Pure and Applied Mathematics Quarterly
Volume 16, Number 3, 659–674, 2020

Curve counting on $\mathcal{A}_n \times \mathbb{C}^2$

Yalong Cao*

To Professor Kyoji Saito with greatest admiration on the occasion of his 75th birthday

Abstract: Let $\mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}$ be the minimal resolution of A_n-singularity and $X = \mathcal{A}_n \times \mathbb{C}^2$ be the associated toric Calabi-Yau 4-fold. In this note, we study curve counting on X from both Donaldson-Thomas and Gromov-Witten perspectives. In particular, we verify conjectural formulae relating them proposed by the author, Maulik and Toda.

Keywords: Curve counting, A_n-surfaces, Calabi-Yau 4-folds.

1. Introduction

There are many studies of curve counting on resolutions of ADE singularities (e.g. [BG1, BG2, BG3, M]). The perspective of this note is to work with a toric Calabi-Yau 4-fold:

$$X = \mathcal{A}_n \times \mathbb{C}^2,$$

where $\pi : \mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}$ is the minimal resolution of A_n-singularity, and to study Donaldson-Thomas and Gromov-Witten invariants on X. As X is non-compact, we define counting invariants using torus localization. Let $T \subseteq (\mathbb{C}^*)^4$ be the 3-dimensional subtorus which preserves the holomorphic volume form on X. It lifts to actions on several moduli spaces:

- moduli space $\overline{M}_{g,0}(X, \beta)$ of genus g stable maps,
- moduli space $M_{X,\beta}$ of one dimensional stable sheaves E with $[E] = \beta$ and $\chi(E) = 1$,
- moduli space $P_m(X, \beta)$ of PT stable pairs $(s : \mathcal{O}_X \to F)$ with $[F] = \beta$ and $\chi(F) = m$.

Received January 30, 2019.
2010 Mathematics Subject Classification: Primary 14N35, 14J32.
*The author is partially supported by the World Premier International Research Center Initiative (WPI), MEXT, Japan, the JSPS KAKENHI Grant Number JP19K23397 and Royal Society Newton International Fellowships Alumni 2019.

They all have finitely many points as torus fixed locus and we may define corresponding counting invariants by localization formulae [GP, CMT1, CMT2, CK1, CK2, CKM, CT1, CT3, CT4]. In particular, we have

$$\mathrm{GW}_{0,\beta}(X),\ \mathrm{DT}_4(X,\beta),\ P_{1,\beta}(X) \in \frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)},$$

which are rational functions in equivariant variables $\{\lambda_i\}_{i=1}^4$ (see Defintion 2.2, 3.2 and 3.6). Note that the last two invariants are well-defined up to a choice of orientation.

As an equivariant analogue of GW/GV/DT_4 conjecture in [CMT1, CMT2], we show the following:

Theorem 1.1 (Theorem 4.1)**.** *Let $X=\mathcal{A}_n\times\mathbb{C}^2$ and $\beta\in H_2(X,\mathbb{Z})$. Then for certain choice of orientation, we have*

$$P_{1,\beta}(X)=\mathrm{DT}_4(X,\beta),$$

and a multiple cover formula

$$\mathrm{GW}_{0,\beta}(X)=\sum_{k\geqslant 1,\, k|\beta}\frac{1}{k^3}\cdot\mathrm{DT}_4(X,\beta/k).$$

When $n=1$, $X=\mathrm{Tot}_{\mathbb{P}^1}(-2,0,0)$ is the total space of canonical bundle of a non-compact Fano 3-fold $\mathrm{Tot}_{\mathbb{P}^1}(0,0)$. In such case, there is a conjecture on fixing the choice of orientation in the above theorem [Cao], which we verify in Corollary 4.2.

In the appendix, we study stable pair moduli spaces $P_m(X,\beta)$ for general m. We define the corresponding stable pair invariants (Definition A.2) and explicitly compute several examples on $X=\mathcal{A}_1\times\mathbb{C}^2$.

Proposition 1.2 (Proposition A.4)**.** *Let $X=\mathcal{A}_1\times\mathbb{C}^2$ and $\beta=d\,[\mathbb{P}^1]\in H_2(X)$. For certain choice of orientation, we have*

$$P_{m,d\,[\mathbb{P}^1]}(X)=0,\ \text{if } m<d;\ P_{1,\,[\mathbb{P}^1]}(X)=\frac{(\lambda_3+\lambda_4)}{\lambda_3\lambda_4};\ P_{m,\,[\mathbb{P}^1]}(X)=0,\ \text{if } m\geqslant 2;$$
$$P_{2,2\,[\mathbb{P}^1]}(X)=\frac{(\lambda_3+\lambda_4)^2}{2\lambda_3^2\lambda_4^2};\quad P_{3,2\,[\mathbb{P}^1]}(X)=0.$$

Finally we remark that one can also relate stable pair invariants discussed above to curve counting invariants defined by the Hilbert schemes $I_n(X,\beta)$ of one dimensional subschemes Z with $[Z]=\beta$ and $\chi(\mathcal{O}_Z)=n$. This is usually referred as the DT/PT correspondence (see conjectures proposed in [CK2, CKM]).

2. Gromov-Witten invariants

2.1. Geometric set-up

Let $\mathbb{Z}_{n+1} \subseteq SU(2)$ be the cyclic group of order $(n+1)$ which acts on $\mathbb{C}^2$ by

$$g \cdot (z_1, z_2) = (g \cdot z_1, g^{-1} \cdot z_2).$$

The algebraic torus $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by the standard diagonal action which commutes with the cyclic group action. The minimal resolution

$$\pi : \mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}$$

is endowed with the induced $(\mathbb{C}^*)^2$-action, which makes it to be a toric Calabi-Yau surface.

