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Berkovich log discrepancies in positive characteristic

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Abstract: We introduce and study a log discrepancy function on the space of semivaluations centered on an integral noetherian scheme of positive characteristic. Our definition shares many properties with the analogue in characteristic zero; we prove that if log resolutions exist in positive characteristic, then our definition agrees with previous approaches to log discrepancies of semivaluations that use these resolutions. We then apply this log discrepancy to a variety of topics in singularity theory over fields of positive characteristic. Strong F-regularity and sharp F-purity of Cartier subalgebras are detected using positivity and non-negativity of log discrepancies of semivaluations, just as Kawamata log terminal and log canonical singularities are defined using divisorial log discrepancies, making precise a long-standing heuristic. We prove, in positive characteristic, several theorems of Jonsson and Mustaţă in characteristic zero regarding log canonical thresholds of graded sequences of ideals. Along the way, we give a valuation-theoretic proof that asymptotic multiplier ideals are coherent on strongly F-regular schemes.

Keywords: Log discrepancy, Berkovich spaces, multiplier ideals, graded sequences of ideals.

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1. Introduction

One of the fundamental ways to study singularities of normal varieties of dimension at least 3 is through the *log discrepancies* of real valuations on the variety. Log discrepancies were discovered by Mori's school in the 1980s as part of the development of the minimal model program; they have since found applications across algebraic geometry and commutative algebra.

During the 1990s and early 2000s, researchers began to realize deep connections between classes of singularities defined using log discrepancies and resolutions of singularities (e.g.: rational, log canonical, and Kawamata log terminal) and singularities appearing in tight closure (e.g.: F-rational, F-pure, and F-regular, resp.) [24, 39, 40, 51, 26, 27]. The groundbreaking connection was made by Hara and Watanabe in [26], where the authors showed that splittings of the Frobenius morphism on normal $\mathbb{Q}$-Gorenstein varieties can be converted into divisors giving log canonical pairs. We build on this connection, extending many of their ideas to the setting of normal F-finite schemes, and beyond.

1.1. Log discrepancies

Let X be a normal variety over an algebraically closed field k. Log discrepancies were classically defined only for divisorial valuations, meaning those of the form $c\,\mathrm{ord}_E$ for a real number $c > 0$ and a prime divisor $E \subset Y$ on a normal variety with a proper birational morphism $\pi : Y \to X$.

Extending the log discrepancy function to the space Val_X of valuations centered on X goes back at least to Favre and Jonsson's *Valuative Tree*, [23], for smooth complex surfaces, where they called it *thinness*. Numerous groups of authors developed the theory more generally in higher dimensions and on singular varieties [18, 30, 8, 9, 7]. The approach taken depends on using resolutions of singularities to find subspaces of Val_X that have simple, cone-like structures, and so is presently unavailable in dimensions greater than three in positive characteristics.

The goals of this paper are to extend log discrepancies to Val_X in characteristic $p > 0$ making systematic use of p^{-e}-linear maps, and to show that this is a good extension by demonstrating that a number of properties enjoyed by log-resolution-based extensions to Val_X are also enjoyed by our definition. In fact, we prove that the two approaches yield the *same* log discrepancy function should one have log resolutions (e.g. for surfaces, or 3-dimensional varieties over a perfect field [14, 15]); see 4.7. This is our main theorem, and should be compared with Mauri, Mazzon, and Stevenson's comparison of log discrepancies in characteristic zero with Temkin's canonical metrics, and Temkin's comparison with Mustaţă and Nicaise's weight metrics [37, 52, 41]. It is interesting to observe that Brezner and Temkin's different function for a finite morphism between Berkovich curves is closely related to our definition when applied to the Frobenius morphism (if this is finite) [11].

1.2. Statement of results

By a pair (X, Δ) we mean X is a normal variety over an algebraically closed field and $\Delta \geq 0$ is a $\mathbb{Q}$-Weil divisor on X. The starting point for this article is the following observation, due to Cascini, Mustaţă, and Schwede, shared with the present author in private correspondence.

Proposition 1.1 (4.1)**.** *Let (X, Δ) be a pair over a field with characteristic $p > 0$. Assume $(1 - p^e)(K_X + \Delta)$ is an integral Cartier divisor for some $e > 0$, and let $\psi_\Delta : \mathcal{O}_X((p^e - 1)\Delta) \to \mathcal{O}_X$ be the associated p^{-e}-linear map. For every divisor E over X, one has*

$$A_{(X,\Delta)}(E) = \sup\{(p^e - 1)^{-1}\mathrm{ord}_E(f) \,:\, f \in k(X), \psi_\Delta(f) = 1\}.$$

This expression is quite similar to the definition of $A_X(E)$ in the approaches to log discrepancies on non-$\mathbb{Q}$-Gorenstein varieties [18, 8]. We model our definition of log discrepancy on this proposition, incorporating also a supremum over $e \geq 1$ with $(1 - p^e)(K_X + \Delta)$ Cartier. The main result of this paper is that this extension to arbitrary valuations matches Jonsson and Mustaţă's in the presence of log resolutions.

Main Theorem (4.7)**.** *Let (X, Δ) be a pair over a field with characteristic $p > 0$. Suppose $(1 - p^e)(K_X + \Delta)$ is an integral Cartier divisor for some $e > 0$, and suppose log resolutions exist for varieties over k of dimension $\dim(X)$. For valuations $v \in \mathrm{Val}_X$, denote by $A_{(X,\Delta)}(v)$ the log discrepancy defined as in [30], and by $A(v; \mathcal{C}^X \cdot \Delta)$ our log discrepancy from Section 3. Then $A_{(X,\Delta)}(v) = A(v; \mathcal{C}^X \cdot \Delta)$ for all $v \in \mathrm{Val}_X$.*

This theorem proves that our approach is the "correct" definition of log discrepancies on Val_X, cf. [37, 41, 52, 11]. Because of this theorem, for the rest of this introduction we write A_X for the quantity that would be written $A(-; \mathcal{C}^X)$ in the notation of §3.

As one might hope, we can extend Hara and Watanabe's result that $\mathbb{Q}$-Gorenstein normal varieties that are F-pure (resp. F-regular) are log canonical (resp. Kawamata log terminal; klt). We no longer need the normal nor $\mathbb{Q}$-Gorenstein assumptions, understanding log canonical (resp. klt) to mean

$A_X(E) \geq 0$ (resp. > 0) for all divisors E over X. In fact, we can say much more using our approach. The new tool is our extension of log discrepancies to the space $X^{\beth}$ of *semi*valuations on X, a compactification of Val_X common to non-archimedean geometry.

Theorem A (5.3, 5.4, cf. [26, Theorem 3.3])**.** *Let X be an F-finite integral scheme.*

1. *If X is F-pure, then X is log canonical.*
2. *If X is F-regular, then X is klt.*
3. *Conversely, if $A_X(\xi) \geq 0$ (resp. > 0) for all $\xi \in X^{\beth}$ besides the trivial valuation on X, then X is F-pure (resp. F-regular).*

A consequence of the third statement is that the F-pure centers of a sharply F-pure variety are identified as those points whose minimal $\beth$-log discrepancy is zero, cf. 5.2 and §6.

One of the most important properties of the extension of log discrepancies to Val_X in characteristic zero is lower-semicontinuity; we prove that our extension also has this property in 6.4. As a corollary, we generalize a result of Ambro over $\mathbb{C}$: the minimal ($\beth$-)log discrepancy is lower-semicontinuous, for any Cartier subalgebra on an integral scheme X of characteristic $p > 0$, if we consider X with the **constructible topology**; cf. 6.17 and [2, Theorem 2.2].

In many ways, the similarities between our log discrepancies and the characteristic zero analogues amount to this shared lower-semicontinuity. As a demonstration of this assertion, we prove, in the new setting of regular F-finite schemes, the main theorems from [30] regarding valuations computing the log canonical threshold $\mathrm{lct}(X, \mathfrak{a}_\star)$ of a multiplicatively graded sequence of ideals. For simplicity, we state our results for a fixed smooth variety X over an algebraically closed field k in this introduction.

Let $\mathfrak{a}_\star$ be a graded sequence of ideals on X. We write $\mathcal{J}(X, \mathfrak{a}_\star^t)$ for the sheaf of ideals whose sections over an affine open $U = \mathrm{Spec}(R) \subseteq X$ are those $f \in R$ satisfying

$$v(f) + A_X(v) - (t/m)\, v(\mathfrak{a}_m) > 0$$

for all $v \in \mathrm{Val}_U$ and all $m \gg 1$. This ideal is well-known (in characteristic zero) as the *asymptotic multiplier ideal* of $(X, \mathfrak{a}_\star^t)$. These are coherent whenever a log resolution exists for all pairs $(X, \mathfrak{a}_m^{t/m})$, being defined in this case as the pushforward of a certain invertible sheaf on any log resolution of $(X, \mathfrak{a}_m^{t/m})$ for $m \gg 0$ divisible enough [20, Definition 1.4], cf. [35, Lemma 11.1.1]. Lacking resolutions in positive characteristics, it was unknown if these ideals were coherent, and a reasonable expectation would be that a proof of coherence would require a sufficient theory of resolutions of singularities. A significant result of this paper is a purely valuative proof that multiplier ideals are coherent on strongly F-regular schemes, e.g. smooth varieties.

Theorem B (7.7)**.** *Let X be a smooth variety of characteristic p. The asymptotic multiplier ideal sheaf $\mathcal{J}(X, \mathfrak{a}_\star^t)$ is coherent for any graded sequence of ideals $\mathfrak{a}_\star$ on X and $t \in [0, \infty)$.*

The main technical statement one needs in this proof is 7.6, which identifies a compact subset of Val_X that is large enough to define these multiplier ideals and minimize the infima in log canonical thresholds. A number of subsets like this are known in the literature, cf. [30, Proposition 5.9] and [8, Lemma 3.4, Theorem 3.1]. If one has such a statement, it is reasonable to expect that our proof of 7.7 may be adapted to that setting; see 7.8.

Once we have 7.7, we can adapt the argument of Jonsson and Mustaţă, with technical modifications, extending the following results to positive characteristics.

Theorem C (7.16; cf [30], Theorem A)**.** *Let X be a smooth variety of characteristic p, and let $\mathfrak{a}_\star$ be a graded sequence of ideals on X. Suppose $\lambda = \mathrm{lct}(\mathfrak{a}_\star) < \infty$. For any generic point x of an irreducible component of $\mathbb{V}(\mathcal{J}(X, \mathfrak{a}_\star^\lambda))$ there exists a valuation with center x computing $\mathrm{lct}(\mathfrak{a}_\star)$, i.e. such that $\lambda = A_X(v)/v(\mathfrak{a}_\star)$.*

These computing valuations are obtained here and in [30] using a compactness argument, and their properties (e.g. Abhyankar) do not seem to be revealed from the proof. Jonsson and Mustaţă conjecture that these valuations must be *quasi-monomial*, a condition equivalent to Abhyankar for excellent schemes in characteristic zero. We state the analogous conjectures here using Abhyankar valuations, all of which are *locally* quasi-monomial [31].

Conjecture (7.17; cf. [30], Conjecture B)**.** Let X be a smooth variety of characteristic p, and let $\mathfrak{a}_\star$ be a graded sequence of ideals on X such that $\mathrm{lct}(\mathfrak{a}_\star) < \infty$.

- **Weak version**: some Abhyankar valuation computes $\mathrm{lct}_B(\mathfrak{a}_\star)$.
- **Strong version**: any valuation computing $\mathrm{lct}_B(\mathfrak{a}_\star)$ is Abhyankar.

Following [30], we reduce this conjecture to what is hopefully a more approachable form on affine spaces.

Conjecture (7.18; cf. [30], Conjecture C)**.** Let $X = \mathbb{A}^n_k$, with $\mathbb{F}_p \subset k = \overline{k}$, and let $\mathfrak{a}_\star$ be a graded sequence of ideals on X with $\mathrm{lct}(\mathfrak{a}_\star) < \infty$, vanishing only at a closed point $x \in X$.

- **Weak version**: some Abhyankar valuation centered at x computes $\mathrm{lct}(\mathfrak{a}_\star)$.
- **Strong version**: any valuation of transcendence degree 0 over $\mathbb{A}_k^n$, centered at x, and computing $\mathrm{lct}(\mathfrak{a}_\star)$ must be Abhyankar.

Theorem D (7.23; cf. [30], Theorem D)**.** *If 7.18 holds for all $n \leq d$, then 7.17 holds for all X with* $\dim(X) \leq d$.

1.3. Structure of the paper

Section 2 introduces notation and some background. In Section 3, we define our log discrepancy for Cartier subalgebras of integral schemes in characteristic $p > 0$, and prove some essential (but elementary) properties. In Section 4, we undertake the proof of our main theorem, starting with Cascini, Mustaţă, and Schwede's proof of their result. Section 5 is a brief study of the relationship between our log discrepancies, F-purity, and strong F-regularity; here we generalize Hara and Watanabe's result. We prove in Section 6 that log discrepancies are lower-semicontinuous on $X^\beth$, and use this to deduce constructible lower-semicontinuity of the minimal log discrepancy on any integral scheme of positive characteristic. In Section 7, we extend to positive characteristics the aforementioned theorems of Jonsson and Mutaţă; we prove along the way that asymptotic multiplier ideals are coherent sheaves on strongly F-regular schemes.

2. Notation and background

Let us first establish conventions, and gather definitions, used throughout the paper.

2.1. Conventions

The following basic terminology is used throughout this article.

1. The letter p will always denote a positive prime number. We usually add $p > 0$ for emphasis.
2. A *ring* always has a multiplicative identity in this paper. Except for Cartier subalgebras, all rings are commutative.
3. The group of units of a (commutative) ring R is denoted $R^\times$.
4. By a *scheme* we (almost) always mean a separated, noetherian, and excellent scheme; in particular, all schemes considered here are quasi-compact. The *sole* exception is that we may consider $\mathrm{Spec}(V)$ for a valuation ring V; we will be explicit about this exception. A point of a scheme refers to any (not necessarily closed) point.

5. A *variety* over a field k is an integral scheme of finite type over k.
6. The *reduction* of a $\mathbb{Q}$-Weil divisor $D = \sum_{i=1}^{m} r_i D_i$, where the D_i are distinct prime divisors (on some normal scheme), is $D_{red} = \sum_{i=1}^{m} D_i$.
7. By an $\mathcal{O}_X$*-module* we mean a quasi-coherent sheaf of $\mathcal{O}_X$-modules. An *ideal of* $\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.
8. If $I \subset \mathcal{O}_X$ is an ideal on X, we denote by $\mathbb{V}(I)$ the closed subscheme of X with structure sheaf $\mathcal{O}_X/I$.
9. *Neighborhoods* of a point in a topological space are open neighborhoods.
10. Valuation rings V are all rank-one, meaning the Krull dimension of V is one. Equivalently, the value group $\mathrm{Frac}(V)^\times/V^\times$ can be taken to be a subgroup of $(\mathbb{R}, +)$ [10]. Valuations take value $+\infty$ on 0.
11. If X is a scheme and $Z \subseteq X$ is an integral subscheme with generic point x, we denote by $\mathcal{O}_{X,Z}$ or $\mathcal{O}_{X,x}$ the local ring at x, and $\kappa(Z)$ or $\kappa(x)$ the residue field $\mathcal{O}_{X,x}/\mathfrak{p}_x$.
12. For local sections $f \in \mathcal{O}_{X,x}$, we write $f(x)$ for the residue of f in $\kappa(x)$.
13. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules, we denote by $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ the $\mathcal{O}_X$-module $(U \mapsto \mathrm{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U))$.
14. If X is a scheme of characteristic $p > 0$, and $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal, then we define the ideal $\mathcal{I}^{[p^e]}$ generated by p^e-th powers of sections of $\mathcal{I}$.
15. To avoid constantly passing between points and the associated sheaves of ideals, we will often make statements like "Let $\mathfrak{m} \in X$ be a point..." understanding schemes to have underlying topological spaces consisting of the set of prime ideal subsheaves of $\mathcal{O}_X$ (that is, $X \cong \mathcal{S}pec(\mathcal{O}_X)$).

2.2. Arithmetic on the extended real line

We will often need the standard extension of arithmetic operations on $\mathbb{R}$ to $\mathbb{R}_{\pm\infty} = \mathbb{R} \cup \{+\infty, -\infty\}$, where $+\infty$ and $-\infty$ satisfy $-\infty < r < +\infty$ for every $r \in \mathbb{R}$. The following expressions are **undefined**:

$$(\pm\infty) + (\mp\infty), (\pm\infty) - (\pm\infty), 0 \cdot (\pm\infty), (\pm\infty) \cdot 0.$$

We otherwise set

1. $r \cdot (\pm\infty) = \pm\infty$ if $r > 0$,
2. $r \cdot (\pm\infty) = \mp\infty$ if $r < 0$,
3. $r + (\pm\infty) = \pm\infty$.

Many technicalities in our definitions in sections 3 and 6 arise from the need to avoid the undefined expressions above.

2.3. SNC divisors

A core concept is that of an *snc divisor* on a regular scheme, short for *simple normal crossings (support)*, a global version of partial regular systems of parameters. Such divisors play a key role in the definition of quasi-monomial valuations and the construction of *retraction morphisms* used to define log discrepancies of arbitrary valuations.

Definition 2.1 (SNC divisor)**.** Let Y be a regular scheme, and let $D_1, \dots, D_N$ be prime divisors on Y. We say the $\mathbb{Q}$-divisor $D = \sum_1^N r_i D_i$, $r_i \in \mathbb{Q}$, is an *snc divisor* if:

1. Each D_i is a regular scheme.
2. For each $y \in Y$, let $D_{i_1}, \dots, D_{i_s}$ be those D_i that contain y. Suppose D_{i_j} corresponds to the prime $(f_{i_j}) \subset \mathcal{O}_{Y,y}$. Then $f_{i_1}, \dots, f_{i_s}$ form part of a regular system of parameters for $\mathcal{O}_{Y,y}$.

2.4. Log resolutions and geometric log discrepancies

We briefly review the definition of log resolutions, and log discrepancies in the classical sense, which we will call *geometric* log discrepancies to distinguish them *a priori* from the ones defined in 3. In the more general setting of F-finite schemes, we prefer the language of Cartier subalgebras to that of pairs.

For this subsection, let k be an algebraically closed field. We start with several standard definitions in birational geometry, taking [32, Notation 0.4] as our primary source (see also [35, Ch. 9] for a simpler setting).

Definition 2.2 (Pair, log $\mathbb{Q}$-Gorenstein)**.** A *pair* (X, Δ) is the data of a normal variety X over k and an effective $\mathbb{Q}$-Weil divisor Δ on X. Fixing a canonical class K_X, we say (X, Δ) is *log* $\mathbb{Q}$*-Gorenstein* if there exists an integer $m > 0$ such that $m(K_X + \Delta)$ is an integral Cartier divisor. The least such m is the *Cartier index* of $K_X + \Delta$.

Definition 2.3 (Strict transform)**.** Let (X, Δ) be a pair, and $\pi : Y \to X$ a proper birational morphism from a normal variety Y that is a isomorphism between open subsets $U \subseteq X$ and $V \subseteq Y$, where $X \setminus U$ has codimension at least 2. Suppose $\Delta = \sum_i r_i \Delta_i$ with Δ_i prime divisors on X. The *strict transform* of Δ on Y, denoted here $\pi_*^{-1}(\Delta)$, is the divisor $\sum_i r_i \widetilde{\Delta}_i$ whose prime components $\widetilde{\Delta}_i$ are the (topological) closures of $\Delta_i \cap U$ in Y, identifying U and V via π.

Definition 2.4 (Log resolution)**.** Let (X, Δ) be a pair. A *log resolution* of (X, Δ) is a proper birational morphism $\pi : Y \to X$, from a smooth variety Y, whose exceptional locus $E \subset Y$ is a divisor, and $E + \pi_*^{-1}(\Delta)_{red}$ is snc.

Definition 2.5 (Geometric log discrepancy)**.** Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair on a normal variety over k with canonical class K_X. Suppose Y is a normal variety with a proper birational morphism $\pi : Y \to X$, let $E \subset Y$ be a prime divisor, and let K_Y be the canonical class on Y with $\pi_* K_Y = K_X$. Define a $\mathbb{Q}$-Weil divisor Δ_Y on Y via

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta).$$

The *geometric log discrepancy of* (X, Δ) *on* E is

$$A_{(X,\Delta)}(E) = 1 - \mathrm{ord}_E(\Delta_Y).$$

Here, by $\mathrm{ord}_E(\Delta_Y)$ we mean the coefficient on E in Δ_Y.

It is well-known that $A_{(X,\Delta)}$ depends only on the valuation ord_E on the fraction field $k(X)$ and not the variety Y on which we have realized E as a prime divisor, see e.g. [32]. In characteristic $p > 0$, this will also follow from Cascini, Mustaţă, and Schwede's result, 1.1.

2.5. Cartier subalgebras, sharp F-purity, and strong F-regularity

Our central organizational objects for birational geometry in positive characteristics are the *Cartier subalgebras* defined by Schwede [48] and Blickle [4]. These can be thought of as a positive characteristic variant of pairs or triples $(X, \Delta, \mathfrak{a}_\star^t)$, though they can be much more general. These are the primary objects of study from Section 3 onwards.

We follow the down-to-earth approach of p^{-e}-linear maps in lieu of the systematic use of pushforwards of sheaves under the Frobenius morphism. For this subsection, X is an integral scheme of characteristic $p > 0$ with function field L. The *Frobenius morphism* on X is denoted $F : X \to X$; this is the identity on the underlying topological space of X, and is the p-th power morphism on the structure sheaf. Some authors call this the absolute Frobenius morphism. We also have the e-th iterated Frobenius $F^e : X \to X$, i.e. the p^e-th power morphism on $\mathcal{O}_X$.

One of the starting points for studying singularities of rings in characteristic $p > 0$ is Kunz's celebrated 1969 result.

Theorem 2.6 ([33])**.** *A Noetherian ring R of characteristic $p > 0$ is regular if and only if it is flat over the subring $R^p = \{f^p : f \in R\}$.*

Notable examples include polynomial rings $k[x_1, \ldots, x_n]$ and power series rings $k[\![x_1, \ldots, x_n]\!]$, where k is a field with $[k : k^p] < \infty$, both of which are in fact *free* over their rings of p-th powers. For example, if k is perfect, then a basis for either of these rings over R^p is given by

$$\{x_1^{a_1} \cdots x_n^{a_n} \ : \ 0 \leq a_i \leq p-1 \ \text{ for all } 1 \leq i \leq n\}.$$

Based on Kunz's theorem, one studies and characterizes singularities of characteristic p rings and schemes in terms of the existence of free R^p-summands

$$R = R^p f \oplus M$$

where M is an R^p-submodule of R. A summand is equivalent to an R^p-linear mapping $\phi : R \to R^p \subset R$. If R is reduced, then $R^p \cong R$ as rings, sending r^p to r, so we can view ϕ as an additive map $\phi : R \to R$ with the additional property $\phi(a^p g) = a\,\phi(g)$ for all $a, g \in R$. Such ϕ are called p^{-1}-linear maps, see (2.8).

Definition 2.7. We say X is *F-finite* if F^e is a finite morphism for some, equivalently any, $e \geq 1$. When $X = \operatorname{Spec}(R)$ is affine, we will say R is F-finite.

Definition 2.8. A *p^{-e}-linear map* on the function field L of X is an additive function $\psi : L \to L$ with the property $\psi(f^{p^e} g) = f\psi(g)$ for all $f, g \in L$. If ψ is a p^{-e}-linear map with $\psi(\mathcal{O}_X) \subseteq \mathcal{O}_X$, then we say ψ is a *p^{-e}-linear map on X.*

Remark 2.9. We have chosen one of several approaches to p^{-e}-linear maps; let us briefly discuss the two other approaches, both for continuity with other literature cited here, and because we need a few consequences of these alternative descriptions.

1. Writing $F^e : L \to L$ for the Frobenius morphism, a p^{-e}-linear map ψ is none other than an L-linear map $F^e_* L \to L$. Similarly, a p^{-e}-linear map on X is an $\mathcal{O}_X$-linear morphism $F^e_* \mathcal{O}_X \to \mathcal{O}_X$.
2. If we view the Frobenius as the *inclusion* $L \subseteq L^{1/p^e}$, taking L^{1/p^e} to be the ring of p^e-th roots of elements of L in some fixed algebraic closure of L, then a p^{-e}-linear map becomes an L-linear mapping $L^{1/p^e} \to L$. Taking $\mathcal{O}_X^{1/p^e}$ to be the integral closure of $\mathcal{O}_X$ in the constant sheaf on X associated to L^{1/p^e}, p^{-e}-linear maps on X are $\mathcal{O}_X$-linear morphisms $\mathcal{O}_X^{1/p^e} \to \mathcal{O}_X$.

The consequences we will need occasionally are the following.

1. Supposing X is F-finite, the sheaf of p^{-e}-linear morphisms $\mathcal{H}om_X(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ is just $(F^e)^!\mathcal{O}_X$ [28, Exercise III.6.10]. Consider an affine open subset $\operatorname{Spec}(R) \subseteq X$, $\mathfrak{p} \in \operatorname{Spec}(R)$, and let $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ be the completion morphism at $\mathfrak{p}$. Then

$$R' \otimes_R \operatorname{Hom}_R(F^e_* R, R) \cong \operatorname{Hom}_{R'}(F^e_* R', R') \tag{1}$$

since R' is faithfully flat and $F^e_* R$ is finitely generated over R.
2. The dual $\operatorname{Hom}_L(L^{1/p^e}, L)$ has the structure of an L^{1/p^e}-vector space. If L is an F-finite field, then $\dim_L \operatorname{Hom}_L(L^{1/p^e}, L) = \dim_L(L^{1/p^e})$. This implies $\operatorname{Hom}_L(L^{1/p^e}, L)$ is one-dimensional over L^{1/p^e}. In particular, given two non-zero maps $\phi, \psi \in \operatorname{Hom}_L(L^{1/p^e}, L)$ there exists a unique $h \in L$ such that $\phi = \psi h^{1/p^e}$. The right hand side of this equality will be written $\psi \cdot h$ in this article, cf. (2.11).

To shorten notation throughout the article, we introduce the notation $(F^e)^!\mathcal{O}_X$ for the collection of p^{-e}-linear maps on X. More generally, we write $(F^e)^!\mathcal{L}$ for the collection of p^{-e}-linear maps $\mathcal{L} \to \mathcal{O}_X$, where $\mathcal{L}$ is a rank-one reflexive sheaf on X. The notation is chosen to accord with Grothendieck duality in the F-finite case (cf. [28, Exercise III.6.10]). We develop what we can without an F-finite assumption, so the endofunctor $(F^e)^!$ on the derived category of X may not even be defined, and the notation $(F^e)^!$ is purely formal.

