  may be reconstructed purely category-theoretically from the category  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . In the following discussion, we will often speak of various properties of objects and morphisms in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  as being “*characterized (or reconstructed) category-theoretically*”. By this, we mean that they are preserved by arbitrary equivalences of categories  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes$  (where  $X'^\otimes$  is another locally noetherian superscheme). For instance, the set of monomorphisms in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  may be *characterized category-theoretically* as the morphisms  $f^\otimes : Z^\otimes \rightarrow Y^\otimes$  such that, for any  $W^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ , the map of sets  $\text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes} (W^\otimes, Z^\otimes) \rightarrow \text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes} (W^\otimes, Y^\otimes)$  given by composing with  $f^\otimes$  is injective. To simplify notation, however, we omit explicit mention of this equivalence  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes$ , of  $X'$ , and of the various “primed” objects and morphisms corresponding to the original objects and morphisms, respectively, in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ .

In this section, *let us fix a locally noetherian superscheme  $X^\otimes$ .*

2.1.

Our tactics for completing the proof of Theorem A (i.e., recognizing the structure of superscheme of  $X^{\otimes}$ ) is, as in [5], to reconstruct step-by-step various partial information of  $X^{\otimes}$  from the categorical structure of  $\mathfrak{Sch}_{/X^{\otimes}}^{\otimes}$ . As the first step, we reconstruct the set of objects in  $\mathfrak{Sch}_{/X^{\otimes}}^{\otimes}$  which are isomorphic to spectrums of fields (cf. Proposition 2.5). Of course, these objects allow us to know the points in the topological space underlying  $X^{\otimes}$ .

For each superalgebra  $R$ , we denote by

$$(47) \quad \text{Spec}(R)^{\otimes}$$

the superspectrum of  $R$ . Let  $k$  be a field and  $M$  a finite-dimensional  $k$ -vector space. We shall equip  $k \oplus M$  with a structure of superalgebra over  $k$  given as follows:

- The bosonic part is the first factor  $k$  and the fermionic part is the second factor  $M$ ;
- The multiplication is given by assigning  $(a, \epsilon_a) \cdot (b, \epsilon_b) := (ab, a\epsilon_b + b\epsilon_a)$  for any  $a, b \in k$  and  $\epsilon_a, \epsilon_b \in M$ .

We shall write

$$(48) \quad \mathbb{A}_k^{0|M} := \text{Spec}(k \oplus M)^{\otimes}.$$

In other words,  $\mathbb{A}_k^{0|M}$  is a unique (up to isomorphism) superscheme satisfying that  $\mathcal{A}_{\mathbb{A}_k^{0|M}} := (\text{Spec}(k), \mathcal{O}_{\text{Spec}(k)} \otimes_k M, 0)$ . In particular,  $\mathbb{A}_k^{0|k} = \mathbb{A}_k^{0|1}$  (cf. (33)). If  $M_1$  and  $M_2$  are finite-dimensional  $k$ -vector spaces, then any morphism  $\mathbb{A}_k^{0|M_1} \rightarrow \mathbb{A}_k^{0|M_2}$  of superschemes over  $k$  coincides with the morphism induced from a unique  $k$ -linear morphism  $M_2 \rightarrow M_1$ . This observation shows the following two lemmas.

**Lemma 2.1.** *Let us write  $\mathfrak{Vec}_k$  for the opposite category of finite-dimensional  $k$ -vector spaces and write*

$$(49) \quad {}^{\circ}\mathfrak{Sch}_{/k}^{\otimes}$$

*for the full subcategory of  $\mathfrak{Sch}_{/k}^{\otimes}$  consisting of superschemes which are isomorphic to  $\mathbb{A}_k^{0|M}$  for some finite-dimensional  $k$ -vector space  $M$ . Then, the functor*

$$(50) \quad \mathfrak{Vec}_k \rightarrow {}^{\circ}\mathfrak{Sch}_{/k}^{\otimes}$$

$$M \mapsto \mathbb{A}_k^{0|M}$$

defines an equivalence of categories.

**Lemma 2.2.** *Let  $M$  be a finite-dimensional  $k$ -vector space. Suppose that  $\text{Spec}(k)$  is an object of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ , in particular, admits a structure morphism  $\text{Spec}(k) \rightarrow X^\otimes$ . (Hence,  $\mathbb{A}_k^{0|M}$  may be thought of as an object of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  by taking account of the composite  $\mathbb{A}_k^{0|M} \rightarrow \text{Spec}(k) \rightarrow X^\otimes$ ). Then, there exists a natural bijection*

$$(51) \quad \text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes}(\mathbb{A}_k^{0|M}, Y^\otimes) \xrightarrow{\sim} \left\{ (s, h) \mid s \in \text{Map}_{\mathfrak{Sch}_{/X_b}}(\text{Spec}(k), Y_b), h \in \text{Hom}_k(s^*(\mathcal{O}_{Y_f}), M) \right\}$$

for any object  $Y^\otimes$  of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ .

*Proof.* The assertion follows directly from the definition of  $\mathbb{A}_k^{0|M}$ . □

Therefore, we have the following proposition.

**Proposition 2.3.** *A morphism  $f^\otimes$  ( $:= (f_b, f^b)$ ) :  $Z^\otimes \rightarrow Y^\otimes$  in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  is a monomorphism (in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ ) if and only if the induced morphism  $f_t : Z_t \rightarrow Y_t$  is a monomorphism in  $\mathfrak{Sch}_{/X_t}$  and  $f^b : f_b^*(\mathcal{O}_{Y^\otimes}) \rightarrow \mathcal{O}_{Z^\otimes}$  is surjective.*

*Proof.* Let  $f^\otimes$  ( $:= (f_b, f^b)$ ) :  $Z^\otimes \rightarrow Y^\otimes$  be a monomorphism in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . Suppose that  $f^b$  is not surjective, or equivalently, its restriction  $f^b|_{f_b^*(\mathcal{O}_{Y_f})} : f_b^*(\mathcal{O}_{Y_f}) \rightarrow \mathcal{O}_{Z_f}$  is not surjective. By Nakayama’s lemma (and the condition that  $Z^\otimes$  is noetherian), there exists a point  $s^\otimes$  ( $:= (s_b, s^b)$ ) :  $\text{Spec}(k) \rightarrow Z^\otimes$  of  $Z^\otimes$  such that  $(f_b \circ s_b)^*(\mathcal{O}_{Y_f}) \rightarrow s_b^*(\mathcal{O}_{Z_f})$  is not surjective. Hence, the induced morphism between  $k$ -vector spaces

$$(52) \quad \text{Hom}_k(s_b^*(\mathcal{O}_{Z_f}), k) \rightarrow \text{Hom}_k((f_b \circ s_b)^*(\mathcal{O}_{Y_f}), k)$$

is not injective. It follows from Lemma 2.2 that the map

$$(53) \quad \text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes}(\mathbb{A}_k^{0|k}, Z^\otimes) \rightarrow \text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes}(\mathbb{A}_k^{0|k}, Y^\otimes)$$

given by composing with  $f^\otimes$  is not injective, and we obtain a contradiction. Thus,  $f^b$  must be surjective.

Next, suppose that  $f_t$  is *not* a monomorphism in  $\mathfrak{Sch}/_{X_t}$ , or equivalently, there exists an object  $W$  of  $\mathfrak{Sch}/_{X_t}$  whose associated map

$$(54) \quad \text{Map}_{\mathfrak{Sch}/_{X_t}}(W, Z_t) \rightarrow \text{Map}_{\mathfrak{Sch}/_{X_t}}(W, Y_t)$$

is not injective. Since  $\tau$  (cf. (16)) is a right adjoint functor of the functor  $\mathfrak{Sch}/_{X_t} \xrightarrow{(17)} \mathfrak{Sch}/_{X^\otimes}^\otimes$ , the non-injective map (54) may be identified with the map

$$(55) \quad \text{Map}_{\mathfrak{Sch}/_{X^\otimes}^\otimes}(W, Z^\otimes) \rightarrow \text{Map}_{\mathfrak{Sch}/_{X^\otimes}^\otimes}(W, Y^\otimes).$$

This contradicts the assumption that  $f^\otimes$  is a monomorphism. Thus,  $f_t$  must be a monomorphism.

The reverse direction may be verified immediately, and hence, we complete the proof of Proposition 2.3. □

- Definition 2.4.** (i) We shall say that an object  $Y^\otimes$  in  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  is **minimal (over  $X^\otimes$ )** if it is nonempty (i.e., not an initial object of  $\mathfrak{Sch}/_{X^\otimes}^\otimes$ ) and any monomorphism  $Z^\otimes \rightarrow Y^\otimes$  from a nonempty object  $Z^\otimes \in \text{Ob}(\mathfrak{Sch}/_{X^\otimes}^\otimes)$  to  $Y^\otimes$  is necessarily an isomorphism.
- (ii) We shall say that an object  $Y^\otimes$  in  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  is **terminally minimal (over  $X^\otimes$ )** if it is minimal over  $X^\otimes$  and any minimal object  $Z^\otimes$  over  $X^\otimes$  with  $Y^\otimes \times_{X^\otimes} Z^\otimes \neq \emptyset$  admits a morphism  $Z^\otimes \rightarrow Y^\otimes$ .

These properties on objects in  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  give a category-theoretic characterization of spectrums of fields, as follows.

**Proposition 2.5** (Characterization of spectrums of fields). *The following assertions (i), (ii) hold.*

- (i) *An object  $Y^\otimes$  of  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  is minimal if and only if  $Y^\otimes$  is isomorphic to  $\text{Spec}(k)$  for some field  $k$ .*
- (ii) *An object  $Y^\otimes$  of  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  is terminally minimal if and only if it is a point of  $X_t$ , considered as an object of  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  via composition with  $\tau_{X^\otimes} : X_t \rightarrow X^\otimes$ .*

*Consequently, the objects of  $\mathfrak{Sch}/_{X^\otimes}^\otimes$  consisting of (super)schemes which are isomorphic to  $\text{Spec}(k)$  for some field (resp., consisting of points of  $X_t$ ) may be reconstructed category-theoretically from the category  $\mathfrak{Sch}/_{X^\otimes}^\otimes$ .*

*Proof.* The assertions are formal consequences of the definitions of being minimal and terminally minimal. □



2.2.

Next, we shall consider the category-theoretic reconstruction of the superschemes  $\mathbb{A}_k^{0|k}$  ( $= \mathbb{A}_k^{0|1}$ ) and  $\mathbb{A}_k^{\varepsilon|0}$  (introduced below) in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . After reconstructing these objects, one may use them to understand the local structure of  $X^\otimes$  (cf. Proposition 2.9 described later).

**Definition 2.6.** We shall say that an object  $Y^\otimes$  of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  is **one-pointed** if its underlying topological space consists precisely of one element.

The following proposition may be immediately verified.

