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On the almost generic covers of the projective plane

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Dedicated to G.-M. Greuel on the occasion of his 75th birthday

Abstract: A finite morphism $f\colon X\to\mathbb{P}^2$ of a a smooth irreducible projective surface X is called an almost generic cover if for each point $p\in\mathbb{P}^2$ the fibre $f^{-1}(p)$ is supported at least on $\deg f-2$ distinct points and f is ramified with multiplicity two at a generic point of its ramification locus R. In the article, the singular points of the branch curve $B\subset\mathbb{P}^2$ of an almost generic cover are investigated and main invariants of the covering surface X are calculated in terms of invariants of the curve B.
Keywords: Covers of the projective plane, monodromy groups of covers.

0. Introduction

Let X be a smooth irreducible projective surface. A finite morphism $f\colon X\to\mathbb{P}^2$, branched along a curve $B\subset\mathbb{P}^2$, is called a *generic cover of the projective plane* if it has the following properties:

(G_1) for each point $p\in\mathbb{P}^2$ the fibre $f^{-1}(p)$ is supported on at least $\deg f-2$ distinct points,
(G_2) f is ramified with multiplicity 2 at a generic point of its ramification locus R,
(G_3) the singular points of B are only the ordinary nodes and ordinary cusps.

In particular, if X is imbedded in some projective space $\mathbb{P}^n$ then it is well known (see, for example, [3]) that the restriction $f\colon X\to\mathbb{P}^2$ to X of a linear projection $\mathrm{pr}\colon\mathbb{P}^n\to\mathbb{P}^2$ generic with respect to the imbedding of X, is a generic cover of $\mathbb{P}^2$.

Properties of generic covers of $\mathbb{P}^2$ were investigated in [5]–[7] and [9].

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The notion of generic covers of $\mathbb{P}^2$ can be generalized as follows. We say that a finite morphism $f\colon X\to\mathbb{P}^2$ is an *almost generic cover* of the projective plane if it satisfies properties (G_1) and (G_2) of generic covers.

Chisini's Conjecture claims that a curve $B\subset\mathbb{P}^2$ satisfying property (G_3) can be the branch curve of at most one generic cover $f\colon X\to\mathbb{P}^2$, $\deg f\geq 5$. This conjecture was proved in [6] for generic linear projections and it was proved in the general case if $\deg f\geq 12$ ([9]). In our opinion, it is of interest to find wider classes of finite covers of the plane whose branching curves uniquely determine the covers of these classes. In particular, it is interesting to check "Chisini's Conjecture" for the branch curves of almost generic covers of $\mathbb{P}^2$. As a first step in this direction, the aim of this paper is to investigate the types of singular points of the branch curve of an almost generic cover and to compute the basic invariants of the covering surface of this cover in terms of invariants of the branch curve and degree of the cover.

A dominant morphism $f\colon X\to\mathbb{P}^2$ defines a homomorphism $f_*\colon\pi_1(\mathbb{P}^2\setminus B,p)\to\mathbb{S}_{\deg f}$ (called the *monodromy* of f) whose image $G_f:=f_*(\pi_1(\mathbb{P}^2\setminus B,p))$ is the *monodromy group* of f and it is a subgroup of the symmetric group $\mathbb{S}_{\deg f}$ acting on the fibre $f^{-1}(p)=\{q_1,\dots,q_{\deg f}\}$.

Let o be a point of a curve $B\subset\mathbb{P}^2$. It is well known that the group $\pi_1^{loc}(B,o):=\pi_1(V\setminus B)$ does not depend on V, where $V\subset\mathbb{P}^2$ is a sufficiently small complex analytic neighbourhood biholomorphic to a ball of small radius centered at o. The image $G_{f,o}:=\operatorname{im} f_*\circ i_*$ is called the *local monodromy group* of f at the point o, where $i_*\colon\pi_1^{loc}(B,o)=\pi_1(V\setminus B,q)\to\pi_1(\mathbb{P}^2\setminus B,p)$ is a homomorphism defined (uniquely up to conjugation) by the imbedding $V\hookrightarrow\mathbb{P}^2$.

A complete description of the monodromy group of an almost generic cover of $\mathbb{P}^2$ and the local monodromy groups at the points of its branch curve is given by the following

Theorem 1. *The monodromy group G_f of an almost generic cover $f\colon X\to\mathbb{P}^2$ coincides with $\mathbb{S}_{\deg f}$.*

The branch curve B of the cover f can have only the singular points of type A_n and the points of B are divided into three types according to the types of singularities of B at these points and properties of the local monodromy groups:

- (i) *$p\in B\setminus\operatorname{Sing}B$ and $G_{f,p}\simeq\mathbb{Z}_2$ is generated by a transposition;*
- (ii) *$p\in\operatorname{Sing}B$ is of type $A_{n,2}$, that is (by definition), B has the singularity of type A_{2n-1} at p, $n\in\mathbb{N}$, and $G_{f,p}\simeq\mathbb{Z}_2\times\mathbb{Z}_2$ is generated by two commuting transpositions;*

(*iii*) $p \in \operatorname{Sing} B$ *is of type* $A_{n,3}$*, that is* (*by definition*), B *has the singularity of type* A_{3n-1} *at* p, $n \in \mathbb{N}$, *and* $G_{f,p} \simeq \mathbb{S}_3$ *is generated by two non-commuting transpositions.*

Proof of Theorem 1 is based on the following complete classification of the germs of three-sheeted smooth finite covers.

