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Meromorphic connections in filtered A_∞ categories

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Dedicated to Professor Kyoji Saito on his 75th birthday

Abstract: In this note, introducing notions of CH module, CH morphism and CH connection, we define a meromorphic connection in the "z-direction" on periodic cyclic homology of an A_∞ category as a connection on cohomology of a CH module. Moreover, we study and clarify compatibility of our meromorphic connections under a CH module morphism preserving CH connections at chain level. Our motivation comes from symplectic geometry. The formulation given in this note designs to fit algebraic properties of open-closed maps in symplectic geometry.

Keywords: Filtered A_∞ category, CH structure, mermorphic connection, Euler connection, Getzler-Gauss-Manin connection, cyclic homology, primitive form.

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1. Introduction

In this note we introduce and study a certain meromorphic connection on periodic cyclic homology of a filtered A_∞ category. In [21] Getzler introduced the Gauss-Manin connection on periodic cyclic homology of an A_∞ algebra, which we call the *Getzler-Gauss-Manin connection* in this note. He constructed the connection at chain level and showed that the curvature is chain homotopic to zero. His connection does not involve derivative of the 'z-direction'. Here the parameter z is the auxiliary variable in the cyclic homology, which is denoted by $-u$ in [21]. In this article we will incorporate the derivative of the 'z-direction' with the connection. This is necessary and important when we study the relationship between the Fukaya category for a general, not necessary Calabi-Yau, symplectic manifold and Kyoji Saito's flat structure [30]. On the other hand, Katzarkov-Kontsevich-Pantev [25] introduced a connection on periodic cyclic homology which contains the derivative of the 'z-direction'. Although the definition of our connection is inspired from their definition and it looks similar to theirs, they are essentially different as we will explain later in Remark 1.1.

Our motivation comes from symplectic geometry. We briefly and informally recall the symplectic geometric background to explain our motivation and aim of this article, though these geometric contents are not used in this note. Let (X, ω) be a closed symplectic manifold. We do *not* assume $c_1(X) = 0$ here. Let

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \to \infty \right\} \tag{1.1}$$

be (a version of) the Novikov ring. We choose and fix a homogeneous basis $T_0 = 1, T_1, \dots, T_m$ of $H^*(X; \Lambda_0)$. (For simplicity, we assume $\deg T_i$ are even.) Let $\mathbf{t} = \sum_{i=0}^m t_i T_i \in H^*(X; \Lambda_0)$ and denote by $*_{\mathbf{t}}$ the quantum product on $H^*(X; \Lambda_0)$ defined by

$$(a *_{\mathbf{t}} b, c)_{PD_X} = \sum_{\alpha \in H_2(X;\mathbb{Z})} \sum_{n=0}^{\infty} \frac{1}{n!} GW^X_{0,\alpha,n+3}(\underbrace{\mathbf{t}, \dots, \mathbf{t}}_{n}, a, b, c) T^{\omega(\alpha)}.$$

Here $(\cdot, \cdot)_{PD_X}$ denotes the Poincaré paring on $H^*(X; \Lambda_0)$ and $GW^X_{0,\alpha,n+3}$ is (the Λ_0 linear extension of) the genus zero $(n+3)$ points Gromov-Witten invariant of X with class α. Dubrovin introduced a meromorphic flat connection ∇^{D} called *Dubrovin's quantum connection* on the trivial $H^*(X; \Lambda_0)$-bundle over $H^*(X; \Lambda_0) \times \mathbb{P}^1$ satisfying

$$\begin{aligned} \nabla^{\mathrm{D}}_{\frac{\partial}{\partial t_i}} &= \frac{\partial}{\partial t_i} + \frac{1}{z}(T_i *_{\mathbf{t}}), \\ \nabla^{\mathrm{D}}_{z\frac{\partial}{\partial z}} &= z\frac{\partial}{\partial z} - \frac{1}{z}(E *_{\mathbf{t}}) + \mu, \end{aligned}$$

where z is the parameter sitting in $\mathbb{P}^1$, $E = c_1(X) + \sum_{i=0}^m (1 - \frac{\deg T_i}{2}) t_i T_i$ and $\mu \in \operatorname{End} H^*(X; \Lambda_0)$ defined by $\mu(a) = \frac{1}{2}(\deg a - \frac{\dim X}{2})a$ for $a \in H^*(X; \Lambda_0)$. See, e.g., [10, Lecture 3] and compare Example 5.10 below.

On the other hand, Lagrangian intersection Floer theory on X provides a filtered A_∞ category, called the *Fukaya category*, denoted by $\mathrm{Fuk}(X)$. The *open-closed map* $\mathfrak{p}$ introduced in [14] is extended in [1] to a $QH^*(X)$-module homomorphism

$$\widehat{\mathfrak{p}} \;:\; HH_*(\mathrm{Fuk}(X); \Lambda_0) \to QH^*(X; \Lambda_0) \tag{1.2}$$

from the Hochschild homology of the Fukaya category. Here $QH^*(X; \Lambda_0)$ denotes the quantum cohomology ring of X with coefficients in Λ_0. If we use the de Rham model, then $\mathrm{Fuk}(X)$ has a structure of cyclic filtered A_∞ category.

(See [11], [12], [1].) The open-closed map $\widehat{\mathfrak{p}}$ will be lifted to a cyclic open-closed map from the the cyclic homology of $\mathrm{Fuk}(X)$ to the (S^1-equivariant) quantum cohomology:

$$\widehat{c\mathfrak{p}} \ : \ CH_*(\mathrm{Fuk}(X);\Lambda_0[[z]]) \to QH^*(X;\Lambda_0[[z]]). \tag{1.3}$$

Then general algebraic theory on the Hochschild chain/cochain of an A_∞ category tells us that the Hochschild chain $CC_*(\mathrm{Fuk}(X);\Lambda_0[[z]])$ has a structure of DGLA module over the Hochschild cochain $CC^*(\mathrm{Fuk}(X);\Lambda_0[[z]])$ which has a structure of DGLA. Moreover it is known that $QH^*(X;\Lambda_0[[z]])$ has a structure of hypercommutative algebra in the sense of [22] (see Example 3.10 for the definition). Furthermore, we note that these maps and categories are bulk-deformed by elements of $H^*(X;\Lambda_0)$. (See [14], [17] for more detailed discussion on bulk-deformations.) In this situation, we like to construct a meromorphic connection on periodic cyclic homology of $\mathrm{Fuk}(X)$ over $H^*(X;\Lambda_0)\times\mathbb{P}^1$ which is compatible with Dubrovin's quantum connection ∇^{D} under the cyclic open-closed map $\widehat{c\mathfrak{p}}$. This is our motivation. In this article we do not touch the geometric part of the cyclic open-closed map. The purpose of this article is to extract and describe the properties that the cyclic open-closed map is supposed to have in terms of purely algebraic language, and to introduce a meromorphic connection containing the derivative of the z-direction as well, and to prove that the connection is compatible with Dubrovin's quantum connection under the cyclic open-closed map in our algebraic formulation. In this sense our formulation provides an algebraic counterpart of the cyclic open-closed map in Lagrangian Floer theory together with our meromorphic connection, and describes the relation of our meromorphic connection in the Fukaya A_∞ category to Dubrovin's quantum connection in quantum cohomology via the cyclic open-closed map. See Proposition 5.21, Corollary 5.22 and Theorem 6.5, and also Remark 5.23 for the variants. The unitarity of the A_∞ category is used in Section 6. We also note that there is a related result on compatibility of the Getzler-Gauss-Manin connection under a certain L_∞ morphism of modules in deformation quantization [5, Proposition 1.4].

In this note a key ingredient to incorporate the derivative of the z-direction with the Getzler-Gauss-Manin connection on periodic cyclic homology of a filtered A_∞ category is to consider a $\mathbb{Z}$-grading. To encode a $\mathbb{Z}$-grading structure in the Fukaya A_∞ category $\mathrm{Fuk}(X)$[1] we will use the universal Novikov ring Λ_0^e (see Example 4.2 (1) for this notation) by adding one formal variable e of degree 2 instead of Λ_0 in (1.1). This formal variable e was

[1]More precisely, we consider anchored/graded Lagrangian submanifolds as objects. See [15], [32] for more details.

originally used in the universal Novikov ring in Lagrangian Floer theory [13], [14] in order to encode the Maslov index of holomorphic disks, while Λ_0 or its quotient field Λ without the formal variable e are frequently used in recent literatures, for example [16], [18], where only $\mathbb{Z}/2\mathbb{Z}$-grading is considered. By using the $\mathbb{Z}$-grading structure we define a meromorphic connection called an *Euler connection* in Definition 5.14, which incorporates the derivative of the z-direction with the Getzler-Gauss-Manin connection.

To establish the compatibility of our Euler connection with Dubrovin's quantum connection under the cyclic open-closed map at cohomology level, we need argument at chain level. In an algebraic aspect, we will study certain algebraic structure keeping track of information of chain homotopy at certain depth. For this purpose we introduce the notion of *CH module* over an L_∞ algebra in Definition 3.6, where 'CH' stands for *Cartan homotopy.* It is a cousin of the notion of calculus algebra in [8], [36]. (We note that the Cartan homotopy formula already played an important role in Getzler's paper [21] to construct the Getzler-Gauss-Manin connection on periodic cyclic homology.) Actually we will see that the structure of calculus algebra gives a typical example of our CH module structure and a hypercommutative algebra mentioned above also provides a typical example of the CH module structure. (See Examples 3.9, 3.10.) In the language of CH-module and CH-morphism we define, we formulate and study the compatibility of our Euler connections under a CH-morphism in Proposition 5.21 and Corollary 5.22, and apply these results to the situation arising from an A_∞ category in Section 6.

Remark 1.1. Our connection formally looks similar to one in [25, p.108] by replacing our variables z, e by their variables u, t respectively. In fact, the variable z plays the same role as the variable u. However there are indeed differences in the following points. Katzarkov-Kontsevich-Pantev consider a $t \in \mathbb{A}_{\mathbb{C}} \setminus \{0\}$-parametrized family of $\mathbb{Z}/2\mathbb{Z}$-graded (DG) algebras (see [4] for its A_∞ version), while we consider a $\mathbb{Z}$-graded family of (DG) algebras. Their variable t stands for a formal abstract parameter and does not have a non-trivial grading (but they consider a 'weight' instead) because of the property $d_{\mathcal{A}_t} = t \cdot d_A$ etc, described in [25, p.108]. On the other hand, our variable e appears in more primitive way and has a geometric meaning in Lagrangian Floer theory as mentioned above, which naturally involves a $\mathbb{Z}$-grading. Furthermore, since our $\mathbb{Z}$-graded family of (DG) algebras is different form one they consider, the Getzler-Gauss-Manin connections associated to these families are, at least a priori, different.

Remark 1.2. There are related works, e.g. in [19], [33] motivated also by symplectic geometry. In [19, Section 4.2] Ganatra-Perutz-Sheridan claims the compatibility of the Getzler-Gauss-Manin connection and Dubrovin's quantum connection under the cyclic open-closed map. Their Geztler-Gauss-Manin connection does not involve the derivative of the z-direction. However they consider the Calabi-Yau case. So it is actually not necessary for the situation they study. Moreover, we note that a $\mathbb{Z}$-grading structure naturally appears in their case.

