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Rank of ordinary webs in codimension one an effective method

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Abstract: We are interested by holomorphic d-webs W of codimension one in a complex n-dimensional manifold M. If they are *ordinary*, i.e. if they satisfy to some condition of genericity (whose precise definition is recalled below), we proved in [CL] that their rank $\rho(W)$ is upper-bounded by a certain number $\pi'(n,d)$ (which, for $n \geq 3$, is strictly smaller than the Castelnuovo-Chern's bound $\pi(n,d)$).

In fact, denoting by $c(n,h)$ the dimension of the space of homogeneous polynomials of degree h with n unknowns, and by h_0 the integer such that

$$c(n,h_0-1) < d \leq c(n,h_0),$$

$\pi'(n,d)$ is just the first number of a decreasing sequence of positive integers

$$\pi'(n,d) = \rho_{h_0-2} \geq \rho_{h_0-1} \geq \cdots \geq \rho_h \geq \rho_{h+1} \geq \cdots \geq \rho_\infty = \rho(W) \geq 0$$

becoming stationary equal to $\rho(W)$ after a finite number of steps. This sequence is an interesting invariant of the web, refining the data of the only rank.

The method is effective: theoretically, we can compute ρ_h for any given h; and, as soon as two consecutive such numbers are equal ($\rho_h = \rho_{h+1},\ h \geq h_0-2$), we can construct a holomorphic vector bundle $R_h \to M$ of rank ρ_h, equipped with a tautological holomorphic connection ∇^h whose curvature K^h vanishes iff the above sequence is stationary from there. Thus, we may stop the process at the first step where the curvature vanishes, and compute the rank without to have to exhibit explicitly independant abelian relations.

Examples will be given.

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1. Introduction

Recall that a totally decomposable holomorphic *d-web* of codimension one without singularity on a complex n-dimensional manifold M is defined by the data of d holomorphic regular foliations $\mathcal{F}_i$ of codimension one on M, $(1 \leq i \leq d)$, any one of them being transverse to each other at any point.

We assume $d > n$ and the web to be at least in *weak general position*: this means that, at any point m, there exists at least n of the foliations among the d's, whose tangent spaces at m are in general position (if any family of n foliations among the d's has this property, the web is said to be in *strong* general position).

An *abelian relation* on an open set U (assumed to be connected and simply connected) of M is then the data of a family $(F_i)_i$ of holomorphic functions on U, $1 \leq i \leq d$, such that the sum $\sum_{i=1}^{d} F_i$ is a constant on U, and F_i is a first integral of $\mathcal{F}_i$ (maybe with singularities) for any i. These first integrals being defined up to an additive constant, we are only interested by their differential $\omega_i = dF_i$, in such a way that we may still define an abelian relation as a family $(\omega_i)_{1 \leq i \leq d}$ of holomorphic 1-forms ω_i on U (maybe with singularities), which are

- (i) closed (hence locally exact): $d\omega_i = 0$,
- (ii) verifying $T\mathcal{F}_i \subset Ker\ \omega_i$ ($T\mathcal{F}_i = Ker\ \omega_i$ at any point where ω_i doesn't vanish),
- (iii) such that $\sum_{i=1}^{d} \omega_i = 0$.

The germs of abelian relations at a point m constitute a vector space, whose dimension is called *the rank* of the web at this point (A. Hénaut ([H2]) proved that this rank doesn't depend on m, as far as the web satisfies to the assumption of strong general position). In case we have only weak general position, we shall define the *rank* of the web as being the highest of the rank at a point).

It will be useful to give an equivalent definition in words of differential operator. Then, denote by $T\mathcal{F}_i$ $(\subset TM)$ the vector bundle of vectors tangent to $\mathcal{F}_i$, and A_i $(\subset T^*M)$ the dual vector bundle of $TM/T\mathcal{F}_i$ (i.e. the vector bundle of holomorphic 1-forms vanishing on $T\mathcal{F}_i$). Let

$$Tr : \oplus_{i=1}^d A_i \to T^*M$$

be the morphism of vector bundles (the *trace*[1]), defined by $Tr\big((\omega_i)_i\big) = \sum_{i=1}^d \omega_i$. The assumption of “at least weak general position” means that Tr has maximal rank n: its kernel

$$A := Ker\ Tr$$

is therefore a holomorphic vector bundle of rank $d-n$. We define a linear differential operator of order one

$$D : J^1 A \to B,$$

where $B = (\wedge^2 T^*M)^{\oplus d}$, by mapping any section $s = (\omega_i)_i$ of A onto the family $(d\omega_i)_i$ of the differentials. Then, an *abelian relation* may be identified with a holomorphic section s of A such that $D(j^1 s) = 0$.

The kernel $R_1 = Ker(D : J^1 A \to B)$ is the vector bundle of formal abelian relations at order one. More generally, the space R_h of *formal abelian relations* at order h is the kernel of the $(h-1)th$-prolongation D_h of the differential operator D $(= D_1)$:

$$R_h = Ker(D_h : J^h A \to J^{h-1} B).$$

For any h $(h \geq 1)$, abelian relations may still be identified with holomorphic sections s of A such that $j^h s$ belongs to R_h.

In ([C]), Chern proved that the maximal rank of a d-web of codimension one in a n-dimensional manifold, satisfying to the assumption of strong general position, is equal to the Castelnuovo number

$$\pi(n,d) := \sum_{h \geq 1} \big(d - h(n-1) - 1\big)^+, \text{ where } a^+ = sup\ (a, 0)$$

(which is also the maximal arithmetical genus of irreducible algebraic curves of degree d in the complex n-dimensional projective space $\mathbb{P}_n$).

[1]The map Tr being a symmetric function of the indices i, the order of the numbering of the foliations doesn’t matter. It is the main reason why the definitions above and results make sense globally, even for webs which are not totally decomposable.