Theorem 2. *Let* (U, o') *and* (V, o) *be two connected germs of smooth complex-analytic surfaces and* $f\colon (U, o') \to (V, o)$ *a finite three-sheeted cover. Then there are local coordinates* z, w *in* (U, o') *and* u, v *in* (V, o), *and a non-negative integer* $n \in \mathbb{Z}_{\geq 0}$ *such that* f *coincides with the cover* $f_n\colon (U, o') \to (V, o)$ *given by*

$$\begin{aligned} u &= z, \\ v &= w^3 - nz^n w, \qquad n \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Theorems 1 and 2 are proved in Section 1.

Let $f\colon X \to \mathbb{P}^2$ be an almost generic cover branched along a curve B. Below, we use the following notations:

- $d := \frac{1}{2}\deg B$;
- n_k, the number of singular points of B of type $A_{2k+1,3}$, $k \in \mathbb{Z}_{\geq 0}$;
- m_k, the number of singular points of B of type $A_{2k,3}$, $k \in \mathbb{N}$;
- t_k, the number of singular points of B of type $A_{k,2}$, $k \in \mathbb{N}$;
- $\mathfrak{c} = \sum_{k=0}^{\infty}((2k+1)n_k + 2km_k)$, $\quad \mathfrak{n} = \sum_{k=1}^{\infty} kt_k$, $\quad \mathfrak{s} = \sum_{k=1}^{\infty} k(n_k + m_k)$.

Note that the numbers $\mathfrak{c}$, $\mathfrak{n}$, and $\mathfrak{s}$ are well-defined, since only for finitely many k the numbers n_k, m_k, and t_k do not vanish.

In Section 2, we compute the squares of canonical class and the Euler characteristic of the structure sheaf of the covering surface of an almost generic cover $f\colon X \to \mathbb{P}^2$ with irreducible branch curve $B \subset \mathbb{P}^2$ in terms of degree, numbers and singularity types of B. The obtained formulas coincide with similar formulas for generic covers of the plane if we replace c (the number of ordinary cusps) with $\mathfrak{c}$ and n (the number of ordinary nodes) with $\mathfrak{n}$ in the formulas for generic covers of the plane in [5] (compare Claim 2.5, Propositions 2.1–2.3, Corollary 2.1, and Claim 2.4 in Section 2 with Lemmas 4, 6–8, Corollary 2, and the formula for the degree of the dual curve of the branch curve B in [5]). Therefore, we call $\mathfrak{c}$ the *number of pseudo-cusps* and $\mathfrak{n}$ the *number of pseudo-nodes* of the branch curve B. And in view of Claim 2.3, the number $\mathfrak{s}$ is called the *superabundance*. Also in Section 2, we investigate the singular points of the Galoisations of almost generic covers of the plane and calculate main invariants of the desingularisations of their covering surfaces.

1. Proof of Theorems 1 and 2

1.1. Covers f_n, $n \geq 1$

Consider a finite cover $f_n \colon \mathbb{C}^2 \to \mathbb{C}^2$, $n \geq 1$, introduced in Theorem 2. If we perform the coordinate change $(z_1, w_1) = (\sqrt[n]{\frac{n}{3}}z, w)$, $(u_1, v_1) = (\sqrt[n]{\frac{n}{3}}u, v)$, then in new coordinates the cover f_n is given by

$$\begin{aligned} u_1 &= z_1, \\ v_1 &= w_1^3 - 3z_1^n w_1, \qquad n \in \mathbb{N}. \end{aligned} \tag{1}$$

Claim 1.1. *The branch curve B_n of the cover f_n, $n \geq 1$, has the singularity of type A_{3n-1} at the point $o = (0, 0)$.*

Proof. The ramification curve R_n of f_n is given by equation

$$J(f_n) := \det \begin{pmatrix} 1 & 0 \\ \frac{\partial v_1}{\partial z_1} & \frac{\partial v_1}{\partial w_1} \end{pmatrix} = 0,$$

i.e., R_n is given by equation

$$w_1^2 - z_1^n = 0. \tag{2}$$

Let $n = 2k + \delta$, where $\delta = 0$ or 1 depending on the parity of n.

If $n = 2k$ is an even number, then R_{2k} consists of two irreducible components, $R_{2k} = R_+ \cup R_-$, given by equations $w = \pm z^k$. Therefore the ramification curve $B_{2k} = B_+ \cup B_-$, where, due to (1), $B_+ = f_n(R_+)$ and $B_- = f_n(R_-)$ parametrically given by equations

$$\begin{aligned} u_1 &= z_1, \\ v_1 &= \mp 2z_1^{3k} \end{aligned}$$

and hence, B_{2k} is given by equation

$$v_1^2 - 4u_1^{6k} = 0 \tag{3}$$

i.e., in the case when $n = 2k$, the point o is the singular point of B_n of type A_{3n-1}.

