Pure and Applied Mathematics Quarterly
Volume 16, Number 4, 1083–1098, 2020

Connectedness of Milnor fibres and Stein factorization of compactifiable holomorphic functions

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Abstract: We start with conditions under which the Milnor fibre of a holomorphic function on a singular space is connected. In this case the special fibre is contractible, hence connected. So we pass to a more general question: compare the number of connected components of the fibres of a holomorphic function. Useful ingredients are local Lefschetz theorems and some kind of a Stein factorization.

A result of classical singularity theory is that the Milnor fibre of a function of $n+1$ complex variables with isolated singularity has the homotopy type of a bouquet of n-spheres, in particular, the Milnor fibre is connected if $n > 0$. G.-M. Greuel asked if there is a weaker hypothesis which ensures connectedness, see [3, Section 8] about the result of the subsequent discussion. Here we weaken the hypothesis even more.

Moreover we can embed the question in a broader context. First of all it is not necessary to restrict to the local situation. Then we may ask whether the general and the special fibre have the same number of connected components. It would be natural to deduce this from the bijectivity of a certain mapping – but which one? In this context it is preferable to look at several mappings: $\pi_0(f^{-1}(t)) \to \pi_0(f^{-1}(V \setminus \{0\})) \to \pi_0(f^{-1}(V))$, V small disk around 0, $t \in V \setminus \{0\}$, resp. $\pi_0(f^{-1}(0)) \to \pi_0(f^{-1}(V))$. This turns out to be related to generalized Stein factorizations. We will discuss different approaches, there does not seem to be a straightforward one.

1. The local case

Let X be a locally closed (reduced) analytic subspace of $\mathbb{C}^N$, Y a (reduced) analytic subspace of X, $0 \in Y$, $f : X \to \mathbb{C}$ holomorphic and open, $f(0) = 0$.

Let $X_\epsilon := X \cap \{\|z\| < \epsilon\}$, $Y_\epsilon := Y \cap X_\epsilon$. $0 < \epsilon \ll 1$. Now replace X, Y by X_ϵ, Y_ϵ. Then we have a generalization of a result of Dimca ([2, Prop. 2.3], see Theorem 2.3 below):

Received January 4, 2019.
2010 Mathematics Subject Classification: Primary 32S55; secondary 32S50, 14B05.

Theorem 1.1. *Assume that $X \setminus (Y \cup f^{-1}(0))$ is connected and that there are irreducible components $D_1, \dots, D_r$, $r > 0$, of $f^{-1}(0) \setminus (Y \cup Sing\, X)$ of multiplicity $d_1, \dots, d_r$ where these numbers are coprime.*

Then $F \setminus Y$ is connected, where $F := f^{-1}(t)$, $0 < |t| \ll \epsilon$.

Proof. Assume that $z \in D_j$. Let $0 < |t| \ll \rho \ll \epsilon' \ll \epsilon$, $U := \{\zeta \in X \mid \|\zeta - z\| < \epsilon'\}$, then U is a neighbourhood of z in $X \setminus (Y \cup Sing\, X)$.

Put $\mathbb{D} = \mathbb{D}_\rho := \{s \in \mathbb{C} \mid |s| < \rho\}$, $\mathbb{D}^* = \mathbb{D}^*_\rho := \mathbb{D}_\rho \setminus \{0\}$.

Now let $z_0 \in F \setminus Y$.

Then there is in $F \setminus Y$ a path from z_0 to some $z_1 \in F \cap U$: (*)

Let $z_2 \in F \cap U$. Note that $f^{-1}(\mathbb{D}^*) \setminus Y$ is connected because it has the same homotopy type as $X \setminus (Y \cup f^{-1}(0))$. This is because both $f^{-1}(\mathbb{D})$ and X are good neighbourhoods of 0 in X with respect to $Y \cup f^{-1}(0)$ in the sense of Prill, see [9, Section B]. So there is a path α from z_0 to z_2 in $f^{-1}(\mathbb{D}^*) \setminus Y$. Then $f \circ \alpha$ is a closed path in $\mathbb{D}^*$. This path can be lifted to a path β in U with end point z_2. Let $z_1 \in F \cap U$ be its initial point and $\gamma := \alpha * \beta^{-1}$, it is a path from z_0 to z_1. Of course, $f \circ \gamma$ is nullhomotopic in $\mathbb{D}^*$, i.e. there is a mapping $\phi : I \times I \to \mathbb{D}^*$: $\phi(0, s) = \phi(1, s) = \phi(s, 1) = t$, $s \in I$, $\phi(s, 0) = (f \circ \gamma)(s)$. Here $I = [0, 1]$.

Since $f : f^{-1}(\mathbb{D}^*) \setminus Y \to \mathbb{D}^*$ is a fibration there is a mapping $\Phi : I \times I \to f^{-1}(\mathbb{D}^*) \setminus Y$ such that $f \circ \Phi = \phi$ and $\Phi(s, 0) = \gamma(s)$. Restricting Φ to the other boundary segments of $I \times I$ we obtain a path from z_0 to z_1 in $F \setminus Y$, so we get (*).

In particular, the natural mapping $H_0(F \cap U) \to H_0(F \setminus Y)$ is surjective (we take homology with integral coefficients). So we have a commutative diagram

$$\begin{array}{ccc} H_0(F \cap U) & \to & H_0(F \cap U) \\ \downarrow & & \downarrow \\ H_0(F \setminus Y) & \to & H_0(F \setminus Y) \end{array}$$

where the vertical arrows are surjective and the upper arrow is $(h_U)_*^{d_j} = id$, the lower one $h_*^{d_j}$.

Here $h : F \to F$ is the monodromy of f; we may assume $h(U) \subset U$, let $h_U : F \cap U \to F \cap U$ be the monodromy of $f|U$.

This implies that $h_*^{d_j} = id$, too. Since $d_1, \dots, d_r$ are relatively prime we obtain $h_* = id$.

Now the Wang sequence yields an exact sequence

$$H_0(F \setminus Y) \overset{h_* - id}{\to} H_0(F \setminus Y) \to H_0(f^{-1}(\mathbb{D}^*) \setminus Y) \to 0$$

The first arrow is 0, hence $H_0(F \setminus Y) \simeq H_0(f^{-1}(\mathbb{D}^*) \setminus Y) \simeq \mathbb{Z}$.