But if the web is ordinary (definition recalled below), we proved in [CL] that its rank is at most equal to the integer

$$\pi'(n,d) := \sum_{h\geq 1} \big(d - c(n,h)\big)^+,$$

strictly smaller than $\pi(n,d)$ for $n \geq 3$, with notation[2]

$$c(n,h) := \frac{(n-1+h)!}{(n-1)!\ h!}.$$

Denoting by $\pi_h : R_h \to R_{h-1}$ the natural projection, we first observed that the elements of R_h which are mapped by π_h onto a given element a_{h-1} of R_{h-1} are the solutions of a linear system $\Sigma_h(a_{h-1})$ of $c(n,h+1)$ equations with d unknowns, whose homogeneous part doesn't depend on a_{h-1}.

Case of ordinary webs: They are the webs for which all of the linear systems $\Sigma_h(a_{h-1})$ above have maximal rank $\inf\big(d, c(n,h+1)\big)$. Denoting by h_0 the integer such that

$$c(n,h_0-1) < d \leq c(n,h_0),$$

it is in fact sufficient that this rank be maximal for $h \leq h_0$, for being maximal for any h.

Since, for $h \leq h_0 - 2$, the set of solutions of $\Sigma_h(a_{h-1})$ is an affine space of dimension $d - c(n,h+1)$, then, for any $k \leq h_0 - 2$, $R_k \to M$ is a holomorphic vector bundle of rank $\rho_k = \sum_{h=1}^{k+1}\big(d - c(n,h)\big)$. In particular,

$$R_{h_0-2} \to M \text{ is a holomorphic vector bundle of rank } \pi'(n,d).$$

But for $h \geq h_0 - 1$, $\Sigma_h(a_{h-1})$ has now rank d, and has at most one solution (since it contains a cramerian sub-system), but maybe no one (since it is over-determined). In general, R_h will still be a vector bundle, but it may happen that the projection $\pi_{h+1} : R_{h+1} \to R_h$ be no more surjective, hence: $\rho_h \geq \rho_{h+1}$.

When $\rho_h = \rho_{h+1}$, $(h \geq h_0 - 1)$, the projection $\pi_{h+1} : R_{h+1} \to R_h$ is an isomorphism of vector bundles. The inverse isomorphism $R_h \overset{\cong}{\to} R_{h+1}$ composed with the natural inclusion $R_{h+1} \subset J^1 R_h$ defines a connection ∇^h on R_h, and abelian relations may be identified with sections s of $A = R_0$ such that $\nabla^h(j^h s) \equiv 0$. Hence the rank $\rho(W)$ of the web will be at most equal to

[2] We prefer this notation to the usual one for the binomial coefficient, because it suggests explicitly that it is the dimension of the vector space of homogeneous polynomials of degree h with n variables, and also because it needs less space.

the rank ρ_h of R_h, and equal iff the curvature K_h of ∇^h vanishes. This proves in particular the inequalities

$$\Big((h+1)d-c(n+1,h+1)+1\Big)^+ \leq \rho_h \leq (h_0-1)d-c(n+1,h_0-1)+1 = \pi'(n,d)$$

for $h \geq h_0 - 2$.

When $d = c(n, h_0)$ (then, d is said to be *calibrated*), $\rho_{h_0-2} = \rho_{h_0-1}$, and ∇^{h_0-2} is the connection defined in [CL]. A program by Maple for computing K_{h_0-2} has been written in [DL].

[If $n = 2$, any planar web is ordinary and calibrated $\big(d = c(2, d-1)\big)$; then, the connection ∇^{d-3} has been defined by Hénaut ([H1]), and independently by Pirio ([Pi]) who related its curvature to invariants defined formerly by Pantazi ([Pa] (the Blaschke-Dubourdieu curvature ([BB]) when $d = 3$). See also Ripoll ([R]).]

Case of non-ordinary webs: In this case, everything may happen: we may not affirm anymore that the sequence of the ρ_h's increases for $h \leq h_0 - 2$ and decreases for h bigger; the rank of the web may be smaller or bigger than $\pi'(n, d)$ (but always at most equal to $\pi(n, d)$). However, if by chance, we can find some h such that $\pi_{h+1} : R_{h+1} \to R_h$ is an isomorphism of vector bundles ($\rho_h = \rho_{h+1}$), then we still can define the connection ∇^h, and it is still true that the vanishing of the curvature K^h implies the equality $\rho(W) = \rho_h$.

Sections 2 and 3 are technical, respectively devoted to the computation of R_h and ∇^h.

In section 4, we sketch an algorithm for the explicit computation of the decreasing sequence (ρ_h), $h \geq h_0 - 2$.

In section 5, we give some examples of application of our methods. In particular, the concept of *ordinary algebraic curve* seems to us very interesting, and we sketch what we know about it. But, unless the curve is simultaneously ordinary and arithmetically Cohen-Macauley (and in this case, $g = \pi'(n, d)$), we cannot say really for the moment that our method allows to compute their arithmetical genus g. Up to now, our applications to Algebraic Geometry remain poor; in the future, we hope to improve the results in this direction.

2. Computation of R_h

Denote by

i an index from 1 to d,
λ, μ,... an index from 1 to n,
$L = (\ell_1, \ell_2, \cdots, \ell_n)$ a multi-index $\ell_\lambda \geq 0$ of n integers, and $|L| := \sum_\lambda \ell_\lambda$ its *degree*.

If $L = (\ell_1, \ell_2, \cdots, \ell_n))$, and $L' = (\ell'_1, \ell'_2, \cdots, \ell'_n))$, $L+L'$ (resp. $L-L'$) denotes $(\ell_1 + \ell'_1, \cdots, \ell_n + \ell'_n)$ (resp $(\ell_1 - \ell'_1, \cdots, \ell_n - \ell'_n))$.

In particular 1_λ denotes the multi-index obtained with 1 at the place λ and 0 elsewhere.

Relatively to local coordinates $x = (x_1, \cdots, x_n)$ in M, we shall denote by $\partial_\lambda a$ or a'_λ the partial derivative $\frac{\partial a}{\partial x_\lambda}$ of a holomorphic function a or of a matrix with holomorphic coefficients.

