Pure and Applied Mathematics Quarterly
Volume 16, Number 5, 1533–1561, 2020

On exchange spectra of valued quivers and cluster algebras

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Abstract: Inspirited by the importance of the spectral theory of graphs, we introduce the spectral theory of the valued quiver of a cluster algebra. Our aim is to characterize a cluster algebra via its spectrum so as to use the spectral theory as a tool.

First, we give relations between exchange spectrum of valued quivers and adjacency spectrum of their underlying valued graphs, and between exchange spectra of valued quivers and their full valued subquivers. The key point is to find some invariants from the spectrum theory under mutations of cluster algebras, which is the second part we discuss. We give two equivalent conditions for a quiver Q without 3-cycles and its mutation to be cospectral. In particular, we prove that Q and $\mu_k(Q)$ are cospectral if and only if k is a sink or a source. Following this discussion, the so-called cospectral subalgebras of cluster algebras are introduced. We study bounds of exchange spectrum radii of quivers and give a characterization of 2-maximal quivers via the classification of oriented graphs of its mutation equivalence. Then as an application, we obtain that the preprojective algebra of a quiver of Dynkin type is representation-finite if and only if the quiver is 2-maximal.

Keywords: Cluster algebra, cluster quiver, skew-symmetrizable matrix, spectrum, mutation.

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky in a series of papers [10, 11, 3, 12] and are thought to be a spectacular advance in mathematics. There are many relations and applications between cluster algebras and other important subjects, such as representations of quivers, combinatorics and quiver gauge theories.

arXiv: 1703.02225
Received March 7, 2017.
2010 Mathematics Subject Classification: Primary 13F60, 58C40, 68R10.
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In the theory of cluster algebras, two vital roles are exchange matrices and mutations of them. Exchange matrices are assumed to be totally sign-skew-symmetric matrices introduced by Fomin and Zelevinsky in [10]. An important class of totally sign-skew-symmetric matrices consists of integer skew-symmetrizable matrices and they can be associated a one-to-one correspondence with valued quivers, which are simple oriented graphs without loops together with a pair of integers $(v(\alpha)_1,\ v(\alpha)_2)$ for each arrow α satisfying some rules. Skew-symmetrizable matrices and their mutations play very important roles in the study of cluster algebras, and many conjectures and problems are usually worked out firstly in this case. In [12], Fomin and Zelevinsky conjectured that the exchange graph of a cluster algebra depends only on its initial exchange matrix, and it had been proved in the case of skew-symmetrizable matrices. Therefore it is meaningful to study skew-symmetrizable matrices, their corresponding valued quivers and mutations of them. Valued quivers can be just regarded as valued oriented graphs and spectral graph theory is one of the major method to study properties of graphs.

In general, to reveal the properties of a graph, we usually associate a graph with some matrices and study these matrices via algebraic methods. In contrast, we can also make use of graph theory to study the properties of some matrices and transformations of matrices. Moreover, spectral graph theory also has universal applications in many areas, such as information science, computer science, and communications. The most common matrices associated to (oriented) graphs are adjacency matrices and Laplacian matrices. There are lots of literature and results on spectral graph theory, especially for unoriented graphs, see e.g [4, 8]. But there are not enough attentions on spectral theory of oriented graphs. Recently, in [7] Chung considered the Laplacians for oriented graphs and studied their spectra, and Bauer introduced normalized Laplacians for weighted oriented graphs and investigated the properties of their spectra in [2]. However, we are more interested in valued quivers and their corresponding skew-symmetrizable matrices, hence we shall develop a novel spectral theory for valued quivers in contrast to the classical spectral graph theory. On one hand, it can be contributed to study exchange matrices and their transformations. On the other hand, it supports a new sight to study spectral graph theory.

In Section 2, we introduce the definition of exchange spectra for valued cluster quivers and other elementary notations and concepts. In Section 3.1, we first give the relation between exchange spectrum of a valued quiver and adjacency spectrum of its underlying valued graph. Then in Section 3.2, we give the relation between exchange spectra of a valued quiver and its full valued subquivers. As a corollary, we claim that the exchange spectrum radius of a quiver is either larger than or equal to that of its full subquiver, see Corollary 3.9.

The cospectral relation of two quivers given in Section 4 is important for our study. Concretely, we give sufficient and necessary conditions for Q without 3-cycles and $\mu_k(Q)$ to be cospectral as follows.

Theorem 4.5. *Let Q be a quiver without 3-cycles and $Q_0 = \{1, 2, \dots, n\}$. Fix a vertex $k \in Q_0$, $B = B(Q) = (b_{ij})_{n\times n}$ and $\mu_k(B) = WBW^T$, where W satisfies the equality (1). The following statements are equivalent:*

(i) Q and $\mu_k(Q)$ are cospectral;
(ii) k is either a sink or a source;
(iii) $W = \begin{bmatrix} I_{k-1} & \varepsilon\zeta & \mathbf{0} \\ \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \varepsilon\theta & I_{n-k} \end{bmatrix}$, where $\zeta = (b_{1,k}, b_{2,k}, \dots, b_{k-1,k})^T$, $\theta = (b_{k+1,k}, b_{k+2,k}, \dots, b_{n,k})^T$, and $\varepsilon \in \{0, 1\}$.

The other main result in this section gives the characterization of 2-maximal connected quivers, as follows.

Theorem 4.14. *Let Q be a connected quiver.*

(i) The quiver Q is 2-maximal if and only if it is mutation equivalent to an orientation of one of X_2, A_1, A_2, A_3, or A_4, where X_2 is a graph with two vertices and two edges.

(ii) If the underlying graph of Q is one of Dynkin diagrams, then the preprojective algebra $\Theta(Q)$ of Q is representation-finite if and only if Q is 2-maximal.

This article is organized as follows. In Section 2, some basic concepts and definitions are given and we characterize acyclicity of valued quivers by its adjacency spectrum. In Section 3, we investigate properties of spectra of exchange matrices of valued quivers. Finally, in Section 4, we study how mutations influence on exchange spectra of quivers.

2. Valued quivers and exchange matrices

2.1. Definitions and notations

We follow [1] for most basic concepts of quivers. A *quiver* is an oriented graph described by a 4-tuple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a set of *vertices*, Q_1 is a set of *arrows*, and s, t are two functions that map each arrow to its *source* and *target*, respectively. We usually label the vertices by natural numbers. A path of length p is a sequence of p arrows $\alpha_1\alpha_2\dots\alpha_p$ satisfying that $s(\alpha_{j+1}) = t(\alpha_j)$, $1 \leqslant j \leqslant p-1$. For a path $\omega = \alpha_1\alpha_2\dots\alpha_p$, define $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_p)$. A quiver Q is said to be *finite* if both Q_0 and Q_1 are finite sets, write $Q_0 = \{1, \cdots, n\}$. In Q, if the multiplicities of arrows are at most 1, then Q is said to be *simply-laced*. A *sink* is a vertex $i \in Q_0$ satisfying that there is no arrow $\alpha \in Q_1$ such that $s(\alpha) = i$ and a *source* is a vertex $j \in Q_0$ satisfying that there is no arrow $\alpha \in Q_1$ such that $t(\alpha) = j$.

A *full subquiver* Q' of a quiver Q is a quiver $Q' = (Q'_0, Q'_1, s', t')$ satisfying that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, $s' = s|_{Q'_1}$, $t' = t|_{Q'_1}$, and $Q'_1 = \{\gamma \in Q_1 \mid s(\gamma),\ t(\gamma) \in Q'_0\}$.

The path algebra KQ of a quiver Q over an algebraically closed field K is the K-vector space KQ whose basis consisting of all paths in Q with multiplication $\cdot$ defined on two basis elements ω_1, ω_2 by

$$\omega_1 \cdot \omega_2 = \begin{cases} \omega_1\omega_2, & \text{if } s(\omega_2) = t(\omega_1); \\ 0, & \text{otherwise.} \end{cases}$$

The underlying graph of a quiver Q is got by forgetting all orientations of arrows and is denoted by $\bar{Q}$. We say a quiver Q to be *connected* if its underlying graph is connected.

A *loop* of a quiver is just an arrow γ such that $s(\gamma) = t(\gamma)$, and a k-cycle of a quiver is a path $\alpha_1\alpha_2\dots\alpha_k$ of length k such that $s(\alpha_1) = t(\alpha_k)$. A chordless k-cycle in a quiver is a k-cycle such that no two vertices of the cycle are connected by an arrow that does not itself belong to the cycle.

A *cluster quiver* is a finite quiver without loops or 2-cycles. For any quiver Q, the degree of any vertex is just its degree in the underlying graph $\bar{Q}$, i. e., the number of edges incident with this vertex in $\bar{Q}$.

A *valued cluster quiver* (Q, v) is a finite quiver Q without loops and at most one arrow between any pair of vertices, together with a *valuing map* $v : Q_1 \to \mathbb{N}^2$ satisfying that there is a map $d : Q_0 \to \mathbb{N}_{>0}$ and for each arrow $\alpha : i \to j$ in Q_1, we have $d(i)v(\alpha)_1 = d(j)v(\alpha)_2$, where the *value* $v(\alpha) = (v(\alpha)_1, v(\alpha)_2)$.

Throughout this paper, all *(valued) quivers* are always assumed to be (valued) cluster quivers unless stated otherwise.

By a little abuse of notation, denote a valued quiver (Q, v) only by Q and its underlying valued graph by $\bar{Q}$. For any arrow $\alpha : i \to j$, the notation (v_{ij}, v_{ji}) is used to replace $(v(\alpha)_1, v(\alpha)_2)$.

If (Q, v) is a valued quiver, (Q', v') is called a *full valued subquiver* of Q if Q' is a full subquiver of Q and $v' = v|_{Q'_1}$. Note that (Q', v') is also a valued quiver.