This proves our statement. □

Remarks:

a) The hypothesis that $X \setminus (Y \cup f^{-1}(\{0\}))$ is connected is guaranteed if X is irreducible, e.g. normal.
b) We may also treat the case where $X \setminus (Y \cup f^{-1}(\{0\}))$ is not connected, similarly to Theorem 4.1 b) below.
c) Without loss of generality we may suppose that $Y \cap f^{-1}(0)$ is nowhere dense in Y. Then in the case X smooth the hypothesis of Theorem 1.1 is also necessary, see section 2.

Important special cases:

a) $Y = \emptyset$: the question is about the connectedness of F, which amounts to ask whether f is simple ("einfach") in the sense of [10].
b) $Y = Sing\, X$: Then we obtain information about irreducibility, because $F \setminus Sing\, X = F \setminus Sing\, F$ is connected if and only if F is irreducible:

Theorem 1.2. *Assume that X is irreducible and that there are irreducible components $D_1, \ldots, D_r$ of $f^{-1}(0) \setminus Sing\, X$ of multiplicity $d_1, \ldots, d_r$ $(r > 0)$ where these numbers are coprime.*

Then F is irreducible, too.

Examples: Let $\mathbb{B}_n := \{z \in \mathbb{C}^n \mid \|z\| < \epsilon\}$.

a) $X = \mathbb{B}_2$, $f(z) := z_1^2 z_2^3$: F irreducible, by Theorem 1.2, hence connected.
b) $X = \mathbb{B}_2$, $f(z) := z_1^2 z_2^4$: $F = \{z \mid z_1 z_2^2 = \pm\sqrt{t}\}$ is not connected, hence reducible, so one cannot drop the condition on $d_1, \ldots, d_r$ in Theorem 1.1.
c) $X := \{z_1 z_2 = 0\} \subset \mathbb{B}_3$, $f(z) := z_3$. Then X is reducible, $X \setminus f^{-1}(0)$ is connected. F is connected but not irreducible.
d) $X := (\mathbb{B}_2 \times \{0\}) \cup (\{0\} \times \mathbb{B}_2) \subset \mathbb{B}_4$, $f(z) := z_1 - z_3$:
Then $f^{-1}(t) = (\{(t, s, 0, 0) \mid s \in \mathbb{C}\} \cup \{(0, 0, -t, s) \mid s \in \mathbb{C}\}) \cap \mathbb{B}_4$. So F is not connected, hence reducible. In fact, $X \setminus f^{-1}(0)$ has two connected components.

2. Use of a kind of Stein factorization

Let us keep the notations of the last section.

There is some weak local version of the Stein factorization theorem which we will take up later on (section 4). It sheds light on Example b) in section 1:

Theorem 2.1. *Suppose that $X \setminus (Y \cup f^{-1}(0))$ is connected. Let d be the number of connected components of $F \setminus Y$. Then there is a continuous weakly holomorphic function $g : f^{-1}(\mathbb{D}_\rho) \setminus Y \to \mathbb{C}$ such that $g|f^{-1}(\mathbb{D}_\rho^*) \setminus Y$ is holomorphic, $f = g^d$ and that the general fibre of g is connected.*

If $X \setminus Y$ is weakly normal, g is holomorphic on $f^{-1}(\mathbb{D}_\rho) \setminus Y$.
If X is normal, g can be holomorphically extended to $f^{-1}(\mathbb{D}_\rho)$.

For the notion of weak normality see [7, 72.3].

Proof. Looking at the connected components of $F \setminus Y$ we get an unramified covering over $\mathbb{D}_\rho^*$ with connected total space, because $X \setminus (Y \cup f^{-1}(0))$, and hence $f^{-1}(\mathbb{D}_\rho^*) \setminus Y$, is connected. It can be identified with $\mathbb{D}_\beta^* \to \mathbb{D}_\rho^* : t \mapsto t^d$, where $\beta^d = \rho$. Then $f|f^{-1}(\mathbb{D}_\rho^*) \setminus Y \to \mathbb{D}_\rho^*$ admits a lifting $g : f^{-1}(\mathbb{D}_\rho^*) \setminus Y \to \mathbb{D}_\beta^*$ which must be holomorphic, namely: $g(z)$ corresponds to the connected component of z in $f^{-1}(f(z)) \setminus Y$. Of course the fibres of this function are connected. We may extend g to a continuous function on $f^{-1}(\mathbb{D}_\rho) \setminus Y$.

Note that g is weakly holomorphic on $f^{-1}(\mathbb{D}_\rho)$, so g can be holomorphically extended to $f^{-1}(\mathbb{D}_\rho)$ if X is normal. □

If we drop the connectedness hypothesis for $X \setminus (Y \cup f^{-1}(0))$ we must replace $\mathbb{D}_\beta$ by some complex curve C with a finite mapping $p : C \to \mathbb{D}_\rho$, then $f = p \circ g$. Note that C is a union of discs identified at their center.

Alternative proof of Theorem 1.1: Let d be chosen as in Theorem 2.1. We must show that $d = 1$. Let $x \in D_j$. Near x the function g must be holomorphic. So $d|d_i$. Since $d_1, \ldots, d_r$ are coprime we have $d = 1$.

Theorem 2.2. *Suppose that X is weakly normal, $X \setminus (Y \cup f^{-1}(0))$ is connected and that $Y \cap f^{-1}(0)$ is nowhere dense in Y. Then the following conditions are equivalent:*

a) $F \setminus Y$ is connected,
b) $f^{-1}(0) \setminus Y$ is not of the form $d\cdot$ divisor of a holomorphic function g on $X \setminus Y$, $d > 1$,
c) there is no $d > 1$, $g : f^{-1}(\mathbb{D}_\rho) \setminus Y \to \mathbb{C}$ holomorphic, such that $f = g^d$.

Proof. a) $\Rightarrow$ c): obvious.

c) $\Rightarrow$ a): If a) did not hold we would get a contradiction because of Theorem 2.1.

b) $\Leftrightarrow$ c): clear. □

In a special case we obtain a result proved by Dimca (he showed a) $\Leftrightarrow$ b)):

Theorem 2.3 (See [2, Prop. 2.3, p. 76])**.** *Let $X = \mathbb{B}_N$, $Y = \emptyset$. Then the following conditions are equivalent:*

a) F is connected,

b) the multiplicities of the irreducible components of $f^{-1}(0)$ are coprime, i.e. we have a prime decomposition $f = f_1^{d_1} \cdots f_r^{d_r}$ where $d_1, \ldots, d_r$ are coprime,

c) there is no $d > 1$, $g : f^{-1}(\mathbb{D}_\rho) \to \mathbb{C}$ holomorphic, such that $f = g^d$.