The product $X = \mathcal{A}_n \times \mathbb{C}^2$ is naturally endowed with a $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ action:

$$t \cdot (x_1, x_2, z_1, z_2) = (t_1^{-1}x_1,\, t_2^{-1}x_2,\, t_3^{-1}z_1,\, t_4^{-1}z_2),$$

which makes X to be a toric Calabi-Yau 4-fold.

We take the Calabi-Yau subtorus

$$T := \Big\{ t \in (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \,|\, t_1t_2t_3t_4 = 1 \Big\},$$

which preserves the holomorphic volume form of X.

Let $\bullet$ be $\operatorname{Spec}\mathbb{C}$ with trivial $(\mathbb{C}^*)^4$-action. Denote $\mathbb{C} \otimes t_i$ to be the 1-dimensional $(\mathbb{C}^*)^4$-representation with weight t_i and write $\lambda_i \in H^*_{(\mathbb{C}^*)^4}(\bullet)$ to be its $(\mathbb{C}^*)^4$-equivariant first Chern class. Then

$$H^*_{(\mathbb{C}^*)^4}(\bullet) \cong \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \lambda_4],$$
$$H^*_T(\bullet) \cong \frac{\mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}.$$

The homology of X satisfies

$$H_*(X, \mathbb{Z}) = \mathbb{Z} \oplus H_2(X, \mathbb{Z}),$$
$$H_2(X, \mathbb{Z}) \cong H_2(\mathcal{A}_n, \mathbb{Z}) = \mathbb{Z}\langle [E_1], [E_2], \cdots, [E_n] \rangle,$$

where $\{E_i\}$ are irreducible (-2)-curves in $\mathcal{A}_n$ such that $E_i \bigcap E_j \neq \emptyset$ iff $|i - j| = 1$.

Using notation from Lie theory, we define

Definition 2.1. *A class $\beta \in H_2(X, \mathbb{Z})$ corresponds to a positive root if*

$$\beta = [E_i] + [E_{i+1}] + \cdots + [E_{j-1}],$$

for some $1 \leqslant i < j \leqslant n+1$.

2.2. GW invariants

A stable map $f : C \to X$ factors through some $f : C \to S \times \{z\} \hookrightarrow X$. The moduli space $\overline{M}_{0,0}(X, \beta)$ of genus zero stable maps to X satisfies

$$\overline{M}_{0,0}(X, \beta) \cong \overline{M}_{0,0}(S, \beta) \times \mathbb{C}^2.$$

Although it is non-compact, the torus T fixed locus is compact. The corresponding Gromov-Witten invariants may be defined using localization formula. We consider diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & S \\ \downarrow{\scriptstyle \pi} & & \\ \overline{M}_{0,0}(S, \beta), & & \end{array}$$

where $\mathcal{C}$ is the universal curve and f is the universal stable map.

Definition 2.2.

$$\mathrm{GW}_{0,\beta}(X) := \int_{[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_T} e(-\mathbf{R}\pi_* f^* N),$$

where $N = \mathcal{O}_S \otimes t_3 \oplus \mathcal{O}_S \otimes t_4$ is the normal bundle of $S \times \{0\} \subseteq X$.

Proposition 2.3. *If $\beta = d \cdot \alpha$ for $d \in \mathbb{Z}_{>0}$ and $\alpha \in H_2(X, \mathbb{Z})$ corresponds to a positive root,*

$$\mathrm{GW}_{0,\beta}(X) = \frac{1}{d^3} \cdot \frac{(\lambda_1 + \lambda_2)}{\lambda_3 \lambda_4}.$$

Otherwise, $\mathrm{GW}_{0,\beta}(X) = 0$.

Proof. A direct calculation (e.g. [M, Lem. 2.1]) shows the T-equivariant virtual class $[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_T$ satisfies

$$[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_T = (\lambda_1+\lambda_2)\cdot[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_{\mathrm{red}},$$

where $[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_{\mathrm{red}} \in A_0(\overline{M}_{0,0}(S,\beta))$ is the reduced virtual class for the moduli space of stable maps to (holomorphic symplectic) surface S. Hence, we have

$$\begin{aligned}\mathrm{GW}_{0,\beta}(X) &= \frac{1}{\lambda_3\lambda_4}\int_{[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_T} 1 \\ &= \frac{(\lambda_1+\lambda_2)}{\lambda_3\lambda_4}\deg[\overline{M}_{0,0}(S,\beta)]^{\mathrm{vir}}_{\mathrm{red}} \\ &= \frac{1}{d^3}\cdot\frac{(\lambda_1+\lambda_2)}{\lambda_3\lambda_4},\end{aligned}$$

where the last equality is by the Aspinwall-Morrison formula (e.g. [M, Theorem 1.1]). ☐

Remark 2.4. *Defining higher genus GW invariants of $X = \mathcal{A}_n \times \mathbb{C}^2$ requires insertions. In fact, a more basic counting invariant is the so-called BPS or Gopakumar-Vafa invariant. Since any curve in X sit inside the surface $\mathcal{A}_n$, whose BPS invariants vanish in higher genus (e.g. [BG2, M]). So we may simply define higher genus ($g \geqslant 1$) Gopakumar-Vafa invariant of X to be zero in accordance with the situation of compact Calabi-Yau 4-folds [CMT1, CMT2, CT2, KP].*

3. DT$_4$ invariants

In the case of a compact Calabi-Yau 4-fold X, there are sheaf theoretical approaches [CMT1, CT2, CMT2] to Klemm-Pandharipande's Gopakumar-Vafa type invariants defined using GW invariants of X [KP]. The relevant moduli spaces are moduli spaces of one dimensional stable sheaves and stable pairs. We study their T-equivariant analogues on toric Calabi-Yau 4-fold $X = \mathcal{A}_n \times \mathbb{C}^2$ in this section.