For a valued quiver (Q, v), if $v(\alpha)_1 = 1 = v(\alpha)_2$ for any arrow $\alpha \in Q_1$, then we call Q a *simple quiver*. Dealing with simple quivers, we usually omit the labels. Trivially, simple quivers are equivalent to simply-laced quivers. We call Q a *tree quiver* if it is a simple quiver and $\bar{Q}$ is a tree.

Throughout this paper, we use the notation $[x]_+ = \max\{x, 0\}$. Let $M = (m_{ij})_{l\times n}$ be a real matrix, then $[M]_+ = ([m_{ij}]_+)_{l\times n}$ is the non-negative matrix defined component-wisely.

Definition 2.1. *Let (Q, v) be a valued quiver with vertex set $Q_0 = \{1, 2, \ldots, n\}$.*

(1) The exchange matrix *$B(Q) = (b_{ij})_{n\times n}$ of Q is the integer matrix defined by the following rule: For $1 \leqslant i, j \leqslant n$, $b_{ij} = v_{ij}$ if there is an arrow $\alpha : i \to j$, $b_{ij} = -v_{ij}$ if there is an arrow $\beta : j \to i$ and otherwise $b_{ij} = 0$.*

(2) The matrix $A(Q) = [B(Q)]_+ = ([b_{ij}]_+)_{n\times n}$ is called the adjacency matrix *of Q, and the matrix $C(Q) = [B(Q)]_+ + [-B(Q)]_+$ is called the adjacency matrix of the underlying valued graph $\bar{Q}$ of Q.*

A square matrix M is *symmetrizable* (*skew-symmetrizable*, resp.) if there exists a diagonal square integer matrix D with positive diagonal entries such that DM is symmetric (skew-symmetric, resp.). Note that exchange matrices of valued quivers are integer skew-symmetrizable matrices. Let $B = (b_{ij})_{n\times n}$ be an integer skew-symmetrizable matrix, we can define a valued quiver $(Q(B), v)$ whose vertex set is $\{1, 2, \ldots, n\}$ as follows. There is an arrow $\alpha : i \to j$ in $Q(B)_1$ whenever $b_{ij} > 0$ and $v(\alpha) = (|b_{ij}|, |b_{ji}|)$. It is clear that there is a bijective correspondence between integer skew-symmetrizable matrices and valued quivers. In particular, skew-symmetric matrices are skew-symmetrizable. In this case, we can use quivers instead of valued quivers to express integer skew-symmetric matrices. Indeed, if $B = (b_{ij})_{n\times n}$ is an integer skew-symmetric matrix, we can construct a quiver Q such that $Q_0 = \{1, 2, \ldots, n\}$ and there shall be b_{ij} arrows from i to j whenever $b_{ij} > 0$ for any $i, j \in Q_0$. Then there is a bijective correspondence between integer skew-symmetric matrices and quivers. Quivers can be considered as a special case of valued quivers if for any quiver Q, we regard the multiplicity v_α of each arrow α as its value, that is, let $v(\alpha)_1 = v(\alpha)_2 = v_\alpha$ for each arrow α. We will emphatically discuss quivers in Section 4.

Remark 2.2. *When we consider skew-symmetric matrices and its corresponding quivers, the adjacency matrix defined above is the same as the definition of the adjacency matrices of (oriented) graphs in [13, 4].*

Since a full valued subquiver of a valued quiver is also a valued quiver, the following relation between adjacency (exchange, resp.) matrices of a valued quiver and its full valued subquivers is obvious.

Lemma 2.3. *Let Q be a valued quiver. Then there is a bijection between principal submatrices of $A(Q)$ $(B(Q),$ resp.) and full valued subquivers of Q. More precisely, each principal submatrix of $A(Q)$ $(B(Q)$, resp.) is just the adjacency (exchange, resp.) matrix of its corresponding full valued subquiver in Q.*

Clearly, for a valued quiver Q, $B(Q)$ is an integer skew-symmetrizable matrices and $C(Q)$ is an integer symmetrizable matrix with respect to the same positive definite diagonal matrix. The spectrum of $A(Q)$ $(B(Q)$, resp.) is called the *adjacency* (*exchange*, resp.) *spectrum* of Q, and the characteristic polynomial of $A(Q)$ $(B(Q)$, resp.) is called the *adjacency* (*exchange*, resp.) *polynomial* of Q. Since $B(Q)$ is similar to a real skew-symmetric matrix, eigenvalues of B appear in complex conjugate pairs, and any eigenvalue of B is either an imaginary number or zero. We usually denote the exchange spectrum of Q by

$$\mathrm{Spec}(B(Q)) = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \\ n_1 & n_2 & \dots & n_m \end{bmatrix},$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are all distinct eigenvalues of the matrix $B(Q)$ such that $-\lambda_1\mathrm{i} < -\lambda_2\mathrm{i} < \cdots < -\lambda_m\mathrm{i}$, and $n_1, n_2, \dots, n_m$ are the corresponding multiplicities of them, where $\mathrm{i} = \sqrt{-1}$. And $-\lambda_m\mathrm{i}$ is called the exchange spectrum radius of the valued quiver Q and denoted by $\mathrm{Radi}(Q) = -\lambda_m\mathrm{i} = |\lambda_m|$. Note that $|\lambda_k| \leq \mathrm{Radi}(Q)$ for any $k \in \{1, 2, \dots, n\}$.

Let Q be a valued quiver and Q'_1, Q'_2, …, and Q'_s be its connected components. Suppose that B, B_1, B_2, …, and B_s are exchange matrices of Q, Q'_1, Q'_2, …, and Q'_s, respectively. Then it is clear that there exists a permutation matrix P such that

$$PBP^T = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_s \end{bmatrix}$$

Thus it is easy to see that

$$\mathrm{Spec}(B) = \bigcup_{i=1}^{s} \mathrm{Spec}(B_i),$$

$$\mathrm{Radi}(Q) = \max \{ \mathrm{Radi}(Q'_k),\ k = 1, 2, \dots, s\}.$$

So in general, we can assume that Q is connected.

For an integer skew-symmetrizable matrix $B = (b_{ij})_{n\times n}$ and any $k \in [1, n]$, in [10, 12], Fomin and Zelevinsky defined the mutation $\mu_k(B) = (b'_{ij})_{n\times n}$ of B at k, which is given by the following formula

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \text{sgn}(b_{ik})\max\{b_{ik}b_{kj}, 0\}, & \text{otherwise.} \end{cases}$$

Note that $\mu_k(B)$ is still an integer skew-symmetrizable matrix. The corresponding mutation of valued quivers can be defined as follows.

Definition 2.4. *Let (Q, v) be a valued quiver with vertex set $Q_0 = \{1, 2, \ldots, n\}$ and $k \in Q_0$ be a fixed vertex. The mutation $(Q', v') = \mu_k(Q, v)$ of (Q, v) at k is defined as follows:*

(1) For every 2-paths $i \xrightarrow[\alpha]{(v_{ik}, v_{ki})} k \xrightarrow[\beta]{(v_{kj}, v_{jk})} j$,

(i) if there exists an arrow $j \xrightarrow[\gamma]{(v_{ji}, v_{ij})} i$, keep this arrow and $v'(\gamma) = (v_{ji} - v_{jk}v_{ki},\, v_{ij} - v_{ik}v_{kj})$ if $v_{ij} > v_{ik}v_{kj}$; delete this arrow if $v_{ij} = v_{ik}v_{kj}$; delete this arrow, and add a new arrow $\gamma' : i \to j$ and $v'(\gamma') = (v_{ik}v_{kj} - v_{ij},\, v_{jk}v_{ki} - v_{ji})$ if $v_{ij} < v_{ik}v_{kj}$;

(ii) if there exists an arrow $i \xrightarrow[\gamma]{(v_{ij}, v_{ji})} j$, keep this arrow and $v'(\gamma) = (v_{ik}v_{kj} + v_{ij},\, v_{jk}v_{ki} + v_{ji})$;

(iii) if there are not any arrows between i and j, just add an arrow $\epsilon : i \to j$ and $v'(\epsilon) = (v_{ik}v_{kj},\, v_{jk}v_{ki})$.

(2) Reverse all arrows incident with k, and $v'(\alpha^{op}) = (v(\alpha)_2,\, v(\alpha)_1)$ for any arrow α incident with k, where α^{op} is the opposite arrow of α;

(3) Keep other arrows and values unchanged.

It is obvious that $v_{ij} = |b_{ij}|$ when $b_{ij} \neq 0$. Furthermore, if there is an arrow from i to j, then $v_{ij} = b_{ij}$ and $v_{ji} = -b_{ji}$. Let $\mu_k(B) = (b'_{ij})_{n\times n}$, then $b'_{ij} = b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+$ for $i, j \neq k$. Therefore $b'_{ij} \neq b_{ij}$ if and only if $b_{ik} > 0$, $b_{kj} > 0$ or $b_{ik} < 0$, $b_{kj} < 0$ whenever $i, j \neq k$. It can be seen that

$$B(\mu_k(Q, v)) = \mu_k(B(Q, v)).$$

Also, for either matrices or valued quivers, the *mutation map* μ_k is always an involution, that is, $\mu_k\mu_k(B) = B$ and $\mu_k\mu_k(Q, v) = (Q, v)$.