In [33] Seidel studies a similar quantum connection on equivariant Hamilton Floer homology and its compatibility with Dubrovin's quantum connection under the PSS map.

The outline of this paper is in order. Since we borrow Tygan's reformulation [37], following Barannikov's idea sketched in [2], of the Getzler-Gauss-Manin connection in terms of L_∞ algebra and L_∞ module, we start with recalling basic definitions concerning L_∞ algebra and L_∞ module in Section 2. In Section 3 we introduce the notion of CH module over an L_∞ algebra L and CH module morphism. For a given L_∞ algebra L over a graded algebra R, we first enhance an L_∞ algebra structure on $\widetilde{L} := L[[z]] \oplus \epsilon L[[z]]$ with new formal variables z and ϵ of $\deg z = 2$ and $\deg \epsilon = 1$, which can be regarded as a mapping cone of the morphism $z \cdot \mathrm{id} \in \mathrm{End}_{R[[z]]}(L[[z]])$. Then a *CH module structure* on a graded $R[[z]]$ module $\widetilde{M}$ over the L_∞ algebra L is defined as an 'ϵ^2-truncated' L_∞ module structure on $\widetilde{M}$ over the cone $\widetilde{L}$. This structure includes information of chain homotopy at certain depth which will be used in later argument at chain level. On the other hand, as we mentioned above, our motivation comes from symplectic geometry, where the symplectic energy plays an important role. To encodes the symplectic energy we will use the Novikov ring/field as coefficients on which the symplectic energy induces a valuation or norm. Thus in Section 4 we define the normed (filtered) version of the notion of CH module. In Section 5, after defining the notion of *CH connection* on a CH module $\widetilde{M}$ over L, we define the Getzler-Gauss-Manin connection on $\widetilde{M}$ for each CH connection and Maurer-Cartan element of the L_∞ algebra L in Definition 5.8. In Subsection 5.4, using the grading operator and the Euler vector field, we incorporate the derivative of the z-direction with the Getzler-Gauss-Manin connection defined above. We call the resulted connection an *Euler connection* on $\widetilde{M}$. Then for any CH module morphism which intertwines CH connections we show compatibility of the Euler connections under the CH module morphism (Proposition 5.21, Corollary 5.22). Now in Section 6, we study the situation of the Hochschild chain/cochain of an A_∞ category $\mathscr{A}$. It is known that the shifted (reduced) Hochschild cochain complex $\overline{CC}^\bullet(\mathscr{A})[1]$ has a structure of DGLA and the shifted (reduced) Hochschild chain complex $\overline{CC}_\bullet(\mathscr{A})[1]$ has a structure of

DGLA module over $\overline{CC}^\bullet(\mathscr{A})[1]$. Moreover we show in Theorem 6.5 that the reduced Hochschild chain $\widetilde{M} := \overline{CC}_\bullet(\mathscr{A})[1][[z]]$ has a CH module structure over the DGLA $L := \overline{CC}^\bullet(\mathscr{A})[1]$. Therefore we can apply the story developed in up to Section 5 to this situation. Thus this derives a Getzler-Gauss-Manin connection together with derivative of the z-direction (an Euler connection) on the periodic cyclic homology of an A_∞ category. Finally in Section 7, we recall the definition of a part of primitive forms and briefly explain relations to filtered A_∞ categories and Euler connections.

2. Preliminaries on L_∞ algebras and L_∞ modules

In this section, we recall some definitions related to L_∞ algebras (e.g. [27]). Let $R = \oplus_{k\in\mathbb{Z}} R^k$ be a ($\mathbb{Z}-$)graded ring. Throughout this paper, we assume that rings are (graded) commutative and contains $\mathbb{Q}$, i.e., $n \cdot 1$ is invertible for each $n \in \mathbb{Z} \setminus \{0\}$. By graded R modules, we mean left graded R modules, which are naturally considered as graded R bimodules. Usually, elements of graded modules are assumed to be homogeneous.

For graded modules, the degree of homogeneous elements are denoted by $|\cdot|$ and set $|\cdot|' := |\cdot| - 1$. For a graded R module V, the one shift $V[1]$ is defined by $V[1]^k := V^{k+1}$. Let $s : V \to V[1]$ be the "identity map" (degree -1). The graded R module structure of $V[1]$ is defined by $rsv := (-1)^{|r|} srv$. To simplify notation, we use the same letter v for sv.

For graded R-modules V, W, the graded tensor product over R is denoted by $V \otimes W$. The symmetric group S_k of degree k naturally acts on $V^{\otimes k}$. The coinvariant of this action is called the graded symmetric tensor product and denoted by $V^{\odot k}$. An element $\sigma \in S_k$ is called an $(i_1, i_2, \ldots, i_l)$-shuffle if $i_1 + \cdots + i_l = k$ and

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i_1), \ldots, \sigma(i_1 + \cdots + i_{l-1} + 1) < \cdots < \sigma(k).$$

The set of $(i_1, i_2, \ldots, i_l)$-shuffles is denoted by $\mathrm{Sh}(i_1, i_2, \ldots, i_l)$. For $x_1, \ldots, x_k \in V$ and $\sigma \in S_k$, the Koszul sign $\epsilon(\sigma)$ is defined by

$$x_1 \odot \cdots \odot x_k = \epsilon(\sigma) x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k)}.$$

Note that $\epsilon(\sigma)$ depends on $x_1, \ldots, x_k$.

Set $E_k V := V[1]^{\odot k}$ and $EV := \oplus_{0\le k} E_k V$. We define a product $EV \otimes EV \to EV$ and a coproduct $\Delta : EV \to EV \otimes EV$ as follows:

$$(y_1 \odot \cdots \odot y_k) \cdot (y'_1 \odot \cdots \odot y'_l) = y_1 \odot \cdots \odot y_k \odot y'_1 \odot \cdots \odot y'_l,$$

$$\Delta(y_1 \odot \cdots \odot y_k) = \sum_{\substack{p+q=k \\ 0 \le p \le k}} \sum_{\sigma \in \mathrm{Sh}(p,q)} \epsilon(\sigma) y_{\sigma(1)} \odot \cdots \odot y_{\sigma(p)} \otimes y_{\sigma(p+1)} \odot \cdots \odot y_{\sigma(k)},$$

where $\epsilon(\sigma)$ is the koszul sign determined by the shifted degree $|\cdot|'$. We define a unit η by the inclusion $R \cong E_0 V \subset EV$ and define a counit ϵ by the projection from EV to R. Then $(EV, \cdot, \Delta, \eta, \epsilon)$ is a bialgebra. We note that η gives a coaugmentation and $(EV, \delta, \eta, \epsilon)$ is a cocomutative conilpotent cofree coalgebra (see, e.g., [29, §1]). For

$$\{f_k\}_{1 \le k} \in \prod_{1 \le k} \mathrm{Hom}^1_R(V[1]^{\odot k}, V[1]),$$

we define $\widehat{f} \in \mathrm{Hom}^1_R(EV, EV)$ by

$$\widehat{f}(y_1, \dots, y_k) = \sum_{\substack{p+q=k \\ 1 \le p \le k}} \sum_{\sigma \in \mathrm{Sh}(p,q)} \epsilon(\sigma) f_p(y_{\sigma(1)}, \dots, y_{\sigma(p)}) \odot \cdots \odot y_{\sigma(k)}.$$

This gives an isomorphism between the set $\prod_{1 \le k} \mathrm{Hom}^1_R(V[1]^{\odot k}, V[1])$ and the set of coderivations of (EV, Δ) such that composition of the coderivation and η is equal to zero. Similarly, for

$$\{f_k\}_{1 \le k} \in \prod_{1 \le k} \mathrm{Hom}^0_R(V[1]^{\odot k}, W[1]),$$

we define $\mathrm{e}^f \in \mathrm{Hom}^0_R(EV, EW)$ by the following:

$$f^{\odot l}(y_1, \dots, y_k) = \sum_{\substack{i_1 + \cdots + i_l = k \\ \sigma \in \mathrm{Sh}(i_1, \dots, i_l)}} \epsilon(\sigma) f_{i_1}(y_{\sigma(1)}, \dots, y_{\sigma(i_1)}) \odot \cdots$$
$$\cdots \odot f_{i_l}(y_{\sigma(i_1 + \cdots + i_{l-1} + 1)}, \dots, y_{\sigma(k)}),$$
$$\mathrm{e}^f = \sum_{l=0}^{\infty} \frac{f^{\odot l}}{l!}.$$

Then this gives an isomorphism between the set $\prod_{1 \le k} \mathrm{Hom}^0_R(V[1]^{\odot k}, W[1])$ and the set of coalgebra morphisms from EV to EW such that the composition of the morphism and η is equal to η.

Definition 2.1. Let L be a graded module over R and

$$\{\ell_k\} \in \prod_{1 \le k} \mathrm{Hom}^1_R(L[1]^{\odot k}, L[1]).$$

The pair $(L, \{\ell_k\})$ is called an L_∞ *algebra over* R if $\{\ell_k\}$ satisfies the relation $\widehat{\ell} \circ \widehat{\ell} = 0$.

Remark 2.2. Let $(L, \{\ell_k\}_{1\le k})$ be an L_∞ algebra. Set

$$\delta = -\ell_1(y),\ \ [y_1, y_2] = (-1)^{|y_1|}\ell_2(y_1, y_2).$$

Assume that $\ell_k = 0$ $(3 \le k)$. Then $(L, [\cdot,\cdot], \delta)$ is a differential graded Lie algebra (DGLA for short). Conversely, a DGLA is naturally considered as an L_∞ algebra.

Definition 2.3. Let $(L, \{\ell_k\}), (L', \{\ell'_k\})$ be L_∞ algebras. A set of morphisms

$$\{f_k\} \in \prod_{1\le k} \mathrm{Hom}^0_R(L[1]^{\odot k}, L'[1])$$

is called an L_∞ *morphism* if it satisfies $\widehat{\ell'} \circ \mathrm{e}^f = \mathrm{e}^f \circ \widehat{\ell}$.