3.1. One dimensional stable sheaves

Let $X = \mathcal{A}_n \times \mathbb{C}^2$ with Calabi-Yau torus $T \subseteq (\mathbb{C}^*)^4$ action. Let $M_{X,\beta}$ denote the moduli scheme of one-dimensional stable sheaves on X with Chern character $(0,0,0,\beta,1)$. It has an induced T-action whose fixed locus is described as follows.

Lemma 3.1. *The torus fixed locus $M^T_{X,\beta}$ satisfies:*
(1) $M^T_{X,\beta}=\{\mathcal{O}_C\}$, if β corresponds to a positive root. Here $C\subseteq S\times\{0\}$ is the unique curve in class β.
(2) $M^T_{X,\beta}=\emptyset$, otherwise.

Proof. It is due to Bryan-Gholampour [BG3, Section 2]. ☐

Following the localization definition in DT_4 theory (e.g. [CMT1, Section 3.3]), we define

Definition 3.2. *If $\beta\in H_2(X,\mathbb{Z})$ corresponds to a positive root,*

$$\mathrm{DT}_4(X,\beta):=\frac{\sqrt{(-1)^{\frac{1}{2}\mathrm{ext}^2_X(\mathcal{O}_C,\mathcal{O}_C)}\cdot e_T\big(\mathrm{Ext}^2_X(\mathcal{O}_C,\mathcal{O}_C)\big)}}{e_T\big(\mathrm{Ext}^1_X(\mathcal{O}_C,\mathcal{O}_C)\big)} \in\frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}.$$

Otherwise, we define $\mathrm{DT}_4(X,\beta):=0$.

Remark 3.3. *The above square root is unique up a sign corresponding to a choice of orientation in defining* DT_4 *invariants (see e.g. [BJ, CGJ, CL1, CL2]).*

Proposition 3.4. *In Definition 3.2, if β corresponds to a positive root, then*

$$\mathrm{DT}_4(X,\beta)=\pm\frac{(\lambda_3+\lambda_4)}{\lambda_3\cdot\lambda_4}.$$

Proof. By adjunction, we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_X(\mathcal{O}_C,\mathcal{O}_C)\simeq{}&\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_C,\mathcal{O}_C)\oplus\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_C,\mathcal{O}_C)[-2]\otimes(t_3\cdot t_4)\oplus\\ &\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_C,\mathcal{O}_C)[-1]\otimes t_3\oplus\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_C,\mathcal{O}_C)[-1]\otimes t_4,\end{aligned}$$

whose cohomology gives

$$\begin{aligned}\mathrm{Ext}^1_X(\mathcal{O}_C,\mathcal{O}_C)&\cong\mathrm{Ext}^1_S(\mathcal{O}_C,\mathcal{O}_C)\oplus\mathrm{Ext}^0_S(\mathcal{O}_C,\mathcal{O}_C)\otimes t_3\oplus\mathrm{Ext}^0_S(\mathcal{O}_C,\mathcal{O}_C)\otimes t_4,\\ \mathrm{Ext}^2_X(\mathcal{O}_C,\mathcal{O}_C)&\cong\mathrm{Ext}^2_S(\mathcal{O}_C,\mathcal{O}_C)\oplus\mathrm{Ext}^0_S(\mathcal{O}_C,\mathcal{O}_C)\otimes(t_3\cdot t_4)\oplus\\ &\quad\mathrm{Ext}^1_S(\mathcal{O}_C,\mathcal{O}_C)\otimes t_3\oplus\mathrm{Ext}^1_S(\mathcal{O}_C,\mathcal{O}_C)\otimes t_4.\end{aligned}$$

Note that $\mathrm{Ext}^0_S(\mathcal{O}_C,\mathcal{O}_C)\cong\mathbb{C}$ is generated by identity map whose T-weight is zero. By T-equivariant Serre duality, we have

$$\mathrm{Ext}^2_S(\mathcal{O}_C, \mathcal{O}_C) \cong \mathbb{C} \otimes (t_1 \cdot t_2)$$
$$\mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C) \cong \mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C)^\vee \otimes (t_1 \cdot t_2).$$

Then, it is easy to obtain

$$e_T(\mathrm{Ext}^2_X(\mathcal{O}_C, \mathcal{O}_C)) = (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) \cdot \left(e_T(\mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C) \otimes t_3)\right)^2.$$

Since T is the CY torus, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, so we can take

$$\sqrt{-e_T(\mathrm{Ext}^2_X(\mathcal{O}_C, \mathcal{O}_C))} = \pm(\lambda_3 + \lambda_4) \cdot e_T(\mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C) \otimes t_3).$$

Therefore

$$\mathrm{DT}_4(X, \beta) = \pm \frac{(\lambda_3 + \lambda_4)}{\lambda_3 \cdot \lambda_4} \cdot \frac{e_T(\mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C) \otimes t_3)}{e_T(\mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C))}.$$

Finally, the conclusion follows from Riemann-Roch computation:

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_S(\mathcal{O}_C, \mathcal{O}_C) = 2 + \beta \cdot \beta = 0. \qquad \square$$

3.2. Stable pairs

In this section, we study stable pair moduli space $P_m(X, \beta)$ in the sense of Pandharipande-Thomas [PT] and its T-equivariant counting invariant. For the purpose of matching with GW invariants [CMT2], we restrict to the case of $m = 0, 1$ here and leave the study of $P_{m \geqslant 2}(X, \beta)$ to the appendix.