The case X singular is more complicated because divisors may not be principal divisors.

Example: $X = \{z_1 z_2 = z_3^2\} \subset \mathbb{B}_3, f(z) = z_2$: $f^{-1}(0)$ is irreducible and has multiplicity 2, yet F is connected. In fact, $\frac{1}{2}f^{-1}(0)$ is a divisor which is not a principal divisor. Cf. [4, II Expl. 6.5.2, p. 133f].

3. Use of local Lefschetz theorems

We have a topological characterization of local irreducibility:

Let X be endowed with a Whitney stratification such that $Sing\, X$ is a union of strata.

Lemma 3.1. *Let Y be a closed analytic subset of X.*

a) Let X be irreducible. Then $X \setminus Y$ is connected, and if $Y \neq X$, $\overline{X \setminus Y} = X$.

b) If C is a connected component of $X \setminus Y$ the closure $\bar{C}$ in X is analytic.

c) The closure of $X \setminus Y$ in X is analytic.

Proof. a) That $X \setminus Y$ is connected is claimed as an exercise in [7, E.49f]. In fact, it is well-known that the irreducibility of X means that $X \setminus Sing\, X$ is connected, see e.g. [7, 49.5]. Then $X \setminus Sing\, X \cup Y$ is connected, too. Let $x \in X \setminus Y$: then there is a connected neighbourhood U in $X \setminus Y$, and U intersects $X \setminus Sing\, X$, i.e. $X \setminus Y \cup Sing\, X$, because $Sing\, X$ is nowhere dense in X. Therefore $X \setminus Y$ is connected, too.

If $Y \neq X$, Y is nowhere dense by [7, Prop. 49.8], hence $\overline{X \setminus Y} = X$.

b) Let $X_i, i \in I$, be the irreducible components of X. Let J be the set of all $i \in I$ such that $C \cap X_i \neq \emptyset$. Since $X_i \setminus Y$ is connected by a), we obtain that $X_i \setminus Y \subset C$. Let X' be the union of all $X_i, i \in J$. Then $X' \setminus Y = C$, so $X' = \bar{C}$, hence $\bar{C}$ is analytic.

c) $\overline{X \setminus Y}$ is the support of the ideal sheaf of Y in X. □

Lemma 3.2. *Suppose that $Y \subset X$ is a closed subset which is a union of strata. Suppose that for all $y \in Y$ the following holds: $U \setminus S$ is connected, i.e. $N \setminus \{y\}$ is connected, where S is the stratum which contains y, U is a good neighbourhood of y in X and N a normal slice at y with respect to S. Then $\pi_0(X \setminus Y) \simeq \pi_0(X)$.*

Proof. Let Y_i be the union of all strata of Y of dimension $\leq i$. Then it is sufficient to show for all i that $\pi_0(X \setminus Y_i) \simeq \pi_0(X \setminus Y_{i-1})$. So it is sufficient to prove the theorem in the case where Y is a union of strata of the same dimension.

Let $i : Y \to X$ and $j : X \setminus Y \to X$ be the inclusions. Then we have a distinguished triangle

$$i_* Ri^! \mathbb{Z}_X \to \mathbb{Z}_X \to Rj_* \mathbb{Z}_{X \setminus Y} \stackrel{[+1]}{\to}$$

so we get an exact sequence

$$i_* i^! \mathbb{Z}_X \to \mathbb{Z}_X \to j_* \mathbb{Z}_{X \setminus Y} \to i_* R^1 i^! \mathbb{Z}_X \to 0$$

Our hypothesis implies that $R^k i^! \mathbb{Z}_X = 0, k = 0, 1$. This implies that $0 = \mathbb{H}^k(Y, Ri^! \mathbb{Z}_X) \simeq H^k(X, X \setminus Y; \mathbb{Z})$, $k = 0, 1$, which leads to our assertion. □

Lemma 3.3. *a) X is locally irreducible $\Leftrightarrow$ for all $x \in X$ the following holds: $U \setminus S$ is connected, i.e. $N \setminus \{x\}$ is connected, where S is the stratum which contains x, U is a good neighbourhood of x in X and N a normal slice at x with respect to S.*

b) If H is transversal to X and X is locally irreducible, then $X \cap H$ is locally irreducible, too.

Proof. a) $\Rightarrow$: Since U is irreducible $U \setminus S$ is connected by Lemma 3.1a).

$\Leftarrow$: We can assume that $Sing\, X$ is a union of strata. Lemma 3.2 implies that $\pi_0(U \setminus Sing\, U) \simeq \pi_0(U) = 0$, so U is irreducible.

b) The normal slices for $X \cap H$ are normal slices for X, too. □

Recall the definition of rectified homotopical depth in [5, Def. 1.1]: $rhd(X) \geq n$ if and only if for all strata S of X and all x in S, the pair $(U, U \setminus S)$ is $n - \dim\, S - 1$-connected, where U is a suitable neighbourhood of x in X. Again we fix here a Whitney stratification of X. We say $\dim^-(X) \geq n$ if all irreducible components of X have dimension $\geq n$.

Lemma 3.4. *a) $rhd(X \setminus X_{n-2}) \geq n \Leftrightarrow \dim^-(X \setminus X_{n-2}) \geq n$.*

b) $rhd(X \setminus X_{n-3}) \geq n$ if $\dim^-(X \setminus X_{n-3}) \geq n$ and $X \setminus X_{n-3}$ is locally irreducible.

Proof. a) $rhd(X \setminus X_{n-2} \geq n$ means that for all strata S of X of dimension $> n - 2$ and all x in S, $(U, U \setminus S)$ is $n - \dim\, S - 1$-connected. We have only to look at strata of dimension $n - 1$, and these should be nowhere dense in X, which means $\dim^-(X \setminus X_{n-2}) \geq n$.

b) In addition we must look at strata of dimension $n-2$, here $U \setminus S$ should be connected. Now apply Lemma 3.3. □

Now let us compare X with the special fibre $f^{-1}(0)$.

Theorem 3.5. *Let* $(X, 0)$ *be connected in dimension* 2*,* $\dim^-(X) \geq 3$*. Then* $f^{-1}(0) \setminus \{0\}$ *is connected.*

Recall that $(X, 0)$ is connected in dimension k if $X \setminus Y$ is connected for every closed analytic subspace Y of dimension $< k$, see [1].