If $n = 2k + 1$ is an odd number, then R_n is an irreducible curve given by equation $w^2 - z^n = 0$. Therefore, R_n can be given parametrically by equations $z = t^2$ and $w = t^{2k+1}$ and, due to (1), the ramification curve B_n parametrically given by equations

$$u_1 = t^2,$$
$$v_1 = -2t^{6k+3}$$

Hence, B_n is given by equation

$$v_1^2 - 4u_1^{6k+3} = 0, \tag{4}$$

i.e., if $n = 2k + 1$ then o is the singular point of B_n of type A_{3n-1}. □

1.2. Proof of Theorem 2

Denote $M_1 = f^{-1}(L_1)$ and $M_2 = f^{-1}(L_2)$ for $L_1 = \{u_0 = 0\}$ and $L_2 = \{v_0 = 0\}$, where u_0, v_0 are some local complex-analytic coordinates in (V, o). Then the local intersection number of the curves M_1 and M_2 at the point o' is equal to $(M_1, M_2)_{o'} = \deg_{o'} f = 3$. Therefore either M_1 or M_2 is a germ of a non-singular curve. Let M_1 be non-singular. Then we can choose local coordinates z_0, w_0 in (U, o') such that $f^*(u_0) = z_0$ and $f^*(v_0) = v_0(z_0, w_0) = \sum_{i=0}^{\infty} a_i(z_0)w_0^i$, where $a_i(z_0) = \sum_{j=0}^{\infty} a_{i,j}z_0^j \in \mathbb{C}[[z_0]]$. Performing the coordinates change $v_0 \leftrightarrow v_0 - a_0(u_0)$, we can assume that $a_0(z_0) \equiv 0$. In addition, we have $a_{1,0} = a_{2,0} = 0$ and can assume that $a_{3,0} = 1$, since $(M_1, M_2)_{o'} = 3$.

Denote $R \subset (U, o')$ the ramification curve of the cover f and $B = f(R) \subset (V, o)$ the branch curve. Note that the curves $R \subset (U, o')$ and $B \subset (V, o)$ depend only on the cover f and do not depend on the choice of coordinates in (U, o') and (V, o). Denote $\mathfrak{m} \subset \mathbb{C}[[z_0, w_0]]$ the maximal ideal in the ring of power series $\mathbb{C}[[z_0, w_0]]$. The curve R is given by equation

$$J(f) := \det \begin{pmatrix} 1 & 0 \\ \frac{\partial v_0}{\partial z_0} & \frac{\partial v_0}{\partial w_0} \end{pmatrix} = 0,$$

i.e., R is given by equation

$$\sum_{i=1}^{\infty} i a_i(z_0) w_0^{i-1} = 0. \tag{5}$$

Let us write equation (5) in the following form

$$a_{1,1}z_0 + a_{1,2}z_0^2 + 2a_{2,1}z_0w_0 + 3w_0^2 + H(z_0, w_0) = 0, \tag{6}$$

where $H(z_0, w_0) \in \mathfrak{m}^3$. It follows from (6) that there are three possibilities: either $R = 2R_1$, where R_1 is a germ of a smooth curve, or $R = R_1 \cup R_2$, where R_1 and R_2 are germs of smooth curves, or R is an irreducible germ and o' is either a smooth point (if $a_{1,1} \neq 0$), or a singular point of multiplicity two (if $a_{1,1} = 0$). In the first case the cover f is ramified along R_1 with multiplicity three.

Claim 1.2. *If the finite cover f is ramified along R_1 with multiplicity three then the branch curve B is smooth and f coincides with the cover f_0.*

Proof. It follows from (6) that the germ R_1 is given by equation of the form

$$w_0 + \alpha z_0 + H_1(z_0, w_0) = 0,$$

where $\alpha \in \mathbb{C}$ and $H_1(z_0, w_0) \in \mathfrak{m}^2$. In the new system of coordinates $z_1 = z_0$, $w_1 = w_0 + \alpha z_0 + H_1(z_0, w_0)$, the cover f is given by functions $u = z_1$ and $v_0 = v_1(z_1, w_1)$, where $v_1(z_1, w_1)$ has the following property:

$$\frac{\partial v_1}{w_1} = w_1^2(3 + H_2(z_1, w_1)),$$

with some $H_2(z_1, w_1) \in \mathfrak{m}$. Therefore

$$v_0 = H_0(z_1) + w_1^3(1 + \frac{1}{w_1^3}\int w_1^2 H_2(z_1, w_1)dw_1),$$

where $\frac{1}{w_1^3}\int w_1^2 H_2(z_1, w_1)dw_1 \in \mathfrak{m}$. Now, it is easy to see that in the new systems of coordinates

$$z = z_1, \ w = w_1\sqrt[3]{1 + \frac{1}{w_1^3}\int w_1^2 H_2(z_1, w_1)dw_1},$$

and $u = u_0$, $v = v_0 - H_0(u_0)$ the cover f coincides with f_0. Note that the branch curve B of the cover f_0 is smooth and it is given by equation $v = 0$. □

Now, we assume that the cover f is ramified along R with multiplicity two.

Claim 1.3. *The restriction $f_{|R}\colon R \to B$ of the cover f to the ramification locus R is one-to-one mapping.*

Proof. Obviously, $3 = \deg f \geq 2 \deg f_{|R}$. Therefore $\deg f_{|R} = 1$. □

Claim 1.4. *Let the branch curve $B \subset (V, o)$ of the cover f have a singularity of type A_m at the point o. Then $m = 3n - 1$ for some $n \in \mathbb{N}$ and f coincides with the cover f_n.*

Proof. The cover $f\colon (U, o') \to (V, o)$ defines a homomorphism

$$f_*\colon \pi_1^{loc}(B, o) = \pi_1(V \setminus B, p) \to \mathbb{S}_3,$$

where $\mathbb{S}_3$ is the symmetric group acting on the fibre $f^{-1}(p)$. Note that the epimorphism f_* is defined uniquely only if we fix a numbering of the points of $f^{-1}(p)$ and in general case it is defined uniquely up to an inner automorphism of $\mathbb{S}_3$.