**Proposition 2.7** (Characterization of one-pointed superschemes). *The one-pointed objects of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  may be characterized category-theoretically as the nonempty objects  $Y^\otimes$  which satisfy the following condition:*

(A) $_{Y^\otimes}$ : *For any two minimal objects  $Z_1^\otimes \rightarrow Y^\otimes$ ,  $Z_2^\otimes \rightarrow Y^\otimes$  over  $Y^\otimes$ , the fiber product  $Z_1^\otimes \times_{Y^\otimes} Z_2^\otimes$  is nonempty.*

For any field  $k$ , we shall write

$$(56) \quad \mathbb{A}_k^{\varepsilon|0} := \text{Spec}(k[\varepsilon]/\varepsilon^2).$$

**Proposition 2.8** (Characterization of  $\mathbb{A}_k^{0|1}$ ). *Suppose that we are given a morphism  $\text{Spec}(k) \rightarrow X^\otimes$  (where  $k$  denotes a field) belonging to  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . (Hence, the category  $\mathfrak{Sch}_{/k}^\otimes$  may be characterized category-theoretically from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, \text{Spec}(k))$ , i.e., a pair consisting of a category and a minimal object of it.) Then, the following assertions (i), (ii) hold.*

(i) *The set consisting of two objects*

$$(57) \quad \{\mathbb{A}_k^{0|1}, \mathbb{A}_k^{\varepsilon|0}\}$$

*of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  may be characterized (up to isomorphism in the evident sense) category-theoretically as the image (via the functor  $\mathfrak{Sch}_{/k}^\otimes \rightarrow \mathfrak{Sch}_{/X^\otimes}^\otimes$  given by composing with  $\text{Spec}(k) \rightarrow X^\otimes$ ) of the set  $\{S^\otimes, T^\otimes\}$  of two one-pointed objects of  $\mathfrak{Sch}_{/k}^\otimes$  satisfying the following two conditions (B) $_{S^\otimes, T^\otimes}$  and (C) $_{S^\otimes, T^\otimes}$ :*

(B) $_{S^\otimes, T^\otimes}$ :  *$S^\otimes$  is not isomorphic to  $T^\otimes$ , and  $\text{Spec}(k)$  is neither isomorphic to  $S^\otimes$  nor to  $T^\otimes$ ;*

$(C)_{S^\otimes, T^\otimes}$ : Let  $V^\otimes$  be a one-pointed object  $V^\otimes$  of  $\mathfrak{Sch}_{/k}^\otimes$  satisfying the following two conditions:

- $V^\otimes$  is not isomorphic to  $\text{Spec}(k)$ ;
- Any terminally minimal object over  $V^\otimes$  (which is uniquely determined up to isomorphism) is isomorphic to the terminal object  $\text{Spec}(k)$ .

Then, there exists either a monomorphism  $S^\otimes \hookrightarrow V^\otimes$  from  $S^\otimes$  or a monomorphism  $T^\otimes \hookrightarrow V^\otimes$  from  $T^\otimes$ .

(ii) Let  $U^\otimes$  be either  $\mathbb{A}_k^{0|1}$  or  $\mathbb{A}_k^{\varepsilon|0}$ , and denote by  $U'^\otimes$  the unique object in  $\{\mathbb{A}_k^{0|1}, \mathbb{A}_k^{\varepsilon|0}\} \setminus \{U^\otimes\}$ . Then,  $U^\otimes$  coincides with  $\mathbb{A}_k^{0|1}$  if and only if for any morphism  $U'^\otimes \times_k U'^\otimes \rightarrow U^\otimes \times_k U^\otimes$  factors through a terminally minimal morphism over  $U^\otimes \times_k U^\otimes$ . In particular, the object  $\mathbb{A}_k^{0|1}$  (resp.,  $\mathbb{A}_k^{\varepsilon|0}$ ) in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  may be reconstructed category-theoretically (up to isomorphism) from the minimal object  $\text{Spec}(k)$  in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ .

*Proof.* First, we shall consider assertion (i). Since the set  $\{\mathbb{A}_k^{0|1}, \mathbb{A}_k^{\varepsilon|0}\}$  is immediately verified to satisfy both the conditions  $(B)_{S^\otimes, T^\otimes}$  and  $(C)_{S^\otimes, T^\otimes}$ , it suffices to prove its reverse direction.

Note that any one-pointed object of  $\mathfrak{Sch}_{/k}^\otimes$  is necessarily isomorphic to the superspectrum of some (local) superalgebra over  $k$ . For a one-pointed object  $W^\otimes$  in  $\mathfrak{Sch}_{/k}^\otimes$ , we shall write

$$(58) \quad \dim_k(W^\otimes) := \dim_k(\Gamma(W_b, \mathcal{O}_{W^\otimes})) (< \infty).$$

Now, let  $\{S^\otimes, T^\otimes\}$  be a set of two one-pointed objects of  $\mathfrak{Sch}_{/k}^\otimes$  satisfying both the conditions  $(B)_{S^\otimes, T^\otimes}$  and  $(C)_{S^\otimes, T^\otimes}$ . Suppose that one of the objects in this set, say  $S^\otimes$ , satisfies the inequality  $\dim_k(S^\otimes) \geq 3$ . By Proposition 2.3, there does not exist a monomorphism from  $S^\otimes$  to  $\mathbb{A}_k^{0|1}$  since  $\dim_k(\mathbb{A}_k^{0|1}) = 2$ . It follows from the condition  $(C)_{S^\otimes, T^\otimes}$  that there exists a monomorphism from  $T^\otimes$  to  $\mathbb{A}_k^{0|1}$ , and hence, that  $\dim_k(T^\otimes) \leq 2$  (by Proposition 2.3 again). Since  $T^\otimes \not\cong \text{Spec}(k)$  and there does not exist a monomorphism  $\mathbb{A}_k^{\varepsilon|0}$  from  $\mathbb{A}_k^{0|1}$ ,  $T^\otimes$  must be isomorphic to  $\mathbb{A}_k^{0|1}$ . On the other hand, by a similar argument where  $\mathbb{A}_k^{0|1}$  is replaced with  $\mathbb{A}_k^{\varepsilon|0}$ ,  $T^\otimes$  must be isomorphic to  $\mathbb{A}_k^{\varepsilon|0}$ , and we obtain a contradiction. Consequently, we have  $\dim_k(S^\otimes) = \dim_k(T^\otimes) = 2$ . This implies that  $S^\otimes$  and  $T^\otimes$  are respectively isomorphic to either  $\mathbb{A}_k^{0|1}$  or  $\mathbb{A}_k^{\varepsilon|0}$ . Thus, we complete the proof of assertion (i).

Assertion (ii) follows directly from the fact that

$$(59) \quad \mathbb{A}_k^{\varepsilon|0} \times_k \mathbb{A}_k^{\varepsilon|0} \cong \text{Spec}(k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_1\varepsilon_2, \varepsilon_2^2))$$

and

$$(60) \quad \mathbb{A}_k^{0|1} \times_k \mathbb{A}_k^{0|1} \cong \text{Spec}(\bigwedge_k^\bullet(k^{\oplus 2}))^\otimes, \quad (\mathbb{A}_k^{0|1} \times_k \mathbb{A}_k^{0|1})_t \cong \text{Spec}(k)$$

(where  $\bigwedge_k^\bullet(k^{\oplus 2})$  denotes the exterior algebra over  $k$  associated with  $k^{\oplus 2}$ , which admits naturally a structure of superalgebra over  $k$ ). □

### 2.3.

Next, we consider the reconstruction of the schematic structure on  $Y_t$  for each  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$  (cf. Corollary 2.10 below) and a topological structure of the underlying space of  $X^\otimes$  (cf. Proposition 2.11 below). First, we observe that there exists, by means of Proposition 2.8, the following category-theoretic criterion for each object  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$  to be a scheme (i.e.,  $\mathcal{O}_{Y_f} = 0$ ).

**Proposition 2.9** (Characterization of schemes). *The objects  $Y^\otimes$  of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  which are schemes (i.e., objects contained in the subcategory  $\mathfrak{Sch}_{/X^\otimes}$ ) may be characterized category-theoretically as those objects which satisfy the following condition:*

(D) <sub>$Y^\otimes$</sub> : *For any minimal object  $W$  over  $X^\otimes$  (hence  $W \cong \text{Spec}(k)$  for some field  $k$  by Proposition 2.5, (i)), the map*

$$(61) \quad \text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes}(W, Y^\otimes) \rightarrow \text{Map}_{\mathfrak{Sch}_{/X^\otimes}^\otimes}(\mathbb{A}_W^{0|1}, Y^\otimes)$$

*induced from the morphism  $\beta_{\mathbb{A}_W^{0|1}} : \mathbb{A}_W^{0|1} \rightarrow W$  is bijective.*

*In particular, the full subcategory  $\mathfrak{Sch}_{/X^\otimes}$  of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  may be reconstructed category-theoretically.*

*Proof.* The assertion is a formal consequence of Nakayama’ lemma and Lemma 2.2. □

Moreover, by the above proposition, one may have, for each  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ , a category-theoretic reconstruction of the schematic structure of  $Y_t$ , as follows.

**Corollary 2.10** (Characterization of  $Y_t$  for  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ ). *Let  $Y^\otimes$  be an object of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ .*

(i) *The object  $Y_t \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$  may be characterized (up to isomorphism) category-theoretically as the object  $Z^\otimes$  of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  which is a scheme (i.e., satisfies the condition  $(D)_{Z^\otimes}$  in Proposition 2.9) and satisfies the following condition:*

$(E)_{Z^\otimes}$ : *For any object  $W$  in  $\mathfrak{Sch}_{/Y^\otimes} (\subseteq \mathfrak{Sch}_{/Y^\otimes}^\otimes)$ , there exists uniquely a morphism  $W \rightarrow Z^\otimes$ .*

(ii) *The schematic structure of  $Y_t$  (i.e., a topological space together with a sheaf of rings on it), as well as the topological structure of (the underlying space of)  $Y_b$  may be reconstructed (up to isomorphism) category-theoretically from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ , i.e., a pair consisting of a category and an object of it.*

*Proof.* Assertion (i) follows from the functorial bijection (18). Assertion (ii) follows from [5], Theorem A, and the fact that the morphism of topological spaces underlying  $\gamma_Y : Y_t \rightarrow Y_b$  is a homeomorphism. Indeed, we may reconstruct (un to equivalence) the category  $\mathfrak{Sch}_{/Y_t} (\cong \mathfrak{Sch}_{/Y^\otimes})$  from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$  by Proposition 2.9 and assertion (i). □

**Proposition 2.11** (Characterization of  $X^\otimes|_U$  for an open  $U$ ). *Let  $Y^\otimes$  be an object of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  and  $\bar{U}$  a quasi-compact open subscheme of  $Y_t$ . Denote by  $U$  the (quasi-compact) open subscheme of  $Y_b$  with  $\gamma_Y^{-1}(U) = \bar{U}$  (cf. (13)). Then, the object  $Y^\otimes|_U$  of  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  may be characterized (up to isomorphism) category-theoretically as the object  $Z^\otimes$  of  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  which satisfies the following condition:*

$(F)_{Z^\otimes, \bar{U}}$ : *For any object  $f^\otimes : W^\otimes \rightarrow Y^\otimes$  of  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  such that the image of  $f_t : W_t \rightarrow Y_t$  lies in  $U$ , there exists uniquely a morphism  $W^\otimes \rightarrow Z^\otimes$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$ .*

*Consequently, the collection of objects in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  consisting of quasi-compact open subschemes of  $X^\otimes$  may be characterized as the objects  $V^\otimes$  such that for any  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ , the fiber product  $V^\otimes \times_{X^\otimes} Y^\otimes$  satisfies the condition  $(F)_{V^\otimes \times_{X^\otimes} Y^\otimes, \bar{U}}$  for some open subscheme  $\bar{U}$  of  $Y_t$ .*

*Proof.* This is a formal consequence of the definition of a quasi-compact open subscheme. □

2.4.