Proof. Take a Whitney stratification of X. Let Y be the union of $\{0\}$ and of the strata of dimension 1 which are not contained in $f^{-1}(0)$. Then $f^{-1}(0) \cap Y = \{0\}$, $rhd(X \setminus Y \cup f^{-1}(\{0\})) \geq 3$, see Lemma 3.4a). By Weak Lefschetz, see [5, Cor. 3.3.4], the pair $(X \setminus Y, f^{-1}(0) \setminus \{0\})$ is 1-connected. By hypothesis, $X \setminus Y$ is connected. So $f^{-1}(0) \setminus \{0\}$ is connected, too. □

Note that in this way we obtain Théorème 1,β, i.e. the key result of [1]: in fact, $(X, 0)$ is connected in dimension 2 if $(X, 0)$ is irreducible and $\dim^- X \geq 3$.

Now we concentrate upon the case where X is locally irreducible, e.g. normal.

If $\Sigma \subset X$ is analytic, $codim^-_X \Sigma \geq k$ means that $\dim X_i - \dim \Sigma_j \geq k$ for all irreducible components X_i of X, Σ_j of Σ such that $\Sigma_j \subset X_i$.

Theorem 3.6. *Assume that* $\dim^- X \geq 3$*,* Y *analytic subset of* X*,* $0 \in Y$*,* $X \setminus (Y \cup f^{-1}(\{0\}))$ *locally irreducible, and that* $f|Y$ *is submersive along* $Y \cap f^{-1}(0) \setminus \{0\}$ *up to a subset* Σ *of* $Y \cap f^{-1}(0) \setminus \{0\}$ *with* $0 \in \Sigma$*,* $codim^-_X \Sigma \geq 3$*.*

Then the connected components of $X \setminus Y$ *correspond bijectively to those of* $f^{-1}(\{0\}) \setminus Y$*.*

More precisely: $H^k(X \setminus Y, f^{-1}(\{0\}) \setminus Y) = 0$*,* $k = 0, 1$*.*

Proof. Remember that X, Y are open subsets of $\mathbb{C}^N$.

a) First we give a proof under the additional assumption that $X \setminus Y$ is locally irreducible.

We may assume that f is linear, so $H = f^{-1}(0)$ is a linear hyperplane.

(i) First assume that $\Sigma = \{0\}$: Then $H^k(X \setminus Y, f^{-1}(\{0\}) \setminus Y) = 0$, $k = 0, 1$, according to [5, Cor. 3.3.4].

(ii) Now assume that L is a linear subspace of $\mathbb{C}^N$ codimension k which is transversal to all strata of $Y \setminus \{0\}$ and $\dim^- X \cap L \geq 3$. Then we can find a chain of linear subspaces $L = L_k \subset L_{k-1} \subset \ldots \subset L_1$ with the same property such that $codim\, L_i = i$. By Lemma 3.3b), $X \cap L_{i-1} \setminus Y$ is locally irreducible. By (i) we obtain inductively that $H^k(X \setminus Y, L \cap X \setminus Y) = 0$, $k = 0, 1$.

(iii) Now return to the hypothesis of the theorem. We can find a linear subspace L of H of codimension $codim^-_X \Sigma$ such that L is transverse to all strata of Y as well as $Y \cap f^{-1}(0)$. By (ii), we have that $(X \setminus Y, X \cap L \setminus Y)$ is

1-connected, as well as $(X \cap f^{-1}(0), \setminus Y, X \cap L \setminus Y)$. Altogether, $(X \setminus Y, X \cap f^{-1}(0) \setminus Y$ is 1-connected.

b) Now we go back to our original assumption.

Without loss of generality assume $0 \in Y$. Induction on dim Σ.

Note that $H^k(X \setminus Y, f^{-1}(\{0\}) \setminus Y) \simeq \mathcal{H}^k(Rj_! Rl_* \mathbb{Z}_{X \setminus Y \cup f^{-1}(\{0\})})_0$, where $j : X \setminus f^{-1}(\{0\}) \to X$ and $l : X \setminus Y \cup f^{-1}(\{0\}) \to X \setminus f^{-1}(\{0\})$ are the inclusions.

Case dim $\Sigma = 0$: Then $f|Y$ is submersive along $(f^{-1}(\{0\}) \cap Y) \setminus \{0\}$. Since $rhd(X \setminus (Y \cup f^{-1}(\{0\}))) \geq 3$ by Lemma 3.4b) we get that $(X \setminus Y, f^{-1}(\{0\}) \setminus Y)$ is 1-connected, by Weak Lefschetz, see [5, Cor. 3.3.4].

Induction on dim Σ: Let $F_0 := f^{-1}(\{0\}), V := \partial X \cap \overline{\{|f| \leq \beta\}}, 0 < \beta \ll \epsilon$. By [5, Theorem 3.3.1] we have: $H^k(\partial X \setminus \partial Y, V \setminus \partial Y) = 0, k = 0, 1$.

Now look at the exact sequence:

$$H^k(\partial X \setminus \partial Y, V \setminus \partial Y) \to H^k(\partial X \setminus \partial Y, \partial F_0 \setminus \partial Y) \to H^k(V \setminus \partial Y, \partial F_0 \setminus \partial Y)$$

It is sufficient to show that $H^k(V \setminus \partial Y, \partial F_0 \setminus \partial Y) = 0, k = 0, 1$.

But this group coincides with $\mathbb{H}^k(V, R(j_\epsilon)_! R(l_\epsilon)_* \mathbb{Z}_{V \setminus \partial Y \cup \partial F_0})$.

Here $j_\epsilon : V \setminus \partial Y \to V$ and $l_\epsilon : V \setminus \partial Y \cup \partial F_0 \to V \setminus \partial F_0$ are the inclusions.

Since the hypercohomology group in question coincides with the group $\mathbb{H}^k(\partial F_0, R(j_\epsilon)_! R(l_\epsilon)_* \mathbb{Z}_{V \setminus \partial Y \cup \partial F_0})$, it is sufficient to show that for all $x \in \partial F_0$ the following holds: $\mathcal{H}^k(R(j_\epsilon)_!(R(l_\epsilon)_* \mathbb{Z}_{V \setminus \partial Y \cup \partial F_0})_x = 0, k = 0, 1$.