It is well known (see, for example, [8]) that if $m = 2k - \delta$, where $\delta = 0$ or 1, then the group $\pi_1^{loc}(B, o)$ is generated by two so called geometric generators γ_1 and γ_2 such that $\pi_1^{loc}(B, o)$ has the following presentation:

$$\pi_1^{loc}(B, o) = \langle \gamma_1, \gamma_2 \mid (\gamma_1\gamma_2)^k\gamma_1^{1-\delta} = (\gamma_2\gamma_1)^k\gamma_2^{1-\delta}\rangle. \tag{7}$$

Denote $\tau_i := f_*(\gamma_i) \in \mathbb{S}_3$, $i = 1, 2$. The branch curve B is singular. Therefore, by Claim 1.2, τ_i are not cycles of length three, i.e., τ_1 and τ_2 are transpositions generating the group $\mathbb{S}_3$, since (U, o') is a germ of an irreducible surface. Without loss of generality, we can assume that $\tau_1 = (1, 3)$ and $\tau_2 = (2, 3)$, i.e., up to conjugation there is the unique epimorphism from $\pi_1^{loc}(B, o)$ to $\mathbb{S}_3$. We have $\tau_1\tau_2 = (1, 2, 3)$, $\tau_2\tau_1 = (1, 3, 2)$ and it follows from (7) that

$$(1, 2, 3)^k(1, 3)^{1-\delta} = (1, 3, 2)^k(2, 3)^{1-\delta}.$$

If $\delta = 1$ then $k = 3k_1$, i.e., $m = 3(2k_1) - 1$ and if $\delta = 0$ then $k = 3k_1 - 2$, i.e., $m = 2(3k_1 - 2) = 3(2k_1 - 1) - 1$, where $k_1 \geq 1$. By Grauert–Remmert–Riemann–Stein Theorem ([11], [2]), the cover $f\colon (U, o') \to (V, o)$ is uniquely defined by the epimorphism $f_*\colon \pi_1^{loc}(B, o) = \pi_1(V \setminus B, p) \to \mathbb{S}_3$. Now, to complete the proof of Claim 1.4, it suffices to apply Claim 1.1. □

It follows from Claim 1.4 that to prove Theorem 2, it suffices to show that in the second and third cases the branch curve B has the singularity of type A_m for some $m \geq 1$. Therefore it suffices to show (see, for example, [1]) that the multiplicity of the singular point o of B is equal to 2.

It follows from (6) that in the second case $a_{1,1} = 0$ and the germ $R = R_1 \cup R_2$ is the union of two curves smooth at o'. In addition, it is easy to see that equations of R_i, $i = 1, 2$, have the following form

$$w_0 + \alpha_i z_0 + H_i(z_0, w_0) = 0,$$

where $H_i(z_0, w_0) \in \mathfrak{m}^2$ and $\alpha_i \in \mathbb{C}$. Therefore the function z_0 is a local parameter at o' for the germs R_i, $i = 1, 2$, i.e., $B_i = f(R_i) \subset V$ are germs of smooth curves at the point o, since $u_0 = z_0$. Hence, the multiplicity of the singular point o of B is equal to 2. As a result, we obtain that B has a singularity of type A_{2m-1} for some m equals to the intersection number $(B_1, B_2)_o$.

In the third case, denote by $\nu\colon \widetilde{R} \to R$ the resolution of singular point o of R (if $a_{1,1} \neq 0$ then $\nu = id$), $p = \nu^{-1}(o')$, and denote by $\mathfrak{m}_{\widetilde{R},p}$ the maximal ideal of the ring of holomorphic functions at p on $\widetilde{R}$.

It follows from (6) that $(L_1, R)_{o'} = 2$. Therefore $\nu^*(u_0) \in \mathfrak{m}^2_{\widetilde{R},p} \setminus \mathfrak{m}^3_{\widetilde{R},p}$ and the function $t = \sqrt{\nu^*(u_0)}$ is a local parameter on $\widetilde{R}$ at p. Let $\nu^*(w_0) = w(t) \in \mathfrak{m}_{\widetilde{R},p}$. Then R is given parametrically by

$$z_0 = t^2, \qquad w_0 = w(t)$$

and therefore B is given by

$$u_0 = t^2, \qquad v_0 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j} t^{2j} w(t)^i, \tag{8}$$

where $v_0(t) \in \mathfrak{m}^3_{\widetilde{R},p}$, since $a_{1,0} = a_{2,0} = 0$. It follows from (8) that the multiplicity of B the point o is equal to 2. Therefore the singularity type of B at o is A_{2m} for some $m \in \mathbb{N}$.

1.3. Proof of Theorem 1

In the beginning, we prove the following

Claim 1.5. *Let (U, o') and (V, o) be two germs of smooth complex-analytic surfaces and $f\colon (U, o') \to (V, o)$ a finite two-sheeted cover. Then there are local coordinates z, w in (U, o) and u, v in (V, o) such that f is given by equations $u = z$, $v = w^2$.*

The ramification locus $R \subset (U, o')$ and the branch curve $B \subset (V, o)$ of the cover f are smooth curves and they are given, respectively, by equations $w = 0$ and $v = 0$.

Proof. As in the proof of Theorem 2, denote $M_1 = f^{-1}(L_1)$ and $M_2 = f^{-1}(L_2)$ for $L_1 = \{u_0 = 0\}$ and $L_2 = \{v_0 = 0\}$, where u_0, v_0 are some local complex-analytic coordinates in (V, o). Then the local intersection number of the curves M_1 and M_2 at the point o' is equal to $(M_1, M_2)_{o'} = \deg_{o'} f = 2$.