The T-fixed locus can be easily determined as follows.

Lemma 3.5. *The torus fixed locus $P_m(X, \beta)^T$ satisfies:*
(1) $P_0(X, \beta)^T = \emptyset$.
(2) $P_1(X, \beta)^T = \{s_C : \mathcal{O}_X \to \mathcal{O}_C\}$, if β corresponds to a positive root. Here $C \subseteq S \times \{0\}$ is the unique curve in class β and s_C is the canonical section of $\mathcal{O}_C$.
(3) $P_1(X, \beta)^T = \emptyset$, otherwise.

Proof. Given a T-fixed stable pair $(s : \mathcal{O}_X \to F)$, we have an exact sequence

$$0 \to I_C \to \mathcal{O}_X \to F \to \mathrm{coker}(s) \to 0,$$

where C be the scheme theoretical support of F and $\mathrm{coker}(s)$ is zero dimensional.

Then $C \subseteq S \times \{0\}$ is a Cohen-Macaulay curve and there exists $\{a_i \geqslant 0\}_{1 \leqslant i \leqslant l}$ such that

$$C = \sum_{i=1}^{l} a_i E_i.$$

Note that

$$\chi(\mathcal{O}_C) = -\frac{1}{2} C \cdot C = \frac{1}{2}\Big(a_1^2 + a_l^2 + \sum_{i=1}^{l-1} (a_i - a_{i+1})^2\Big). \tag{1}$$

Thus

$$\chi(F) = \chi(\mathcal{O}_C) + \chi(\mathrm{coker}(s)) > 0, \text{ for } \beta \neq 0.$$

Hence $P_0(X, \beta)^T = \emptyset$.

If $\chi(F) = 1$, we have $\mathrm{coker}(s) = 0$ and $F \cong \mathcal{O}_C$. By (1), it is elementary to show $\chi(\mathcal{O}_C) = 1$ is equivalent to the condition β corresponds to a positive root. □

Stable pair invariants can be defined similarly by torus localization.

Definition 3.6. *(1) $P_{0,\beta}(X) := 0$. (2) If β corresponds to a positive root,*

$$P_{1,\beta}(X) := \frac{\sqrt{(-1)^{\frac{1}{2}\mathrm{ext}^2_X(I_C, I_C)_0} \cdot e_T\big(\mathrm{Ext}^2_X(I_C, I_C)_0\big)}}{e_T\big(\mathrm{Ext}^1_X(I_C, I_C)_0\big)} \in \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}.$$

Otherwise, $P_{1,\beta}(X) := 0$.

Proposition 3.7. *In Definition 3.6, if β corresponds to a positive root, then*

$$P_{1,\beta}(X) = \pm \frac{(\lambda_3 + \lambda_4)}{\lambda_3 \cdot \lambda_4}.$$

Proof. From the distinguished triangle

$$\mathcal{O}_C \to I_C[1] \to \mathcal{O}_X[1],$$

we have a diagram

$$\begin{array}{ccccc} & & \mathbf{R}\Gamma(\mathcal{O}_X)[1] & = & \mathbf{R}\Gamma(\mathcal{O}_X)[1] \\ & & \downarrow & & \downarrow \\ \mathbf{R}\mathrm{Hom}_X(I_C, \mathcal{O}_C) & \longrightarrow & \mathbf{R}\mathrm{Hom}_X(I_C, I_C)[1] & \longrightarrow & \mathbf{R}\mathrm{Hom}_X(I_C, \mathcal{O}_X)[1] \\ & & \downarrow & & \downarrow \\ & & \mathbf{R}\mathrm{Hom}_X(I_C, I_C)_0[1] & & \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_C, \mathcal{O}_X)[2], \end{array}$$

where the horizontal and vertical arrows are distinguished triangles. By taking cones, we obtain a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_X(I_C, \mathcal{O}_C) \to \mathbf{R}\mathrm{Hom}_X(I_C, I_C)_0[1] \to \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_C, \mathcal{O}_X)[2]. \tag{2}$$

Combining with the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{O}_C, \mathcal{O}_C) \to \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_C) \to \mathbf{R}\mathrm{Hom}_X(I_C, \mathcal{O}_C),$$

we obtain T-equivariant isomorphisms

$$\mathrm{Ext}^1_X(I_C, I_C)_0 \cong \mathrm{Ext}^1_X(\mathcal{O}_C, \mathcal{O}_C),$$
$$\mathrm{Ext}^2_X(I_C, I_C)_0 \cong \mathrm{Ext}^2_X(\mathcal{O}_C, \mathcal{O}_C).$$

So our calculation reduces to Proposition 3.4. □

4. GW/GV/DT$_4$ comparison

By Remark 2.4, higher genus ($g > 0$) Gopakumar-Vafa invariants of X are zero (in particular, by Lemma 3.5 and Definition 3.6, $P_{0,\beta}(X) = 0$, which matches with the conjecture in the compact setting [CMT2]), so here we concentrate on the genus zero comparison.

