Pure and Applied Mathematics Quarterly
Volume 16, Number 5, 1585–1617, 2020

Regularity of fully non-linear elliptic equations on Kähler cones

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Abstract: We derive quantitative boundary estimates, and then solve the Dirichlet problem for a general class of fully non-linear elliptic equations on annuli of Kähler cones over closed Sasakian manifolds. This extends extensively a result concerning the geodesic equations in the space of Sasakian metrics due to Guan-Zhang. Our results show that the solvability is deeply affected by the transverse Kähler structures of Sasakian manifolds. We also discuss possible extensions of the results to equations with right-hand side depending on unknown solutions.

Keywords: Dirichlet problem, degenerate fully non-linear elliptic equations, quantitative boundary estimate, gradient estimate, cone condition, Sasakian manifolds.

1. Introduction

In Kähler geometry, Donaldson conjectured in [8] that the space of Kähler metrics is geodesic convex by smooth geodesic and that it is a metric space. By the observation of [8, 24, 27] the geodesic equation in the space of Kähler metrics can be deduced to a homogeneous complex Monge-Ampère equation on a manifold of one dimension higher. In [3], Chen proved the existence of $C^{1,\alpha}$-geodesics ($\forall 0 < \alpha < 1$) in the space of Kähler metrics, and then solved the second part of Donaldson's conjecture.

In the setting of Sasaki geometry, which can be viewed as an odd dimensional counterpart of Kähler geometry, Guan-Zhang [21] studied the corresponding geodesic equation in the space of Sasakian metrics $\mathcal{H}$ and partially verified the counterpart to Donaldson's conjecture in Sasakian setting. As in

Received July 31, 2019.
2010 Mathematics Subject Classification: Primary 35J15; secondary 58J05, 53C25, 35B45.
*The author is partially supported by the National Natural Science Foundation of China (Grant No. 11801587).

[21], such a geodesic equation connecting the potentials φ_1 and φ_2 in $\mathcal{H}$ is similarly equivalent to

$$(1)\qquad \begin{cases} (\Omega_u)^n & = 0, \text{ in } S \times (1, \frac{3}{2}), \\ u|_{r=1} & = \varphi_1, \\ u|_{r=\frac{3}{2}} & = \varphi_2 + 4\log(\frac{3}{2}), \end{cases}$$

where (S, ξ, η, Φ, g) is a closed Sasakian manifold of dimension $(2n-1)$, $\Omega_u = \bar{\omega} + \frac{r^2}{2}\sqrt{-1}(\partial\overline{\partial}u - \frac{\partial u}{\partial r}\partial\overline{\partial}r)$, $\bar{\omega} = \frac{1}{2}d(r^2\eta)$ is the Kähler form of the Kähler cone $(C(S), \bar{g}) = (S \times \mathbb{R}^+, r^2 g + dr^2)$, r is the coordinate on $\mathbb{R}^+$. Here

$$\mathcal{H} := \left\{ v \in C_B^\infty(S) : (\eta + d_B^c v) \wedge (d\eta + \sqrt{-1}\partial_B\overline{\partial}_B v)^{n-1} \neq 0 \right\},$$

and $d_B^c = \frac{\sqrt{-1}}{2}(\overline{\partial}_B - \partial_B)$ (where $\partial_B, \overline{\partial}_B$ defined as below). Furthermore, they obtained the uniqueness of transverse Kähler metric with constant scalar curvature in each **basic** Kähler class if first basic Chern class is non-positive.

In this paper, our aim is to study a class of fully nonlinear elliptic equations on an annulus $M := S \times (a, b)$ $(0 < a < b < +\infty)$,

$$(2)\qquad \begin{cases} F(\mathfrak{g}[u]) := f(\lambda(\mathfrak{g}[u])) & = \psi, \text{ in } M, \\ u|_{r=a} & = \varphi_a, \\ u|_{r=b} & = \varphi_b, \end{cases}$$

where ψ and φ_a, φ_b are prescribed functions with appropriate regularities, $\mathfrak{g} = \mathfrak{g}[u] = \chi + \sqrt{-1}(\partial\overline{\partial}u - \frac{\partial u}{\partial r}\partial\overline{\partial}r)$, χ is a smooth real $(1,1)$-form, $\lambda(\mathfrak{g}) = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of $\mathfrak{g}$ with respect to $\bar{\omega}$, and f is a smooth symmetric function defined in a convex $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and boundary $\partial\Gamma \neq \emptyset$,

$$\Gamma_n := \{\lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0 \text{ for each } 1 \leq k \leq n\} \subseteq \Gamma,$$

where $\sigma_k(\lambda)$ is the k-th elementary function

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

For convenience we define $\sigma_0(\lambda) \equiv 1$.

The most important equation is complex Monge-Ampère equation corresponding to $f(\lambda) = (\sigma_n(\lambda))^{1/n}$ with $\Gamma = \Gamma_n$, as the relation to the representation of Ricci curvature on Kähler manifolds. The complex Monge-Ampère equation thus plays important roles in the existence of canonical Kähler metrics in complex geometry. A celebrated work is due to Yau [36], in which he proved Calabi's conjecture and showed that the existence of Kähler-Einstein metrics on closed Kähler manifolds of vanishing or negative first Chern class. The existence of Kähler-Einstein metric on the closed Kähler manifold with negative first Chern class was also proved by Aubin [1] independently.

In the setting of real variables, the study of equations of this type can be traced back to the work of Caffarelli-Nirenberg-Spruck [2] on the Dirichlet problem on bounded domains $\Omega \subset \mathbb{R}^n$. In order to study the equation within the framework of elliptic equations, we solve the equations in the class of *admissible* functions satisfying $\lambda(\mathfrak{g}[u]) \in \Gamma$; moreover, f satisfies the following standard and fundamental conditions:

$$f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \ 1 \leq i \leq n, \tag{3}$$

$$f \text{ is concave in } \Gamma, \tag{4}$$

$$\inf_{\bar{M}} \psi > \sup_{\partial \Gamma} f, \tag{5}$$

where

$$\sup_{\partial\Gamma} f := \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda), \text{ and } \bar{M} := S \times [a, b].$$

We also denote $\delta_{\psi,f} := \inf_{\bar{M}} \psi - \sup_{\partial\Gamma} f$ by the constant which measures if the equation is degenerate.

Furthermore, we assume

$$\text{For any } \sigma < \sup_{\Gamma} f \text{ and } \lambda \in \Gamma \text{ we have } \lim_{t \to +\infty} f(t\lambda) > \sigma. \tag{6}$$

The above condition allows one to derive gradient estimate by using the blow-up argument used in [30]. Typical examples satisfying (3), (4) and (6) are as the following: the corresponding cone of f is $\Gamma = \Gamma_n$, or if f is homogeneous of degree one with $f > 0$ in Γ.

We further present some additional materials: A Sasakian structure (ξ, η, Φ, g) consists of a Reeb field ξ, a contact 1-form η with $\eta(X) = g(\xi, X)$, and a tensor Φ with $\Phi(X) = \nabla_X \xi$. Also, the metric cone $(C(S), \bar{g})$ is a Kähler manifold with the compatible complex structure J given by

$$J(X) = \Phi(X) - \eta(X) r \frac{\partial}{\partial r}, \ J(r\frac{\partial}{\partial r}) = \xi.$$

There is an important fact: Φ determines a complex structure on the contact subbundle $\mathcal{D} = \ker\{\eta\}$, and $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$ further provides S with a transverse

Kähler structure admitting with Kähler form $\omega^T = \frac{1}{2}d\eta$. The complexification $\mathcal{D}^{\mathbb{C}}$ of the sub-bundle $\mathcal{D}$ can be decomposed into its eigenspaces with respect to $\Phi|_{\mathcal{D}}$ as $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. A class of C^1-smooth functions v with $\xi v \equiv 0$ are called **basic**. The basic functions play crucial roles in the theory of Sasakian geometry, as they are invariant along the Reeb field ξ. Let's denote

$$C^k_B(S) = \left\{u \in C^k(S) : \xi u = 0\right\}, C^{k,\alpha}_B(S) = \left\{u \in C^{k,\alpha}(S) : \xi u = 0\right\},$$
$$C^k_B(M) = \left\{u \in C^k(M) : \xi u = 0\right\}, C^{k,\alpha}_B(M) = \left\{u \in C^{k,\alpha}(M) : \xi u = 0\right\}, \text{ etc.}$$

It is easy to see that the exterior differential preserves **basic** forms. The transverse complex structure follows the splitting of the complexification of the bundles of the sheaf of germs of **basic** p-forms $\Lambda^p_B(S)$ on S,

$$\Lambda^p_B(S) \otimes \mathbb{C} = \oplus_{i+j=p} \Lambda^{i,j}_B(S),$$

where $\Lambda^{i,j}_B(S)$ denotes the bundle of **basic** forms of type (i,j). Set $d|_B = d|_{\Lambda^p_B}$, we can decompose $d|_B = \partial_B + \overline{\partial}_B$, where $\partial_B : \Lambda^{i,j}_B \longrightarrow \Lambda^{i+1,j}_B$, $\overline{\partial}_B : \Lambda^{i,j}_B \longrightarrow \Lambda^{i,j+1}_B$. Furthermore, $\partial^2_B = \overline{\partial}^2_B = 0$, $\partial_B\overline{\partial}_B + \overline{\partial}_B\partial_B = 0$. From now on, $P^*\eta$ and $P^*d\eta$ will be used to denote pull-backs by η and $d\eta$, respectively, where $P : C(S) \to S$ is the natural projective map. Given a real $(1,1)$-form $\mathfrak{g}$, let

$$\mathfrak{g}^T(\cdot,\cdot) = \mathfrak{g}|_{P^*\mathcal{D}^{1,0} \times P^*\mathcal{D}^{0,1}} : P^*\mathcal{D}^{1,0} \times P^*\mathcal{D}^{0,1} \to \mathbb{C},$$

and $\lambda'(\mathfrak{g}[v]^T) = (\lambda'_1, \cdots, \lambda'_{n-1})$ be the eigenvalues of $\mathfrak{g}[v]^T$ with respect to $r^2\omega^T$. In particular, $\mathfrak{g}[v]^T(\cdot) = \chi^T(\cdot) + \sqrt{-1}\partial_B\overline{\partial}_B v(\cdot)$ for $v \in C^2_B(\bar{M})$.

Observing that equation (2) involves the radial derivation of the unknown solution u, say $\frac{\partial u}{\partial r}$, we know that it is much more complicated than the standard one and draws a hard difficulty due to the two different types of complex derivatives.

