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A note on finite determinacy of matrices

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Dedicated to Professor Gert-Martin Greuel on the occasion of his seventy-fifth birthday

Abstract: In this note, we give a necessary and sufficient condition for a matrix $A \in M_{2,2}$ to be finitely G-determined, where $M_{2,2}$ is the ring of 2×2 matrices whose entries are formal power series over an infinite field, and G is a group acting on $M_{2,2}$ by change of coordinates together with multiplication by invertible matrices from both sides.
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1. Introduction

Throughout this paper let K be an infinite field of arbitrary characteristic and

$$R := K[[\mathbf{x}]] = K[[x_1, \dots, x_s]]$$

the formal power series ring over K in s variables with maximal ideal $\mathfrak{m} = \langle x_1, \dots, x_s \rangle$. We denote by

$$M_{m,n} := Mat(m, n, R)$$

the set of all $m \times n$ matrices with entries in R. Let G denote the group

$$G := (GL(m, R) \times GL(n, R)^{\text{op}}) \rtimes Aut(R),$$

where $GL(n, R)^{\text{op}}$ is the opposite group of the group $GL(n, R)$ and $Aut(R)$ is the group of automorphisms defined on R. The group G acts on $M_{m,n}$ as follows

$$(U, V, \phi, A) \mapsto U \cdot \phi(A) \cdot V,$$

where $A = [a_{ij}(\mathbf{x})] \in M_{m,n}$, $U \in GL(m, R)$, $V \in GL(n, R)^{\text{op}}$, and $\phi(A) := [\phi(a_{ij}(\mathbf{x}))] = [a_{ij}(\phi(\mathbf{x}))]$ with $\phi(\mathbf{x}) := (\phi_1, \dots, \phi_s)$, $\phi_i := \phi(x_i) \in \mathfrak{m}$ for all

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$i = 1, \ldots, s$. Two matrices $A, B \in M_{m,n}$ are called *G-equivalent*, denoted $A \overset{G}{\sim} B$, if B lies in the orbit of A. We say that $A \in M_{m,n}$ is *G k-determined* if for each matrix $B \in M_{m,n}$ with $B - A \in \mathfrak{m}^{k+1} \cdot M_{m,n}$, we have $B \overset{G}{\sim} A$, i.e. if A is G-equivalent to every matrix which coincides with A up to and including terms of order k. A matrix A is called *finitely G-determined* if there exists a positive integer k such that it is G k-determined.

Over the complex numbers, the classical criterion for finite determinacy says that a matrix $A \in M_{m,n}$ is finitely G-determined if and only if the tangent space at A to the orbit GA has finite codimension in $M_{m,n}$ (see [2, 7, 9]). Over fields of arbitrary characteristic, finite G-determinacy was first studied for one power series, i.e. $m = n = 1$, as a key ingredient for classification of singularities (see [1, 3]), and has been developed to matrices of power series recently in [4, 5, 8]. It was shown in [4, 6] that in positive characteristic the tangent space to the orbit GA in general does not coincide with the image of the tangent map of the orbit map. For $A \in M_{m,n}$ instead of the tangent space we consider the R-submodule of $M_{m,n}$

$$\widetilde{T}_A(GA) := \langle E_{m,pq} \cdot A \rangle + \langle A \cdot E_{n,hl} \rangle + \mathfrak{m} \cdot \left\langle \frac{\partial A}{\partial x_\nu} \right\rangle,$$

which is the image of the tangent map of the orbit map $G \to GA$, and call it the *tangent image* at A to the orbit GA in [4]. Here $\langle E_{m,pq} \cdot A \rangle$ is the R-submodule generated by $E_{m,pq} \cdot A$, $p, q = 1, \ldots, m$, with $E_{m,pq}$ the (p, q)-th canonical matrix of $Mat(m, m, R)$ (1 at place (p, q) and 0 elsewhere) and $\left\langle \frac{\partial A}{\partial x_\nu} \right\rangle$ is the R-submodule generated by the matrices $\frac{\partial A}{\partial x_\nu} = \left[\frac{\partial a_{ij}}{\partial x_\nu}(\mathbf{x}) \right]$, $\nu = 1, \ldots, s$. By replacing $\mathfrak{m}$ by R in $\widetilde{T}_A(GA)$ we call the corresponding submodule $\widetilde{T}_A^e(GA)$ the *extended tangent image* at A to the orbit GA. In arbitrary characteristic, the following equivalent sufficient conditions for finite determinacy were obtained in [4, Proposition 4.2 and Theorem 4.3].