Therefore either M_1 or M_2 is a germ of a non-singular curve. Let M_1 be non-singular. Then we can choose local coordinates z_0, w_0 in (U, o') such that

$$f^*(u_0) = z_0 \quad \text{and} \quad f^*(v_0) = v_0(z_0, w_0) = \sum_{i=0}^{\infty} a_i(z_0) w_0^i, \tag{9}$$

where $a_i(z_0) = \sum_{j=0}^{\infty} a_{i,j} z_0^j \in \mathbb{C}[[z_0]]$. Performing the coordinates change $v_0 \leftrightarrow v_0 - a_0(u_0)$, we can assume that $a_0(z_0) \equiv 0$. In addition, we have $a_{1,0} = 0$ and can assume that $a_{2,0} = 1$, since $(M_1, M_2)_{o'} = 2$.

The ramification curve $R \subset (U, o')$ of the cover f is given by equation

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} i a_{i,j} z_0^j w_0^{i-1} = 0. \tag{10}$$

It follows from (10) that (R, o') is the germ of a smooth curve, since $a_{2,0} \neq 0$. In addition, the function z_0 is a local parameter at o' for R. Therefore, by (9), the branch curve $B = f(R) \subset (V, o)$ is also smooth at the point o. To complete the proof of Claim 1.5, it suffices to apply Grauert–Remmert–Riemann–Stein Theorem. □

Let $V \subset \mathbb{P}^2$ be a sufficiently small neighbourhood of a point q of the branch curve B of an almost generic cover f. Let $\deg f = N$. It follows from property (G_1) that there are three possibilities:

(1) $f^{-1}(V)$ is a disjoint union of $N-1$ neighbourhoods $U_1, \dots, U_{N-1}$ such that $f\colon U_1 \to V$ is a two-sheeted cover and $f\colon U_i \to V$ are biholomorphic maps for $i = 2, \dots, N-1$;

(2) $f^{-1}(V)$ is a disjoint union of $N-2$ neighbourhoods $U_1, \dots, U_{N-2}$ such that $f\colon U_1 \to V$ and $f\colon U_2 \to V$ are two-sheeted covers and $f\colon U_i \to V$ are biholomorphic maps for $i = 3, \dots, N-2$;

(3) $f^{-1}(V)$ is a disjoint union of $N-2$ neighbourhoods $U_1, \dots, U_{N-1}$ such that $f\colon U_1 \to V$ is a three-sheeted cover and $f\colon U_i \to V$ are biholomorphic maps for $i = 2, \dots, N-1$.

Claim 1.6. *Let B_1 be an irreducible component of the branch curve $B \subset \mathbb{P}^2$ of an almost generic cover f and $R_1 = f^{-1}(B_1) \cap R$. Then the restriction $f_{|R_1}\colon R_1 \to B_1 = f(R_1)$ of f to R_1 is a birational morphism.*

Proof. It follows from property (G_1) that $\deg f_{|R_1} \leq 2$. Assume that $\deg f_{|R_1} = 2$. Then it follows from Claims 1.5 and 1.3 that B_1 is a smooth curve and $f_{|R_1}\colon R_1 \to B_1$ is unramified two-sheeted cover. Applying property (G_1), we obtain that $B = B_1$, since any two curves in $\mathbb{P}^2$ have non-empty intersection.

Therefore, by Zariski Theorem, the group $\pi_1(\mathbb{P}^2 \setminus B)$ is cyclic. Hence, the monodromy group $G_f \simeq \mathbb{Z}_2$ is generated in $\mathbb{S}_{\deg f}$ by product of two commuting transpositions. Therefore G_f can not act transitively on the set $f^{-1}(p)$ which contradicts the irreducibility of X. □

The surface X is irreducible. Therefore $X \setminus f^{-1}(B)$ is connected and hence, G_f acts transitively on the fibre $f^{-1}(p)$ over the base point p of the group $\pi_1(X \setminus f^{-1}(B), p)$. It follows from property (G_2) and Claim 1.6 that the monodromy group $G_f \subset \mathbb{S}_{\deg f}$ of an almost generic cover $f\colon X \to \mathbb{P}^2$ is generated by transpositions. Therefore the monodromy group G_f coincides with $\mathbb{S}_{\deg f}$.

It follows from Claim 1.5 that in case (1) locally at the point q, the branch curve B is smooth (the germ (B, q) is the branch curve of the restriction of f to the neighbourhood U_1). By Claim 1.5, the local fundamental group $\pi_1^{loc}(B, q)$ is generated by circuit γ around B and the local monodromy group $G_{f,q}$ is generated by transposition $\tau = f_*(\gamma)$.

It follows from Claim 1.5 that in case (2) locally at the point q, the branch curve B consists of two smooth components (denote them by B_1 and B_2), B_1 is the branch curve of the restriction of f to the neighbourhood U_1 and B_2 is the branch curve of the restriction of f to the neighbourhood U_2. Note that by Claim 1.6, we have $B_1 \neq B_2$. Therefore B at the point q has the singularity of type A_{2k-1}, where $k = (B_1, B_2)_q$. The local fundamental group $\pi_1^{loc}(B, q)$ is generated by circuits γ_1 around B_1 and γ_2 around B_2 and the local monodromy group $G_{f,q}$ is generated by commuting transpositions $\tau_1 = f_*(\gamma_1)$ and $\tau_2 = f_*(\gamma_2)$. Therefore q is a point of type $A_{k,2}$.