A feasible approach to overcoming the difficulty is to complexify the radial derivation in equation. To do this we need to ensure $J(\frac{\partial}{\partial r})u = \frac{1}{r}\xi u = 0$ (i.e. u is **basic**) under the following condition

$$\nabla_\xi \chi = 0, \text{ and } \varphi_a, \varphi_b, \psi \text{ are all } \textbf{basic}, \tag{7}$$

where ∇ is Chern connection of $\bar{g}$. Such a condition is very natural from the view-point of Sasakian geometry. The following lemma states that every *admissible* solution of Dirichlet problem (2) is **basic** with assuming (7) holds.

Lemma 1.1 ([21, 26])**.** *Suppose* (3), (5) *and* (7) *hold. Let* $u \in C^3(M) \cap C^1(\bar{M})$ *be an admissible solution of Dirichlet problem* (2)*, then* $\xi u \equiv 0$ *in* $\bar{M}$.

The above lemma was first proved by Guan-Zhang [21] for complex Monge-Ampère type equation with $\chi = \frac{2}{r^2}\bar{\omega}$, and further by Qiu and the author [26] for general fully nonlinear elliptic equations. Moreover, the author showed in [37] that condition (7) can be removed for certain fully nonlinear elliptic equations, including Dirichlet problem of complex Monge-Ampère type equation for $n = 2$.

A powerful notion of a subsolution is developed to study Dirichlet problem for fully nonlinear elliptic equations and the related geometric problems (cf. [3, 13, 14, 18, 19, 20, 21, 22] and references therein).

For purpose of constructing the desired **basic** *admissible* subsolution, we assume that there exists a **basic** function $\underline{v} \in C^{4,\alpha}_B(\bar{M})$ with $\underline{v}|_{r=a} = \varphi_a, \underline{v}|_{r=b} = \varphi_b$ such that

$$\lambda'(\mathfrak{g}[\underline{v}]^T) \in \Gamma_\infty, \lim_{R\to+\infty} f(\lambda'(\mathfrak{g}[\underline{v}]^T), R) > \psi, \text{ in } \bar{M}. \tag{8}$$

As in [32], $\Gamma_\infty := \{(\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{R}^{n-1} : (\lambda_1, \cdots, \lambda_{n-1}, R) \in \Gamma$ for some $R > 0\}$ stands for the projection of Γ onto $\mathbb{R}^{n-1}$. Such a condition is very closely related to the transverse Kähler structure of the underlying Sasakian manifold (S, ξ, η, Φ, g).

The following theorem shows that the solvability of Dirichlet problem (2) is heavily determined by cone condition (8), thereby being deeply affected by the transverse Kähler structures of underlying Sasakian manifolds.

Theorem 1.2. *Let $\psi \in C^{k,\alpha}_B(\bar{M})$, $\varphi_a, \varphi_b \in C^{k+2,\alpha}_B(S)$, $k \geq 2$, $0 < \alpha < 1$. Assume (3), (4), (5), (6), (7) and (8) hold. Then Dirichlet problem (2) has a unique basic admissible solution $u \in C^{k+2,\alpha}_B(\bar{M})$.*

We stress that condition (8) can be viewed as a kind of cone condition. Moreover, in contrast to the cone conditions in [10, 28, 29] as well as the notion of $\mathcal{C}$-subsolution in [30], our cone condition is much more easier to check. In particular, if $\chi^T(\cdot, r) = \chi^T(\cdot)$ (i.e. χ^T is invariant with varying radial variable r) and

$$\lim_{t\to+\infty} f(\lambda', t) > \sigma, \quad \text{for any } \sigma < \sup_\Gamma f, \ \lambda' \in \Gamma_\infty, \tag{9}$$

then condition (8) can be replaced by

$$\lambda'(\chi^T + \sqrt{-1}\partial_B\overline{\partial}_B\varphi_a), \lambda'(\chi^T + \sqrt{-1}\partial_B\overline{\partial}_B\varphi_b) \in \Gamma_\infty, \text{ in } S, \tag{10}$$

which is completely determined by the given basic boundary data. Consequently, we get

Theorem 1.3. *In addition to* (3), (4), (5), (6), (7) *and* (9), *we assume* $\chi^T(\cdot, r) = \chi^T(\cdot)$ *and* $\psi \in C^\infty_B(\bar{M})$ *is a smooth basic function. Then the Dirichlet problem* (2) *with basic smooth boundary data* φ_a, φ_b *satisfying* (10) *has a unique smoothly basic solution* $u \in C^\infty_B(\bar{M})$.

It is noteworthy that such a condition does not rely on the right-hand side of the equation, and the condition holds for ψ with $\sup_{\partial\Gamma} f < \psi < \sup_\Gamma f$ which is necessary for the solvability. There is no any other restriction to the upper bound of the right-hand side. Hence, if (9) holds, we can solve the Dirichlet problem with only assuming necessary fundamental assumptions mentioned in this context. We should figure out a simple fact that, even for the existence of a C^2-smoothly basic *admissible* function with the given basic boundary data, such a condition is necessary and needed. Therefore, it is necessary for the solvability of Dirichlet problem (2) in the class of basic *admissible* functions.

For degenerate equations, we have the following existence results.

Theorem 1.4. *Let* $\psi \in C^{2,\gamma}_B(\bar{M})$ *be a basic function with* $\delta_{\psi,f} = 0$ *for some* $\gamma \in (0,1)$, *and* $\varphi_a, \varphi_b \in C^{4,\gamma}_B(S)$. *Suppose* (3), (4), (6), (7) *and* (8) *hold. We further assume* $f \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$, $\bar{\Gamma} = \Gamma \cup \partial\Gamma$. *Then there exists a (weak) basic solution* $u \in C^{1,\alpha}_B(\bar{M})$, $\forall 0 < \alpha < 1$, *with* $\lambda(\mathfrak{g}[u]) \in \bar{\Gamma}$ *and* $\Delta u \in L^\infty(\bar{M})$ *to the Dirichlet problem* (2) *for degenerate fully nonlinear elliptic equations. Moreover, the solution is* $C^{4,\gamma}$*-smooth in* $M^+ := \{z \in \bar{M} : \psi(z) > \sup_{\partial\Gamma} f\}$.

Theorem 1.5. *Suppose* (3), (4), (6), (7) *and* (9) *hold. Assume* $\chi^T(\cdot, r) = \chi^T(\cdot)$ *and* $\psi \in C^\infty_B(\bar{M})$ *is a smooth basic function with* $\delta_{\psi,f} \geq 0$. *Then the Dirichlet problem* (2) *with basic smooth boundary data* φ_a, φ_b *satisfying* (10) *has a (weak) basic solution* $u \in C^{1,\alpha}_B(\bar{M})$, $\forall 0 < \alpha < 1$, *with* $\lambda(\mathfrak{g}[u]) \in \bar{\Gamma}$ *and* $\Delta u \in L^\infty(\bar{M})$. *Moreover, the solution is smooth in the subset* M^+.

Our result is applicable for some geometric problems, as condition (10) originates naturally from Sasakian geometry. A typical and important example satisfying (10) is $\chi^T \equiv d\eta$ and $\varphi_a, \varphi_b \in \mathcal{H}$, in which $d\eta + \sqrt{-1}\partial_B\overline{\partial}_B\varphi_a > 0$ and $d\eta + \sqrt{-1}\partial_B\overline{\partial}_B\varphi_b > 0$. This example can be applied to the above-mentioned geodesic equations in the space of Sasakian metrics $\mathcal{H}$ which were studied in [21]. Applying Theorem 1.5 or 1.4 to Dirichlet problem (1) for complex Monge-Ampère type equation, one immediately obtains Guan-Zhang's result in [21]: the existence of a **basic** weak solution of the geodesic equation in the space of Sasakian metrics. Furthermore, the results obtained in this paper remove some additional assumptions in [26] and so extensively extend main theorems proved there. Moreover, condition (9) allows many interesting symmetric functions. In addition to a class of functions $f = \sum_{j=1}^N g_j$ with the cone $\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0 \text{ for } 1 \le i \le k\}$, where k is a fixed integer with $2 \le k \le n$,

$$g_j = \prod_{i=1}^{N_j} \left(c_i + \sum_{l=0}^{k-1} c_{i,l} (\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}} \right)^{\alpha_i},$$

where $c_i, c_{i,l}$ and α_i are all nonnegative constants with $\sum_{i=1}^{N_j} \alpha_i c_{i,0} > 0$, $\sum_{i=1}^{N_j} \alpha_i = 1$, $c_i + \sum_{l=0}^{k-1} c_{i,l} > 0$ for each i, condition (9) also allows

$$\log P_m(\lambda) = \sum_{1 \le i_1 < \cdots < i_m \le n} \log(\lambda_{i_1} + \cdots + \lambda_{i_m})$$

with the cone $\mathcal{P}_m := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_m} > 0 \text{ for any } 1 \le i_1 < \cdots < i_m \le n\}$. The function $\log P_{n-1}$ recently has received attention, as it is related to Gauduchon's conjecture which can be reduced to solving $\log P_{n-1}(\lambda(\tilde{\chi} + \sqrt{-1}\partial\overline{\partial}u + W(\cdot, \partial u, \overline{\partial}u))) = \psi$ on closed Hermitian manifolds (see [25, 34]), where $W(\cdot, \partial u, \overline{\partial}u)$ is a certain real $(1,1)$-form depending linearly on ∂u and $\overline{\partial}u$. Please refer to [31, 33, 34] for series works on the Gauduchon's conjecture, and to [16] for related work.

It would be worthwhile to note that the constant C in (14) of Section 2 is a uniform constant depending only on $|\varphi|_{C^{2,1}(\bar{M})}$, $|\psi|_{C^{1,1}(\bar{M})}$ and other known data (but not on $(\delta_{\psi,f})^{-1}$). The boundary estimates of this type depending on the $C^{2,1}$-norm of boundary data is proved for equation (16) on compact Hermitian manifolds with certain boundary, thereby posing in [39, 40] some new phenomenon on regularity assumptions on boundary and boundary data. Unfortunately, I don't know if there is such new phenomenon on regularity assumptions, since the second order estimate in Theorem 2.1 above relies heavily on condition (7) in a significant way and the approximation method may not work any more in the class of **basic** functions. More precisely, at least to the best of my knowledge, I don't know if every basic $C^{1,1}$-function (respectively, basic $C^{2,1}$-functions) can be approximated by certain basic $C^{2,\gamma}$-functions (respectively, basic $C^{4,\gamma}$-functions) in the sense of $C^{1,1}$-norm (respectively, $C^{2,1}$-norm).

The rest of this paper is organized as follows. In Section 2 we outline the proof of quantitative boundary estimates. In Section 3, the desired **basic** *admissible* subsolution will be constructed there with assuming cone condition (8) holds. In Section 4 we establish the quantitative boundary estimates. In Section 5 we further study fully nonlinear elliptic equations with the right-hand side depending on the unknown solutions. In Appendix A, we finally append the proof of Lemma 2.5.