We next recall the definition of L_∞ modules over an L_∞ algebra $(L, \{\ell_k\})$. Let M be a graded R-module. We consider an EL comodule $EL \otimes M[1]$. By the construction, there exists an isomorphism between the set

$$\mathrm{Hom}^1_R(EL \otimes M[1], M[1])$$

and the set of coderivations of $EL \otimes M[1]$, where $(EL, \widehat{\ell})$ is considered as a differential graded coalgebra. For an element

$$\{\ell^M_k\} \in \prod_{0\le k} \mathrm{Hom}^1_R(L[1]^{\odot k} \otimes M[1], M[1]),$$

the corresponding coderivation is denoted by $\widehat{\ell}^M \in \mathrm{End}^1_R(EL\otimes M[1])$. Namely, writing $\widehat{\ell}^M(y_1 \odot \cdots \odot y_k \otimes m)$ as $\widehat{\ell}^M(y_1, \ldots, y_k|m)$, we have

$$\begin{aligned}
&\widehat{\ell}^M(y_1, \ldots, y_k|m)\\
&= \sum_{\substack{p+q=k\\1\le p\le k}} \sum_{\sigma\in \mathrm{Sh}(p,q)} \epsilon(\sigma)\ell_p(y_{\sigma(1)}, \ldots, y_{\sigma(p)}) \odot \cdots \odot y_{\sigma(k)} \otimes m \\
&\quad + \sum_{\substack{p+q=k\\0\le p\le k}} \sum_{\sigma\in \mathrm{Sh}(p,q)} \epsilon'(\sigma) y_{\sigma(1)} \odot \cdots \odot y_{\sigma(p)} \otimes \ell^M_q(y_{\sigma(p+1)}, \ldots, y_{\sigma(k)}|m),
\end{aligned} \tag{2.1}$$

where $\epsilon'(\sigma) = (-1)^{|y_{\sigma(1)}|'+\cdots+|y_{\sigma(p)}|'}\epsilon(\sigma)$. Note that $\widehat{\ell}^M$ depends on the L_∞ algebra structure of L.

Definition 2.4. $(M, \{\ell_k^M\}_{0\le k})$ is called an L_∞ *module over* L if $\widehat{\ell}^M \circ \widehat{\ell}^M = 0$.

Remark 2.5. Let L be a DGLA and (M, d) be a DGLA module over the DGLA L. The action of $y \in L$ is denoted by $\mathcal{L}_y \in \mathrm{End}_R^{|y|}(M)$. We consider L as an L_∞ algebra (see Remark 2.2). Set

$$\ell_0^M = -d, \quad \ell_1^M(y|m) = (-1)^{|y|}\mathcal{L}_y(m), \quad \ell_k^M = 0 \ (k \ge 2).$$

Then $(M, \{\ell_k^M\}_{0\le k})$ is an L_∞ module. In this way, a DGLA module $(M, d, \mathcal{L})$ over a DGLA L can be regarded as an L_∞ module.

We recall the definition of morphisms of L_∞ modules. Let $(N, \{\ell_k^N\}_{0\le k})$ be another L_∞ module over L. Then there exists an isomorphism between the set $\mathrm{Hom}_R^0(EL \otimes M[1], N[1])$ and the set of comodule morphisms from $EL \otimes M[1]$ to $EL \otimes N[1]$. For an element

$$\{f_k\}_{0\le k} \in \prod_{0\le k} \mathrm{Hom}_R^0(L[1]^{\odot k} \otimes M[1], N[1]),$$

the corresponding comodule morphism is denoted by $\check{f}$.

Definition 2.6. $\{f_k\}_{0\le k}$ is called an L_∞ *module morphism* if it satisfies $\widehat{\ell}^N \circ \check{f} = \check{f} \circ \widehat{\ell}^M$.

Remark 2.7. Let L and L' be L_∞ algebras over R, $\{f_k\}_{1\le k}$ be an L_∞ morphism from L to L', and $(M, \{\ell_k^M\}_{0\le k})$ be an L_∞ module over L'. Then $(\mathrm{e}^{-f} \otimes \mathrm{id}) \circ \widehat{\ell}^M \circ (\mathrm{e}^f \otimes \mathrm{id})$ is a coderivation and this coderivation makes M into an L_∞ module over L. Using this construction, we can define L_∞ module morphisms between L_∞ modules defined over different L_∞ algebras. Similar remarks are applied to the cases of morphisms of CH-modules (Definition 3.11) and normed CH-modules (Definition 4.9).

3. CH structures

3.1. Notations on formal power series

Let M be a graded additive group and let $t_1, t_2, \ldots, t_k$ be formal variables with degree $d_1, d_2, \ldots, d_k \in \mathbb{Z}$ respectively. The space of degree d formal power series of $t_1, \ldots, t_k$ (these variables are graded commutative) with coefficients in M is denoted by $M[[t]]^d$ and set $M[[t]] := \oplus_{d\in\mathbb{Z}} M[[t]]^d$. Note that the degrees of coefficients also contribute to the degrees of elements of $M[[t]]$.

The space of formal Laurent power series $M((t)) = \oplus_{d\in\mathbb{Z}} M((t))^d$ is defined similarly. These are naturally considered as graded additive groups.

A morphism from M to another graded additive group N naturally extends to a morphism from $M[[t]]$ (resp. $M((t))$) to $N[[t]]$ (resp. $N((t))$). By abuse of notation, these extended morphisms are also denoted by the same symbols.

We use multi-index notation, i.e., $t^\alpha := t_1^{\alpha_1}\cdots t_k^{\alpha_k}$ for $\alpha = (\alpha_1,\dots,\alpha_k) \in \mathbb{Z}^k$. If R is a graded ring and M is a graded module over R, then $R[[t]]$ (resp. $R((t))$) has a natural graded ring structure and $M[[t]]$ (resp. $M((t))$) has a natural (left) graded module structure over $R[[t]]$ (resp. $R((t))$). Note that the ring structure and module structure are determined by the following sign rules:

$$t^\alpha r = (-1)^{|r||t^\alpha|} r t^\alpha,\ t^\alpha m = (-1)^{|m||t^\alpha|} m t^\alpha, \quad (r\in R, m\in M).$$

3.2. CH structures

In §3.2, we introduce CH structures. This construction is inspired by [37]. Let $(L, \{\ell_k\}_{1\le k})$ be an L_∞ algebra over a graded algebra R. Let z and ϵ be formal variables with degree

$$|z| = 2, \quad |\epsilon| = 1.$$

We introduce a “mapping cone” of the morphism $z\cdot \mathrm{id} \in \mathrm{End}(L[[z]])$. Set

$$\widetilde{L} := L[[z]] \oplus \epsilon L[[z]].$$

Then $\widetilde{L}[1]$ is naturally identified with $L[1][[z]] \oplus \epsilon L[1][[z]]$. Note that $\epsilon\circ s = -s\circ\epsilon$, where ϵ is the multiplication by ϵ. We define operations $\widetilde{\ell}_k$ on $\widetilde{L}[1]$ by the following equations:

$$\begin{aligned}&\widetilde{\ell}_k(y_1+\epsilon y_1',\dots,y_k+\epsilon y_k')\\ &:=\ell_k(y_1,\dots,y_k) - \epsilon\sum_{i=1}^k (-1)^{\#}\ell_k(y_1,\dots,y_i',\dots,y_k) + \begin{cases} z y_1' & \text{if } k=1,\\ 0 & \text{if } k\ge 2,\end{cases}\end{aligned} \tag{3.1}$$

where $\# := |y_1|' + \cdots + |y_{i-1}|'$ and $|y_i|' = |y_i|+1$ for $y_i \in L[[z]]$.

Proposition 3.1. *$(\widetilde{L}, \{\widetilde{\ell}_k\}_{1\le k})$ is an L_∞ algebra over $R[[z]]$.*

Proof. We put

$$\widetilde{\ell}_k'(y_1+\epsilon y_1',\dots,y_k+\epsilon y_k') := \ell_k(y_1,\dots,y_k) - \epsilon\sum_{i=1}^k(-1)^{\#}\ell_k(y_1,\dots,y_i',\dots,y_k).$$

Then it is easy to see that they satisfy the L_∞ relations. Hence it is sufficient to show that the coefficient of z in the formula

$$\widetilde{\ell}_1\widetilde{\ell}_k(y_1+\epsilon y_1',\dots,y_k+\epsilon y_k')+\sum_{i=1}^k(-1)^{\#}\widetilde{\ell}_k(y_1+\epsilon y_1',\dots,\widetilde{\ell}_1(y_i+\epsilon y_i'),\dots,y_k+\epsilon y_k')$$

is equal to zero, where $\# := |y_1+\epsilon y_1'|'+\cdots+|y_{i-1}+\epsilon y_{i-1}'|'$. This statement follows by direct calculation. $\square$

Let $(L', \{\ell_k'\}_{1\le k})$ be another L_∞ algebra. For en element

$$\{f_k\} \in \prod_{1\le k} \mathrm{Hom}^0_R(L[1]^{\odot k}, L'[1]),$$

we set

$$\begin{aligned}&\widetilde{f}_k(y_1+\epsilon y_1',\dots,y_k+\epsilon y_k')\\ :=&f_k(y_1,\dots,y_k)+\epsilon\sum_{i=1}^k(-1)^{\#}f_k(y_1,\dots,y_i',\dots,y_k),\end{aligned} \tag{3.2}$$

where $\# := |y_1+\epsilon y_1'|'+\cdots+|y_{i-1}+\epsilon y_{i-1}'|'$.

Proposition 3.2. *If $\{f_k\}_{1\le k}$ is an L_∞ morphism from L to L', then $\{\widetilde{f}_k\}_{1\le k}$ is also an L_∞ morphism from $\widetilde{L}$ to $\widetilde{L'}$.*

Proof. It is sufficient to show that the coefficient of z in the formula

$$\widetilde{\ell}_1\widetilde{f}_k(y_1+\epsilon y_1',\dots,y_k+\epsilon y_k')-\sum_{i=1}^k(-1)^{\#}\widetilde{f}_k(y_1+\epsilon y_1',\dots,\widetilde{\ell}_1(y_i+\epsilon y_i'),\dots,y_k+\epsilon y_k')$$

is equal to zero, where $\# := |y_1+\epsilon y_1'|'+\cdots+|y_{i-1}+\epsilon y_{i-1}'|'$. This follows by direct calculation. $\square$

Now we consider the following decreasing filtration on $\widetilde{L}[1]$ which is defined by the number of ϵ:

$$\cdots F^{-2} := \widetilde{L}[1] \supseteq F^{-1} := \widetilde{L}[1] \supset F^0 := L[1][[z]] \supset F^1 := 0 \supseteq \cdots.$$

Let $\widetilde{M}$ be a graded $R[[z]]$ module. Then $\widetilde{M}[1]$ is equipped with the following filtration:

$$\cdots F^{-1} := \widetilde{M}[1] \supseteq F^0 := \widetilde{M}[1] \supset F^1 := 0 \supseteq \cdots.$$

We note that tensor products of modules with filtrations are naturally equipped with filtrations. We consider a set of morphisms

$$\ell_k^{\widetilde{M}} \in \mathrm{Hom}^1_{R[[z]]}(\widetilde{L}[1]^{\odot k} \otimes \widetilde{M}[1], \widetilde{M}[1]) \quad (0 \le k).$$

Note that $\odot$ and $\otimes$ are defined over $R[[z]]$. We denote by

$$\widehat{\ell}^{\widetilde{M}} \in \mathrm{Hom}^1_{R[[z]]}(E\widetilde{L} \otimes \widetilde{M}[1], E\widetilde{L} \otimes \widetilde{M}[1])$$

the coderivation corresponding to $\ell_k^{\widetilde{M}}$.