Next, we consider the reconstruction (cf. Proposition 2.12, Lemma 2.13, and Lemma 2.14 below) of the ring object  $\mathbb{A}_{X^\otimes}^{1|0}$  over  $X^\otimes$  (more precisely, the objects  $\mathbb{A}_{Y^\otimes}^{1|0}$  for various  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ ) corresponding to the ring structure of  $\mathcal{O}_{X_b}$ .

**Proposition 2.12** (Characterization of  $\mathbb{A}_{Y^\otimes}^{1|0}$  for  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ ). *Let  $Y^\otimes$  be an object of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ . Also, let*

$$(62) \quad \mathfrak{z} := (Z^\otimes, \sigma^{0\otimes}, \sigma^{1\otimes})$$

*be a triple consisting of an object  $Z^\otimes$  of  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  and two sections  $\sigma^{0\otimes}, \sigma^{1\otimes} : Y^\otimes \rightarrow Z^\otimes$  of the structure morphism  $Z^\otimes \rightarrow Y^\otimes$  of  $Z^\otimes$ . Then,  $\mathfrak{z}$  is isomorphic to the triple*

$$(63) \quad \mathfrak{a}_Y := (\mathbb{A}_{Y^\otimes}^{1|0}, \sigma_{Y^\otimes}^{[0]}, \sigma_{Y^\otimes}^{[1]})$$

*(more precisely, there exists an isomorphism  $h^\otimes : Z^\otimes \xrightarrow{\sim} \mathbb{A}_{Y^\otimes}^{1|0}$  over  $Y^\otimes$  satisfying the equalities  $h^\otimes \circ \sigma^{0\otimes} = \sigma_{Y^\otimes}^{[0]}$  and  $h^\otimes \circ \sigma^{1\otimes} = \sigma_{Y^\otimes}^{[1]}$ ) if and only if it satisfies the following three conditions  $(G)_\mathfrak{z}$ - $(I)_\mathfrak{z}$ :*

- $(G)_\mathfrak{z}$ : *The fiber product  $Z^\otimes \times_{Y^\otimes} Y_t$  is isomorphic (over  $Y_t$ ) to the scheme  $\mathbb{A}_{Y_t}^{1|0}$  (which may be reconstructed by Corollary 2.10, (ii));*
- $(H)_\mathfrak{z}$ : *Suppose that we are given an arbitrary commutative square diagram*

$$(64) \quad \begin{array}{ccc} W_0^\otimes & \longrightarrow & Z^\otimes \\ \downarrow & & \downarrow \\ W_1^\otimes & \longrightarrow & Y^\otimes \end{array}$$

*in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  such that  $W_1^\otimes$  is one-pointed and  $W_0^\otimes$  is terminally minimal over both  $W_1^\otimes$  and  $Z^\otimes$ . Then, there exists a morphism  $g^\otimes : W_1^\otimes \rightarrow Z^\otimes$  making the diagram*

$$(65) \quad \begin{array}{ccc} W_0^\otimes & \longrightarrow & Z^\otimes \\ \downarrow & \nearrow g^\otimes & \downarrow \\ W_1^\otimes & \longrightarrow & Y^\otimes \end{array}$$

*commute;*

(I)<sub>3</sub>: The fiber product  $Y^{\otimes} \times_{\sigma^0, Z^{\otimes}, \sigma^1} Y^{\otimes}$  is empty.

*Proof.* One may verify immediately that the triple  $\mathfrak{a}_Y$  satisfies the three conditions (G)<sub>a<sub>Y</sub></sub>, (H)<sub>a<sub>Y</sub></sub>, and (I)<sub>a<sub>Y</sub></sub>. Hence, it suffices to prove its reverse direction.

Let  $\mathfrak{z} := (Z^{\otimes}, \sigma^{0\otimes}, \sigma^{1\otimes})$  be a triple satisfying the required three conditions. To begin with, we shall prove the claim that  $Z^{\otimes}$  is, Zariski locally on  $Y_b$ , isomorphic to  $\mathbb{A}_{Y^{\otimes}}^{1|0}$ . Let  $y$  be a closed point of  $Y_b$  and write  $Y'^{\otimes} := Y^{\otimes} \times_{Y_b} \text{Spec}(\mathcal{O}_{Y_b, y})$  and  $Z'^{\otimes} := Z^{\otimes} \times_{Y_b} \text{Spec}(\mathcal{O}_{Y_b, y})$ . By the condition (G)<sub>z</sub>, the fiber of the natural morphism  $Z'^{\otimes} \rightarrow Y'^{\otimes}$  at  $y$  is isomorphic to  $\mathbb{A}_y^{1|0}$ . Let us take a morphism  $f^{\otimes} : Z'^{\otimes} \rightarrow \mathbb{A}_{Y'^{\otimes}}^{1|0}$  over  $Y'^{\otimes}$  whose restriction to the fibers at  $y$  is an isomorphism. (Such a morphism necessarily exists due to the universal property of the polynomial ring  $\mathcal{O}_{Y'^{\otimes}}[t]$  with coefficients in  $\mathcal{O}_{Y'^{\otimes}}$ .) To complete the proof of the claim, it suffices to prove that  $f^{\otimes}$  is an isomorphism. Let  $z$  be an arbitrary point of  $Z'_b$  lying over  $y$ . Write

$$(66) \quad f_z^b : (\mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)}, \mathfrak{m}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)}) \rightarrow (\mathcal{O}_{Z'^{\otimes}, z}, \mathfrak{m}_{Z'^{\otimes}, z})$$

(where  $\mathfrak{m}_{(-)}$  denotes the maximal ideal) for the homomorphism of local rings defined by  $f^{\otimes}$  and (for each  $i \geq 1$ ) write

$$(67) \quad f_z^{b,i} : \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)} / \mathfrak{m}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)}^i \rightarrow \mathcal{O}_{Z'^{\otimes}, z} / \mathfrak{m}_{Z'^{\otimes}, z}^i$$

for the induced homomorphism. By the definition of  $f^{\otimes}$  and Nakayama's lemma for noncommutative rings, all  $f_z^b$  and  $f_z^{b,i}$  ( $i = 1, 2, \dots$ ) are surjective. We shall show that  $f_z^b$  is also injective. Suppose that  $f_z^b$  is not injective. One verifies, like as the case of commutative rings, that  $\bigcap_{i \geq 1} \mathfrak{m}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)}^i = 0$ .

Hence, there exists  $i \geq 1$  for which  $f_z^{b,i}$  is not injective. By the condition (H)<sub>z</sub>, there exists a homomorphism

$$(68) \quad g^b : \mathcal{O}_{Z'^{\otimes}, z} \rightarrow \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)} / \mathfrak{m}_{Y_b, y}^i \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)}$$

which makes the following diagram

$$(69) \quad \begin{array}{ccc} \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)} / \mathfrak{m}_{Y_b, y} \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)} & \longleftarrow & \mathcal{O}_{Z'^{\otimes}, z} \\ \uparrow & \swarrow g^b & \uparrow \\ \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)} / \mathfrak{m}_{Y_b, y}^i \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0}, f_b(z)} & \longleftarrow & \mathcal{O}_{Y'^{\otimes}, y} \end{array}$$

commute, where the upper horizontal arrow denotes the composite of the quotient  $\mathcal{O}_{Z'^{\otimes},z} \twoheadrightarrow \mathcal{O}_{Z'^{\otimes},z}/\mathfrak{m}_{Z'^{\otimes},z}$  and the isomorphism  $(f_z^{b,1})^{-1}$ . This homomorphism  $g^b$  factors through the quotient  $\mathcal{O}_{Z'^{\otimes},z} \twoheadrightarrow \mathcal{O}_{Z'^{\otimes},z}/\mathfrak{m}_{Z'^{\otimes},z}^i$ . The resulting homomorphism

$$(70) \quad \mathcal{O}_{Z'^{\otimes},z}/\mathfrak{m}_{Z'^{\otimes},z}^i \rightarrow \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0},f_b(z)}/\mathfrak{m}_{\mathbb{A}_{Y'^{\otimes}}^{1|0},f_b(z)}^i$$

becomes a split injection of  $f_z^{b,i}$ . Thus, we have

$$(71) \quad \mathcal{O}_{\mathbb{A}_{Y'^{\otimes}}^{1|0},f_b(z)}/\mathfrak{m}_{\mathbb{A}_{Y'^{\otimes}}^{1|0},f_b(z)}^i \cong (\mathcal{O}_{Z'^{\otimes},z}/\mathfrak{m}_{Z'^{\otimes},z}^i) \oplus \text{Ker}(f_z^{b,i}),$$

which contradicts the fact that  $f_z^{b,1}$  is an isomorphism. Consequently,  $f_z^b$  is an isomorphism (for any  $z$ ), that is to say,  $f^{\otimes}$  is an isomorphism. This completes the proof of the claim.