More generally, a'_L denotes the partial derivative $\frac{\partial^{|L|} a}{(\partial x_1)^{\ell_1} \cdots (\partial x_n)^{\ell_n}}$ of order $|L|$.

We assume that each foliation $\mathcal{F}_i$ is defined by a first integral u_i without singularity. The data of another first integral $F_i = G_i(u_i)$ up to an additive constant is equivalent to the data of the derivative $g_i = (G_i)'$. Each vector bundle A_i being now trivialized by du_i, we set $\omega_i = g_i(u_i)\ du_i$ (such a 1-form is automatically closed). The data of an abelian relation is now equivalent to the data of a family (g_i) of holomorphic functions of one variable $(1 \leq i \leq d)$ such that $\sum_i g_i(u_i)\ du_i \equiv 0$, or equivalently:

$$(E_\lambda) \qquad \sum_i (u_i)'_\lambda\ g_i(u_i) \equiv 0 \qquad \text{for any } \lambda,$$

which can still be written $< P_1\ ,\ f > \equiv 0$, where $P_1 := \frac{D(u_1, \cdots, u_d)}{D(x_1, \cdots, x_n)}$ denotes the jacobian matrix and f the d-vector $\big(g_1 \circ u_1, \cdots, g_d \circ u_d\big)$, the functions u_i being given and the functions g_i unknown.

Coefficients $C^h_L(u)$ and matrices $M^{(h)}_j$: For any $h \geq 0$, $g^{(h)}_i$ will denote the $h-th$ derivative of g_i (with the convention $g^{(0)}_i := g_i$);

We set:

$f_i := g_i \circ u_i$ and
$f^{(h)}_i := g^{(h)}_i \circ u_i$,
$f :=$ d-vector $(f_1, f_2, \cdots, f_d)$, and $f^{(h)} :=$ d-vector $(f^{(h)}_1, f^{(h)}_2, \cdots, f^{(h)}_d)$.

For any integer k $(k \geq 0)$, a k-jet of abelian relation at a point m of M is defined by the family

$$\Big(f^{(h)}_i(m) = (g^{(h)}_i \circ u_i)(m)\Big)_{i,h}, \quad (0 \leq h \leq k,\ 1 \leq i \leq d).$$

The partial derivatives of the relations (E_λ) will make us able to compute locally R_h. In fact, the functions $C^h_{i,L}$ will be defined by iteration on $|L|$ in such a way that

$$\Big(f_i.(u_i)'_\lambda\Big)'_L \equiv \sum_{h=0}^{|L|} C^h_{i,L+1_\lambda}\ .\ f_i^{(h)}$$

as far as $(f_i\ du_i)_i$ is an abelian relation.

Lemma 2-1: *For any holomorphic function u of n variables, and any holomorphic function g of one variable,*

(*i*) *The derivatives $\Big((g\circ u)\ u'_\lambda\Big)'_L$ are linear combinations*

$$\Big((g\circ u)\ u'_\lambda\Big)'_L = \sum_{h=0}^{|L|} C^h_{L+1_\lambda}(u)\ .\ (g^{(h)}\circ u)$$

of the successive derivatives $g^{(h)}$ of g (we set: $g^{(0)} = g$), whose coefficients $C^h_{L'}(u) = C^h_{L+1_\lambda}(u)$ depend only on u and on the multi-index $L' = L + 1_\lambda$, and not on its decomposition under the shape $L + 1_\lambda$.

(*ii*) *They can be computed by iteration on $|L|$, using the formula*

$$\begin{aligned} C^0_{1_\lambda}(u) &= u'_\lambda\ , \\ C^0_{L+1_\mu}(u) &= \partial_\mu C^0_L(u), \\ C^h_{L+1_\mu}(u) &= \partial_\mu C^h_L(u) + C^{h-1}_L(u)\ .\ u'_\mu \quad \textit{for } 1\le h\le |L|-1, \\ C^{|L|}_{L+1_\mu}(u) &= C^{|L|-1}_L(u)\ .\ u'_\mu. \end{aligned}$$

The 1-form $d\big(G(u)\big)$ is closed, where G denotes a primitive of g. The formulae above are then obtained by iteration on $|L|$.

For a web locally defined by the functions u_i, we set:

$$C^h_{i,L} = C^h_L(u_i).$$

We check in particular $C^0_{i,L} = (u_i)'_L$, and $C^{|L|-1}_{i,L} = \prod_{\lambda=1}^n \big((u_i)'_\lambda\big)^{\ell_\lambda}$ for $L = (\ell_1,\ell_2,\cdots,\ell_n)$.

We set:

Θ^r denotes the trivial holomorphic bundle of rank r,
$\beta_k := c(n+1,k) - 1\ \ \big(= \sum_{h=1}^k c(n,h)\big)$,

$M_j^{(h)}$ denotes the matrix $((C^h_{i,L}))_{(i,|L|=j)}$ of size $c(n,j)\times d$, $(1\le j, 0\le h)$, (with $M_j^{(h)}=0$ for $h\ge j$),
$P_j := M_j^{(j-1)}$ (symbol of the differential operator D_{j-1}),
$\mathcal{M}_k$ denotes the matrix of size $\beta_k\times kd$ built with the blocks $M_j^{(h)}$ for $1\le j\le k$ and $0\le h\le k-1$, where the block $M_j^{(h+1)}$ is on the right of $M_j^{(h)}$, and $M_{j+1}^{(h)}$ below,

and Q_{k+1} denotes the sub matrix of size $c(n,k+1)\times kd$ in $\mathcal{M}_{k+1}$ built with the blocks $M_{k+1}^{(h)}$ for $0\le h\le k-1$:

$$\mathcal{M}_k =$$
$$\begin{pmatrix} M_1^{(0)}=P_1 & 0 & 0 & .... & .... & 0 & 0\\ M_2^{(0)} & M_2^{(1)}=P_2 & 0 & .... & .... & 0 & 0\\ .... & .... & .... & .... & .... & 0 & 0\\ M_{k-1}^{(0)} & M_{k-1}^{(1)} & M_{k-1}^{(2)} & .... & .... & M_{k-1}^{(k-2)}=P_{k-1} & 0\\ M_k^{(0)} & M_k^{(1)} & M_k^{(2)} & M_4^{(3)} & .... & M_k^{(k-2)} & M_k^{(k-1)}=P_k \end{pmatrix}$$

$$Q_{k+1} =$$
$$\begin{pmatrix} M_{k+1}^{(0)} & M_{k+1}^{(1)} & M_{k+1}^{(2)} & M_{k+1}^{(3)} & .... & M_{k+1}^{(k-2)} & M_{k+1}^{(k-1)} \end{pmatrix}$$