Definition 2.10. The *Cartier algebra of X* is the graded sheaf

$$\mathcal{C}^X = \oplus_{e \geq 0}(F^e)^!\mathcal{O}_X.$$

We will often use $\mathcal{C}^X_e$ as a synonym for $(F^e)^!\mathcal{O}_X$.

Definition 2.11. The Cartier algebra of X is a sheaf of non-commutative graded $\mathbb{F}_p$-algebras on X via composition. Indeed, if ψ_i is p^{-e_i}-linear, $i = 1, 2$, then $\psi_1 \circ \psi_2$ is $p^{-(e_1+e_2)}$ linear:

$$(\psi_1 \circ \psi_2)(f^{p^{e_1+e_2}} g) = \psi_1(f^{p^{e_1}} \psi_2(g)) = f\, \psi_1(\psi_2(g)).$$

To emphasize that we are thinking of $\psi_1 \circ \psi_2$ as the result of a graded multiplication (i.e. in $\mathcal{C}^X_{e_1+e_2}$), we write $\psi_1 \cdot \psi_2$, or ψ^2 if $\psi_1 = \psi = \psi_2$.

As a special case of this, where one $e_i = 0$, we see that each $\mathcal{C}^X_e$ has *distinct* right and left structures as an $\mathcal{O}_X$-module. Working affine-locally on $\operatorname{Spec}(R) \subseteq X$, suppose $\psi : R \to R$ is p^{-e}-linear, and let $f \in R$. Then f

is p^0-linear (i.e. R-linear), so both $f \cdot \psi$ and $\psi \cdot f$ are p^{-e}-linear. They are, however, not equal:

$$(f \cdot \psi)(1) = f\,\psi(1) = \psi(f^{p^e})$$

which is not $\psi(f) = (\psi \cdot f)(1)$, generally.

Example 2.12. Let us describe the Cartier algebra of a complete F-finite regular local ring. We will need this description many times in the subsequent sections.

By Cohen's structure theorem, any complete ring containing a field is isomorphic to one of the form $R := k[\![x_1, \dots, x_n]\!]$. Since R is F-finite, any quotient ring of R is also (using the classes of the same generators) so we know k is F-finite.

Suppose $\{\lambda_i\}_{i=1}^{[k:k^p]}$ is a basis for k over k^p, with $\lambda_1 = 1$. Then R is F-finite, and is free over R^p with basis

$$\mathcal{B} = \{\lambda_i(x_1^{a_1} \cdots x_n^{a_n}) \, : \, 0 \leq a_j \leq p-1, \, 1 \leq i \leq [k : k^p]\}.$$

It is well-known that $\mathcal{C}^R$ is canonically generated, as a ring, by the projection Φ onto the $(x_1 \cdots x_n)^{p-1}$-basis element (see, e.g., [12, Chapter 1] or [5, Example 3.0.5]). Thus, given any $e > 0$ and $\psi \in \mathcal{C}^R_e$, there exist $r_\psi \in R$ such that $\psi = \Phi^e \cdot r_\psi$. The map Φ^e is projection onto the $(x_1 \cdots x_n)^{p^e-1}$-summand of the free R^{p^e}-module R.

Moreover, the elements in $\mathcal{B}$ also give a basis for L over L^p, so given any element $f \in L$, we can uniquely write

$$f = \sum_{i=1}^{[k:k^p]} \sum_{j=1}^{n} \sum_{a_j=0}^{p-1} f^p_{(i,a_1,\dots,a_n)} \lambda_i x_1^{a_1} \cdots x_n^{a_n},$$

with $f^p_{(i,a_1,\dots,a_n)} \in L^p$. It follows from L-linearity that $\Phi(f) = f_{(1,p-1,\dots,p-1)}$.

We can also use this to describe $\mathcal{C}^R$ for F-finite regular local rings $(R, \mathfrak{m})$. Indeed, writing $\hat{R}$ for the $\mathfrak{m}$-adic completion of R, we have seen in 2.9(1) that $\mathcal{C}^{\hat{R}} \cong \mathcal{C}^R \otimes_R \hat{R}$. Therefore, after fixing a regular system of parameters for R we have a canonical choice of generator Φ for $\mathcal{C}^R$ (as a ring).

This will be highly useful in the sequel, because it allows us to relate p^{-e}-linear maps on F-finite local rings $(A, \mathfrak{n})$ to rational sections of $\mathcal{C}^R$ for regular local rings R birational to A (obtained, e.g., by blowing-up points of Spec(A) or via Knaf-Kuhlmann local monomializations). See §§3, 4, and 7 for many examples.

Definition 2.13. A *Cartier subalgebra on X* is a quasi-coherent sheaf of graded subrings $\mathcal{D} = \oplus_{e \geq 0} \mathcal{D}_e \subseteq \mathcal{C}^X$. We ask that $\mathcal{D}_0 = \mathcal{O}_X = \mathcal{C}_0^X$, which implies $\mathcal{D}_e \subseteq \mathcal{C}_e^X$ is an $\mathcal{O}_X$-submodule under both module structures, cf. 2.11.

Definition 2.14. Given $0 \neq \psi \in \Gamma(U, \mathcal{C}_e^X)$, there is an associated Cartier subalgebra $\{\!\{\psi\}\!\}$ on U. This is non-zero only in degrees ne, $n \geq 0$, and is the $\mathcal{O}_U$-module $\psi^n \cdot \mathcal{O}_U$ in degree ne.

Definition 2.15 (cf. [29, 26, 45])**.** Suppose X is an F-finite integral scheme, and $\mathcal{D}$ is a Cartier subalgebra on X.

1. We say $\mathcal{D}$ is *sharply F-pure* at $x \in X$ if there exists $\psi \in (\mathcal{D}_e)_x$ that is surjective, as a function $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$, for some $e \geq 1$. If $\mathcal{C}^X$ is sharply F-pure at x, then we say *X is F-pure at x*.
2. We say $\mathcal{D}$ is *strongly F-regular* at $x \in X$ if the following condition is satisfied. For all $f \in \mathcal{O}_{X,x}$ non-zero, there exists $e \geq 1$ and $\psi \in (\mathcal{D}_e)_x$ such that $\psi(f) = 1$. If $\mathcal{C}^X$ is strongly F-regular at x, we say *X is F-regular at x*.

If X is F-pure (or F-regular) at every $x \in X$, then we say X is F-pure (resp., F-regular).

Remark 2.16. Following a growing consenus among experts, we drop the adjectives "sharply" and "strongly" from F-purity and F-regularity of F-finite schemes.

Remark 2.17. There is a well-known, heuristic, correspondence between F-pure and log canonical varieties, and F-regular and klt varieties. See, e.g., [26] and 5.3.

2.6. Divisors and p^{-e}-linear maps

We now recall the close connection between p^{-e}-linear maps and certain divisors on a normal variety X. The ideas present here go back at least to Mehta-Ramanathan [38] and Ramanan-Ramanathan [43], though the most direct origin of the technique is Hara and Watanabe's [26, Lemma 3.4]. We refer the reader also to [12, Ch. 1] and the excellent surveys [50, 5, 42].

Let X be a normal variety of dimension n over an algebraically closed field k of characteristic $p > 0$; this implies X is F-finite. Let ω_X be the canonical bundle on X, the rank-one reflexive sheaf agreeing with $\wedge^n \Omega_{X/k}$ on the smooth locus of X. Fix a line bundle $\mathcal{L}$ on X. Grothendieck duality for the Frobenius morphism $F : X \to X$ provides an isomorphism of reflexive, coherent right $\mathcal{O}_X$-modules

$$(F^e)^! \mathcal{L} \cong \mathcal{L}^{-1} \otimes \omega_X^{\otimes(1-p^e)}.$$

As a consequence, a globally defined p^{-e}-linear map $\mathcal{L} \to \mathcal{O}_X$ is equivalent, up to a unit of $\Gamma(X, \mathcal{O}_X)$, to an effective Cartier divisor D with

$$\mathcal{O}_X(D) \cong \mathcal{L}^{-1} \otimes \omega_X^{\otimes(1-p^e)}. \tag{2}$$

Of course, D is unchanged if we multiply ψ (on the right) by a unit of $\Gamma(X, \mathcal{O}_X)$. If we set $\Delta = \frac{1}{p^e-1} D$, and choose a canonical class K_X on X, then the isomorphism (2) can be re-formulated as saying: Δ is an effective $\mathbb{Q}$-Weil divisor on X such that $m(K_X + \Delta)$ is Cartier for some m prime to p, i.e. (X, Δ) is a log-$\mathbb{Q}$-Gorenstein pair, and the Cartier index of $K_X + \Delta$ is not divisible by p. The normalization by $(p^e - 1)$ is useful because $\psi^m \in (F^{me})^! \mathcal{L}^{\otimes(p^{(m-1)e} + \cdots + p^e + 1)}$ has the same associated divisor as ψ for all $m \geq 1$.

Definition 2.18. Let Δ be an effective divisor on X, and assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier, with Cartier index not divisible by p. We define $\mathcal{C}^X \cdot \Delta$ to be

$$\oplus_e (F^e)^! \mathcal{O}_X((p^e - 1)\Delta)$$

where $e \geq 0$ ranges over values for which $(p^e - 1)(K_X + \Delta)$ is Cartier.

Lemma 2.19 (cf. [47, Lemma 2.8]). *Let Δ be as in the previous definition. Then $\mathcal{C}^X \cdot \Delta$ is a Cartier subalgebra on X. If ψ_Δ any p^{-e}-linear map on an open subset $U \subseteq X$ corresponding to a Cartier divisor $(1 - p^e)(K_X + \Delta) \cap U$ then $(F^e)^! \mathcal{O}_X((p^e - 1)\Delta)|_U = \{\!\{\psi_\Delta\}\!\}_e = \psi_\Delta \cdot \mathcal{O}_U$, see (2.14).*

Proof. We refer the reader to [47] for the proof that $\mathcal{C}^X \cdot \Delta$ is a Cartier subalgebra. The second claim, that $(F^e)^! \mathcal{O}_X((p^e-1)\Delta)|_U = \{\!\{\psi_\Delta\}\!\}_e = \psi_\Delta \cdot \mathcal{O}_U$ for any $\psi_\Delta \in \Gamma(U, \mathcal{C}^X)$ with divisor $(1 - p^e)(K_X + \Delta) \cap U$, is a re-statement of the association of an invertible sheaf to a Cartier divisor [28, §II.6]. □

2.7. Berkovich spaces

Because we expect this article to be of interest to researchers unfamiliar with constructions in non-archimedean geometry, we provide a summary of the Berkovich space theory as we need it. We use Thuillier's $\beth$-spaces ([53]) and not the entire Berkovich analytification [3], because they are compact and their points are more closely tied to the birational geometry of a given (possibly non-proper) variety.

We provide sketches of proofs when the ideas involved are necessary for later sections, deferring to [3] for more detailed developments of the following material. When terminology is already developed in birational geometry that conflicts with Berkovich's terminology for these concepts, we use the birational language.

Definition 2.20. Let X be a scheme over a field k. Recall that we assume X is separated and noetherian.

1. A *semivaluation on* X is a pair $\zeta = (w, x)$ consisting of a point $x \in X$, with closure $Z = \overline{\{x\}}$, and a valuation w on $\kappa(x)$ that is *centered on* X, meaning w is non-negative on some local ring $\mathcal{O}_{Z,z} \subseteq \kappa(x) = \kappa(Z)$, $z \in Z$. If $f \in \mathcal{O}_{X,x}$, we define $\mathrm{ev}_f(\zeta) = \zeta(f) := w(f(x)) \in [0, \infty]$. Note that being centered on X forces $w(u) = 0$ for every $u \in k \setminus \{0\}$, i.e. w restricts to the *trivial* valuation on k.
2. We denote by $X^{\beth}$ the set of all semivaluations on X. The point x of $\zeta = (w, x) \in X^{\beth}$ is called the *home* of ζ (on X), and the *home function* $h_X : X^{\beth} \to X$ is $h_X(\zeta) = x$. The home map is also commonly called the *kernel* map, e.g., in [3].
3. If X is integral with generic point η_X, we define Val_X to be $h_X^{-1}(\eta_X)$, i.e. valuations on $\kappa(X)$ having center on X.
4. Continuing with an integral X, the *divisorial valuations* are those $v \in \mathrm{Val}_X$ whose value group, considered as a subgroup of $(\mathbb{R}, +)$, is discrete in the usual topology of $\mathbb{R}$. We denote the collection of divisorial valuations by X^{div}.
5. The *semivaluation ring* of $(w, x) \in X^{\beth}$ is $w^{-1}[0, \infty] \subseteq \kappa(x)$, denoted throughout this article as $\mathcal{O}_w$. Semivaluation rings of valuations are called valuation rings.
6. For any $\zeta = (w, x) \in X^{\beth}$, there is a unique morphism $i_\zeta : \mathrm{Spec}(\mathcal{O}_w) \to X$ extending the natural map $\mathrm{Spec}(\kappa(x)) \to X$; here, $\mathcal{O}_w = w^{-1}[0, \infty]$ is the semivaluation ring of ζ. The point $c_X(\zeta) := i_\zeta(\mathfrak{m}_w)$ is called the *center of* ζ *on* X (where $\mathfrak{m}_w \subset \mathcal{O}_w$ is the maximal ideal). In the more general setting of [3], $c_X(\zeta)$ is called the *reduction* of ζ, denoted there $\mathrm{red}(\zeta)$.
7. We will write $\mathcal{O}_{X,c(\zeta)}$ for $\mathcal{O}_{X,c_X(\zeta)}$. We follow a similar convention with $h_X(\zeta)$.

We put no finite type hypothesis on the scheme X above, so we can take $k = \mathbb{Q}$ or $\mathbb{F}_p$, and in some sense we are just using k to force X to have equal characteristic.

Remark 2.21. Integral subschemes of X and X_{red} are the same, and so $X^{\beth} = (X_{\mathrm{red}})^{\beth}$ as sets. Moreover, any nilpotent section f is sent to $+\infty$ by all $\zeta \in X^{\beth}$ whose home is in an open subset where f is regular, so $\mathrm{ev}_f = \mathrm{ev}_0$. It will be clear from our construction that $X^{\beth}$ and $(X_{\mathrm{red}})^{\beth}$ agree as topological spaces, too.

We now topologize the set $X^{\beth}$ to define the $\beth$-*space* of X.

Definition 2.22. Suppose $X = \mathrm{Spec}(R)$ is a reduced affine scheme. Consider

$$\mathrm{Spec}(R)^{\beth} \hookrightarrow \prod_{f\in R}[0,\infty] \quad \text{defined by} \quad \zeta \mapsto \prod_{f\in R}\zeta(f).$$

Here $[0,\infty]$ has the topology making it homeomorphic to $[0,1]$. We give $\mathrm{Spec}(R)^{\beth}$ the subspace topology via this injection. One checks easily that the image of $\mathrm{Spec}(R)^{\beth}$ is a closed subspace of this product space, so is compact by Tychonoff's theorem. The Hausdorff property is inherited.

Remark 2.23. It is equivalent to give the set $\mathrm{Spec}(R)^{\beth}$ the weakest topology such that ev_f is continuous for all $f \in R$.

The following lemma is well-known, see e.g. [3, Corollary 2.4.2], [30, Remark 4.2]. We include a proof for convenience, and to illustrate the basics of working with the center and home functions.

Lemma 2.24. *For any affine* $X = \mathrm{Spec}(R)$, $h_X : X^{\beth} \to X$ *is continuous. On the other hand,* c_X *is anti-continuous, in the sense that if* $U \subseteq X$ *is Zariski open,* $c_X^{-1}(U) \subseteq X^{\beth}$ *is closed.*

Proof. We first check that $h_X^{-1}(\mathbb{V}(I))$ is closed for all proper ideals $I \subset R$. If $\mathfrak{p} = h_X(\zeta) \in \mathbb{V}(I)$ then $I \subseteq \mathfrak{p}$. Thus $\zeta(I) = \{+\infty\}$. We conclude $h_X^{-1}(\mathbb{V}(I)) = \bigcap_{f\in I} \mathrm{ev}_f^{-1}(+\infty)$, which is closed by continuity.

Considering $c_X^{-1}(\mathbb{V}(I))$, note that $f \in P$, $P \in \mathrm{Spec}(R)$ if and only if $\zeta(f) > 0$ when $c_X(\zeta) = P$. Thus, $c_X^{-1}(\mathbb{V}(I)) = \bigcup_{f\in I} \mathrm{ev}_f^{-1}(0,\infty]$, which is open in $X^{\beth}$. □

Remark 2.25. Any ring homomorphism $\pi : R \to S$ induces a continuous map $\pi_* : \mathrm{Spec}(S)^{\beth} \to \mathrm{Spec}(R)^{\beth}$ by $\pi_*(\zeta)(f) = \zeta(\pi(f))$, $f \in R$. We see that when π gives an open immersion, the subspace topology on (the compact subset) $\pi_*(\mathrm{Spec}(S)^{\beth})$ agrees with the one defined directly on $\mathrm{Spec}(R)^{\beth}$ as above. We use this observation to define the $\beth$-space of an arbitrary (non-affine) scheme.

Definition 2.26 ($\beth$-space of a scheme)**.** Let X be a scheme over a field k, and let U and V be two affine open subschemes of X. Since X is separated,

$U \cap V$ is again affine, and the topology on $(U \cap V)^{\beth}$ is identical to the subspace topology induced from either $U^{\beth}$ or $V^{\beth}$. Thus, there is a unique topology on $X^{\beth}$ whose open subsets are those subsets $\mathcal{U} \subset X^{\beth}$ such that $\mathcal{U} \cap U^{\beth}$ is open in the topology from 2.22 for every affine open subscheme $U \subseteq X$.

Remark 2.27. The topology of $X^{\beth}$ is entirely determined by the topology of $U^{\beth}$ as U ranges over affine subschemes $U \subset X$, and a finite affine open cover $\{U_1, \ldots, U_t\}$ of X leads to a finite cover of $X^{\beth}$ by the **compact** subsets $U_i^{\beth}$. Thus, c_X remains anticontinuous and h_X continuous for any X, since they are so on $U^{\beth}$ for any affine open $U \subseteq X$.

Remark 2.28. As a word of caution, it is eminently not true that $U^{\beth} = h_X^{-1}(U)$ for open subsets U of X. Indeed: supposing that X is integral for simplicity, the home of every valuation having center on X is the generic point of X, and so $\mathrm{Val}_X \subseteq h_X^{-1}(U)$. However, not every valuation having center on X necessarily has center on U. We do have $U^{\beth} = c_X^{-1}(U)$ for $U \subseteq X$ open, and $(X \setminus U)^{\beth} = h_X^{-1}(X \setminus U)$.

2.8. Retractions and Abhyankar valuations

We recall some of the basic numerics of valuations, and the construction of monomialization retractions, in the setting of integral excellent schemes over a field. We need this material for the proof of our main theorem (where the more typical setting of varieties over algebraically closed fields suffices) and also in the final section (developed in the general setting of strongly F-regular F-finite schemes).

Let X be an integral excellent scheme over a field k with function field L, and $v \in \mathrm{Val}_X$. The *value group of* v is $\Gamma_v = v(L^{\times}) \subseteq \mathbb{R}$. The *rational rank of* v is $\mathrm{ratrk}(v) = \dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$. If $(B, \mathfrak{m})$ is a local subring of L dominated by the valuation ring $(\mathcal{O}_v, \mathfrak{m}_v)$, then $(B/\mathfrak{m}) =: \ell \subseteq \kappa(v) := \mathcal{O}_v/\mathfrak{m}_v$ and we set the *transcendence degree of* v *over* B to be $\mathrm{tr.deg}_B(v) = \mathrm{tr.deg}(\kappa(v) \mid \ell)$. When $c_X(v) = x$, we define $\mathrm{tr.deg}_X(v) = \mathrm{tr.deg}_{\mathcal{O}_{X,x}}(v)$. The fundamental estimate is *Abhyankar's inequality*:

$$\mathrm{ratrk}(v) + \mathrm{tr.deg}_X(v) \leq \dim(\mathcal{O}_{X,x}).$$

Valuations achieving equality are called *Abhyankar valuations*. We refer the reader to [54, Théorème 9.2] for this result in our generality here.

Definition 2.29. One easy way to obtain an Abhyankar valuation is via the following construction, outlined more carefully in §3.1 of [30]. Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring containing a field. For any regular system of parameters

$r_1, \dots, r_d$ for R, there is an isomorphism $\hat{R} \cong \kappa[\![r_1, \dots, r_d]\!]$, and so we may view $f \in R \subseteq \hat{R}$ as having an expansion of the form $f = \sum_{u \in \mathbb{N}_0^d} c_u r^u$, where $r^{(u_1,\dots,u_d)} = r_1^{u_1} \cdots r_d^{u_d}$ and $c_u \in \kappa$. Jonsson and Mustaţă prove in Proposition 3.1 of [30] that for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d_{\geq 0}$, there is a unique valuation val_α on $\hat{R}$ such that $\mathrm{val}_\alpha(f) = \min\{\sum_{j=1}^d \alpha_j u_j : c_u \neq 0\}$; such valuations are sometimes called *Gauss* or *monomial* valuations. Restricting val_α to R now gives an Abhyankar valuation v. After blowing-up appropriately, there exists a regular local ring $(R', \mathfrak{m}')$, dominating and birational to $(R, \mathfrak{m})$, with Krull dimension $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_d) = \mathrm{ratrk}(\mathrm{val}_\alpha)$, on which v is centered [30, Proposition 3.6(ii)]. Moreover, the residue field $\kappa(v)$ must be an algebraic extension of $R'/\mathfrak{m}'$, thanks to the *dimension formula* [36, Theorem 15.2] (using that X is excellent); see the discussion at the beginning of [30, §3.2].

Definition 2.30 (Monomialization retraction, cf. [30])**.** Fix an snc divisor D on a regular excellent scheme Y with a proper birational morphism $\pi : Y \to X$. Let $v \in \mathrm{Val}_X$ with $c_Y(v) = y$, and suppose $D \cap \mathrm{Spec}(\mathcal{O}_{Y,y}) = \mathrm{div}(z_1 \cdots z_t)$, with $z_i \in \mathfrak{m}_y$; extend these to a generating set $z_1, \dots, z_d$ for $\mathfrak{m}_y$. The $\mathfrak{m}_y$-adic completion of $\mathcal{O}_{Y,y}$ is isomorphic to $\hat{R} := \kappa(y)[\![z_1, \dots, z_d]\!]$ by Cohen's structure theorem, and one has an inclusion $\iota : \mathcal{O}_{Y,y} \hookrightarrow \hat{R}$ sending z_i to z_i. On $\hat{R}$, we have the monomial valuation w with $w(z_i) = v(D_i)$ for $1 \leq i \leq t$ and $w(z_i) = 0$ for $t < i \leq d$. We define $r_{(Y,D)}(v) := \iota_*(w) \in \mathrm{Val}_X$.

Remark 2.31. Note that, by construction and super-additivity of valuations, $v(f) \geq r_{(Y,D)}(v)(f)$ for all $f \in \mathcal{O}_{Y,y}$. Consequently, $c_Y(v) \in \overline{\{c_Y(r_{(Y,D)}(v))\}}$.

For regular excellent schemes over $\mathbb{Q}$, every Abhyankar valuation is obtained as the result of some monomialization retraction, induced from a proper birational $Y \to X$ and some SNC divisor H on Y (see [20] and Proposition 3.7 in [30]). If X is a variety and k has characteristic $p > 0$, Knaf and Kuhlmann show in [31] that an Abhyankar valuation whose residue field is separable over k admits a *local monomialization* over X, meaning that if v is an Abhyankar valuation centered at $x \in X$, then there exists an open neighborhood $U \subset X$ of x, a proper birational morphism $\pi : Y \to U$ from a regular variety Y, and an snc divisor D on Y, such that $v = r_{(Y,D)}(v)$.

3. Log discrepancies of Cartier subalgebras

This is the foundational section of this article; the subsequent sections are applications of the ideas developed here.

Let X be an integral scheme of prime characteristic $p > 0$. We denote the function field of X by L thoughout this section. Recall from Section 2.6

that when X is a normal variety, there is essentially a bijection between log $\mathbb{Q}$-Gorenstein pairs (X, Δ) with $(1-p^e)(K_X+\Delta)$ Cartier for some $e>0$ and p^{-e}-linear maps ψ on open sets U where $(1-p^e)(K_X+\Delta)$ is principal, furnished by an isomorphism $(F^e)^! \mathcal{O}_X((p^e-1)\Delta)|_U \cong \{\!\{\psi\}\!\}_U$.

Our primary goal is to define the log discrepancy of Cartier subalgebras on X, with the fundamental definition being that of log discrepancies of p^{-e}-linear maps on *the function field L of X*. Much of this section goes through without an assumption of F-finiteness; because this may be of interest, e.g. for varieties over non-F-finite fields, we do what we can without assuming X is F-finite. Of course, for many of our more precise results, we need this assumption.

Definition 3.1. Let $\mathcal{D}$ be a Cartier subalgebra on X, and $Z \subseteq X$ a closed integral subscheme with ideal $\mathcal{I}_Z \subseteq \mathcal{O}_X$. We say that Z is *uniformly $\mathcal{D}$-compatible* if $\psi(\mathcal{I}_{Z,x}) \subseteq \mathcal{I}_{Z,x}$ for each $x \in X$, $e \geq 1$, and $\psi \in (\mathcal{D}_e)_x$. In this case, each ψ induces a well-defined p^{-e}-linear map on Z, which we denote by $\psi\|_Z$. The collection of all $\psi\|_Z$ give a Cartier subalgebra $\mathcal{D}\|_Z$ on Z, which we call the *exceptional restriction*. The usual restriction $\oplus_{e\geq 0}(\mathcal{O}_Z \otimes_X \mathcal{D}_e)$ is not useful for us.

3.1. The definition of log discrepancy

Definition 3.2 (Log discrepancy of p^{-e}-linear maps and Cartier subalgebras)**.** We define the log discrepancy of p^{-e}-linear maps and Cartier subalgebras at a semivaluation in four steps.