Definition 3.3. Let $(L, \{\ell_k\}_{1 \le k})$ be an L_∞ algebra over a graded algebra R and $\widetilde{M}$ a graded $R[[z]]$ module. Let $n \in \mathbb{Z}_{\ge 0}$. A set of morphisms $\{\ell_k^{\widetilde{M}}\}_{0 \le k}$ is called an L_∞ *module structure on* $\widetilde{M}$ *over* $\widetilde{L}$ mod ϵ^n if it satisfies

$$(\widehat{\ell}^{\widetilde{M}} \circ \widehat{\ell}^{\widetilde{M}})(F^a) \subset F^{a+n} \tag{3.3}$$

for any $a \in \mathbb{Z}$. Here F^a is the filtration on $E\widetilde{L} \otimes \widetilde{M}[1]$ induced by the filtrations on $\widetilde{L}[1]$ and $\widetilde{M}[1]$ defined as above.

Remark 3.4. Let $\epsilon : E\widetilde{L} \to R[[z]]$ be the counit. By the explicit formula (2.1) (modify the sign $\epsilon'(\sigma)$ to $\epsilon(\sigma)$), the morphism $\epsilon \otimes \mathrm{id}$ gives an isomorphism from the set of coderivations (of degree 2) of $E\widetilde{L} \otimes \widetilde{M}[1]$ to $\mathrm{Hom}^2_R(E\widetilde{L} \otimes \widetilde{M}[1], \widetilde{M}[1])$ and this isomorphism preserves the filtrations. Moreover

$$F^{a+n}\widetilde{M}[1] = \begin{cases} \widetilde{M}[1] & (a+n \le 0), \\ 0 & (a+n \ge 1). \end{cases}$$

Hence the condition $(\widehat{\ell}^{\widetilde{M}} \circ \widehat{\ell}^{\widetilde{M}})(F^a) \subset F^{a+n}$ is equivalent to the condition

$$((\epsilon \otimes \mathrm{id}) \circ \widehat{\ell}^{\widetilde{M}} \circ \widehat{\ell}^{\widetilde{M}})(F^{1-n}) = 0.$$

Remark 3.5. When $n = 1$, the condition (3.3) yields $\ell^{\widetilde{M}} \circ \widehat{\ell}^{\widetilde{M}}(F^0) = 0$. Thus an L_∞ module structure on $\widetilde{M}$ over $\widetilde{L}$ mod ϵ is nothing but an L_∞ module structure on $\widetilde{M}$ over $L[[z]]$. When $n = 2$, the condition (3.3) implies $\ell^{\widetilde{M}} \circ \widehat{\ell}^{\widetilde{M}}(F^{-1}) = 0$.

Definition 3.6. An L_∞ module structure on $\widetilde{M}$ over $\widetilde{L}$ mod ϵ^2 is called a *CH structure on* $\widetilde{M}$ *over* L. A graded $R[[z]]$ module $\widetilde{M}$ with a CH structure over L is called a *CH module over* L.

Let $(\widetilde{M}, \{\ell_k^{\widetilde{M}}\}_{0\le k})$ be a CH module over an L_∞ algebra $(L, \{\ell_k\}_{0\le k})$. We put

$$\delta := -\ell_1, \quad [y_1, y_2] := (-1)^{|y_1|}\ell_2(y_1, y_2) \text{ for all } y_1, y_2 \in L, \tag{3.4}$$

and

$$d := -\ell_0^{\widetilde{M}}, \quad \mathcal{L}_y := (-1)^{|y|}\ell_1^{\widetilde{M}}(y|\cdot) \text{ for all } y \in L[[z]] \subset \widetilde{L}. \tag{3.5}$$

Moreover, associated to the CH module structure, we define maps I_{y_1} and ρ_{y_1,y_2} for $y_1, y_2 \in L[[z]]$ by

$$I_{y_1} := (-1)^{|y_1|+1}\ell_1^{\widetilde{M}}(\epsilon y_1|\cdot), \qquad \rho_{y_1,y_2} := (-1)^{|y_1|+|y_2|}\ell_2^{\widetilde{M}}(y_1, \epsilon y_2|\cdot). \tag{3.6}$$

Now we write down explicitly the relations of the CH module structure under the following conditions:

$$\begin{cases} \ell_k = 0 \ (3 \le k), \\ \ell_k^{\widetilde{M}} = 0 \ (3 \le k), \quad \ell_2^{\widetilde{M}}(y_1, y_2|\cdot) = 0 \text{ for all } y_1, y_2 \in L[[z]] \subset \widetilde{L}. \end{cases} \tag{3.7}$$

Note that $(L, \delta, [\cdot,\cdot])$ is a DGLA over R and $(\widetilde{M}, d, \mathcal{L})$ is a DGLA module over the DGLA $L[[z]]$, where the DGLA structure on L linearly extends to $L[[z]]$.

Remark 3.7. We will consider $\mathcal{L}$ (resp. I) as an element of

$$\mathrm{Hom}_{R[[z]]}(L[[z]], \mathrm{End}_{R[[z]]}(\widetilde{M}))$$

of degree 0 (resp. degree 1). Similarly, we will consider

$$\rho \in \mathrm{Hom}^0_{R[[z]]}(L[[z]] \otimes L[[z]], \mathrm{End}_{R[[z]]}(\widetilde{M})).$$

The L_∞ relations $\mod \epsilon^2$ yield the following relations among the operators (see Remark 3.4):

$$[d, I_y] + I_{\delta y} + z\mathcal{L}_y = 0, \tag{3.8}$$

$$I_{[y_1,y_2]} - (-1)^{|y_1|}[\mathcal{L}_{y_1}, I_{y_2}] + [d, \rho_{y_1,y_2}] - \rho_{\delta y_1,y_2} - (-1)^{|y_1|}\rho_{y_1,\delta y_2} = 0, \tag{3.9}$$

$$\begin{aligned} \rho_{[y_1,y_2],y_3} - \rho_{y_1,[y_2,y_3]} + (-1)^{|y_1||y_2|}\rho_{y_2,[y_1,y_3]} - [\mathcal{L}_{y_1}, \rho_{y_2,y_3}] \\ + (-1)^{|y_1||y_2|}[\mathcal{L}_{y_2}, \rho_{y_1,y_3}] = 0. \end{aligned} \tag{3.10}$$

Conversely, we easily see the following:

Proposition 3.8. *Let* $(L,\delta,[\cdot,\cdot])$ *be a DGLA over* R *and* $(\widetilde{M},d,\mathcal{L})$ *be a DGLA module over the DGLA* $L[[z]]$ *equipped with morphisms* I *and* ρ *where*

$$I \in \operatorname{Hom}^1_{R[[z]]}(L[[z]], \operatorname{End}_{R[[z]]}(\widetilde{M})),$$
$$\rho \in \operatorname{Hom}^0_{R[[z]]}(L[[z]] \otimes L[[z]], \operatorname{End}_{R[[z]]}(\widetilde{M})).$$

Suppose that these morphisms satisfy the relations (3.8), (3.9) *and* (3.10). *Then, by the formulas* (3.4), (3.5) *and* (3.6), *these morphisms give a CH structure on* $\widetilde{M}$ *over* L *which satisfies the conditions* (3.7).

Example 3.9 (Calculus algebra)**.** To give an example of CH module, we recall the definition of calculus (e.g., [8], [36]). Let $(V,\wedge)$ be a graded commutative algebra over R, W be a graded module over V with the structure morphism $\iota_\bullet : V \otimes W \to W$ and $B \in \operatorname{End}_R^{-1}(W)$ be a degree -1 morphism with $B^2 = 0$. Assume $V[1]$ is equipped with a graded Lie algebra structure $[\cdot,\cdot]$ and $W[1]$ is equipped with a graded Lie module structure $\mathcal{L}_\bullet : V[1] \otimes W[1] \to W[1]$. For $x \in V$, the morphisms $(-1)^{|x|'}[x,\cdot] \in \operatorname{End}_R(V)$ and $(-1)^{|x|'}\mathcal{L}_x \in \operatorname{End}_R(W)$ are denoted by l_x. The 7-tuple $(V,W,\wedge,[\cdot,\cdot],\iota_\bullet,\mathcal{L}_\bullet,B)$ is called a *calculus* if they satisfy

$$l_{x_1}(x_2 \wedge x_3) = (l_{x_1}x_2) \wedge x_3 + (-1)^{(|x_1|+1)|x_2|} x_2 \wedge l_{x_1}x_3$$

and

$$l_{x_1 \wedge x_2} = l_{x_1} \circ \iota_{x_2} + (-1)^{|x_1|} \iota_{x_1} \circ l_{x_2}, \quad \iota_{l_{x_1}x_2} = [l_{x_1}, \iota_{x_2}], \quad l_x = [B, \iota_x].$$

For a calculus, set

$$L := V[1],\ \widetilde{M} := W[1][[z]],\ d := zB,\ \mathcal{L}_x := \mathcal{L}_x,\ I_x := (-1)^{|x|}\iota_x\ (x \in V).$$

Then $\widetilde{M}$ is a CH module over L with $\ell_k^{\widetilde{M}} = 0,\ (k \geq 2)$.

Example 3.10 (Hypercommutative algebra)**.** We recall the definition of hypercommutative algebra [22]. Let A be a graded R module equipped with a symmetric k-ary operation $(\cdot,\ldots,\cdot) : A^{\odot k} \to A$ of degree $4-2k$ for each $k \geq 2$. For a subset $S = \{i_1,i_2,\ldots,i_k\} \subset \mathbb{N}$ $(i_1 < \cdots < i_k)$ and $x_{i_1},\ldots,x_{i_k} \in A$, we will denote $x_{i_1} \odot \cdots \odot x_{i_k}$ by x_S. We call A a *hypercommutative algebra* if

$$\sum_{S_1 \sqcup S_2 = \{3,4,\ldots,k-1\}} \pm(x_1, x_{S_1}, (x_2, x_{S_2}, x_k)) = \sum_{S_1 \sqcup S_2 = \{3,4,\ldots,k-1\}} \pm(x_2, x_{S_1}, (x_1, x_{S_2}, x_k)),$$

where $\pm$ are the Koszul signs. These equations are called the *WDVV equations*.

Set $L := A[1]$, which is considered as an abelian graded Lie algebra. We also set $\widetilde{M} := A[[z]]$. We define $\ell_k^{\widetilde{M}}$ by

$$\ell_k^{\widetilde{M}}(x_1,\dots,x_k|m) := 0,$$
$$\ell_k^{\widetilde{M}}(\epsilon x_1,x_2,\dots,x_k|m) := (-1)^{1+|x_1|+|x_2|+\cdots+|x_k|}(x_1,x_2,\dots,x_k,m),$$

where $x_1,\dots,x_k,m \in A[[z]]$. Note that the double suspensions s^2x_i are also denoted by the same symbol x_i. Then $\widetilde{M}$ is a CH module over L. Moreover, if we define $\ell_k^{\widetilde{M}}(\epsilon x_1,\epsilon x_2,\dots,x_k|m) = 0$, we obtain a trivially extended L_∞ module mod ϵ^3 structure in this example by using the $WDVV$ equations.