Using Theorem 2 and Claim 1.1, it is easy to see that in case (3) the point q is of type $A_{k,3}$ for some $k \in \mathbb{N}$.

2. Relations between invariants of covering surfaces of almost generic covers and invariants of their branch curves

2.1. Invariants of the branch and ramification curves

Let $f\colon X \to \mathbb{P}^2$ be an almost generic cover. We will assume that the branch curve $B \subset \mathbb{P}^2$ of f is irreducible and let $R \subset X$ be the ramification locus of f.

Claim 2.1. *The degree of B is an even number,* $\deg B = 2d$, $d \in \mathbb{N}$.

Proof. Let L be a line in $\mathbb{P}^2$ generic with respect to B and $M = f^{-1}(L)$. By property (G_2), the restriction $f_{|M}\colon M \to L$ of f to M is a generic cover branched over the common points of L and B and by Hurwitz formula, a generic cover of $\mathbb{P}^1$ is branched over even number of points. Therefore $\deg B = (L,B)_{\mathbb{P}^2}$ is an even number. □

The following claim is well known.

Claim 2.2. *If $s \in \operatorname{Sing} B$ is of type $A_{k,2}$ then its δ-invariant is equal to k; if s is of type $A_{2k+1,3}$, $k \geq 0$, then its δ-invariant is equal to $3k+1$; if s is of type $A_{2k,3}$, $k \geq 1$, then its δ-invariant is equal to $3k$.*

Claim 2.3. *The geometric genus $g(B)$ of B is equal to*

$$g(B) = (2d-1)(d-1) - \mathfrak{c} - \mathfrak{n} - \mathfrak{s}. \tag{11}$$

Proof. By Theorem 1, the singular points of the curve B are of the types $A_{n,2}$ and $A_{n,3}$, $n \in \mathbb{N}$. Therefore Claim 2.3 follows from Claim 2.2. □

Claim 2.4. *The degree $\hat{d} = \deg \hat{B}$ of the dual curve $\hat{B}$ of B is equal to*

$$\hat{d} = 2d(2d-1) - 3\mathfrak{c} - 2\mathfrak{n}.$$

Proof. If $s \in \operatorname{Sing} B$ is of type $A_{k,2}$ then its number of virtual cusps vanishes and the number of its virtual nodes is equal to k; if s is of type $A_{2k+1,3}$, $k \geq 0$, then its number of virtual cusps is equal to 1 and its number of virtual nodes is equal to $3k$; if s is of type $A_{2k,3}$, $k \geq 1$, then its number of virtual cusps vanishes and its number of virtual nodes is equal to $3k$. Now, Claim 2.4 follows from generalized Plükker's formula (see [4]). □

Claim 2.5. *The self-intersection number $(R^2)_X$ of R is positive and it is equal to*

$$(R^2)_X = 2d^2 - \mathfrak{c} - \mathfrak{n}. \tag{12}$$

Proof. It follows from Theorems 1 and 2, and equation (2) that s is a singular point of the curve R (and its singular type is A_{n-1}) iff $f(s)$ is a point of type $A_{n,3}$, $n \geq 1$. Therefore

$$2(g(R)-1) = (K_X + R, R)_X - 2\sum_{k=1}^{\infty} k(n_k + m_k). \tag{13}$$

By Claim 1.6, we have $g(B) = g(R)$. In addition, $K_X = f^*(K_{\mathbb{P}^2}) + R$ and

$$(f^*(K_{\mathbb{P}^2}), R)_X = -3 \deg B = -6d.$$

Therefore, equality (12) follows from (11) and (13) and it follows from Claim 2.4 that $(R^2)_X > 0$, since $\hat{d} > 0$. □

2.2. Invariants of the covering surfaces

Let N be the degree of an almost generic cover $f\colon X \to \mathbb{P}^2$.

Proposition 2.1. *The self-intersection number K_X^2 is equal to*

$$K_X^2 = 9N + 2(d^2 - 6d) - \mathfrak{c} - \mathfrak{n}. \tag{14}$$

Proof. We have $K_X = f^*(K_{\mathbb{P}^2}) + R$. Therefore, by Claim 2.5,

$$\begin{aligned} K_X^2 = (f^*(K_{\mathbb{P}^2}), f^*(K_{\mathbb{P}^2}))_X + 2(f^*(K_{\mathbb{P}^2}), R)_X + R^2 = \\ 9N - 12d + 2d^2 - \mathfrak{c} - \mathfrak{n}. \end{aligned}$$ □

Denote by $e(M) = \sum_i (-1)^i \dim H^i(M, \mathbb{Q})$ the topological Euler characteristic of a topological space M.