Mutations of an integer skew-symmetrizable matrix can be written in matrix form, see [3]. In particular, mutations of an integer skew-symmetric matrix is the same as a congruent transformation. Indeed, in the proof of [[3], Lemma 3.2], let $B_{n\times n}$ be an integer skew-symmetric matrix and $W=(w_{ij})_{n\times n}$ be a matrix of the following form

$$W=\begin{bmatrix} I_{k-1} & \xi & \mathbf{0} \\ \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \eta & I_{n-k} \end{bmatrix} \tag{1}$$

and the two vectors ξ and η are given by $\xi=([b_{1,k}]_+,[b_{2,k}]_+,\dots,[b_{k-1,k}]_+)^T$ and $\eta=([b_{k+1,k}]_+,[b_{k+2,k}]_+,\dots,[b_{n,k}]_+)^T$, where I_m denotes the identity matrix of order m. It is easy to check that $det(W)=-1$ and $\mu_k(B)=WBW^T$.

Two integer skew-symmetrizable matrices (respectively, valued quivers) are said to be *mutation equivalent* if one can be obtained by a sequence of mutations of the other. It is easy to see that this defines a equivalence relation. The *mutation class* of Q consists of all valued quivers mutation equivalent to Q and is usually denoted by $Mut(Q)$. We use the notation $Q\sim Q'$ ($B\sim B'$, resp.) to denote that Q and Q' (B and B', resp.) are mutation equivalent.

Let $\mathbb{P}$ be a semifield which is an abelian multiplicative group endowed with an auxiliary addition $\oplus$ which is associative, commutative, and distributive with respect to the multiplication in $\mathbb{P}$. Let F be a field which is isomorphic to the field of rational functions in n indeterminates with the coefficients from the field of fractions of $\mathbb{ZP}$. Following [12], a seed is a triple $\Sigma=(\mathbf{x},\mathbf{y},B)$ such that $B=(b_{ij})_{n\times n}$ is an integer skew-symmetrizable matrix, $\mathbf{y}=(y_1,\dots,y_n)$ is an n-tuple of elements of $\mathbb{P}$, and $\mathbf{x}=(x_1,\dots,x_n)$ is an n-tuple of a free generating set of F. In [12], for $k\in[1,n]$, $(\mathbf{x}',\mathbf{y}',B')=\mu_k(\mathbf{x},\mathbf{y},B)$ is obtained by the following rules:

(1) $\mathbf{x}'=(x'_1,\dots,x'_n)$ is given by $x'_kx_k=\frac{y_k\prod x_i^{[b_{ik}]_+}+\prod x_i^{[-b_{ik}]_+}}{y_k\oplus 1}$ and $x'_i=x_i$ for $i\neq k$;
(2) $\mathbf{y}'=(y'_1,\dots,y'_n)$ is given by $y'_i=y_k^{-1}$ for $i=k$; and otherwise $y'_i=y_iy_k^{[b_{ki}]_+}(y_k\oplus 1)^{-b_{ki}}$;
(3) $B'=\mu_k(B)$.

Note that we can also use $(\mathbf{x},\mathbf{y},Q(B))$ instead of $(\mathbf{x},\mathbf{y},B)$. For every seed $(\tilde{\mathbf{x}},\tilde{\mathbf{y}},\tilde{B})$ obtained from the seed $\Sigma=(\mathbf{x},\mathbf{y},B)$ by a sequence of mutations, we call $\tilde{\mathbf{x}}$ a *cluster* and its elements are called *cluster variables*. The *cluster algebra* $\mathcal{A}(\Sigma)$ of rank n associated to a seed $\Sigma=(\mathbf{x},\mathbf{y},B)$ is the $\mathbb{ZP}$-subalgebra of F generated by all cluster variables. In this paper, we mainly focus on mutations of quivers.

2.2. A characterization of acyclic valued quivers

A valued quiver Q is called *acyclic*, if it has no k-cycles in Q for any $k \geqslant 1$. The fact whether a valued quiver is acyclic will be influential for the corresponding cluster algebra. Indeed, some important conjectures were proved to be true in the case cluster algebras have acyclic valued quivers; otherwise, however, they would face great difficult for affirmation. In this section, we give a criterion of acyclicity. The following lemma is easy to see:

Lemma 2.5. *Let Q be a valued quiver and $A = A(Q)$ be its adjacency matrix. Suppose π is a permutation of $\{1, 2, \ldots, n\}$, and $A' = (a'_{ij})$, where $a'_{ij} = a_{\pi(i)\pi(j)}$. If P is the corresponding permutation matrix of π, then $PAP^T = A'$. In particular, $det(A) = det(A')$.*

Proposition 2.6. *Let Q be a valued quiver. Then the following statements are equivalent:*

(i) Q is acyclic.
(ii) The principal minors of $A(Q)$ are zeros.
(iii) The eigenvalues of $A(Q)$ are zeros.

Proof. (i)⇒(ii): Since Q is a finite acyclic quiver, there exists a bijection between Q_0 and $\{1, 2, \ldots, n\}$ such that if we have an arrow $j \to i$, then $j < i$. Hence by the Lemma 2.5, there exists a permutation matrix P such that $PA(Q)P^T = A'(Q)$, where $A'(Q)$ is a strictly upper triangular matrix. The principal minors of $A'(Q)$ are zeros, so are the principal minors of $A(Q)$.

(ii)⇒(i): Assume that Q is not acyclic, then there exists at least one cycle in Q. Let Q' be a full valued subquiver of Q such that Q' has a cycle. We may, without loss of generality, assume Q' to be minimal with this property. Then Q' must be a chordless k-cycle. Suppose $Q'_0 = \{j_1, j_2, \ldots, j_k\}$ with $k \geqslant 3$, and there only exist arrows from j_s to j_{s+1} $(1 \leqslant s \leqslant k-1)$ and from j_k to j_1. It follows from the Lemma 2.5 that the adjacency matrix $A(Q')$ of Q' is similar to the matrix

$$\begin{bmatrix} 0 & c_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & c_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & c_{k-1} \\ c_k & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

where c_i equals to the first element $v(\alpha_i)_1$ of $v(\alpha_i)$, where α_i is the arrow from j_i to j_{i+1} for $1 \leqslant i \leqslant k-1$, and c_k equals the first element $v(\alpha_k)_1$ of $v(\alpha_k)$, where α_k is the arrow from j_k to j_1. Hence $det(A(Q')) = (-1)^{1+k} c_1 \ldots c_k \neq 0$.

Then the principal minor of $A(Q)$ indexed by $\{j_1, j_2, \ldots, j_k\}$ is not zero, which is a contradiction.

(ii) $\Rightarrow$ (iii): Suppose that $f(\lambda) = |\lambda I_n - A(Q)| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$. Since $(-1)^s a_s$ is the sum of all principal minors of order s, we get $f(\lambda) = \lambda^n$, then (iii) holds.

(iii) $\Rightarrow$ (ii): We will prove any principal minor of the matrix $A(Q)$ of order k is zero by induction on k. In the cases of $k = 1$ and $k = 2$, the conclusion follows from the definition of valued cluster quivers. Now we assume that this conclusion holds for any m, where $m \leqslant k-1 \leqslant n$. We consider the case of $m = k$. Because the principal minors of $A(Q)$ which has l rows and l columns are zeros for each $l \in [1, k-1]$, Q has not any l-cycles for any $l \in [1, k-1]$ and so are all of its full valued subquivers of order k. Let Q' be any full valued subquiver of order k. If the full valued subquiver Q' is acyclic, then the corresponding principal minor is zero. Otherwise Q' has a cycle and hence Q' must be a chordless k-cycle. Similar to the proof in (ii)$\Rightarrow$(i), we deduce that the corresponding principal minor has the sign $(-1)^{k+1}$. Then all of the nonzero principal minors of order k share the same sign. Because the eigenvalues of $A(Q)$ are zeros, the sum of all of principal minors of order k is zero. Now it is obvious that any principal minor of the matrix $A(Q)$ of order k is zero. We finish the proof. □

As the converse-negative result of Proposition 2.6, we have:

Corollary 2.7. *Assume that $A_{i_1 i_2 \ldots i_k}$ is a principal submatrix of $A(Q)$ indexed by $i_1, i_2, \ldots, i_k$. If $det A_{i_1 i_2 \ldots i_k} \neq 0$, then there exists a full subquiver Q' of Q which is a cycle such that $Q'_0 \subseteq \{i_1, i_2, \ldots, i_k\}$.*

3. Spectra of exchange matrices

In this section, we discuss firstly the relations between exchange spectrum of a valued quiver and adjacency spectrum of its underlying valued graph, and secondly the relations between exchange spectrum of a valued quiver and that of its full valued subquivers.

Let Q be a valued quiver, we now turn on the properties of the exchange matrix $B(Q)$. Since $B(Q)$ is skew-symmetrizable, there is a diagonal matrix D with positive diagonal entries such that $DB(Q)$ is skew-symmetric. It is easy to check that $D^{\frac{1}{2}} B(Q) D^{-\frac{1}{2}}$ is real, skew-symmetric and similar to $B(Q)$. We will use this property frequently and some well-known properties for real skew-symmetric matrices are given as follows.

Lemma 3.1. *Let B be a real skew-symmetric matrix of order n, then following assertions hold:*

(i) *$det(B) \geq 0$. Moreover, if n is odd, then $det(B) = 0$.*
(ii) *The eigenvalues of B appear in complex conjugate pairs, and any eigenvalue of B is either an imaginary number or zero.*
(iii) *The sum of all the eigenvalues of B is zero, and equals the sum of all the imaginary parts of the eigenvalues of B.*
(iv) *There exists an orthogonal matrix P such that*

$$PBP^T = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & B_1 & & \\ & & & & \ddots & \\ & & & & & B_s \end{bmatrix},$$

where $B_k = \begin{bmatrix} 0 & b_k \\ -b_k & 0 \end{bmatrix}$, $b_k \in \mathbb{R}$, $1 \leqslant k \leqslant s$, and P^T is the transpose of P.
(v) *There exists an unitary matrix U such that UBU^* is a diagonal matrix, where U^* is the conjugate transpose of U.*

3.1. Relations between exchange spectrum and adjacency spectrum

The largest eigenvalue of the adjacency matrix of an unoriented graph has been studied well and this investigation is still active. It is also interesting to consider the exchange spectrum radius of a valued quiver. Here we introduce some notations for the sake of the following proofs. Let $M = (m_{ij})_{l\times s}$ and $N = (n_{ij})_{l\times s}$ be two real matrices of the same size, we say $M > N$ ($M \geqslant N$, resp.) if $m_{ij} > n_{ij}$ ($m_{ij} \geqslant n_{ij}$, resp.) for all $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant s$. For a real matrix $M = (m_{ij})_{m\times n}$, if $m_{ij} > 0$ ($\geqslant 0$, resp.) for any $i \in [1, m]$, $j \in [1, n]$, M is said to be positive (nonnegative, resp.) and is denoted by $M > 0$ ($M \geqslant 0$, resp.). And $\mathrm{i} = \sqrt{-1}$.