We next define morphisms of CH modules. Let $\widetilde{M}$ and $\widetilde{N}$ be CH modules over L. We consider a set of morphisms

$$\{f_k\}_{0\le k} \in \prod_{0\le k} \mathrm{Hom}^0_{R[[z]]}(\widetilde{L}[1]^{\odot k}\otimes \widetilde{M}[1],\widetilde{N}[1]).$$

Definition 3.11. The set $\{f_k\}_{0\le k}$ is called a *CH module morphism from $\widetilde{M}$ to $\widetilde{N}$* if it satisfies

$$(\widehat{\ell}^{\widetilde{N}}\circ \check{f} - \check{f}\circ \widehat{\ell}^{\widetilde{M}})(F^a) \subset F^{a+2}$$

for all $a\in\mathbb{Z}$.

Remark 3.12. Let L, L' be L_∞ algebras and $\{f_k\}_{1\le k}$ be an L_∞ morphism from L to L'. Then the morphism $\mathrm{e}^{\widetilde{f}}$ preserves the filtrations. Hence a CH module over L' naturally gives a CH module over L (see also Remark 2.7).

Let $\{f_k\}_{0\le k}$ be a CH module morphism from $\widetilde{M}$ to $\widetilde{N}$. For $y\in L[[z]]$ and $m\in\widetilde{M}$ we put

$$F_y(m) := (-1)^{|y|}f_1(y|m),\quad F^\epsilon_y(m) := (-1)^{|y|+1}f_1(\epsilon y|m). \tag{3.11}$$

Thus we have $F, F^\epsilon \in \mathrm{Hom}_{R[[z]]}\Big(L[[z]],\mathrm{End}_{R[[z]]}(\widetilde{M},\widetilde{N})\Big)$. Then we find

$$I_y\circ f_0 - f_0\circ I_y = d\circ F^\epsilon_y - (-1)^{|y|}F^\epsilon_y\circ d - F^\epsilon_{\delta y} - zF_y. \tag{3.12}$$

Here $f_0 \in \mathrm{End}_{R[[z]]}(\widetilde{M}[1],\widetilde{N}[1])$ is naturally considered as an element of $\mathrm{End}_{R[[z]]}(\widetilde{M},\widetilde{N})$.

4. Normed objects

4.1. Preliminaries on norms

A main reference of this subsection is [3]. We basically follow the terminology used there. Let $V = \oplus_{k\in\mathbb{Z}} V^k$ be an graded additive group. A (non-archimedean) norm on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ with the following properties:

- $\|v_1 - v_2\| \leq \max\{\|v_1\|, \|v_2\|\}$ for all $v_1, v_2 \in V$.
- $\|v\| = 0$ if and only if $v = 0$.
- $\|v\| = \max_{k\in\mathbb{Z}} \|v_k\|$, where $v = \sum v_k$ $(v_k \in V^k)$.

A graded additive group with a norm is called a *graded normed group*. A graded normed group V is said to be *complete* if each V^k is a complete metric space with respect to the norm. Set $\|V\| := \sup\{\|v\| \mid v \in V\}$. A graded normed group V is said to be *bounded* if $\|V\| \leq C$ for some constant $C \in \mathbb{R}_{\geq 0}$.

Let W be another graded normed group. Then $V \otimes W$ is equipped with a norm which is defined by $\|\sum_{i=1}^k v_i \otimes w_i\| = \sup_{1\leq i\leq k} \|v_i\| \cdot \|w_i\|$. The completion of $(V \otimes W)^k$ is denoted by $(V \widehat{\otimes} W)^k$. The completed tensor product is defined by $V \widehat{\otimes} W := \oplus (V \widehat{\otimes} W)^k$. The completed symmetric tensor product $\widehat{\odot}$ and the completed direct sum $\widehat{\oplus}$ are defined similarly.

A group morphism $f : V \to W$ is said to be *contractive* if $\|f(v)\| \leq \|v\|$ for all $v \in V$.

Let R be a graded ring and $\|\cdot\|$ be a norm on R (as a graded additive group). The norm $\|\cdot\|$ is called a *ring norm* if it satisfies $\|r_1 r_2\| \leq \|r_1\|\|r_2\|$ $(r_1, r_2 \in R)$ and $\|1\| = 1$. A graded ring equipped with a ring norm is called a *graded normed ring*.

Let V be a graded module over a graded normed ring R and $\|\cdot\|$ be a norm on V. The norm $\|\cdot\|$ is called a *module norm* if it satisfies $\|rv\| \leq \|r\|\|v\|$ $(r \in R, v \in V)$. A graded module equipped with a module norm is called a *graded normed module*.

Let $\mathbb{K}$ be a normed ring (ungraded, i.e., concentrated in degree zero). Let R be a graded normed algebra over $\mathbb{K}$, i.e., R is a graded algebra over $\mathbb{K}$ equipped with a norm $\|\cdot\|$ such that $\|\cdot\|$ is a ring norm and a $\mathbb{K}$ module norm.

Example 4.1. Here are some examples of normed rings $\mathbb{K}$.

1. A field with the trivial valuation.

2. The universal Novikov field

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; \begin{array}{l} a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim\limits_{i\to\infty} \lambda_i = \infty \\ \lambda_0 < \lambda_1 < \lambda_2 < \cdots \end{array} \right\}.$$

The norm on Λ is defined by $\|\sum_{i=0}^{\infty} a_i T^{\lambda_i}\| := \mathrm{e}^{-\lambda_0}$. Note that Λ is a non-archimedean valuation field.

3. The Novikov ring $\Lambda_0 = \{x \in \Lambda \mid \|x\| \leq 1\}$ (the valuation ring of Λ), where the norm is induced from Λ.

Example 4.2. Here are some examples of graded normed algebras and graded normed modules.

1. Let R be an (ungraded) normed algebra over $\mathbb{K}$ (e.g., $R = \Lambda_0, \mathbb{K} = \mathbb{C}$). Set $R^e := R((e))$, where e is a formal variable of degree 2. Note that $R((e)) = \oplus_k R((e))^k = R[e, e^{-1}]$ (see §3.1). Set

$$\| \sum_{k\in\mathbb{Z}} r_k e^k \| := \max_{k\in\mathbb{Z}} \|r_k\|, \quad (r_k \in R).$$

Then R^e is a graded normed algebra over $\mathbb{K}$.

2. Let R be a graded normed algebra over $\mathbb{K}$ and z be a formal variable of degree 2. Assume that R is bounded. We define a norm on $R[[z]]$ by

$$\| \sum_{k=0}^{\infty} r_k z^k \| := \sup_{k\in\mathbb{Z}_{\geq 0}} \|r_k\|.$$

Then $R[[z]]$ is a graded normed algebra.
Let M be a graded bounded normed module over R. Similar to $R[[z]]$, $M[[z]]$ is also equipped a norm and $M[[z]]$ is a graded normed module over $R[[z]]$.

3. Let R be a graded normed algebra over $\mathbb{K}$ and let $t^1, \ldots, t^m$ be formal variables with degree $d_1, \ldots, d_m \in \mathbb{Z}$. Assume that R is bounded. Take some constant $C \in \mathbb{R}_{>1}$. We define a norm on $R[[t]]$ by

$$\| \sum_{\alpha} r_\alpha t^\alpha \| := \sup_{\alpha} \|r_\alpha\| \|t^\alpha\|,$$

where $\|t^\alpha\| := C^{-\alpha_1 - \cdots - \alpha_m}$. Then $R[[t]]$ is a graded normed algebra over $\mathbb{K}$.

Let M be a graded bounded normed module over R. Similar to $R[[t]]$, $M[[t]]$ is also equipped with a norm and $M[[t]]$ is a graded normed module over $R[[t]]$.

Remark 4.3. If a graded bounded normed group V is complete, then $V[[z]], V[[t]]$ are also complete.

4.2. Normed L_∞ algebras and modules

Let R be a graded normed algebra over a normed ring $\mathbb{K}$. Assume that R is complete. Let L be an L_∞ algebra equipped with a complete R module norm. Set $\widehat{E}L := \widehat{\oplus} L[1]^{\widehat{\odot} k}$. As in the case of EL, we can define morphisms $\cdot, \Delta, \eta, \epsilon$ on $\widehat{E}L$. Morphisms $\ell_k \in \mathrm{Hom}^1_R(L[1]^{\odot k}, L[1])$ $(1 \le k)$ with $\|\ell_k(y_1, \dots, y_k)\| \le \|y_1\| \cdots \|y_k\|$ naturally extend to a contractive coderivation $\widehat{\ell}$ of $\widehat{E}L$ with $\widehat{\ell} \circ \eta = 0$. Also as in the case of EL, this correspondence gives an isomorphism. A set of morphisms $\{\ell_k\}_{1\le k}$ corresponding to a contractive morphism $\widehat{\ell}$ is also said to be *contractive*. Similarly, we can define contractive morphisms $\mathrm{e}^f, \widehat{\ell}^M, \check{f}$ (see §2) and corresponding sets of morphisms are also said to be contractive.

Definition 4.4. A *normed L_∞ algebra* is a pair $(L, \{\ell_k\}_{1\le k})$, where

- L is a graded complete normed R-module.
- $\{\ell_k\}$ is an L_∞-structure on L.
- $\{\ell_k\}$ is contractive.

Definition 4.5. A morphism between normed L_∞ algebras is defined by a contractive L_∞ morphism, which is called a *normed L_∞ morphism*.

Definition 4.6. A *normed L_∞ module* over a normed L_∞ algebra L is a pair $(M, \{\ell^M_k\}_{0\le k})$, where

- M is a graded complete normed R-module.
- $\{\ell^M_k\}$ is an L_∞-module structure on M over $(L, \{\ell_k\}_{1\le k})$.
- $\{\ell^M_k\}$ is contractive.

Definition 4.7. A morphism between normed L_∞ modules is defined by a contractive L_∞ module morphism, which is called a *normed L_∞ module morphism*.

4.3. Normed CH modules

We assume $\|R\| \le 1$. Let L be a normed L_∞ algebra with $\|L\| \le 1$. We define a norm on $\widetilde{L}$ by $\|y + \epsilon y'\| := \max\{\|y\|, \|y'\|\}$ (see Example 4.2 (2) for the definition of the norm on $L[[z]]$). Then $(\widetilde{L}, \{\widetilde{\ell}_k\})$ is a normed L_∞ algebra.

Definition 4.8. A *normed CH module* over a normed L_∞ algebra L is a pair $(\widetilde{M}, \{\ell_k^{\widetilde{M}}\}_{0\le k})$, where

- $\widetilde{M}$ is a graded complete normed $R[[z]]$ module.
- $\{\ell_k^{\widetilde{M}}\}$ is a CH structure on $\widetilde{M}$.
- $\{\ell_k^{\widetilde{M}}\}$ is contractive.

Definition 4.9. A morphism between normed CH modules is defined by a contractive CH module morphism, which is called a *normed CH morphism*.