Theorem 2-2:

(*i*) *Locally, R_k is the kernel of $\mathcal{M}_{k+1}$ (included into the trivial bundle $\Theta^{(k+1)d}$). Hence, when the matrix $\mathcal{M}_{k+1}$ has constant rank (always true for $k\le h_0-2$), $R_k\to M$ is a holomorphic vector bundle of rank*

$$\rho_k=(k+1)d-rank(\mathcal{M}_{k+1}).$$

(*ii*) *If an element $a_{k-1}\in R_{k-1}$ (imbedded into Θ^{kd}) is defined by the family f of d-vectors $(f^{(h)})_h$, $(0\le h\le k-1)$, the elements of R_k which project onto a_{k-1} are those whose last component $f^{(k)}$ is solution of the linear system $\Sigma_h(a_{h-1})$:*

$$\begin{aligned} < P_{k+1}\ , f^{(k)} > &= \quad - < Q_{k+1}\ ,\ f >,\\ &= -\textstyle\sum_{h=0}^{k-1} < M_{k+1}^{(h)}\ ,\ f^{(h)} > . \end{aligned}$$

Proof: A k-jet of abelian relation at a point $m\in M$ is then represented by its components $j^h_m(u_i\circ g_i)$ in J^hA_i, and each of them is completely defined by the family of the numbers

$$\Big(f_i^{(h)} = (g_i^{(h)} \circ u_i)(m)\Big)_{0\leq h\leq k}.$$

Thus a family of numbers $\big(w_i^{(h)}\big)_{i,h}$ belongs to the set R_k of formal abelian relations at order k (may be bigger than the jets of the true abelian relations), if it satisfies to any of the equations

$$(E_L) \qquad \sum_{i=1}^{d} \sum_{h=0}^{|L|-1} C_{iL}^{h} \, . \, w_i^{(h)} = 0. \qquad \text{QED}$$

Estimation of the ranks ρ_h: The assumption for the web to be ordinary means that the matrices $P_j := M_j^{(j-1)}$ have all maximal rank, that is $c(n,j)$ for $j \leq h_0 - 1$, and d for $j \geq h_0$.

Lemma 2-3: *If P_j has maximal rank $c(n,j)$ for $1 \leq j \leq h_0$, it has maximal rank d for any $j \geq h_0$.*

Proof: The meaning of the lemma not depending on the local coordinates, we may assume that all foliations are transversal to the x_n-axis near a point; therefore all derivatives $(u_i)'_n$ are not zero. The formula

$$\big(P_{j+1}\big)_{i,L+1_n} = (u_i)'_n . \big(P_j\big)_{i,L}$$

proves that the rank of P_{j+1} is at least equal to that of P_j, thus is equal if j is big enough for this rank to be stationary equal to d. QED

Theorem 2-4 ([CL])**:** *Let ρ_k be the rank of R_k.*

For $k \leq h_0 - 2$, $R_k \to M$ is a holomorphic vector bundle of rank

$$\rho_k = (k+1)d - \beta_{k+1}.$$

In particular, $\rho_{h_0-2} = \pi'(n,d)$.

Proof: In fact, the matrices $\mathcal{M}_h$, of size $\beta_h \times hd$, are triangular by blocks, and the diagonal blocks are the P_j's. Since the rank of P_j is $c(n,j)$ for $j \leq h_0 - 1$, $\mathcal{M}_{k+1}$ has maximal rank $\beta_{k+1} = \sum_{h=1}^{k+1} c(n,h)$ in this range. Thus R_k ($=$ Ker $\mathcal{M}_{k+1}$) has there rank $(k+1)d - \beta_{k+1}$. QED

The sequence $(hd - \beta_h)_h$ becomes decreasing for $h \geq h_0$ Then, for $h \geq h_0$, it may be no more true that $\mathcal{M}_{h+1}$ has maximal rank β_{h+1}, so that the rank

ρ_h of R_h may be now bigger than $(h+1)d-\beta_{h+1}$ (but remains at most equal to $\pi'(n,d) = (h_0-1)\ d-\beta_{h_0-1}$). Thus, we get:

Theorem 2-5: *Assuming that π_{h+1} has a constant rank for $h \geq h_0-2$, the sequence $(\rho_h)_{h\geq h_0-2}$ is decreasing from $\pi'(n,d)$ to the rank $\rho(W)$ of the web, and satisfies to the inequalities*

$$(h+2)d-\beta_{h+2} \leq \rho_{h+1} \leq \rho_h \leq \pi'(n,d).$$

3. The connections ∇^h

In this section, we assume:

> $h \geq h_0-2$,
> and $\rho_h = \rho_{h+1}$, $\pi_{h+1} : R_{h+1} \overset{\cong}{\to} R_h$ being an isomorphism of vector bundles.

If $\rho_h = 0$, then $\rho(W) = 0$. If $\rho_h > 0$, we shall define a connection ∇^h on R_h, whose curvature vanishes iff $\rho_h = \rho(W)$.

We recall that R_{h+1} is the intersection of $J^1(R_h)$ and $J^{h+1}R_0$ into $J^1(J^hR_0)$:

$$R_{h+1} = J^1R_h \cap J^{h+1}R_0.$$

Denote by

> $\epsilon_h : R_{h+1} \hookrightarrow J^1R_h$ the natural inclusion,
> and by $v_h : R_h \to R_{h+1}$ the inverse isomorphism.