Lemma 3.2 ([5], Theorem 3.6.2)**.** *The eigenvalues of the complex matrix $M = (m_{ij})_{n\times n}$ of order n lie in the region of the complex plane determined by the union of the n-closed discs*

$$T_i = \{x|\ |x - m_{ii}| \leq t_i\}, \quad i = 1, 2, \ldots, n.$$

where $t_i = \sum_{j=1,j\neq i}^{n} |m_{ij}|$, $i = 1, 2, \ldots, n$.

Proposition 3.3. *Suppose that Q is a valued quiver with $n = |Q_0|$ and the connected components Q'_1, Q'_2, ..., Q'_s. Let $B = B(Q) = (b_{ij})_{n\times n}$ be the exchange matrix of Q and $C = C(Q) = (c_{ij})_{n\times n}$ be the adjacency matrix of the underlying valued graph $\bar{Q}$. Let $h_i = \sum_{j=1}^{n}|b_{ij}| = \sum_{j=1}^{n} c_{ij}$ be the degree of the vertex $i \in Q_0$ and $h = \max\{h_1, h_2, \ldots, h_n\}$. Then for the exchange spectrum radius λ of Q and the adjacency spectrum radius μ of the valued graph $\bar{Q}$, the following is satisfied that*

$$0 \leqslant \lambda \leqslant \mu \leqslant h = r,$$

for $r_p = \max_{i\in(Q'_p)_0}\sum_{j\in(Q'_p)_0}|b_{ij}|$, $1 \leqslant p \leqslant s$, and $r = \max\{r_1, r_2, \ldots, r_s\}$.

Proof. $0 \leqslant \lambda$ and $h = r$ are obvious by definitions.

$\mu \leqslant h$ is just a corollary of Lemma 3.2 by letting $M = C(Q)$ in Lemma 3.2.

Let us show that the inequality $\lambda \leqslant \mu$ holds. Since B is a skew-symmetrizable matrix, there exists a diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ with $d_i > 0$ for $1 \leqslant i \leqslant n$ such that DB is skew-symmetric. Then it is clear that $B' = (b'_{ij}) = D^{\frac{1}{2}}BD^{-\frac{1}{2}}$ is skew-symmetric, $C' = (c'_{ij}) = D^{\frac{1}{2}}CD^{-\frac{1}{2}}$ is symmetric and $c'_{ij} = |b'_{ij}|$ for any i, $j \in [1, n]$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of B' corresponding to λi, say, $B'x = (\lambda\mathrm{i})x$. Thus for any $j \in [1, n]$, we have

$$\lambda|x_j| = |\lambda\mathrm{i}x_j| = |\sum_{s=1}^{n} b'_{js}x_s| \leqslant \sum_{s=1}^{n}|b'_{js}||x_s| = \sum_{s=1}^{n} c'_{js}|x_s|.$$

Let $y = (|x_1|, |x_2|, \ldots, |x_n|)^T$, we get $C'y \geqslant \lambda y \geqslant 0$, and $y \geqslant 0, y \neq 0$. Therefore, $\lambda y^T y \leqslant y^T C'y$.

Now let z be an eigenvector of C' corresponding to μ. By the Rayleigh theorem, we have

$$\lambda \leqslant \frac{y^T C'y}{y^T y} \leqslant \frac{z^T C'z}{z^T z} = \mu.$$

□

Proposition 3.4. *With the notations above, if $\lambda = h$, then there exists a full valued subquiver Q' of Q which is also a connected component of Q such that $h_i = \sum_{j\in(Q')_0}|b_{ij}| = h$, for each vertex $i \in (Q')_0$.*

Proof. Assume that hi is an eigenvalue of $B(Q)$. If the set of arrows $Q_1 = \varnothing$, it is obvious. Now we assume that $Q_1 \neq \varnothing$ so that $h \neq 0$. Let Q'_1, Q'_2, ..., Q'_s be all connected components of the valued quiver Q and $B_i = B(Q'_i)$ be the exchange matrix of Q'_i for $i = 1, 2, \ldots, s$. Then we have

$$\mathrm{Spec}(B(Q)) = \bigcup_{i=1}^{s}\mathrm{Spec}(B_i).$$

We may assume that $h\mathrm{i}$ is an eigenvalue of B_k for some $k \in [1, s]$. Without loss of generality, we assume that $(Q'_k)_0 = \{1, 2, \dots, m\}$, $m \geqslant 2$, $B_k = (b_{ij})_{m\times m}$, and $C_k = C(Q'_k) = (c_{ij})_{m\times m}$. Since h is the exchange spectrum radius of Q'_k, it follows from Proposition 3.3 that h is the largest eigenvalue of C_k. It is also clear that

$$C_k(1, 1, \dots, 1)^T \leqslant h(1, 1, \dots, 1)^T.$$

Since $\bar{Q}'_k$ is connected, the symmetrizable matrix C_k is irreducible and $C_k \geqslant 0$. It follows from the Perron-Frobenius theorem that

$$C_k(1, 1, \dots, 1)^T = h(1, 1, \dots, 1)^T.$$

Now it is easy to see that the connected component Q'_k is what we need. □

From Proposition 3.3, for any valued quiver, we know its exchange spectrum radius is not more than the adjacency spectrum radius of its underlying valued graph. In particular, when its underlying graph is a tree, we can say more.

Proposition 3.5. *Let Q be a valued quiver with $Q_0 = n$ and its underlying graph $\bar{Q}$ be a tree. Assume that $f(x)$ and $g(x)$ are the exchange polynomial of Q and the adjacency polynomial of the underlying valued graph $\bar{Q}$ respectively, that is, $f(x) = |xI_n - B(Q)|$, $g(x) = |xI_n - C(Q)|$. Then, for $\lambda \in \mathbb{R}$, $f(\lambda\mathrm{i}) = 0$ if and only if $g(\lambda) = 0$. Moreover, it holds that*

$$\mathrm{Spec}(B(Q)) = \begin{bmatrix} \lambda_0\mathrm{i} & \lambda_1\mathrm{i} & \dots & \lambda_p\mathrm{i} \\ n_0 & n_1 & \dots & n_p \end{bmatrix}$$

if and only if

$$\mathrm{Spec}(C(Q)) = \begin{bmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_p \\ n_0 & n_1 & \dots & n_p \end{bmatrix},$$

where $\lambda_0 < \lambda_1 < \dots < \lambda_p$, and $n_0 + n_1 + \dots + n_p = n$.

Proof. Note that $B = B(Q) = (b_{ij})_{n\times n}$ is skew-symmetrizable and $C = C(Q) = (c_{ij})_{n\times n}$ is symmetrizable with the same diagonal matrix D, and it is easy to see that $|b_{ij}| = c_{ij}$. We have $D^{\frac{1}{2}}BD^{-\frac{1}{2}}$ is skew-symmetric and $D^{\frac{1}{2}}CD^{-\frac{1}{2}}$ is symmetric.

At first, we prove that $det(B) = (-1)^{\frac{n}{2}}det(C)$.

Note that we have that

$$det(B) = \sum_{\pi\in S_n} \mathrm{sgn}(\pi)b_{1\pi1}b_{2\pi2}\dots b_{n\pi n},\ det(C) = \sum_{\pi\in S_n} \mathrm{sgn}(\pi)c_{1\pi1}c_{2\pi2}\dots c_{n\pi n},$$

where S_n means the permutation group of $\{1, 2, \dots, n\}$. Because of the definition of B, $b_{ij} \neq 0$ if and only if the two vertices i and j are adjacent in $\bar{Q}$. In particular, if $i = \pi i$, then $b_{i\pi i} = 0$. If π is not the identity, then π can be uniquely expressed to be a product of disjoint cycles of length at least two. Let the cycle $(spr \dots t)$ be a factor of length more than two of π, then it corresponds to the factor $b_{sp}b_{pr} \dots b_{ts}$ of the term $\text{sgn}(\pi)b_{1\pi 1} \dots b_{n\pi_n}$. And $b_{sp}b_{pr} \dots b_{ts} \neq 0$ if and only if the pairs $\{s, p\}$, $\{p, r\}$, $\dots$, $\{t, s\}$ are adjacent pairs in $\bar{Q}$. In this case, the induced subgraph of $\bar{Q}$ determinded by $\{s, p, r, \dots, t\}$ admits a cycle of length more than two. But we know that there are no k-cycles for $k \geqslant 3$ in $\bar{Q}$. Thus if the term $\text{sgn}(\pi)b_{1\pi 1}b_{2\pi 2} \dots b_{n\pi n}$ does not vanish, π must be a product of disjoint cycles of length two. The same statements hold for C.