5. Getzler-Gauss-Manin connections

5.1. Connections on CH modules

Let R be a graded algebra over a ring $\mathbb{K}$. In this subsection R and $\mathbb{K}$ are not assumed to be normed. The graded Lie algebra of derivations of R is denoted by $\mathrm{Der}_{\mathbb{K}}(R)$, i.e.,

$$\begin{aligned}
&\mathrm{Der}_{\mathbb{K}}(R) := \oplus_k \mathrm{Der}^k_{\mathbb{K}}(R),\\
&\mathrm{Der}^k_{\mathbb{K}}(R) := \left\{ X \in \mathrm{End}^k_{\mathbb{K}}(R) \;\middle|\; X(rr') = X(r)r' + (-1)^{k|r|} rX(r') \right\}.
\end{aligned}$$

Definition 5.1. We define $E \in \mathrm{Der}^0_{\mathbb{K}}(R)$ by $E(r) := \frac{1}{2}|r|r$. This vector field is called an *Euler vector field*.

For a graded R module V, a *connection on* V is a (degree preserving) morphism

$$\nabla : \mathrm{Der}_{\mathbb{K}}(R) \to \mathrm{End}_{\mathbb{K}}(V)$$

such that $\nabla_X(rv) = (Xr)v + (-1)^{|X||r|} r\nabla_X v$, for $X \in \mathrm{Der}_{\mathbb{K}}(R), r \in R, v \in V$. For a connection ∇ on V, a connection on $V[1]$ is defined by the formula $(-1)^{|X|}\nabla_X$. For another graded R module W with a connection ∇, graded R modules $V \oplus W, V \otimes W, \mathrm{Hom}_R(V,W)$ are also equipped with connections. These connections are also denoted by ∇. Explicitly, the connections on $V\otimes W$ and $\mathrm{Hom}_R(V,W)$ are defined by the following formulas:

$$\begin{aligned}
\nabla_X(v \otimes w) &= (\nabla_X v) \otimes w + (-1)^{|X||v|} v \otimes (\nabla_X w),\\
(\nabla_X f)(v) &= \nabla_X(f(v)) - (-1)^{|f||X|} f(\nabla_X v).
\end{aligned}$$

The curvature $R^\nabla(X,Y)$ is defined by

$$\nabla_X\nabla_Y - (-1)^{|X||Y|}\nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

A connection ∇ is called *flat* if ∇ is an endomorphism of a graded Lie algebra, i.e., $R^\nabla = 0$.

Let L be an L_∞ algebra over R and ∇ be a connection on the graded R module L. Then ∇ naturally gives connections on EL and $\mathrm{End}_R(EL)$, which are also denoted by ∇. A connection ∇ is called an L_∞ *connection* if $\nabla_X(\widehat{\ell}) = 0$ for any $X \in \mathrm{Der}_{\mathbb{K}}(R)$. Explicitly, this equality is written as follows:

$$\nabla_X(\ell_k(y_1,\dots,y_k)) = (-1)^{|X|}\sum_{i=1}^{k}(-1)^{(|y_1|'+\cdots+|y_{i-1}|')|X|}\ell_k(y_1,\dots,\nabla_X y_i,\dots y_k).$$

For a connection ∇ on L, we define a morphism

$$\widetilde{\nabla} : \mathrm{Der}_{\mathbb{K}}(R) \to \mathrm{End}_{\mathbb{K}[[z]]}(\widetilde{L})$$

by

$$\widetilde{\nabla}_X(y+\epsilon y') := \nabla_X(y) + (-1)^{|X|}\epsilon\nabla_X(y'). \tag{5.1}$$

Note that ∇_X linearly extends as a $\mathbb{K}[[z]]$ module morphism.

Definition 5.2. Let $\widetilde{M}$ be a CH module over an L_∞ algebra L and ∇ be an L_∞ connection on L. A *CH connection on* $\widetilde{M}$ is a morphism

$$\widetilde{\nabla} : \mathrm{Der}_{\mathbb{K}}(R) \to \mathrm{End}_{\mathbb{K}[[z]]}(\widetilde{M})$$

such that $\widetilde{\nabla}$ is a connection on $\widetilde{M}$ regarded as a graded R module and satisfies the following relations:

$$\begin{aligned}&\widetilde{\nabla}_X(\ell_k^{\widetilde{M}}(y_1,\dots,y_k|m))\\ =&(-1)^{|X|}\sum_{i=1}^{k}(-1)^{(|y_1|'+\cdots+|y_{i-1}|')|X|}\ell_k^{\widetilde{M}}(y_1,\dots,\widetilde{\nabla}_X y_i,\dots,y_k|m)\\ &+(-1)^{(1+|y_1|'+\cdots+|y_k|')|X|}\ell_k^{\widetilde{M}}(y_1,\dots,y_k|\widetilde{\nabla}_X m)\end{aligned}$$

for $y_i \in \widetilde{L}$ (not $L[[z]]$). Here $\widetilde{\nabla}$ in the first term on the right hand side is the connection induced by the L_∞ connection ∇ on L as in (5.1).

Remark 5.3. For a connection $\widetilde{\nabla}$ on $\widetilde{M}$, we define a connection on $\widetilde{M}[1]$ by $(-1)^{|X|}\widetilde{\nabla}$. We use the same symbol by abuse of notation.

Definition 5.4. Let $(\widetilde{M}, \widetilde{\nabla})$ and $(\widetilde{N}, \widetilde{\nabla})$ be CH modules with CH connections over an L_∞ algebra L with an L_∞ connection ∇. We say that a CH module morphism $\{f_k\}_{0\le k}$ from $\widetilde{M}$ to $\widetilde{N}$ *preserves the CH connections* if

$$\widetilde{\nabla}_X \circ \check{f} = \check{f} \circ \widetilde{\nabla}_X$$

for any $X \in \mathrm{Der}_{\mathbb{K}}(R)$. Here $\check{f} \in \mathrm{Hom}^0_{R[[z]]}(E\widetilde{L} \otimes \widetilde{M}[1], E\widetilde{L} \otimes \widetilde{N}[1])$ is the comodule morphism corresponding to $\{f_k\}_{0\le k}$ and $\widetilde{\nabla}_X$ are the $\mathbb{K}[[z]]$ linear morphisms induced from the L_∞ connection and CH connections.

5.2. Maurer-Cartan elements

Let L be a normed L_∞ algebra over a graded complete normed ring R. Let $\gamma \in L^1 \cong L[1]^0$. If $\|\gamma\| < 1$, then we can define

$$\mathrm{e}^\gamma := \sum_{k=0}^{\infty} \frac{1}{k!} \overbrace{\gamma \widehat{\odot} \cdots \widehat{\odot} \gamma}^{k} \in \widehat{E}L^0.$$

This is a group like element, i.e., $\Delta \mathrm{e}^\gamma = \mathrm{e}^\gamma \widehat{\otimes} \mathrm{e}^\gamma$ and $\epsilon(\mathrm{e}^\gamma) = 1$. Moreover, the morphism $\mathrm{e}^\gamma \cdot$ is an isomorphism with the inverse $\mathrm{e}^{-\gamma} \cdot$.

Definition 5.5. $\gamma \in L^1$ is called a *Maurer-Cartan element* of L if $\|\gamma\| < 1$ and γ satisfies $\widehat{\ell}(\mathrm{e}^\gamma) = 0$, i.e.,

$$\sum_{k=1}^{\infty} \ell_k(\gamma, \ldots, \gamma)/k! = 0.$$

This equation is called a *Maurer-Cartan equation*.

For a Maurer-Cartan element γ of L, we can twist L_∞ structure of L by

$$\widehat{\ell}^\gamma := \mathrm{e}^{-\gamma} \widehat{\ell} \mathrm{e}^\gamma.$$

Explicitly, the corresponding set of morphisms $\{\ell^\gamma_k\}_{1\le k}$ is written as follows:

$$\ell^\gamma_k(y_1, \ldots y_k) := \sum_{i=0}^{\infty} \frac{1}{i!} \ell_{k+i}(\overbrace{\gamma, \ldots, \gamma}^{i}, y_1, \ldots, y_k).$$

We note that the Maurer-Cartan equation for γ implies "$\ell^\gamma_0 = 0$". By construction, we easily see that $\{\ell^\gamma_k\}_{1\le k}$ gives an L_∞ algebra structure. To simplify notation, this L_∞ algebra is denoted by L^γ.

Let L' be another normed L_∞ algebra and $\{f_k\}_{0\le k}$ be a normed L_∞ morphism from L to L'. For $\gamma \in L^1$ with $\|\gamma\| < 1$, we see that $\mathrm{e}^f(\mathrm{e}^\gamma)$ is a group like element. Hence there exists a unique element $f_*\gamma \in EL'^0$ such that $\|f_*\gamma\| < 1$ and $\mathrm{e}^f(\mathrm{e}^\gamma) = \mathrm{e}^{f_*\gamma}$. Explicitly, we have

$$f_*\gamma = \sum_{k=1}^{\infty} \frac{1}{k!} f_k(\gamma, \ldots, \gamma).$$

Moreover, if γ is a Maurer-Cartan element, then $f_*\gamma$ is also a Maurer-Cartan element. Set $\mathrm{e}^{f,\gamma} := \mathrm{e}^{-f_*\gamma}\mathrm{e}^f\mathrm{e}^\gamma$, then we easily see that $\mathrm{e}^{f,\gamma}$ is a coalgebra morphism with $\mathrm{e}^{f,\gamma} \circ \eta_{EL} = \eta_{EL'}$. (Here η_{EV} denotes a unit of EV. See the beginning of Section 2.) The corresponding set of morphisms is denoted by $\{f_k^\gamma\}_{0\le k}$. If γ is a Maurer-Cartan element, then $\{f_k^\gamma\}_{0\le k}$ gives a normed L_∞ morphism from L^γ to $L'^{f_*\gamma}$.

We next consider twists of normed CH modules. Assume $\|R\|, \|L\| \le 1$. Then $\widetilde{L}$ is also a normed L_∞ algebra. If γ is a Maurer-Cartan element of L, then we easily see that γ is also a Maurer-Cartan element of $\widetilde{L}$. Let $(\widetilde{M}, \{\ell_k^{\widetilde{M}}\}_{0\le k})$ be a normed CH module over L. By the left multiplication, e^γ gives an automorphism of $\widehat{E}\widetilde{L}\widehat{\otimes}\widetilde{M}[1]$. We define $\{\ell_k^{\widetilde{M},\gamma}\}_{0\le k}$ by

$$\widehat{\ell}^{\widetilde{M},\gamma} := \mathrm{e}^{-\gamma}\widehat{\ell}^{\widetilde{M}}\mathrm{e}^\gamma.$$

Note that $\mathrm{e}^{-\gamma}\widehat{\ell}^{\widetilde{M}}\mathrm{e}^\gamma$ is a contractive coderivation of $\widehat{E}\widetilde{L}^\gamma\widehat{\otimes}\widetilde{M}[1]$.