2. Outline of proof of main estimate

The central issue for proving Theorem 1.2 is to derive the *a priori* estimates for the complex Hessian of the solution, so that equation (2) becomes to be a uniform concave elliptic equation. The uniform bound of the real Hessian can be derived directly as in [15], since C^2 estimates yield that the equation is uniformly elliptic. Finally, one can use Evans-Krylov theorem [9, 23] and Schauder theory to establish the higher order regularity.

The proof of gradient estimate for the solutions of fully nonlinear elliptic equations is exceedingly hard in complex setting. Our approach to deriving gradient estimate is based on a blow-up argument developed by Dinew-Kołodziej [7], and further by Székelyhidi [30]. To achieve it we need to prove

$$\sup_M |\Delta u| \leq C(1+\sup_M |\nabla u|^2). \tag{11}$$

The following second order estimate was established in [26].

Theorem 2.1 ([26]). *In addition to (3), (4), (5) and (7), we assume that there is a basic admissible subsolution $\underline{u} \in C^2_B(\bar{M})$ for Dirichlet problem (2) with $\psi \in C^2_B(M) \cap C^{1,1}_B(\bar{M})$ and $\varphi_a, \varphi_b \in C^2_B(S)$. Then for any admissible solution $u \in C^4(M) \cap C^2(\bar{M})$ of the Dirichlet problem, there exists a uniformly positive constant C depending on $|u|_{C^0(\bar{M})}$, $|\psi|_{C^{1,1}(\bar{M})}$, $|\underline{u}|_{C^2(\bar{M})}$, $|\chi|_{C^2(\bar{M})}$ and other data under control (but not on $\sup_{\bar{M}} |\nabla u|$), such that*

$$\sup_M |\Delta u| \leq C(1+\sup_M |\nabla u|^2 + \sup_{\partial M} |\Delta u|). \tag{12}$$

Moreover, the constant C is independent of $(\delta_{\psi,f})^{-1}$.

Remark 2.2. Throughout this paper we say that a constant C does not depend on $(\delta_{\psi,f})^{-1}$ if C remains uniformly bounded as $\delta_{\psi,f}$ tends to zero, while we say a constant κ depends not on $\delta_{\psi,f}$ if κ has a uniformly positive lower bound as $\delta_{\psi,f} \to 0$.

With Theorem 2.1 at hand, the main estimate in this paper is to derive the quantitative boundary estimates as follows.

Theorem 2.3. *Let $\psi \in C^1_B(M) \cap C^{0,1}_B(\bar{M})$, $\varphi_a, \varphi_b \in C^{2,1}_B(S)$, and*

$$\varphi(\cdot, r) = \frac{1}{b-a}\left((b-r)\varphi_a + (r-a)\varphi_b\right). \tag{13}$$

Suppose conditions (3), (4), (5), (7) and (8) hold. Then for any admissible solution $u \in C^3(M) \cap C^2(\bar{M})$ of Dirichlet problem (2), we have

$$\sup_{\partial M} |\Delta u| \leq C(1 + \sup_M |\nabla u|^2), \tag{14}$$

where C is a uniformly positive constant depending on $|\psi|_{C^{0,1}(\bar{M})}$, $|\varphi|_{C^{2,1}(\bar{M})}$ and other known data. Furthermore, the constant C is independent of $(\delta_{\psi,f})^{-1}$.

The proof of quantitative boundary estimates is based on the following proposition.

Proposition 2.4. *Fix $p = (q, \cdot) \in S \times \{a\} \cup S \times \{b\}$. Let X_i be the vectors given by (26) and (27), in which we choose the local coordinate around q such that (24) and (25) hold. Let's denote $\mathfrak{g}_{i\bar{j}} = \mathfrak{g}[u](X_i, J\bar{X}_j)$ for the solution u. Then*

$$\mathfrak{g}_{n\bar{n}}(p) \leq C(1 + \sum_{\alpha=1}^{n-1} |\mathfrak{g}_{\alpha\bar{n}}(p)|^2).$$

The key ingredient in the proof of Proposition 2.4 is the following lemma proposed in [39], which is a quantitative version of Lemma 1.2 in [2].

Lemma 2.5 ([39])**.** *Let A be an $n \times n$ Hermitian matrix*

$$\begin{pmatrix} d_1 & & & & a_1 \\ & d_2 & & & a_2 \\ & & \ddots & & \vdots \\ & & & d_{n-1} & a_{n-1} \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & \mathbf{a} \end{pmatrix}$$

with $d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1}$ fixed, and with $\mathbf{a}$ variable. Denote $\lambda_1, \cdots, \lambda_n$ by the eigenvalues of A. Let $\epsilon > 0$ be a fixed constant. Suppose that the parameter $\mathbf{a}$ in A satisfies the quadratic growth condition

$$\mathbf{a} \geq \frac{2n-3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n-1) \sum_{i=1}^{n-1} |d_i| + \frac{(n-2)\epsilon}{2n-3}. \tag{15}$$

Then the eigenvalues (possibly with an order) behavior like

$$\begin{aligned} &|d_\alpha - \lambda_\alpha| < \epsilon, \forall 1 \leq \alpha \leq n-1, \\ &0 \leq \lambda_n - \mathbf{a} < (n-1)\epsilon. \end{aligned}$$

Remark 2.6. We follow the argument developed in [39, 40], in which the author derive quantitative boundary estimates for equations on Hermitian manifolds (X, ω),

$$f(\lambda(\chi + \sqrt{-1}\partial\overline{\partial}u + \sqrt{-1}\partial u \wedge \overline{\eta^{1,0}} + \sqrt{-1}\eta^{1,0} \wedge \overline{\partial}u)) = \psi \text{ in } X \tag{16}$$

with Dirichlet boundary $u|_{\partial X} = \varphi$, where $\eta^{1,0}$ is a smooth $(1,0)$-form. When $\eta^{1,0} = 0$, this is the standard case and was studied by the author in [39]. The corresponding proposition analogous to Proposition 2.4 was proved by the author in [39, 40]. When (X, ω) is a closed Hermitian manifold, equation (16) is also studied in [40]; while the special case of equation (16) corresponding to $f = (\sigma_n)^{\frac{1}{n}}$ (complex Monge-Ampère equation) is also studied by Tosatti-Weinkove [35] independently. As a corollary, this partially extends some results in [37]. Moreover, as in Remark 1.2 of [35], equation (16) can be used to study the deformation of Aeppli cohomology class of bidegree $(1,1)$.

Remark 2.7. We shall remark here that, when the boundary of background Hermitian manifold is supposed to satisfy the condition in main theorem of [39], inspired by the work [39], we can try to give a proof of gradient estimate for Dirichlet problem of the equation corresponding to Gauduchon's conjecture by applying blow-up argument used in literature (cf. [7, 30, 31, 34]) in which the underlying manifolds are all closed without boundary. This is discussed in future work*. To do this, as in [39], we only need to prove analogous Proposition 2.4 in this context.

Remark 2.8. Shortly after T. Collins and S. Picard posted their paper [5] to arXiv.org, T. Collins also informed me Lemma 2.5 (in a different and weak form) was also proved independently in [6] (see the Lemma 6.3 there). Indeed the Lemma 6.3 in [6] was used in a different way than our application. I want to thank T. Collins for informing me the Lemma 6.3 in their work.

Remark 2.9. We can use Proposition 5.1 of [4] in place of Székelyhidi's Liouville type theorem to derive the gradient estimate and so prove Theorem 1.2 for certain equations without obeying the additional assumption (6) but with $\Gamma^{\underline{\psi},\overline{\psi}} \subseteq -K_0\vec{1} + \Gamma_n$ for some $K_0 \geq 0$, which includes specified Lagrangian phase equation and deformed Hermitian-Yang-Mills equation (with supercritical phase condition) as a special case. Here $\vec{1} = (1, \cdots, 1) \in \mathbb{R}^n$ and

$$\Gamma^{\underline{\psi},\overline{\psi}} := \left\{\lambda \in \Gamma : \inf_{\bar{M}} \psi \leq f(\lambda) \leq \sup_{\bar{M}} \psi\right\}.$$

*Recently, this was done by the author, and the proof is appended in part II of new version of [40].

The direct gradient estimate for deformed Hermitian-Yang-Mills equation (with supercritical phase condition) can be also derived by the argument used in [37], as it satisfies the condition (3.5) there (see also Remark 1.7 and proof of Theorem 1.2 in [37]).

3. The construction of a (strictly) basic *admissible* subsolution

Under the cone condition (8), we can construct a basic *admissible* subsolution of the Dirichlet problem. Together with Lemma 1.2 of [2] or Lemma 2.5, condition (8) also implies the following condition in [26],

$$\begin{aligned} &\lambda(\mathfrak{g}[\underline{v}] + \sqrt{-1}R\theta^n \wedge \bar{\theta}^n) \in \Gamma \text{ for some } R > 0, \\ &\lim_{R\to+\infty} f(\lambda(\mathfrak{g}[\underline{v}] + \sqrt{-1}R\theta^n \wedge \bar{\theta}^n)) > \psi, \end{aligned} \tag{17}$$

where $\theta^n = dr + \sqrt{-1}r\eta$ and $\bar{\theta}^n = dr - \sqrt{-1}r\eta$ defined as in (28).

Given the function $\underline{v} \in C_B^{4,\alpha}(\bar{M})$ satisfying condition (8), we set

$$\underline{u} = \underline{v} + A(r-a)(r-b). \tag{18}$$

It is easy to verify that $\underline{u} \leq \underline{v}$ in $\bar{M}$. By a simple computation, we have

$$\sqrt{-1}\left(\partial\bar{\partial} - \partial\bar{\partial}r\frac{\partial}{\partial r}\right)((r-a)(r-b)) = \frac{\sqrt{-1}}{2}\theta^n \wedge \bar{\theta}^n. \tag{19}$$

By setting $A \gg 1$, $\underline{u}$ is an appropriate (strictly) basic *admissible* subsolution of Dirichlet problem (2). Namely,

$$\begin{cases} f(\lambda(\mathfrak{g}[\underline{u}])) > \psi \text{ in } \bar{M}, \\ \quad \underline{u}|_{r=a} = \varphi_a, \\ \quad \underline{u}|_{r=b} = \varphi_b. \end{cases} \tag{20}$$

4. Proof of the quantitative boundary estimates

In this section, we will give the proof of quantitative boundary estimates, which also extensively extends certain results in [21, 26].

The proof of quantitative boundary estimates consists of two steps:

- The proof of Proposition 2.4.
- The proof of quantitative boundary estimates for mixed derivatives.