Theorem 4.1. *Let $X = \mathcal{A}_n \times \mathbb{C}^2$ and $\beta \in H_2(X, \mathbb{Z})$. Then for certain choice of orientation, we have*

$$P_{1,\beta}(X) = \mathrm{DT}_4(X, \beta),$$

and a multiple cover formula

$$\mathrm{GW}_{0,\beta}(X) = \sum_{k\geqslant 1,\, k|\beta} \frac{1}{k^3} \cdot \mathrm{DT}_4(X, \beta/k).$$

Proof. It is a combination of Proposition 2.3, 3.4 and 3.7. □

Concerning the choice of orientation, there is a conjecture when X is the total space of canonical bundle of a Fano 3-fold [Cao]. We restrict to $X = \mathcal{A}_1 \times \mathbb{C}^2$, i.e. $X = \mathrm{Tot}_{\mathbb{P}^1}(-2,0,0)$. In this case, we have $X = K_Y$, where $Y = \mathrm{Tot}_{\mathbb{P}^1}(0,0)$ is a non-compact Fano 3-fold.

More specifically, in [Cao, Sect. 3.2], we defined twisted DT_3 invariants of $Y = \mathrm{Tot}_{\mathbb{P}^1}(l_1,l_2)$ with $l_1 + l_2 \geqslant -1$ to be

$$\mathrm{DT}_3^{\mathrm{twist}}(Y, d\,[\mathbb{P}^1]) := (-1)^{d(l_1+l_2)-1} \int_{[M_{Y,d}^{T_0}]^{\mathrm{vir}}} e_{T_0}(N^{\mathrm{vir}}) \in \mathbb{Q}(\lambda_3,\lambda_4), \tag{3}$$

where $T_0 \subseteq T$ is the two dimensional subtorus acting trivially on the base $\mathbb{P}^1$, $M_{Y,d}$ is the moduli scheme of one dimensional stable sheaves F on Y with $[F] = d\,[\mathbb{P}^1]$, $\chi(F) = 1$ and N^{vir} is the virtual normal bundle of $M_{Y,d}^{T_0} \hookrightarrow M_{Y,d}$. The above signs $(-1)^{d(l_1+l_2)-1}$ correspond to the choice of orientation which conjecturally match with GW invariants.

Corollary 4.2. *Conjecture 4.10 of [CMT1] and Conjecture 3.8 of [Cao] are true for* $X = \mathrm{Tot}_{\mathbb{P}^1}(-2,0,0)$*, i.e.*

$$\mathrm{GW}_{0,d\,[\mathbb{P}^1]}(X) = \sum_{k\geqslant 1,\, k|d} \frac{1}{k^3}\cdot \mathrm{DT}_3^{\mathrm{twist}}\left(Y, \frac{d}{k}[\mathbb{P}^1]\right) \in \mathbb{Q}(\lambda_3,\lambda_4),$$

where $Y = \mathrm{Tot}_{\mathbb{P}^1}(0,0)$*.*

Proof. Similarly as the proof of Proposition 3.4, we have

$$\begin{aligned}
\mathrm{DT}_3^{\mathrm{twist}}(Y, d\,[\mathbb{P}^1]) &= 0, \quad \text{if } d > 1. \\
\mathrm{DT}_3^{\mathrm{twist}}(Y, [\mathbb{P}^1]) &= (-1)\cdot \frac{e_{T_0}(\mathrm{Ext}^2_Y(\mathcal{O}_{\mathbb{P}^1},\mathcal{O}_{\mathbb{P}^1}))}{e_{T_0}(\mathrm{Ext}^1_Y(\mathcal{O}_{\mathbb{P}^1},\mathcal{O}_{\mathbb{P}^1}))} \\
&= (-1)\cdot \frac{e_{T_0}(\mathbb{C}\otimes(t_3\cdot t_4))}{e_{T_0}(\mathbb{C}\otimes t_3 \oplus \mathbb{C}\otimes t_4)} \\
&= -\frac{(\lambda_3+\lambda_4)}{\lambda_3\lambda_4}.
\end{aligned}$$

By $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0$ and Proposition 2.3, we are done. □

Appendix A. Stable pair invariants

For general $m \geqslant 0$, torus fixed stable pairs $(s : \mathcal{O}_X \to F) \in P_m(X,\beta)^T$ are classified in [CK2, Section 2.2]. Moreover, by [CK2, Proposition 2.6], we have:

Proposition A.1. *For any* $I = (s : \mathcal{O}_X \to F) \in P_m(X, \beta)^T$*, we have* $\mathrm{Ext}^1(I, I)_0^T = 0$.

Therefore we may define

Definition A.2. *Let* $X = \mathcal{A}_n \times \mathbb{C}^2$ *and* $\beta \in H_2(X, \mathbb{Z})$*. Then*

$$(4) \qquad P_{m,\beta}(X) := \sum_{[I] \in P_m(X,\beta)^T} (-1)^{n_I} \frac{\sqrt{(-1)^{\frac{1}{2}\mathrm{ext}^2_X(I,I)_0} \cdot e_T\big(\mathrm{Ext}^2_X(I,I)_0\big)}}{e_T\big(\mathrm{Ext}^1_X(I,I)_0\big)} \in \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)},$$

where $n_I = 0$ *or* 1 *for each* $I \in P_m(X, \beta)^T$ *and the sign corresponds to a choice of orientation.*

We define $P_{m,\beta}(X) = 0$ *if* $P_m(X, \beta)^T = \emptyset$.

Below, we study stable pair invariants on $X = \mathcal{A}_1 \times \mathbb{C}^2$, i.e. $X = \mathrm{Tot}_{\mathbb{P}^1}(-2, 0, 0)$. In this case, the T-fixed locus $P_m(X, d\,[\mathbb{P}^1])^T$ can also be described as follows (ref. [CMT2, Section 4.2]).