Let U be a suitable neighbourhood of x in V. Then the latter group coincides with $H^k(U \setminus \partial Y, U \setminus \partial Y \cup \partial F_0) \simeq H^k(N \setminus \partial Y, N \setminus \partial Y \cup \partial F_0) \simeq \mathcal{H}^k(R(j_N)_! R(l_N)_* \mathbb{Z}_{N \setminus \partial Y \cup F_0}))_x$, where N is a suitable neighbourhood of x in a general hyperplane section of X at x, $j_N : N \setminus \partial F_0 \to N \setminus \partial Y$ and $l_N : N \setminus \partial Y \cup \partial F_0 \to N \setminus \partial F_0$ are the inclusions. This is true due to transversality. By induction hypothesis, $\mathcal{H}^k(R(j_N)_!(R(l_N)_* \mathbb{Z}_{N \setminus \partial Y \cup \partial F_0})_x = 0, k = 0, 1$, because dim $N \cap \Sigma <$ dim Σ. So we obtain our assertion. □

A special case is $Y = \{0\}$ where the submersiveness condition is automatically fulfilled.

Similarly, we can compare with the general fibre:

Theorem 3.7. *Suppose that* $\dim^- X \geq 2$, *Y analytic subset of X, $X \setminus Y$ locally irreducible, and that $f^{-1}(0) \setminus \Sigma$ is a smooth divisor, where $\Sigma \subset f^{-1}(0)$, $codim_X^- \Sigma \geq 2$.*

Then the connected components of $X \setminus Y$ correspond bijectively to those of $F \setminus Y$, where $F := f^{-1}(t), t \neq 0$ small.

More precisely: $H^k(X \setminus Y, F \setminus Y) = 0, k = 0, 1$.

Proof. We use the Strong local Lefschetz theorem, see [5, Th. 4.2.1], instead of the Weak one. We have $rhd(X \setminus Y) \geq 2$, by Lemma 3.1 and 3.2.

We can assume that $X \setminus (Y \cup f^{-1}(\{0\}))$ is dense, by Lemma 3.1c).

Furthermore we may assume that $0 \in Y$: otherwise look at the exact sequence

$$H^k(X, X \setminus \{0\}) \to H^k(X, F) \to H^k(X \setminus \{0\}, F)$$

and note that $H^k(X, X \setminus \{0\}) = 0, k = 0, 1$, because $rhd(X) \geq 2$ by Lemma 3.1 and 3.2.

And $H^k(X \setminus Y, F \setminus Y) = \mathcal{H}^k(\Phi_f Rl_* \mathbb{Z}_{X \setminus Y})_0$, where $l : X \setminus Y \to X$ is the inclusion.

Induction on dim Σ:

Case dim $\Sigma = 0$: Then $f^{-1}(\{0\}) \setminus \{0\}$ is smooth, so our claim follows from loc. cit. (recall that we may assume $0 \in Y$).

Induction step: Again assume $0 \in Y$, and put $V = \partial X \cap \{|f| \leq \beta\}$. By the theorem loc. cit., we have $H^k(\bar{X} \setminus \bar{Y}, \bar{F} \cup V \setminus \bar{Y}) = 0$, $k = 0, 1$.

Look at the exact sequence:

$$H^k(\bar{X} \setminus \bar{Y}, \bar{F} \cup V \setminus \bar{Y}) \to H^k(\bar{X} \setminus \bar{Y}, \bar{F} \setminus \bar{Y}) \to H^k(\bar{F} \cup V \setminus \bar{Y}, \bar{F} \setminus \bar{Y})$$

where $H^k(\bar{F} \cup V \setminus \bar{Y}, \bar{F} \setminus \bar{Y}) \simeq H^k(V \setminus \partial Y, \partial F_0 \setminus \partial Y) \simeq \mathbb{H}^k(\partial F_0, \Phi_{f_\epsilon} R(l_\epsilon)_* \mathbb{Z}_{V \setminus \partial Y})$. Here $f_\epsilon = f|V, l_\epsilon : V \setminus \partial Y \to V$ is the inclusion.

So we show for $x \in V \cap \partial F_0$: $\mathcal{H}^k(\Phi_{f_\epsilon} R(l_\epsilon)_* \mathbb{Z}_{V \setminus \partial Y})_x = 0, k = 0, 1$.

This group coincides with $\mathcal{H}^k(\Phi_{f_N} R(l_N)_* \mathbb{Z}_{N \setminus \partial Y})_x$, where N is a general hyperplane section of X at x, $f_N := f|N$, $l_N : N \setminus \partial Y \to N$ the inclusion. Now we may apply the induction hypothesis. □

4. The global case

Here, we look at a mapping which is compactifiable in a certain sense. An important application is given in the algebraic case: Suppose that $f : X \to S$ is a morphism of (not necessarily irreducible) complex algebraic varieties. Then it is well-known that f is compactifiable: Let $\bar{X}$ be a compactification of X. Let $\Gamma \subset X \times S$ be the graph of f and $\bar{\Gamma}$ its closure in $\bar{X} \times S$. If we restrict the projection of $\bar{X} \times S \to S$ to $\bar{\Gamma}$ we get the desired compactification.

Now turn back to the analytic case. We assume that S is a Riemann surface and that $\bar{X}$ is a complex space, $\bar{f} : \bar{X} \to S$ holomorphic and proper, X_∞ a closed complex subspace of $\bar{X}$, $X := \bar{X} \setminus X_\infty$, $f := \bar{f}|X$.

We have the following global analogue of Theorem 1.1, where we drop the connectedness assumption:

Theorem 4.1. *Let $s \in S$, V a good small neighbourhood of s, $s' \in V \setminus \{s\}$ and $F := f^{-1}(s')$.*

a) The mapping $\pi_0(F) \to \pi_0(f^{-1}(V \setminus \{s\}))$ is surjective.

b) Suppose that for all $s \in S$ the following holds: Let V be a good neighbourhood of s. Then for every connected component C of $f^{-1}(V \setminus \{s\})$ there are irreducible components $D_1, \ldots, D_r$ of $\bar{C} \cap f^{-1}(s) \setminus Sing\, X$ of multiplicity $d_1, \ldots, d_r$, $r > 0$, where these numbers are coprime.

Then the mapping $\pi_0(F) \to \pi_0(f^{-1}(V \setminus \{s\}))$ is bijective.

Proof. a) clear.

b) If $f^{-1}(V \setminus \{s\})$ is connected argue as in the proof of Theorem 1.1 where it is not important that X is a space germ.