The composed map $\xi_h := \epsilon_h \circ v_h$ is a splitting of the exact sequence

$$0 \to T^*(V) \otimes R_h \to J^1R_h \overset{\xi_h}{\underset{\longrightarrow}{\longleftarrow}} R_h \to 0$$

and defines consequently a holomorphic connection on R_h, whose covariant derivative is:

$$\nabla^h\sigma = j^1\sigma - < \xi_h, \sigma > .$$

Since the abelian relations may be identified by the map $s \to j^{h+1}s$ to the sections s of R_0 $(= A)$ such that $j^{h+1}s$ belong to R_{h+1}, and since ξ_h factorizes through R_{h+1}, the following assertions are equivalent:

(i) s is an abelian relation,
(ii) $\nabla^h(j^hs) \equiv 0$.

Since the framework is holomorphic, $\rho_\infty = \rho(W)$, and we get therefore the

Theorem 3-1:

(i) *A section s of A $(= R_0)$ is an abelian relation iff $j^h s$ is a section of R_h and $\nabla^h(j^h s) \equiv 0$.*

(ii) *The rank $\rho(W)$ of the web is at most equal to the rank ρ_h of the bundle R_h.*

(iii) *There exists an integer h_1 such that*
- *either $\rho_{h_1} = 0$ and then $\rho(W) = 0$,*
- *or $\rho_{h_1} = \rho_{h_1+1}$ $(\neq 0)$, the curvature K^{h_1} vanishes, and then $\rho(W) = \rho_{h_1}$.*

Remark: If the web is not-ordinary, we still may define the connection ∇^h, as far as we can find some h for which the projection $R_{h+1} \to R_h$ is an isomorphism of vector bundles, whatever be h. And it remains true that the vanishing of its curvature K^h implies $\rho(W) = \rho_h$.

4. Algorithm

From the previous sections, we deduce the following procedure for computing the rank, even when it is not maximal, without having to exhibit explicit abelian relations.

First, we compute the successive symbols $P_1, \cdots, P_{h_0}$, and check that the rank of P_j is equal to $c(n, j)$ for $1 \leq j \leq h_0 - 1$ and to d for $j = h_0$, (condition for the web to be ordinary).

We then compute $\mathcal{M}_{h_0}$.

We define a loop $\mathcal{L}$(h) (from $h = h_0 - 2$), by computing $\mathcal{M}_{h+2}$ (and its sub-matrix $\mathcal{M}_{h+1}$), and by computing $\rho_h = (h+1)\ d -$ Rank $(\mathcal{M}_{h+1})$ and ρ_{h+1}. Then

- if $\rho_h > \rho_{h+1}$, we go to $\mathcal{L}(\mathrm{h}+1)$,
- if $\rho_h = \rho_{h+1}$, we compute ∇^h and K^h;
- if $K^h \neq 0$, we still go to $\mathcal{L}(\mathrm{h}+1)$, else, $\rho(W) = \rho_h$.

Theoretically, this algorithm always works for any ordinary web. But it may need a long time of computer. Practically, in some cases, considerations specific to each example may be used for making the process shorter, some of them being sketched below:

1- When $\rho_h = \rho_{h+1}$, it is often useful to check immediately if there would not be some k, $k > h$, such that $\rho_k > \rho_{k+1}$. In this case, we know a priori that K^h doesn't vanish, without to have to compute it.

2- There are usually two ways for computing ρ_h: the first one, used in the algorithm above, consists in computing the kernel of the matrix $\mathcal{M}_{h+1}$ of size$\big(c(n+1,h+1)-1\big) \times (h+1)d$:

$$\rho_h = (h+1)\ d - Rank(\mathcal{M}_{h+1}).$$

This size increases more rapidly with h than the size $c(n,h+1) \times d$ of the matrix P_{h+1} of the linear system $\Sigma_h(a)$ (essentially because the process uses the knowledge of R_{h-1} that we got previously, which is not true for the first process). Thus, for h big enough, knowing already ρ_{h-1} and a trivialization $(\epsilon_s)_s$ of R_{h-1}, the following process may need a shorter time of computer than the previous one, despite of the fact that there are more operations to be done:

- choose a $d \times d$ invertible sub-matrix P^0_{h+1} of P_{h+1},
- solve the corresponding cramerian sub-system of $\Sigma_h(a)$,
- for each line ℓ among the $c(n,h+1) - d$ deleted for getting P^0_{h+1} from P_{h+1}, and for each ϵ_s belonging to the trivialization of R_{h-1}, build the characteristic determinant $\Delta(s,\ell)$ whose vanishing asserts the compatibility of the new equation ℓ with the cramerian sub-system,
- then the kernel of the matrix $\Delta^h := \big(\!\big(\Delta(s,\ell)\big)\!\big)$ of size $\big(c(n,h+1)-d\big) \times \rho_{h-1}$ defines the projection of R_h onto R_{h-1}, and

$$\rho_h = \rho_{h-1} - Rank(\Delta^h).$$

5. Examples

The process described in the algorithm above works for any (n,d,h). However, most of our examples are relative to low values of these integers: in fact, the size of the involved matrices becomes very rapidly huge, and would often need in practice more powerful computers than our small portable.

5.1. Case $n = 2$, $d = 3$

There is no hope to refine the classification of the non-hexagonal planar 3-webs by the order of the step from which the sequence of the ρ_h's vanishes. In fact, we can prove easily that the sequence of the ρ_h's becomes immediately stationary after the first step, and there are only two possibilities:

- sequence $(1, 1, \cdots, 1 = \rho_\infty)$ if the Blaschke-Dubourdieu curvature K^0 vanishes (hexagonal case),
- sequence $(1, 0, \cdots, 0 = \rho_\infty)$ if $K^0 \neq 0$.