Let $p : X \to \mathbb{P}^1$ be the projection, then for $I = (s : \mathcal{O}_X \to F) \in P_m(X, d\,[\mathbb{P}^1])^T$, we have $(\mathbb{C}^*)^3$-weight decompositions

$$p_*F = \bigoplus_{(i_0,i_1,i_2) \in \mathbb{Z}^3} F^{i_0,i_1,i_2},$$

$$p_*\mathcal{O}_X = \bigoplus_{(i_0,i_1,i_2) \in \mathbb{Z}^3_{\geqslant 0}} L_0^{-i_0} \otimes L_1^{-i_1} \otimes L_2^{-i_2},$$

where $L_0 = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes t_2$, $L_1 = \mathcal{O}_{\mathbb{P}^1} \otimes t_3$ and $L_2 = \mathcal{O}_{\mathbb{P}^1} \otimes t_4$.

The T-equivariance of s implies morphisms

$$s^{i_0,i_1,i_2} : L_0^{-i_0} \otimes L_1^{-i_1} \otimes L_2^{-i_2} \to F^{-i_0,-i_1,-i_2}$$

in $\mathrm{Coh}(\mathbb{P}^1)$ which are surjective in one dimension, so either $F^{-i_0,-i_1,-i_2} = \emptyset$ or

$$F^{-i_0,-i_1,-i_2} = L_0^{-i_0} \otimes L_1^{-i_1} \otimes L_2^{-i_2} \otimes \mathcal{O}_{\mathbb{P}^1}(Z_{i_0,i_1,i_2})$$

for some T-fixed divisor $Z_{i_0,i_1,i_2} \subseteq \mathbb{P}^1$. The $\mathcal{O}_X$-module structure implies

$$Z_{i,j,k} \subseteq Z_{i+1,j,k},\ Z_{i,j+1,k},\ Z_{i,j,k+1}$$

as closed subschemes of $\mathbb{P}^1$. So the set

$$\Delta_F := \Big\{(i,j,k) \in \mathbb{Z}^3_{\geqslant 0} \,\big|\, F^{-i,-j,-k} \neq 0\Big\}$$

is a (finite) three dimensional Young diagram with d number of boxes.

As Z_{i_0,i_1,i_2} is T-fixed, it is supported on 0 or $\infty \in \mathbb{P}^1$ and determined uniquely by its length $n^0_{i_0,i_1,i_2}$, $n^{\infty}_{i_0,i_1,i_2}$ at 0 and ∞ respectively. Thus a T-fixed stable pair $I \in P_m(X, d\,[\mathbb{P}^1])^T$ can be characterized by two sequences of nonnegative integers $\left\{n^0_{i_0,i_1,i_2}\right\}_{(i_0,i_1,i_2)\in\Delta_F}$, $\left\{n^{\infty}_{i_0,i_1,i_2}\right\}_{(i_0,i_1,i_2)\in\Delta_F}$, such that

$$n^*_{i,j,k} \leqslant n^*_{i+1,j,k}\,,\ n^*_{i,j+1,k}\,,\ n^*_{i,j,k+1}, \quad * = 0 \text{ or } \infty,$$
$$\sum_{(i,j,k)\in\Delta_F} (2i + n^0_{i,j,k} + n^{\infty}_{i,j,k}) = (m-d),$$

where the last equation is deduced from $\chi(F) = m$.

As for the stable pair invariant, we note that

$$P_{m,\beta}(X) = \sum_{[I]\in P_m(X,\beta)^T} (-1)^{n_I} \sqrt{(-1)^m \cdot e_T(\chi_X(I,I)_0)},$$

where $\chi_X(-,-)$ is the Euler pairing on X. For $I = (s : \mathcal{O}_X \to F) \in P_m(X,\beta)^T$, we have

$$\chi_X(I,I)_0 = \chi_X(F,F) - \chi_X(\mathcal{O}_X, F) - \chi_X(F, \mathcal{O}_X)$$

in the T-equivariant K-theory $K^T_0(\bullet)$ of one point.

To choose a square root of its Euler class, we can first choose a 'square root' of $\chi_X(I,I)_0$, i.e. finding $\chi_X(I,I)_0^{\frac{1}{2}} \in K^T_0(\bullet)$ such that

$$\chi_X(I,I)_0 = \chi_X(I,I)_0^{\frac{1}{2}} + \overline{\chi_X(I,I)_0^{\frac{1}{2}}} \in K^T_0(\bullet),$$

where $\overline{(\cdot)}$ denotes the involution on $K^T_0(\bullet)$ induced by $\mathbb{Z}$-linearly extending the map

$$t_1^{w_1} t_2^{w_2} t_3^{w_3} t_4^{w_4} \mapsto t_1^{-w_1} t_2^{-w_2} t_3^{-w_3} t_4^{-w_4}.$$

By Serre duality, we then have

$$e_T(\chi_X(I,I)_0^{\frac{1}{2}}) = \pm\sqrt{(-1)^m \cdot e_T\left(\chi_X(I,I)_0\right)}.$$