If $n = |Q_0|$ is odd, then $det(B) = 0$ follows from Lemma 3.1(1) and the fact that $D^{\frac{1}{2}}BD^{-\frac{1}{2}}$ is skew-symmetric. For any $\pi \in S_n$, if the term $\text{sgn}(\pi)c_{1\pi 1}c_{2\pi 2} \dots c_{n\pi n} \neq 0$, then it implies $\pi i \neq i$ for any $i \in [1, n]$ and π is a product of disjoint cycles of length two for $\bar{Q}$ is a tree, which means it is impossible for $n = |Q_0|$ to be odd. Thus, $\text{sgn}(\pi)c_{1\pi 1}c_{2\pi 2} \dots c_{n\pi n} = 0$ for any π. Hence $det(B) = 0 = det(C)$.

If $n = |Q_0|$ is even, it is easy to see that for $\pi \in S_n$, $b_{1\pi 1}b_{2\pi 2} \dots b_{n\pi n} \neq 0$ if and only if $c_{1\pi 1}c_{2\pi 2} \dots c_{n\pi n} \neq 0$ and in this case, π is a product of disjoint cycles of length two. Note that $b_{ij}b_{ji} = -c_{ij}c_{ji}$, we have $\text{sgn}(\pi)b_{1\pi 1}b_{2\pi 2} \dots b_{n\pi n} = (-1)^{\frac{n}{2}}\text{sgn}(\pi)c_{1\pi 1}c_{2\pi 2} \dots c_{n\pi n}$. for any $\pi \in S_n$. Thus we prove that $det(B) = (-1)^{\frac{n}{2}}det(C)$.

Because the underlying graphs of full valued subquivers of Q do not have l-cycles for $l \geqslant 3$ either, then for any full valued subquiver Q' of Q of order r, we have $det(B(Q')) = (-1)^{\frac{r}{2}}det(C(Q'))$. By the relations between coefficients of characteristic polynomials and principal minors, we have the following statements.

When n is even, let $n = 2m$. Note that all principal minors of B of odd orders are zeros, we may assume that

$$f(\lambda) = \lambda^{2m} + v_2\lambda^{2m-2} + v_4\lambda^{2m-4} + \cdots + v_{2m-2}\lambda^2 + v_{2m},$$

where $(-1)^k v_k$ is the sum of all of principal minors of B of order k. Then

$$g(\lambda) = \lambda^{2m} + (-1)v_2\lambda^{2m-2} + (-1)^2 v_4\lambda^{2m-4} + \cdots + (-1)^m v_{2m}.$$

It is easy to see that $f(\lambda \mathrm{i}) = (-1)^m g(\lambda)$.

When n is odd, let $n = 2m + 1$. Similarly, we may assume that

$$f(\lambda) = \lambda^{2m+1} + v_2\lambda^{2m-1} + v_4\lambda^{2m-3} + \cdots + v_{2m-2}\lambda^3 + v_{2m}\lambda,$$

and then

$$g(\lambda) = \lambda^{2m+1} + (-1)^2 v_4 \lambda^{2m-3} + \cdots + (-1)^m v_{2m} \lambda.$$

It is clear that $f(\lambda \mathrm{i}) = \mathrm{i}(-1)^m g(\lambda)$.

Thus in all cases, we have that $f(\lambda \mathrm{i}) = 0$ if and only if $g(\lambda) = 0$.

Moreover, suppose that $f(\lambda) = (\lambda^2 + q_1)(\lambda^2 + q_2)\dots(\lambda^2 + q_s)\lambda^{n-2s}$ for $0 < q_1 \leqslant q_2 \leqslant \cdots \leqslant q_s$.

When $n = 2m$, we have

$$\begin{aligned}(-1)^m g(\lambda) = f(\lambda \mathrm{i}) &= (\lambda^2 - q_1)(\lambda^2 - q_2)\dots(\lambda^2 - q_s)\lambda^{n-2s}(-1)^s(\mathrm{i})^{\mathrm{n-2s}} \\ &= (\lambda^2 - q_1)(\lambda^2 - q_2)\dots(\lambda^2 - q_s)\lambda^{n-2s}(-1)^m.\end{aligned}$$

Thus, $g(\lambda) = (\lambda^2 - q_1)(\lambda^2 - q_2)\dots(\lambda^2 - q_s)\lambda^{n-2s}$.

When $n = 2m + 1$, we have

$$\begin{aligned}\mathrm{i}(-1)^m g(\lambda) = f(\lambda \mathrm{i}) &= (\lambda^2 - q_1)(\lambda^2 - q_2)\dots(\lambda^2 - q_s)\lambda^{n-2s}(-1)^s(\mathrm{i})^{\mathrm{n-2s}} \\ &= (\lambda^2 - q_1)(\lambda^2 - q_2)\dots(\lambda^2 - q_s)\lambda^{n-2s}(-1)^m \mathrm{i}.\end{aligned}$$

Thus, $g(\lambda) = (\lambda^2 - q_1)(\lambda^2 - q_2)\dots(\lambda^2 - q_s)\lambda^{n-2s}$.

So $\mathrm{Spec}(f) = \begin{bmatrix}\lambda_0 \mathrm{i} & \lambda_1 \mathrm{i} & \dots & \lambda_p \mathrm{i} \\ n_0 & n_1 & \dots & n_p\end{bmatrix}$ implies $\mathrm{Spec}(g) = \begin{bmatrix}\lambda_0 & \lambda_1 & \dots & \lambda_p \\ n_0 & n_1 & \dots & n_p\end{bmatrix}$.

The proof of the converse statement is similar. □

A valued quiver (Q', v') is said to be obtained by re-orienting an arrow α from a valued quiver (Q, v) if $Q'_1 = \{\alpha^{op}\} \bigcup Q_1 \setminus \{\alpha\}$, $(v'(\alpha^{op})_1, v'(\alpha^{op})_2) = (v(\alpha)_2, v(\alpha)_1)$ and $v'(\beta) = v(\beta)$ for $\beta \in Q_1 \setminus \{\alpha\}$, where α^{op} is the opposite arrow of α. Re-orientations of a valued quiver by re-orienting a set of arrows are defined step by step. Then we have the following corollary.

Corollary 3.6. *Let T be a full valued subquiver of a connected valued quiver Q such that:*

(i) The underlying graph $\bar{T}$ of T is a tree.
(ii) There is only one vertex $x \in T_0$ connecting with the vertices in $Q_0 \setminus T_0$.

Then all re-orientations of the valued quiver Q by re-orienting T and maintaining T' unchanged share the same exchange polynomial, where T' is a full valued subquiver of Q determined by $Q_0 \setminus T_0$. In particular, if the underlying graph $\bar{Q}$ of Q is a tree, then all re-orientations of Q share the same exchange polynomial.

Proof. Without loss of generality, we may suppose that $T'_0 = \{1, 2, \dots, m\}$, $T_0 = \{m+1, m+2, \dots, m+n\}$ and $x = m+1$. The exchange matrices of T' and $T \setminus \{x\}$ are assumed to be $X_{m\times m}$ and $Y_{(n-1)\times(n-1)}$, respectively. Then the exchange matrix of Q will be the following form:

$$\begin{bmatrix} X & -w^T & 0 \\ \alpha & 0 & \beta \\ 0 & -y^T & Y \end{bmatrix},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$, $w = (w_1, w_2, \dots, w_m)$, and $y = (y_1, y_2, \dots, y_{n-1})$. Assume that characteristic polynomials of X and Y are $X(\lambda)$ and $Y(\lambda)$, respectively. Then the exchange polynomial of Q is

$$\begin{aligned} &\begin{vmatrix} \lambda I_m - X & w^T & 0 \\ -\alpha & \lambda & -\beta \\ 0 & y^T & \lambda I_{n-1} - Y \end{vmatrix} \\ &= (\alpha_1 w_1 X_1(\lambda) + \alpha_2 w_2 X_2(\lambda) + \cdots + \alpha_m w_m X_m(\lambda))Y(\lambda) \\ &\quad + \lambda X(\lambda)Y(\lambda) + [\beta_1 y_1 Y_1(\lambda) + \beta_2 y_2 Y_2(\lambda) + \cdots + \beta_{n-1} y_{n-1} Y_{n-1}(\lambda)]X(\lambda), \end{aligned}$$

where $X_k(\lambda)$ is the determinant of the principal submatrix of the matrix $\lambda I_m - X$ obtained by deleting the k-th row and k-th column, and $Y_j(\lambda)$ is the determinant of the principal submatrix of $\lambda I_{n-1} - Y$ obtained by deleting the j-th row and j-th column for any $k \in [1, m]$, $j \in [1, n-1]$. Reorientations of Q with T' unchanged will keep $X(\lambda)$, $X_1(\lambda)$, $\dots$, $X_m(\lambda)$, $\alpha_1 w_1$, $\dots$, $\alpha_m w_m$, $\beta_1 y_1$, $\dots$, $\beta_{n-1} y_{n-1}$ unchanged. It only needs to show that $Y(\lambda)$, $Y_1(\lambda)$, $\dots$, $Y_{n-1}(\lambda)$ stay unchanged, and this follows immediately from Proposition 3.5. □

For any orientation of a tree, we may get a tree quiver. Recall that an *induced subgraph* (or say, a *full subgraph*) of a graph is a subgraph obtained from the original graph by keeping an arbitrary subset of vertices together with all the edges that have both endpoints in this subset. We have the following results for (tree) quivers on exchange spectrum radii.

Corollary 3.7. *The following assertions hold:*

(1) Let Q be a tree quiver, then

(i) The exchange spectrum radius of Q is less than two if and only if the underlying graph of Q is one of Dynkin diagrams (see Figure 1).