Firstly, we derive *a priori* C^0-estimate and gradient estimate on the boundary. Let w be a C^2 solution to

$$\begin{cases} \mathrm{tr}_{\bar{\omega}}(\mathfrak{g}[w]) = 0 \text{ in } M, \\ w|_{r=a} = \varphi_a, \\ w|_{r=b} = \varphi_b. \end{cases} \tag{21}$$

The solvability of (21) can be found in [12]. Let $u \in C^2(\bar{M})$ be an *admissible* solution for Dirichlet problem (2), then $\mathrm{tr}_{\bar{\omega}}(\mathfrak{g}[u]) > 0$. Therefore the maximum principle, together with the boundary value condition, yields

$$\underline{u} \leq u \leq w, \text{ in } M. \tag{22}$$

Hence there is a positive constant C^* depending only on $|\underline{u}|_{C^1(\bar{M})}$ and $|w|_{C^1(\bar{M})}$, such that

$$\sup_{\bar{M}} |u| + \sup_{\partial M} |\nabla u| \leq C^*.$$

Given a point $p \in \partial M$, let $\rho(z) = \mathrm{dist}_{\bar{M}}(z, p)$ be the distance function from z to p and

$$\Omega_\delta \equiv \{z \in M : \rho(z) < \delta\}, 0 < \delta \ll 1. \tag{23}$$

As in the Kähler setting, the Sasakian metric can be locally generated by a free real function of $2(n-1)$ variables. For the given point $p = (q, a) \in S \times \{a\}$ (or $p = (q, b) \in S \times \{b\}$), we may pick a local coordinate chart $(z^1, \cdots, z^{n-1}, x)$, $z^i = x^i + \sqrt{-1}y^i$. Around q, there is a local **basic** function h and a local coordinate chart $(z^1, \cdots, z^{n-1}, x) \in \mathbb{C}^{n-1} \times \mathbb{R}$ on a small neighborhood U around q such that

$$\xi = \frac{\partial}{\partial x}, \; g = \eta \otimes \eta + 2h_{i\bar{j}} dz^i d\bar{z}^j, \; \eta = dx - \sqrt{-1}(h_j dz^j - h_{\bar{j}} d\bar{z}^j), \tag{24}$$

$$h_i(q) = 0, \; h_{i\bar{j}}(q) = \delta_{ij} \text{ and } dh_{i\bar{j}}|_q = 0,$$

$$\frac{1}{4}\delta_{ij} \leq h_{i\bar{j}}(z) \leq \delta_{ij}, \; \sum_{i=1}^{n-1} |h_i|^2(z) \leq 1, \forall z \in U \subset S. \tag{25}$$

Moreover, $\mathcal{D} \otimes \mathbb{C}$ is spanned by

$$X_i = \frac{\partial}{\partial z^i} + \sqrt{-1} h_i \frac{\partial}{\partial x}, \; \bar{X}_i = \frac{\partial}{\partial \bar{z}^i} - \sqrt{-1} h_{\bar{i}} \frac{\partial}{\partial x}, \; 1 \leq i \leq n-1. \tag{26}$$

Let

$$X_n = \frac{1}{2}(\frac{\partial}{\partial r} - \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial x}),\ \bar{X}_n = \frac{1}{2}(\frac{\partial}{\partial r} + \sqrt{-1}\frac{1}{r}\frac{\partial}{\partial x}). \tag{27}$$

Then

$$JX_i = \sqrt{-1}X_i,\ J\bar{X}_i = -\sqrt{-1}\bar{X}_i \text{ for } i = 1, \cdots, n.$$

Hence $\{X_1, \cdots, X_{n-1}, X_n\}$ is a basis of $T^{1,0}M$. Let $\{\theta^1, \cdots, \theta^n\}$ be the dual basis

$$\theta^i = dz^i, 1 \le i \le n-1,\ \theta^n = dr + \sqrt{-1}r\eta. \tag{28}$$

The Kähler form $\bar{\omega}$ of $(C(S), \bar{g})$ can be written as

$$\bar{\omega} = \sqrt{-1}(\sum_{i,j=1}^{n-1} r^2 h_{i\bar{j}} dz^i \wedge d\bar{z}^j + \frac{1}{2}\theta^n \wedge \bar{\theta}^n). \tag{29}$$

We refer the reader to [11] for more details.

By the computation in [21], for $w \in C^2(\bar{M})$ one has

$$\partial\overline{\partial}w - 2(\bar{X}_n w)\partial\overline{\partial}r = \sum_{i,j=1}^{n}(X_i\bar{X}_j w)\theta^i \wedge \bar{\theta}^j.$$

Moreover, if w is basic then

$$\partial\overline{\partial}w - \frac{\partial w}{\partial r}\partial\overline{\partial}r = \sum_{i,j=1}^{n}(X_i\bar{X}_j w)\theta^i \wedge \bar{\theta}^j.$$

(See (2.12) of [21]). From now on, let's denote $w_{i\bar{j}} = \sqrt{-1}\partial\overline{\partial}w(X_i, J\bar{X}_j)$, and

$$F^{i\bar{j}} = \frac{\partial F}{\partial a_{i\bar{j}}}((\mathfrak{g}_{p\bar{q}})).$$

The linearized operator $\mathcal{L}$ is given by

$$\mathcal{L}h = F^{i\bar{j}}(h_{i\bar{j}} - \frac{\partial h}{\partial r} r_{i\bar{j}}). \tag{30}$$

In addition, if h is basic then

$$\mathcal{L}h = F^{i\bar{j}} X_i \bar{X}_j h. \tag{31}$$

The second order boundary estimates for pure tangential derivatives is standard. The boundary value condition implies that there exists a uniformly positive constant C_1' depending on $\sup_{\partial M} |\frac{\partial(u-\underline{u})}{\partial r}|$ and other known data under control, such that for any $1 \leq i, j \leq n-1$ one has

$$\left|\frac{\partial^2 u}{\partial x^i \partial y^j}(p)\right| \leq C_1', \quad \left|\frac{\partial^2 u}{\partial x^i \partial x^j}(p)\right| \leq C_1', \quad \left|\frac{\partial^2 u}{\partial y^i \partial y^j}(p)\right| \leq C_1'. \tag{32}$$

Moreover, C_1' is independent of $(\delta_{\psi,f})^{-1}$.

4.1. Proof of Proposition 2.4

The proof follows the outline of the proof of corresponding proposition in [39]. Fix $p \in \partial M = S \times \{a\} \cup S \times \{b\}$. In what follows the discussion will be given at p, and the Greek letters α, β range from 1 to $n-1$, we can assume further that $\{\underline{\mathfrak{g}}_{\alpha\bar{\beta}}\}$ is diagonal at p (otherwise we can make a suitable transformation for $\{\underline{\mathfrak{g}}_{\alpha\bar{\beta}}\}$ at p). It follows from the boundary value condition that

$$\mathfrak{g}_{\alpha\bar{\beta}} = \underline{\mathfrak{g}}_{\alpha\bar{\beta}}, \text{ at } p, \tag{33}$$

where $1 \leq \alpha, \beta \leq n-1$, $\underline{\mathfrak{g}} = \mathfrak{g}[\underline{u}]$.

Firstly, we claim that there exist two uniformly positive constants ε_0, R_0 depending on $\underline{\mathfrak{g}}$ and f, such that

$$f(\underline{\mathfrak{g}}_{1\bar{1}} - \varepsilon_0, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} - \varepsilon_0, R_0) \geq \psi \tag{34}$$

and $(\underline{\mathfrak{g}}_{1\bar{1}} - \varepsilon_0, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} - \varepsilon_0, R_0) \in \Gamma$. We leave the proof of (34) at the end of the proof of this proposition.

Next, we apply Lemma 2.5 together with (34) to establish the quantitative boundary estimates for double normal derivative. Let's denote

$$A(R) = \begin{pmatrix} \mathfrak{g}_{1\bar{1}} & & & & \mathfrak{g}_{1\bar{n}} \\ & \mathfrak{g}_{2\bar{2}} & & & \mathfrak{g}_{2\bar{n}} \\ & & \ddots & & \vdots \\ & & & \mathfrak{g}_{(n-1)\overline{(n-1)}} & \mathfrak{g}_{(n-1)\bar{n}} \\ \mathfrak{g}_{n\bar{1}} & \mathfrak{g}_{n\bar{2}} & \cdots & \mathfrak{g}_{n\overline{(n-1)}} & R \end{pmatrix}$$

and

$$\underline{A}(R) = \begin{pmatrix} \underline{\mathfrak{g}}_{1\bar{1}} & & & & \mathfrak{g}_{1\bar{n}} \\ & \underline{\mathfrak{g}}_{2\bar{2}} & & & \mathfrak{g}_{2\bar{n}} \\ & & \ddots & & \vdots \\ & & & \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} & \mathfrak{g}_{(n-1)\bar{n}} \\ \mathfrak{g}_{n\bar{1}} & \mathfrak{g}_{n\bar{2}} & \cdots & \mathfrak{g}_{n\overline{(n-1)}} & R \end{pmatrix}.$$

Let's pick $\epsilon = \frac{\varepsilon_0}{128}$ in Lemma 2.5, and we assume

$$R_c = \frac{128(2n-3)}{\varepsilon_0} \sum_{\alpha=1}^{n-1} |\mathfrak{g}_{\alpha\bar{n}}|^2 + (n-1) \sum_{\alpha=1}^{n-1} |\underline{\mathfrak{g}}_{\alpha\bar{\alpha}}| + \frac{(n-2)\varepsilon_0}{128(2n-3)} + R_0.$$

It follows from Lemma 2.5 that the eigenvalues of $\underline{A}(R_c)$ (possibly with an order) shall behavior like

$$\lambda(\underline{A}(R_c)) \in (\underline{\mathfrak{g}}_{1\bar{1}} - \frac{\varepsilon_0}{128}, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} - \frac{\varepsilon_0}{128}, R_c) + \overline{\Gamma}_n \subset \Gamma. \tag{35}$$

Applying (3), (33), (34) and (35), one hence has

$$\begin{aligned} F(A(R_c)) &= F(\underline{A}(R_c)) \\ &\geq f(\underline{\mathfrak{g}}_{1\bar{1}} - \frac{\varepsilon_0}{128}, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} - \frac{\varepsilon_0}{128}, R_c) \\ &> f(\underline{\mathfrak{g}}_{1\bar{1}} - \varepsilon_0, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} - \varepsilon_0, R_c) \geq \psi. \end{aligned}$$