1. Suppose ψ is a p^{-e}-linear map on L for some $e \geq 1$, let $0 \neq f \in L$, and $v \in \mathrm{Val}_X$. We define

$$E(f, \psi, v) = \frac{v(f) - p^e v(\psi(f))}{p^e - 1} \in [-\infty, \infty). \tag{3}$$

   Note that if $\psi(f)$ is a unit of the valuation ring $\mathcal{O}_v$, then $E(f, \psi, v) = (p^e-1)^{-1} v(f)$. The notation is meant to suggest the coefficient of $K_Y - \pi^*(K_X+\Delta)$ on divisors $E \subset Y$ over a variety X, when Δ corresponds to ψ via (2.6).
   If $g := \psi(f)$ is nonzero, then $\psi(fg^{-p^e}) = g^{-1}\psi(f) = 1$ and $v(fg^{-p^e}) = v(f) - p^e v(g) = v(f) - p^e v(\psi(f))$. Thus, given any $f \in L$ with $\psi(f) \neq 0$, there is some other $h \in L$ with $E(h, \psi, v) = E(f, \psi, v)$ and $\psi(h) = 1$. On the other hand, 3 allows us to consider the full range of values of $E(f, \psi, v)$, as f ranges over nonzero $f \in L$, by restricting to any subring $R \subseteq L$ with $L = \mathrm{Frac}(R)$.

2. The *log discrepancy* of $0 \neq \psi : L \to L$ at $v \in \mathrm{Val}_X$ is
$$A(v;\psi) = \sup_{f\neq 0, n\geq 1} E(f,\psi^n,v).$$
Note that, per the comment at the end of the previous item, we can (and often do) restrict this supremum to those $f \neq 0$ with $\psi^n(f) = 1$, or $E(f;\psi^n,v)$ for $f \in R$ for some fixed ring R with $\mathrm{Frac}(R) = L$, e.g. $\mathcal{O}_{X,x}$ for $x \in X$, or valuation rings $\mathcal{O}_v$.
3. Let $\mathcal{D}$ be a Cartier subalgebra on X. The *log discrepancy* of $\mathcal{D}$ at $v \in \mathrm{Val}_X$ centered at x is
$$A(v;\mathcal{D}) = \sup_{e\geq 1}\left(\sup_{0\neq\psi\in(\mathcal{D}_e)_x} A(v;\psi)\right).$$
4. When $\zeta \in X^{\beth} \setminus \mathrm{Val}_X$, we define $E(f,\psi,\zeta)$ for nonzero p^{-e}-linear maps ψ, when $\zeta(f)$ and $\zeta(\psi(f))$ are not both $+\infty$.
5. Let $x \in X$, with closure $Z = \overline{\{x\}}$ that is uniformly compatible with $\mathcal{D}$. We define
$$A(\zeta;\mathcal{D}) = A(\zeta;\mathcal{D}\|_Z)$$
for every $\zeta \in \mathrm{Val}_Z = h_X^{-1}(x) \subseteq X^{\beth}$. For $\zeta \in X^{\beth}$ whose home is not uniformly $\mathcal{D}$-compatible, we set $A(\zeta;\mathcal{D}) = +\infty$.

We immediately prove several very useful ways to simplify the calculation of log discrepancies. 3.3 presents an important proof method we build on several times in this section; this was inspired by (and our proof is a solution to) [5, Exercise 4.11]. 3.4 is used constantly throughout the paper.

Proposition 3.3. *Let $v \in \mathrm{Val}_X$, $\psi : L \to L$, and $f \in L$, where both ψ and f are nonzero. Suppose $A(v;\psi) < \infty$. Then*
$$A(v;\psi\cdot f) + (p^e-1)^{-1}v(f) = A(v;\psi).$$

Here, $\psi \cdot f$ denotes the product in the Cartier algebra of $\mathrm{Spec}(L)$ (2.11).

Proof. Define $f_n = f^{(p^{ne}-1)/(p^e-1)}$ for all $n \geq 1$. The following observations are easy to check:

1. $(\psi\cdot f)^n = \psi^n \cdot f_n$.
2. $\psi^n(h) = 1$ if and only if
$$(\psi\cdot f)^n\left(f_n^{-1}h\right) = 1.$$
3. $E(h,\psi^n,v) = E(f_n^{-1}h,\psi^n,v) + (p^e-1)^{-1}v(f)$.

Thus, there is a bijection between $h \in L$ with $\psi^n(h) = 1$ and $g \in L$ with $(\psi \cdot f)^n(g) = 1$ given by multiplication by f_n. Considering then the definitions of $A(v; \psi \cdot f)$ and $A(v; \psi)$, and applying the third observation, gives the claimed expression for $A(v; \psi \cdot f)$. $\square$

Proposition 3.4. *Let $\mathcal{D}$ be a Cartier subalgebra on X and $x \in X$. Suppose that $\mathcal{D}_x = \{\!\{\psi\}\!\}_x \subset \mathcal{C}_x^X$ for some $\psi \in (\mathcal{D}_e)_x$, cf. (2.14). Then $A(v; \mathcal{D}) = A(v; \psi)$ for every $v \in \mathrm{Val}_X$ with $c_X(v) = x$.*

Proof. By definition, $A(v; \mathcal{D}) \geq A(v; \psi)$. On the other hand, the assumption $\mathcal{D}_x = \{\!\{\psi\}\!\}_x$ implies $(\mathcal{D}_m)_x = 0$ unless $m = ne$, and any $\phi \in (\mathcal{D}_{ne})_x$ can be written as $\psi^n \cdot f$ for some $f \in \mathcal{O}_{X,x}$. Now (3.3) shows $A(v; \phi) = A(v; \psi) + (1 - p^e)^{-1} v(f)$. Thus, $A(v; \phi) \leq A(v; \psi)$ since $c_X(v) = x$, which implies $v(f) \geq 0$. Therefore, $A(v; \mathcal{D}) = A(v; \psi)$. $\square$

We now prove some easy consequences of (3.3), inspired by [5, Exercise 4.12], cf. [49, Lemma 4.9(i)]. The first, we attribute to Cascini, Mustaţă, and Schwede, since the key claim in the middle of the proof was shared with the author by Karl Schwede in private correspondence. 3.9 generalizes this result.

Corollary 3.5 (Cascini-Mustaţă-Schwede)**.** *Let v be a discrete valuation on L whose associated valuation ring R is F-finite, and let $\varpi \in R$ be any generator for the maximal ideal. Let $\mathcal{C}^R$ be the Cartier algebra of* $\mathrm{Spec}(R)$*. Then $A(v; \mathcal{C}^R) = v(\varpi)$.*

Proof. It is well-known that R is a free R^{p^e}-module (of rank $p^{e[L:L^p]}$) with a basis containing $\varpi^{(p^e-1)}$, and that $\mathcal{C}_e^R = \Phi^e \cdot R$, where Φ^e is the p^{-e}-linear projection $\Phi : R \to R$ onto $\varpi^{(p^e-1)}$. 3.4 proves $A(v; \mathcal{C}^R) = A(v; \Phi)$.

Because $\Phi(\varpi^{(p-1)}) = 1$, $A(v; \Phi) \geq v(\varpi) = E(\varpi, \Phi, v)$. Now suppose $f \in R$ has $\Phi^e(f) = 1$ and $E(f, \Phi^e, v) \geq E(\varpi, \Phi, v)$. Write $f = u\varpi^s$, with $u \in R^\times$, and $s \geq p^e - 1$. I claim that in fact $s = p^e - 1$. More generally:

Claim. Define $s' = \lceil p^{-e}(s - p^e + 1) \rceil$. Then $\Phi^e(\varpi^s R) = \varpi^{s'} R$.

Proof of claim. The claim is clear when $p^e \mid (s - p^e + 1)$: $\varpi^{(p^e-1)}$ is sent to 1 by Φ^e, and Φ^e is p^{-e}-linear, so

$$\begin{aligned} \Phi^e(\varpi^s) &= \Phi^e(\varpi^{s-p^e+1} \varpi^{(p^e-1)}) \\ &= \varpi^{s'}. \end{aligned}$$

More generally, for $f \in R$, we have

$$v(\Phi^e(f \varpi^{-np^e}))) = v(\Phi^e(f)) - v(\varpi^n)$$

so $\varpi^{(n+1)p^e-1} R$ is the smallest ideal of R sent into $\varpi^n R$. $\boxtimes$

Finishing the proof, we see that if $\Phi(f) = 1$, then $s' = 0$. This is not the case unless $s = p^e - 1$. □

Corollary 3.6 (cf. [49, Lemma 4.9(i)])**.** *Suppose v is a discrete valuation on L whose valuation ring R is F-finite. Let $0 \neq \psi_i : L^{1/p^{e_i}} \to L$, $i = 1, 2$. Suppose $\min\{e_1, e_2\} \geq 1$. Then*

$$(1-\varepsilon)A(v;\psi_1) + \varepsilon A(v;\psi_2) = A(v;\psi_1 \cdot \psi_2) \tag{4}$$

where $\varepsilon = (p^{e_2} - 1)/(p^{(e_1+e_2)} - 1)$.

We expect that when R is not F-finite $A(v;\psi) = +\infty$ for every p^{-e}-linear map. This is true when X is a variety over a perfect field, see (3.7).

Proof. Suppose $\mathcal{C}_1^R = \Phi \cdot R$ as in the last proof. Considering Φ, ψ_1, and ψ_2 as elements of $\mathrm{Hom}_L(L^{1/p^a}, L) \cong L^{1/p^a}$, for $a \in \{1, e_1, e_2\}$ (respectively), then there exist $h_i \in L$ with $\Phi^{e_i} \cdot h_i = \psi_i$ for $i = 1, 2$, see 2.9(2). Now (3.3) and (3.5) imply

$$A(v;\psi_i) = v(\varpi) + \frac{1}{1 - p^{e_i}} v(h_i).$$

Additionally,

$$\begin{aligned}\psi_1 \cdot \psi_2 &= (\Phi^{e_1} \cdot h_1) \cdot (\Phi^{e_2} \cdot h_2)\\ &= \Phi^{e_1+e_2} \cdot (h_1^{p^{e_2}} h_2)\end{aligned}$$

so another application of (3.3) gives the value

$$\begin{aligned}v(\varpi) - A(v;\psi_1 \cdot \psi_2) &= \frac{1}{p^{(e_1+e_2)} - 1}(p^{e_2} v(h_1) + v(h_2).\\ &= (1-\varepsilon)(v(\varpi) - A(v;\psi_1)) + \varepsilon(v(\varpi) - A(v;\psi_2))\\ &= v(\varpi) - (1-\varepsilon)A(v;\psi_1) - \varepsilon A(v;\psi_2).\end{aligned}$$

Re-arranging the terms of these equations, we are left with (4). □

Example 3.7. Datta and Smith carefully studied p^{-e}-linear maps on valuation rings inside function fields [16, 17]. Among other things, they prove that if a valuation ring in a function field is F-finite, then the associated valuation must be an Abhyankar discrete valuation. Let us re-interpret their results as statements about log discrepancies of non-Abhyankar discrete valuations.

Suppose v is a non-Abhyankar discrete valuation with value group equal to $\mathbb{Z}$ on the function field L of some variety X over a perfect field, with valuation ring R, and let $\varpi \in R$ with $v(\varpi) = 1$. Datta and Smith proved that $(F^e)^! R = 0$ for all $e \geq 1$ [16, Corollary 4.2.2 and Lemma 4.2.4], cf. [17, Theorem 0.1]). I claim that this implies that $A(v; \psi) = +\infty$ for any nonzero p^{-e}-linear map $\psi : L \to L$. Indeed, the set of real numbers $\{v(\psi(r)) : r \in R, \psi(r) \neq 0\}$ must be unbounded below, for if $\psi(r) \geq -M = v(\varpi^{-M})$ for all $r \in R$, then $\varpi^M \psi$ gives a nonzero element of $(F^e)^! R$. Therefore, the set $\{E(r, \psi, v) : r \in R\}$ is unbounded above, so $A(v; \psi) = +\infty$.

Proposition 3.8. *Suppose $(R, \mathfrak{m}, k)$ is a regular F-finite ring and v is a valuation centered on $\mathfrak{m}$ and monomial with respect to some regular system of parameters $x_1, \ldots, x_n$ for R. Then $A(v; \mathcal{C}^R) = \sum_{i=1}^n v(x_i)$.*

Proof. Since R is regular and F-finite, the Cohen structure theorem yields an isomorphism $\hat{R} \cong k[\![x_1, \ldots, x_n]\!]$, with k an F-finite field. Recalling 2.9(1), and letting $\hat{v}$ be the $\mathfrak{m}$-adic extension of v to $\hat{R}$, we see $A(v; \mathcal{C}^R) = A(\hat{v}; \mathcal{C}^{\hat{R}})$. Thus, we have reduced to the case $R = k[\![x_1, \ldots, x_n]\!]$ for an F-finite field k, and $v = \mathrm{val}_{\mathbf{r}}$ for some $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n_{\geq 0}$, cf. (2.29). Recall our description of $\mathcal{C}^R$ from 2.12, and let Φ be the generator for $\mathcal{C}^R$ that splits $(x_1 \cdots x_n)^{p-1}$. Applying 3.4, we see that

$$A(v_{\mathbf{r}}; \mathcal{C}^R) = A(v_{\mathbf{r}}; \Phi) = \sum_{j=1}^{d} r_j.$$

To see the rightmost equality, we can argue as in (3.5): localizing R at (x_i) gives an F-finite DVR, and we conclude that if $\Phi(f) = 1$ then $\mathrm{ord}_{x_i}(f) = p-1$ for all $1 \leq i \leq n$. $\square$

Recall from the last paragraph of §2 that $v \in \mathrm{Val}_X$ is called *locally quasi-monomial* if there exists a neighborhood U of $x = c_X(v)$ and a proper birational morphism $\pi : Y \to U$ from a regular scheme such that $v = r_{(Y,D)}(v)$ for some snc divisor D on Y.

Proposition 3.9. *Suppose ψ is a p^{-e}-linear map on an integral F-finite scheme X. Suppose $v \in \mathrm{Val}_X$ is locally quasi-monomial with center x, and $\pi : Y \to U$ is a proper birational morphism for which $v = r_{(Y,D)}(v)$. Let $\psi = \Phi_y^e \cdot h$ where Φ_y is a chosen algebra generator for $\mathcal{C}_y^Y$ near $y = c_Y(v)$; see (2.12) and (3.8). Then*

$$A(v; \psi) = v(D) + \frac{v(h)}{1 - p^e}.$$

Proof. Applying 3.4, we see $A(v; \mathcal{C}^Y) = A(v; \Phi_y)$. Now applying 3.3 to $\psi = \Phi_y^e \cdot h$, and 3.8 to $\mathcal{O}_{Y,y}$, we see that $A(v; \psi) = v(D) + (1 - p^e)^{-1} v(h)$. $\square$

We will return to the previous proposition, giving more precise information about $\mathrm{div}_Y(h)$ in 7.21 and 7.24 with added assumptions on X.

Proposition 3.10. *Let $\mathcal{D}$ be a Cartier subalgebra on X, fix $x \in X$, and let $Z \subseteq X$ be a uniformly $\mathcal{D}$-compatible subscheme passing through x. Denote $\mathcal{O}_{X,x}$ by R, and let $\mathfrak{p} \in \mathrm{Spec}(R)$ be the prime corresponding to Z. Then for all $\psi \in \mathcal{D}_x$,*

$$A(\zeta; \psi\|_Z) = \sup_{n,f}\{E(f, \psi^n, \zeta) \,:\, \psi^n(f) \in R \setminus \mathfrak{p}\}.$$

Proof. The condition $\psi^n(f) \in R\setminus\mathfrak{p}$ is equivalent to $(\psi\|_Z)^n(f(\mathfrak{p})) \neq 0 \in \kappa(\mathfrak{p})$, thus also equivalent to $\zeta(\psi^n(f)) < \infty$. The claimed expression for $A(\zeta; \psi\|_Z)$ is then just unwinding the definition of $A(\zeta; \psi\|_Z)$, recalling that $\zeta(f)$ is defined to be $\zeta(f(\mathfrak{p}))$, see Convention 2.1(12). ☐

Proposition 3.11. *Let $\mathcal{D}$ be a Cartier subalgebra on X and suppose $\zeta \in X^{\beth}$ has uniformly $\mathcal{D}$-compatible home. Let $x = c_X(\zeta)$, $R = \mathcal{O}_{X,x}$, and $\mathfrak{p} = h_X(\zeta) \in \mathrm{Spec}(R)$.*

1. *For $\psi \in (\mathcal{D}_e)_x$,*

$$A(\zeta; \psi) = \limsup_{n\to\infty}\left(\sup_f \{E(f, \psi^n, \zeta) \,:\, f \in R, \psi^n(f) \in R \setminus \mathfrak{p}\}\right).$$

2. *Similarly, we have an equality*

$$A(\zeta; \mathcal{D}) = \limsup_{e\to\infty} \sup_{\psi\in(\mathcal{D}_e)x} A(\zeta; \psi).$$

Proof. If we replace the limit supremum with supremum in the first statement, then we are left with 3.10. Thus, it suffices to show that $\{E(f, \psi^n, \zeta) \,:\, f \in R, \psi^n(f) \in R\setminus\mathfrak{p}\}$ is contained in $\{E(f, \psi^{nm}, \zeta) \,:\, f \in R, \psi^{nm}(f) \in R\setminus\mathfrak{p}\}$ for all $m \geq 1$. The key idea here is similar to the observations in the proof of 3.3. Suppose $f \in R$ has the property $\psi^n(f) \in R \setminus \mathfrak{p}$. I claim that for each $m \geq 1$, there is some $f_m \in R$ with $\psi^{nm}(f_m) \in R \setminus \mathfrak{p}$ and $E(f_m, \psi^{nm}, v) = E(f, \psi^n, v)$.

Localizing at $\mathfrak{p}$, we are looking for $f_m \in R$ so that $\psi^{nm}(f_m)$ is a unit of $R_\mathfrak{p}$. By assumption, $\psi^n(f) = u$ is a unit of $R_\mathfrak{p}$ that is contained in R. Thus, as a p^{-ne}-linear map $\psi^n : L \to L$, $\psi^n(u^{-p^{ne}} f) = 1$. Define $g_m = (u^{-p^{ne}} f)^{(p^{nme}-1)/(p^{ne}-1)}$ and $u_m = u^{p^{ne}(p^{nme}-1)/(p^{ne}-1)}$; note $\psi^{nm}(g_m) =$

$\psi^n(u^{-p^{ne}}f) = 1$ for all $m \geq 1$. Moreover, the idea for proving the first observation in the proof of (3.3) can be used to show

$$\psi^{nm}(u_m g_m) = u$$

for each m. Set $f_m = u_m g_m \in R$. By construction, $E(f_m, \psi^{nm}, v) = E(g_m, \psi^{nm}, v)$, and also

$$\frac{v(g_m)}{p^{nme} - 1} = \frac{v(f) - p^{ne}v(u)}{p^{ne} - 1}.$$

The right-hand expression is precisely $E(f, \psi^n, v)$. This proves that we can find the claimed f_m, and that the supremum over n in the definition of $A(\zeta; \psi)$ is equal to the limit supremum over n.

For the second claim, about $A(\zeta; \mathcal{D})$, note that by definition (or the first part of this proposition) $A(v; \psi) = A(v; \psi^n)$ for all n. Thus, given $\psi \in (\mathcal{D}_e)_x$ we have $\psi^n \in (\mathcal{D}_{ne})_x$ with the same log discrepancy, and $A(\zeta; \mathcal{D})$ can be calculated as a limit supremum over $e \geq 1$ instead of a supremum. ☐

3.2. Multiplicatively and F-graded sequences of ideals

We now study how Cartier subalgebras can be twisted by sequences of ideals with multiplicative structures, proving formulas similar to 3.3. Let $\mathbb{N}_0 = \mathbb{Z} \cap [0, \infty)$ and $\{\mathfrak{a}_e : e \in \mathbb{N}_0\}$ be a sequence of non-zero ideals on X. We say that this collection is *multiplicatively graded* (or simply *graded*) if $\mathfrak{a}_s \mathfrak{a}_t \subseteq \mathfrak{a}_{s+t}$ for all $s, t \in \mathbb{N}_0$, and that this sequence is *F-graded* if $\mathfrak{a}_1 = \mathcal{O}_X$ and $\mathfrak{a}_s^{[p^t]} \mathfrak{a}_t \subseteq \mathfrak{a}_{s+t}$ for all $s, t \geq 0$. While it is standard to use the notation $\mathfrak{a}_\bullet$ for both of these, we have need to use these two concepts together and so we will **always** interpret $\mathfrak{a}_\bullet$ as an F-graded sequence and $\mathfrak{a}_\star$ as a multiplicatively graded sequence.

Interesting examples of (multiplicatively) graded sequences of ideals arise as base ideals of tensor powers of a line bundle on X ([34, Def. 1.1.18]), and symbolic powers of a fixed ideal. Another common source of graded sequences of ideals, especially relevant here, are *valuation ideals* associated to valuations v on X, defined as

$$\mathfrak{a}_s(v) = \{f \in \mathcal{O}_X : v(f) \geq s\} \text{ for } s \in \mathbb{N}_0.$$

We write $\mathfrak{a}_\star(v)$ for this graded sequence.

The F-graded condition is precisely what is needed to make new Cartier subalgebras from old as described below. Every Cartier subalgebra on a Gorenstein scheme arises in this way, cf. [4, 6].

Definition 3.12. Let $\mathcal{D}$ be a Cartier subalgebra, and $\mathfrak{a}_\bullet$ an F-graded sequence, on X. We define a Cartier subalgebra $\mathcal{D}\cdot\mathfrak{a}_\bullet \subset \mathcal{D}$ by

$$\begin{aligned}\mathcal{D}\cdot\mathfrak{a}_\bullet &= \oplus_{e\geq 0}\left(\mathcal{D}_e\cdot\mathfrak{a}_e\right)\\ &= \oplus_{e\geq 0}\left\{\sum(\phi_i\cdot a_i)\,:\,a_i\in\mathfrak{a}_e \text{ and } \phi_i\in\mathcal{D}_e\right\}.\end{aligned}$$

A special case of interest is constructed from the data of a Cartier subalgebra $\mathcal{D}$ on X, an ideal $\mathfrak{a}\subseteq\mathcal{O}_X$, and a real number $t\geq 0$. Setting $\mathfrak{a}_e=\mathfrak{a}^{\lceil t(p^e-1)\rceil}$, we get an F-graded sequence $\mathfrak{a}_\bullet$. We define

$$\mathcal{D}\cdot\mathfrak{a}^t := \mathcal{D}\cdot\mathfrak{a}_\bullet.$$

These F-graded sequences seem to first appear in the theory of tight closure with respect to an ideal [27], see also [45]. Their study was key to the development of sharp F-purity, and Schwede's approach to F-singularities of pairs and triples [45, 46, 47, 48].

Finally, let $\mathfrak{a}_\star$ be a graded sequence of ideals on X. For $t\in[0,\infty)$ we define the Cartier subalgebra

$$\mathcal{D}\cdot\mathfrak{a}_\star^t = \sum_{m\geq 1}\mathcal{D}\cdot\mathfrak{a}_m^{t/m}.$$

For $\zeta\in X^\beth$, by definition (3.1) we have

$$A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\star^t) = \sup_{m\geq 1} A(\zeta;\mathcal{D}\cdot\mathfrak{a}_m^{t/m}).$$

Suppose $\mathfrak{a}\subseteq\mathcal{O}_X$ is an ideal sheaf on X, and $\zeta\in X^\beth$ has center x. We define

$$\zeta(\mathfrak{a}) = \min\{\zeta(f)\,:\,f\in\mathfrak{a}_x\}.$$

The following lemma allows us to evaluate semivaluations on sequences of ideals in two ways. The existence of the limits for multiplicative graded sequences is well-known; the F-graded case is similar, but we provide a short proof for completeness.

Lemma 3.13 (cf. [19, 30]). *Let $\zeta\in X^\beth$ and $\{\mathfrak{a}_e\}_{e\in\mathbb{N}_1}$ be a sequence of ideals on X. If $\mathfrak{a}_\bullet$ is F-graded, we have the limit*

$$\zeta_F(\mathfrak{a}_\bullet) := \lim_{e\to\infty}\frac{\zeta(\mathfrak{a}_e)}{p^e-1} = \inf_{e\geq 1}\frac{\zeta(\mathfrak{a}_e)}{p^e-1}.$$

Similarly, if $\mathfrak{a}_\star$ *is a graded sequence, then*

$$\zeta(\mathfrak{a}_\star) := \lim_{e\to\infty} \frac{\zeta(\mathfrak{a}_e)}{e} = \inf_{e\geq 1} \frac{\zeta(\mathfrak{a}_e)}{e}.$$

Proof. Both cases are essentially an application of Fekete's lemma; we refer the reader to [30, Lemma 2.3] for the multiplicatively graded case. Since the F-graded case is not precisely the situation of Fekete's lemma, let us show directly that $\{\zeta(\mathfrak{a}_e)/(p^e-1)\}_{e=1}^\infty$ is a non-increasing sequence, bounded below by 0, so has a limit (maybe infinity) that agrees with the infimum.

Suppose that $\mathfrak{a}_\bullet$ is F-graded. Note that $\mathfrak{a}_s^{[p^t]}\mathfrak{a}_t \subseteq \mathfrak{a}_{s+t}$ implies $\zeta(\mathfrak{a}_{s+t}) \leq p^t\zeta(\mathfrak{a}_s) + \zeta(\mathfrak{a}_t)$. Therefore,

$$\frac{\zeta(\mathfrak{a}_{s+t})}{p^{s+t}-1} \leq \frac{p^t\zeta(\mathfrak{a}_s)}{p^{s+t}-1} + \frac{\zeta(\mathfrak{a}_t)}{p^{s+t}-1} \leq \frac{p^t\zeta(\mathfrak{a}_s)}{p^t(p^s-1)} + \frac{\zeta(\mathfrak{a}_t)}{p^t-1}.$$

We conclude $\{\zeta(\mathfrak{a}_e)/(p^e-1)\}_{e=1}^\infty$ is a non-increasing sequence, bounded below by 0, so its infimum is the limit. □

We close this section by proving some statements analogous to 3.3, but with Cartier subalgebras and F-graded sequences of ideals in place of single p^{-e}-linear maps and function field elements.