Finally, it follows immediately from the condition  $(I)_3$  and a standard argument that  $Z^{\otimes}$  is isomorphic to  $\mathbb{A}_{Y^{\otimes}}^{1|0}$ . This completes the proof of Proposition 2.12.  $\square$

Let  $Y^{\otimes}$  be an object of  $\mathfrak{Sch}_{/X^{\otimes}}^{\otimes}$ . We shall define a functor

$$(72) \quad (\mathbb{G}_m)_{Y^{\otimes}} : \mathfrak{Sch}_{/Y^{\otimes}}^{\otimes} \rightarrow \mathfrak{Grp}$$

(where  $\mathfrak{Grp}$  denotes the category of groups) to be the functor which, to any object  $Z^{\otimes}$  of  $\mathfrak{Sch}_{/Y^{\otimes}}^{\otimes}$ , assigns the group of automorphisms of  $\mathbb{A}_{Z^{\otimes}}^{1|0}$  over  $Z^{\otimes}$  that are compatible with  $\sigma_{Z^{\otimes}}^{[0]} : Z^{\otimes} \rightarrow \mathbb{A}_{Z^{\otimes}}^{1|0}$ . It may be represented uniquely (up to a canonical isomorphism) by an object of  $\mathfrak{Sch}_{/Y^{\otimes}}^{\otimes}$ , which we also denote by  $(\mathbb{G}_m)_{Y^{\otimes}}$  by abuse of notation. (Indeed, the open subsuperscheme  $\mathbb{A}_{Y^{\otimes}}^{1|0}|_{\mathbb{A}_{Y^{\otimes}}^{1|0} \setminus \text{Im}((\sigma_{Y^{\otimes}}^{[0]})_b)}$  of  $\mathbb{A}_{Y^{\otimes}}^{1|0}$  represents this functor.) Write

$$(73) \quad \mu_{Y^{\otimes}}^{\mathbb{G}_m} : (\mathbb{G}_m)_{Y^{\otimes}} \times_{Y^{\otimes}} (\mathbb{G}_m)_{Y^{\otimes}} \rightarrow (\mathbb{G}_m)_{Y^{\otimes}}$$

for the multiplication morphism of  $(\mathbb{G}_m)_{Y^{\otimes}}$ , and write

$$(74) \quad \mu_{Y^{\otimes}}^{\dagger} : (\mathbb{G}_m)_{Y^{\otimes}} \times_{Y^{\otimes}} \mathbb{A}_{Y^{\otimes}}^{1|0} \rightarrow \mathbb{A}_{Y^{\otimes}}^{1|0}$$

for the natural action of  $(\mathbb{G}_m)_{Y^{\otimes}}$  on  $\mathbb{A}_{Y^{\otimes}}^{1|0}$ . The morphism  $\mu_{Y^{\otimes}}^{\dagger}$  induces a morphism

$$(75) \quad \nu_{Y^\otimes} \left( := \mu_{Y^\otimes}^\dagger \circ (\text{id}_{(\mathbb{G}_m)_{Y^\otimes}} \times \sigma_{Y^\otimes}^{[1]}) \right) : (\mathbb{G}_m)_{Y^\otimes} \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}$$

which is an open immersion. It follows from Proposition 2.12 that the group object  $(\mathbb{G}_m)_{Y^\otimes}$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  and the morphisms  $\mu_{Y^\otimes}^\dagger$  and  $\nu_{Y^\otimes}$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  may be reconstructed (up to isomorphism) category-theoretically from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ . The following two lemmas will be used in the proof of Corollary 2.15 below.

**Lemma 2.13.** *Denote by*

$$(76) \quad \mu_{Y^\otimes}^{1|0} : \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}$$

*the morphism corresponding to the multiplication of  $\mathcal{O}_{Y_b}$  (via the functorial bijection (34)). Then, a morphism  $\mu^\otimes : \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  coincides with  $\mu_{Y^\otimes}^{1|0}$  if and only if it satisfies the following condition:*

*(J) $_{\mu^\otimes}$ : the equality*

$$(77) \quad \mu^\otimes \circ (\nu_{Y^\otimes} \times \nu_{Y^\otimes}) = \nu_{Y^\otimes} \circ \mu_{Y^\otimes}^{\mathbb{G}_m}$$

*of morphisms  $(\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} (\mathbb{G}_m)_{Y^\otimes} \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}$  holds;*

*Consequently, the morphism  $\mu_{Y^\otimes}^{1|0}$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  may be reconstructed category-theoretically (up to isomorphism) from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ .*

*Proof.* Since the equality  $\mu_{Y^\otimes}^{1|0} \circ (\nu_{Y^\otimes} \times \nu_{Y^\otimes}) = \nu_{Y^\otimes} \circ \mu_{Y^\otimes}^{\mathbb{G}_m}$  holds, the assertion follows directly from the fact that  $\nu_{Y^\otimes} \times \nu_{Y^\otimes}$  is an epimorphism in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$ . □

**Lemma 2.14.** *Denote by*

$$(78) \quad \alpha_{Y^\otimes}^{1|0} : \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}$$

*the morphism corresponding to the addition of  $\mathcal{O}_{Y_b}$  (via the functorial bijection (34)). Then, a morphism  $\alpha^\otimes : \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \rightarrow \mathbb{A}_{Y^\otimes}^{1|0}$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  coincides with  $\alpha_{Y^\otimes}^{1|0}$  if and only if it satisfies the following two conditions  $(K)_{\alpha^\otimes}$  and  $(L)_{\alpha^\otimes}$ :*



$(K)_{\alpha^\otimes}$ : The square diagram

(79)

$$\begin{array}{ccc}
 (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} & \xrightarrow{\text{id}_{(\mathbb{G}_m)_{Y^\otimes}} \times \alpha^\otimes} & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \\
 (\mu_{Y^\otimes}^\dagger \times \mu_{Y^\otimes}^\dagger) \circ \lambda^\otimes \downarrow & & \downarrow \mu_{Y^\otimes}^\dagger \\
 \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} & \xrightarrow{\alpha^\otimes} & \mathbb{A}_{Y^\otimes}^{1|0}
 \end{array}$$

is commutative, where  $\lambda^\otimes$  denotes the morphism

(80)

$$\begin{array}{ccc}
 (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} & \rightarrow & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \\
 (g, a_1, a_2) & \mapsto & (g, a_1, g, a_2)
 \end{array}$$

over  $Y^\otimes$ .

$(L)_{\alpha^\otimes}$ : We have the equalities

$$(81) \quad \alpha^\otimes \circ (\sigma_{Y^\otimes}^{[0]} \times \text{id}_{\mathbb{A}_{Y^\otimes}^{1|0}}) = \alpha^\otimes \circ (\text{id}_{\mathbb{A}_{Y^\otimes}^{1|0}} \times \sigma_{Y^\otimes}^{[0]}) = \text{id}_{\mathbb{A}_{Y^\otimes}^{1|0}}.$$

of endomorphisms of  $\mathbb{A}_{Y^\otimes}^{1|0}$ .

Consequently, the morphism  $\alpha_{Y^\otimes}^{1|0}$  in  $\mathfrak{Sch}_{Y^\otimes}^\otimes$  may be reconstructed category-theoretically (up to isomorphism) from the data  $(\mathfrak{Sch}_{X^\otimes}^\otimes, Y^\otimes)$ .

*Proof.* Let  $\alpha^\otimes$  be a morphism satisfying the conditions  $(K)_{\alpha^\otimes}$  and  $(L)_{\alpha^\otimes}$ . We write  $\alpha^\flat : \mathcal{O}_{Y^\otimes}[t] \rightarrow \mathcal{O}_{Y^\otimes}[t] \otimes_{\mathcal{O}_{Y^\otimes}} \mathcal{O}_{Y^\otimes}[t]$  for the homomorphism of superalgebras over  $\mathcal{O}_{Y^\otimes}$  corresponding to  $\alpha^\otimes$ . The condition  $(K)_{\alpha^\otimes}$  implies that  $\alpha^\flat$  is given by  $t \mapsto a \cdot t \otimes 1 + b \cdot 1 \otimes t$  for some  $a, b \in \Gamma(Y_b, \mathcal{O}_{Y_b})$ . Moreover, the equalities in  $(L)_{\alpha^\otimes}$  imply that  $a = b = 1$ , that is to say,  $\alpha^\otimes = \alpha_{Y^\otimes}^{1|0}$ . Thus, we complete the proof of Lemma 2.14.  $\square$

### 2.5.

By combining the results in §2.3 and §2.4, one may reconstruct category-theoretically the schematic structure of  $X_b$  as follows.

**Corollary 2.15** (Characterization of  $Y_b$  for  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{X^\otimes}^\otimes)$ ). *Let  $Y^\otimes$  be an object of  $\mathfrak{Sch}_{X^\otimes}^\otimes$ . Then, the schematic structure of  $Y_b$  (i.e., a topological*

space together with a sheaf of rings on it) may be reconstructed category-theoretically (up to isomorphism) from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ . Moreover, this reconstruction is functorial (in the evident sense) in  $Y^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ ; strictly speaking, if we are given a morphism  $f^\otimes : Z^\otimes \rightarrow Y^\otimes$  in  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ , then (the two schemes  $Y_b, Z_b$  and) its underlying morphism  $f_b : Z_b \rightarrow Y_b$  may be reconstructed category-theoretically.

*Proof.* By Corollary 2.10 and Proposition 2.11, one may reconstruct (up to equivalence) category-theoretically the topological structure of  $X_b$  and the full subcategory of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$  whose objects are the collection

$$(82) \quad \left\{ X^\otimes|_U \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes) \mid U \text{ is a quasi-compact open subscheme of } X_b \right\}.$$

Moreover, it follows from Proposition 2.12, Lemma 2.13, and Lemma 2.14 that one may reconstruct ring objects  $\mathbb{A}_{X^\otimes|_U}^{1|0} \in \text{Ob}(\mathfrak{Sch}_{X^\otimes}^\otimes)$  (for each quasi-compact open  $U$  in  $X_b$ ) over  $X^\otimes|_U$  corresponding to  $\mathcal{O}_U$ . By considering the set of various sections  $X^\otimes|_U \rightarrow \mathbb{A}_{X^\otimes|_U}^{1|0}$ , we obtain the ring structure of  $\Gamma(U, \mathcal{O}_{X_b})$  that is compatible with restriction to open subschemes of  $U$ . Consequently, the schematic structure of  $X_b$  may be reconstructed, as desired. The latter assertion follows from this reconstruction procedure.  $\square$

### 2.6.

In this subsection, we consider the reconstruction of the various  $\mathbb{A}^{0|1}$ -twists associated with fermionic twists of  $X^\otimes$ , together with the multiplication and addition maps. As a consequence, one may reconstruct (cf. Corollary 2.20) the schematic structure of superschemes  $Z^\otimes$  with  $Z^\otimes \overset{f}{\sim} X^\otimes$ .

Let us fix an object  $Y^\otimes$  of  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ .

**Proposition 2.16** (Characterization of  $\mathbb{A}^{0|1}$ -twists). *Let  $(Z^\otimes, \sigma^\otimes)$  be a pair consisting of an object  $Z^\otimes$  of  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  (i.e., a morphism  $f^\otimes : Z^\otimes \rightarrow Y^\otimes$ ) and a morphism  $\sigma^\otimes : Y^\otimes \rightarrow Z^\otimes$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  (i.e., a section  $\sigma^\otimes$  of  $f^\otimes$ ). Then, the pair  $(Z^\otimes, \sigma^\otimes)$  forms an  $\mathbb{A}^{0|1}$ -twists over  $Y^\otimes$  if and only if it satisfies the following three conditions  $(M)_{Z^\otimes, \sigma^\otimes} - (O)_{Z^\otimes, \sigma^\otimes}$ :*

$(M)_{Z^\otimes, \sigma^\otimes}$ : *The underlying morphism  $f_b : Z_b \rightarrow Y_b$  of schemes (which may be reconstructed category-theoretically from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, f^\otimes)$  by Corollary 2.15) is finite;*

- (N)<sub>Z<sup>⊗</sup>,σ<sup>⊗</sup></sub>: For each minimal object W<sup>⊗</sup> over Y<sup>⊗</sup>, the fiber product Z<sup>⊗</sup> ×<sub>Y<sup>⊗</sup></sub> W<sup>⊗</sup> is isomorphic to  $\mathbb{A}_{W^{\otimes}}^{0|1}$  (which may be reconstructed category-theoretically from the data (Sch<sub>Y<sup>⊗</sup></sub><sup>⊗</sup>, W<sup>⊗</sup>) by Proposition 2.8);
- (O)<sub>Z<sup>⊗</sup>,σ<sup>⊗</sup></sub>: Let Y'<sup>⊗</sup> be an open subscheme of Y<sup>⊗</sup> (i.e., an object Y'<sup>⊗</sup> of Sch<sub>Y<sup>⊗</sup></sub><sup>⊗</sup> satisfying the condition (F)<sub>Y'<sup>⊗</sup>, $\bar{U}$</sub>  in Proposition 2.11 for some open subscheme  $\bar{U}$  of Y<sub>t</sub>). Also, let (Z'<sup>⊗</sup>, σ'<sup>⊗</sup>) be a pair, where Z'<sup>⊗</sup> denotes an object in Sch<sub>Y'<sup>⊗</sup></sub><sup>⊗</sup> and σ'<sup>⊗</sup> denotes a morphism Y'<sup>⊗</sup> → Z'<sup>⊗</sup> in Sch<sub>Y'<sup>⊗</sup></sub><sup>⊗</sup>, satisfying the conditions (M)<sub>Z'<sup>⊗</sup>,σ'<sup>⊗</sup></sub> and (N)<sub>Z'<sup>⊗</sup>,σ'<sup>⊗</sup></sub>. Then, there exists an open subscheme Y''<sup>⊗</sup> of Y'<sup>⊗</sup> and a monomorphism h<sup>⊗</sup> : Z'<sup>⊗</sup> ×<sub>Y'<sup>⊗</sup></sub> Y''<sup>⊗</sup> → Z<sup>⊗</sup> ×<sub>Y<sup>⊗</sup></sub> Y''<sup>⊗</sup> in Sch<sub>Y''<sup>⊗</sup></sub><sup>⊗</sup> satisfying the equality of morphisms

$$(83) \quad h^{\otimes} \circ (\sigma'^{\otimes} \times \text{id}_{Y''^{\otimes}}) = \sigma^{\otimes} \times \text{id}_{Y''^{\otimes}} : Y''^{\otimes} \rightarrow Z^{\otimes} \times_{Y^{\otimes}} Y''^{\otimes}.$$

Consequently, the collection of objects in Sch<sub>Y<sup>⊗</sup></sub><sup>⊗</sup> which are isomorphic to  $\mathbb{A}^{0|1}$ -twists over Y<sup>⊗</sup> may be reconstructed category-theoretically (up to isomorphism) from the data (Sch<sub>X<sup>⊗</sup></sub><sup>⊗</sup>, Y<sup>⊗</sup>).

*Proof.* Let (Z<sup>⊗</sup>, σ<sup>⊗</sup>) be a pair satisfying the required three conditions. By the existence of a section σ<sup>⊗</sup> and the condition (N)<sub>Z<sup>⊗</sup>,σ<sup>⊗</sup></sub>, the underlying continuous map of f<sup>⊗</sup> is a homeomorphism (hence, we consider  $\mathcal{O}_{Z^{\otimes}}$  as a sheaf on the underlying topological space of Y<sup>⊗</sup>). The conditions (M)<sub>Z<sup>⊗</sup>,σ<sup>⊗</sup></sub> implies that  $\mathcal{O}_{Z^{\otimes}}$  is a finite  $\mathcal{O}_{Y^{\otimes}}$ -module. It follows from the condition (N)<sub>Z<sup>⊗</sup>,σ<sup>⊗</sup></sub> and Nakayama’s lemma that one may find, locally on Y<sub>b</sub>, an isomorphism  $\mathcal{O}_{Z^{\otimes}} \xrightarrow{\sim} \mathcal{O}_{Y^{\otimes}} \oplus (\mathcal{O}_{Y^{\otimes}}/\mathcal{I})$  of  $\mathcal{O}_{Y^{\otimes}}$ -superalgebras, where the multiplication of the right-hand side is given by  $(a, \bar{b}) \cdot (c, \bar{d}) = (ac, a\bar{d} + c\bar{b})$ . Moreover, the universal property described in (O)<sub>Z<sup>⊗</sup>,σ<sup>⊗</sup></sub> implies that  $\mathcal{I} = 0$ . Consequently, (Z<sup>⊗</sup>, σ<sup>⊗</sup>) forms an  $\mathbb{A}^{0|1}$ -twist over Y<sup>⊗</sup>. Since the reverse direction of this assertion may be verified immediately, we complete the proof of Proposition 2.16. □

Next, let us fix an  $\mathbb{A}^{0|1}$ -twist (Z<sup>⊗</sup>, σ<sub>Z<sup>⊗</sup>/Y<sup>⊗</sup></sub>) over Y<sup>⊗</sup>.

**Lemma 2.17.** *We shall write*

$$(84) \quad \text{Aut}_{Y^{\otimes}}(Z^{\otimes}, \sigma_{Z^{\otimes}/Y^{\otimes}}) : \text{Sch}_{Y^{\otimes}}^{\otimes} \rightarrow \mathfrak{Grp}$$

for the functor which, to any W<sup>⊗</sup> ∈ Ob(Sch<sub>Y<sup>⊗</sup></sub><sup>⊗</sup>), assigns the automorphism group of the  $\mathbb{A}^{0|1}$ -twist (Z<sup>⊗</sup> ×<sub>Y<sup>⊗</sup></sub> W<sup>⊗</sup>, σ<sub>Z<sup>⊗</sup>/Y<sup>⊗</sup></sub> × id<sub>W<sup>⊗</sup></sub>) over W<sup>⊗</sup>. Consider

the isomorphism

$$(85) \quad \eta_{Z^\otimes} : (\mathbb{G}_m)_{Y^\otimes} \xrightarrow{\sim} \mathcal{A}ut_{Y^\otimes}(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$$

which, to any automorphism in  $(\mathbb{G}_m)_{Y^\otimes}(W^\otimes)$  (where  $W^\otimes \in \text{Ob}(\mathfrak{Sch}_{Y^\otimes}^\otimes)$ ) corresponding to the automorphism of  $\mathcal{O}_{W^\otimes}[t]$  determined by  $t \mapsto g \cdot t$  (where  $g \in \Gamma(W_b, \mathcal{O}_{W_b}^\times)$ ), assigns the automorphism of  $(Z^\otimes \times_{Y^\otimes} W^\otimes, \sigma_{Z^\otimes/Y^\otimes} \times \text{id}_{W^\otimes})$  corresponding to the automorphism of  $\mathcal{O}_{Z^\otimes \times_{Y^\otimes} W^\otimes}$  (which is locally isomorphic to  $\mathcal{O}_{W^\otimes}[\psi]$ ) determined by  $\psi \mapsto g \cdot \psi$ . Then, an isomorphism  $\eta^\otimes : (\mathbb{G}_m)_{Y^\otimes} \xrightarrow{\sim} \mathcal{A}ut_{Y^\otimes}(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  coincides with  $\eta_{Z^\otimes}$  if and only if it satisfies the following condition:

(P) $_{\eta^\otimes}$ : Let  $W^\otimes$  be an object of  $\mathfrak{Sch}_{Y^\otimes}^\otimes$  and  $h^\otimes$  an automorphism in  $(\mathbb{G}_m)_{Y^\otimes}(W^\otimes)$  such that the induced automorphism of  $\mathcal{O}_{W_b}[t]$  ( $= \mathcal{O}_{(\mathbb{A}_W^{1|0})_b}$ ) is given by  $t \mapsto g \cdot t$  for some  $g \in \Gamma(\mathcal{O}_{W_b}, \mathcal{O}_{W_b}^\times)$ . (Such a pair  $(W^\otimes, h^\otimes)$  may be characterized category-theoretically thanks to Corollary 2.15.) Here, note that the section

$$(86) \quad (\sigma_{Z^\otimes/Y^\otimes}|_{W^\otimes}, \sigma_{Z^\otimes/Y^\otimes}|_{W^\otimes}) : W^\otimes \rightarrow Z_W^\otimes \times_{W^\otimes} Z_W^\otimes$$

(where  $Z_W^\otimes := Z^\otimes \times_{Y^\otimes} W^\otimes$ ) determines a decomposition  $\mathcal{O}_{(Z_W^\otimes \times_{W^\otimes} Z_W^\otimes)_b} \cong \mathcal{O}_{W_b} \oplus \mathcal{O}_{W_b} \epsilon$ , where the multiplication of the right-hand side is given by  $(a, b\epsilon) \cdot (c, d\epsilon) = (ac, (bc + ad)\epsilon)$ . Then, the automorphism  $\eta^\otimes(h^\otimes) \times \eta^\otimes(h^\otimes)$  of  $Z_W^\otimes \times_{W^\otimes} Z_W^\otimes$  induces the automorphism of  $\mathcal{O}_{W_b} \oplus \mathcal{O}_{W_b} \epsilon$  given by assigning  $(a, b\epsilon) \mapsto (a, g^2 \cdot b\epsilon)$ .

Consequently, the morphism  $\eta_{Z^\otimes}$  in  $\mathfrak{Sch}_{Y^\otimes}^\otimes$  may be reconstructed category-theoretically (up to isomorphism) from  $(\mathfrak{Sch}_{X^\otimes}^\otimes, Y^\otimes, (Z^\otimes, \sigma_{Z^\otimes/Y^\otimes}))$ , i.e., a collection of data consisting of a category  $\mathfrak{Sch}_{X^\otimes}^\otimes$ , an object  $Y^\otimes$  of it, and a pair  $(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  satisfying the conditions described in Proposition 2.16.

*Proof.* The assertion follows from the various definitions involved.  $\square$

We shall write

$$(87) \quad \mu\eta_{Z^\otimes}^\dagger : (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}$$

for the action of  $(\mathbb{G}_m)_{Y^\otimes}$  on  $\mathbb{A}_{Z^\otimes}^{1|0}$  ( $\cong \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} Z^\otimes$ ) defined by

$$(88) \quad \begin{aligned} (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} Z^\otimes &\rightarrow \mathbb{A}_{Y^\otimes}^{1|0} \times_{Y^\otimes} Z^\otimes \\ (g, a, b) &\mapsto (\mu_{Y^\otimes}^\dagger(g, a), \eta_{Z^\otimes}(g, b)). \end{aligned}$$

According to Proposition 2.12, Lemma 2.17, and the discussion preceding Lemma 2.13, this action may be reconstructed category-theoretically from  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes, (Z^\otimes, \sigma_{Z^\otimes/Y^\otimes}))$ .