5.2. Example $n = 2$, $d = 4$ $(\pi'(2, 4) = 3)$

We recall that all planar webs are ordinary, and calibrated with $h_0 = d - 1$. Moreover $\pi'(2, d)$ is then equal to $\pi(2, d)$ $\left(= \frac{(d-1)d-2)}{2}\right)$.

For the planar 4-web

$$(x, y, x + y + xy, x - y + x^5),$$

we have an obvious abelian relation $f \circ u_1 - u_2 - u_4 \equiv 0$, with $f(x) := x + x^5$. Thus, we know already

$$1 \leq \rho(W) \leq 3.$$

Computing ρ_k, we get

$$\rho_1 = \rho_2 = 3 > \rho_3 = \rho_4 = 2.$$

Since $\rho_3 < \rho_2$, we are sure that the curvature K^1 doesn't vanish, without to have to compute it. We get $K^3 = 0$. Therefore, the sequence of the ρ_i's is necessarily stationary equal to 2 from ρ_3:

$$\rho(W) = 2.$$

We are sure that there is another abelian relation independant on the obvious one, without to have to exhibit it.

5.3. Example $n = 2$, $d = 8$ $(\pi'(2, 8) = 21)$

Let W be the planar 8-web

$$x, y, x + y, x - y, xy, x^2 + y^2, x^2 - y^2, x^4 + y^4.$$

We observe that its curvature K^5 doesn't vanish, but that the space generated by the 19 first columns is preserved by the connection form ω^5 relative to some "adapted" trivialization (matrix of size 21×21), and that the restriction of the curvature K^5 to this subspace vanishes. Consequently, the rank of W is at least 19, and at most 20: in fact, $\rho_5 = 21$, $\rho_6 = 20$ and $\rho_7 = 19$. Thus

$$\rho(W) = 19.$$

By the way, we check that the sub-webs generated by the 5, 6 and 7 first functions have maximal rank (respectively 6, 10 and 15). This has already been quoted by Pirio. In particular, exhibiting a basis of algebraic abelian relations ([Pi]), he proved that the conjecture by Chern and Griffiths (according which polylogarithms must necessarily occur in the abelian relations of an exceptional web) was wrong.

5.4. Case $d = n + 1$, $n > 2$ ($\pi'(n, n + 1) = 1$)

Denoting by $(x_1, \ldots, x_n)$ local coordinates, we consider the $(n+1)$-web W defined by the functions $(x_1, \ldots, x_n, F(x_1, \cdots, x_n))$. Relatively to a convenient order of the multi-indices L, the matrix $\mathcal{M}_2$ has the shape

$$\begin{pmatrix} I_n & F^{(1)} & 0 & 0 \\ 0 & F^{(2)} & I_n & F^2 \\ 0 & G^{(2)} & 0 & G^2 \end{pmatrix},$$

where

I_n is the identity $n \times n$-matrix,
$F^{(1)}$ is the column of the $(F'_i)_{1 \leq i \leq n}$, (with n rows),
$F^{(2)}$ is the column of the $(F''_{ii})_{1 \leq i \leq n}$, (with n rows),
F^2 is the column of the $((F'_i)^2)_{1 \leq i \leq n}$, (with n rows),
$G^{(2)}$ is the column which has coefficients F''_{ij} ($i \neq j$, (with $c(n, 2) - n$ rows),
and G^2 is the column which has coefficients $F'_i F'_j$ ($i \neq j$, (with $c(n, 2) - n$ rows, the pairs (i, j) being in the same order of as for $G^{(2)}$).

The sub-matrix $\mathcal{M}_1$ has always rank n, hence $\rho_0 = 1$, while $\mathcal{M}_2$ has generally rank $2n + 2$; hence, in general ρ_1 $(= \rho(W))$ $= 0$, and there is no abelian relation.

The exceptional case (Rank$(\mathcal{M}_2) = 2n + 1$, and $\rho_1 = 1$) happens iff $G^{(2)}$ and G^2 are collinear. This means the set of relations

$$\frac{F''_{ij}}{F'_i F'_j} \equiv \frac{F''_{rs}}{F'_r F'_s}$$

for any i, j, r and s with $j \neq i$ and $s \neq r$.

We shall now study this case by mean of the connection ∇^0. A trivialization of $R_0 = \text{Ker } \mathcal{M}_1$ is then given by the $(n+1)$-vector

$$f^{(0)} = (-F'_1, -F'_2, \cdots, -F'_n, 1)$$

and a trivialization of $R_1 = \text{Ker } \mathcal{M}_2$ is given by some $(n+1)$-vector

$$f^{(1)} = (X_1, X_2, \cdots X_{n+1})$$

satisfying in particular to the identities

$$X_{n+1} = -\frac{F''_{ij}}{F'_i F'_j} \quad \text{whatever be } i, j, \ (i \neq j).$$

Denoting by Δ_i the $(n+1)\times(n+1)$ diagonal matrix built on the $(n+1)$-vector

$$(0, \cdots, 0, 1, 0, \cdots, 0, F'_i),$$

with 1 as $i-th$ component, F'_i as $(n+1)-th$ component and 0 elsewhere, the connexion ∇^0 on R_0 is then defined by

$$\nabla^0_i f^{(0)} = \partial_i f^{(0)} - < \Delta_i, f^{(1)} >,$$

where ∂_i (resp. ∇^0_i) means the partial derivative (resp. the covariant derivative) with respect to $\frac{\partial}{\partial x_i}$. Thus, we get:

$$\nabla^0_i f^{(0)} = -F'_i X_{n+1} f^{(0)}.$$

The curvature has then components

$$K^0_{ij} = \partial_i(F'_j X_{n+1}) - \partial_j(F'_i X_{n+1}).$$

Fix a pair (i, j) and choose an index k different from i and j. We get:

$$F'_i X_{n+1} = -F''_{ik}/F'_k,$$

hence

$$\begin{array}{rcl} \partial_j(F'_i X_{n+1})) & = & -\partial_j(F''_{ik}/F'_k) \\ & = & -\partial_j(\partial_i \ln(F'_k)) \end{array}.$$

This gives $K^0_{ij} = 0$. So, if we set

$$L_{ij} = ln(F'_i/F'_i),$$

we have proved the following proposition:

Proposition 5-2: *The web W has a non-trivial abelian relation iff*

$$(L_{ij})'_k = 0,$$

for any triple i, j, k of indices, each one being different to each other.