In general look at each connected component of $f^{-1}(V \setminus \{s\})$. □

Lemma 4.2. *a) If f is open we have that $\pi_0(f^{-1}(V \setminus \{s\})) \to \pi_0(f^{-1}(V))$ is surjective.*

b) If X is locally irreducible we have that $\pi_0(f^{-1}(V \setminus \{s\})) \to \pi_0(f^{-1}(V))$ is injective.

Proof. We may assume $X = f^{-1}(V), s = 0$.

a) obvious.

b) (i) Let X_0 be the closure of $X \setminus f^{-1}(0)$ in X. It is an analytic subspace of X which is not only closed but open, too, because of local irreducibility. Hence $\pi_0(X_0) \to \pi_0(X)$ is injective.

(ii) Let C_1, C_2 be two different connected components of $X \setminus f^{-1}(0)$. Look at the closure $\overline{C}_i$ of C_i in X. This is an analytic subset of X_0. Then we must have $\overline{C}_1 \cap \overline{C}_2 = \emptyset$:

Otherwise let us choose a Whitney regular stratification of X_0 which is compatible with $\overline{C}_1$ and $\overline{C}_2$. Let S be a maximal stratum of $\overline{C}_1 \cap \overline{C}_2$, $x \in S$. Take a small neighbourhood U of x in X. Since X is locally irreducible, $U \setminus S$ is connected. Furthermore, C_1 and C_2 intersect $U \setminus S$. This yields a contradiction.

Now let $C_1, \ldots, C_n$ be the different connected components of $X \setminus f^{-1}(0)$. Note that X_0 is the union of the closures $\overline{C}_i$.

We know that the $\overline{C}_i$ are connected, closed and open (the complement in X_0 is closed), so $\overline{C}_i$ is an irreducible component of X_0.

So $\pi_0(X \setminus f^{-1}(0)) \to \pi_0(X_0)$ is bijective: it maps C_i onto $\overline{C}_i$. This implies our statement, using (i). □

From now on assume $S = \mathbb{D}_\rho$.

Analogue of Theorem 3.6:

Theorem 4.3. *Assume that* $\dim^- X \geq 3$, *X locally irreducible, and that* $\bar{f}|X_\infty$ *is submersive along* $\bar{f}^{-1}(\{0\}) \cap X_\infty$ *up to a subset of* $\bar{f}^{-1}(\{0\}) \cap X_\infty$ *of codimension* ≥ 2.

Then the connected components of X correspond bijectively to those of $f^{-1}(\{0\})$.

More precisely: $H^k(X, f^{-1}(\{0\})) = 0, k = 0, 1$.

Proof. Let $l : X \setminus f^{-1}(\{0\}) \to \bar{X} \setminus \bar{f}^{-1}(\{0\})$ and $\bar{j} : \bar{X} \setminus \bar{f}^{-1}(\{0\}) \to \bar{X}$ be the incusions. Then $H^k(X, f^{-1}(\{0\})) = \mathbb{H}^k(\bar{X}, R\bar{j}_! Rl_* \mathbb{Z}_{X \setminus f^{-1}(\{0\})})$; we can replace $\bar{X}$ by $\bar{f}^{-1}(\{0\})$. But according to Theorem 3.6, we have for every $x \in \bar{f}^{-1}(\{0\})$: $\mathcal{H}^k(R\bar{j}_! Rl_* \mathbb{Z}_{X \setminus f^{-1}(\{0\})})_x = 0, k = 0, 1$. This implies our statement. □

Analogue of Theorem 3.7:

Theorem 4.4. *Assume that* $\dim^- X \geq 2$, *X locally irreducible, and that* $\bar{f}^{-1}(\{0\})$ *is generically smooth.*

Then the connected components of X correspond bijectively to those of F.

More precisely: $H^k(X, F) = 0, k = 0, 1$.

Proof. It is sufficient to show that $H^k(X, F; \mathbb{Z}) = 0, k = 0, 1$.

Let $id : \mathbb{D}_\rho \to \mathbb{D}_\rho$ be the identity, $j : X \to \bar{X}$ the inclusion, and $\bar{f}_0 : \bar{f}^{-1}(\{0\}) \to \{0\}$ the projection.

Then $H^k(X, F; \mathbb{Z}) = \mathcal{H}^k(\Phi_{id} Rf_* \mathbb{Z}_X)_0 = \mathcal{H}^k(\Phi_{id} R\bar{f}_* Rj_* \mathbb{Z}_X)_0 = \mathcal{H}^k(R(\bar{f}_0)_* \Phi_{\bar{f}} Rj_* \mathbb{Z}_X)_0 = \mathbb{H}^k(\bar{f}^{-1}(0), \Phi_{\bar{f}} Rj_* \mathbb{Z}_X) = 0, k = 0, 1$, since $\mathcal{H}^k(\Phi_{\bar{f}} Rj_* \mathbb{Z}_X) = 0, k = 0, 1$, by Theorem 3.7. □

A stronger result can be obtained if we apply Theorem 4.1 and Lemma 4.2 as soon as we renounce to the statement about $H^k(X, F), k = 0, 1$:

Suppose that X is locally irreducible, f is open and that the assumption of Theorem 4.1b) holds. Then $\pi_0(F) \to \pi_0(X)$ is bijective.

5. Relation to Stein factorization

Recall the Stein factorization theorem, cf. [7, 49.A.22]:

Suppose that $f : X \to S$ is a proper holomorphic mapping between complex spaces. Then there is a complex space Z, a proper holomorphic mapping $g : X \to Z$ with connected fibres and a finite (= proper holomorphic with finite fibres) mapping $p : Z \to S$ such that $f = p \circ g$.

For a generalization to non-proper continuous mappings see [8].

Now we pass to the set-up of the last section, i.e. f is compactifiable, S a Riemann surface.

Note that f defines a topological fibre bundle over $S_0 := S \setminus D$, D being a suitable discrete subset of S (discriminant).