Proposition 2.2. *The topological Euler characteristic $e(X)$ is equal to*

$$e(X) = 3N + 2d(2d - 3) - 3\mathfrak{c} - 2\mathfrak{n}.$$

Proof. We have $e(X) = N(e(\mathbb{P}^2) - e(B)) + (N-1)e(B \setminus \operatorname{Sing} B) + (N-2)e(\operatorname{Sing} B)$. Therefore Proposition 2.2 follows from (11) and equalities

$$e(B) = (2 - 2g(B)) - \sum_{k=1}^{\infty}(m_k + t_k), \tag{15}$$

$$e(B \setminus \operatorname{Sing} B) = (2 - 2g(B)) - n_0 - \sum_{k=1}^{\infty}(n_k + 2m_k + 2t_k), \tag{16}$$

$$e(\operatorname{Sing} B) = n_0 + \sum_{k=1}^{\infty}(n_k + m_k + t_k). \tag{17}$$ □

Proposition 2.3. *The Euler characteristic $\chi(\mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$ equals*

$$\chi(\mathcal{O}_X) = N + \frac{d(d-3)}{2} - \frac{\mathfrak{c}}{3} - \frac{\mathfrak{n}}{4}.$$

Proof. Proposition 2.3 follows from Noether's formula $K_X^2 + e(X) = 12\chi(\mathcal{O}_X)$. □

Corollary 2.1. *We have*

$$\mathfrak{c} \equiv 0 \ (\bmod 3), \quad \mathfrak{n} \equiv 0 \ (\bmod 4).$$

Proposition 2.4. *The following inequality holds*

$$N \leq \frac{4d^2}{2d^2 - \mathfrak{c} - \mathfrak{n}}.$$

Proof. Let $M = f^{-1}(L)$, where L is a line in $\mathbb{P}^2$. We have $(M^2)_X = N$ and $(M, R)_X = \deg f = 2d$. Applying Hodge Index Theorem to M and R and applying Claim 2.5, we obtain inequality

$$N(2d^2 - \mathfrak{c} - \mathfrak{n}) - 4d^2 \leq 0. \tag{18}$$

□

Claim 2.6. *If* $N \geq 6$ *then* $\mathfrak{n} > 0$.

Proof. Transforming the left side of inequality (18), we obtain the inequality

$$\frac{1}{2}[(N-6)(2d(2d-1)+2d-3\mathfrak{c}-2\mathfrak{n}+\mathfrak{c})+4(2d(2d-1)-3\mathfrak{c}-2\mathfrak{n}+2d)-4\mathfrak{n}] \leq 0$$

and applying Claim 2.4, we have

$$(N-6)(\hat{d} + 2d + \mathfrak{c}) + 4(\hat{d} + 2d) \leq 4\mathfrak{n}.$$

Now, Claim 2.6 follows from inequalities $\hat{d} > 0$ and $d > 0$. □

2.3. Galoisations of almost generic covers

The Cayley imbedding $c\colon G_f = \mathbb{S}_N \hookrightarrow \mathbb{S}_{N!}$, defined by the action of $\mathbb{S}_N$ on itself by multiplication from the right side, defines the Galois finite cover $g\colon Y \to \mathbb{P}^2$ branched along the curve B, $\deg g = N!$. For a point $p \in \mathbb{P}^2$ the fibre $g^{-1}(p)$ is supported on $M_{f,p} = \frac{N!}{|G_{f,p}|}$ distinct points, where $|G_{f,p}|$ is the order of the local monodromy group $G_{f,p} = G_{g,p}$, and if $V \subset \mathbb{P}^2$ is a sufficiently small complex-analytic neighbourhood of the point p, then $g^{-1}(V) = \bigsqcup_{i=1}^{M_{f,p}} W_i$ is a disjoint union of $M_{f,p}$ complex-analytic normal varieties W_i biholomorphic to each other and the restriction $g_{|W_i}\colon W_i \to V$ to each W_i is the Galois cover with the Galois group isomorphic to $G_{f,p}$.

If $p \in B \setminus \operatorname{Sing} B$ then for each $i = 1, \dots, M_{f,p} = \frac{N!}{2}$ the cover $g_{|W_i}\colon W_i \to V$ is a two-sheeted cover branched over the non-singular curve $B \cap V$.

If $p \in \operatorname{Sing} B$ and B has the singularity of type $A_{n,2}$ at p, then for each $i = 1, \dots, M_{f,p} = \frac{N!}{4}$ the cover $g_{|W_i}\colon W_i \to V$ is the Galois cover with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$ branched along $B \cap V$ with multiplicity two.

Claim 2.7. *If B has the singularity of type $A_{n,2}$ at p, then for $n \geq 2$ the point $g^{-1}_{|W_i}(p)$ is a singular point of W_i of type A_{n-1} and it is smooth point if $n = 1$.*

Proof. Let B_1 and B_2 be two irreducible branches of the curve $B \cap V$. Without loss of generality, we can assume that $u = 0$ is the equation of B_1 and $u - v^n = 0$ is the equation of B_2. Then W_i is biholomorphic to a neighbourhood of the point $o = (0,0,0,0)$ in the surface given in $\mathbb{C}^4$ by equations $z^2 = u$ and $w^2 = u - v^n$. Therefore, it is biholomorphic to a neighbourhood of the point $o' = (0,0,0)$ in the surface given in $\mathbb{C}^3$ by equation $w^2 = z^2 - v^n$. □

If $p \in \operatorname{Sing} B$ and B has the singularity of type $A_{n,3}$ at p, then for each $i = 1, \dots, M_{f,p} = \frac{N!}{6}$ the cover $g_{|W_i} \colon W_i \to V$ is the Galois cover with Galois group $\mathbb{S}_3$ branched along $B \cap V$ with multiplicity two.