Notice that, when W is in strong general position, the existence of an abelian relation is equivalent to the fact that we can choose new coordinates $(\overline{x}_i)_i$ such that

$$F(x_1, \dots, x_n) \equiv \overline{x}_1 + \cdots + \overline{x}_n.$$

Thus, the existence of an abelian relation is equivalent for the web to be "parallelisable".

5.5. An example $n = 3,\ d = 5\ (\pi'(3, 5) = 2)$

Denoting by (x, y, z) local coordinates, and defining the web by the functions $\big(x, y, z, x + y + z, F(x, y, z)\big)$, assume that F depends only on $x + y$ and z:

$$F(x, y, z) \equiv g(x + y, z) \text{ for some function } g.$$

We set:

$$\begin{aligned}
&u := x + y,\\
&p := g'_u,\ q := g'_z,\\
&r := g''_{u^2},\ s := g''_{uz},\ t := g''_{z^2},\\
&a := g'''_{u^3},\ b := g'''_{u^2z},\ c := g'''_{uz^2},\ e := g'''_{z^3}.
\end{aligned}$$

We consider $\mathcal{M}_1$, $\mathcal{M}_2$ and Q_3 as sub-matrices of the matrix $\mathcal{M}_3$ described below relatively to a convenient order of the multi-indices L. We can check that $\mathcal{M}_1$, $\mathcal{M}_2$ below have respectively rank 3, 8; thus

$$\rho_0 = 2\ (= 5 - 3),\ \text{and}\ \rho_1 = 2\ (= 10 - 8).$$

In general $\mathcal{M}_3$ has rank 14 and $\rho_2 = 1$ $(= 15 - 14)$. But it may happen that $\mathcal{M}_3$ has rank 13 and $\rho_2 = 2$ for exceptional g's. This can be seen by computing the curvature K_0.

A basis for $R_0 = \text{Ker } \mathcal{M}_1$ is

$$f_1 = \Big(-1, -1, -1, 1, 0\Big) \quad , \quad f_2 = \Big(-p, -p, -q, 0, 1\Big),$$

and

$$\mathcal{M}_3 = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & p \\ 0 & 1 & 0 & 1 & p \\ 0 & 0 & 1 & 1 & q \end{pmatrix} & 0 & 0 \\ \begin{pmatrix} 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & r \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 1 & p^2 \\ 0 & 1 & 0 & 1 & p^2 \\ 0 & 0 & 1 & 1 & q^2 \\ 0 & 0 & 0 & 1 & pq \\ 0 & 0 & 0 & 1 & pq \\ 0 & 0 & 0 & 1 & p^2 \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & a \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 3pr \\ 0 & 0 & 0 & 0 & 3pr \\ 0 & 0 & 0 & 0 & 3qt \\ 0 & 0 & 0 & 0 & 2ps+rq \\ 0 & 0 & 0 & 0 & 2ps+rq \\ 0 & 0 & 0 & 0 & 2ps+rq \\ 0 & 0 & 0 & 0 & 2qs+pt \\ 0 & 0 & 0 & 0 & 2qs+pt \\ 0 & 0 & 0 & 0 & 3pr \\ 0 & 0 & 0 & 0 & 3pr \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 1 & p^3 \\ 0 & 1 & 0 & 1 & p^3 \\ 0 & 0 & 1 & 1 & q^3 \\ 0 & 0 & 0 & 1 & p^2q \\ 0 & 0 & 0 & 1 & p^2q \\ 0 & 0 & 0 & 1 & p^2q \\ 0 & 0 & 0 & 1 & pq^2 \\ 0 & 0 & 0 & 1 & pq^2 \\ 0 & 0 & 0 & 1 & p^2q \\ 0 & 0 & 0 & 1 & p^2q \end{pmatrix} \end{pmatrix}$$

The lines 7 and 8 of $\mathcal{M}_2$ being the same, we may ignore the line 8 in the computation of $R_1 = \text{Ker } \mathcal{M}_2$. We assume $p \neq q$, in such a way that the sub-matrix P_2^0 of P_2 that we get in forgetting its line 5 is invertible. Thus, R_1 has rank $\rho_1 = 2$, and we can lift f_1 and f_2 in R_1, defining $f_1^{(1)} = - < (P_2^0)^{-1}.\ M_2^{(0)}\ ,\ f_1 >$, and $f_2^{(1)} = - < (P_2^0)^{-1}.\ M_2^{(0)}\ ,\ f_2 >$. We get:

$$f_1^{(1)} = \Big(0,0,0,0,0\Big)\ \ ,\ \ f_2^{(1)} = \Big(0,0,Z,T,U\Big),$$

where Z, T and U are solution of the cramerian linear system

$$\begin{array}{ccccccccc} & & T & + & p^2U & + & r & = & 0 \\ & & T & + & pqU & + & s & = & 0 \\ Z & + & T & + & q^2U & + & t & = & 0 \end{array}$$

Denoting respectively by Δ_x, Δ_y, and Δ_x, the 5×5 diagonal matrices built with $(1,0,0,1,p)$, $(0,1,0,1,p)$, and $(0,0,1,1,q)$, the connection ∇^0 on R_0 is then given by the formulae:

$$\begin{aligned}
&\nabla^0 f_1 \equiv 0,\\
&\nabla^0_x f_2 \equiv \tfrac{\partial}{\partial x} f_2 - < \Delta_x,\ f_2^{(1)} >,\\
&\nabla^0_y f_2 \equiv \tfrac{\partial}{\partial y} f_2 - < \Delta_y, f_2^{(1)} >,\\
&\nabla^0_z f_2 \equiv \tfrac{\partial}{\partial z} f_2 - < \Delta_z, f_2^{(1)} >,
\end{aligned}$$

where ∇^0_x, ∇^0_y and ∇^0_z denote the covariant derivative with respect to $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$. The connection form relative to (f_1, f_2) is then

$$\omega_0 = \begin{pmatrix} 0 & -T(du+dz) \\ \\ 0 & -U(p\ du + q\ dz) \end{pmatrix},$$

and the curvature is

$$K^0 = \begin{pmatrix} 0 & T'_z - T'_u + (q-p)TU \\ \\ 0 & pU'_z - qU'_u \end{pmatrix} (dx+dy) \wedge dz.$$

If this curvature vanishes (according to g), $\rho(W) = 2$. Otherwise, $\rho(W) = 1$. (The rank may not be zero, because of the obvious non-trivial abelian relation $(x) + (y) + (z) - (x+y+z) \equiv 0$).