Lemma 5.1. *There is a Riemann surface Z_0 and an unramified holomorphic covering $p_0 : Z_0 \to S_0$ such that $p_0^{-1}(s) = \pi_0(f^{-1}(\{s\}), s \in S_0$.*

Proof. It is clear how we must define Z_0 as a set. Let $s_0 \in S_0$ and V a contractible neighbourhood of s_0 in S_0. Let $C \in p_0^{-1}(s_0)$, i.e. a connected component of $f^{-1}(s_0)$. Choose $x_{s_0} \in C$. If $s \in V$ we choose a corresponding element $x_s \in f^{-1}(\{s\})$ as follows: Let γ be a path from s_0 to s in V. We can lift it to a path in $f^{-1}(V)$ with initial point x_{s_0}, let x_s be the end point. Then the connected component $[x_s]$ of x_s in $f^{-1}(\{s\})$ depends only on the connected component $[x_{s_0}]$ of x_{s_0} in $f^{-1}(\{s_0\})$. Put $\tilde{V}_C := \{[x_s] \mid s \in V\} \subset p_0^{-1}(V)$. Then $V \to \tilde{V}_C : s \mapsto [x_s]$ is bijective. Choose the topology and the holomorphic structure of $\tilde{V}_C$ such that it is biholomorphic. Now $p_0^{-1}(V)$ is the disjoint union of all $\tilde{V}_C$. After all, p_0 is an unramified holomorphic covering. □

Now we will introduce extensions $p_i : Z_i \to S$, $i = 1, 2, 3$, of $p_0 : Z_0 \to S_0$, i.e. $p_i^{-1}(S_0) = Z_0, p_i|Z_0 = p_0, i = 1, 2, 3$:

Let $s \in D$ and V a suitable neighbourhood of s in S.

Put $p_1^{-1}(s) := \pi_0(f^{-1}(V \setminus \{0\}))$. This defines Z_1 as a set. If we extend p_0 to a holomorphic branched covering $p_1' : Z_1' \to S$ with a smooth complex curve Z_1' we have a bijection $Z_1 \to Z_1'$, so we obtain on Z_1 the structure of a smooth complex curve, too, and p_1 is a holomorphic branched covering.

Now put $p_2^{-1}(s) := \pi_0(f^{-1}(V))$. Then we obtain Z_2 as a set, and we have a mapping $Z_1 \to Z_2$. Then we get Z_2 from Z_1 by adjoining an isolated point for each element of $\pi_0(f^{-1}(V))$ not contained in the image of $\pi_0(V \setminus f^{-1}(\{s\}))$ and identifying all points which correspond to elements of $\pi_0(V \setminus f^{-1}(\{s\}))$ which are mapped to the same element of $\pi_0(V)$. If we have identified r points a neighbourhood of their image is of the following form: $\mathbb{D} \times \{1, \dots, r\}/\sim$ where $(z_1, k) \sim (z_2, l)$ iff $z_1 = z_2$ and: $k = l$ whenever $z_1 \neq 0$. The canonical projection can be identified with $\mathbb{D} \times \{1, \dots, r\} \to \mathbb{D}_r := \{z \in \mathbb{D}^r \mid z_l = 0$ for at least $r - 1$ values of $l\}$: $(z, k) \mapsto (z_1, \dots, z_r)$ with $z_k = z$ and $z_l = 0$ for $l \neq k$. We define a complex structure on Z_2 by using the last mapping. Then Z_2 is weakly normal and p_2 holomorphic.

We have a mapping $g_2 : X \to Z_2$: let $g_2(x)$ be the connected component of $f^{-1}(V)$ which contains x. In fact, Z_2 has the quotient topology with respect to g_2, so g_2 is continuous, and $g_2|X_0$ is holomorphic, where $X_0 = f^{-1}(S \setminus D)$. Note that $g_2^{-1}(z)$ is connected for $z \in Z_2$.

And $p_2 \circ g_2 = f$.

If X is weakly normal we obtain that g_2 is holomorphic.

Finally put $p_3^{-1}(s) := \pi_0(f^{-1}(s))$. Then we have Z_3 as a set; we have a mapping $Z_3 \to Z_2$, define $U \subset Z_3$ to be open if its image in Z_2 is open. For an element of $\pi_0(f^{-1}(V))$ not contained in the image of $\pi_0(f^{-1}(s))$ take off the corresponding point of Z_2. If ρ elements of $\pi_0(f^{-1}(s))$ are mapped to the same element of $\pi_0(f^{-1}(V))$ replace the point of Z_2 which corresponds to the latter by ρ points and identify the corresponding punctured neighbourhoods. In fact this mapping corresponds locally to some canonical projection $\mathbb{D}_r \times \{1, \ldots, \rho\} \to \mathbb{D}_{r,\rho}$, where $\mathbb{D}_{r,\rho}$ is the quotient space with respect to the following equivalence relation: $(z_1, k) \sim (z_2, l)$ iff $z_1 = z_2$ and: $k = l$ whenever $z_1 = 0$. Note that this mapping is a local homeomorphism.

The mapping $Z_3 \to Z_2$ is a local homeomorphism, so we get on Z_3 the structure of a weakly normal, not necessarily Hausdorff complex space Z_3 (i.e. a ringed space where in the definition of a complex space we drop the assumption to be Hausdorff). Note that anyhow Z_3 is a T_1-space, so indeed locally a complex space. We have a mapping $g_3 : X \to Z_3$: let $g_3(x)$ be the connected component of $f^{-1}(s)$ which contains x. Again $g_3^{-1}(z)$ is connected if $z \in Z_3$. And $p_3 \circ g_3 = f$.

If X is weakly normal, g_3 is holomorphic.

So with these definitions we obtain:

Theorem 5.2. *a) Z_1 is a smooth complex curve, $p_1 : Z_1 \to S$ is a finite holomorphic mapping such that $\#p_1^{-1}(\{s\})$ is the number of connected components of $f^{-1}(V \setminus \{s\})$ whenever $s \in S$, V small suitable neighbourhood of s.*

*b) Let X be weakly normal. Z_2 is a weakly normal complex space, $g_2 : X \to Z_2$ holomorphic, $p_2 : Z_2 \to S$ is a finite holomorphic mapping such that $f = p_2 \circ g_2$ and $g_2^{-1}(V)$ is connected whenever $s \in Z_2$, V are as in a).
So $\#p_2^{-1}(\{s\})$ is the number of connected components of $f^{-1}(V)$ whenever $s \in S$, V small suitable neighbourhood neighbourhood of s.*

*c) Let X be weakly normal. Z_3 is a weakly normal, not necessarily Hausdorff complex space (i.e. a ringed space where in the definition of a complex space we drop the assumption to be Hausdorff), $g_3 : X \to Z_3$ is a holomorphic mapping and $p_3 : Z_3 \to S$ a quasi-finite holomorphic mapping (i.e. with finite fibres) such that $f = p_3 \circ g_3$ and $g_3^{-1}(\{s\})$ is connected for all $s \in Z_3$.
$\#p_3^{-1}(\{s\})$ is the number of connected components of $f^{-1}(\{s\})$ whenever $s \in S$.*

Lemma 5.3. *a) Z_2 and Z_3 do not contain isolated points as soon as f is open (e.g. flat).*

b) We have holomorphic mappings $Z_1 \to Z_2 \leftarrow Z_3$.