Proposition 5.6. *For a Maurer-Cartan element γ of L, the pair $(\widetilde{M}, \{\ell_k^{\widetilde{M},\gamma}\})$ is a normed CH module over the normed L_∞ algebra L^γ.*

Proof. By the definition, we have $\widehat{\ell}^{\widetilde{M},\gamma} \circ \widehat{\ell}^{\widetilde{M},\gamma} = \mathrm{e}^{-\gamma}\widehat{\ell}^{\widetilde{M}} \circ \widehat{\ell}^{\widetilde{M}}\mathrm{e}^\gamma$. Since the automorphism e^γ preserves the filtration on $\widehat{E}\widetilde{L}^\gamma\widehat{\otimes}\widetilde{M}[1]$, we see that $\widehat{\ell}^{\widetilde{M},\gamma} \circ \widehat{\ell}^{\widetilde{M},\gamma}(F^a) \subset F^{a+2}$. $\square$

This normed CH module is denoted by $\widetilde{M}^\gamma$. Accordingly, the corresponding maps in (3.5), (3.6) are denoted by

$$d^\gamma,\ \mathcal{L}_y^\gamma,\ I_{y_1}^\gamma,\ \rho_{y_1,y_2}^\gamma. \tag{5.2}$$

Let $\widetilde{N}$ be another normed CH module over L and $\{f_k\}_{0\le k}$ be a normed CH morphism from $\widetilde{M}$ to $\widetilde{N}$. Set

$$\check{f}^\gamma := \mathrm{e}^{-\gamma}\check{f}\mathrm{e}^\gamma. \tag{5.3}$$

This is a contractive comodule morphism and the corresponding set of morphisms is denoted by $\{f_k^\gamma\}_{0\le k}$.

Proposition 5.7. *$\{f_k^\gamma\}_{0\le k}$ is a normed CH morphism from $\widetilde{M}^\gamma$ to $\widetilde{N}^\gamma$.*

Proof. By definition, we have

$$\widehat{\ell}^{\widetilde{N},\gamma} \circ \check{f}^\gamma - \check{f}^\gamma \circ \widehat{\ell}^{\widetilde{M},\gamma} = \mathrm{e}^{-\gamma}(\widehat{\ell}^{\widetilde{N}} \circ \check{f} - \check{f} \circ \widehat{\ell}^{\widetilde{M}})\mathrm{e}^\gamma,$$

which implies the proposition. $\square$

For the normed CH morphism $\{f_k^\gamma\}_{0\le k}$, the corresponding maps in (3.11) are denoted by

$$F_y^\gamma,\ F_y^{\epsilon,\gamma}. \tag{5.4}$$

5.3. Getzler-Gauss-Manin connections

In [21], Getzler constructed a connection on periodic cyclic homology of A_∞ algebras. Following Barannikov's idea outlined in [2, Remark 3.3], Tsygan reformulated this connection in terms of L_∞ modules in [37]. We adapt Tsygan's reformulation and introduce similar connections on our normed CH modules.

In the rest of Section 5, let $\mathbb{K}$ be a complete normed ring and R be a graded complete normed algebra over $\mathbb{K}$ unless otherwise mentioned. Let L be a normed L_∞ algebra over R with an L_∞ connection ∇. We assume that $\|\mathbb{K}\|, \|R\|, \|L\| \le 1$. Let $\widetilde{M}$ be a normed CH module over L with a CH connection $\widetilde{\nabla}$. Let $\gamma \in L^1$ be a Maurer-Cartan element.

Definition 5.8. We define a morphism $z\nabla^{\mathrm{G}} : \mathrm{Der}_{\mathbb{K}}(R) \to \mathrm{End}_{\mathbb{K}[[z]]}(\widetilde{M})$ as follows:

$$z\nabla_X^{\mathrm{G}} := z\widetilde{\nabla}_X - (-1)^{|X|} I^\gamma_{\nabla_X\gamma}.$$

This morphism is called a *Getzler-Gauss-Manin connection* (GGM connection for short).

Remark 5.9. See (3.6) and Remark 3.7 for the definition of I. Since I is a morphism from $L \otimes \widetilde{M}$, the connection ∇ in the term $I^\gamma_{\nabla_X\gamma}$ should be considered as a connection on L (not a connection on $L[1]$).

Since $I^\gamma_{\nabla_X\gamma} \in \mathrm{End}^{2+|X|}_{R[[z]]}(\widetilde{M})$, we see that $z\nabla^{\mathrm{G}}$ satisfies the "Leibniz rule" in the following form:

$$z\nabla_X^{\mathrm{G}}(rm) = z(Xr)m + (-1)^{|X||r|} r z\nabla_X^{\mathrm{G}} m.$$

Thus the GGM connection $z\nabla^{\mathrm{G}}$ itself is not a connection in the sense of the beginning of Subsection 5.1, but later $\nabla^{\mathrm{G}} := (z\nabla^{\mathrm{G}})/z$ will be regarded as a meromorphic connection and also called a GGM connection.

Example 5.10 (Continued from Example 3.10)**.** Let A be a finite-dimensional graded vector space over a field $\mathbb{K}$ and choose a homogeneous basis $\{T_0, \dots, T_m\}$. Assume that $A^e := A((e))$ be a hypercommutative algebra over $\mathbb{K}^e := \mathbb{K}((e))$. (See Example 4.2 (1) for the notation.) Choose formal variables $t_0, \dots, t_m$ with degrees $2 - |T_0|, \dots, 2 - |T_m|$ respectively. Set $R := \mathbb{K}^e[[t]]$. Then $A^e[[t]]$ is naturally considered as a hypercommutative algebra over R equipped with the trivial connection. By Example 3.10, $A^e[[t]]$ is a CH module over $A^e[[t]][1]$. We define a Maurer Cartan element γ to be $\mathbf{t} = \sum_{i=0}^m T_i t_i$. Set

$$(T_{i_1}, \dots, T_{i_k})^\gamma := \sum_{l=0}^\infty \frac{1}{l!}(\overbrace{\gamma, \dots, \gamma}^{l}, T_{i_1}, \dots, T_{i_k}),\ T_i *_{\mathbf{t}} T_j := (T_i, T_j)^\gamma.$$

Then $A^e[[t]]$ with operators $(\cdot, \dots, \cdot)^\gamma$ is a hypercommutative algebra. In this setting we see that $z\nabla^{\mathrm{G}}_{\frac{\partial}{\partial t_i}} = z\frac{\partial}{\partial t_i} + T_i *_{\mathbf{t}}$. This gives a reinterpretation of Dubrovin's connection in [9].

To show that GGM connection defines a morphism on cohomology, we use the following lemmas:

Lemma 5.11. *$\ell_1^\gamma(\nabla_X \gamma) = 0$.*

Proof. Since $|\gamma| = 1$, we easily see that

$$\nabla_X \ell_k(\gamma, \dots, \gamma) = (-1)^{|X|} k \ell_k(\gamma, \dots, \gamma, \nabla_X \gamma).$$

Hence we have

$$\begin{aligned}\ell_1^\gamma(\nabla_X\gamma) &= \sum_{k=0}^\infty \ell_{1+k}(\gamma, \dots, \gamma, \nabla_X\gamma)/k! = (-1)^{|X|}\nabla_X\left(\sum_{k=1}^\infty \ell_k(\gamma, \dots, \gamma)/k!\right) \\ &= 0.\end{aligned}$$

□

Lemma 5.12. *$[\widetilde{\nabla}_X, \ell_0^{\widetilde{M},\gamma}] = (-1)^{|X|}\ell_1^{\widetilde{M},\gamma}(\nabla_X\gamma | \cdot)$.*

Proof. We see that

$$[\widetilde{\nabla}_X, \ell_0^{\widetilde{M},\gamma}] = [\widetilde{\nabla}_X, \sum_{k=0}^\infty \ell_k^{\widetilde{M}}(\gamma, \dots, \gamma | \cdot)/k!\,]$$

$$
\begin{aligned}
&= (-1)^{|X|} \sum_{k=0}^{\infty} \ell_{1+k}^{\widetilde{M}}(\gamma, \dots, \gamma, \nabla_X \gamma | \cdot)/k! \\
&= (-1)^{|X|} \ell_1^{\widetilde{M}}(\nabla_X \gamma | \cdot). \qquad \square
\end{aligned}
$$

Proposition 5.13. $[d^\gamma, z\nabla_X^{\mathrm{G}}] = 0$.

Proof. By Equation (3.8) and Lemma 5.11, we have $[d^\gamma, I^\gamma_{\nabla_X \gamma}] + z\mathcal{L}^\gamma_{\nabla_X \gamma} = 0$. By Lemma 5.12, we have $[d^\gamma, z\widetilde{\nabla}_X] = (-1)^{1+|X|} z\mathcal{L}^\gamma_{\nabla_X\gamma}$. This proposition follows from these equations. $\square$

5.4. Euler connections

In [25, §2.2.5], Katzarkov-Kontsevich-Pantev introduced a connection in the "u-direction" on a cyclic homology (see also [34]). Inspired by their construction (but different from theirs as noted in Remark 1.1), we extend a GGM connection to a "connection in the z-direction". In different contexts, similar constructions have been considered by many people (e.g., [23, §4.1], [33] and see also Remark 1.2).

In §5.4, we continue to use the same notations as in §5.3. We note that

$$
\mathrm{Der}_{\mathbb{K}}(R) \cong \{X \in \mathrm{Der}_{\mathbb{K}}(R[[z]]) \mid Xz = 0\} \subset \mathrm{Der}_{\mathbb{K}}(R[[z]])
$$

and $\mathrm{Der}_{\mathbb{K}}(R[[z]]) = \mathrm{Der}_{\mathbb{K}}(R) \oplus \langle \frac{d}{dz} \rangle$. Let $\deg \in \mathrm{End}^0_{\mathbb{K}}(\widetilde{M})$ be the degree operator, i.e., $\deg m := |m|m$.

Definition 5.14. We define a morphism $z^2\nabla^{\mathrm{E}} : \mathrm{Der}_{\mathbb{K}}(R[[z]]) \to \mathrm{End}_{\mathbb{K}}(\widetilde{M})$ as follows:

$$
\begin{aligned}
z^2\nabla^{\mathrm{E}}_X &:= z^2\nabla^{\mathrm{G}}_X, \\
z^2\nabla^{\mathrm{E}}_{\frac{d}{dz}} &:= \frac{z}{2}\deg - z\nabla^{\mathrm{G}}_E,
\end{aligned}
$$

where $X \in \mathrm{Der}_{\mathbb{K}}(R)$ and E is the Euler vector of R. This morphism is called an *Euler connection*.

Since $\deg(rm) = 2(Er)m + 2(z\frac{d}{dz}r)m + r\deg m$ and $z\nabla^{\mathrm{G}}_E rm = z(Er)m + rz\nabla^{\mathrm{G}}_E m$, we see that $\nabla^{\mathrm{E}}_{\frac{d}{dz}}$ satisfies the following "Leibniz rule":

$$
z^2\nabla^{\mathrm{E}}_{\frac{d}{dz}}(rm) = (z^2\frac{d}{dz}r)m + rz^2\nabla^{\mathrm{E}}_{\frac{d}{dz}}m.
$$

Proposition 5.15 (cf. [34, §2.2])**.** $[z^2\nabla^{\mathrm{E}}_{\frac{d}{dz}}, d^\gamma] = \frac{z}{2}d^\gamma$.