Lemma 3.14. *Let* $\mathcal{D}$ *and* $\mathcal{R}$ *be Cartier subalgebras on* X*, and fix* $\zeta \in X^\beth$ *whose home is uniformly* $\mathcal{D}$*-compatible.*

1. ***Monotonicity:*** *If* $\mathcal{D} \subseteq \mathcal{R}$ *then* $A(\zeta;\mathcal{D}) \leq A(\zeta;\mathcal{R})$*.*
2. ***Conservation:*** *Let* $\mathfrak{a}_\bullet$ *be an* F*-graded sequence of ideals on* X*. If* $\zeta_F(\mathfrak{a}_\bullet)$*,* $A(\zeta;\mathcal{D})$*, and* $A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\bullet)$ *are all finite, then* $A(\zeta;\mathcal{D}) = A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\bullet) + \zeta_F(\mathfrak{a}_\bullet)$*.*

Proof. Monotonicity follows directly from the definition; we therefore begin with conservation. Towards this end, let $x = h_X(\zeta)$ and $Z = \overline{\{x\}}$. Then $\mathcal{D}\cdot\mathfrak{a}_\bullet \subseteq \mathcal{D}$ implies that Z is uniformly compatible with $\mathcal{D}\cdot\mathfrak{a}_\bullet$. By passing to $\mathfrak{a}_\bullet\mathcal{O}_Z$ and $\mathcal{D}\|_Z$, and re-setting notation (replacing Z with X), we assume $\zeta = v$ is a valuation, with valuation ring $\mathcal{O}_v \subset L$. Furthermore, the definitions of log discrepancy and $\zeta_F(\mathfrak{a}_\bullet)$ are local near the center $z = c_X(v) \in X$, so we restrict our attention to $R := \mathcal{O}_{X,x}$ and write $\mathcal{D}$ for the stalk $\mathcal{D}_x$. Since R is noetherian, $\mathfrak{a}_e$ is finitely generated, and so $\mathfrak{a}_e\mathcal{O}_v$ is principally generated, say by $g_e \in \mathfrak{a}_e$. Possible generators are characterized by $v(g_e) = v(\mathfrak{a}_e)$.

Let $\psi \in \mathcal{D}_e$ be nonzero. Then 3.3 tells us

$$A(v;\psi^n) - A(v;\psi^n \cdot g_{ne}) = \frac{v(g_{ne})}{p^{ne}-1} = \frac{v(\mathfrak{a}_e)}{p^{ne}-1}$$

so $\lim_{n\to\infty} A(v;\psi^n) - A(v;\psi^n \cdot g_{ne}) = v_F(\mathfrak{a}_\bullet)$. On the other hand, $A(v;\psi^n) = A(v;\psi)$ for all n (3.11), so

$$A(v;\psi) = v_F(\mathfrak{a}_\bullet) + \lim_{n\to\infty} A(v;\psi^n \cdot g_{ne}).$$

Since $\psi^n \cdot g_{ne} \in (\mathcal{D}\cdot\mathfrak{a}_\bullet)_{ne}$, applying 3.11(2) gives

$$A(v;\psi) \le v_F(\mathfrak{a}_\bullet) + A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet)$$

for all $\psi \in \mathcal{D}_e$. Now taking a supremum over $\psi \in (\mathcal{D}_e)_x$, and $e \ge 1$, shows $A(v;\mathcal{D}) \le v_F(\mathfrak{a}_\bullet) + A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet)$.

We now establish the reversed inequality. By 3.13, it suffices to show that for all $\varepsilon > 0$ there exists $e > 0$ such that

$$A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet) + \frac{1}{p^e-1} v_F(\mathfrak{a}_e) < A(v;\mathcal{D}) + \varepsilon.$$

From the definition of $A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet)$, we know there exists $e > 0$, $\psi \in (\mathcal{D}\cdot\mathfrak{a}_\bullet)_{e,x}$, and $f \in L$ such that $\psi(f) = 1$ and

$$A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet) - \varepsilon < \frac{1}{p^e-1} v(f) = E(f,\psi,v). \tag{5}$$

By definition of $\mathcal{D}\cdot\mathfrak{a}_\bullet$, there exist $\psi_1,\dots\psi_n \in \mathcal{D}_{e,x}$ and $a_1,\dots,a_n \in \mathfrak{a}_{e,x}$ such that

$$\psi(f) = \sum_1^n \psi_i(a_i f) = 1.$$

Let $-c = \min_{1\le i\le n}\{v(\psi_i(a_i f))\} \le v(\psi(f)) = 0$. By reindexing, we may assume that $-c = v(\psi_1(a_1 b))$ so that $\psi_1(ug^{p^e} a_1 f) = 1$ for some unit $u \in \mathcal{O}_v^\times$ and $g = \psi_1(a_1 f) \in \mathcal{O}_v$ (note $v(g) = c$). Since $a_1 \in \mathfrak{a}_e$, it follows that $v(\mathfrak{a}_e) \le$

$v(a_1)$, so

$$\begin{aligned} A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet)-\varepsilon+\frac{1}{p^e-1}v(\mathfrak{a}_e) &\le A(v;\mathcal{D}\cdot\mathfrak{a}_\bullet)-\varepsilon+\frac{1}{p^e-1}v(a_1)\\ &< E(f,\psi,v)+\frac{1}{p^e-1}v(a_1)\\ &\le E(f,\psi,v)+\frac{1}{p^e-1}v(a_1)+\frac{p^e}{p^e-1}cp^e\\ &= E(ug^{p^e}a_1f;\psi,v)\\ &\le A(v;\mathcal{D}). \end{aligned}$$

Here we are using (5), and $c \ge 0$, between the first and second lines, and second and third, respectively. Finally, we used that $\psi_1 \in \mathcal{D}_e$ and $\psi_1(ug^{p^e}a_1f) = 1$, so $E(ug^{p^e}a_1f,\psi_1,v) \le A(v;\mathcal{D})$. ☐

Remark 3.15. The assumption that $\zeta_F(\mathfrak{a}_\bullet)$, $A(\zeta;\mathcal{D})$, and $A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\bullet)$ are finite is essential. Indeed, for a fairly trivial counterexample without these assumptions (which demonstrates the main trouble), we can take $\zeta = \mathrm{triv}_X$, $\mathcal{D} = \mathcal{C}_X$, and $\mathfrak{a}_e = 0$ for $e > 1$. Then $\zeta_F(\mathfrak{a}_\bullet) = +\infty$, $A(\zeta;\mathcal{D}) = 0$, and $A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\bullet) = -\infty$.

Alternatively, suppose X is F-finite and regular, $Z \subset X$ is a proper integral subscheme of dimension at least one, let $\mathfrak{p}_Z$ be the associated prime ideal sheaf, and set $\mathfrak{a}_e = (\mathfrak{p}_Z^{[p^e]} : \mathfrak{p}_Z)$ for $e > 1$. Then $\mathfrak{p}_Z$ is compatible with $\mathcal{D} = \mathcal{C}_X\cdot\mathfrak{a}_\bullet$, and $\mathcal{D}\|_Z = \mathcal{C}_Z$ [24, Corollary to Lemma 1.6]. Let $z \in Z_{reg}$ be a closed point, and let $\zeta = \mathrm{ord}_z$ be the valuation corresponding to the exceptional fiber of $\mathrm{Bl}_z(Z) \to Z$. Then $A(\zeta;\mathcal{D}) = A(\zeta;\mathcal{C}_Z) = \dim(Z)$ while $\zeta_F(\mathfrak{a}_\bullet) = +\infty = A(\zeta;\mathcal{C}_X)$.

We also have conservation for graded sequences. We return to studying asymptotic invariants of graded sequences of ideals in §7.

Corollary 3.16. *Let $\mathcal{D}$ be a Cartier subalgebra on X, $t \in [0,\infty)$, and $\zeta \in X^{\beth}$ whose home Z is $\mathcal{D}$-compatible. Suppose $\mathfrak{a}_\star \subseteq \mathcal{O}_X$ is a multiplicatively graded sequence of ideals on X such that $\zeta(\mathfrak{a}_\star) < \infty$. If $A(\zeta;\mathcal{D})$, $A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\star^t)$, and $\zeta(\mathfrak{a}_\star)$ are finite, then*

$$A(\zeta;\mathcal{D}) = A(\zeta;\mathcal{D}\cdot\mathfrak{a}_\star^t) + t\,\zeta(\mathfrak{a}_\star).$$

Proof. We apply 3.14 to the sequence of Cartier subalgebras $\mathcal{D}\cdot\mathfrak{a}_m^{t/m}$. By definition, $\zeta(\mathfrak{a}_m^{\lceil (t/m)(p^e-1)\rceil}) = \lceil (t/m)(p^e-1)\rceil\zeta(\mathfrak{a}_m)$, so $\zeta_F(\{\mathfrak{a}_m^{\lceil (t/m)(p^e-1)\rceil}\}_e) =$

$(t/m)\zeta(\mathfrak{a}_m)$. Taking suprema over $m \geq 1$ gives

$$\begin{aligned} A(\zeta; \mathcal{D} \cdot \mathfrak{a}_\star^t) &= \sup_{m \geq 1} A(\zeta; \mathcal{D} \cdot \mathfrak{a}_m^{t/m}) \\ &= A(\zeta; \mathcal{D}) - tv(\mathfrak{a}_\star). \end{aligned}$$

□

4. Proof of the main theorem

For this section, we fix a normal variety X over an algebraically closed field k of characteristic $p > 0$. We also fix a canonical Weil divisor K_X on X, which fixes a canonical divisor K_Y on every normal variety Y with a proper birational morphism $\pi : Y \to X$ by requiring $\pi_* K_Y = K_X$. To state and prove our main theorem, we review the construction of log discrepancies of arbitrary valuations of log $\mathbb{Q}$-Gorenstein pairs (X, Δ). But first, let us give the proof of (1.1), using (3.5), a corollary of which is a description of $A(v; \mathcal{C}^X)$ similar to the theory of log discrepancies initiated in [18]. Recall the notation $\mathcal{C}^X \cdot \Delta$ from 2.18.

Proposition 4.1 (Cascini-Mustaţă-Schwede)**.** *Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair, and assume the Cartier index of $K_X + \Delta$ is not divisible by p. Then $A_{(X,\Delta)}(v) = A(v; \mathcal{C}^X \cdot \Delta)$ for every divisorial valuation $v \in X^{\mathrm{div}}$.*

Proof. We assert that the valuation ring R of v must be F-finite. Indeed, every variety over a perfect field is F-finite. Since v is divisorial, there must exist some prime divisor E on a normal variety Y with a proper birational morphism $\pi : Y \to X$, and $C \in (0, \infty)$, such that $v = C \operatorname{ord}_E$. Then $\mathcal{O}_{Y,E} \cong R$ is F-finite. Thus, (3.5) gives $A(v; \mathcal{C}^Y) = A(v; \mathcal{C}^R) = C$. If $\mathcal{C}^R = \{\!\{\Phi_E\}\!\}$ (see proof of 3.5), then the fact that $\mathcal{C}_1^Y \cong \omega_Y^{\otimes(1-p)}$ implies Φ_E must correspond to a generator for $\omega_{Y,E}^{\otimes(1-p)}$ over R.

Now suppose $(1-p^e)(K_X+\Delta)$ is Cartier. The definitions of both $A_{(X,\Delta)}(v)$ and $A(v; \mathcal{C}^X \cdot \Delta)$ are local near $x = c_X(v)$, so by restricting to some neighborhood of x we may assume there is some $\psi_\Delta \in \Gamma(X, (F^e)^! \mathcal{O}_X((p^e-1)\Delta))$ corresponding to Δ as in (2.6); ψ_Δ is unique up to a unit of $\Gamma(X, \mathcal{O}_X)$. Then $\{\!\{\psi\}\!\} \cong (F^e)^! \mathcal{O}_X((p^e-1)\Delta)$ (see 2.14 and 2.19).

We define Δ_Y as in Section 2.6: $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. Since $(1 - p^e)(K_X + \Delta)$ is Cartier, so too is $(1 - p^e)(K_Y + \Delta_Y)$. Thus, in some neighborhood V of the generic point of E there is some corresponding p^{-e}-linear map $\psi_{\Delta_Y} \in \Gamma(V, (F^e)^! \mathcal{O}_Y((p^e-1)\Delta_Y))$. I claim that we can take $\psi_{\Delta_Y} = \psi_\Delta$. Indeed, up to units, ψ_{Δ_Y} agrees with ψ_Δ on the dense open subset $V' \subseteq V$

where π is an isomorphism. Normality of Y lets us shrink V and assume that V is smooth, so $(p^e-1)\Delta_Y$ is principal on V, say $(p^e-1)\Delta_Y \cap V = \mathrm{div}_V(h_\Delta)$. Letting $K_V = K_Y \cap V$, we then have

$$V \cap (1-p^e)(K_Y+\Delta_Y) = (1-p^e)K_V - \mathrm{div}_V(h_\Delta)$$

so ψ_Δ gives a generator of the line bundle

$$\mathcal{O}_V((1-p^e)K_V - \mathrm{div}_V(h_\Delta)) = h_\Delta \cdot \omega_V^{\otimes(1-p^e)}.$$

Thus, $\psi_\Delta = \Phi_E^e \cdot h_\Delta$ via the $\mathcal{O}_V$-linear isomorphism $\omega_V^{\otimes(1-p^e)} \cong (F^e)^! \mathcal{O}_V$. Applying (3.3) and (3.4) now gives

$$\begin{aligned} A(v;\mathcal{C}^X \cdot \Delta) &= A(v;\psi_\Delta) \\ &= A(v;\Phi_E) - \frac{v(h_\Delta)}{p^e-1} \\ &= C(1-\mathrm{ord}_E(\Delta_Y)) \\ &= A_{(X,\Delta)}(v). \end{aligned}$$

□

Corollary 4.2 (cf. [18, 8])**.** *Let X be a normal variety over an algebraically closed field k of characteristic $p>0$. Then*

$$A(v;\mathcal{C}^X) = \sup A_{(U,\Delta)}(v)$$

for every $v \in X^{\mathrm{div}}$, where the supremum is over effective log-$\mathbb{Q}$-Gorenstein pairs (U,Δ) on neighborhoods U of $c_X(v) \in X$ such that the Cartier index of $K_U+\Delta$ is not divisible by p.

Proof. By definition, $A(v;\mathcal{C}^X)$ is the supremum of $A(v;\psi)$ over p^{-e}-linear maps $\psi \in \mathcal{C}_x^X$, where $x = c_X(v)$. Recall from 2.6 that there is a bijection between such ψ (up to units of $\Gamma(U,\mathcal{O}_X)$) and log-$\mathbb{Q}$-Gorenstein pairs (U,Δ) for neighborhoods U of x on which $\psi \in \Gamma(U,\mathcal{C}^X)$. Applying 2.19 to the statement of 4.1, we have the claimed equality. □

The remainder of this section builds on (4.1). We must first review and extend the method of Jonsson and Mustaţă to our setting.

4.1. Log smooth pairs and domination

A *log smooth pair over* X consists of a smooth variety Y with a proper birational morphism $\pi : Y \to X$ and a reduced snc divisor $D = \sum_i D_i$ on Y; we will say $\pi : (Y, D) \to X$ is a log smooth pair over X. We would like to follow [30] in the construction of a partial order on log smooth pairs over X; since in our setting X is not assumed to be $\mathbb{Q}$-Gorenstein, we must handle boundary divisors, so our definition becomes more involved. At the beginning of this section, we have fixed a canonical class K_Y on each normal variety admitting a proper birational morphism $\pi : Y \to X$ by requiring $\pi_* K_Y = K_X$. This choice of K_Y implies that it is supported on $E + \pi_*^{-1}(K_X)$, where E is the exceptional locus of π.

Definition 4.3. Fix a log $\mathbb{Q}$-Gorenstein pair (X, Δ). We will say a log smooth pair $\pi : (Y, D) \to X$ *dominates* (X, Δ), and write $(Y, D) \succeq (X, \Delta)$, if $\pi : Y \to X$ is a log resolution with the two properties below.

1. The support of $\pi^*(K_X + \Delta)$ is contained in the support of $K_Y + D$.
2. The support of $E + (\pi_*^{-1}(\Delta))$ is contained in the support of D.

When $(Y, D) \succeq (X, \Delta)$, we call π *the domination morphism*.

Following Jonsson and Mustaţă, we extend $\succeq$ to a partial order on log smooth pairs over X as follows. If (Y', D') and (Y, D) are two log smooth pairs dominating (X, Δ), with domination morphisms $\pi' : Y' \to X$ and $\pi : Y \to X$, we will write $(Y', D') \succeq (Y, D)$ whenever π' factors as $\pi \circ \mu$ for a proper birational morphism $\mu : Y' \to Y$, and $\mu^* D$ is supported on D'.

4.2. Comparison of retractions

For every log smooth pair (Y, D) over X, recall from 2.30 the retraction morphism $r_{(Y,D)} : \mathrm{Val}_X \to \mathrm{Val}_X$. By definition, for any $v \in \mathrm{Val}_X$ the image $r_{(Y,D)}(v)$ is monomial with respect to D, meaning monomial on the completion of $\mathcal{O}_{Y,y}$ after choosing a regular system of parameters at $c_Y(v) = y \in Y$ as in 2.30; if y is not in the support of D, then $r_{(Y,D)}(v)$ is the trivial valuation. Since the image of $r_{(Y,D)}$ consists of precisely those valuations on X that are quasi-monomial with respect to (Y, D), this image is denoted here by $\mathrm{QM}(Y, D)$. Some authors further filter $\mathrm{QM}(Y, D)$ by the so-called *strata* of D, but we do not need this level of precision and refer the interested reader to Remark 3.4, and the paragraph immediately before it, in [30].

Suppose $D = \sum_i D_i$, with each D_i a prime divisor on Y. Following [30, Proposition/Definition 5.1], for $(Y, D) \succeq (X, \Delta)$ and for $v \in \mathrm{QM}(Y, D)$ we define

$$A_{(X,\Delta)}(v) = \sum_i v(D_i) A_{(X,\Delta)}(D_i). \tag{6}$$

The Proposition portion of Jonsson and Mustaţă's (5.1) shows that this definition is independent of (Y, D). Our first result towards agreement of our log discrepancy with established approaches is that the expression (6) is the value of $A(v; \mathcal{C}_X \cdot \Delta)$ for $v \in \mathrm{QM}(Y, D)$. The result, and proof, is quite similar to (3.9).

Lemma 4.4. *Let $\Delta \geq 0$ be a $\mathbb{Q}$-Weil divisor on X such that $(1-p^e)(K_X+\Delta)$ is Cartier for some $e > 0$. Suppose $(Y, D) \succeq (X, \Delta)$ and $v \in \mathrm{QM}(Y, D)$. Then $A_{(X,\Delta)}(v) = A(v; \mathcal{C}_X \cdot \Delta)$.*

Proof. Fix $v \in \mathrm{QM}(Y, D)$ and define $x = c_X(v)$, $y = c_Y(v)$. Let $\pi : Y \to X$ be the domination morphism. By assumption, $\pi^*(K_X + \Delta) - K_Y = \Delta_Y$ is supported on D. Suppose $\Delta_Y = \sum_{i=1}^t \frac{b_i}{p^e-1} D_i$, where $b_i \in \mathbb{Z}$.

We work locally near the centers $x = c_X(v)$ and $y = c_Y(v)$ so we can associate p^{-e}-linear maps to divisors. Since $(1-p^e)(K_X+\Delta)$ is Cartier, there is a corresponding $\psi_\Delta \in (F^e)^! \mathcal{O}_X((p^e-1)\Delta)_x$; similarly, $(1-p^e)(K_Y+\Delta_Y) = \pi^*((1-p^e)(K_X+\Delta))$ corresponds to some $\psi_{\Delta_Y} \in (F^e)^! \mathcal{O}_Y((p^e-1)\Delta_Y)_y$. Since these p^{-e}-linear maps must agree on any dense open neighborhood of y where π is an isomorphism, they must be $\mathcal{O}_{Y,y}$-unit multiples of the same map. Therefore,

$$A(v; \mathcal{C}^X \cdot \Delta) = A(v; \psi_\Delta) = A(v; \psi_{\Delta_Y}).$$

The first equality follows from 3.4 and the end of 2.19. The second follows from $\psi_\Delta = \psi_{\Delta_Y}$.

Let $f_1, \ldots, f_t \in \mathcal{O}_{Y,y}$ be such that $D_i = \mathrm{div}(f_i)$ in some neighborhood of y. By re-numbering the D_i if necessary, we can assume that $f_1, \ldots, f_s$ give a regular system of parameters for $\mathcal{O}_{Y,y}$ (so for $s < i \leq t$, the divisor D_i does not contain y). The monomials $(f_1^{n_1} \cdots f_s^{n_s})$, with $0 \leq n_i \leq p^e - 1$, give a free basis for $\mathcal{O}_{Y,y}$ over $\mathcal{O}_{Y,y}^{p^e}$. The projection Φ_y^e onto the basis element $(f_1 \cdots f_s)^{(p^e-1)}$ gives a generator for $\mathcal{C}_y^Y$ over $\mathcal{O}_{Y,y}$. Since $y \notin D_i$ for $s < i \leq t$, we see that $u := f_{s+1}^{b_{s+1}} \cdots f_t^{b_t}$ is a unit in $\mathcal{O}_{Y,y}$.

Recalling that $\Delta_Y = \sum_1^t \frac{b_i}{p^e-1} D_i$, there is a unit $v \in \mathcal{O}_{Y,y}^\times$ such that we have

$$\psi_{\Delta_Y} = \Phi_y^e \cdot (uv f_1^{b_1} \cdots f_s^{b_s}).$$

For example, $v = 1$ when the generator Φ_y is chosen to correspond to the specific embedding $\omega_{Y,y}^{\otimes(1-p^e)} \cong \mathcal{O}_Y((1-p^e)K_Y)_y \subseteq L$ furnished by our choice of the Weil divisor K_Y.

Now using 3.9 and 3.3, we get

$$\begin{aligned} A(v;\psi_{\Delta_Y}) &= A(v;\Phi_y^e) - \frac{1}{p^e-1}\left(\sum_{i=1}^{s} b_i v(f_i)\right) \\ &= \left(\sum_{i=1}^{s} v(f_i)\right) - \frac{1}{p^e-1}\left(\sum_{i=1}^{s} b_i v(f_i)\right) \\ &= \sum_{i=1}^{s} v(f_i)\left(1 - \frac{b_i}{p^e-1}\right) \\ &= \sum_{i=1}^{s} v(D_i)(1-\mathrm{ord}_{D_i}(\Delta_Y)) \\ &= \sum_{i=1}^{s} v(D_i)A_{(X,\Delta)}(D_i) \\ &= A_{(X,\Delta)}(v) \text{ by definition.} \end{aligned}$$

□

Lemma 4.5. *Let (X,Δ) be a log $\mathbb{Q}$-Gorenstein pair. Suppose (Y',D') and (Y,D) are log smooth pairs over X, and $(Y',D') \succeq (Y,D) \succeq (X,\Delta)$. Then for all $v \in \mathrm{Val}_X$,*

$$A_{(X,\Delta)}(r_{(Y,D)}(v)) \leq A_{(X,\Delta)}(r_{(Y',D')}(v)).$$

Proof. This follows from [30, Lemma 1.5(i)]. In their setting they work with *sheaves of special differentials*, but the ideas in the proof of Lemma 1.5(i) are equally applicable to the Kähler and canonical bundles of smooth varieties. Note that the claimed inequalities are checked on the smooth varieties Y and Y'. □

Corollary 4.6. *With the notation as in the previous proposition,*

$$A(r_{(Y,D)}(v);\mathcal{C}^X\cdot\Delta) \leq A(r_{(Y',D')}(v);\mathcal{C}^X\cdot\Delta).$$

4.3. Log discrepancies of arbitrary valuations

Assume for this subsection that log resolutions of Weil divisors on varieties birational to X exist, so that the collection of log smooth pairs $(Y,D) \succeq (X,\Delta)$

is non-empty, and Abhyankar valuations admit global monomializations. Still following Jonsson and Mustaţă, we define the log discrepancy of arbitrary $v \in \mathrm{Val}_X$ to be:

$$A_{(X,\Delta)}(v) = \sup_{(Y,D)\succeq(X,\Delta)} A_{(X,\Delta)}(r_{(Y,D)}(v)). \tag{7}$$

Note that this is well defined, thanks to (4.5). There are numerous reasons to believe this is the correct extension of $A_{(X,\Delta)}$ from X^{div} to all of Val_X. For example, this extension is the maximal lower-semicontinuous extension of the log discrepancy on X^{div}. In characteristic zero, Mauri, Mazzon, and Stevenson [37] recently proved that this definition coincides with Temkin's pluricanonical metric from [52]; their approach relates log discrepancies to the *weight metrics* of Mustaţă and Nicaise [41], which give an analogous function for discretely valued ground fields.

Our main theorem is that in the case log resolutions exist, defining $A_{(X,\Delta)}(v)$ for valuations v using (7) gives the same function on Val_X as 3.2.

Theorem 4.7. *Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair, and suppose that the Cartier index of $K_X + \Delta$ is not divisible by p. Suppose log resolutions exist for Weil divisors on varieties birational to X. Then $A_{(X,\Delta)}(v) = A(v; \mathcal{C}^X \cdot \Delta)$ for all $v \in \mathrm{Val}_X$.*

Proof. Fix $v \in \mathrm{Val}_X$. Since $r_{(Y,D)}(v) \in \mathrm{QM}(Y, D)$, 4.4 provides an equality $A_{(X,\Delta)}(r_{(Y,D)}(v)) = A(r_{(Y,D)}(v); \mathcal{C}^X \cdot \Delta)$ for every $(Y, D) \succeq (X, \Delta)$.

All of our considerations are local near $x = c_X(v)$, so we restrict our attention to $R = \mathcal{O}_{X,x}$, denoting by $\mathfrak{m} = \mathfrak{m}_x$ the maximal ideal of R. The prime ideal $\mathfrak{p}$ of R associated to $c_X(r_{(Y,D)}(v)) \in X$ is contained in $\mathfrak{m}$ for every $(Y, D) \succeq (X, \Delta)$.

Claim ($\star$). For every $(Y, D) \succeq (X, \Delta)$, there exists $(Y', D') \succeq (Y, D)$ such that $c_X(r_{(Y',D')}(v)) = \mathfrak{m}$.