**Lemma 2.18.** *Let  $\mu^\otimes : \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}$  be a morphism in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  and consider the following condition concerning  $\mu^\otimes$ :*

$(Q)_{\mu^\otimes}$ : *There exists a fermionic twist  $W^\otimes$  of  $Y^\otimes$  satisfying that the  $\mathbb{A}^{0|1}$ -twist  $(\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}, \sigma_{\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}})$  (cf. (42)) over  $Y^\otimes$  associated with  $W^\otimes$  coincides with  $(Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  and the equality  $\mu^\otimes = \mu_{Y^\otimes \rightsquigarrow W^\otimes}$  holds.*

*Then, the above condition  $(Q)_{\mu^\otimes}$  is equivalent that  $\mu^\otimes$  satisfies the following four conditions  $(R)_{\mu^\otimes}$ - $(U)_{\mu^\otimes}$ :*

$(R)_{\mu^\otimes}$ : *The square diagram*

(89)

$$\begin{array}{ccc} (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\text{id}_{(\mathbb{G}_m)_{Y^\otimes}} \times \mu^\otimes} & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \\ (\mu\eta_{Z^\otimes}^\dagger \times \mu\eta_{Z^\otimes}^\dagger) \circ \lambda^\otimes \downarrow & & \downarrow \mu\eta_{Z^\otimes}^\dagger V \\ \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\mu^\otimes} & \mathbb{A}_{Z^\otimes}^{1|0} \end{array}$$

*is commutative, where  $\mu\eta_{Z^\otimes}^\dagger$  denotes the action of  $(\mathbb{G}_m)_{Y^\otimes}$  on  $\mathbb{A}_{Z^\otimes}^{1|0}$  given by  $(g, a) \mapsto \mu\eta_{Z^\otimes}^\dagger(g^2, a)$  and  $\lambda^\otimes$  denotes the morphism*

(90)

$$\begin{array}{ccc} \lambda^\otimes : (\mathbb{G}_m)_Y \times_Y \mathbb{A}_Z^{1|0} \times_Y \mathbb{A}_Z^{1|0} & \rightarrow & (\mathbb{G}_m)_Y \times_Y \mathbb{A}_Z^{1|0} \times_Y (\mathbb{G}_m)_Y \times_Y \mathbb{A}_Z^{1|0} \\ (g, a_1, a_2) & \mapsto & (g, a_1, g, a_2); \end{array}$$

$(S)_{\mu^\otimes}$ : *The square diagrams*

(91)

$$\begin{array}{ccc} (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\text{id}_{(\mathbb{G}_m)_{Y^\otimes}} \times \mu^\otimes} & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \\ \mu\eta_{Z^\otimes}^\dagger \times \text{id}_{\mathbb{A}_{Z^\otimes}^{1|0}} \downarrow & & \downarrow \mu\eta_{Z^\otimes}^\dagger \\ \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\mu^\otimes} & \mathbb{A}_{Z^\otimes}^{1|0} \end{array}$$

and

(92)

$$\begin{array}{ccc}
 (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\text{id}_{(\mathbb{G}_m)_{Y^\otimes}} \times \mu^\otimes} & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \\
 (\mu\eta^\dagger_{Z^\otimes} \times \text{id}_{\mathbb{A}_{Z^\otimes}^{1|0}}) \circ \theta^\otimes \downarrow & & \downarrow \mu\eta^\dagger_{Z^\otimes} \\
 \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\mu^\otimes} & \mathbb{A}_{Z^\otimes}^{1|0}
 \end{array}$$

are commutative, where  $\theta^\otimes$  denotes the isomorphism

$$\begin{array}{ccc}
 (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\sim} & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \\
 (g, a_1, a_2) & \mapsto & (g, a_2, a_1);
 \end{array}$$

$(T)_{\mu^\otimes}$ : Let us write

$$(94) \quad p^\otimes := \sigma_{Y^\otimes}^{[1]} \times \sigma_{Z^\otimes/Y^\otimes} : Y^\otimes \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}, \quad q^\otimes := \sigma_{Y^\otimes}^{[0]} \times \text{id}_{Z^\otimes} : Z^\otimes \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}.$$

Then, the following equalities hold:

$$\begin{array}{l}
 (95) \quad \mu^\otimes \circ (p^\otimes \times p^\otimes) = p^\otimes : Y^\otimes \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}; \\
 (96) \quad \mu^\otimes \circ (p^\otimes \times q^\otimes) = q^\otimes : Z^\otimes \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}; \\
 (97) \quad \mu^\otimes \circ (q^\otimes \times p^\otimes) = q^\otimes : Z^\otimes \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}.
 \end{array}$$

Also, it holds the equality

$$(98) \quad \mu^\otimes \circ (q^\otimes \times q^\otimes) = \sigma_{Z^\otimes}^{[0]} \circ \sigma_{Z^\otimes/Y^\otimes} \circ (h^\otimes \times h^\otimes)$$

of morphisms  $Z^\otimes \times_{Y^\otimes} Z^\otimes \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}$ , where  $h^\otimes$  denotes the structure morphism  $Z^\otimes \rightarrow Y^\otimes$  of  $Z^\otimes$ ;

$(U)_{\mu^\otimes}$ : The morphism

$$(99) \quad ((\text{id}_{\mathbb{A}_{Y^\otimes}^{1|0}} \times h^\otimes) \circ \mu^\otimes \circ (q^\otimes \times q^\otimes))_b : (Z^\otimes \times_{Y^\otimes} Z^\otimes)_b \rightarrow (\mathbb{A}_{Y^\otimes}^{1|0})_b$$

is a closed immersion of schemes.

Moreover, if these equivalent conditions are satisfied, then the fermionic twist  $W^\otimes$  resulting from  $(Q)_{\mu^\otimes}$  is uniquely determined up to isomorphism.

Consequently, the objects  $\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}$  (where  $W^\otimes$  is any fermionic twist of  $Y^\otimes$ ) together with morphisms  $\sigma_{\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}}$  and  $\mu_{Y^\otimes \rightsquigarrow W^\otimes}$  may be reconstructed (up to isomorphism) category-theoretically from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ .

*Proof.* Let  $\mu^\otimes$  be a morphism satisfying the required four conditions. It corresponds, Zariski locally on  $Y_b$ , to a homomorphism

$$(100) \quad \mu^\flat : \mathcal{O}_{Y^\otimes}[t, \psi] \rightarrow \mathcal{O}_{Y^\otimes}[t, \psi] \otimes_{\mathcal{O}_{Y^\otimes}} \mathcal{O}_{Y^\otimes}[t, \psi]$$

of  $\mathcal{O}_{Y^\otimes}$ -superalgebras. By the conditions  $(R)_{\mu^\otimes}$  and  $(S)_{\mu^\otimes}$ ,  $\mu^\flat$  may be given by

$$(101) \quad t \mapsto a_1 \cdot t \otimes t + a_2 \cdot \psi \otimes \psi + b_1 \cdot \psi \otimes t + b_2 \cdot t \otimes \psi$$

and

$$(102) \quad \psi \mapsto b_3 \cdot t \otimes t + b_4 \cdot \psi \otimes \psi + a_3 \cdot \psi \otimes t + a_4 \cdot t \otimes \psi,$$

where  $a_i \in \Gamma(Y_b, \mathcal{O}_{Y_b})$  and  $b_i \in \Gamma(Y_b, \mathcal{O}_{Y_b})$  ( $1 \leq i \leq 4$ ). The equality (95) implies that  $a_1 = 1$  and  $b_3 = 0$ . The equality (96) implies that  $b_2 = 0$  and  $a_4 = 1$ . The equality (97) implies that  $b_1 = 0$  and  $a_3 = 1$ . The equality (98) implies that  $b_4 = 0$ . Hence, the morphism (99) corresponds to the homomorphism  $\mathcal{O}_{Y_b}[t] \rightarrow \mathcal{O}_{Y_b} \oplus (\mathcal{O}_{Y_b} \cdot \psi \otimes \psi)$  of  $\mathcal{O}_{Y_b}$ -algebras given by  $t \mapsto a_2 \cdot \psi \otimes \psi$ . But, the condition  $(U)_{\mu^\otimes}$  implies that  $a_2 \in \Gamma(Y_b, \mathcal{O}_{Y_b}^\times)$ . Thus, there exists a Zariski open covering  $\{U_\alpha\}_{\alpha \in I}$  of  $Y_b$  such that the pair  $(\mathbb{A}_{Z^\otimes}^{1|0}, \mu^\otimes)$  may be obtained by gluing the pairs  $(\mathbb{A}_{Y^\otimes|U_\alpha}^{1|1}, \mu_\alpha^\otimes)$  together, where  $\mu_\alpha^\otimes$  denotes the morphism  $\mathbb{A}_{Y^\otimes|U_\alpha}^{1|1} \times_{Y^\otimes|U_\alpha} \mathbb{A}_{Y^\otimes|U_\alpha}^{1|1} \rightarrow \mathbb{A}_{Y^\otimes|U_\alpha}^{1|1}$  corresponding to the homomorphism

$$(103) \quad \begin{aligned} \mathcal{O}_{Y^\otimes|U_\alpha}[t, \psi] &\rightarrow \mathcal{O}_{Y^\otimes|U_\alpha}[t, \psi] \otimes_{\mathcal{O}_{Y^\otimes|U_\alpha}} \mathcal{O}_{Y^\otimes|U_\alpha}[t, \psi] \\ t &\mapsto t \otimes t + s_\alpha \cdot \psi \otimes \psi, \\ \psi &\mapsto t \otimes \psi + \psi \otimes t. \end{aligned}$$

(for some  $s_\alpha \in \Gamma(U_\alpha, \mathcal{O}_{U_\alpha}^\times)$ ). If  $U_{\alpha,\beta} := U_\alpha \cap U_\beta \neq \emptyset$ , then the gluing automorphism  $\xi_{\alpha,\beta}^\otimes$  of  $\mathbb{A}_{Y^\otimes|U_{\alpha,\beta}}^{1|1}$  (over  $\mathbb{A}_{Y^\otimes|U_{\alpha,\beta}}^{1|0}$ ) is given by  $\psi \mapsto t_{\alpha,\beta} \cdot \psi$  for some  $t_{\alpha,\beta} \in \Gamma(U_{\alpha,\beta}, \mathcal{O}_{U_{\alpha,\beta}}^\times)$ . Since  $\xi_{\alpha,\beta}^\otimes$  is compatible with  $\mu_\alpha^\otimes$  and  $\mu_\beta^\otimes$ , we have the equality  $s_\alpha = t_{\alpha,\beta}^2 \cdot s_\beta$ . Hence, we obtain a collection of data  $(\{U_\alpha\}_\alpha, \{s_\alpha\}_\alpha, \{t_{\alpha,\beta}\}_{\alpha,\beta})$  representing an element  $a$  of  $H_{\text{fppf}}^1(Y_b, \mu_2)$  (cf. (28)). One verifies immediately that  $W^\otimes := {}^a Y^\otimes$  becomes the required fermionic twist of  $Y^\otimes$ . This completes the proof of Lemma 2.18.  $\square$