Therefore,

$$\mathfrak{g}_{n\bar{n}}(p) \leq R_c.$$

To finish the proof of Proposition 2.4, what is left to prove is the key inequality (34). We propose two proofs of (34). Writing

$$\underline{B}(R) = \begin{pmatrix} \underline{\mathfrak{g}}_{1\bar{1}} & & & & \underline{\mathfrak{g}}_{1\bar{n}} \\ & \underline{\mathfrak{g}}_{2\bar{2}} & & & \underline{\mathfrak{g}}_{2\bar{n}} \\ & & \ddots & & \vdots \\ & & & \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}} & \underline{\mathfrak{g}}_{(n-1)\bar{n}} \\ \underline{\mathfrak{g}}_{n\bar{1}} & \underline{\mathfrak{g}}_{n\bar{2}} & \cdots & \underline{\mathfrak{g}}_{n\overline{(n-1)}} & R \end{pmatrix}.$$

The first proof is as in the following: For $R > \sup_{\partial M} |\underline{\mathfrak{g}}|$, one has

$$f(\lambda(\underline{B}(R))) > \psi \text{ on } \partial M. \tag{36}$$

It follows from (3), (4), (36) and the openness of Γ that

$$f(\lambda(\underline{B}(R_1)) - \frac{\epsilon_0}{2}\vec{1}) > \psi \text{ and } (\lambda(\underline{B}(R_1)) - \frac{\epsilon_0}{2}\vec{1}) \in \Gamma, \tag{37}$$

where $\vec{1} = (1, \cdots, 1) \in \mathbb{R}^n$ defined as in Section 2, and ϵ_0 (small enough) and R_1 (large enough) are two uniformly positive constants depending only on $\underline{\mathfrak{g}}$ and other known data. Moreover, by applying Lemma 2.5 to the matrix $\underline{B}(R)$ (by setting the parameter $\epsilon = \frac{\epsilon_0}{128}$ in Lemma 2.5), we know that the eigenvalues $\lambda(\underline{B}(R_2)) = (\mu_1, \cdots, \mu_n)$ ($R_2 = O(\frac{128(2n-3)}{\epsilon_0}|\underline{\mathfrak{g}}|^2)$) behavior as

$$\underline{\mathfrak{g}}_{\alpha\bar{\alpha}} - \frac{\epsilon_0}{128} < \mu_\alpha < \underline{\mathfrak{g}}_{\alpha\bar{\alpha}} + \frac{\epsilon_0}{128}, \ R_2 \leq \mu_n < R_2 + \frac{(n-1)\epsilon_0}{128}.$$

Combining it with (37) we can derive (34) by setting $\varepsilon_0 = \frac{63}{128}\epsilon_0$. We shall point out that in this proof (4) may be replaced by the convexity of the level sets of f. Moreover, condition (37) can be also derived from

$$\lim_{R\to+\infty} f(\lambda(\underline{B}(R))) > \psi \text{ on } \partial M.$$

This condition can be achieved by the boundary data φ according to Lemma 2.5 and (33). Also, this condition is satisfied by a $\mathcal{C}$-subsolution $\underline{u}$ with the same boundary value condition $\underline{u}|_{\partial M} = \varphi$.

The second proof is the following: Applying Lemma 2.5 to $\underline{B}(R)$ we can prove there is a uniformly positive constant R_3 depending on $\underline{\mathfrak{g}}$ such that

$$(\underline{\mathfrak{g}}_{1\bar{1}}, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}}, R_3) \in \Gamma.$$

Here we also use the fact that Γ is an open set. The ellipticity and concavity of equation (2), couple with Lemma 6.2 in [2], therefore yield that

$$F(A) - F(B) \geq F^{i\bar{j}}(A)(a_{i\bar{j}} - b_{i\bar{j}})$$

for the Hessian matrices $A = \{a_{i\bar{j}}\}$ and $B = \{b_{i\bar{j}}\}$ with $\lambda(A)$, $\lambda(B) \in \Gamma$. Thus, there exists a uniformly positive constant $R_4 \geq R_3$ depending only on $\underline{\mathfrak{g}}$ such that

$$\begin{aligned} f(\underline{\mathfrak{g}}_{1\bar{1}}, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}}, R_4) &= F(\mathrm{diag}(\underline{\mathfrak{g}}_{1\bar{1}}, \cdots, \underline{\mathfrak{g}}_{(n-1)\overline{(n-1)}}, R_4)) \\ &> F(\underline{\mathfrak{g}}) \geq \psi. \end{aligned}$$

Thus one can derive (34). We thus complete the proof of Proposition 2.4.

4.2. Proof of quantitative boundary estimates for mixed derivatives

In this subsection we establish the quantitative boundary estimates for mixed (tangential-normal) derivatives. It is an extension of Proposition 1 in [26].

Proposition 4.1. *Let $u \in C^3(M) \cap C^2(\bar{M})$ be any admissible solution to Dirichlet problem (2). Let $\psi \in C^1_B(M) \cap C^{0,1}(\bar{M})$, we also assume that conditions (3), (4), (5) and (8) hold. Then for any $1 \leq \alpha \leq n-1$ there is a positive constant C depending on $|\underline{u}|_{C^2(\bar{M})}$, $|\varphi|_{C^{2,1}(\bar{M})}$, $|\psi|_{C^{0,1}(\bar{M})}$ and other known data, such that*

$$\left|\frac{\partial^2 u}{\partial r \partial z^\alpha}(p)\right| \leq C(1 + \sup_{\bar{M}} |\nabla u|). \tag{38}$$

Moreover, the constant C in (38) does not depend on $(\delta_{\psi,f})^{-1}$.

Proposition 4.1 yields that the quantitative boundary estimates for the mixed (tangential-normal) derivatives

$$|\mathfrak{g}_{\alpha\bar{n}}(p)|, \ |\mathfrak{g}_{n\bar{\alpha}}(p)| \leq C(1 + \sup_{\bar{M}} |\nabla u|), \tag{39}$$

for any $1 \leq \alpha \leq n-1$. Here we use the fact that u is basic according to Lemma 1.1.

The quantitative boundary estimates for mixed derivatives will be proved by constructing barrier functions. This type of construction of barrier functions originally follows from [22, 18, 13].

Let $b_1 = \sqrt{1 + 2\sup_{\bar{M}} |\nabla u|^2 + 2\sup_{\bar{M}} |\nabla \varphi|^2}$. The barrier function is given by

$$\Psi = A_1 b_1 v - A_2 b_1 \rho^2 + \frac{A_3}{b_1} \sum_{\tau=1}^{n-1} |X_\tau (u - \varphi)|^2, \tag{40}$$

where A_1, A_2, A_3 are positive uniform constants to be determined, $\rho(z) = \mathrm{dist}_{\bar{M}}(z, p)$ is as defined before (23), and v is the function defined by

$$v = \begin{cases} \underline{u} - u + \dfrac{N}{2}(r-a)^2 - t(r-a) \text{ in } S \times (a, a+\delta), \\ \underline{u} - u + \dfrac{N}{2}(b-r)^2 - t(b-r) \text{ in } S \times (b-\delta, b). \end{cases} \tag{41}$$

We choose δ small enough ($\delta \leq \frac{2t}{N}$), such that $v \leq 0$ in $M_\delta := S \times (a, a+\delta) \cup S \times (b-\delta, b)$. Moreover, $|\nabla r| = \frac{1}{2}$, $\mathcal{L}(r) = 0$ in M_δ, and

$$\mathcal{L}v = \mathcal{L}(\underline{u} - u) + \frac{N}{4} F^{n\bar{n}}. \tag{42}$$

Next, we are going to prove

Lemma 4.2. *Given some constants $A_1 \gg A_2 \gg A_3 \gg 1$ and small $\delta > 0$, one has $\Psi(p) = 0$ and*

$$\begin{cases} \mathcal{L}(\Psi) \geq 1 + b_1 F^{i\bar{j}} \bar{g}_{i\bar{j}} & \text{in } \Omega_\delta, \\ \Psi \leq 0 & \text{on } \partial\Omega_\delta \cap \partial M, \\ \Psi \leq -b_1 & \text{on } \partial\Omega_\delta \cap \bar{\Omega}_\delta. \end{cases} \tag{43}$$

A key ingredient in the proof of Lemma 4.2 is the following lemma proved in [17].

Lemma 4.3 ([17])**.** *$\varepsilon' > 0$ such that when $|\nu_\mu - \nu_\lambda| \geq \beta$ Suppose that f satisfies (3) and (4). Let K be a compact subset of Γ and $\beta > 0$. There is a constant $\varepsilon > 0$ such that, for any $\mu \in K$ and $\lambda \in \Gamma$, when $|\nu_\mu - \nu_\lambda| \geq \beta$,*

$$\sum_{i=1}^{n} f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon(1 + \sum_{i=1}^{n} f_i(\lambda)) \tag{44}$$

where $\nu_\lambda = Df(\lambda)/|Df(\lambda)|$ denotes the unit normal vector to the level surface of f through λ.

When equation (2) becomes degenerate ($\delta_{\psi,f} = 0$), we shall use the following observation [17, 38].

Lemma 4.4. *If (3)-(4) hold, then for any $\lambda \in \Gamma$ with $|\lambda| \leq R$ one has*

$$\sum_{i=1}^{n} f_i(\lambda) \geq \frac{1}{1+2R}(f((1+R)\vec{1}) - f(R\vec{1})) > 0. \tag{45}$$

Proof. Using the formula

$$t \sum_{i=1}^{n} f_i(\lambda) \geq \sum_{i=1}^{n} f_i(\lambda)\lambda_i + f(t\vec{1}) - f(\lambda), \tag{46}$$

we can derive that $t\sum_{i=1}^{n} f_i(\lambda) \geq -R\sum_{i=1}^{n} f_i(\lambda) + f(t\vec{1}) - f(|\lambda|\vec{1})$. Then one has (45) by setting $t = 1 + R$. □

Proof of Lemma 4.2. It is easy to verify that $F^{i\bar{j}}\bar{g}_{i\bar{j}} = \sum_{i=1}^{n} f_i$. We know that there is a uniform positive constant C_ρ such that

$$|\mathcal{L}(\rho^2)| \leq C_\rho \sum f_i \text{ in } \Omega_\delta. \tag{47}$$

The key step of the proof is to estimate

$$\mathcal{L}(\sum_{\tau=1}^{n-1} |X_\tau(u-\varphi)|^2) = \sum_{\tau=1}^{n-1} \mathcal{L}(|X_\tau(u-\varphi)|^2).$$