Proposition 1.1. 1. *Let $A \in \mathfrak{m} \cdot M_{m,n}$. Then A is finitely G-determined if one of the following equivalent statements holds:*

(i) $\dim_K \left(\mathfrak{m} \cdot M_{m,n} / \widetilde{T}_A(GA) \right) =: d < \infty$.

(ii) $\dim_K M_{m,n} / \widetilde{T}_A^e(GA) =: d_e < \infty$.

(iii) $\mathfrak{m}^k \subset I_{mn} \left(\Theta_{(G,A)} \right)$ *for some positive integer k, where*

$$R^t \xrightarrow{\Theta_{(G,A)}} M_{m,n} \to M_{m,n} / \widetilde{T}_A^e(GA) \to 0$$

is a presentation of $M_{m,n}/\widetilde{T}_A^e(GA)$ and $I_{mn}(\Theta_{(G,A)})$ is the ideal of $mn \times mn$ minors of $\Theta_{(G,A)}$.

Furthermore, if the condition (i) (resp. (ii) and (iii)) above holds then A is G $(2c - \operatorname{ord}(A) + 2)$-determined, where $c = d$ (resp. d_e and k) and $\operatorname{ord}(A)$ is the minimum of the orders of entries of A.

2. *If $\operatorname{char}(K) = 0$ then the converse of 1. also holds.*

The question whether in positive characteristic the finite codimension of $\widetilde{T}_A(GA)$ is necessary for a matrix $A \in M_{m,n}$ to be finitely G-determined for arbitrary m and n remains open (see [5, Conjecture 1.3]). For the case of a one column matrix $A \in M_{m,1}$, the finite codimension of $\widetilde{T}_A(GA)$ is equivalent to finite G-determinacy of A in arbitrary characteristic (see [5]).

The above question is answered positively for the case of 2×2 matrices in this short note, where we prove that the finite codimension of $\widetilde{T}_A(GA)$ is a necessary and sufficient criterion for a matrix $A \in M_{2,2}$ to be finitely G-determined. In order to do that we first prove that there exist finitely G-determined 2×2 matrices A of homogeneous polynomials of arbitrarily high order by showing that the ideal generated by the maximal minors of a presentation matrix $\Theta_{(G,A)}$ of the R-module $M_{2,2}/\widetilde{T}^e_A(GA)$ is Artinian (Proposition 2.1).

The main result of the paper is the following theorem.

Theorem 1.2. *Let $A \in \mathfrak{m} \cdot M_{2,2}$. Then the following are equivalent:*

1. *A is finitely G-determined.*
2. $\dim_K M_{2,2}/\widetilde{T}_A(GA) < \infty$.

2. Proof of Theorem 1.2

We show in Proposition 2.1 the important fact that there exist finitely G-determined matrices in $M_{2,2}$ of arbitrarily high order in arbitrary characteristic. Its proof shows furthermore that the coefficients of the entries of such a matrix belong to a Zariski open subset of K^{4s}.

Proposition 2.1. *Let $\operatorname{char}(K) = p \geq 0$ and $B = \left[\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right]$, where*

$$f_{ij} = c_{ij}^{(1)} x_1^N + \cdots + c_{ij}^{(s)} x_s^N \in K[[\mathbf{x}]],$$

$p \nmid N$ if $p > 0$, and $c_{ij}^{(k)}$ are general elements in K. Then

$$\dim_K M_{2,2}/\widetilde{T}^e_B(GB) < \infty$$

and B is finitely G-determined.

Proof. We first claim that the ideal of 4×4 minors of the presentation matrix

$$M := \Theta_{(G,B)} = \begin{bmatrix} f_{11} & f_{12} & 0 & 0 & f_{11} & f_{21} & 0 & 0 & \frac{\partial f_{11}}{\partial x_1} & \cdots & \frac{\partial f_{11}}{\partial x_s} \\ f_{21} & f_{22} & 0 & 0 & 0 & 0 & f_{11} & f_{21} & \frac{\partial f_{21}}{\partial x_1} & \cdots & \frac{\partial f_{21}}{\partial x_s} \\ 0 & 0 & f_{11} & f_{12} & f_{12} & f_{22} & 0 & 0 & \frac{\partial f_{12}}{\partial x_1} & \cdots & \frac{\partial f_{12}}{\partial x_s} \\ 0 & 0 & f_{21} & f_{22} & 0 & 0 & f_{12} & f_{22} & \frac{\partial f_{22}}{\partial x_1} & \cdots & \frac{\partial f_{22}}{\partial x_s} \end{bmatrix}$$

in $K[\mathbf{x}]$ is Artinian.