**Lemma 2.19.** *We shall assume that there exist a fermionic twist  $W^\otimes$  of  $Y^\otimes$  and an isomorphism  $h^\otimes : (\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}, \sigma_{\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}}) \xrightarrow{\sim} (Z^\otimes, \sigma_{Z^\otimes/Y^\otimes})$  of  $\mathbb{A}^{0|1}$ -twists. (This assumption may be characterized category-theoretically thanks to Lemma 2.18.) Let  $\alpha^\otimes : \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \rightarrow \mathbb{A}_{Z^\otimes}^{1|0}$  be a morphism in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$ . Then,  $\alpha^\otimes$  coincides with  $\alpha_{Y^\otimes \rightsquigarrow W^\otimes}$  (cf. (45)) via the isomorphism  $h^\otimes \times \text{id}_{\mathbb{A}^{1|0}} : \mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{1|1} \xrightarrow{\sim} \mathbb{A}_{Z^\otimes}^{1|0}$  if and only if  $\alpha^\otimes$  satisfies the following two conditions  $(V)_{\alpha^\otimes}$  and  $(W)_{\alpha^\otimes}$ :*

$(V)_{\alpha^\otimes}$ : *The square diagram*

(104)

$$\begin{array}{ccc}
 (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\text{id}_{(\mathbb{G}_m)_{Y^\otimes}} \times \alpha^\otimes} & (\mathbb{G}_m)_{Y^\otimes} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} \\
 (\mu\eta_{Z^\otimes}^\dagger \times \mu\eta_{Z^\otimes}^\dagger) \circ \lambda^\otimes \downarrow & & \downarrow \mu\eta_{Z^\otimes}^\dagger \\
 \mathbb{A}_{Z^\otimes}^{1|0} \times_{Y^\otimes} \mathbb{A}_{Z^\otimes}^{1|0} & \xrightarrow{\alpha^\otimes} & \mathbb{A}_{Z^\otimes}^{1|0}
 \end{array}$$

is commutative, where  $\lambda^\otimes$  is as defined in (90).

$(W)_{\alpha^\otimes}$ : *We have the equalities*

(105)

$$\alpha^\otimes \circ ((\sigma_{Z^\otimes}^{[0]} \circ \sigma_{Z^\otimes/Y^\otimes}) \times \mathbb{A}_{Z^\otimes}^{1|0}) = \alpha^\otimes \circ (\mathbb{A}_{Z^\otimes}^{1|0} \times (\sigma_{Z^\otimes}^{[0]} \circ \sigma_{Z^\otimes/Y^\otimes})) = \text{id}_{\mathbb{A}_{Z^\otimes}^{1|0}},$$

of endomorphisms of  $\mathbb{A}_{Z^\otimes}^{1|0}$ .

Consequently, the objects  $\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}$  (where  $W^\otimes$  is any fermionic twist of  $Y^\otimes$ ) together with morphisms  $\sigma_{\mathbb{A}_{Y^\otimes \rightsquigarrow W^\otimes}^{0|1}}$  and  $\alpha_{Y^\otimes \rightsquigarrow W^\otimes}$  may be reconstructed (up to isomorphism) category-theoretically from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ .

*Proof.* The assertion follows from an argument similar to the argument in the proof of Lemma 2.14. □

**Corollary 2.20** (Characterization of fermionic twists over  $Y^\otimes$ ). *The collection of fermionic twists over  $Y^\otimes$  (i.e., a collection of topological spaces together with a sheaf of superalgebras) are reconstructed category-theoretically (up to isomorphism) from the data  $(\mathfrak{Sch}_{/X^\otimes}^\otimes, Y^\otimes)$ . Moreover, this reconstruction is functorial (in the evident sense) in  $Y^\otimes \in \mathfrak{Sch}_{/X^\otimes}^\otimes$ .*

*Proof.* The assertion follows from Proposition 2.11, Lemma 2.18, Lemma 2.19, and the discussion in § 1.7 (especially, the isomorphism (46)). □



2.7.

We turn to the proof of the main result of the present paper, i.e., Theorem A. Before beginning the proof, let us first mention the following rigidity property concerning  $\mathfrak{Sch}_{/X^\otimes}^\otimes$ .

**Proposition 2.21.** *Let  $X^\otimes$  and  $X'^\otimes$  be two locally noetherian superschemes. Let*

$$(106) \quad \mathfrak{Isom}(\mathfrak{Sch}_{/X^\otimes}^\otimes, \mathfrak{Sch}_{/X'^\otimes}^\otimes)$$

*denotes the category of equivalences  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes$  and*

$$(107) \quad \text{Isom}(\mathfrak{Sch}_{/X^\otimes}^\otimes, \mathfrak{Sch}_{/X'^\otimes}^\otimes)$$

*denotes the set of isomorphism classes of equivalences  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes$  (i.e., the set of isomorphism classes of objects in  $\mathfrak{Isom}(\mathfrak{Sch}_{/X^\otimes}^\otimes, \mathfrak{Sch}_{/X'^\otimes}^\otimes)$ ). Also, let*

$$(108) \quad \text{Isom}(X'^\otimes, X^\otimes)$$

*denotes the set of isomorphisms of superschemes  $X'^\otimes \xrightarrow{\sim} X^\otimes$ . Consider the map of sets*

$$(109) \quad \text{Isom}(X'^\otimes, X^\otimes) \rightarrow \text{Isom}(\mathfrak{Sch}_{/X^\otimes}^\otimes, \mathfrak{Sch}_{/X'^\otimes}^\otimes)$$

*which, to any isomorphism  $f^\otimes : X'^\otimes \xrightarrow{\sim} X^\otimes$ , assigns (the isomorphism class of) the equivalence  $\mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes$  given by base-change via  $f^\otimes$ . Then, this map (109) is injective.*

*Proof.* The assertion follows immediately from the functorial bijection (34) and the various reconstruction procedures involved. □

**Remark 2.22.** Unlike the case of schemes proved in [5], Theorem 1.7 (ii), the map (109) may not be surjective. Indeed, suppose that  $X^\otimes = X'^\otimes = Y$  for some scheme  $Y$  and there exists a nonzero element  $a \in H_{\text{fppf}}^1(Y, \mu_2)$ . Then, the assignment  $Z^\otimes \mapsto {}^aZ^\otimes$  defines an autoequivalence  ${}^a\phi : \mathfrak{Sch}_{/Y}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/Y}^\otimes$ . Since  $Z^\otimes$  is, in general, not isomorphic to  ${}^aZ^\otimes$ ,  ${}^a\phi$  is not isomorphic to the identity functor. But, one may verify immediately that  ${}^a\phi$  cannot arise from the base-change via any automorphism of  $Y$ . This implies that the isomorphism class of  ${}^a\phi$  does not lie in the image of the map (109).

Finally, by applying the results obtained so far, we prove the remaining portion of Theorem A (cf. Proposition 1.9) as follows:

*Proof of Theorem A.* Suppose that we are given an equivalence of categories:

$$(110) \quad \phi : \mathfrak{Sch}_{/X^\otimes}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/X'^\otimes}^\otimes.$$

Let us take a Zariski open covering  $\{U_\alpha\}_{\alpha \in I}$  of  $X_b$ , where each  $U_\alpha$  is quasi-compact, i.e.,  $X^\otimes|_{U_\alpha} \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ . The image  $\phi(X^\otimes|_{U_\alpha})$  of  $X^\otimes|_{U_\alpha}$  (for each  $\alpha \in I$ ) is isomorphic (as an object of  $\mathfrak{Sch}_{/X'^\otimes}^\otimes$ ) to  $X'^\otimes|_{U'_\alpha}$  for some quasi-compact open subscheme  $U'_\alpha$  of  $X'_b$  (cf. Proposition 2.11). It follows from Corollary 2.20 (and the various reconstruction procedures involved) that one may find an isomorphism  $\iota_\alpha^\otimes : Z_\alpha^\otimes \xrightarrow{\sim} X^\otimes|_{U_\alpha}$  of superschemes, where  $Z_\alpha^\otimes$  denotes a fermionic twist of  $X'^\otimes|_{U'_\alpha}$ ; such an isomorphism  $\iota_\alpha^\otimes$  is uniquely determined (thanks to Proposition 2.21) by the condition that the functor  $\mathfrak{Sch}_{/X^\otimes|_{U_\alpha}}^\otimes \xrightarrow{\sim} \mathfrak{Sch}_{/Z_\alpha^\otimes}^\otimes$  given by base-change via  $\iota_\alpha^\otimes$  is isomorphic to the composite functor

$$(111) \quad \iota_\alpha^{\mathfrak{Sch}} : \mathfrak{Sch}_{/X^\otimes|_{U_\alpha}}^\otimes \xrightarrow{\phi|_{U_\alpha}} \mathfrak{Sch}_{/X'^\otimes|_{U'_\alpha}}^\otimes \xrightarrow{(25)} \mathfrak{Sch}_{/Z_\alpha^\otimes}^\otimes,$$

where the first arrow denotes the restriction of  $\phi$  to  $\mathfrak{Sch}_{/X^\otimes|_{U_\alpha}}^\otimes$ . For any pair  $(\alpha, \beta) \in I \times I$  with  $U_{\alpha, \beta} := U_\alpha \cap U_\beta \neq \emptyset$ , we obtain an isomorphism  $\iota_{\alpha, \beta}^\otimes := (\iota_\beta^\otimes)^{-1} \circ \iota_\alpha^\otimes : Z_\alpha^\otimes|_{U_{\alpha, \beta}} \xrightarrow{\sim} Z_\beta^\otimes|_{U_{\alpha, \beta}}$ . Proposition 2.21 implies that the collection of isomorphisms  $\{\iota_{\alpha, \beta}^\otimes\}_{\alpha, \beta}$  satisfies the cocycle condition (in the evident sense), and hence, the superschemes  $\{Z_\alpha^\otimes\}_{\alpha \in I}$  may be glued (by means of  $\{\iota_{\alpha, \beta}^\otimes\}_{\alpha, \beta}$ ) together to obtain a superscheme  $Z^\otimes$ . By construction,  $Z^\otimes$  is a fermionic twist of  $X'^\otimes$  and the isomorphisms  $\{\iota_\alpha^\otimes\}_{\alpha \in I}$  may be glued together to an isomorphism  $\iota^\otimes : Z^\otimes \xrightarrow{\sim} X^\otimes$ . Consequently, we have  $X^\otimes \stackrel{f}{\sim} X'$ . This completes the proof of Theorem A.  $\square$

### 3. Further rigidity properties

In this final section, we propose further rigidity properties concerning the category of superschemes.

**Proposition 3.1.** *Let  $X^\otimes$  and  $Y^\otimes$  be two locally noetherian superschemes. Also, let  $f^\otimes (:= (f_b, f^b)) : Y^\otimes \rightarrow X^\otimes$  be a morphism of superschemes such that  $f_b$  is quasi-compact. We shall write*

$$(112) \quad \mathfrak{Sch}_{f^\otimes}^\otimes : \mathfrak{Sch}_{/X^\otimes}^\otimes \rightarrow \mathfrak{Sch}_{/Y^\otimes}^\otimes$$

for the functor induced by base-change via  $f^\otimes$ . Then, the following properties hold.