There are many ways to choose a square root $\chi_X(I,I)_0^{\frac{1}{2}}$, for example, when $F = j_*F_0 \oplus j_*F_1$, where $j : \mathbb{P}^1 \hookrightarrow X$ is the zero section, we can take

$$\chi_X(I,I)_0^{\frac{1}{2}} = \chi_X(j_*F_0, j_*F_0) + \chi_X(j_*F_0, j_*F_1) - \chi_X(\mathcal{O}_X, j_*F_0 \oplus j_*F_1),$$

since we have (e.g. [CMT1, Lemma 4.1]):

$$\begin{aligned}\chi_X(j_*F_0, j_*F_1) &= \chi_{\mathbb{P}^1}(F_0, F_1) - \chi_{\mathbb{P}^1}(F_0, F_1 \otimes N_{\mathbb{P}^1/X}) \\ &\quad + \chi_{\mathbb{P}^1}(F_0, F_1 \otimes \wedge^2 N_{\mathbb{P}^1/X}) - \chi_{\mathbb{P}^1}(F_0, F_1 \otimes \wedge^3 N_{\mathbb{P}^1/X}),\end{aligned}$$

where

$$N_{\mathbb{P}^1/X} = \mathcal{O}_{\mathbb{P}^1}(-2Z_\infty) \otimes t_2 \oplus \mathcal{O}_{\mathbb{P}^1} \otimes t_3 \oplus \mathcal{O}_{\mathbb{P}^1} \otimes t_4.$$

Note also from equivariant Riemann-Roch formula, for any T-fixed divisor $(aZ_0 + bZ_\infty) \subset \mathbb{P}^1$,

$$\mathrm{ch}\left(\chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty))\right) = \frac{e^{a\lambda_1}}{(1 - e^{-\lambda_1})} + \frac{e^{-b\lambda_1}}{(1 - e^{\lambda_1})},$$

from which we obtain the following identities (ref. [CT1, Lemma 6.3]).

Lemma A.3. *As elements in $K_0^T(\bullet)$, we have*

$$\begin{aligned}&\chi(\mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty)) \\ &\quad = \begin{cases} t_1^{-b} + \cdots + t_1^{-1} + 1 + t_1 + \cdots + t_1^{a}, & \text{if } a,\, b \geqslant 0, \\ -t_1\,, & \text{if } a = 0,\, b = -2. \end{cases}\end{aligned}$$

We apply Lemma A.3 to explicitly compute stable pair invariants $P_{m,\beta}(X)$ for some small degree curve classes.

Proposition A.4. *Let $X = \mathcal{A}_1 \times \mathbb{C}^2$ and $\beta = d\,[\mathbb{P}^1] \in H_2(X)$. For certain choice of orientation, we have*

$$P_{m,d\,[\mathbb{P}^1]}(X) = 0, \text{ if } m < d;\ P_{1,\,[\mathbb{P}^1]}(X) = \frac{(\lambda_3 + \lambda_4)}{\lambda_3\lambda_4};\ P_{m,\,[\mathbb{P}^1]}(X) = 0, \text{ if } m \geqslant 2;$$

$$P_{2,2\,[\mathbb{P}^1]}(X) = \frac{(\lambda_3 + \lambda_4)^2}{2\lambda_3^2\lambda_4^2};\ P_{3,2\,[\mathbb{P}^1]}(X) = 0.$$

Proof. From the description of T-fixed locus, $P_m(X, \beta)^T = \emptyset$ if $m < d$, so invariants are zero.

For $d = 1$, $m = n + 1 > 0$, we have m possibilities of F:

$$F = \mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty), \quad \text{where } a, b \in \mathbb{Z}_{\geqslant 0},\ a + b = n,$$

for which we can choose

$$\chi_X(I, I)_0^{\frac{1}{2}} = 1 - t_3 - t_4 + t_3t_4 - \chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty)).$$

Combining with Lemma A.3, we have

$$\begin{aligned} e_T\left(\chi_X(I,I)_0^{\frac{1}{2}}\right) &= \frac{(\lambda_3+\lambda_4)}{\lambda_1^n\lambda_3\lambda_4}\cdot\frac{1}{a(a-1)\cdots\widehat{(a-a)}\cdots(a-n)} \\ &= \frac{(\lambda_3+\lambda_4)}{\lambda_1^n\lambda_3\lambda_4}\cdot\frac{(-1)^{a-n}}{a\,!\,(n-a)!}. \end{aligned}$$

By taking sum over $0\leqslant a\leqslant n$, we obtain

$$\sum_{[I]\in P_m(X,\beta)^T} e_T\left(\chi_X(I,I)_0^{\frac{1}{2}}\right) = \frac{(\lambda_3+\lambda_4)}{\lambda_3\lambda_4}, \quad \text{if } m=1.$$
$$\sum_{[I]\in P_m(X,\beta)^T} e_T\left(\chi_X(I,I)_0^{\frac{1}{2}}\right) = 0, \quad \text{if } m>1.$$

For $d=2$, $m=2$, the only possibilities for F are

$$F=\mathcal{O}_{\mathbb{P}^1}+\mathcal{O}_{\mathbb{P}^1}\otimes t_3^{-1} \text{ or } F=\mathcal{O}_{\mathbb{P}^1}+\mathcal{O}_{\mathbb{P}^1}\otimes t_4^{-1}.$$

The corresponding Euler class satisfies

$$e_T\left(\chi_X(I,I)_0^{\frac{1}{2}}\right) = \frac{(\lambda_3+\lambda_4)(2\lambda_3+\lambda_4)}{2\lambda_3^2\lambda_4(\lambda_4-\lambda_3)}, \quad \frac{(\lambda_3+\lambda_4)(2\lambda_4+\lambda_3)}{2\lambda_3\lambda_4^2(\lambda_3-\lambda_4)},$$

whose sum gives the answer.