Proof of claim. Fix some $(Y, D) \succeq (X, \Delta)$ with domination morphism $\pi : Y \to X$, and assume $\mathfrak{p} = c_{(Y,D)}(v) \subsetneq \mathfrak{m}$. Then we can choose $f_1 \in \mathfrak{m}\backslash\mathfrak{p}$; this element is defined in some neighborhood U of $\mathfrak{m}$ in X, and we define N to be the scheme-theoretic closure of $\mathrm{div}_U(f)$ in X, which is an effective Weil divisor. Take $(Y^{(1)}, D^{(1)})$ to be a log resolution of $(X, \Delta+N)$ dominating (Y, D); note that such a log resolution exists by taking a log resolution $\mu : Y^{(1)} \to Y$ of $(Y, D + \pi_*^{-1}(\Delta + N)_{red})$. If $\mu^{(1)} : Y^{(1)} \to X$ is the domination morphism, then by definition the strict transform $(\mu^{(1)})_*^{-1}N$ is supported on $D^{(1)}$. In particular, $r_{(Y^{(1)},D^{(1)})}(v)(f) > 0$. Therefore,

$$\mathfrak{p} \subsetneq c_X(r_{(Y^{(1)},D^{(1)})}(v)) \subseteq \mathfrak{m}.$$

Call $\mathfrak{p}_1 = c_X(r_{(Y^{(1)},D^{(1)})}(v))$. If there exists $f^{(2)} \in \mathfrak{m} \setminus \mathfrak{p}_1$, we repeat this argument with $(Y^{(1)}, D^{(1)})$ in place of (Y, D), giving $(Y^{(2)}, D^{(2)}) \succeq (Y^{(1)}, D^{(1)})$ and a new center $\mathfrak{p}_2 = c_X(r_{(Y^{(2)},D^{(2)})}(v))$. Thus, we get a chain

$$\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \mathfrak{m}$$

with $\cup_i \mathfrak{p}_i = \mathfrak{m}$. Since X is noetherian, this chain must stabilize, meaning $\mathfrak{p}_i = \mathfrak{m}$ for $i \gg 0$. Taking $(Y', D') = (Y^{(i)}, D^{(i)})$ gives the desired pair. ⊠

If (Y, D) is a pair dominating (X, Δ) with $c_X(r_{(Y,D)}(v)) = c_X(v) = \mathfrak{m}$, we write $(Y, D) \gtrdot (X, \Delta)$. Using (4.4), (4.5), and Claim ($\star$) above, we see

$$\begin{aligned} A_{(X,\Delta)}(v) :&= \sup_{(Y,D)\succeq(X,\Delta)} A_{(X,\Delta)}(r_{(Y,D)}(v)) \\ &= \sup_{(Y,D)\gtrdot(X,\Delta)} A_{(X,\Delta)}(r_{(Y,D)}(v)) \\ &= \sup_{(Y,D)\gtrdot(X,\Delta)} A(r_{(Y,D)}(v); \mathcal{C}^X \cdot \Delta) \end{aligned}$$

The Cartier subalgebra $\mathcal{C}^X \cdot \Delta$ is generated near $\mathfrak{m}$ by a single p^{-e}-linear map ψ; see (2.6). 3.4 implies $A(w; \mathcal{C}^X \cdot \Delta) = A(w; \psi)$ for every $w \in \mathrm{Val}_X$ with $c_X(w) = \mathfrak{m}$, so we would be done if we could show

$$\sup_{(Y,D)\gtrdot(X,\Delta)} A(r_{(Y,D)}(v); \psi) = A(v; \psi). \tag{8}$$

We cannot follow a definitional proof of this because the expressions $E(f, \psi^n, v)$ used to define $A(v; \psi)$ are *not necessarily non-decreasing* along retractions, i.e. do not satisfy an analogue of (4.5). We therefore proceed by careful estimates of $A(v; \psi)$, and by using ideas similar to those in the proofs of (3.9) and (4.4). To simplify the notation going forward, call $\mathcal{S}$ the supremum on the left in (8).

For any fixed $f \in R$, [30, Lemma 4.7] shows $v(f) = r_{(Y,D)}(f)$ whenever (Y, D) gives a log resolution of $(X, \overline{\mathrm{div}(f)})$. Using ideas similar to those in the proof of Claim ($\star$), any log resolution for the closure $N_{f,n}$ of the divisor $\mathrm{div}(f) + \mathrm{div}(\psi^n(f))$ is dominated by some $(Y, D) \gtrdot (X, \Delta)$. When $(Y, D) \gtrdot (X, \Delta)$ is a log resolution of $N_{f,n}$, we do have $E(f, \psi^n, v) = E(f, \psi^n, r_{(Y,D)}(v))$.

Let $\varepsilon > 0$, and choose $n \geq 1$ and $f \in R$ so that

$$A(v; \psi) - \varepsilon < E(f, \psi^n, v).$$

Suppose $(Y, D) \gtrdot (X, \Delta)$ is also a log resolution of the Weil divisor $N_{f,n}$ defined in the previous paragraph. Then $E(f, \psi^n, v) = E(f, \psi^n, r_{(Y,D)}(v)) \leq A(r_{(Y,D)}(v); \psi)$, so

$$A(v; \psi) - \varepsilon < A(r_{(Y,D)}(v); \psi) \leq \mathcal{S}.$$

Since this was true for all ε, we conclude $A(v; \psi) \leq \mathcal{S}$.

For the reversed inequality, we use (3.9). Let $(Y, D) \gtrdot (X, \Delta)$ and $w = r_{(Y,D)}(v)$. We prove $A(w; \psi) \leq A(v; \psi)$. Call $\eta = c_Y(w)$, and $y = c_Y(v)$. If we number the components D_i of D so that $v(D_i) > 0$ if and only if $1 \leq i \leq s$, then local equations $f_1, \ldots, f_s \in \mathcal{O}_{Y,\eta}$ for $D_1, \ldots, D_s$ (resp.) generate the maximal ideal of $\mathcal{O}_{Y,\eta}$. This generating set can be extended by $g_1, \ldots, g_{n-s} \in \mathcal{O}_{Y,y}$ to a set of generators for the maximal ideal of $\mathcal{O}_{Y,y}$. Fix the generator Φ_y for $\mathcal{C}_y^Y$ that projects onto $(f_1 \cdots f_s g_1 \cdots g_{n-s})^{(p-1)}$. Then $\psi = \Phi_y^e \cdot h$ for some $h \in L = \kappa(X)$. We have assumed (Y, D) gives a log resolution for (X, Δ), so in particular $\pi^*(K_X + \Delta)$ is snc and supported on $K_Y + D$. This implies that, after possibly multiplying Φ_y^e on the right by a unit $u \in \mathcal{O}_{Y,y}^\times$, we have $h = \prod_{i=1}^s f_i^{c_i}$ for some $c_i \in \mathbb{Z}$. See the proof of (4.4) for more details. 3.9 shows

$$A(w; \psi) = w(D) - \frac{w(h)}{p^e - 1} = \frac{v(D) - v(h)}{p^e - 1} = \left(\sum_i v(D_i)(p^e - 1 - c_i) \right).$$

Suppose we can find $f \in L$ such that $\psi(f)$ is a unit in $\mathcal{O}_{Y,y}$ and $E(f, \psi, v) \geq A(w; \psi)$; from this it follows $A(v; \psi) \geq A(w; \psi)$. The choice for f is clear: $f = (\prod_{j=1}^{n-s} g_j^{p^e-1})(\prod_{i=1}^s f_i^{p^e-1-c_i})$. By construction,

$$\psi(f) = \Phi_y^e(u(f_1 \cdots f_s g_1 \cdots g_{n-s})^{(p^e-1)}) \in \mathcal{O}_{Y,y}^\times.$$

Recall that to get h to be monomial, we may have had to multiply Φ_y^e by some $u \in \mathcal{O}_{Y,y}^\times$, so $\psi(f)$ may not be 1. We also see:

$$\begin{aligned}(p^e - 1)E(f, \psi, v) &= \left(\sum_{j=1}^{n-s} (p^e - 1)v(g_i) \right) + \left(\sum_{j=1} (p^e - 1 - c_j)v(f_i) \right) \\ &\geq (p^e - 1)A(w; \psi)\end{aligned}$$

since $v(g_j) > 0$ for all j. We conclude that (8) is true, which completes the proof. ∎

5. Connections with *F*-singularities

In this section, we briefly explore the relationship between our log discrepancies, sharp F-purity, and strong F-regularity. These results are of independent interest, and are also important in Section 7 to prove, e.g., that asymptotic multiplier ideals of graded sequences on strongly F-regular schemes are coherent.

We prove that sharply F-pure and strongly F-regular Cartier subalgebras are characterized as non-negativity (resp. positivity) of log discrepancies on $\beth$-spaces. This builds on the heuristic correspondence between sharply F-pure and log canonical singularities (resp. strongly F-regular and klt). In particular, our result greatly generalizes Hara and Watanabe's theorem [26, Theorem 3.3].

Throughout, we fix a Cartier subalgebra $\mathcal{D} \subseteq \mathcal{C}_X$ on an integral F-finite scheme X. To avoid trivialities, we assume $\mathcal{D}_e \neq 0$ for some $e > 0$.

Definition 5.1. The *splitting prime* of $\mathcal{D}$ at $x \in X$ is the ideal

$$\mathcal{P}(\mathcal{D}_x) = \{ f \in \mathcal{O}_{X,x} \,:\, \psi(f) \in \mathfrak{m} \text{ for all } \psi \in (\mathcal{D}_e)_x, \text{ for all } e \geq 1 \}.$$

Standard facts about $\mathcal{P} = \mathcal{P}(\mathcal{D}_x)$ include (see [1, 6]):

1. $\mathcal{P} \neq \mathcal{O}_X$ if and only if $\mathcal{D}$ is sharply F-pure at x.
2. As suggested by the name, $\mathcal{P}$ is prime whenever it is proper.
3. When $\mathcal{P}$ is proper, no prime $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathcal{P} \subsetneq \mathfrak{p}$ can be $\mathcal{D}_x$-compatible. In particular, $\mathcal{P} = 0$ if and only if $\mathcal{D}$ is strongly F-regular at x.
4. The restriction $\mathcal{D}_x\|_{\mathcal{P}}$ is strongly F-regular if $\mathcal{D}$ is sharply F-pure at x.

Lemma 5.2. *Let $x \in X$. Denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by* triv_x *the trivial valuation $\kappa(x)^\times \to \{0\}$. There are three possible values for* $A(\operatorname{triv}_x, \mathcal{D})$*:*

$$A(\operatorname{triv}_x, \mathcal{D}) = \begin{cases} -\infty & \textit{iff } \mathcal{D}_x \textit{ is not sharply } F\textit{-pure.} \\ 0 & \textit{iff } \mathcal{D}_x \textit{ is sharply } F\textit{-pure and } \mathcal{P}(\mathcal{D}_x) = \mathfrak{m}_x. \\ +\infty & \textit{iff } \mathcal{D}_x \textit{ is sharply } F\textit{-pure and } \mathcal{P}(\mathcal{D}_x) \neq \mathfrak{m}_x. \end{cases}$$

Proof. Restricting our attention to $R := \mathcal{O}_{X,x}$, let us write $\mathcal{D}$ for $\mathcal{D}_x$, $X = \operatorname{Spec}(R)$, and x for the closed set $\{x\} \subset X$.

Suppose $\mathcal{D}$ is not sharply F-pure. Then no $\psi \in \mathcal{D}_e$ is surjective for $e > 0$, so in particular $\psi(\mathfrak{m}_x) \subseteq \mathfrak{m}_x$. Thus, x is uniformly $\mathcal{D}$-compatible, and $\psi(R) \subseteq$

$\mathfrak{m}_x$ for all $\psi \in \mathcal{D}_{>0}$ implies the exceptional restriction $\mathcal{D}\|_x$ is zero. But then

$$A(\mathrm{triv}_x; \mathcal{D}) = A(\mathrm{triv}_x; 0) = \sup \varnothing = -\infty.$$

If $\mathcal{D}$ is sharply F-pure and $\mathcal{P}(\mathcal{D}) = \mathfrak{m}_x$, then x is again $\mathcal{D}$-compatible. In this case, one has non-zero elements of $\mathcal{D}\|_x$ in positive degrees, corresponding to surjective maps $\psi \in \mathcal{D}_e$. If $\psi\|_x \neq 0$, then $A(\mathrm{triv}_x; \psi\|_x) = 0$. Thus, $A(\mathrm{triv}_x; \mathcal{D}) = \sup\{0\} = 0$.

In the last case, we see x is not uniformly $\mathcal{D}$-compatible, and we defined $A(\mathrm{triv}_x; \mathcal{D}) = +\infty$. □

The center map admits a section $\mathrm{triv} : X \to X^{\beth}$, sending $x \in X$ to triv_x. Let us write X^{triv} for the image of this map and triv_X for the image of the generic point of X. Let $X^{\beth,*} = X^{\beth} \setminus \{\mathrm{triv}_X\}$, and $X^{\mathrm{triv},*} = X^{\mathrm{triv}} \cap X^{\beth,*}$. Our main theorem in this section is:

Theorem 5.3. *Let X be an F-finite integral scheme, and let $\mathcal{D}$ be a Cartier subalgebra on X. Then $\mathcal{D}$ is:*

1. *sharply F-pure if and only if $A(\zeta; \mathcal{D}) \geq 0$ for all $\zeta \in X^{\beth}$.*
2. *strongly F-regular if and only if $A(\zeta; \mathcal{D}) > 0$ for all $\zeta \in X^{\beth,*}$.*

Moreover, it suffices to check these statements on X^{triv} and $X^{\mathrm{triv},*}$, respectively.

Proof. Sharp F-purity, and strong F-regularity, are conditions $\mathcal{D}$ must satisfy at each point of X, and $A(\zeta; \mathcal{D})$ depends only on $\phi \in \mathcal{D}_{c(\zeta)}$. Therefore, we assume $X = \mathrm{Spec}(R)$, R is local with maximal ideal $\mathfrak{m}$, and $c_X(\zeta) = \mathfrak{m}$. Simplifying notation, let $\mathcal{D} = \mathcal{D}_{\mathfrak{m}}$.

Suppose first that $\mathcal{D}$ is sharply F-pure and let $\psi \in \mathcal{D}_e$ be surjective with $\psi(f) = 1$. Then for all $\zeta \in c_X^{-1}(\mathfrak{m})$, we have

$$A(\zeta; \mathcal{D}) \geq E(f, \psi, \zeta) \geq 0.$$

On the other hand, if $\mathcal{D}$ is not sharply F-pure, then (5.2) shows $A(\mathrm{triv}_{\mathfrak{m}}; \mathcal{D}) = -\infty$, so $A(\zeta; \mathcal{D}) < 0$ for some $\zeta \in X^{\beth}$.

Suppose then $\mathcal{D}$ is strongly F-regular, and let $\zeta \in X^{\beth,*}$ with $c_X(\zeta) = \mathfrak{m}$. If $h_X(\zeta) \neq (0)$, then $h_X(\zeta)$ is not uniformly $\mathcal{D}$-compatible, and we have defined $A(\zeta; \mathcal{D}) = +\infty > 0$. If $h_X(\zeta) = (0)$, meaning $\zeta \in \mathrm{Val}_X$, take $f \in \mathfrak{m}$ and $\psi \in \mathcal{D}_e$ with $\psi(f) = 1$. Then $A(\zeta; \mathcal{D}) \geq E(f, \psi, v) > 0$. Contrapositively, suppose $\mathcal{D}$ is not strongly F-regular. If $\mathcal{D}$ is not even sharply F-pure, then the previous case shows $A(\mathrm{triv}_{\mathfrak{m}}; \mathcal{D}) = -\infty \leq 0$. We therefore assume $\mathcal{D}$ is sharply F-pure. Then $\mathcal{P}(\mathcal{D}) =: \mathfrak{p}$ is a nonzero prime ideal of R, and $\mathcal{D}_{\mathfrak{p}}$ is a sharply F-pure Cartier subalgebra on $R_{\mathfrak{p}}$ with $\mathcal{P}(\mathcal{D}_{\mathfrak{p}}) = \mathfrak{p}R_{\mathfrak{p}}$. We have seen $A(\mathrm{triv}_{\mathfrak{p}}; \mathcal{D}) = 0$.

To complete the proof, we note that in both cases, the points of X^{triv} gave semivaluations with negative (resp. non-positive) log discrepancy. ☐

Corollary 5.4 (cf. [26])**.** *Let X be an integral F-finite scheme.*

1. *If X is F-pure, then $A(E; \mathcal{C}^X) \geq 0$ for all divisors E over X.*
2. *If X is F-regular, then $A(E; \mathcal{C}^X) > 0$ for all divisors E over X.*

Question 5.4.1. Suppose $\mathcal{D}$ is sharply F-pure but not strongly F-regular at $x \in X$. Does there exist a non-trivial $\zeta \in X^{\beth}$ with $c_X(\zeta) = x$ and $A(\zeta; \mathcal{D}) = 0$?

Question 5.4.2. What is the relationship between the F-signature $s(D_x)$ [6] and $A_X(\zeta; \mathcal{D})$, for various $\zeta \in X^{\beth}$ centered at x? Cf. [13, Theorem 3.1]; note that the limit function in that theorem factors as $\left(\frac{1}{2}\mathrm{mld}(R, f^t)\right)^2$.

6. Lower-semicontinuity

For this section, let X be an integral scheme of characteristic $p > 0$. We show that $A(-; \mathcal{D})$ is lower-semicontinuous on $X^{\beth}$ for any Cartier subalgebra $\mathcal{D}$ on X. As a first application, we deduce that the minimal log discrepancy function derived from log discrepancies on $X^{\beth}$ is lsc on X considered with **the constructible topology**. In Section 7, we give our major applications of lower-semicontinuity of $A(-; \mathcal{D})$: coherence of asymptotic multiplier ideals, and existence of valuations calculating log canonical thresholds, on regular F-finite schemes.

Definition 6.1. Recall that a function $f : Y \to \mathbb{R}_{\pm\infty} = [-\infty, \infty]$ on a topological space Y is *lower-semicontinuous* (lsc) at $y_0 \in Y$ if $f(y_0) = -\infty$ or one of the following equivalent conditions holds:

1. For every convergent net $y_\alpha \to y_0$, $f(y_0) \leq \liminf_\alpha f(y_\alpha)$.
2. For every $\varepsilon > 0$ there exists an open neighborhood $U \subseteq Y$ of y_0 such that $f(y_0) - \varepsilon < f(y)$ for all $y \in U$;
3. If we consider $\mathbb{R}_{\pm\infty}$ with the topology whose open subsets are of the form $(a, \infty]$, $a \in \mathbb{R}$, and $[-\infty, \infty]$, then f is continuous.

Remark 6.2. Let us make some comments on topological notions used here.

1. Since $X^{\beth}$ is not generally first countable, we use nets and not sequences for questions of convergence and compactness.

2. If a topological space Y is not Hausdorff, there may be more than one limit point of a convergent net. We write $\lim y_\beta$ for the **set of limit points** of a convergent net y_β. In the case $\lim y_\beta$ consists of one point y_*, per usual we write $y_* = \lim y_\beta$.

The following lemma is a technical generalization of a classical way of producing a new lsc function from a given collection of lsc functions. The author thanks Kevin Tucker for suggesting this approach, which leads to a much simpler proof of lower-semicontinuity than the author's original.

Lemma 6.3 (cf. [25], Proposition 7.11(c))**.** *Let Y be a topological space and let $\mathcal{G}$ be a sheaf of lsc functions on Y, meaning for every open subset $V \subseteq Y$, $\mathcal{G}(V)$ is a (possibly empty) collection of $\mathbb{R}_{\pm\infty}$-valued lsc functions defined on V. Then $a(y) := \sup\{g(y) : g \in \mathcal{G}_y\}$ is lsc.*

Proof. The proof is very similar to one in [25]. We use definition 6.1(3).

Fix $r \in \mathbb{R}$ and $(r, \infty] \subseteq \mathbb{R}_{\pm\infty}$, and let $G = \bigsqcup_{y : a(y) > r} \mathcal{G}_y$. Each $g \in \mathcal{G}_y \subset G$ represents an equivalence class of lsc functions $g_U : U \to \mathbb{R}_{\pm\infty}$ on open subsets U containing y. If $r < g(y)$, then $r < g_U(y)$ for each U and g_U that g represents, so $g_U^{-1}(r, \infty]$ is a non-empty open subset of U (so also of X). Write $g \sim (U, g_U)$ to mean g is the image of g_U in $\mathcal{G}_y$. I claim that

$$a^{-1}(r, \infty] = \cup_{g \in G} \cup_{g \sim (U, g_U)} \; g_U^{-1}(r, \infty].$$

From this, it will follow that $a^{-1}(r, \infty]$ is open.

If $a(y) > r$, then $\mathcal{G}_y \subseteq G$ and there exists some $g \in \mathcal{G}_y$ with $g(y) > r$. Therefore, $y \in g_U^{-1}(r, \infty]$ for any g_U with $g \sim (U, g_U)$. On the other hand, if $y \in \cup_{g \in G} \cup_{g \sim (U, g_U)} g_U^{-1}(r, \infty]$, then $y \in g_U^{-1}(r, \infty]$ for some $(U, g_U) \sim g \in \mathcal{G}_y$ and some $U \ni y$. Thus, $r < g_U(y) \leq a(y)$, so $y \in a^{-1}(r, \infty]$. □

Theorem 6.4. *For every Cartier subalgebra $\mathcal{D}$ on an integral scheme X of positive characteristic, the log discrepancy $A(-; \mathcal{D})$ is lsc on $X^{\beth}$.*

Proof. To make the notation for preimages of sets of $\mathbb{R}_{\pm\infty}$ under $A(-; \mathcal{D})$ more sensible, let us simply write $A(\zeta)$ for $A(\zeta; \mathcal{D})$. We first reduce to the affine case. Suppose $\{U_i = \operatorname{Spec}(R_i)\}_{i=1}^N$ is an affine open cover of X. We then have a corresponding finite, compact cover $\{U_i^{\beth}\}_{i=1}^N$ of $X^{\beth}$. Suppose when we restrict A to each U_i, we have an lsc function. I claim this implies A is lsc. Indeed, fix $r \in \mathbb{R}$, and let $\mathcal{V}_i \subset U_i^{\beth}$ be $A|_{U_i^{\beth}}^{-1}(r, \infty]$, which is an open subset of $U_i^{\beth}$. Then $\mathcal{V} = \cup_i \mathcal{V}_i$ is an open subset, since $\mathcal{V} \cap U_i^{\beth} = \mathcal{V}_i$ is open for all i. By definition, $\mathcal{V} = A^{-1}(r, \infty]$. Thus, it suffices to check that A is lower-semicontinuous when $X = \operatorname{Spec}(R)$ is affine. Let $D = \oplus_e D_e = \Gamma(X, \mathcal{D})$. Each $\psi \in D_e$ has globally defined log-discrepancy $A_\psi : X^\beth \to \mathbb{R}_{\pm\infty}$, and A is built from these A_ψ as in (6.3), taking $\mathcal{G}$ to be the constant sheaf associated to $(\mathcal{U} \mapsto \sqcup_e\{A_\psi : \psi \in D_e\})$, where $\mathcal{U} \subset X^\beth$ is an open subset. Therefore, it suffices to fix $0 \neq \psi \in D_e$, for some $e > 0$, and prove A_ψ is lsc.

Recall the notation $\mathrm{ev}_g : X^\beth \to [0, \infty]$ for $(\zeta \mapsto \zeta(g))$, $g \in R$. By definition of the topology on $X^\beth$ (2.22), all ev_g are continuous. Thus, $\mathrm{ev}_v^{-1}(\mathbb{R}) = \mathrm{ev}_g^{-1}[0, \infty)$ is an open subset of $X^\beth$, and we can identify $\mathrm{ev}_g^{-1}(\mathbb{R})$ as $h_X^{-1}(\operatorname{Spec}(R_g))$. For any $f \in R$, the function $E(f, \psi, -)$ is continuous on $\mathrm{ev}_{\psi(f)}^{-1}(\mathbb{R})$, taking values in $(-\infty, \infty]$: it is the sum of the continuous functions $\frac{1}{p^e-1}\mathrm{ev}_f$ and $\frac{-p^e}{p^e-1}\mathrm{ev}_{\psi(f)}$.

Applying (3.10), we see that for $\zeta \in X^\beth$ with $\mathfrak{p} = h_X(\zeta)$,

$$A_\psi(\zeta) = \sup_{n,f}\{E(f, \psi^n, \zeta) : \mathrm{ev}_{\psi(f)}^{-1}(\mathbb{R}) \ni \zeta\}$$

We then apply (6.3) to the sheaf $\mathcal{E}$ of continuous functions

$$\Gamma(\mathcal{U}, \mathcal{E}) = \cup_{n,f}\{E(f, \psi^n, -) : \mathcal{U} \subseteq \mathrm{ev}_{\psi^n(f)}^{-1}(\mathbb{R})\} \tag{9}$$

on an open subset $\mathcal{U} \subseteq X^\beth$, noting that each $E(f, \psi^n, -)$ is continuous on any open subset $\mathcal{U}$ for which $\mathcal{U} \subseteq \mathrm{ev}_{\psi^n(f)}^{-1}(\mathbb{R})$. □

Remark 6.5. The sheaf $\mathcal{E}$ from (9) may have stalks $\mathcal{E}_\zeta$ that are empty: let $Z \subset X$ be the subset where $\mathcal{D}$ is not sharply F-pure; it is easy to see that Z is closed. For any $\zeta \in h_X^{-1}(Z)$, the stalk $\mathcal{E}_\zeta = \varnothing$ since $\zeta(\psi(f)) = +\infty$ for all $f \in R$. In this case, $A_\psi(\zeta) = -\infty$. Thus, A_ψ is automatically lsc on the closed set $h_X^{-1}(Z)$ (recall h_X is continuous).

Remark 6.6. If $A(v; \psi) \in \mathbb{R}$ for $v \in \mathrm{Val}_X$, and $\varepsilon > 0$, we can be more precise, demonstrating a basic open subset $\mathcal{U} \subseteq \mathrm{Val}_X$ with $A(v; \psi) - \varepsilon < A(w; \psi)$ for all $w \in \mathcal{U}$. Indeed, keeping notation from the proof, suppose $A(v; \psi) - \varepsilon/2 < E(f, \psi^n, \zeta)$ for some $f \in L$ with $\psi^n(f) = 1$. One can check that $\mathcal{U} = \{w \in \mathrm{Val}_X : |v(f) - w(f)| < (p^{en} - 1)\varepsilon/2\}$ has the desired property.

6.1. $\mathfrak{D}$-spaces and minimal $\beth$ log discrepancies

The remainder of this section is devoted to proving the minimal $\beth$ log discrepancy is lower-semicontinuous in the constructible topology of X. The key ingredient is 6.15, which is also applicable to the log discrepancy in characteristic zero on excellent regular schemes, or singular complex varieties. The resulting constructible lower-semicontinuity in characteristic zero greatly generalizes Ambro's theorem for complex varieties [2, Theorem 2.2]. This section is independent of the rest of the paper, and the $\mathfrak{D}(X)$-spaces introduced do not come up again. The reader may safely skip to 7, if they like.

Definition 6.7. The *constructible topology* on X is the minimal topology containing both Zariski open and Zariski closed subsets. This is equivalent to the topology with basis consisting of finite unions of locally closed subsets.

Definition 6.8. We will say that $w \in \mathrm{Val}_X$ is a *$\mathbb{Z}$-valuation* if $\mathrm{im}(w) \subseteq \mathbb{Z}$; denote this set by $\mathrm{Val}_X^{\mathbb{Z}}$. The *$\mathbb{Z}$-semivaluations on X* are the elements in the closure of $\mathrm{Val}_X^{\mathbb{Z}}$ inside $X^{\sqsupset}$, which we denote $\mathfrak{D}(X)$.