From Lemma 1.1 and the construction of φ as in (13), we know that both u and φ are basic, and so $X_k(u-\varphi) = \frac{\partial}{\partial z^k}(u-\varphi)$, $\bar{X}_i(u-\varphi) = \frac{\partial}{\partial \bar{z}^k}(u-\varphi)$ for each $1 \leq k \leq n-1$. Similar to the computation in (4.9) of Guan-Zhang [21], fix $1 \leq k \leq n-1$,

$$\begin{aligned}
|\mathcal{L}(X_k(u-\varphi))| &= |F^{i\bar{j}} X_i \bar{X}_j X_k u - \mathcal{L}(X_k\varphi)| \\
&= |F^{i\bar{j}} X_k X_i \bar{X}_j u - \mathcal{L}(X_k\varphi)| \\
&= |F^{i\bar{j}} X_k (\mathfrak{g}_{i\bar{j}} - \chi_{i\bar{j}}) - \mathcal{L}(X_k\varphi)| \\
&\leq |X_k\psi| + C\sum_{i,j=1}^{n} F^{i\bar{j}} g_{i\bar{j}},
\end{aligned}$$

and $|\mathcal{L}(\bar{X}_k(u-\varphi))| \leq |\bar{X}_k\psi| + C\sum_{i,j=1}^{n} F^{i\bar{j}} g_{i\bar{j}}$. In the proof, the second equality follows from the basic of solution u, and

$$\begin{aligned}
&[X_n, \bar{X}_n] = -\frac{1}{2}\sqrt{-1}r^{-2}\xi,\ [X_\alpha, \bar{X}_\beta] = -2\sqrt{-1}h_{\alpha\bar{\beta}}\xi, \\
&[X_\alpha, X_\beta] = [X_\alpha, X_n] = [\bar{X}_\alpha, \bar{X}_\beta] = [X_\alpha, \bar{X}_n] = 0
\end{aligned}$$

for $\alpha, \beta < n$. Thus

$$\begin{aligned}
\mathcal{L}(|X_k(u-\varphi)|^2) &= X_k(u-\varphi)\mathcal{L}(\bar{X}_k(u-\varphi)) + \bar{X}_k(u-\varphi)\mathcal{L}(X_k(u-\varphi)) \\
&\quad + F^{i\bar{j}}(X_i X_k(u-\varphi))(\bar{X}_j \bar{X}_k(u-\varphi)) \\
&\quad + F^{i\bar{j}}(X_i \bar{X}_k(u-\varphi))(\bar{X}_j X_k(u-\varphi)) \\
&\geq \frac{1}{2}F^{i\bar{j}}\mathfrak{g}_{i\bar{k}}\mathfrak{g}_{k\bar{j}} - Cb_1(1+\sum f_i),
\end{aligned} \tag{48}$$

where we use the inequality $F^{i\bar{j}}(a_i - b_i)(\bar{a}_j - \bar{b}_j) \geq \frac{1}{2}F^{i\bar{j}} a_i \bar{a}_j - F^{i\bar{j}} b_i \bar{b}_j$.

As in Proposition 2.19 of [14], there is an index r such that

$$\sum_{\tau=1}^{n-1} F^{i\bar{j}} \mathfrak{g}_{i\bar{\tau}} \mathfrak{g}_{\tau\bar{j}} \geq \frac{1}{4} \sum_{i\neq r} f_i \lambda_i^2. \tag{49}$$

For completeness we leave the proof of (49) at the end of the proof.

Therefore, it follows from (47), (48) and (49) that

$$\mathcal{L}\left(\Psi\right) \geq A_1 b_1 \mathcal{L} v + \frac{A_3}{8b_1} \sum_{i\neq r} f_i \lambda_i^2 - A_3 C_1 - (A_2 C_\rho b_1 + A_3 C_1) \sum_{i=1}^{n} f_i. \tag{50}$$

Denote by $\lambda[v] := \lambda(\mathfrak{g}[v])$ for convenience. For the *admissible* subsolution $\underline{u}$, $\lambda[\underline{u}]$ falls in a compact subset of Γ,

$$\beta := \frac{1}{2} \min_{\bar{M}} dist(\nu_{\lambda[\underline{u}]}, \partial\Gamma_n) > 0. \tag{51}$$

Case I: Lemma 4.3 implies that when $|\nu_{\lambda[u]} - \nu_{\lambda[\underline{u}]}| \geq \beta$, we have

$$\mathcal{L}(\underline{u} - u) \geq \varepsilon(1 + \sum_{i=1}^{n} f_i),$$

where ε is the positive constant in Lemma 4.3 determined by f, β and $\lambda[\underline{u}]$.

Let $A_1 \gg A_2 \gg A_3 \gg 1$. By (50) and (42), we have

$$\mathcal{L}(\Psi) \geq b_1(1 + \sum_{i=1}^{n} f_i).$$

Case II: Suppose that $|\nu_{\lambda[u]} - \nu_{\lambda[\underline{u}]}| < \beta$. Then $\nu_{\lambda[u]} - \beta\vec{1} \in \Gamma_n$ and

$$f_i \geq \frac{\beta}{\sqrt{n}} \sum_{j=1}^{n} f_j.$$

Combining it with (42) one derives

$$\mathcal{L} v \geq \frac{\beta N}{4\sqrt{n}} \sum_{i=1}^{n} f_i,$$

where we use $\mathcal{L}(\underline{u} - u) \geq 0$ in M.

As in [17] there exist two uniformly positive constants c_0 and C_0, such that

$$\sum_{i\neq r} f_i\lambda_i^2 \geq c_0|\lambda|^2\sum_{i=1}^n f_i - C_0\sum_{i=1}^n f_i, \tag{52}$$

where c_0 depends only on β and n, while C_0 depends only on β, n and $|\lambda[\underline{u}]|$. By setting A_1 large such that

$$\frac{A_1 N\beta}{\sqrt{n}} - 8A_2C_\rho - 8A_3C_1 - \frac{A_3C_0}{b_1} \geq 0,$$

then we have

$$\begin{aligned} \mathcal{L}(\Psi) &\geq \Big(\frac{A_1N\beta b_1}{4\sqrt{n}} - A_2C_\rho b_1 - A_3C_1 - \frac{A_3C_0}{8b_1}\Big)\sum_{i=1}^n f_i \\ &\quad + \frac{A_3c_0}{8b_1}|\lambda|^2\sum_{i=1}^n f_i - A_3C_1 \\ &\geq \frac{A_1N\beta}{16\sqrt{n}}b_1\sum_{i=1}^n f_i + \frac{1}{8}\sqrt{\frac{A_1A_3c_0N\beta}{2b_1\sqrt{n}}}|\lambda|\sum_{i=1}^n f_i - A_3C_1, \end{aligned} \tag{53}$$

where we use the elementary inequality $a+b\geq 2\sqrt{ab}$ for $a,b\geq 0$.

We know that if $|\lambda|\geq R_0 \equiv 1+\sup_{\bar{M}}|\lambda[\underline{u}]|$, then

$$|\lambda|\sum_{i=1}^n f_i(\lambda) \geq b_0, \tag{54}$$

where $b_0 \equiv \frac{1}{2}\{f(R_0\vec{1}) - \sup_{\bar{M}} f(\lambda[\underline{u}])\}$. Let us verify it here. By the concavity of f,

$$|\lambda|\sum f_i(\lambda) \geq f(|\lambda|\vec{1}) - f(\lambda[\underline{u}]) - |\lambda|\sum f_i(\lambda),$$

thus

$$|\lambda|\sum f_i(\lambda) \geq \frac{1}{2}(f(|\lambda|\vec{1}) - f(\lambda[\underline{u}])).$$

So (54) holds by setting $|\lambda|\geq R_0$.

Therefore, by using (54) and Lemma 4.4 we derive that if $A_1 \gg A_2 \gg A_3 \gg 1$ then

$$\mathcal{L}(\Psi) \geq 1 + b_1\sum_{i=1}^n f_i.$$

Moreover,

$$\begin{cases} \Psi \leq 0 & \text{on } \partial\Omega_\delta \cap \partial M, \\ \Psi \leq -b_1 & \text{on } \partial\Omega_\delta \cap \Omega_\delta. \end{cases}$$

We sketch the proof of (49) to complete the proof: Let $U = (a_{ij})$ be a $n \times n$ unitary matrix that simultaneously diagonalizes $(F^{i\bar{j}})$ and $(\mathfrak{g}_{i\bar{j}})$ at a fixed point. That is

$$\left(F^{i\bar{j}}\right) = U^* \mathrm{diag}(f_1, \cdots, f_n) U, \ \left(\mathfrak{g}_{i\bar{j}}\right) = U^* \mathrm{diag}(\lambda_1, \cdots, \lambda_n) U.$$

Here $U^* = (b_{ij})$, $b_{ij} = \overline{a_{ji}}$. Since U is unitary, $U^* = U^{-1}$. Thus $\left(\mathfrak{g}_{i\bar{j}}\right) \cdot (F^{i\bar{j}}) \cdot \left(\mathfrak{g}_{i\bar{j}}\right) = U^* \mathrm{diag}(f_1 \lambda_1^2, \cdots, f_n \lambda_n^2) U$, which implies

$$\sum_{i,j=1}^{n} F^{i\bar{j}} \mathfrak{g}_{i\bar{\tau}} \mathfrak{g}_{\tau\bar{j}} = \sum_{k=1}^{n} f_k \lambda_k^2 |a_{k\tau}|^2$$

for fixed τ. So

$$\sum_{\tau=1}^{n-1} \sum_{i,j=1}^{n} F^{i\bar{j}} \mathfrak{g}_{i\bar{\tau}} \mathfrak{g}_{\tau\bar{j}} = \sum_{k=1}^{n} f_k \lambda_k^2 (1 - |a_{kn}|^2) \geq \frac{n-1}{n} \sum_{k \neq r} f_k \lambda_k^2 \geq \frac{1}{4} \sum_{k \neq r} f_k \lambda_k^2,$$

where $|a_{rn}|^2 \leq \frac{1}{n}$ corresponding to r (such a_{rn} exists). $\square$

The following lemma can be found in [26].

Lemma 4.5. *Let φ be the function which is defined in (13). There is a positive constant C depending on $|\chi|_{C^{0,1}(\bar{M})}$, $|\varphi|_{C^{2,1}(\bar{M})}$ and other known data, such that*

$$|\mathcal{L} X_i (u - \varphi)| \leq \sup_{\Omega_\delta} |X_i \psi| + C F^{i\bar{j}} \bar{g}_{i\bar{j}} \ \textit{in } \Omega_\delta \tag{55}$$

for $1 \leq i \leq n$. Moreover, C is independent of $(\delta_{\psi,f})^{-1}$.

Combining Lemma 4.2 with Lemma 4.5, we obtain Proposition 4.1.