- (i) If there exists an open subscheme  $U$  of  $X_b$  such that  $2$  is invertible in  $\Gamma(U, f_{b*}(\mathcal{O}_{Y_b}))$  (in particular,  $\Gamma(U, f_{b*}(\mathcal{O}_{Y_b})) \neq 0$ ), then the functor  $\mathfrak{Sch}_{f^\otimes}^\otimes$  has no nontrivial automorphisms.
- (ii) If  $Y^\otimes$  is a scheme (i.e.,  $\mathcal{O}_{Y_f} = 0$ ), then each automorphism of  $\mathfrak{Sch}_{f^\otimes}^\otimes$  corresponds to the collection of automorphisms  $\{\langle \zeta \rangle_{Z^\otimes \times_{X^\otimes} Y^\otimes}\}_{Z^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)}$  for some  $\zeta \in \mu_2(Y_b)$  (cf. (20)).

*Proof.* First, let us make the following observation. Let  $\kappa$  be an automorphism of  $\mathfrak{Sch}_{f^\otimes}^\otimes$ , which consists of automorphisms

$$(113) \quad \kappa_{Z^\otimes}^\otimes \left( := (\kappa_{Z,b}, \kappa_Z^b) \right) : Y^\otimes \times_{X^\otimes} Z^\otimes \xrightarrow{\sim} Y^\otimes \times_{X^\otimes} Z^\otimes$$

in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  that are functorial in  $Z^\otimes \in \text{Ob}(\mathfrak{Sch}_{/X^\otimes}^\otimes)$ . If  $\mathfrak{Sch}_{f_b} : \mathfrak{Sch}_{/X_b} \rightarrow \mathfrak{Sch}_{/Y_b}$  denotes the functor defined by base-change via  $f_b : Y_b \rightarrow X_b$ , then it makes the following square diagram commute

$$(114) \quad \begin{array}{ccc} \mathfrak{Sch}_{/X_b} & \xrightarrow{\mathfrak{Sch}_{f_b}} & \mathfrak{Sch}_{/Y_b} \\ \mathfrak{Sch}_{\beta_{X^\otimes}} \downarrow & & \downarrow \mathfrak{Sch}_{\beta_{Y^\otimes}} \\ \mathfrak{Sch}_{/X^\otimes}^\otimes & \xrightarrow{\mathfrak{Sch}_{f^\otimes}^\otimes} & \mathfrak{Sch}_{/Y^\otimes}^\otimes, \end{array}$$

where the left-hand and right-hand vertical arrows arise from base-change via  $\beta_{X^\otimes}$  and  $\beta_{Y^\otimes}$  respectively. Since  $(W \times_{X_b} Y^\otimes)_b = W \times_{X_b} Y_b$  (for any  $W \in \text{Ob}(\mathfrak{Sch}_{/X_b})$ ), the automorphism  $\kappa$  restricts to an automorphism  $\kappa|_{\mathfrak{Sch}_{/X_b}}$  of  $\mathfrak{Sch}_{f_b}$ , which is given by  $\{(\kappa_{W \times_{X_b} X^\otimes})_b\}_{W \in \text{Ob}(\mathfrak{Sch}_{/X_b})}$ . By [5], Theorem 1.7, (i), we have  $(\kappa_{W \times_{X_b} X^\otimes})_b = \text{id}_{W \times_{X_b} Y_b}$  for any  $W \in \text{Ob}(\mathfrak{Sch}_{/X_b})$ . In particular, the equality  $(\kappa_{\mathbb{A}_{X^\otimes}^{1|0}})_b = \text{id}_{\mathbb{A}_{Y_b}^{1|0}}$  implies the equality

$$(115) \quad \kappa_{\mathbb{A}_{X^\otimes}^{1|0}}^\otimes = \text{id}_{\mathbb{A}_{Y^\otimes}^{1|0}}.$$

Next, let us denote by  $\gamma_1^\otimes$  (resp.,  $\gamma_2^\otimes$ ) the morphism  $\mathbb{A}_{Y^\otimes}^{0|2} \rightarrow \mathbb{A}_{Y^\otimes}^{0|1}$  in  $\mathfrak{Sch}_{/Y^\otimes}^\otimes$  corresponding to the homomorphism  $\mathcal{O}_{Y^\otimes}[\psi] \rightarrow \mathcal{O}_{Y^\otimes}[\psi] \otimes_{\mathcal{O}_{Y^\otimes}} \mathcal{O}_{Y^\otimes}[\psi]$  given by  $\psi \mapsto \psi \otimes 1$  (resp.,  $\psi \mapsto 1 \otimes \psi$ ). Note that the automorphism  $\kappa_{\mathbb{A}_{X^\otimes}^{0|1}}^\otimes$  of  $\mathbb{A}_{Y^\otimes}^{0|1}$  is given by  $\psi \mapsto \zeta \cdot \psi + a$  for some  $\zeta \in \Gamma(Y_b, \mathcal{O}_{Y_b}^\times)$  and  $a \in \Gamma(Y_b, \mathcal{O}_{Y_f}^\times)$ .

Since  $\gamma_{\square}^{\otimes} : \mathbb{A}_{Y^{\otimes}}^{0|2} \rightarrow \mathbb{A}_{Y^{\otimes}}^{0|1}$  (for each  $\square = 1, 2$ ) is compatible with  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^{\otimes}$  and  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|1}}^{\otimes}$  (due to the functoriality of  $\kappa_{Z^{\otimes}}^{\otimes}$  with respect to  $Z^{\otimes}$ ),  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^{\otimes}$  is given by  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^b(\psi \otimes 1) = (\zeta \cdot \psi + a) \otimes 1$  and  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^b(1 \otimes \psi) = 1 \otimes (\zeta \cdot \psi + a)$ . It follows that

$$(116) \quad \kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^b(\psi \otimes \psi) = \zeta^2 \cdot \psi \otimes \psi + \zeta \cdot \psi \otimes a + \zeta \cdot a \otimes \psi + a \otimes a.$$

Here, for any superscheme  $Z^{\otimes}$ , we shall write  $\mathbb{A}_{Z^{\otimes}}^{\epsilon|0} := Z^{\otimes} \times \text{Spec}(\mathbb{Z}[\epsilon]/\epsilon^2)$ . Since  $\mathbb{A}_{Y^{\otimes}}^{\epsilon|0}$  lies in the essential image of the composite  $\mathfrak{Sch}_{\beta_{Y^{\otimes}}} \circ \mathfrak{Sch}_{f_b}$ , we have  $(\kappa_{\mathbb{A}_{X^{\otimes}}^{\epsilon|0}})_b = \text{id}_{(\mathbb{A}_{Y^{\otimes}}^{\epsilon|0})_b}$ . But, a morphism  $\gamma_{\epsilon}^{\otimes} : \mathbb{A}_{Y^{\otimes}}^{0|2} \rightarrow \mathbb{A}_{Y^{\otimes}}^{\epsilon|0}$  over  $Y^{\otimes}$  given by assigning  $\epsilon \mapsto \psi \otimes \psi$  is compatible with  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^{\otimes}$  and  $\kappa_{\mathbb{A}_{X^{\otimes}}^{\epsilon|0}}^{\otimes}$ . This implies that  $\kappa_{\mathbb{A}_{X^{\otimes}}^{0|2}}^b(\psi \otimes \psi) = \psi \otimes \psi$ , i.e.,  $a = 0$  and  $\zeta^2 = 1$  ( $\Leftrightarrow \zeta \in \mu_2(Y_b)$ ). Since we have obtained the equality (115),  $\kappa_{\mathbb{A}_{X^{\otimes}}^{1|1}}^{\otimes}$  corresponds to the automorphism of the  $\mathcal{O}_{Y^{\otimes}}$ -superalgebra  $\mathcal{O}_{Y^{\otimes}}[t, \psi]$  given by  $t \mapsto t$  and  $\psi \mapsto \zeta \cdot \psi$ . Hence, by the discussion in §1.6 (especially, the composite bijection (34)) and the functoriality of  $Z^{\otimes} \mapsto \kappa_{Z^{\otimes}}^{\otimes}$ , the equality  $\kappa_{Z^{\otimes}}^{\otimes} = \langle \zeta \rangle_{Z^{\otimes} \times_{X^{\otimes}} Y^{\otimes}}$  holds for any  $Z^{\otimes} \in \text{Ob}(\mathfrak{Sch}_{/X^{\otimes}}^{\otimes})$ .

Now, we shall prove assertion (i) and (ii). Assertion (ii) follows directly from the above observation, so it suffices to consider assertion (i). Let  $U$  be an open subscheme of  $X_b$  satisfying the assumption. Then, since there is no nontrivial automorphism of the initial object  $Y^{\otimes}$  in  $\mathfrak{Sch}_{/Y^{\otimes}}^{\otimes}$ ,  $\kappa_{X^{\otimes}|U}^{\otimes}$  must be the identity morphism of  $Y^{\otimes}|_{f_b^{-1}(U)}$ . In particular, the fermionic part of  $\kappa_{X^{\otimes}|U}^b$  coincides with the identity morphism of  $\mathcal{O}_{Y_f}|_{f_b^{-1}(U)}$ . It follows from the assumption in (ii) that  $\zeta = 1$ , i.e.,  $\kappa$  coincides with the identity morphism. This completes the proof of Proposition 3.1.  $\square$

**Proposition 3.2.** *Let  $X^{\otimes}$  be a locally noetherian superscheme. Suppose that for any  $Y^{\otimes} \in \text{Ob}(\mathfrak{Sch}_{/X^{\otimes}}^{\otimes})$ , one has an automorphism  $\kappa_{Y^{\otimes}}^{\otimes}$  of  $Y^{\otimes}$  (which is not necessarily over  $X^{\otimes}$ ) and for any morphism  $f^{\otimes} : Y_1^{\otimes} \rightarrow Y_2^{\otimes}$  in  $\mathfrak{Sch}_{/X^{\otimes}}^{\otimes}$ , one has a commutative square diagram:*

$$(117) \quad \begin{array}{ccc} Y_1^{\otimes} & \xrightarrow{\kappa_{Y_1^{\otimes}}^{\otimes}} & Y_1^{\otimes} \\ f^{\otimes} \downarrow & & \downarrow f^{\otimes} \\ Y_2^{\otimes} & \xrightarrow{\kappa_{Y_2^{\otimes}}^{\otimes}} & Y_2^{\otimes} \end{array}.$$

Then, there exists  $\zeta \in \mu_2(X_b)$  such that  $\kappa_{Y^\circledast}^\circledast = \langle \zeta \rangle_{Y^\circledast}$  for all  $Y^\circledast \in \text{Ob}(\mathfrak{Sch}_{/X^\circledast}^\circledast)$ .

*Proof.* The assertion follows immediately from an argument similar to the argument in the proof of Proposition 3.1.  $\square$

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