Remark 6.9. We define $\mathfrak{D}(X)$ to be the closure of $\mathrm{Val}_X^{\mathbb{Z}}$, rather than the set of all $\zeta \in X^{\sqsupset}$ with image contained in $\mathbb{Z}$, because we wish to approximate $\mathbb{Z}$-semivaluations by $\mathbb{Z}$-valuations. Our interest in $\mathbb{Z}$-valuations, as opposed to, e.g., all divisorial or discrete valuations, stems from the uninteresting nature of minimizing log discrepancies on X^{div}: log discrepancies are homogeneous for the $\mathbb{R}_{>0}$ scaling action, meaning $A(c\,\zeta;\mathcal{D}) = c\,A(\zeta;\mathcal{D})$ for $c \in (0,\infty)$, so the minimum of $A(\zeta;\mathcal{D})$ for ζ in some $\mathbb{R}_{>0}$-invariant subset $S \subseteq X^{\sqsupset}$ (e.g. X^{div}, or fibers of h_X or c_X) is either 0 or $-\infty$.

Remark 6.10. A natural basis for the topology near a given $\zeta \in \mathfrak{D}(X)$ over an open affine $\mathrm{Spec}(R) \subseteq X$ is given by finite intersections of sets of the form

$$\mathrm{ev}_{f,\mathfrak{D}}^{-1}(-\varepsilon + \zeta(f), \zeta(f) + \varepsilon) = \{\alpha \in \mathfrak{D}(\mathrm{Spec}(R)) \,:\, |\zeta(f) - \alpha(f)| < \varepsilon\}$$

or $\mathrm{ev}_{f,\mathfrak{D}}^{-1}(\varepsilon,\infty]$, where ε is a positive real number and $f \in R$; the first type of set is used in the case $\zeta(f) < \infty$ and the second when $\zeta(f) = \infty$. An important property of $\mathfrak{D}(X)$ not shared by $X^{\sqsupset}$ is that $\mathrm{ev}_f^{-1}(a,b) = \cup_{n\in(a,b)\cap\mathbb{Z}_{\geq 0}} \mathrm{ev}_f^{-1}(n)$.

Lemma 6.11. *Let $U \subseteq X$ be an open subscheme. Then $\mathfrak{D}(U) = \mathfrak{D}(X) \cap U^{\sqsupset}$*

Proof. First, note that for any open subscheme $U \subseteq X$ we have $U^{\sqsupset} = c_X^{-1}(U)$, and so in particular if $U = \mathrm{Spec}(R)$ is an affine open subscheme of X then $\mathrm{Val}_U^{\mathbb{Z}} = \mathrm{Val}_X^{\mathbb{Z}} \cap U^{\sqsupset}$, as both sets are described as $\mathbb{Z}$-valuations on the function field L of X that are non-negative on R. Since $U^{\sqsupset}$ is closed in $X^{\sqsupset}$, the closure of $\mathrm{Val}_U^{\mathbb{Z}}$ in $X^{\sqsupset}$ must be contained in $U^{\sqsupset}$, and so agrees with $\mathfrak{D}(U)$. This implies that $\mathfrak{D}(U) \subseteq \mathfrak{D}(X) \cap U^{\sqsupset}$. For the reverse inclusion, let $\zeta \in \mathfrak{D}(X) \cap U^{\sqsupset}$ and let $\{v_\alpha\} \subset \mathrm{Val}_X^{\mathbb{Z}}$ be a convergent net with limit ζ. Since $c_X(\zeta) \in U$, we know $\zeta(f) \geq 0$ for all $f \in R$, and so $v_\alpha \to \zeta$ implies that there exists α_0 such that for $\alpha \geq \alpha_0$ we have $v_\alpha(f) > -1$. Since v_α takes values only in $\mathbb{Z}$, we conclude that $v_\alpha(f) \geq 0$ for all $\alpha \geq \alpha_0$ and $f \in R$, which is to say that $\{v_\alpha\}_{\alpha\geq\alpha_0} \subset \mathrm{Val}_U^{\mathbb{Z}}$, so $\zeta \in \mathfrak{D}(U)$. ∎

It seems much more difficult to determine if $\mathfrak{D}(Y) = \mathfrak{D}(X) \cap Y^{\beth}$ when $Y \subsetneq X$ is a proper closed subscheme. In general, neither inclusion is clear to the author.

Lemma 6.12. *The center function $c_{\mathfrak{D}} = c_X|_{\mathfrak{D}} : \mathfrak{D}(X) \to X$ is continuous.*

Proof. We may assume X is affine, $X = \operatorname{Spec}(R)$. Let $f \in R$ and $\zeta \in \mathfrak{D}(X)$. Then $c_{\mathfrak{D}}(\zeta) \in \mathbb{V}(f)$ if and only if $\zeta(f) > 0$. Since $\zeta \in \mathfrak{D}(X)$, $\operatorname{im}(\zeta) \subseteq \mathbb{Z}$, and so $\zeta(f) > 0$ is equivalent to $\zeta(f) \geq 1$. Therefore, $c_{\mathfrak{D}}^{-1}(\mathbb{V}(f)) = \operatorname{ev}_f^{-1}[1, \infty] \cap \mathfrak{D}(X)$ is closed. □

Lemma 6.13. *For all $x \in X$, the fiber $c_{\mathfrak{D}}^{-1}(x)$ is closed and compact.*

Proof. Let $U = \operatorname{Spec}(R)$ be an affine neighborhood of x and suppose $x \in \operatorname{Spec}(R)$ corresponds to the prime $\mathfrak{p} \subset R$. Since $c_{\mathfrak{D}}^{-1}(x) \subseteq \mathfrak{D}(U)$, we may assume that $X = \operatorname{Spec}(R)$. Now, $c_X(\zeta) = x$ for $\zeta \in \mathfrak{D}(X)$ if and only if $\zeta(g) = 0$ for all $g \in R \setminus \mathfrak{p}$ and $\zeta(f) \geq 1$ for all $f \in \mathfrak{p}$. These are both closed conditions. □

Proposition 6.14. *The image of a basic open subset of $\mathfrak{D}(X)$ under $c_{\mathfrak{D}} = c_X|_{\mathfrak{D}(X)}$ is a finite union of (Zariski) locally closed subsets, and $c_{\mathfrak{D}}$ induces the constructible topology as its the quotient topology.*

Proof. We assume that $X = \operatorname{Spec}(R)$ and that our basic open subset is of the form $\mathfrak{U} := \cap_{i=1}^{s} \left[\cup_{j=1}^{t_i} \operatorname{ev}_{f_i}^{-1}(n_{i,j})\right]$ for some $f_i \in R$ and $n_{i,j} \in \mathbb{Z}_{\geq 0}$; the case involving $\operatorname{ev}_{f_i}^{-1}[n, \infty]$ is very similar. Suppose $\alpha \in \operatorname{ev}_{f_i}^{-1}(n)$. The condition that $\mathfrak{p} = c_{\mathfrak{D}}(\alpha)$ is equivalent to $\mathfrak{p} \in \mathbb{D}(f_i) := \mathbb{V}(f_i)^c$ when $n = 0$, and $\mathfrak{p} \in \mathbb{V}(f_i)$ when $n > 0$. For each $1 \leq i \leq s$, we re-number so that $n_{i,j} = 0$ for $1 \leq j \leq t_i'$, and $n_{i,j} > 0$ for $t_i' < j \leq t_i$. Then the image under $c_{\mathfrak{D}}$ is:

$$c_{\mathfrak{D}}(\mathfrak{U}) = \bigcap_{i=1}^{s} \left[\Big(\bigcup_{j=1}^{t_i'} \mathbb{D}(f_i) \Big) \bigcup \Big(\bigcup_{j > t_i'} \mathbb{V}(f_i) \Big), \right]$$

which is a finite union of locally closed sets, proving our first claim.

One checks directly that the primage of $c_{\mathfrak{D}}(\mathfrak{U})$ is a basic open subset of $\mathfrak{D}(X)$ (involving various $\operatorname{ev}_f^{-1}(-1/2, 1/2)$ and $\operatorname{ev}_f^{-1}(0, \infty]$). Therefore, the quotient topology on X induced by $c_{\mathfrak{D}}$ is the constructible topology. □

We use the following lemma to pass lsc functions from $\mathfrak{D}(X)$ to X by minimizing on fibers of $c_{\mathfrak{D}}$. These semicontinuity results are typically deduced for excellent schemes over $\mathbb{Q}$ using log resolutions (*e.g.* [2, Theorem 2.2]); such results become special corollaries of our lemma.

Lemma 6.15. *Let $f : Y \to Z$ be a continuous surjective function between topological spaces. Assume that Y is compact and Z is Hausdorff. Fixing $a : Y \to \mathbb{R}_{\pm\infty}$, define a function on Z by*

$$m(z) = \inf_{f(y)=z} a(y).$$

Then m is lsc on Z whenever a is lsc on Y.

Proof. If $m(z) = -\infty$ then lower-semicontinuity is automatic at z, so we fix $z_* \in Z$ with $m(z_*) > -\infty$. Suppose there exists a convergent net $z_\nu \to z_*$, indexed by a directed set $\mathcal{N}$, with $\liminf_\nu m(z_\nu) < m(z_*)$. Then for some fixed $0 < \varepsilon \ll 1$, there exists $\nu_0 \in \mathcal{N}$ with the property that for every $\nu \geq \nu_0$, there is some $\mu \geq \nu$ making

$$m(z_\mu) + \varepsilon < m(z_*).$$

We may therefore select, for each $\nu \geq \nu_0$, $w_\nu \in \{z_\mu\}_{\mu \geq \nu}$ with $m(w_\nu) < m(z_*)$; note $w_\nu \to z_*$. Set $\mathcal{N}' = (\nu \geq \nu_0) \subset \mathcal{N}$. By construction, $m(w_\nu) + \varepsilon < m(z_*)$ for all $\nu \in \mathcal{N}'$.

By definition of $m(w_\nu) = \inf_{f(y)=w_\nu} a(y)$, for each $\nu \in \mathcal{N}'$ there must exist $y_\nu \in Y$ with $f(y_\nu) = w_\nu$ and $a(y_\nu) < m(w_\nu) + \varepsilon < m(z_*)$. Compactness of Y allows us to pass to a convergent subnet $\{y_\beta\}_{\beta \in \mathcal{B}}$ of $\{y_\nu\}_{\nu \in \mathcal{N}'}$; note that $\{f(y_\beta)\}_{\mathcal{B}}$ is a sub-net of $\{f(y_\nu)\}_{\mathcal{N}'} = \{w_\nu\}_{\mathcal{N}'}$, so $z_* \in \lim_\beta f(y_\beta)$. If $y_* \in \lim_\beta y_\beta$, then $a(y_*) \leq \liminf_{\mathcal{B}} a(y_\beta) < m(z_*)$ because a is lsc on Y (6.1(1)). This gives a contradiction:

$$m(z_*) \leq a(y_*) \leq \liminf_{\mathcal{B}} a(y_\beta) < m(z_*).$$

Therefore, m must be lsc on Z. □

Definition 6.16. The *minimal* ℶ *log discrepancy* of $\mathcal{D}$ at $x \in X$ is defined to be

$$\mathrm{mld}_{\beth}(x; \mathcal{D}) = \min A(\zeta; \mathcal{D}),$$

where we minimize over $\zeta \in \mathfrak{D}(X)$ with $c_{\mathfrak{D}}(\zeta) = x$. 6.13 and 6.4 implies this minimum is achieved.

Note that if $\mathrm{mld}_{\beth}(x; \mathcal{D}) < \infty$ then any ζ achieving this minimum must have a $\mathcal{D}$-compatible home Z.

Theorem 6.17 (cf. [2])**.** *For any integral scheme X of characteristic $p > 0$, $\mathrm{mld}_{\beth}(-; \mathcal{D}) : X \to \mathbb{R}_{\pm\infty}$ is lsc in the constructible topology on X. Explicitly, for any $x \in X$ and $\varepsilon > 0$, there is a locally closed subset $G \subset X$ containing x such that $\mathrm{mld}_{\beth}(x'; \mathcal{D}) > \mathrm{mld}_{\beth}(x; \mathcal{D}) - \varepsilon$ for all $x' \in G$.*

Proof. The theorem follows directly from Theorem 6.15, since $\mathfrak{D}(X)$ is compact, X^{Constr} is Hausdorff, $c_{\mathfrak{D}} : \mathfrak{D}(X) \to X^{Constr}$ is continuous, and $A(-, \mathcal{D}) : X^{\beth} \to \mathbb{R}_{\pm\infty}$ is lsc (6.4). □

Remark 6.18. Log discrepancies on Val_X and $X^{\beth}$ are also lsc over fields of characteristic zero, so the same proof recovers Ambro's result over $\mathbb{C}$, and applies more generally [30, 8, 7].

Remark 6.19. It is a major open problem in the (characteristic zero) minimal model program to determine if the (usual) minimal log discrepancy is lsc in the *Zariski* topology on the set of closed points of a variety. Some results in this direction are known for complex varieties, cf. [22, 21].

Another natural function to consider is the appropriate version of the log canonical threshold of graded sequences of ideals with respect to strongly F-regular Cartier subalgebras. Many of the proofs found in the final sections of [30] can be adapted to our setting, and we devote §7 to this function.

7. Log canonical thresholds of graded sequences of ideals

Let us fix an integral, F-finite, strongly F-regular scheme X with fraction field L. We note that F-finiteness implies excellence. We also fix a strongly F-regular Cartier subalgebra $\mathcal{D}$ and a nonzero ideal $\mathfrak{q}$ on X. Starting with Subsection 7.3, we assume that X is regular. We skip these hypotheses when stating most lemmas, but make them explicit in theorems and some definitions for emphasis, clarity, and ease of reference. Denote by triv_X the trivial valuation $L^\times \to \{0\}$ and by $\mathrm{Val}_X^* := \mathrm{Val}_X \setminus \{\mathrm{triv}_X\}$.

In this final section, we study log canonical thresholds of graded sequences of ideals in positive characteristics, proving the positive characteristic versions of several theorems of Jonsson and Mustaţă along the way. We follow the same general strategy as [30]. There are a number of points where we must replace parts of their approach, especially those making use of log resolutions, with arguments involving p^{-e}-linear maps, and more topological arguments involving valuation spaces. An interesting outcome of this approach is we give the first proof that asymptotic multiplier ideals associated to multiplicatively graded sequences are coherent sheaves of ideals on strongly F-regular schemes (7.7). Our approach applies also to klt varities in characteristic zero, giving an alternative to the usual description involving log resolutions; see (7.8).

Convention: Unless explicitly stated otherwise, all schemes in this section are understood to have characteristic $p > 0$. Recall from §3.2 that we write $\mathfrak{a}_\star$ for (multiplicatively) graded sequences of ideals, and $\mathfrak{b}_\bullet$ for an F-graded sequence. All elements of a (multiplicatively/F-) graded sequence of ideals are assumed to be nonzero.

7.1. Log canonical thresholds of graded sequences

Any $v \in \mathrm{Val}_X^*$ defines a graded sequence of ideals, denoted here by $\mathfrak{a}_\star(v)$. On an open affine chart $\mathrm{Spec}(R) \subset X$ containing $c_X(v)$, $\Gamma(\mathrm{Spec}(R), \mathfrak{a}_s(v))$ is the ideal

$$\mathfrak{a}_s(v) = \{f \in R : v(f) \geq s\}; \quad s \in \mathbb{N}_0.$$

If $c_X(v) \notin \mathrm{Spec}(R)$, we set $\Gamma(\mathrm{Spec}(R), \mathfrak{a}_s(v)) = R$ for all $s \geq 0$.

Jonsson and Mustaţă prove the following statement showing that $w(\mathfrak{a}_\star(v))$ compares the values of w and v, asymptotically.

Lemma 7.1 (cf. Lemma 2.4 [30])**.** *Let $v \in \mathrm{Val}_X^*$ be nontrivial. Then*

$$w(\mathfrak{a}_\star(v)) = \inf \frac{w(\mathfrak{b})}{v(\mathfrak{b})}$$

for every $w \in \mathrm{Val}_X$, where $\mathfrak{b}$ ranges over ideals on X such that $v(\mathfrak{b}) > 0$.

Recall from (3.12)

$$\mathcal{D} \cdot \mathfrak{a}_\star^t = \sum_{m \geq 1} \mathcal{D} \cdot \mathfrak{a}_m^{t/m} \quad \text{and} \quad A(\zeta; \mathcal{D} \cdot \mathfrak{a}_\star^t) = \sup_m A(\zeta; \mathcal{D} \cdot \mathfrak{a}_m^{t/m}).$$

We now introduce the central topic of study in this subsection, log canonical thresholds of $X, \mathcal{D}$, and $\mathfrak{a}_\star$ with respect to the nonzero ideal $\mathfrak{q}$ on X; recall our standing assumptions on X, $\mathcal{D}$, and $\mathfrak{a}_\star$.

Definition 7.2 (Log canonical threshold of graded sequence, cf. [30, 8])**.** Let $v \in \mathrm{Val}_X^*$ with $v(\mathfrak{a}_\star) > 0$ and $A(v; \mathcal{D}) < \infty$. The *log canonical threshold* of $\mathfrak{a}_\star$ with respect to $\mathfrak{q}$, $\mathcal{D}$, and $v \in \mathrm{Val}_X^*$ is

$$\mathrm{lct}^{\mathfrak{q}}(v; \mathcal{D}, \mathfrak{a}_\star) = \frac{A(v; \mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)}.$$

When $v(\mathfrak{a}_\star) = 0$, or $A(v; \mathcal{D}) = +\infty$, we define $\mathrm{lct}^{\mathfrak{q}}(v; \mathcal{D}, \mathfrak{a}_\star) = +\infty$. The *log canonical threshold* of $\mathfrak{a}_\star$ with respect to $\mathfrak{q}$ and $\mathcal{D}$ is

$$\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) = \inf_{v \in \mathrm{Val}_X^*} \mathrm{lct}^{\mathfrak{q}}(v; \mathcal{D}, \mathfrak{a}_\star).$$

Remark 7.3. If $v \in \mathrm{Val}_X^*$ has $A(v; \mathcal{D}) < \infty$ and $0 < v(\mathfrak{a}_\star)$, then (3.16) implies

$$\mathrm{lct}^{\mathfrak{q}}(v; \mathcal{D}, \mathfrak{a}_\star) = \sup\{t \geq 0 : v(\mathfrak{q}) + A(v; \mathcal{D} \cdot \mathfrak{a}_\star^t) > 0\}, \tag{10}$$

which is equivalent to

(11) $$\operatorname{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) = \sup\{t \geq 0 \,:\, v(\mathfrak{q}) + A(v;\mathcal{D}\cdot\mathfrak{a}_m^{t/m}) > 0 \text{ for some } m\}.$$

These expressions are quite closely related to the expressions for log discrepancies in terms of sub-canonical divisors in [18]. We note that (11) implies that $\operatorname{lct}^{\mathfrak{q}}(-;\mathcal{D},\mathfrak{a}_\star)$ is lower-semicontinuous on Val_X^*.

Remark 7.4. In previous versions of this article, we introduced log canonical thresholds with respect to semivaluations $\zeta \in X^{\beth}$ with $\zeta(\mathfrak{q}) < \infty$, taking (10) for the defintion. We only treated this on Val_X^* in any depth, and the extension to $X^{\beth}$ we proposed was very technical, so have chosen to focus on Val_X^* in revisions.

7.2. Asymptotic multiplier ideals of Cartier subalgebras

We define sheaves of ideals $\mathcal{J}(\mathcal{D}\cdot\mathfrak{a}_\star^t)$ containing information about values, and in particular minima, of $A(-;\mathcal{D}\cdot\mathfrak{a}_\star^t)$. We model our definition on the valuation-theoretic description of asymptotic multiplier ideals, see e.g. [35, 30, 8].

Definition 7.5 (Asymptotic multiplier ideal of $(\mathcal{D}\cdot\mathfrak{a}_\star^t)$)**.** Consider X, $\mathcal{D}$, and $\mathfrak{a}_\star$ as before. For $t \in \mathbb{R}_{\geq 0}$ and an affine open $U = \mathrm{Spec}(R) \subseteq X$, the *asymptotic multiplier ideal* of $(\mathcal{D}\cdot\mathfrak{a}_\star^t)$ on U is

$$\begin{aligned}\Gamma(U,\mathcal{J}(\mathcal{D}\cdot\mathfrak{a}_\star^t)) &= \bigcap_{v\in\mathrm{Val}_U^*}\{f\in R \,:\, v(f) + A(v;\mathcal{D}\cdot\mathfrak{a}_\star^t) > 0\} \\ &= \bigcap_{v\in\mathrm{Val}_U^*}\{f\in R \,:\, v(f) + A(v;\mathcal{D}\cdot\mathfrak{a}_m^{t/m}) > 0 \text{ for some } m\geq 1\}.\end{aligned}$$

We now prove that this sheaf of abelian groups gives a coherent sheaf of ideals. The following lemma is used to make compactness arguments several times throughout this section. Compactness statements of this form are well-known to experts, as is their usefulness when applying valuation spaces to the study of multiplier ideals; cf. [30, Proposition 5.9] [8, Theorem 3.1(c)]. We choose to re-state the hypotheses on X and $\mathcal{D}$, since this lemma is cited frequently (and in the introduction).

Lemma 7.6 (cf. [30, 8])**.** *Let X be an integral, F-finite, strongly F-regular scheme, $\mathcal{D}$ a strongly F-regular Cartier subalgebra on X, and $\mathfrak{a} \neq 0$ an ideal on X. For any $t \in [0,\infty)$, the set*

$$\mathcal{V}_t := \{\zeta \in X^{\beth} \,:\, \zeta(\mathfrak{a}) = 1, \quad A(\zeta;\mathcal{D}) \leq t\}$$

is a compact subset of Val_X^*.

Proof. This set is closed, since $(\mathfrak{a} \mapsto \zeta(\mathfrak{a}))$ is continuous on $X^{\sqsupset}$ and $A(-;\mathcal{D})$ is lower-semicontinuous; thus, $\mathcal{V}_t$ is compact since $X^{\sqsupset}$ is. Note also that $\mathcal{V}_t \subseteq \mathrm{Val}_X^*$, since strong F-regularity of $\mathcal{D}$ implies $A(\zeta;\mathcal{D}) = +\infty$ for every $\zeta \in X^{\sqsupset}$ with $h_X(\zeta) \neq \eta_X$, and $\zeta(\mathfrak{a}) = 1$ implies that $\zeta \neq \mathrm{triv}_X$. Thus, $\mathcal{V}_t$ is a compact subset of $X^{\sqsupset}$ contained in Val_X^*. □

Theorem 7.7. *Suppose X is an integral, F-finite, strongly F-regular scheme and $\mathcal{D}$ is a strongly F-regular Cartier subalgebra on X. The multiplier ideal $\mathcal{J}(\mathcal{D}\cdot\mathfrak{a}_\star^t)$ is a coherent sheaf for every graded sequence of ideals $\mathfrak{a}_\star$ on X and every $t \in [0,\infty)$.*

Proof. Coherence is a property on each affine chart of X, so we reduce to the case $X = \mathrm{Spec}(R)$, and only must check that the multiplier ideal is preserved by localizing at a single element of R. If $t < \mathrm{lct}(X;\mathcal{D},\mathfrak{a}_\star)$ then the multiplier ideals in question are $\mathcal{O}_X$, so we assume $\mathrm{lct}(X;\mathcal{D},\mathfrak{a}_\star) < \infty$, and $t \geq \mathrm{lct}(X;\mathcal{D},\mathfrak{a}_\star)$.

I claim that it is enough to show the theorem when $\mathfrak{a}_\star$ is a constant sequence, meaning $\mathfrak{a}_s = \mathfrak{a}$ for all $s \geq 1$ and some fixed $\mathfrak{a} \subseteq \mathcal{O}_X$. To ease notation in proving this claim, for localizations R_g of R we define $\mathcal{J}_\star(R_g) = \Gamma(\mathrm{Spec}(R_g), \mathcal{J}(\mathcal{D}\cdot\mathfrak{a}_\star^t))$ (resp. $\mathcal{J}_m(R_g) = \Gamma(\mathrm{Spec}(R_g), \mathcal{J}(\mathcal{D}\cdot\mathfrak{a}_m^{t/m}))$. If $\mathcal{J}(R_g)_m = \mathcal{J}(R)_m R_g$ for all $m \geq 1$, then

$$\mathcal{J}(R)_\star R_g = \left(\sum_m \mathcal{J}(R)_m\right) R_g = \sum_m (\mathcal{J}(R)_m R_g) = \sum_m \mathcal{J}(R_g)_m = \mathcal{J}(R_g)_\star.$$

Thus, we simplify the setting and notation, writing $\mathcal{J}(R_g)$ for the ideal associated to $\mathcal{E} := \mathcal{D}\cdot\mathfrak{a}^t$ as above. We must prove that $\mathcal{J}(R_g) = \mathcal{J}(R)R_g$ for every $g \in R$. Of course, if $g \in \sqrt{\mathcal{J}(R)}$, then $R_g = \mathcal{J}(R)R_g \subseteq \mathcal{J}(R_g) \subseteq R_g$ since $\mathcal{J}(-)$ is a sheaf on X. Thus, we fix $g \in R \setminus \sqrt{\mathcal{J}(R)}$ and set $U := \mathrm{Spec}(R_g)$.

We wish to show that $y \notin \mathcal{J}(R)R_g$ implies $y \notin \mathcal{J}(R_g)$, and it is enough to check this for $y \in R$. We do this by showing that there exists $w \in \mathrm{Val}_X^*$ such that $w(g) = 0$ and $w(y) + A(w;\mathcal{E}) \leq 0$, which implies $w \in \mathrm{Val}_U^*$ and $y \notin \mathcal{J}(R_g)$. Since $y \notin \mathcal{J}(R)R_g$, we know that $g^n \notin (\mathcal{J}(R) : y)$ for any $n \geq 0$, so by definition for each $n \geq 0$ there exists some $w_n \in \mathrm{Val}_X^*$ such that

$$(\dagger_n) \qquad w_n(g^n y) + A(w_n;\mathcal{E}) \leq 0.$$

Being a sum of a continuous and lower-semicontinuous function,

$$w \mapsto (w(g^n y) + A(w;\mathcal{E})) : \mathrm{Val}_X^* \to \mathbb{R} \cup \{+\infty\}$$

is also lower-semicontinuous. Thus, if we denote by $\mathcal{W}_n$ the set of all $w \in \mathrm{Val}_X^*$ satisfying $(\dagger_n)$, each $\mathcal{W}_n$ is a closed subset of Val_X^*. Because $w \in \mathcal{W}_n$ are centered on X, $w(g^{n-1}y) \leq w(g^n y)$, so $\mathcal{W}_n \subseteq \mathcal{W}_{n-1}$ for all $n \geq 1$.