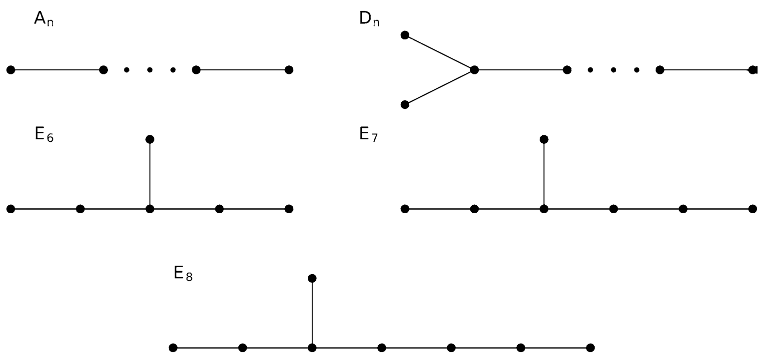

Figure 1: Dynkin diagrams A_n, D_n, E_6, E_7, and E_8.

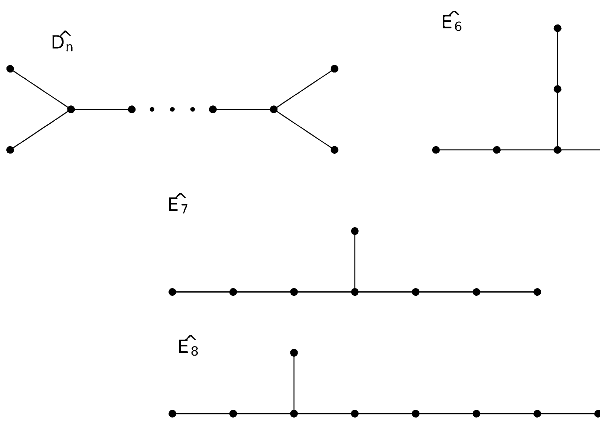

Figure 2: Acyclic extended Dynkin diagrams $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$, and $\hat{E}_8$.

(ii) The exchange spectrum radius of Q is 2 if and only if the underlying graph of Q is one of the graphs $\hat{D}_n$ ($n \geqslant 4$, with $n+1$ vertices in it), $\hat{E}_6$, $\hat{E}_7$, or $\hat{E}_8$ (see Figure 2).

(iii) The exchange spectrum radius of Q is more than two if and only if the underlying graph of Q contains $\hat{D}_n$ ($n \geqslant 4$), $\hat{E}_6$, $\hat{E}_7$, or $\hat{E}_8$ as a proper induced subgraph.

(2) Let Q be an arbitrary quiver with exchange spectrum radius more than two, then $\bar{Q}$ either contains X_2, $\hat{A}_n$ ($n \geqslant 2$), $\hat{D}_n$ ($n \geqslant 4$), $\hat{E}_6$, $\hat{E}_7$, or $\hat{E}_8$ as a proper induced subgraph, or contains X_n ($n \geqslant 3$) as an induced subgraph, where $\hat{A}_n$ is a simple chordless $(n+1)$-cycle, and X_n is a graph with two vertices and n edges.

Proof. (1) It follows from the Proposition 3.5 that the exchange spectrum radius of a valued quiver equals to the adjacency spectrum radius of its underlying graph if its underlying graph is a tree. Then the conclusion follows from results in [15] (refer also to Theorem 3.1.3 in [4]).

(2) This assertion follows from (1), Proposition 3.3 and results in [15]. □

Dynkin diagrams have appeared in many branches of mathematics, for example, in the classification of finite type of cluster algebras, finite dimensional associated algebras, and Lie algebras. It is also interesting to see they can be associated with exchange matrices of valued quivers.

3.2. Exchange spectra of a valued quiver and full valued subquivers

In this subsection, we make use of Cauchy's interlacing theorem for symmetric matrices to prove a similar result for skew-symmetrizable matrices, and we use this result to compare the exchange radii of valued quivers and their full valued subquivers.

Theorem 3.8. *Let Q be a valued quiver with $|Q_0| = n$ and Q' be a full valued subquiver order $n-1$ of Q. If the eigenvalues of $B(Q)$ are $\lambda_1\mathrm{i}$, $\lambda_2\mathrm{i}$, $\dots$, $\lambda_n\mathrm{i}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and the eigenvalues of $B(Q')$ are $\gamma_2\mathrm{i}$, $\gamma_3\mathrm{i}$, $\dots$, $\gamma_n\mathrm{i}$ with $\gamma_2 \geqslant \gamma_3 \geqslant \cdots \geqslant \gamma_n$, then $\lambda_1 \geqslant \gamma_2 \geqslant \lambda_2 \geqslant \gamma_3 \geqslant \lambda_3 \geqslant \cdots \geqslant \gamma_n \geqslant \lambda_n$.*

Proof. Since $B = B(Q)$ is a real skew-symmetrizable matrix and $B' = B(Q')$ is a principal submatrix of order $n-1$ of $B(Q)$, there exists a diagonal matrix D with positive diagonal entries such that DB is skew-symmetric. It is clear that $B_1 = D^{\frac{1}{2}}BD^{-\frac{1}{2}}$ and $B_1' = D_1^{\frac{1}{2}}B'D_1^{-\frac{1}{2}}$ are skew-symmetric, where D_1 is the corresponding principal submatrix of D such that D_1B' is skew-symmetric. It is also obvious that there is a permutation matrix P such that

$$PB_1P^T = \begin{bmatrix} 0 & \alpha^* \\ -\alpha & B_1' \end{bmatrix},$$

where α^* is the conjugate transpose of the vector α. Since B_1' is a real skew-symmetric matrix, there exists an unitary matrix T such that $TB_1'T^*$ is a diagonal matrix, i. e.,

$$TB_1'T^* = \begin{bmatrix} \gamma_2\mathrm{i} & & & \\ & \gamma_3\mathrm{i} & & \\ & & \ddots & \\ & & & \gamma_n\mathrm{i} \end{bmatrix},$$

where T^* is the conjugate transpose of T. Now let a matrix $H = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}$. Then we have

$$\begin{aligned} HPB_1P^TH^* &= \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} 0 & \alpha^* \\ -\alpha & B_1' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & T^* \end{bmatrix} \\ &= \begin{bmatrix} 0 & \alpha^*T^* \\ -T\alpha & TB_1'T^* \end{bmatrix} = \begin{bmatrix} 0 & \beta^* \\ -\beta & TB_1'T^* \end{bmatrix}, \end{aligned}$$

where $\beta = T\alpha = (\beta_2, \beta_3, \dots, \beta_n)^T$. Assume that the characteristic polynomial of $HPB_1P^TH^*$ is $f(\lambda)$, then

$$\begin{aligned} f(\lambda) &= |\lambda I_n - HPB_1P^TH^*| \\ &= \begin{vmatrix} \lambda & -\beta^* \\ \beta & \lambda I_{n-1} - TB_1'T^* \end{vmatrix} \\ &= \begin{vmatrix} \lambda & -\bar{\beta}_2 & -\bar{\beta}_3 & \dots & -\bar{\beta}_n \\ \beta_2 & \lambda - \gamma_2 \mathrm{i} & 0 & \dots & 0 \\ \beta_3 & 0 & \lambda - \gamma_3 \mathrm{i} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & 0 & 0 & \dots & \lambda - \gamma_n \mathrm{i} \end{vmatrix}. \end{aligned}$$

Expanding the above determinant along the first row, we get

$$f(\lambda) = \lambda(\lambda - \gamma_2\mathrm{i}) \dots (\lambda - \gamma_n\mathrm{i}) + \sum_{k=2}^{n} |\beta_k|^2 (\lambda - \gamma_2\mathrm{i}) \dots (\widehat{\lambda - \gamma_k}\mathrm{i}) \dots (\lambda - \gamma_n\mathrm{i}),$$

where $(\widehat{x - \gamma_k})$ means deleting this term. Thus it follows that

$$\begin{aligned} f(x\mathrm{i}) =& (x\mathrm{i}) \dots (x\mathrm{i} - \gamma_n\mathrm{i}) + \sum_{k=2}^{n} |\beta_k|^2 (x\mathrm{i} - \gamma_2\mathrm{i}) \dots (\widehat{x\mathrm{i} - \gamma_k}\mathrm{i}) \dots (x\mathrm{i} - \gamma_n\mathrm{i}) \\ =& \mathrm{i}^n x \dots (x - \gamma_n) + \mathrm{i}^{n-2} \sum_{k=2}^{n} |\beta_k|^2 (x - \gamma_2) \dots (\widehat{x - \gamma_k}) \dots (x - \gamma_n) \\ =& \mathrm{i}^n g(x), \end{aligned}$$

where $g(x) = x(x-\gamma_2) \dots (x-\gamma_n) - \sum_{k=2}^{n} |\beta_k|^2 (x-\gamma_2) \dots (\widehat{x - \gamma_k}) \dots (x-\gamma_n)$. It is easy to see that $g(x)$ is the characteristic polynomial of the real symmetric matrix M defined as follows.

$$M = \begin{bmatrix} 0 & |\beta_2| & \dots & |\beta_n| \\ |\beta_2| & \gamma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ |\beta_n| & 0 & \dots & \gamma_n \end{bmatrix}.$$

Similar to the proof of Proposition 3.5, it is not difficult to see that the eigenvalues of the matrix B are λ_1i, λ_2i, ..., λ_ni if and only if the eigenvalues of the matrix M are λ_1, λ_2, ..., λ_n. Recall that Cauchy's interlacing theorem for Hermitian matrices (see e.g. [[14], Theorem 4.3.17]) claims that if $A_{n\times n}$ is a Hermitian matrix with eigenvalues $s_1 \geq s_2 \geq \cdots \geq s_n$ and $A'_{(n-1)\times(n-1)}$ is its principal submatrix with eigenvalues $t_2 \geq t_3 \geq \cdots \geq t_n$, then $s_i \leq t_i \leq s_{i-1}$ for any $i \in \{2, 3, \ldots, n\}$. Apply this result to M and its principal submatrix indexed by $\{2, 3, \ldots, n\}$, our proof is finished. □

Corollary 3.9. *Let Q be a valued quiver with $|Q_0| = n$ and Q' be a full valued subquiver of order m $(< n)$ of Q. Suppose that the eigenvalues of $B(Q)$ are λ_1i, λ_2i, ..., λ_ni with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and the eigenvalues of $B(Q')$ are γ_1i, γ_2i, ..., γ_mi with $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_m$. Then $\lambda_j \geqslant \gamma_j \geqslant \lambda_{n-m+j}$ for $j = 1, 2, \ldots, m$.*

In particular, the exchange spectrum radius of Q is either larger than or equal to that of Q'.