Indeed, let $P = (a_1, \ldots, a_s)$ be a point where all 4×4 minors of M vanish. Denote by $M_{i_1 i_2 i_3 i_4}$ the 4×4 minor of M obtained from columns $i_1, \ldots, i_4$. Given the generality of the coefficients $c_{ij}^{(k)}$, we may assume that the determinant of B does not vanish in any of the points $(1,0,\ldots,0), \ldots, (0,\ldots,0,1)$. We may also assume that all minors (of any size) of the matrix

$$\begin{bmatrix} c_{11}^{(1)} & \cdots & c_{11}^{(s)} \\ c_{21}^{(1)} & \cdots & c_{21}^{(s)} \\ c_{12}^{(1)} & \cdots & c_{12}^{(s)} \\ c_{22}^{(1)} & \cdots & c_{22}^{(s)} \end{bmatrix}$$

are non-zero.

First note that $M_{1,2,3,4} = \det(B)^2$, so the determinant of B must vanish at P. Now observe that if $s \geq 4$ then a 4×4 minor of M taken from four of the last s columns can be written as

$$M_{i_1 i_2 i_3 i_4} = \begin{vmatrix} c_{11}^{(i_1)} & c_{11}^{(i_2)} & c_{11}^{(i_3)} & c_{11}^{(i_4)} \\ c_{21}^{(i_1)} & c_{21}^{(i_2)} & c_{21}^{(i_3)} & c_{21}^{(i_4)} \\ c_{12}^{(i_1)} & c_{12}^{(i_2)} & c_{12}^{(i_3)} & c_{12}^{(i_4)} \\ c_{22}^{(i_1)} & c_{22}^{(i_2)} & c_{22}^{(i_3)} & c_{22}^{(i_4)} \end{vmatrix} \cdot N^4\, x_{i_1}^{N-1}\, x_{i_2}^{N-1}\, x_{i_3}^{N-1}\, x_{i_4}^{N-1}.$$

Since the determinant is non-zero, at least one of each four coordinates of P must be zero. In other words, P has at most 3 non-zero coordinates. Without loss of generality, we may assume that $a_4 = \cdots = a_s = 0$. Suppose that P is non-zero. Then at least two of a_1, a_2, and a_3 are non-zero, given our assumptions on the vanishing of $\det(B)$, and we may assume that $a_1 a_2 \neq 0$. Note that for $1 \leq i_1 < i_2 \leq 8$,

$$M_{i_1,i_2,9,10} = F \cdot N^2 x_1^{N-1} x_2^{N-1},$$

where F is a form in degree $2N$, involving only the $\binom{s+1}{2}$ monomials of type $x_i^N x_j^N$, with $i \leq j$. Writing $y_{ij} = x_i^N x_j^N$, we can regard F as a linear form on the variables y_{ij}. Now, since $a_1 a_2 \neq 0$, we see that F vanishes on P. Together with $\det(B)$, minors $M_{1,5,9,10}$, $M_{2,4,9,10}$, $M_{3,7,9,10}$, $M_{6,8,9,10}$, and $M_{3,6,9,10}$ form a system of 6 linear equations on the variables y_{ij}. Since P satisfies $y_{ij} = 0$ for $j > 3$, we can regard this as a system on the 6 variables $y_{11}, y_{12}, y_{13}, y_{22}, y_{23}$, and y_{33}. Therefore if the system is independent, the only solution is zero. We can check that this is indeed the case by taking one parameter a, and assigning in the system of equations $c_{11}^{(1)} = a$, $c_{12}^{(3)} = c_{21}^{(2)} = c_{22}^{(1)} = c_{22}^{(2)} = c_{22}^{(3)} = 1$, and $c_{ij}^{(k)} = 0$ otherwise. Then the determinant of the system is $a^7 + a^6$, which is non-zero. This implies that $P = 0$ for a general choice of $c_{ij}^{(k)}$, which finishes the proof of the claim.