Note also that if w satisfies $(\dagger_n)$ then so does βw for all $\beta \in \mathbb{R}_{>0}$, so $\mathbb{R}_{>0} \cdot \mathcal{W}_n = \mathcal{W}_n$ for each n. We assumed that $\mathcal{D}$ is strongly F-regular, so $A(w;\mathcal{D}) > 0$ for every $w \in \mathrm{Val}_X^*$. This implies that if $w \in \mathcal{W}_0$, then $w(\mathfrak{a}) > 0$: indeed, by $(\dagger_n)$ with $n = 0$, we have

$$0 < A(w;\mathcal{D}) \leq t\, w(\mathfrak{a}) - w(y) \leq t\, w(\mathfrak{a}), \tag{12}$$

Therefore, there exists an $\mathbb{R}_{>0}$ multiple of w with $w(\mathfrak{a}) = 1$. Now considering $w \in \mathcal{W}_0$ with $w(\mathfrak{a}) = 1$, (12) tells us $A(w;\mathcal{D}) \leq t$. It follows that $\mathcal{W}_n \cap \mathcal{V}_t$ is non-empty for all $n \geq 0$, where

$$\mathcal{V}_t := \{\zeta \in X^{\beth} : \zeta(\mathfrak{a}) = 1, \quad A(\zeta;\mathcal{D}) \leq t\} \subseteq \mathrm{Val}_X^*, \text{ as in (7.6).}$$

Being the intersection of two non-empty compact sets, $\widetilde{\mathcal{W}_n} := \mathcal{W}_n \cap \mathcal{V}_t$ is also a non-empty compact subset of Val_X^*. A descending chain of non-empty compact subsets has non-empty intersection, hence there exists $w_\infty \in \cap_n \widetilde{\mathcal{W}_n}$. By construction, w_∞ satisfies the inequalities $(\dagger_n)$ for all $n \geq 1$, which is impossible if $w_\infty(g) > 0$. Thus, $w_\infty(g) = 0$, or equivalently $c_X(w_\infty) \in U$. □

Remark 7.8. As mentioned in the introduction, our argument above can be used to prove that asymptotic multiplier ideals are coherent whenever one has a statement such as 7.6.

For example, suppose X is a normal variety over $\mathbb{C}$, or is regular and excellent over $\mathbb{Q}$. Take a graded sequence of ideals $\mathfrak{a}_\star$ on X and suppose $N \subseteq X$ is a closed, proper subscheme containing the singular locus of X and the support of $\mathfrak{a}_1$ (hence of all $\mathfrak{a}_s$). We may reduce, as above, to the case that all $\mathfrak{a}_s = \mathfrak{a} \subseteq \mathcal{O}_X$, and following that argument can produce the closed subsets $\mathcal{W}_n$. Then [30, Proposition 5.9] and [8, Theorem 3.1(iii)] prove

$$\mathcal{V}_t = \{v \in \mathrm{Val}_X^* : A_X(v) \leq t \ \text{ and } \ v(I_N) = 1\}$$

is a compact subset for all t, so $0 < v(\mathfrak{a}) \leq v(I_N)$ implies that $\mathcal{W}_n \cap \mathcal{V}_t$ is non-empty for all n. Thus, our argument above proves that $\mathcal{J}(X, \mathfrak{a}_\star^t)$ is coherent. We remark, however, that this compactness requires simplicial decompositions of Val_X, in [30], and $\{v : v(I_N) = 1\}$ in [8]. These are provided by log resolutions.

Question 7.8.1. Suppose X is a connected normal klt variety of positive characteristic, in the sense that $A(\mathrm{ord}_E;\mathcal{C}_X) > 0$ for every divisor $E \subset Y$ on a normal variety admitting a proper birational morphism $Y \to X$. Are the sets $\mathcal{V}_t \subseteq X^{\beth}$ defined as in (7.6), with $\mathcal{D} = \mathcal{C}_X$, contained in Val_X^*? Note that while $\mathcal{V}_t$ is always a compact subset of $X^{\beth}$, by lower-semicontinuity of $A(-;\mathcal{C}_X)$, there are now uniformly $\mathcal{C}_X$-compatible subschemes when X is not strongly F-regular.

7.3. The conjectures of Jonsson and Mustaţă

For the remainder of this section, **we assume X is a regular F-finite scheme**. Suppose $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) < \infty$. Our first goal is to prove that there exist valuations $v \in \mathrm{Val}_X^*$ with $\mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D}, \mathfrak{a}_\star) = \mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star)$; we say any such v *computes* this lct. We then prove the implications between the conjectures numbered 7.4 and 7.5 in [30] hold also in our setting. These theorems ultimately reduce to affine-local considerations, and so we assume $X = \mathrm{Spec}(R)$. To simplify notation, we write $\mathcal{J}_t$ for $\Gamma(X, \mathcal{J}(\mathcal{D} \cdot \mathfrak{a}_\star^t))$.

If $v \in \mathrm{Val}_X^*$ computes $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) =: \lambda < \infty$, I claim $c_X(v) \in \mathbb{V}(\mathcal{J}_\lambda : \mathfrak{q})$, cf. the beginning of [30, §7]. Indeed, if v computes $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star)$, then $v(\mathfrak{q}) + A(v;\mathcal{D} \cdot \mathfrak{a}_\star^\lambda) = 0$. Now consider $f \in (\mathcal{J}_\lambda : \mathfrak{q})$. Since $f\mathfrak{q} \subseteq \mathcal{J}_\lambda$, the definition of $\mathcal{J}_\lambda$ forces $(v(f) + 0) = (v(f) + v(\mathfrak{q}) + A(v;\mathcal{D} \cdot \mathfrak{a}_\star^\lambda)) > 0$. Therefore, $v(f) > 0$ and $c_X(v) \in \mathbb{V}(\mathcal{J}_\lambda : \mathfrak{q})$.

The following lemmas allow us to pass Abhyankar valuations on an F-finite regular scheme to an appropriate affine space via completion. The proofs for these lemmas given in [30] are independent of characteristic.

Lemma 7.9 (cf. [30, Lemma 3.10])**.** *Let $\mathfrak{m} \in X$ and consider the completion morphism $X' = \mathrm{Spec}(R') \to X$, where $R' = \widehat{\mathcal{O}_{X,\mathfrak{m}}}$. If $v' \in \mathrm{Val}_{X'}$ is centered at the closed point and $v = v'|_X$, then $\kappa(v) = \kappa(v')$ and the value groups of v and v' are equal. In particular,* $\mathrm{tr.deg}_{X'}(v') = \mathrm{tr.deg}_X(v)$ *and* $\mathrm{ratrk}(v') = \mathrm{ratrk}(v)$.

Lemma 7.10 (cf. [30, Lemma 3.11])**.** *Let $k \subseteq K$ be an algebraic field extension and $\phi : \mathbb{A}_K^n \to \mathbb{A}_k^n$ the corresponding morphism of affine spaces. Suppose that v' is a valuation on $K(x_1, \ldots, x_n)$ with center $0 \in \mathbb{A}_K^n$, and let v be its restriction to $k(x_1, \ldots, x_n)$. Then v is centered at $0 \in \mathbb{A}_k^n$,* $\mathrm{tr.deg}_{\mathbb{A}_K^n}(v') = \mathrm{tr.deg}_{\mathbb{A}_k^n}(v)$, *and* $\mathrm{ratrk}(v') = \mathrm{ratrk}(v)$.

Jonsson and Mustaţă prove the next lemma for the more general setting of a regular morphism, but the only case we ever apply it to here is that of completion of local rings of X, so content ourselves to prove it in this case. In this subsection, we have assumed X is regular, hence Gorenstein, so $\mathcal{D} \subseteq \mathcal{C}^X$ is equal to $\mathcal{C}^X \cdot \mathfrak{b}_\bullet$ for some F-graded sequence of ideals. Indeed, $\mathcal{C}_e^X \cong \mathrm{Hom}_R(F_*^e R, R)$ with its right R-module structure (i.e. the $F_*^e R$-module structure) is a canonical module for R, thus isomorphic to R.

Lemma 7.11. *Let* $\mathfrak{m} \in X$ *and let* $X' = \operatorname{Spec}(R') \to X$ *be the completion morphism at* $\mathfrak{m}$. *Set* $\mathfrak{a}'_\star = \{\mathfrak{a}_s R'\}_{s\geq 1}$ *and* $\mathfrak{b}'_\bullet = \{\mathfrak{b}_e R'\}_{e\geq 0}$, *and extend* $\mathcal{D} = \mathcal{C}_X \cdot \mathfrak{b}_\bullet$ *to* $\mathcal{D}' = \mathcal{C}_{X'} \cdot \mathfrak{b}'_\bullet$. *Suppose* $\lambda := \operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) < \infty$. *Let* $\mathcal{H}_\lambda = \mathcal{J}(X'; \mathcal{D}' \cdot (\mathfrak{a}'_\star)^\lambda)$.

1. *For any* $v' \in \operatorname{Val}_{X'}$ *and* $t \geq 0$, *we have* $A(v'|_R; \mathcal{D} \cdot \mathfrak{a}_\star^t) = A(v'; \mathcal{D}' \cdot \mathfrak{a}'^t_\star)$.
2. $\operatorname{lct}^{\mathfrak{q}R'}(\mathcal{D}', \mathfrak{a}'_\star) \geq \lambda$ *with equality when* $\mathfrak{m}$ *is a minimal prime of* $(\mathcal{J}_\lambda : \mathfrak{q})$.
3. *If* $\mathfrak{m}$ *is a minimal prime of* $(\mathcal{J}_\lambda : \mathfrak{q})$, *then* $\sqrt{(\mathcal{J}_\lambda :_R \mathfrak{q})R'} = \sqrt{(\mathcal{H}_\lambda :_{R'} \mathfrak{q}R')}$.

Proof. We start with (1). Fix $v' \in \operatorname{Val}_{X'}$ and let $v = v'|_R$. By definition, $v_F(\mathfrak{b}_\bullet) = v'_F(\mathfrak{b}'_\bullet)$ and $v(\mathfrak{a}_\star) = v'(\mathfrak{a}'_\star)$. Since X is F-finite and R' is a faithfully flat R-algebra, $R' \otimes_R (\mathcal{C}_X)_e \cong (\mathcal{C}_{X'})_e$ for all $e \geq 1$, see (1). A direct calculation, using (3.16), shows that $A(v; \mathcal{D} \cdot \mathfrak{a}_\star^t) = A(v'; \mathcal{D}' \cdot \mathfrak{a}'^t_\star)$.

Moving on to (2) and (3), we just proved $\operatorname{lct}^{\mathfrak{q}R'}(v'; \mathcal{D}', \mathfrak{a}'_\star) = \operatorname{lct}^{\mathfrak{q}}(v'|_R; \mathcal{D}, \mathfrak{a}_\star)$ for all $v' \in \operatorname{Val}^*_{X'}$. Taking infima over all $v' \in \operatorname{Val}^*_{X'}$ and $v \in \operatorname{Val}_X$, we arrive at the inequality claimed in (2).

Now assume $\mathfrak{m}$ is a minimal prime of $(\mathcal{J}_\lambda : \mathfrak{q})$, so $\sqrt{(\mathcal{J}_\lambda : \mathfrak{q})R_\mathfrak{m}} = \mathfrak{m}R_\mathfrak{m}$. Since $R \to R'$ is flat,

$$(\mathcal{J}_\lambda : \mathfrak{q})R' = (\mathcal{J}_\lambda R' : \mathfrak{q}R'), \text{ hence } \sqrt{(\mathcal{J}_\lambda R' : \mathfrak{q}R')} = \mathfrak{m}R'.$$

Note also that $\mathcal{J}_\lambda R' \subseteq \mathcal{H}_\lambda$: if $f \in \mathcal{J}_\lambda$, then $v(f) + A(v; \mathcal{D} \cdot \mathfrak{a}_\star^\lambda) > 0$ for all $v \in \operatorname{Val}^*_X$. In particular, this is true for all v of the form $v'|_R$, $v' \in \operatorname{Val}^*_{X'}$, so $f \in \mathcal{H}_\lambda$. Therefore, to show equality of radicals, it suffices to prove $1 \notin (\mathcal{H}_\lambda : \mathfrak{q}R')$. This is easily seen to be equivalent to $\lambda \geq \operatorname{lct}^{\mathfrak{q}R'}(\mathcal{D}', (\mathfrak{a}'_\star))$, so we reduce to proving the equality of log canonical thresholds in (2).

The key observation is that any $v \in \operatorname{Val}_X$ with $c_X(v) = \mathfrak{m}$ has an extension, by $\mathfrak{m}$-adic continuity, to $\hat{v} \in \operatorname{Val}_{X'}$. A corollary is that $(v' \mapsto v'|_R)$ gives a surjection

$$(\operatorname{Val}_{X'} \cap c_{X'}^{-1}(\mathfrak{m}R')) \to (\operatorname{Val}_X \cap c_X^{-1}(\mathfrak{m})).$$

Since $1 \notin (\mathcal{J}_\lambda : \mathfrak{q})R_\mathfrak{m}$, and the radical of this colon is $\mathfrak{m}R_\mathfrak{m}$, we know $1 \in (\mathcal{J}_\lambda : \mathfrak{q})R_\mathfrak{p}$ for all $\mathfrak{p} \subsetneq \mathfrak{m}$. It must thus be the case that $v(\mathfrak{q}) + A(v; \mathcal{D} \cdot \mathfrak{a}_\star^\lambda) \leq 0$ for some $v \in \operatorname{Val}_X \cap c_X^{-1}(\mathfrak{m})$. But then $\hat{v}(\mathfrak{q}) + A(\hat{v}; \mathcal{D} \cdot \mathfrak{a}_\star^\lambda) \leq 0$, where $\hat{v}$ is the continuous extension of v to R'. This implies $1 \notin (\mathcal{H}_\lambda : \mathfrak{q}R')$. ☐

Remark 7.12. Jonsson and Mustaţă show that in fact $\mathcal{H}_t = \mathcal{J}_t R'$ for all $t \geq 0$, but their proof uses that one can base change log resolutions (used to compute $\mathcal{J}_t$) along regular morphisms. Lacking this technique for computing $\mathcal{J}_t$, we do not know if this holds in our setting.

Theorem 7.13 (cf. [30, Theorem 7.8])**.** *Let* $v \in \operatorname{Val}^*_X$ *with* $A(v; \mathcal{D}) < \infty$. *The following assertions are equivalent:*

1. *There exists* $\mathfrak{a}_\star$ *on* X *such that* v *computes* $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) < \infty$.
2. *For every* $w \in \operatorname{Val}_X^*$ *with* $w(\mathfrak{a}) \geq v(\mathfrak{a})$ *for every ideal* $\mathfrak{a} \subset \mathcal{O}_X$, $A(w; \mathcal{D}) + w(\mathfrak{q}) \geq A(v; \mathcal{D}) + v(\mathfrak{q})$.
3. *The valuation* v *computes* $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star(v)) < \infty$.

Our proof is very similar to the one found in [30]. Notice, however, that we do not have part (ii) of their theorem, since the multiplier ideals $\mathcal{J}_t$ are not known to be subadditive. A proof of subadditivity in characteristic $p > 0$ would have to be quite different from the characteristic zero case, since Kawamata-Viehweg Vanishing is used in the proof, which is known to be **false** in positive characteristics, see [44, 55].

Proof. Certainly (3) implies (1). We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3). For the first implication, suppose v computes $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) < \infty$, so that in particular $v(\mathfrak{a}_\star) > 0$. If $w(\mathfrak{a}) \geq v(\mathfrak{a})$ for all ideals $\mathfrak{a}$ on X, then (7.1) shows $w(\mathfrak{a}_\star) \geq v(\mathfrak{a}_\star)$. Since v minimizes $\operatorname{lct}^{\mathfrak{q}}(-; \mathcal{D}, \mathfrak{a}_\star)$, we know

$$\frac{A(w; \mathcal{D}) + w(\mathfrak{q})}{w(\mathfrak{a}_\star)} \geq \frac{A(v; \mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)}$$

implying that

$$\frac{A(w; \mathcal{D}) + w(\mathfrak{q})}{A(v; \mathcal{D}) + v(\mathfrak{q})} \geq \frac{w(\mathfrak{a}_\star)}{v(\mathfrak{a}_\star)} \geq 1$$

proving (2).

Supposing (2) holds, we prove (3). We see from the definition that $v(\mathfrak{a}_\star(v)) = 1$, and so assertion (3) is equivalent to proving

$$\operatorname{lct}^{\mathfrak{q}}(w; \mathcal{D}, \mathfrak{a}_\star(v)) \geq A(v; \mathcal{D}) + v(\mathfrak{q}).$$

If $w(\mathfrak{a}_\star(v)) = 0$ then $\operatorname{lct}^{\mathfrak{q}}(w; \mathcal{D}, \mathfrak{a}_\star(v)) = +\infty$ and this is trivial. We therefore assume $w(\mathfrak{a}_\star(v)) > 0$, in which case the desired inequality is

$$(*) \qquad \frac{A(w; \mathcal{D}) + w(\mathfrak{q})}{w(\mathfrak{a}_\star(v))} \geq A(v; \mathcal{D}) + v(\mathfrak{q}).$$

The left hand side is invariant under $\mathbb{R}_{>0}$-scaling on w, and so we may assume $w(\mathfrak{a}_\star(v)) = 1$. Since $\mathfrak{a}_\star(v)$ is the sequence of valuation ideals for v, if $w(\mathfrak{a}_\star(v)) = 1$ then $w(\mathfrak{a}) \geq v(\mathfrak{a})$ for all ideals $\mathfrak{a}$. But then $(*)$ holds by assumption (2). $\square$

Still following the approach of Jonsson and Mustaţă, we now prove that we may modify $\mathfrak{a}_1$ and $\mathfrak{q}$ so that they are both locally primary to a chosen minimal prime of $(\mathcal{J}_\lambda : \mathfrak{q})$.

Lemma 7.14. *For $s \in \mathbb{N}_1$ and an ideal $\mathfrak{m}$ on X, define*

$$\mathfrak{c}_j = \sum_{i=0}^{j} \mathfrak{a}_i \mathfrak{m}^{s(j-i)} \tag{13}$$

and $\mathfrak{c}_\star = \{c_j\}_{j\geq 1}$. Then $\mathfrak{c}_\star$ is a graded sequence, and $v(\mathfrak{c}_\star) = \min\{v(\mathfrak{a}_\star), v(\mathfrak{m}^s)\}$.

Proof. That $\mathfrak{c}_\star$ is graded follows definitionally from the graded property of $\mathfrak{a}_\star$.

Let us then prove $v(\mathfrak{c}_\star) = \min\{v(\mathfrak{a}_\star), v(\mathfrak{m}^s)\}$. Note that $\mathfrak{a}_j + \mathfrak{m}^{sj} \subseteq \mathfrak{c}_j$ for all $j \geq 1$, so $v(\mathfrak{c}_\star) \leq \min\{v(\mathfrak{a}_\star), v(\mathfrak{m}^s)\}$ for all s and $v \in \mathrm{Val}_X$.

- If $v(\mathfrak{m}^s) \leq v(\mathfrak{a}_\star)$, then $-v(\mathfrak{a}_i) \leq -i\, v(\mathfrak{m}^s)$ for all $i \geq 1$. Therefore,

$$v(\mathfrak{m}^s) = \frac{j\, v(\mathfrak{m}^s) - v(\mathfrak{a}_i) + v(\mathfrak{a}_i)}{j} \leq \frac{(j-i)\, v(\mathfrak{m}^s) + v(\mathfrak{a}_i)}{j}.$$

  We then see that

$$\frac{v(\mathfrak{c}_j)}{j} = \min_{0\leq i\leq j} \left\{ \frac{(j-i)v(\mathfrak{m}^s) + v(\mathfrak{a}_i)}{j} \right\} = v(\mathfrak{m}^s)$$

  giving $v(\mathfrak{c}_\star) = v(\mathfrak{m}^s)$.
- If $v(\mathfrak{m}^s) > v(\mathfrak{a}_\star)$, then $j\, v(\mathfrak{m}^s) > v(\mathfrak{a}_j)$ for all $j \geq j_0 \gg 0$. Now, for $n > 2j_0$ we have

$$\frac{v(\mathfrak{a}_n)}{n} \leq \frac{v(\mathfrak{a}_j) + v(\mathfrak{a}_{n-j})}{n} < \frac{v(\mathfrak{a}_j) + (n-j)v(\mathfrak{m}^s)}{n} < v(\mathfrak{m}^s).$$

  Therefore, $v(\mathfrak{c}_\star) = v(\mathfrak{a}_\star)$, since

$$v(\mathfrak{c}_n) = \min_j\{v(\mathfrak{a}_j), (n-j)v(\mathfrak{m}^s)\}$$

  for all n. ☐

Lemma 7.15 (cf. [30])**.** *Assume $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) = \lambda < \infty$ and let $\mathfrak{m}$ be the generic point of an irreducible component of $\mathbb{V}(\mathcal{J}_\lambda : \mathfrak{q})$. Defining $\mathfrak{c}_\star$ as in (13) with $s \gg 0$ gives $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) = \mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{c}_\star)$. If $v \in \mathrm{Val}_X^*$ computes $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{c}_\star)$, then v computes $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star)$.*

Proof. Note that $\sqrt{(\mathcal{J}_\lambda : \mathfrak{q})} \subseteq \mathfrak{m}$. First consider the special case $\mathfrak{m} = \sqrt{(\mathcal{J}_\lambda : \mathfrak{q})}$ and let $n \in \mathbb{N}_1$ such that $\mathfrak{m}^n\mathfrak{q} \subseteq \mathcal{J}_\lambda$. Define $\lambda' = \mathrm{lct}^{\mathfrak{m}^n\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) > \lambda$, and define $\mathfrak{c}_\star$ using any $s \gg 0$ with $n/s < (\lambda' - \lambda)$. Fix $0 < \varepsilon \ll 1$ with $n/s < (\lambda'(1-\varepsilon) - \lambda)$.

We denote by W the set of $v \in \mathrm{Val}_X^*$ with $\mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) < \infty$, and $W_\varepsilon = \left\{v \in W \,:\, \mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) \leq \frac{\lambda}{1-\varepsilon}\right\}$. Since $\mathfrak{m}^n \subseteq (\mathcal{J}_\lambda : \mathfrak{q})$ and $v(\mathcal{J}_\lambda : \mathfrak{q}) > 0$, it follows $v(\mathfrak{m}) > 0$. Therefore,

$$
\begin{aligned}
\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{c}_\star) &= \inf_{v\in W} \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{\min\{v(\mathfrak{a}_\star), v(\mathfrak{m}^s)\}} \\
&\leq \inf_{v\in W_\varepsilon} \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{\min\{v(\mathfrak{a}_\star), v(\mathfrak{m}^s)\}} \\
&= \inf_{v\in W_\varepsilon} \left(\frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} \max\left\{1, \frac{v(\mathfrak{a}_\star)}{v(\mathfrak{m}^s)}\right\}\right) \qquad (14)
\end{aligned}
$$

When $v \in W_\varepsilon$ we have

$$
\begin{aligned}
\lambda' - \mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) &\geq \frac{(1-\varepsilon)\lambda' - \lambda}{1-\varepsilon} \\
&> (1-\varepsilon)\lambda' - \lambda > n/s,
\end{aligned}
$$

so in particular

$$
\frac{n}{s}\left(\lambda' - \mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star)\right)^{-1} < 1. \qquad (15)
$$

Additionally,

$$
\lambda' - \mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) \leq \frac{n\, v(\mathfrak{m})}{v(\mathfrak{a}_\star)} = \frac{n\, v(\mathfrak{m}^s)}{s\, v(\mathfrak{a}_\star)}
$$

for all $v \in W$, so if $v(\mathfrak{m}) > 0$ we can re-arrange this estimate and apply (15):

$$
\frac{v(\mathfrak{a}_\star)}{v(\mathfrak{m}^s)} \leq \frac{n}{s}\left(\lambda' - \mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star)\right)^{-1} < 1. \qquad (16)
$$

Thus, by applying (16) to (14) we have:

$$
\begin{aligned}
\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{c}_\star) &\leq \inf_{v\in W_\varepsilon} \left(\frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} \max\left\{1, \frac{v(\mathfrak{a}_\star)}{v(\mathfrak{m}^s)}\right\}\right) \\
&= \inf_{v\in W_\varepsilon} \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} \\
&= \mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star).
\end{aligned}
$$

We conclude $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{c}_\star) \leq \mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$. The other inequality follows from monotonicity, 3.14(1).

Now treating the general case of $\mathfrak{m}$ a minimal prime of $(\mathcal{J}_\lambda : \mathfrak{q})$, we complete X at $\mathfrak{m}$ without changing the log canonical thresholds 7.11(2). After completing, $\sqrt{(\mathcal{J}_\lambda : \mathfrak{q})} = \mathfrak{m}$, so we reduce to the previous case.

Finally, suppose v computes $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{c}_\star)$. Then

$$\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) = \operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{c}_\star) = \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{c}_\star)},$$

and from $v(\mathfrak{c}_\star) \leq v(\mathfrak{a}_\star)$ we get

$$\begin{aligned}\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) &\geq \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} \\ &\geq \operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star).\end{aligned}$$

Therefore, the inequalities are equalities, proving that v also computes $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star)$. ☐

We are ready to prove the following existence result using essentially the same method as Jonsson and Mustaţă.

Theorem 7.16 (cf. [30, Theorem 7.3])**.** *Let* $\lambda = \operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) < \infty$. *For any generic point* $\mathfrak{m}$ *of an irreducible component of* $\mathbb{V}(\mathcal{J}_\lambda : \mathfrak{q})$ *there exists a valuation with center* $\mathfrak{m}$ *computing* $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star)$.