Proof. Because the exchange matrix of the full valued subquiver Q' is a principal submatrix of the exchange matrix of Q, the conclusion follows from iterated applications of Theorem 3.8. □

4. Mutation invariant of spectrum of a cluster quiver

In this section, we study mutation invariants of skew-symmetric matrices and quivers under the meaning of spectrum.

As a special case of Definition 2.4, mutation of quivers can be equivalently defined as follows.

Definition 4.1 ([16])**.** *Let Q be a quiver and $k \in Q_0$ be a fixed vertex. The mutation $\mu_k(Q)$ of Q at k is defined as follows:*

(1) For every 2-path $i \to k \to j$, add a new arrow $i \to j$;
(2) Reverse all arrows incident with k;
(3) Delete a maximal collection of 2-cycles from those created in (1)*.*

Note that parallel arrows are considered as different arrows in the first step in Definition 4.1.

Figure 3: An example.

4.1. Cospectral relationship of quivers and seeds

Two quivers Q and Q' are called *cospectral* if they share the same exchange polynomial. Two mutation equivalent seeds $\Sigma = (\mathbf{x}, \mathbf{y}, Q)$ and $\Sigma' = (\mathbf{x}', \mathbf{y}', Q')$ are said to be *cospectral* if Q and Q' are cospectral. In this case, we call clusters $\mathbf{x}$ and $\mathbf{x}'$ *cospectral* and denoted by $\mathbf{x} \sim_c \mathbf{x}'$.

Firstly, we consider the condition for quivers to be cospectral in a mutation class.

Lemma 4.2. *Let Q be a quiver and B is its exchange matrix. If $k \in Q_0$ is a sink or source, then the quivers Q and $\mu_k(Q)$ are cospectral.*

Proof. If $k \in Q_0$ is a sink or source, there are no 2-paths of the form $i \to k \to j$, $\mu_k(Q)$ is obtained from Q just by reversing all arrows incident with k. Then the exchange matrix of $\mu_k(Q)$ is given by

$$\mu_k(B) = J_k B J_k,$$

where J_k is the diagonal matrix obtained from the identity matrix by replacing the (k, k)-entry by -1. Obviously, the quivers Q and $\mu_k(Q)$ are cospectral. □

The following example shows that in general, the converse of Lemma 4.2 is not true.

Example 4.3. *Let us consider the quiver in Figure 3(a): Mutating at vertex 2, we get a quiver in Figure 3(b). It is obvious that $\mu_2(Q)$ and Q have the same exchange polynomial, see Remark 4.13. However the vertex 2 is neither a sink nor a source.*

If we consider quivers without 3-cycles, we have a better result. To prove the desired result, we need the following lemma.

Lemma 4.4. *Let $B(Q) = (b_{ij})_{n\times n}$ be the exchange matrix of a quiver Q. The exchange polynomial of Q is $f(\lambda) = |\lambda I_n - B(Q)| = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n$, then*

(i) $b_{2k-1} = 0$, $1 \le k \le \frac{n+1}{2}, k \in \mathbb{Z}$; *(ii)* $b_2 = \sum_{i<j} b_{ij}^2 = \sum_{j<i} b_{ij}^2$.

In particular, if Q is a simply-laced quiver, then b_2 equals to the number of arrows in Q.

Proof. Since the coefficient b_k of the characteristic polynomial equals to the sum of all principal minors of order k multiplying by $(-1)^k$, by Lemma 3.1, the determinants of skew-symmetric matrices of odd orders are zeros, then (i) is true. The principal minor of order two must be of the form

$$\begin{vmatrix} 0 & b_{ij} \\ -b_{ij} & 0 \end{vmatrix},$$

where $1 \leqslant i < j \leqslant n$. Then (ii) is also true. For a simply-laced quiver, the nonzero principal minors of order two must be

$$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \quad or \quad \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}.$$

And there is a bijection between the nonzero principal minors of order two and arrows in Q_1. Then the last statement follows. □

Let Q be a quiver and $B = B(Q)$. Given the matrix W at k satisfying the equality (1), we have $\mu_k(B) = WBW^T$.

Theorem 4.5. *Let Q be a quiver without 3-cycles and $Q_0 = \{1, 2, \dots, n\}$. Fix a vertex $k \in Q_0$, $B = B(Q) = (b_{ij})_{n\times n}$ and $\mu_k(B) = WBW^T$, where W satisfies the equality (1). The following statements are equivalent:*

(i) *Q and $\mu_k(Q)$ are cospectral;*
(ii) *k is either a sink or a source;*
(iii) $W = \begin{bmatrix} I_{k-1} & \varepsilon\zeta & \mathbf{0} \\ \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \varepsilon\theta & I_{n-k} \end{bmatrix}$, *where* $\zeta = (b_{1,k}, b_{2,k}, \dots, b_{k-1,k})^T$, $\theta = (b_{k+1,k}, b_{k+2,k}, \dots, b_{n,k})^T$, *and* $\varepsilon \in \{0, 1\}$.

Proof. (i)⇒(ii): Suppose that Q and $\mu_k(Q)$ share the same exchange polynomial, but k is neither a sink nor a source in Q. Since Q does not have 3-cycles, by the definition of mutation of quivers, when we mutate Q at k, the multiplicities of arrows between any two vertices either increase or keep intact. And because k is not a sink or source, there precisely exists some arrow whose multiplicity increases. Then the sum of all of principal minors of order two will changed after mutating at k, thus by Lemma 4.4 the exchange polynomial will changed, which is a contradiction.

(ii)⇒(i): It follows from Lemma 4.2.

(ii)⇔(iii): $k \in Q_0$ is a source if and only if $b_{ik} \leqslant 0$ for any $i \in Q_0$; and k is a sink if and only if $b_{jk} \geqslant 0$ for any $j \in Q_0$. Then it follows through comparing the definition of W in (1) and (iii). □

The following conjecture asserts that cospectral quivers form a finite connected subgraph of the exchange graph (see [12]), and one of them can be obtained by mutation at sinks and sources from the other.

Conjecture 4.6. *Let Q be a quiver. Then $Q', Q'' \in Mut(Q)$ are cospectral if and only if there exists a quiver $R \in Mut(Q)$ such that Q' and R are isomorphic and Q'' can be obtained from R by mutation at sinks and sources.*

For any seed $\Sigma = (\mathbf{x}, \mathbf{y}, Q)$ with Q a quiver, let $S(\Sigma) = \bigcup_{\mathbf{x}' \sim_c \mathbf{x}} \mathbf{x}'$. We call the subalgebra of the cluster algebra $\mathcal{A}(\Sigma)$ generated by $S(\Sigma)$ a *cospectral subalgebra* corresponding to Σ, written as $\mathcal{A}_c(\Sigma)$. If $\mathcal{A}(\Sigma) = \mathcal{A}_c(\Sigma)$, we say this cluster algebra $\mathcal{A}(\Sigma)$ to be a *cospectral cluster algebra*.

Clearly, $0 \neq \mathbb{ZP}[\mathbf{x}] \subsetneqq \mathcal{A}_c(\Sigma) \subsetneqq \mathcal{A}(\Sigma) \subsetneqq F$.

Let $M(\Sigma)$ denote the set of all seeds mutation equivalent to the seed Σ. Cospectral relation $\sim_c$ for seeds in $M(\Sigma)$ is an equivalence relation whose equivalence class for a seed Σ' is denoted by $[\Sigma']$, then we have $\mathcal{A}(\Sigma) = \sum_{[\Sigma'] \in M(\Sigma)/\sim_c} \mathcal{A}_c(\Sigma')$ as $\mathbb{ZP}$-algebras.

Example 4.7. *Some examples of cospectral subalgebra are given as follows.*

(i) Let $\Sigma = (\mathbf{x}, \mathbf{y}, Q)$ be a seed and Q is a quiver whose underlying graph is A_3. Then any quiver in $Mut(Q)$ is either an oriented 3-cycle or a quiver whose underlying graph is A_3 (see Lemma 4.11). Since all orientations of A_3 are cospectral (see Corollary 3.6) and exchange matrices of oriented 3-cycles are similar for they differ by a permutation, there are exactly two cospectral equivalence classes in $M(\Sigma)$. Let $\Sigma' = (\mathbf{x}', \mathbf{y}', Q')$ be a seed in $M(\Sigma)$ such that Q' is a 3-cycle, then we have $\mathcal{A}(\Sigma) = \mathcal{A}_c(\Sigma) + \mathcal{A}_c(\Sigma')$ as $\mathbb{ZP}$-algebras.

(ii) Any cluster algebra of rank two associated with a seed whose matrix is skew-symmetric is a cospectral cluster algebra.

4.2. Bounds of exchange spectrum radii of quivers

In the rest of this section, we consider the bounds of exchange spectrum radii of all quivers in a mutation class. Recall that Fomin and Zelevinsky introduced 2-finite matrices to study finite type classification of cluster algebras, see [11]. For our purposes, we just consider skew-symmetric matrices. For an integer skew-symmetric matrices B, B is said to be 2-*finite* if any matrix $B' = (b'_{ij})_{n\times n}$ mutation equivalent to B satisfies that $|b'_{ij}b'_{ji}| \leqslant 3$ for any $i, j \in [1, n]$. Equivalently, any quiver Q' mutation equivalent to the quiver $Q(B)$ is simply-laced.