For example, if $g(u,z) = u^2 + 2\lambda uz + \mu z^2$, $(\lambda, \mu \in \mathbb{C})$, we can affirm that there is no other independant abelian relation if $\lambda \neq 1$. If $\lambda = 1$, we have a vanishing curvature, corresponding to the second abelian relation $u_5 \equiv (u_4)^2 + (\mu - 1)(u_3)^2$.

5.6. An example $n = 3$, $d = 11$ ($\pi'(3, 11) = 14$)

Let W be the 11-web (quasi-parallel: all u_i's but one are affine functions):

$$x,\ y,\ z,\ x+y+z,\ x+2y+z,\ x+3y+z,\ x+y+5z,$$
$$x+y+7z,\ x+11y+z,\ 19x+y+z,\ x+yz.$$

We get $\rho_2 = 14 > \rho_3 = \rho_4 = 13$, and $K^3 = 0$. Hence

$$\rho(W) = 13.$$

5.7. Parallelisable webs

These are webs such that all u_i's are affine functions relatively to some system of local coordinates. Then, with these coordinates, the only blocks $M_k^{(h)}$ which are not zero in the matrices $\mathcal{M}_k$ are the diagonal blocks $P_k = M_k^{(k-1)}$, and the rank of $\mathcal{M}_k$ is equal to $\sum_{h=1}^{k} Rank(P_h)$. Thus

$$\rho_{h+1} = \rho_h + \big(d - Rank(P_{h+2})\big).$$

In particular, if the web is ordinary, $\rho_{h+1} = \rho_h$ for $h \geq h_0 - 2$. Therefore *all ordinary parallelisable webs have maximal rank* $\pi'(n, d)$ $(= \rho_{h_0-2})$. (This has already been quoted in [CL] (theorem 6-5) by other considerations.)

If a parallelisable web is now *not ordinary*, and if there exists some h_1 $(\geq h_0 - 2)$ such that $\rho_{h_1+1} = \rho_{h_1}$, then the sequence of the ρ_h's is stationary from there because of the lemma 2-3 above, and then

$$\rho(W) = \rho_{h_1} \ \big(> \pi'(n, d)\big).$$

Such an example is given below.

5.8. Non ordinary example $n = 3$, $d = 10$

Let W_{10} be the parallel 10-sub-web of the ordinary 11-web above, obtained by deleting u_{11}. It is not ordinary (since P_3 has rank 9, not 10). We then get:

$$\rho_3 < \rho_4 = \rho_5 = 12 = \rho(W_{10}).$$

The rank is then strictly bigger than $\pi'(3, 10) = 11$, but smaller of course than the Castelnuovo number $\pi(3, 10) = 16$.

5.9. Ordinary algebraic curves

Recall that a non-degenerate algebraic curve Γ of degree d in the complex projective space $\mathbb{P}_n$ induces a linear d-web $\mathcal{W}_\Gamma$ of codimension one on the dual projective space $\check{\mathbb{P}}_n$, a generic hyperplane H intersecting Γ in d distinct points. The curve is said to be ordinary if this web is ordinary, which means that, for any generic hyperplane H, and for any integer $h \geq 1$, the restriction map

$$H^0(\mathcal{O}_H(-h)) \to H^0(\mathcal{O}_{H\cap\Gamma}(-h))$$

has maximal rank (geometrically, the d points of $H \cap \Gamma$ are in "general position": the space of algebraic hypersurfaces of H going through d' points among the $d's$ of $H \cap \Gamma$ is a projective space of dimension $c(n,h) - 1 - d'$ if $d' \leq c(n,h) - 1$, and is empty if $d' \geq c(n,h)$). If Γ is irreducible, it is known that its arithmetical genus g is equal to the rank of the web. Therefore, the arithmetical genus of a ordinary algebraic curve is upper-bounded by the number $\pi'(n,d)$.

Theoretically, we could compute the genus g of a ordinary curve by the general method described above. But unfortunately, unless the curve is rational (and then we have generally simpler methods for computing g), it seems to be difficult to write a good program by Maple, which will not need a too long time of computer.

However, we proved in [GHL] that the number $\pi'(n,d)$ is also a lower-bound for the genus g of the curves of degree d in $\mathbb{P}_n$ which are arithmetically Cohen-Macaulay (acm) (they are the curves for which the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}_n \cap \Gamma}(-h)) \to H^0(\mathcal{O}_{H \cap \Gamma}(-h))$$

is always surjective, whatever be the generic hyperplane H, and whatever be h: geometrically, any algebraic hypersuface of H going through $H \cap \Gamma$ is the intersection with H of a algebraic hypersurface in $\mathbb{P}_n$ going through Γ); we gave examples of curves which are both ordinary and acm, and have therefore a genus g equal to $\pi'(n,d)$. On the other hand, up to the exception of the elliptic quartic (intersection of two quadrics in $\mathbb{P}_3$), the algebraic curves which are complete intersections of $n-1$ algebraic hypersurfaces in $\mathbb{P}_n$ are never ordinary, but are all acm, and have therefore a genus g strictly bigger than $\pi'(n,d)$ (but equal to $\pi'(3,4) = 1$ for the elliptic quartic).

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