Proof. Applying 7.11, we may assume $(R, \mathfrak{m})$ is local and $\mathfrak{m} = \sqrt{(\mathcal{J}_\lambda : \mathfrak{q})}$. Enlarging $\mathfrak{a}_\star$ using the previous proposition, we also assume that $\mathfrak{m}^s \subseteq \mathfrak{a}_1$ for some $s \gg 0$. Since $\lambda < \infty$, we can fix $M \in \mathbb{R}$ with $\lambda < M$, and $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star)$ is unchanged by considering only those $v \in \operatorname{Val}_X^*$ such that $v(\mathfrak{a}_\star) > 0$ and

$$\operatorname{lct}^{\mathfrak{q}}(v; \mathcal{D}, \mathfrak{a}_\star) = \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} \leq M,$$

or equivalently

$$A(v;\mathcal{D}) + v(\mathfrak{q}) \leq M\, v(\mathfrak{a}_\star).$$

Because $\mathfrak{m}^s \subseteq \mathfrak{a}_1$, $\mathfrak{m}^{st} \subseteq \mathfrak{a}_t$ for all $t \geq 1$, and we see $0 < v(\mathfrak{a}_\star) \leq v(\mathfrak{m}^s)$, i.e. $c_X(v) = \mathfrak{m}$. Then $A(v;\mathcal{D}) \leq A(v;\mathcal{D}) + v(\mathfrak{q}) \leq M v(\mathfrak{a}_\star) \leq M v(\mathfrak{m}^s)$. Re-scaling v if necessary, we may assume $v(\mathfrak{m}) = 1$, which now gives $A(v;\mathcal{D}) \leq N := Ms$. Thus, $v \in \mathcal{V}_N$ as defined in (7.6). We see that

$$\operatorname{lct}^{\mathfrak{q}}(\mathcal{D}, \mathfrak{a}_\star) = \inf_{v \in \mathcal{V}_N} \operatorname{lct}^{\mathfrak{q}}(v; \mathcal{D}, \mathfrak{a}_\star),$$

and $\mathrm{lct}^{\mathfrak{q}}(-;\mathcal{D},\mathfrak{a}_\star)$ is lsc (11), so compactness yields $v \in \mathcal{V}_N$ achieving this minimum. □

We do not get very much information about the valuation computing $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ above. In analogy with the discrete valuation case, where $\mathrm{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) = +\infty$ for non-divisorial discrete valuations on varieties (3.7), one might expect these valuations to be Abhyankar. Jonsson and Mustaţă conjecture exactly this, calling Abhyankar valuations *quasi-monomial*, since their schemes are excellent over $\mathbb{Q}$.

Conjecture 7.17 (cf. [30, Conjecture 7.4])**.** Suppose $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star) = \lambda < \infty$.

- **Weak version**: for any generic point $\mathfrak{m}$ of an irreducible component of $\mathbb{V}(\mathcal{J}_\lambda : \mathfrak{q})$ there exists a Abhyankar valuation $v \in \mathrm{Val}_X^*$ computing $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ with $c_X(v) = \mathfrak{m}$.
- **Strong version**: any valuation computing $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ must be Abhyankar.

Jonsson and Mustaţă reduce this conjecture to the (potentially) simpler case of affine space.

Conjecture 7.18 (cf. [30, Conjecture 7.5])**.** Suppose $X = \mathbb{A}_k^n$ with $n \geq 1$ and k algebraically closed of characteristic $p > 0$. Suppose the graded sequence of ideals $\mathfrak{a}_\star$ vanishes only at a closed point $x \in X$, and $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star) < \infty$. Then:

- **Weak version**: there is a Abhyankar valuation v computing $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ with $c_X(v) = x$.
- **Strong version**: any valuation with transcendence degree 0 over k computing $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ must be Abhyankar.

As expected from [30], 7.17 may be reduced to 7.18. To make reductions as they do, we need their second "enlarging lemma."

Lemma 7.19 (cf. [30, Proposition 7.15])**.** *Suppose that* $\lambda = \mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star) < \infty$ *and let* $\mathfrak{m} \in X$ *with* $\mathfrak{m}^s \subseteq \mathfrak{a}_1$*. If* $N > \lambda s$ *and* $\mathfrak{r} = \mathfrak{q} + \mathfrak{m}^N$ *then* $\mathrm{lct}^{\mathfrak{r}}(\mathcal{D},\mathfrak{a}_\star) = \lambda$*. Furthermore,* $v \in \mathrm{Val}_X^*$ *computes* $\mathrm{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ *if and only if* v *computes* $\mathrm{lct}^{\mathfrak{r}}(\mathcal{D},\mathfrak{a}_\star)$*.*

Proof. As noted previously, $\mathfrak{q} \not\subseteq \mathcal{J}_\lambda$ but $\mathfrak{q} \subseteq \mathcal{J}_t$ for all $t < \lambda$. To prove $\mathrm{lct}^{\mathfrak{r}}(\mathcal{D},\mathfrak{a}_\star) = \lambda$, it therefore suffices to show that $\mathfrak{m}^N \subseteq \mathcal{J}_\lambda$. This, of course, follows from our choice of $N > \lambda s$: we want to check that

$$(*) \qquad v(\mathfrak{m}^N) + A(v;\mathcal{D},\mathfrak{a}_\star^\lambda) > 0 \text{ for every valuation } v \in \mathrm{Val}_X^*.$$

If $v(\mathfrak{m}) = 0$, then $0 \leq v(\mathfrak{a}_\star) \leq v(\mathfrak{m}^s) = 0$ and $A(v;\mathcal{D} \cdot \mathfrak{a}_\star^\lambda) = A(v;\mathcal{D}) > 0$ by strong F-regularity, see (5.3). We therefore assume $v(\mathfrak{m}) > 0$. Rescaling v does not change the truth of $(*)$, and so we assume $v(\mathfrak{m}) = 1$. Our assumed inclusion $\mathfrak{m}^s \subseteq \mathfrak{a}_1$ gives $\lambda\, v(\mathfrak{a}_\star) \leq s\lambda < N$. Now,

$$v(\mathfrak{m}^N)+A(v;\mathcal{D},\mathfrak{a}_\star^\lambda) = N+A(v;\mathcal{D},\mathfrak{a}_\star^\lambda) = N+A(v;\mathcal{D})-\lambda v(\mathfrak{a}_\star) > A(v;\mathcal{D}) > 0.$$

We conclude that $(*)$ holds.

We now prove that v computes $\operatorname{lct}^{\mathfrak{q}}(\mathcal{D},\mathfrak{a}_\star)$ if and only if it computes $\operatorname{lct}^{\mathfrak{r}}(\mathcal{D},\mathfrak{a}_\star)$. The inclusion $\mathfrak{q} \subseteq \mathfrak{r}$ implies that $v(\mathfrak{r}) \leq v(\mathfrak{q})$, so if $\lambda = \operatorname{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star)$, then

$$\operatorname{lct}^{\mathfrak{r}}(\mathcal{D},\mathfrak{a}_\star) = \lambda \leq \operatorname{lct}^{\mathfrak{r}}(v;\mathcal{D},\mathfrak{a}_\star) = \frac{A(v;\mathcal{D}) + v(\mathfrak{r})}{v(\mathfrak{a}_\star)} \leq \frac{A(v;\mathcal{D}) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} = \lambda$$

and v also computes $\operatorname{lct}^{\mathfrak{r}}(\mathcal{D},\mathfrak{a}_\star)$. To finish the proof, it suffices to prove that $v(\mathfrak{q}) = v(\mathfrak{r})$ whenever $\operatorname{lct}^{\mathfrak{r}}(v;\mathcal{D},\mathfrak{a}_\star) = \lambda$. Recalling that $v(\mathfrak{a}_\star) \leq v(\mathfrak{m}^s)$, we have

$$v(\mathfrak{m}) \geq \frac{v(\mathfrak{a}_\star)}{s} = \frac{A(v;\mathcal{D}) + v(\mathfrak{r})}{\lambda s} > \frac{v(\mathfrak{r})}{N}.$$

This implies that $v(\mathfrak{m}^N) > v(\mathfrak{r}) = \min\{v(\mathfrak{q}), v(\mathfrak{m}^N)\}$ and completes the proof. □

Lemma 7.20. *Suppose $X = \mathbb{A}^n_k$ with k an F-finite field, let $k \subseteq K$ be an algebraic extension, and let $\mathbb{A}^n_K \to \mathbb{A}^n_k$ be the associated morphism. Extending scalars, the Cartier subalgebra $\mathcal{D}$, graded sequence of ideals $\mathfrak{a}_\star$, and ideal $\mathfrak{q}$ on $\mathbb{A}^n_k$ give $\mathcal{D}', \mathfrak{a}'_\star, \mathfrak{q}'$ on $\mathbb{A}^n_K$. For every valuation $v' \in \operatorname{Val}(\mathbb{A}^n_K)$ centered at 0 with restriction v to $\mathbb{A}^n_\kappa$, $\operatorname{lct}^{\mathfrak{q}}(v;\mathcal{D},\mathfrak{a}_\star) = \operatorname{lct}^{\mathfrak{q}'}(v';\mathcal{D}',\mathfrak{a}'_\star)$.*

Proof. The proof is very similar to (7.11). Specifically, we know $A(v;\mathcal{D}) = A(v';\mathcal{D})$ and $v(\mathfrak{a}_\star) = v'(\mathfrak{a}'_\star)$, so $A(v;\mathcal{D}\cdot\mathfrak{a}_\star^t) = A(v',\mathcal{D}'\cdot(\mathfrak{a}'_\star)^t)$ for all $t \geq 0$. As before, $v(\mathfrak{q}) = v'(\mathfrak{q}')$, so the log canonical thresholds are equal. □

Lemma 7.21. *Suppose $(R,\mathfrak{m})$ and $(S,\mathfrak{n})$ are regular F-finite rings, $\dim(R) \geq 2$, and let $\pi : Y = \operatorname{Spec}(S) \to X = \operatorname{Spec}(R)$ be a birational morphism with $\pi(\mathfrak{n}) = \mathfrak{m}$. Fix generators Φ_R and Φ_S for $\mathcal{C}^X$ and $\mathcal{C}^Y$, respectively, and suppose $\Phi_R = \Phi_S \cdot h_{S/R}$ for $h_{S/R} \in \operatorname{Frac}(R)$. Then $-\operatorname{div}_Y(h_{S/R}) \geq 0$.*

Proof. Consider a prime divisor G on S. If $\pi_* G \neq 0$, then $\operatorname{ord}_G(h_{S/R}) = 0$ since Φ_R is a unit multiple of Φ_S at the generic point of G. Therefore, we may assume $\pi_* G = 0$, and replace R and S with their localizations at the generic points of the closure of $\pi(G)$ and G, respectively. Then π factors through the blow-up of X at $\mathfrak{m}$, so $A(\mathrm{ord}_G;\mathcal{C}^X) \geq A(\mathrm{ord}_{\mathfrak{m}};\mathcal{C}^X) = \dim(R) \geq 2$. Applying (3.8) to S,

$$A(\mathrm{ord}_G;\Phi_R) = A(\mathrm{ord}_G;\Phi_S \cdot h_{S/R}) = A(\mathrm{ord}_G;\mathcal{C}^Y) - \frac{\mathrm{ord}_G(h_{S/R})}{p-1}$$
$$= 1 - \frac{\mathrm{ord}_G(h_{S/R})}{p-1}.$$

Putting this all together, $-\mathrm{ord}_G(h_{S/R}) \geq (p-1) = (p-1)(2-1)$. □

Remark 7.22. Lemma 7.21 generalizes the well-known effectivity of the relative canonical divisor for a proper birational morphism between smooth varieties.

Theorem 7.23 (cf. [30, Theorem 7.5])**.** *If the weak (resp. strong) version of 7.18 holds for every $n \leq N$ and $\mathcal{D} = \mathcal{C}^{\mathbb{A}^n}$, then the weak (resp. strong) version of 7.17 holds for all X with $\dim(X) \leq N$ and $\mathcal{D} = \mathcal{C}^X$.*

Proof. Let us write $\mathrm{lct}^{\mathfrak{q}}(X, \mathfrak{a}_\star)$ for simplicity.

We begin with the weak versions. Suppose $\lambda = \mathrm{lct}^{\mathfrak{q}}(X, \mathfrak{a}_\star) < \infty$ and let $\mathfrak{m}$ be a minimal prime of $(\mathcal{J}_\lambda : \mathfrak{q})$. Applying Lemmas 7.15, 7.19, and 7.11, we may assume $R = k[\![x_1, \ldots, x_n]\!]$, $\mathfrak{m} = (x_1, \ldots, x_n)$, and that $\mathfrak{m}^s \subset \mathfrak{a}_1 \cap \mathfrak{q}$. Since R is F-finite, k is also F-finite.

We now wish to apply 7.11 "in reverse" to reduce to the case of $\mathbb{A}^n_k$. Write $S = k[y_1, \ldots, y_n]$, $\mathfrak{n} = (y_1, \ldots, y_n)$, and identify the $\mathfrak{n}$-adic completion of S with R.

I claim that $\mathfrak{q}$ and each ideal $\mathfrak{a}_s$ of $\mathfrak{a}_\star$ has a generating set contained in S. For any $g \in R$ there exists a sequence $\{g_m\}_{m\geq 1} \subset S$ with $\lim_{m\to\infty} g_m = g$ in the $\mathfrak{m}$-adic topology of R. We have $\mathfrak{m}^s \subseteq \mathfrak{q}$, so if g is a generator for $\mathfrak{q}$, and $g_m - g \in \mathfrak{m}^s$ for $g_m \in S$, then we can replace g with g_m without changing $\mathfrak{q}$. Therefore, we can assume $\mathfrak{q}$ is generated by elements of S. Similarly, $\mathfrak{m}^s \subseteq \mathfrak{a}_1$, so $\mathfrak{m}^{st} \subseteq \mathfrak{a}_t$ for all $t \geq 1$, and each ideal in $\mathfrak{a}_\star$ has a generating set contained in S.

Since $\mathfrak{q}$ and $\mathfrak{a}_\star$ are generated by elements of S, we can view all of these ideals as being extended from $\mathfrak{q}^\flat \subset S$ and $\mathfrak{a}^\flat_\star$ on $\mathrm{Spec}(S) = \mathbb{A}^n_k$, where $\mathfrak{n}^s \subseteq \mathfrak{q}^\flat \cap \mathfrak{a}^\flat_1$ still holds. Now 7.11 implies $\mathrm{lct}^{\mathfrak{q}}(X, \mathfrak{a}_\star) = \mathrm{lct}^{\mathfrak{q}^\flat}(\mathbb{A}^n_k, \mathfrak{a}^\flat_\star)$. Extending scalars from k to $\overline{k}$ leaves this log canonical threshold unchanged (7.20), so we are reduced to the setting of 7.18. We are thereby furnished with a Abhyankar valuation $\overline{v}$ on $\mathbb{A}^n_{\overline{k}}$, centered at 0 computing the log canonical threshold of interest. Restricting $\overline{v}$ to S, then extending $\mathfrak{n}$-adically to R preserves the Abhyankar property (cf. (7.9), (7.10)), so 7.17 holds in this case.

We proceed to prove the implication between strong versions. Our strategy will be similar, the crux being twice applying 7.11 to reduce to the case of affine space. Suppose $v \in \mathrm{Val}_X^*$ computes $\mathrm{lct}^{\mathfrak{q}}(X, \mathfrak{a}_\star) = \lambda < \infty$. We wish to show that v is Abhyankar. If $\dim \mathcal{O}_{X,c_X(v)} = 1$, then $\mathrm{lct}^{\mathfrak{q}}(v; X, \mathfrak{a}_\star) < \infty$ implies $A(v; \mathcal{C}^X, \mathfrak{a}_\star) < \infty$ so v must be divisorial (3.7), hence Abhyankar. Therefore, we may assume $\dim \mathcal{O}_{X,c_X(v)} \geq 2$.

Applying 7.13, we assume that $\mathfrak{a}_\star = \mathfrak{a}_\star(v)$, implying $\mathfrak{m}^s \subseteq \mathfrak{a}_1$ for all $s \gg 0$. Now applying 7.19 we assume that $\mathfrak{m}^N \subseteq \mathfrak{q}$ for some $N > \lambda s$. After these replacements of $\mathfrak{q}$ and $\mathfrak{a}_\star$, $\mathfrak{m}$ is a minimal prime of $(\mathcal{J}_\lambda : \mathfrak{q})$. Indeed, $(\mathcal{J}_\lambda : \mathfrak{q}) \subseteq \mathfrak{m}$ by the same argument as before, and if $\mathfrak{p} \subsetneq \mathfrak{m}$ then $\mathfrak{m}^a \not\subseteq \mathfrak{p}$ for any $a > 0$. This implies $\mathfrak{a}_s = \mathfrak{a}_s(v) \not\subseteq \mathfrak{p}$ for any s, so if $w \in \mathrm{Val}_X$ has center $\mathfrak{p}$ then $w(\mathfrak{a}_\star) = 0$, thus $A(w; \mathcal{C}^X \cdot \mathfrak{a}_\star^\lambda) = A(w; \mathcal{C}^X) > 0$ (5.3). We now reduce to the case $\mathrm{tr.deg}_X(v) = 0$.

The dimension formula [36, Theorem 15.6] implies that $\mathrm{tr.deg}_X(v)$ is the maximum of $\dim(\mathcal{O}_{X,\mathfrak{m}}) - \dim(\mathcal{O}_{Y,c_Y(v)})$ over all birational (but possibly non-proper) morphisms $Y \to X$, with Y a regular scheme on which v is centered; fix some $Y \to X$ achieving this maximum. Localizing at the centers of v on Y and X, setting $R = \mathcal{O}_{X,\mathfrak{m}}$ and $S = \mathcal{O}_{Y,c_Y(v)}$, we assume $\pi : Y \to X$ corresponds to a local, birational extension $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$. Fixing generators Φ_R and Φ_S for $\mathcal{C}^X$ and $\mathcal{C}^Y$, resp., we define $h_{S/R}$ as in (7.21); that lemma implies $g := h_{S/R}^{-1} \in S$ since $\mathrm{div}_Y(g) \geq 0$ and S is normal. If we replace $\mathfrak{q} \subseteq R$ by $\mathfrak{q}' = \mathfrak{q} \cdot \mathcal{O}_Y(-\frac{1}{p-1}\mathrm{div}_Y(g)) \subseteq S$, and write $\mathfrak{a}_\star^S$ for the graded sequence of valuation ideals of v on S (an $\mathfrak{n}$-primary sequence, by construction of S), then a direct calculation (using (3.3)) proves $\mathrm{lct}^{\mathfrak{q}'}(\mathcal{C}^S; \mathfrak{a}_\star^S) = \mathrm{lct}^{\mathfrak{q}}(\mathcal{C}^R; \mathfrak{a}_\star) = \lambda$, and v computes $\mathrm{lct}^{\mathfrak{q}'}(\mathcal{C}^S; \mathfrak{a}_\star^S)$; cf. [30, Corollary 1.8, Lemma 7.11].

Therefore, by replacing X with Y, $\mathfrak{q}$ with $\mathfrak{q}'$, and $\mathfrak{a}_\star$ with $\mathfrak{a}_\star^S$, we may assume $\mathrm{tr.deg}_X(v) = 0$. We apply 7.11 with $\mathfrak{m} = c_X(v)$ and reduce to the case $X = \mathrm{Spec}(\kappa[\![x_1, \ldots, x_d]\!])$ with κ an F-finite field, v is a valuation centered at the closed point of X, computing $\mathrm{lct}^{\mathfrak{q}}(X, \mathfrak{a}_\star)$, and such that $\kappa(v)$ is algebraic over κ. Applying (7.11) again in reverse, followed by (7.20) to assume $\kappa = \overline{\kappa}$, we conclude $v = v'|_Y$ must be Abhyankar, so v' is, too. $\square$

7.4. The monomial case

We now prove the strong form of 7.18 when each $\mathfrak{a}_s$ is generated by monomials. First, we need a complementary result to 4.6, which we can prove using a more direct approach in this special setting.

Lemma 7.24. *Suppose k is algebraically closed of characteristic p, $X = \mathbb{A}_k^n = \mathrm{Spec}(k[x_1, \ldots, x_n])$, and $H = \mathrm{div}_X(x_1 \cdots x_n)$. Then for all $v \in \mathrm{Val}_X^*$ centered* *at* $\mathfrak{m} = (x_1, \ldots, x_n)$, $A(r_{(X,H)}(v); \mathcal{C}^X) \le A(v; \mathcal{C}^X)$, *with strict inequality when* $r_{(X,H)}(v) \neq v$.

Proof. If $n = 1$, then $v = r_{(X,H)}(v)$, so there is nothing to show. Let us then assume $n \geq 2$. Set $w = r_{(X,H)}(v)$.

Fixing the generator Φ for $\mathcal{C}^X$ that projects onto $(x_1 \cdots x_n)^{p-1}$, using 3.4 we consider only $A(v; \Phi)$ and $A(w; \Phi)$. Setting $\phi = \Phi \cdot (x_1 \cdots x_n)^{p-1}$, we see $A(w; \phi) = 0$ using (3.3) and (3.8). Moreover, the Cartier subalgebra $\{\!\{\phi\}\!\} \subseteq \mathcal{C}^X$ is sharply F-pure, so $A(v; \phi) \geq 0$. But $w(H) = v(H)$, so using 3.3 again we see $A(w; \Phi) \leq A(v; \Phi)$.

Assume now $v \neq w$, and let us show $A(w; \Phi) < A(v; \Phi)$. Since w is monomial on (X, H), there exists a log smooth pair $\pi : (Y, D) \succeq (X, H)$ so that the local ring at $y_0 := c_Y(w)$ has dimension ratrk(w), and so that $y_1 := c_Y(v)$ is contained in the closure Z of $\{y_0\}$ in Y. Such pairs exist over arbitrary ground fields, cf. [30, Lemma 3.6(ii)], which is a stronger, but more specialized, version of local monomialization (cf. [31]).

I claim $v \neq w$ implies $c_Y(v)$ is not the generic point y_0 of Z. We proceed by contradiction. To see this, we note that ratrk$(w) \leq$ ratrk(v), since the value group of w is contained in the value group of v. Supposing $y_1 := c_Y(v) = y_0$, the residue field $\kappa(y_0)$ is a sub-field of $\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$ (where $\mathcal{O}_v$ is the valuation ring of v and $\mathfrak{m}_v$ is the maximal ideal), so tr.deg$(\kappa(y_0)/k) \leq$ tr.deg$_Y(v)$. But then Abhyankar's inequality forces v to be Abhyankar, thus equal to w:

$$n = (\mathrm{ratrk}(w) + \mathrm{tr.deg}(\kappa(Z)/k)) \leq \mathrm{ratrk}(v) + \mathrm{tr.deg}_Y(v) \leq n.$$

Since we are assuming $v \neq w$, we thus conclude $y_1 \neq y_0$.

Let $z \in \mathfrak{m}_{y_1} \setminus \mathfrak{m}_{y_1}^2$ that is a unit in $\mathcal{O}_{Y,y_0} = \mathcal{O}_{Y,c(w)}$, so the divisor div$_U(z)$ is smooth in some neighborhood U of y_1. Writing G for the closure of div$_U(z)$ in Y, we can still define the retraction $r_{(Y,H+G)}(v) =: w'$, since $v \in \mathrm{Val}_U^*$ and $(H + G) \cap U$ is snc on U. By construction,

$$A(w; \mathcal{C}^Y) < A(w'; \mathcal{C}^Y), \tag{17}$$

and the same idea we used to show $A(w; \mathcal{C}^X) \leq A(v; \mathcal{C}^X)$ can be used to conclude $A(w'; \mathcal{C}^Y) \leq A(v; \mathcal{C}^Y)$. To complete the proof, we must only study the transformation of log discrepancies from X to Y. Let $S = \mathcal{O}_{Y,c(v)}$, $R = k[x_1, \ldots, x_n]_{\mathfrak{m}}$, fix a generator Φ_S for $\mathcal{C}^S = \mathcal{C}^Y_{c(v)}$, and define $g = h_{S/R}^{-1}$ via $\Phi = \Phi_S \cdot h_{S/R}$. By (7.21), $g \in S$ (since S is normal). Moreover, div$_{\mathrm{Spec}(S)}(g)$ is supported on the exceptional divisor E of π, since it corresponds to the relative canonical divisor $K_{Y/\mathbb{A}^n} = K_Y$ near y_1. In particular, g is a monomial in some regular system of parameters for $\mathcal{O}_{Y,y_0}$, hence $w(g) = w'(g) = v(g)$. Applying (3.3), which says

$$A(u; \mathcal{C}^X) = A(u; \mathcal{C}^Y) + \frac{u(g)}{p-1}$$

for all $u \in \{w, w', v\}$, to (17) now proves $A(w; \mathcal{C}^X) < A(v; \mathcal{C}^X)$. □

Proposition 7.25 (cf. [30], Proposition 8.1)**.** *Suppose k is algebraically closed of characteristic p. Let $\mathfrak{a}_\star$ be a graded sequence of monomial ideals on $X = \mathbb{A}^n_k = \operatorname{Spec}(k[x_1, \ldots, x_n])$, vanishing only at $\mathfrak{m} = (x_1, \ldots, x_n)$, and with $\operatorname{lct}(\mathcal{C}^X, \mathfrak{a}_\star) < \infty$. For any nonzero ideal $\mathfrak{q}$ on X, there exists a monomial valuation computing $\operatorname{lct}^{\mathfrak{q}}(\mathcal{C}^X, \mathfrak{a}_\star)$, and any valuation computing $\operatorname{lct}^{\mathfrak{q}}(\mathcal{C}^X, \mathfrak{a}_\star)$ is monomial.*

Proof. Let $D = \operatorname{div}(x_1 \cdots x_n)$ and $v \in \operatorname{Val}^*_X$. If $c_X(v) \neq \mathfrak{m}$, then $v(\mathfrak{a}_\star) = 0$, hence $\operatorname{lct}^{\mathfrak{q}}(v; \mathcal{C}^X, \mathfrak{a}_\star) = +\infty$ by definition. Therefore, we restrict our attention to $v \in \operatorname{Val}^*_X \cap c_X^{-1}(\mathfrak{m})$. The retraction $\overline{v} := r_{(X,H)}(v)$ satisfies $\overline{v}(\mathfrak{q}) \leq v(\mathfrak{q})$ and agrees with v on monomials in $x_1, \ldots, x_n$, so $v(\mathfrak{a}_\star) = \overline{v}(\mathfrak{a}_\star)$. Applying (7.24), we see

$$\operatorname{lct}^{\mathfrak{q}}(\overline{v}; \mathcal{C}^X, \mathfrak{a}_\star) = \frac{A(\overline{v}; \mathcal{C}^X) + \overline{v}(\mathfrak{q})}{\overline{v}(\mathfrak{a}_\star)} \leq \frac{A(v; \mathcal{C}^X) + v(\mathfrak{q})}{v(\mathfrak{a}_\star)} = \operatorname{lct}^{\mathfrak{q}}(v; \mathcal{C}^X, \mathfrak{a}_\star) \tag{18}$$

for every ideal $\mathfrak{q}$, with strict inequality when $v \neq \overline{v}$. Thus we see (using 7.16) there must be a monomial valuation computing $\operatorname{lct}^{\mathfrak{q}}(\mathcal{C}^X, \mathfrak{a}_\star)$, and (18) shows any computing valuation is monomial. □

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