Claim 2.8. *If B has the singularity of type $A_{n,3}$ at p, then for $n \geq 2$ the point $g^{-1}_{|W_i}(p)$ is a singular point of W_i of type A_{n-1} and it is smooth point if $n = 1$.*

Proof. The local fundamental group $\pi_1^{loc}(B,p)$ is generated by two circuits γ_1 and γ_2 around the curve B (see subsection 1.1) and the Galois cover $g_{|W_i} \colon W_i \to V$ is defined by epimorphism $g_* \colon \pi_1^{loc}(B,p) \to \mathbb{S}_3$ sending γ_i, $i = 1, 2$, to transpositions. The cover $g_{|W_i} \colon W_i \to V$ can be decomposed into a composition $g_{|W_i} = f_n \circ h_n$, where $h_n \colon W_i \to W_i/\langle\tau\rangle \simeq U \subset \mathbb{C}^2$ is the factor-map under the action of a subgroup $\langle\tau\rangle$ of the group $\mathbb{S}_3$ generated by a transposition $\tau \in \mathbb{S}_3$ and by Theorem 2, $f_n \colon U \to V$ is the restriction of the cover $f_n \colon \mathbb{C}^2 \to \mathbb{C}^2$ (see subsection 1.1) to some neighbourhood $U \subset \mathbb{C}^2$.

It is easy to see that $h_n \colon W_i \to U \subset \mathbb{C}^2$ is a two-sheeted cover branched along the curve $C_n \cap U$, where C_n is the complement to the ramification divisor R_n of the cover $f_n \colon \mathbb{C}^2 \to \mathbb{C}^2$ in the total inverse image of B_n, $f_n^*(B_n) = 2R_n + C_n$. Let $F_n(z_1, w_1) = 0$ be an equation of C_n. Then it follows from (2)–(4) that

$$(w_1^3 - 3z_1^n w_1)^2 - 4z_1^{3n} = (w_1^2 - z_1^n)^2 F_n(z_1, w_1) = 0.$$

Therefore we have $F_n(z_1, w_1) = w_1^2 - 4z_1^n$ and hence the surface W_i is isomorphic to a hypersurface in $\mathbb{C}^1 \times U$ given by $y^2 = w_1^2 - 4z_1^n$. □

It follows from Claims 2.7 and 2.8 that the covering surface Y of the Galoisation of an almost generic cover $f \colon X \to \mathbb{P}^2$ has

$$S = N!\Big(\frac{1}{6}\sum_{k=1}^{\infty}(n_k + m_k) + \frac{1}{4}\sum_{k=2}^{\infty} t_k\Big)$$

singular points.

Let $\nu\colon Z \to Y$ be the minimal resolution of singular points of the covering surface Y. It is well known that the inverse image $\nu^{-1}(s)$ of a point $s \in \operatorname{Sing} Y$ of singularity type A_k is the chain $E_1 \cup \dots \cup E_k$ of (-2)-curves E_i. Therefore, by Claims 2.7 and 2.8, the number of (-2)-curves contracted by ν is

$$M = \frac{N!}{6}\sum_{k=1}^{\infty}(2kn_k + (2k-1)m_k) + \frac{N!}{4}\sum_{k=2}^{\infty}(2k-1)t_k$$

and if we denote $\widetilde{g} := g \circ \nu\colon Z \to \mathbb{P}^2$, then

$$e(\widetilde{g}^{-1}(\operatorname{Sing} B)) = \frac{N!}{6}\sum_{k=0}^{\infty}((2k+1)n_k + 2km_k) + \frac{N!}{4}\Big(t_1 + \sum_{k=2}^{\infty} 2kt_k\Big). \tag{19}$$

Proposition 2.5. *The canonical class K_Z of the surface Z is equal to*

$$K_Z = (d-3)\widetilde{g}^*(L),$$

where L is a line in $\mathbb{P}^2$, and its self-intersection number K_Z^2 is equal to

$$K_Z^2 = (d-3)^2 N!. \tag{20}$$

Proof. Let $\nu^{-1}(\operatorname{Sing} Y) = \bigcup_{i=1}^{M} E_i$. We have

$$K_Z = \widetilde{g}^*(K_{\mathbb{P}^2}) + \frac{1}{2}\widetilde{g}^*(B) + \sum_{j=1}^{M}\alpha_j E_j,$$

since $\widetilde{g}$ is ramified over B with multiplicity two. In addition, we have $(K_Z, E_i)_Z = 0$ for each i, since E_i are rational curves with self-intersection number $(E_i^2)_Z = -2$, and $(\widetilde{g}^*(K_{\mathbb{P}^2}), E_i)_Z = (\widetilde{g}^*(B), E_i)_Z = 0$, since $\widetilde{g}(E_i)$ are points. Therefore $(\sum_{j=1}^{M}\alpha_j E_j, E_i)_Z = 0$ for each i and hence $\sum_{i=1}^{M}\alpha_i E_i = 0$, since the intersection matrix $E = ((E_j, E_i)_Z)$ is negatively defined. Now, to complete the proof of Proposition 2.5, notice that $K_{\mathbb{P}^2} = -3L$, the divisor B is equivalent to $2dL$, and $\deg \widetilde{g} = N!$. □

Proposition 2.6. *The topological Euler characteristic $e(Z)$ is equal to*

$$e(Z) = N!\Big[3 + d(2d-3) - \frac{1}{6}\sum_{k=0}^{\infty}((19k+5)n_k + 16km_k) - \frac{3}{4}t_1 - \frac{1}{2}\sum_{k=2}^{\infty}kt_k\Big].$$

Proof. We have

$$e(Z) = N!(e(\mathbb{P}^2) - e(B)) + \frac{N!}{2}e(B \setminus \operatorname{Sing} B) + e(\widetilde{g}^{-1}(\operatorname{Sing} B))$$

and Proposition 2.6 follows from (11), (15), (16), and (19). □

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