Applying now Proposition 1.1, the statements follow. □

We need in addition the semi-continuity of the K-dimension of a 1-parameter family of modules over a power series ring, which was obtained in [5, Proposition 3.4].

Proposition 2.2. *Let $P = K[t][[\mathbf{x}]]$, where $\mathbf{x} = (x_1, \ldots, x_s)$, and M a finitely generated P-module. For $t_0 \in K$, set*

$$M(t_0) := M \underset{K[t]}{\otimes} (K[t]/\langle t - t_0\rangle) \cong M/\langle t - t_0\rangle \cdot M.$$

Then there is a nonempty open neighborhood U of 0 in $\mathbb{A}^1_K$ such that for all $t_0 \in U$, we have

$$\dim_K M(t_0) \leq \dim_K M(0).$$

Proof of Theorem 1.2. By Proposition 1.1, it suffices to prove the implication (1. ⇒ 2.). Assume that A is G k-determined. By finite determinacy we may assume that A is a matrix of polynomials. Let $\mathrm{char}(K) = p > 0$ and $N \in \mathbb{N}$ such that $N > k$ and $p \nmid N$. Let $B \in M_{2,2}$ be a matrix as in Proposition 2.1. Consider

$$B_t = B + tA \in Mat(2, 2, K[t][\mathbf{x}])$$

and the $K[t][[\mathbf{x}]]$-module

$$\widetilde{T}^e_{B_t}(GB_t) = \langle E_{ij} \cdot B_t, i, j = 1, 2\rangle + \langle B_t \cdot E_{ij}, i, j = 1, 2\rangle + \left\langle \frac{\partial B_t}{\partial x_1}, \ldots, \frac{\partial B_t}{\partial x_s} \right\rangle.$$

Then by Proposition 2.2 there is a nonempty open subset $U \subset \mathbb{A}^1_K$ such that for all $t_0 \in U$ we have

$$\dim_K\big(M_{2,2}/\widetilde{T}^e_{B_{t_0}}(GB_{t_0})\big) \leq \dim_K M_{2,2}/\widetilde{T}^e_B(GB) < \infty,$$

where the second inequality follows from Proposition 2.1. Let $t_0 \in U$ and $t_0 \neq 0$. Since $A \overset{G}{\sim} t_0 A \overset{G}{\sim} B_{t_0}$, we have

$$\dim_K M_{2,2}/\widetilde{T}^e_A(GA) = \dim_K\big(M_{2,2}/\widetilde{T}^e_{B_{t_0}}(GB_{t_0})\big) < \infty,$$

which is equivalent to the finiteness of the codimension of $\widetilde{T}_A(GA)$. □

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References

[1] Y. Boubakri, G.-M. Greuel and T. Markwig, Invariants of hypersurface singularities in positive characteristic. *Rev. Mat. Complut.* **25** (2012), 61–85. MR2876917

[2] G.R. Belitskii and D. Kerner, Finite determinacy of matrices over local rings. II. Tangent modules to the miniversal deformations for group-actions involving the ring automorphisms. arXiv:1604.06247 (2016).

[3] G.-M. Greuel and H. Kröning, Simple singularities in positive characteristic. *Math. Z.* **203** (1990), 339–354. MR1033443

[4] G.-M. Greuel and T.H. Pham, On finite determinacy for matrices of power series. *Math. Z.* **290** (2018), 759–774. MR3856832

[5] G.-M. Greuel and T.H. Pham, Finite determinacy of matrices and ideals. *J. Algebra* **530** (2019), 195–214. MR3942718

[6] G.-M. Greuel and T.H. Pham, Algorithms for group actions in arbitrary characteristic and a problem in singularity theory. *Appl. Algebra Engrg. Comm. Comput.* **31** (2020), 87–100. MR4062467

[7] J.N. Mather, Stability of C^∞ mappings. III. Finitely determined map-germs. *Inst. Hautes Études Sci. Publ. Math.* **35** (1968), 279–308. MR0275459

[8] T.H. Pham, On finite determinacy of hypersurface singularities and matrices in arbitrary characteristic. Ph.D. thesis, University of Kaiserslautern (2016).

[9] C.T.C. Wall, Finite determinacy of smooth map-germs. *Bull. London Math. Soc.* **13** (1981), 481–539. MR0634595

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