5. The equations with right-hand side depending on unknown solutions

Let's turn our attention to Dirichlet problem for the equations with right-hand side $\psi[u] = \psi(z, u)$ with $\psi_u \geq 0$,

$$\begin{cases} f(\lambda(\mathfrak{g}[u])) &= \psi[u] \text{ in } M, \\ \quad u|_{r=a} &= \varphi_a, \\ \quad u|_{r=b} &= \varphi_b. \end{cases} \tag{56}$$

It is similar to the proof of Lemma 1.1, we can conclude that every *admissible* solution $u \in C^3(M) \cap C^1(\bar{M})$ of equation (56) is **basic** with assuming

$$\nabla_\xi \chi = 0, \nabla'_\xi \psi = 0, \text{ and } \ \varphi_a, \varphi_b \text{ are both } \textbf{basic} \tag{57}$$

where $\nabla'_\xi \psi$ denotes the partial covariant derivative (along the Reeb field ξ) of $\psi[u]$ when viewed as depending on $z \in M$ only.

If f satisfies (9) and (64), then any C^2-*admissible* function $\underline{u} \in C^2(\bar{M})$ must satisfy

$$\lim_{t \to +\infty} f(\lambda(\mathfrak{g}[\underline{u}]) + te_i) > \psi[u] \text{ in } \bar{M} \text{ for each } i, \tag{58}$$

where e_i is the i-th standard basis vector, and u is the unknown *admissible* solution. Such a condition is hard to verify, since the right hand side depends on the unknown solution u. It is a slight modification of the notion of a $\mathcal{C}$-subsolution introduced by Székelyhidi [30], thereby allowing one to use the following lemma, according to Székelyhidi's insight.

Lemma 5.1. *Suppose that there exists a $\mathcal{C}$-subsolution $\underline{u} \in C^2(\bar{M})$. Then there exist two positive constants R_0 and ε with the following property. If $|\lambda| \geq R_0$, then either*

$$F^{i\bar{j}}(\underline{\mathfrak{g}}_{i\bar{j}} - \mathfrak{g}_{i\bar{j}}) \geq \varepsilon' F^{i\bar{j}} \bar{g}_{i\bar{j}} \tag{59}$$

or

$$F^{i\bar{j}} \geq \varepsilon'(F^{p\bar{q}} \bar{g}_{p\bar{q}}) \bar{g}^{i\bar{j}}. \tag{60}$$

In this paper, the lower bound of $\sum_{i=1}^n f_i$

$$\sum_{i=1}^n f_i \geq \kappa(\sigma) \text{ in } \partial\Gamma^\sigma := \{\lambda \in \Gamma : f(\lambda) = \sigma\}, \tag{61}$$

plays an important role in the proof of (67), as $\sum_{i=1}^n f_i(\underline{\lambda}_i - \lambda_i) < 0$ may occur at some points when $\psi_u \geq 0$. Part (b) of Lemma 9 in Székelyhidi [30] states (61) holds if f satisfies (3), (4) and (6). In a previous paper [39] the author gave a characterization of level sets of f satisfying (3), (4) and (6) and then proved the following Lemma 5.2 which slightly extends (8)′ of Caffarelli-Nirenberg-Spruck [2], thereby giving a new proof of (61); moreover, couple with (46), we can prove that $\kappa(\sigma)$ can be chosen as $\kappa(\sigma) = \frac{f((1+c_\sigma)\vec{1})-\sigma}{1+c_\sigma}$ which is independent of $\delta_{\psi,f}$, where $\vec{1} = (1, \cdots, 1) \in \mathbb{R}^n$ and c_σ is the positive constant such that $f(c_\sigma \vec{1}) = \sigma$.

Lemma 5.2. *Assume f satisfies (3), (4) and (6). Then $\sum_{i=1}^n f_i \lambda_i > 0$ in Γ.*

To construct the *admissible* subsolution or (modified) $\mathcal{C}$-subsolution obeying (58), we hence need assuming that there is a **basic** function $\underline{v} \in C_B^{4,\alpha}(\bar{M})$ such that for large R, one has

$$\underline{v}|_{r=a} = \varphi_a, \underline{v}|_{r=b} = \varphi_b, \psi[\underline{v}] < \sup_{\Gamma} f, \lambda(\mathfrak{g}[\underline{v}] + \sqrt{-1} R \theta^n \wedge \bar{\theta}^n) \in \Gamma. \tag{62}$$

By the maximum principle, one derives $\underline{u} \leq \underline{v}$. Based on $\underline{v}$, as in (18), we can construct the desired *admissible* subsolution $\underline{u}$ if (9) holds. (Here we also use $\psi[\underline{u}] \leq \psi[\underline{v}]$ which follows from $\underline{u} \leq \underline{v}$ and $\psi_u \geq 0$). Namely, there is a **basic** *admissible* function $\underline{u} \in C_B^{4,\alpha}(\bar{M})$ with

$$\begin{cases} f(\lambda(\mathfrak{g}[\underline{u}])) \geq \psi[\underline{u}] \text{ in } M, \\ u|_{r=a} = \varphi_a, \\ u|_{r=b} = \varphi_b. \end{cases} \tag{63}$$

Let w be the supersolution obeying (21). Then the comparison principle implies (22) holds in this case, and $\psi[w] \geq \psi[u] \geq \psi[\underline{u}]$ as $\psi_u \geq 0$. We need moreover assuming

$$\psi[w] < \sup_{\Gamma} f \tag{64}$$

so that one can construct (modified) $\mathcal{C}$-subsolutions by using (9) and (62).

Theorem 5.3. *Let $\psi[u] = \psi(z, u)$ be a smooth function with $\psi_u \geq 0$ and*

$$\inf_{z \in \bar{M}} \psi(z, t) > \sup_{\partial \Gamma} f \text{ for any fixed } -\infty < t < +\infty \tag{65}$$

Suppose that (3), (4), (6), (57), (9), (62) and (64) hold. Then Dirichlet problem (56) is uniquely solvable in class of smoothly basic admissible functions.

Proof. Let $\underline{u}$ be the *admissible* subsolution satisfying (63) which is constructed in (18), and we denote $\underline{\lambda} = \lambda(\mathfrak{g}[\underline{u}])$. We need only to prove (39) for *admissible* solutions of Dirichlet problem (56), as the global second order estimate

$$\sup_{\bar{M}} |\Delta u| \leq C(1 + \sup_{\bar{M}} |\nabla u|^2 + \sup_{\partial M} |\Delta u|)$$

can be derive by a slight modification of the proof of second order estimate in [26].

We assume $|\lambda| \geq R_0$ (otherwise the equation is uniformly elliptic and the proof is trivial). If (59) holds then the proof is almost same as that in Proposition 4.1. As above, (52) is important for the boundary estimates when (60) holds, while the original proof of (52) uses $\sum_{i=1}^n f_i(\underline{\lambda}_i - \lambda_i) \geq 0$, which can be derived from (4) and $\psi_u = 0$. However, in our case that $\psi_u \geq 0$, one has

$$\sum_{i=1}^{n} f_i(\underline{\lambda}_i - \lambda_i) \geq \psi[\underline{u}] - \psi[u], \tag{66}$$

which implies that $\sum_{i=1}^n f_i(\underline{\lambda}_i - \lambda_i) < 0$ may occur. Next, we give a proof of

$$\sum_{i\neq r} f_i \lambda_i^2 \geq c_0' |\lambda|^2 \sum_{i=1}^{n} f_i - C_0' \sum_{i=1}^{n} f_i \tag{67}$$

without using $\sum_{i=1}^n f_i(\underline{\lambda}_i - \lambda_i) \geq 0$ but with using (61).

If $\lambda_r \leq 0$ then $\sum_{i\neq r} \lambda_i > |\lambda_r|$ and $\lambda_r^2 \leq (n-1)\sum_{i\neq r}\lambda_i^2$. So $\sum_{i\neq r}\lambda_i^2 \geq \frac{1}{n}|\lambda|^2$ and (67) holds for $c_0' = \frac{\varepsilon'}{n}$ and $C_0' = 0$ (where ε' is the constant in Lemma 5.1). If $\lambda_r > 0$, then one has

$$f_r^2\lambda_r^2 \leq 4(\psi[u] - \psi[\underline{u}])^2 + 2\sup_{\bar{M}} |\underline{\lambda}|^2 (\sum_{i=1}^{n} f_i)^2 + 2(n-1)\sum_{i\neq r} f_i^2 \lambda_i^2,$$

here we use (66) and Cauchy-Schwarz inequality. Thus

$$\sum_{i\neq r} f_i\lambda_i^2 \geq \frac{\varepsilon'^2}{2n-1}|\lambda|^2 \sum_{i=1}^{n} f_i - \frac{2\sup_{\bar{M}}|\underline{\lambda}|^2}{2n-1}\sum_{i=1}^{n} f_i - \frac{4(\psi[u]-\psi[\underline{u}])^2}{(2n-1)\sum_{i=1}^n f_i},$$

and (67) holds by using (61).

Following the line of proof of Proposition 4.1 we can derive (39), and then obtain quantitative boundary estimates (14). □

Appendix A

In this appendix, we will give the proof of Lemma 2.5 building on [39].

We start with the case of $n = 2$. In this case, we prove that if $\mathbf{a} \geq \frac{|a_1|^2}{\epsilon} + d_1$ then

$$0 \leq d_1 - \lambda_1 = \lambda_2 - \mathbf{a} < \epsilon.$$

Let's briefly present the discussion as follows: For $n = 2$, the eigenvalues of A are $\lambda_1 = \frac{\mathbf{a}+d_1-\sqrt{(\mathbf{a}-d_1)^2+4|a_1|^2}}{2}$ and $\lambda_2 = \frac{\mathbf{a}+d_1+\sqrt{(\mathbf{a}-d_1)^2+4|a_1|^2}}{2}$. We can assume $a_1 \neq 0$; otherwise we are done. If $\mathbf{a} \geq \frac{|a_1|^2}{\epsilon} + d_1$ then one has

$$0 \leq d_1 - \lambda_1 = \lambda_2 - \mathbf{a} = \frac{2|a_1|^2}{\sqrt{(\mathbf{a}-d_1)^2 + 4|a_1|^2} + (\mathbf{a} - d_1)} < \frac{|a_1|^2}{\mathbf{a} - d_1} \leq \epsilon.$$

Here we use $a_1 \neq 0$ to verify that the strictly inequality in the above formula holds. We hence obtain Lemma 2.5 for $n = 2$.

The following lemma enables us to count the eigenvalues near the diagonal elements via a deformation argument. It is an essential ingredient in the proof of Lemma 2.5 for general n.