From Theorem 5.2a) and Theorem 4.1b) we obtain:

Theorem 5.4. *Under the hypothesis of Theorem 4.1b), p_1 is an unramified covering.*

By Lemma 4.2:

Lemma 5.5. *a) If X is locally irreducible then we have that $Z_1 \to Z_2$ is injective.*

b) If X is locally irreducible and f is open then we have $Z_1 \simeq Z_2$, so there is a holomorphic mapping $g_1 : X \to Z_1$: $f = p_1 \circ g_1$.

Remark: If f is proper we have $Z_2 = Z_3$, so we are not obliged to permit non-Hausdorff spaces. Furthermore, by the proof of the Stein factorization theorem, $f_* \mathcal{O}_X = \mathcal{O}_{Z_2}$.

If moreover $Z_1 \simeq Z_2$ we get that $f_* \mathcal{O}_X$ is locally free.

Now we pass to geometric notions:

Since we are mainly interested in the local case assume that $S = \mathbb{C}$ or that S is an open disc around 0 in $\mathbb{C}$, let $S^* := S \setminus \{0\}$, $X^* := f^{-1}(S^*)$. Assume that f defines a topological fibre bundle over S^*.

Definition: Let $(t_n)_{(n=1,\ldots)}$ be a sequence in S^*, $t := t_1$, $t_n \to 0$. Let C_{t_n} be a connected component of F_{t_n}, C_0 a connected component of F_0.

So C_{t_n}, C_0 represent elements of Z_3.

Assume that $C_{t_n} \to C_0$: then there is an N such that the image of C_{t_n} in $\pi_0(X^*)$ does not depend on n if $n > N$.

So we may assume that the image of C_{t_n} in $\pi_0(X^*)$ does not depend on n at all.

Note that the question whether $C_{t_n} \to C_0$ then depends only on this image, hence on C_{t_1}.

a) We say that C_{t_n} tends to infinity if there is no such C_0. This means that the image of C_t in $\pi_0(X)$ does not belong to the image of $\pi_0(F_0)$. See also [6, Def. 2.1].
b) We say that C_{t_n} splits if there are at least two such C_0, i.e. that the image of C_{t_n} in $\pi_0(X)$ is the image of several (i.e. at least two) elements of $\pi_0(F_0)$.
c) We say that C_0 is not a limit (or: isolated) if there is no such (C_{t_n}), i.e. the image of C_0 in $\pi_0(X)$ does not belong to the image of $\pi_0(X^*)$.
d) We say that C_0 is a multiple limit if there are (C_{t_n}) which corresponds to different components of X^*, i.e. there are several elements of $\pi_0(X^*)$ which have the same image in $\pi_0(X)$ as C_0.

Examples:

a) $f : \mathbb{C}^2 \to \mathbb{C} : f(z) := z_1(z_1z_2 - 1)$ (Broughton). Then $Z_1 = Z_2 = \mathbb{C}$, $Z_3 =$ two copies of $\mathbb{C}$, glued together along $\mathbb{C}^*$.
F_1 is connected and splits into the two components $z_1 = 0$ resp. $z_1z_2 = 1$.

b) $f : \{z \in \mathbb{C}^2 \mid z_1z_2 = 0, z \neq 0\} \to \mathbb{C} : z \mapsto z_1$: $Z_1 = \mathbb{C}$, $Z_2 = \mathbb{C} \dot{\cup} \{pt\}$, $Z_3 = \mathbb{C}^* \dot{\cup} \{pt\}$.
F_1 is connected and tends to ∞; F_0 is connected and isolated.

c) $f : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C} : z \mapsto z_1z_2$: $Z_1 = Z_2 = \mathbb{C}$, $Z_3 =$ two copies of $\mathbb{C}$, glued together along $\mathbb{C}^*$.
F_1 is connected and splits into $\mathbb{C}^* \times \{0\}$ and $\{0\} \times \mathbb{C}^*$.

d) $f : \{z \in \mathbb{C}^3 \setminus (\{0\} \times \mathbb{C}) \mid z_1z_2 = 0\} \to \mathbb{C} : z \mapsto z_3$: $Z_1 = Z_2 = Z_3 = \mathbb{C} \dot{\cup} \mathbb{C}$.
Here we have a continuous behaviour.

e) $f : (\mathbb{C}^2 \times \{(0,0)\}) \cup (\{(0,0)\} \times \mathbb{C}^2 \to \mathbb{C} : z \mapsto z_1 - z_3$: $Z_1 = \mathbb{C} \dot{\cup} \mathbb{C}$, $Z_2 = Z_3 = \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$.
$F_0 = \{0\} \times \mathbb{C} \times \{(0,0)\} \cup \{(0,0,0)\} \times \mathbb{C}$ is a multiple limit: of $\{t\} \times \mathbb{C} \times \{(0,0)\} \subset F_t$ as well as of $\{(0,0)\} \times \{-t\} \times \mathbb{C} \subset F_t$.

f) $f : \mathbb{C}^2 \to \mathbb{C} : z \mapsto z_1^2$: $Z_1 = Z_2 = Z_3 = \mathbb{C}$, $p_1(t) = t^2$.

g) $f : \mathbb{C}^* \to \mathbb{C}$ inclusion: $Z_1 = Z_2 = \mathbb{C}$, $Z_3 = \mathbb{C}^*$.
F_1 is connected and tends to ∞.

h) $f : \{z \in \mathbb{C}^3 \setminus \{0\} \mid z_1z_2 = z_3^2\} \to \mathbb{C}$: $z \mapsto z_3^2$.
$Z_1 = Z_2 = \mathbb{C}$, Z_3 arises from two copies of $\mathbb{C}$ which are glued along $\mathbb{C}^*$, $p_i(z) = z^2$.
The connected components of F_1 split, each connected component of F_0 is a double limit.

Note that in the cases b) and h) the Euler characteristic of the fibres is constant but this does not mean that the connected components of the fibres behave continuously!

Instead of a compactifiable mapping f we may also treat the local case: Let $f : X \setminus Y \to \mathbb{C}$ be as in section 1–3, $S = \mathbb{D}_\rho$. Then the results of this section still hold.

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