For $d=2$, $m=3$, we have four possibilities of F:

$$\begin{aligned} F&=\mathcal{O}_{\mathbb{P}^1}+\mathcal{O}_{\mathbb{P}^1}(Z_0)\otimes t_3^{-1}, \quad F=\mathcal{O}_{\mathbb{P}^1}+\mathcal{O}_{\mathbb{P}^1}(Z_\infty)\otimes t_3^{-1}, \\ F&=\mathcal{O}_{\mathbb{P}^1}+\mathcal{O}_{\mathbb{P}^1}(Z_0)\otimes t_4^{-1}, \quad F=\mathcal{O}_{\mathbb{P}^1}+\mathcal{O}_{\mathbb{P}^1}(Z_\infty)\otimes t_4^{-1}, \end{aligned}$$

where $Z_0=\{0\}$, $Z_\infty=\{\infty\}$ are torus fixed points of $\mathbb{P}^1$.

By Lemma A.3, we have

$$\chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(Z_0))=1+t_1, \quad \chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(Z_\infty))=1+t_1^{-1}.$$

This enables us to obtain the corresponding Euler class $e_T(\chi_X(I,I)_0^{\frac{1}{2}})$:

$$\begin{aligned} &\frac{(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)^2}{\lambda_1\lambda_3^2\lambda_4(\lambda_1+\lambda_4-\lambda_3)(\lambda_3-\lambda_4)}, \quad \frac{(\lambda_4-\lambda_1)(\lambda_3+\lambda_4)^2}{\lambda_1\lambda_3^2\lambda_4(\lambda_1+\lambda_3-\lambda_4)(\lambda_3-\lambda_4)}, \\ &\frac{(\lambda_1+\lambda_3)(\lambda_3+\lambda_4)^2}{\lambda_1\lambda_3\lambda_4^2(\lambda_1+\lambda_3-\lambda_4)(\lambda_4-\lambda_3)}, \quad \frac{(\lambda_3-\lambda_1)(\lambda_3+\lambda_4)^2}{\lambda_1\lambda_3\lambda_4^2(\lambda_1+\lambda_4-\lambda_3)(\lambda_4-\lambda_3)}, \end{aligned}$$

whose sum is zero by a direct calculation. □

References

[BJ] D. Borisov and D. Joyce, Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds. *Geom. Topol.* (**21**) (2017) 3231–3311. MR3692967

[BG1] J. Bryan and A. Gholampour, Root Systems and the Quantum Cohomology of ADE resolutions. *Algebra Number Theory* (**2**)(4) (2008) 369–390. MR2411404

[BG2] J. Bryan and A. Gholampour, The Quantum McKay Correspondence for polyhedral singularities. *Invent. Math.* **178**(3) (2009) 655–681. MR2551767

[BG3] J. Bryan and A. Gholampour, BPS invariants for resolutions of polyhedral singularities. *Selecta Math. (N.S.)* (**15**)(4) (2009) 521–533. MR2565049

[Cao] Y. Cao, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds II: Fano 3-folds. *Commun. Contemp. Math.* **22**(7) (2020) 1950060, 25 pages. MR4135009

[CGJ] Y. Cao, J. Gross, and D. Joyce, Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi-Yau 4-folds. *Adv. Math.* **368** (2020) 107134. MR4085139

[CK1] Y. Cao and M. Kool, Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds. *Adv. Math.* **338** (2018) 601–648. MR3861713

[CK2] Y. Cao and M. Kool, Curve counting and DT/PT correspondence for Calabi-Yau 4-folds. *Adv. Math.* **375** (2020) 107371. MR4135421

[CKM] Y. Cao, M. Kool, and S. Monavari, K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds. arXiv:1906.07856.

[CL1] Y. Cao and N. C. Leung, Donaldson-Thomas theory for Calabi-Yau 4-folds. arXiv:1407.7659. MR3660236

[CL2] Y. Cao and N. C. Leung, Orientability for gauge theories on Calabi-Yau manifolds. *Adv. Math.* **314** (2017) 48–70. MR3658712

[CMT1] Y. Cao, D. Maulik, and Y. Toda, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds. *Adv. Math.* **338** (2018) 41–92. MR3861701

[CMT2] Y. Cao, D. Maulik, and Y. Toda, Stable pairs and Gopakumar-Vafa type invariants for Calabi-Yau 4-folds. arXiv:1902.00003. To appear in *J. Eur. Math. Soc. (JEMS)*.

[CT1] Y. Cao and Y. Toda, Curve counting via stable objects in derived categories of Calabi-Yau 4-folds. arXiv:1909.04897.

[CT2] Y. Cao and Y. Toda, Gopakumar-Vafa type invariants on Calabi-Yau 4-folds via descendent insertions. arXiv:2003.00787. To appear in *Comm. Math. Phys.*

[CT3] Y. Cao and Y. Toda, Tautological stable pair invariants of Calabi-Yau 4-folds. arXiv:2009.03553.

[CT4] Y. Cao and Y. Toda, Counting perverse coherent systems on Calabi-Yau 4-folds. arXiv:2009.10909.

[GP] T. Graber and R. Pandharipande, Localization of virtual classes. *Invent. Math.* **135** (1999) 487–518. MR1666787

[KP] A. Klemm and R. Pandharipande, Enumerative geometry of Calabi-Yau 4-folds. *Comm. Math. Phys.* **281** (2008) 621–653. MR2415462

[M] D. Maulik, Gromov-Witten theory of $\mathcal{A}_n$-resolutions. *Geom. Topol.* (**13**) (2009) 1729–1773. MR2496055

[PT] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category. *Invent. Math.* **178** (2009) 407–447. MR2545686

Yalong Cao
Kavli Institute for the Physics and Mathematics of the Universe (WPI)
The University of Tokyo Institutes for Advanced Study
The University of Tokyo
Kashiwa, Chiba 277-8583
Japan
E-mail: yalong.cao@ipmu.jp