Lemma A.1 ([39]). *Let* A *be an* $n \times n$ *Hermitian matrix*

$$\begin{pmatrix} d_1 & & & & a_1 \\ & d_2 & & & a_2 \\ & & \ddots & & \vdots \\ & & & d_{n-1} & a_{n-1} \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & \mathbf{a} \end{pmatrix}$$

with $d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1}$ *fixed, and with* $\mathbf{a}$ *variable. Denote* $\lambda_1, \cdots, \lambda_n$ *by the eigenvalues of* A *with the order* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. *Fix a positive constant* ϵ. *Suppose that the parameter* $\mathbf{a}$ *in the matrix* A *satisfies the following quadratic growth condition*

$$\mathbf{a} \geq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + \sum_{i=1}^{n-1} [d_i + (n-2)|d_i|] + (n-2)\epsilon. \tag{68}$$

Then for any λ_α $(1 \leq \alpha \leq n-1)$ *there exists an* d_{i_α} *with lower index* $1 \leq i_\alpha \leq n-1$ *such that*

$$|\lambda_\alpha - d_{i_\alpha}| < \epsilon, \tag{69}$$

$$0 \leq \lambda_n - \mathbf{a} < (n-1)\epsilon + |\sum_{\alpha=1}^{n-1}(d_\alpha - d_{i_\alpha})|. \tag{70}$$

Proof. Without loss of generality, we assume $\sum_{i=1}^{n-1}|a_i|^2 > 0$ and $n \geq 3$ (otherwise we are done, since A is diagonal or $n = 2$). Note that in the assumption of the lemma the eigenvalues have the order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. It is well known that, for a Hermitian matrix, any diagonal element is less than or equals to the largest eigenvalue. In particular,

$$\lambda_n \geq \mathbf{a}. \tag{71}$$

We only need to prove (69), since (70) is a consequence of (69), (71) and

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}(\mathrm{A}) = \sum_{\alpha=1}^{n-1} d_\alpha + \mathbf{a}. \tag{72}$$

Let's denote $I = \{1, 2, \cdots, n-1\}$. We divide the index set I into two subsets by

$$\mathbf{B} = \{\alpha \in I : |\lambda_\alpha - d_i| \geq \epsilon, \ \forall i \in I\}$$

and $\mathbf{G} = I \setminus \mathbf{B} = \{\alpha \in I : \text{There exists an } i \in I \text{ such that } |\lambda_\alpha - d_i| < \epsilon\}$.

To complete the proof we need to prove $\mathbf{G} = I$ or equivalently $\mathbf{B} = \emptyset$. It is easy to see that for any $\alpha \in \mathbf{G}$, one has

$$|\lambda_\alpha| < \sum_{i=1}^{n-1} |d_i| + \epsilon. \tag{73}$$

Fix $\alpha \in \mathbf{B}$, we are going to give the estimate for λ_α. The eigenvalue λ_α satisfies

$$(\lambda_\alpha - \mathbf{a}) \prod_{i=1}^{n-1} (\lambda_\alpha - d_i) = \sum_{i=1}^{n-1} (|a_i|^2 \prod_{j \neq i} (\lambda_\alpha - d_j)). \tag{74}$$

By the definition of $\mathbf{B}$, for $\alpha \in \mathbf{B}$, one then has $|\lambda_\alpha - d_i| \geq \epsilon$ for any $i \in I$. We therefore derive

$$|\lambda_\alpha - \mathbf{a}| = \left| \sum_{i=1}^{n-1} \frac{|a_i|^2}{\lambda_\alpha - d_i} \right| \leq \sum_{i=1}^{n-1} \frac{|a_i|^2}{|\lambda_\alpha - d_i|} \leq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2, \text{ if } \alpha \in \mathbf{B}. \tag{75}$$

Hence, for $\alpha \in \mathbf{B}$, we obtain

$$\lambda_\alpha \geq \mathbf{a} - \frac{1}{\epsilon}\sum_{i=1}^{n-1}|a_i|^2. \tag{76}$$

For a set $\mathbf{S}$, we denote $|\mathbf{S}|$ the cardinality of $\mathbf{S}$. We shall use proof by contradiction to prove $\mathbf{B} = \emptyset$. Assume $\mathbf{B} \neq \emptyset$. Then $|\mathbf{B}| \geq 1$, and so $|\mathbf{G}| = n - 1 - |\mathbf{B}| \leq n - 2$.

In the case of $\mathbf{G} \neq \emptyset$, we compute the trace of the matrix A as follows:

$$\begin{aligned}
\mathrm{tr(A)} &= \lambda_n + \sum_{\alpha\in\mathbf{B}}\lambda_\alpha + \sum_{\alpha\in\mathbf{G}}\lambda_\alpha \\
&> \lambda_n + |\mathbf{B}|(\mathbf{a} - \frac{1}{\epsilon}\sum_{i=1}^{n-1}|a_i|^2) - |\mathbf{G}|(\sum_{i=1}^{n-1}|d_i| + \epsilon) \\
&\geq 2\mathbf{a} - \frac{1}{\epsilon}\sum_{i=1}^{n-1}|a_i|^2 - (n-2)(\sum_{i=1}^{n-1}|d_i| + \epsilon) \\
&\geq \sum_{i=1}^{n-1} d_i + \mathbf{a} = \mathrm{tr(A)},
\end{aligned} \tag{77}$$

where we use (68), (71), (73) and (76). This is a contradiction.

In the case of $\mathbf{G} = \emptyset$, one knows that

$$\mathrm{tr(A)} \geq \mathbf{a} + (n-1)(\mathbf{a} - \frac{1}{\epsilon}\sum_{i=1}^{n-1}|a_i|^2) > \sum_{i=1}^{n-1} d_i + \mathbf{a} = \mathrm{tr(A)}. \tag{78}$$

Again, it is a contradiction.

We now prove $\mathbf{B} = \emptyset$. Therefore, $\mathbf{G} = I$ and the proof is complete. □

Consequently we can apply it to prove Lemma 2.5 via a deformation argument.

Proof of Lemma 2.5. Without loss of generality, we assume $n \geq 3$ and $\sum_{i=1}^{n-1}|a_i|^2 > 0$ (otherwise $n = 2$ or the matrix A is diagonal, and then we are done). Fix $a_1, \cdots, a_{n-1}$, $d_1, \cdots, d_{n-1}$. Denote $\lambda_1(\mathbf{a}), \cdots, \lambda_n(\mathbf{a})$ by the eigenvalues of A with the order $\lambda_1(\mathbf{a}) \leq \cdots \leq \lambda_n(\mathbf{a})$. Clearly, the eigenvalues $\lambda_i(\mathbf{a})$ are all continuous functions in $\mathbf{a}$. For simplicity, we write $\lambda_i = \lambda_i(\mathbf{a})$.

Fix $\epsilon > 0$. Let $I'_\alpha = (d_\alpha - \frac{\epsilon}{2n-3}, d_\alpha + \frac{\epsilon}{2n-3})$ and

$$P'_0 = \frac{2n-3}{\epsilon}\sum_{i=1}^{n-1}|a_i|^2 + (n-1)\sum_{i=1}^{n-1}|d_i| + \frac{(n-2)\epsilon}{2n-3}.$$

In what follows we assume $\mathbf{a} \geq P_0'$ (i.e. (15) holds). The connected components of $\bigcup_{\alpha=1}^{n-1} I_\alpha'$ are as in the following:

$$J_1 = \bigcup_{\alpha=1}^{j_1} I_\alpha', J_2 = \bigcup_{\alpha=j_1+1}^{j_2} I_\alpha' \cdots, J_i = \bigcup_{\alpha=j_{i-1}+1}^{j_i} I_\alpha' \cdots, J_m = \bigcup_{\alpha=j_{m-1}+1}^{n-1} I_\alpha'.$$

(Here we denote $j_0 = 0$ and $j_m = n-1$). Moreover,

$$J_i \bigcap J_k = \emptyset, \text{ for } 1 \leq i < k \leq m.$$

Let

$$\widetilde{\mathbf{Card}}_k : [P_0', +\infty) \to \mathbb{N}$$

be the function that counts the eigenvalues which lie in J_k. (Note that when the eigenvalues are not distinct, the function $\widetilde{\mathbf{Card}}_k$ denotes the summation of all the multiplicities of distinct eigenvalues which lie in J_k). This function measures the number of the eigenvalues which lie in J_k.

The crucial ingredient is that Lemma A.1 yields the continuity of $\widetilde{\mathbf{Card}}_i(\mathbf{a})$ for $\mathbf{a} \geq P_0'$. More explicitly, by using Lemma A.1 and

$$\lambda_n \geq \mathbf{a} \geq P_0' > \sum_{i=1}^{n-1} |d_i| + \frac{\epsilon}{2n-3}$$

we conclude that if $\mathbf{a}$ satisfies the quadratic growth condition (15) then

$$\begin{aligned} \lambda_n &\in \mathbb{R} \setminus (\bigcup_{k=1}^{n-1} \overline{I_k'}) = \mathbb{R} \setminus (\bigcup_{i=1}^{m} \overline{J_i}), \\ \lambda_\alpha &\in \bigcup_{i=1}^{n-1} I_i' = \bigcup_{i=1}^{m} J_i \text{ for } 1 \leq \alpha \leq n-1. \end{aligned} \tag{79}$$

Hence, $\widetilde{\mathbf{Card}}_i(\mathbf{a})$ is a continuous function in the variable $\mathbf{a}$. So it is a constant. Together with the line of the proof Lemma 1.2 of Caffarelli-Nirenberg-Spruck [2] we see that $\widetilde{\mathbf{Card}}_i(\mathbf{a}) = j_i - j_{i-1}$ for sufficiently large $\mathbf{a}$. The constant of $\widetilde{\mathbf{Card}}_i$ therefore follows that

$$\widetilde{\mathbf{Card}}_i(\mathbf{a}) = j_i - j_{i-1}.$$

We thus know that the $(j_i - j_{i-1})$ eigenvalues

$$\lambda_{j_{i-1}+1}, \lambda_{j_{i-1}+2}, \cdots, \lambda_{j_i}$$

lie in the connected component J_i. Thus, for any $j_{i-1}+1 \le \gamma \le j_i$, we have $I'_\gamma \subset J_i$ and λ_γ lies in the connected component J_i. Therefore,

$$|\lambda_\gamma - d_\gamma| < \frac{(2(j_i - j_{i-1}) - 1)\epsilon}{2n-3} \le \epsilon.$$

Here we also use the fact that d_γ is midpoint of I'_γ and every $J_i \subset \mathbb{R}$ is an open subset.

To be brief, if for fixed index $1 \le i \le n-1$ the eigenvalue $\lambda_i(P'_0)$ lies in J_α for some α, then Lemma A.1 implies that, for any $\mathbf{a} > P'_0$, the corresponding eigenvalue $\lambda_i(\mathbf{a})$ lies in the same interval J_α. Adapting the line of the proof Lemma 1.2 of Caffarelli-Nirenberg-Spruck [2] to our context, we get the asymptotic behavior as $\mathbf{a}$ goes to infinity. □

Acknowledgements

I wish to thank Professors Bo Guan, Chunhui Qiu and Xi Zhang for their encouragement. Also, I would like to thank the referee for the careful reading and valuable comments.

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