Definition 4.8. *A valued quiver Q is called* r-maximal ($r > 0$) *if any quiver Q' mutation equivalent to Q has exchange spectrum radius no more than r.*

Note that a quiver is r-maximal if and only if so are all of its connected components. It follows from Corollary 3.9 that any full subquiver of a r-maximal quiver is r-maximal and any quiver contains a full subquiver which is not r-maximal is not r-maximal.

The following lemmas are well-known.

Lemma 4.9 ([11])**.** *All orientations of A_n (respectively, D_n, E_6, E_7, or E_8) are mutation equivalent.*

By Lemma 4.9, we use $Mut(A_n)$ to denote the mutation class of any quivers whose underlying graphs are A_n.

Lemma 4.10 ([11])**.** *Any* 2*-finite connected quiver is mutation equivalent to an orientation of a Dynkin diagram.*

Lemma 4.11 ([6])**.** *Let $Mut(A_p)$ be the mutation class of A_p. Then the class consists of connected quivers satisfying that:*

(i) All nontrivial cycles are oriented 3*-cycles.*
(ii) The degree of any vertex is less than five.
(iii) If a vertex has degree four, then two of its adjacent arrows belong to one 3*-cycle, and the other two belong to another* 3*-cycle.*
(iv) If a vertex has degree three, then two of its adjacent arrows belong to a 3*-cycle, and the third arrow does not belong to any* 3*-cycle.*

Note that a cycle in the first condition means a cycle in the underlying graph, not passing through the same vertex twice.

The following lemma is a simple observation for the case of quivers whose underlying graphs contain no 4-cycles.

Lemma 4.12. *Let $B(Q)$ be the exchange matrix of a simply-laced quiver Q with $|Q_0| \geqslant 4$. If the underlying graph $\bar{Q}$ of Q contains no 4-cycles, then the sum of all principal minors of $B(Q)$ of order four equals to the number of pairs of disadjacent arrows (i.e., without common vertices).*

Proof. The principal minor of $B(Q)$ of order four equals to the determinant of the exchange matrix of its corresponding full subquiver. Let us compute the determinant of the exchange matrix $R = (r_{ij})_{4\times 4}$ of a full subquiver Q' of order four. Write $det(R) = \sum_{\pi} \mathrm{sgn}(\pi) r_{1\pi 1} r_{2\pi 2} r_{3\pi 3} r_{4\pi 4}$.

Since the underlying graph $\bar{Q}$ of Q contains no 4-cycles, so does the underlying graph of Q'. If the term $\mathrm{sgn}(\pi) r_{1\pi 1} r_{2\pi 2} r_{3\pi 3} r_{4\pi 4}$ is not zero, π must be a composition of two disjoint 2-cycles and $\mathrm{sgn}(\pi) r_{1\pi 1} r_{2\pi 2} r_{3\pi 3} r_{4\pi 4} = 1$. Since each nonzero term corresponds to a pair of disadjacent arrows in a full subquiver of order 4 in Q, thus the sum of all principal minors of $B(Q)$ of order four equals to the number of pairs of disadjacent arrows in Q. □

Remark 4.13. *Let Q be a quiver of order 4, and the underlying graph of Q contains no 4-cycles. It is easy to see the determinant of $B(Q)$ does not depend on the orientations of $\bar{Q}$ from the proof of Lemma 4.12. Therefore its exchange polynomial just depends on its underlying graph. Since it follows from Lemma 4.4, the exchange polynomial of any valued quiver of order 3 does not depend on the orientations, then it is easy to compute the exchange polynomials of quivers of order less than 5.*

We recall preprojective algebras following from [9]. Let Q be a quiver and $\tilde{Q}$ be a quiver obtained from Q by adjoining an arrow $\sigma(\alpha) : j \to i$ for each arrow $\alpha : j \to i$. The preprojective algebra $\Theta(Q)$ of Q is the quotient of the path algebra of $\tilde{Q}$ modulo the ideal generated by the elements

$$\sum_{t(\beta)=i} \sigma(\beta)\beta, \quad i \in \tilde{Q}_0.$$

Then we have the following result.

Theorem 4.14. *Let Q be a connected quiver.*

(i) The quiver Q is 2-maximal if and only if it is mutation equivalent to an orientation of one of X_2, A_1, A_2, A_3, or A_4, where X_2 is a graph with two vertices and two edges.

(ii) If the underlying graph of Q is one of Dynkin diagrams, then the preprojective algebra $\Theta(Q)$ of Q is representation-finite if and only if Q is 2-maximal.

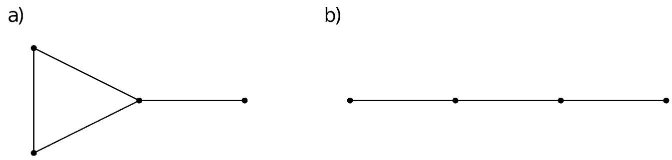

Figure 4: The underlying graphs in $Mut(A_4)$.

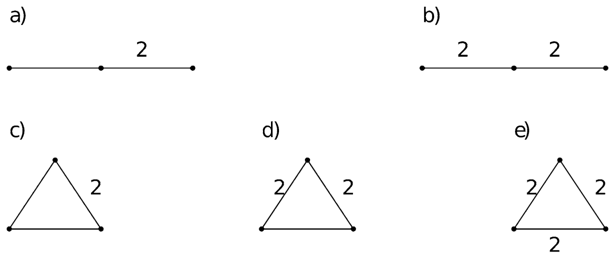

Figure 5: The number beside an edge means the multiplicity of the edge.

Proof. (i) By Lemma 4.11, the underlying graph of any quiver $Q' \in Mut(A_4)$ must be one of graphs in Figure 4. These two underlying graphs do not have 4-cycles, we may compute the exchange radii by using any orientation of them by Lemma 4.12 and Remark 4.13. In any case, it is not difficult to know the exchange spectrum radius is not more than two.

Let Q be a connected 2-maximal quiver. The multiplicities of arrows must be less than three, otherwise there will be a full subquiver whose exchange spectrum radius is more than two. If there exist arrows, whose multiplicities equal to two in Q and $\bar{Q}$, is not X_2, then there exists a full subquiver of Q whose underlying graph is one of the five graphs in Figure 5. In any case, the exchange spectrum radius of this full valued subquiver is more than 2. Thus Q must be mutation equivalent to an orientation of X_2.

Now we suppose that Q is a simply-laced 2-maximal quiver. Since Q is 2-maximal and connected, any quiver $Q' \sim Q$ must be a simply-laced quiver. Hence Q is 2-finite. By Lemma 4.10, Q is mutation equivalent to an orientation of one of Dinkin diagrams. Since all orientations of a Dynkin diagram are mutation equivalent and share the same exchange polynomial by Lemma 4.9 and Corollary 3.6, respectively. Let us consider the quiver Q_4 in Figure 6(a) whose underlying graph is D_4. The exchange spectrum radius of $\mu_1(Q_4)$ (see Figure 6(b)) is $\sqrt{5}$ which is more than two. Since D_n $(n \geq 4)$, E_6, E_7, and E_8 contains D_4 as an induced subgraph, it follows that Q cannot be mutation equivalent to any orientation of one of D_n $(n \geq 4)$, E_6, E_7, or E_8.

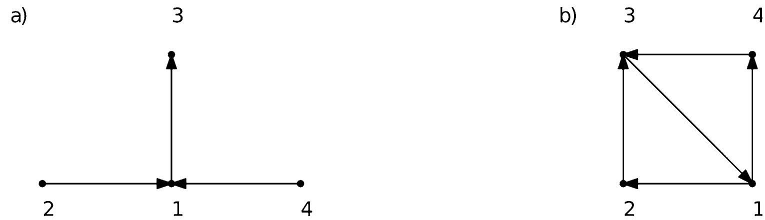

Figure 6: Q_4 and $\mu_1(Q_4)$.

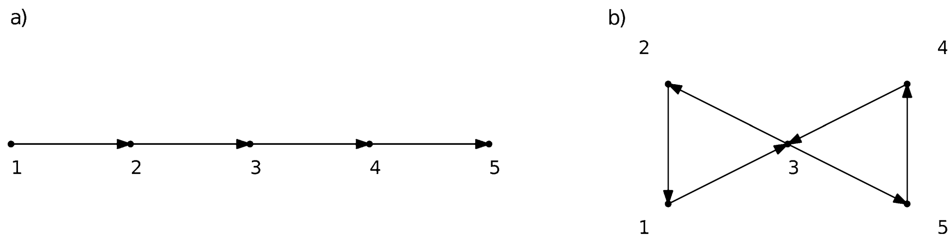

Figure 7: Q_5 and $\mu_2\mu_4(Q_5)$.

Finally we consider the quiver Q_5 in Figure 7(a) whose underlying graph is A_5. Mutate the quiver at the vertex 2 after mutating at the vertex 4, we get a quiver $\mu_2\mu_4(Q_5)$ (see Figure 7(b)) whose exchange spectrum radius is $\sqrt{5}$. In summary, we prove the conclusion.

(ii) It follows from [9] that $\Theta(Q)$ is representation-finite if and only if $\bar{Q}$ is of type A_1, A_2, A_3 or A_4. Thus the conclusion follows from (i). □

Note that since quivers in this paper are not allowed to have 2-cycles, the orientation of X_2 in Theorem 4.14 is just the Kronecker quiver.

Acknowledgements

This project is supported by the National Natural Science Foundation of China (No. 12071422) and the Zhejiang Provincial Natural Science Foundation of China (No. LY19A010023).

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