Proof. Since d^γ is a degree one operator, we see $[\deg, d^\gamma] = d^\gamma$. Hence this proposition follows from Proposition 5.13. □

From this lemma, we easily see that $z^2\nabla^{\mathrm{E}}$ descends to a morphism on cohomology. Set $\widetilde{M}((z)) := \widetilde{M} \otimes_{R[[z]]} R((z))$. By abuse of notation, we consider $\nabla^{\mathrm{G}}, \nabla^{\mathrm{E}}$ as meromorphic connections (on complexes or cohomology).

Remark 5.16. The Euler connection has an irregular singularity at $z = 0$, but formal in the z-direction. Hence we can not consider the "Stokes structure" of the Euler connection.

Proposition 5.17. *If ∇^{G} is flat on cohomology, then ∇^{E} is also flat on cohomology.*

Proof. By the assumption, we have

$$\begin{aligned}
[\nabla^{\mathrm{E}}_{\frac{d}{dz}}, \nabla^{\mathrm{E}}_X] &= [\frac{1}{2z}\deg, \nabla^{\mathrm{G}}_X] - [\frac{1}{z}\nabla^{\mathrm{G}}_E, \nabla^{\mathrm{G}}_X] \\
&= \frac{1}{2z}|X|\nabla^{\mathrm{G}}_X - \frac{1}{z}\nabla^{\mathrm{G}}_{[E,X]} - \frac{1}{z}R^{\nabla^{\mathrm{G}}}(E, X) \\
&= -\frac{1}{z}R^{\nabla^{\mathrm{G}}}(E, X),
\end{aligned}$$

where we use $[\deg, \nabla^{\mathrm{G}}_X] = |X|\nabla^{\mathrm{G}}_X$ and $[E, X] = \frac{1}{2}|X|X$. Hence we have

$$\begin{aligned}
[\nabla^{\mathrm{E}}_{\frac{d}{dz}}, \nabla^{\mathrm{E}}_{\frac{d}{dz}}] &= [\nabla^{\mathrm{E}}_{\frac{d}{dz}}, \frac{1}{z}\left(\frac{1}{2}\deg - \nabla^{\mathrm{E}}_E\right)] \\
&= -\frac{1}{z}\nabla^{\mathrm{E}}_{\frac{d}{dz}} + \frac{1}{z}[\nabla^{\mathrm{E}}_{\frac{d}{dz}}, \frac{1}{2}\deg - \nabla^{\mathrm{E}}_E] \\
&= -\frac{1}{z}\nabla^{\mathrm{E}}_{\frac{d}{dz}} + \frac{1}{z}\nabla^{\mathrm{E}}_{\frac{d}{dz}} + \frac{1}{z^2}R^{\nabla^{\mathrm{G}}}(E, E) \\
&= \frac{1}{z^2}R^{\nabla^{\mathrm{G}}}(E, E).
\end{aligned}$$

This proposition follows from these equations. □

Remark 5.18. By a direct calculation, we see that

$$\begin{aligned}
&R^{\nabla^{\mathrm{G}}}(X, Y) \\
=&R^{\widetilde{\nabla}}(X, Y) \\
&- \frac{1}{z}\Big((-1)^{|X|+|Y|}I^\gamma_{R^\nabla(X,Y)\gamma} + (-1)^{|Y|}\rho^\gamma_{\nabla_X\gamma, \nabla_Y\gamma} - (-1)^{|X|+|X||Y|}\rho^\gamma_{\nabla_Y\gamma, \nabla_X\gamma}\Big) \\
&+ \frac{1}{z^2}(-1)^{|X|+|Y|}[I^\gamma_{\nabla_X\gamma}, I^\gamma_{\nabla_Y\gamma}].
\end{aligned} \tag{5.5}$$

On the other hand, if the CH structure extends to an L_∞ module structure mod ϵ^3, then the L_∞ module relations mod ϵ^3 implies

$$\begin{aligned}
&\ell_0^{\widetilde{M},\gamma}\ell_2^{\widetilde{M},\gamma}(\epsilon y_1, \epsilon y_2|m) + (-1)^{|y_1|+|y_2|}\ell_2^{\widetilde{M},\gamma}(\epsilon y_1, \epsilon y_2|\ell_0^{\widetilde{M},\gamma}m)\\
&+(-1)^{|y_1|}\ell_1^{\widetilde{M},\gamma}(\epsilon y_1|\ell_1^{\widetilde{M},\gamma}(\epsilon y_2|m)) + (-1)^{|y_1||y_2|+|y_2|}\ell_1^{\widetilde{M},\gamma}(\epsilon y_2|\ell_1^{\widetilde{M},\gamma}(\epsilon y_1|m))\\
&-\ell_2^{\widetilde{M},\gamma}(\epsilon\ell_1^{\gamma}y_1, \epsilon y_2|m) - (-1)^{|y_1||y_2|}\ell_2^{\widetilde{M},\gamma}(\epsilon\ell_1^{\gamma}y_2, \epsilon y_1|m)\\
&+\ell_2^{\widetilde{M},\gamma}(zy_1, \epsilon y_2|m) + (-1)^{|y_1||y_2|}\ell_2^{\widetilde{M},\gamma}(zy_2, \epsilon y_1|m)\\
&=0,
\end{aligned}$$

where $y_1, y_2 \in L[[z]]$. Set $\mathcal{L}^{\gamma}_{y_1,y_2} := (-1)^{|y_1|+|y_2|}\ell_2^{\widetilde{M},\gamma}(\epsilon y_1, \epsilon y_2|\cdot) \in \mathrm{End}(\widetilde{M})$. Then the above formula is written as follows:

$$\begin{aligned}
&[d, \mathcal{L}^{\gamma}_{y_1,y_2}] + \mathcal{L}^{\gamma}_{\delta y_1,y_2} + (-1)^{|y_1|}\mathcal{L}^{\gamma}_{y_1,\delta y_2}\\
=&(-1)^{|y_1|}[I^{\gamma}_{y_1}, I^{\gamma}_{y_2}] + z(\rho^{\gamma}_{y_1,y_2} + (-1)^{|y_1||y_2|}\rho^{\gamma}_{y_2,y_1}).
\end{aligned}$$

Combing (5.5) with this formula, we find that ∇^{G} is flat on cohomology if ∇ and $\widetilde{\nabla}$ are flat and the CH structure extends to an L_∞ module structure mod ϵ^3.

Remark 5.19. In Example 5.20 below, we will use the following variant of the GGM connection. Let L be an L_∞ algebra (not normed) over a graded ring R and $\widetilde{M}$ a CH module over L, and let ∇ be a connection on L (not necessary an L_∞ connection) and $\widetilde{\nabla} : \mathrm{Der}_{\mathbb{K}}(R) \to \mathrm{End}_{\mathbb{K}[[z]]}(\widetilde{M})$ a connection on $\widetilde{M}$ regarded as a graded R module (not necessary a CH-connection). Suppose that we have a degree one morphism

$$c_1 : \mathrm{Der}_{\mathbb{K}}(R) \to L,$$

for which ∇ satisfies

$$(\nabla_X \ell_k)(y_1, \dots, y_k) = (-1)^{|X|}\ell_{k+1}(c_1(X), y_1, \dots, y_k) \tag{5.6}$$

and $\widetilde{\nabla}$ satisfies

$$\begin{aligned}
&(\widetilde{\nabla}_X \ell_k^{\widetilde{M}})(y_1 + \epsilon y_1', \dots, y_k + \epsilon y_k'|m)\\
=&(-1)^{|X|}\ell_{k+1}^{\widetilde{M}}(c_1(X), y_1 + \epsilon y_1', \dots, y_k + \epsilon y_k'|m).
\end{aligned} \tag{5.7}$$

Then the connections ∇ and $\widetilde{\nabla}$ satisfy the properties in Lemma 5.11 and Lemma 5.12. Thus in Definition 5.8, replacing the CH connection by the connection $\widetilde{\nabla}$ above and $\nabla_X\gamma$ by $c_1(X)$, we can define a (variant of) GGM connection on cohomology. Similarly, we can define a (variant of) Euler connection as well in this setting.

Example 5.20 (Continued from Example 5.10)**.** Suppose that there exists an element

$$c_1 \in A^2 \subset A^{e,2}$$

with

$$(c_1, x_1, \ldots, x_k) = e\frac{d}{de}(x_1, \ldots, x_k). \tag{5.8}$$

We define a degree one R module morphism

$$c_1 : \mathrm{Der}_{\mathbb{K}}(R) \to A^e[1]$$

by $c_1(e\frac{d}{de}) := c_1$ and $c_1(\frac{\partial}{\partial t_i}) := (-1)^{|T_i|}T_i$. Then the trivial connections ∇ and $\widetilde{\nabla}$ satisfy the condition of Remark 5.19. Set

$$E := c_1 + \sum_{i=0}^{m}(-1)^{|T_i|}(\frac{2-|T_i|}{2})T_it_i \in A^e[[t]].$$

Then we easily see that

$$\nabla^{\mathrm{E}}_{\frac{\partial}{\partial t_i}} = \frac{\partial}{\partial t_i} + \frac{1}{z}T_i*_{\mathbf{t}}, \quad \nabla^{\mathrm{E}}_{\frac{d}{dz}} = \frac{d}{dz} + \frac{\deg}{2z} - \frac{E*_{\mathbf{t}}}{z^2},$$

where $\mathbf{t} = \sum_{i=0}^{m} T_it_i$ and deg is the degree operator of A.

Now let $\widetilde{N}$ be another normed CH module equipped with a CH connection and $\{f_k\}_{0\le k}$ be a morphism of normed CH modules from $\widetilde{M}$ to $\widetilde{N}$ preserving the CH connections on $\widetilde{M}$ and $\widetilde{N}$ in the sense of Definition 5.4. Then recalling (5.2), (5.3), (5.4), we have

Proposition 5.21.

$$\nabla^{\mathrm{G}}_X \circ f_0^\gamma - f_0^\gamma \circ \nabla^{\mathrm{G}}_X = (-1)^{|X|+1}d^\gamma \circ F^{\epsilon,\gamma}_{\nabla^{\mathrm{G}}_X\gamma} - F^{\epsilon,\gamma}_{\nabla^{\mathrm{G}}_X\gamma} \circ d^\gamma.$$

Proof. Since the morphism preserves the CH connections, we see that

$$\widetilde{\nabla}_X \circ f_0^\gamma = f_0^\gamma \circ \widetilde{\nabla}_X - (-1)^{|X|}F^\gamma_{\nabla_X\gamma}.$$

By Equation 3.12 and Lemma 5.11, we have

$$zF^{\gamma}_{\nabla_X\gamma} + I^{\gamma}_{\nabla_X\gamma} \circ f^{\gamma}_0 = f^{\gamma}_0 \circ I^{\gamma}_{\nabla_X\gamma} + d^{\gamma} \circ F^{\epsilon,\gamma}_{\nabla_X\gamma} - (-1)^{|X|+1} F^{\epsilon,\gamma}_{\nabla_X\gamma} \circ d^{\gamma}.$$

The proposition follows from these equations. □

Since f^{γ}_0 is degree zero, we obtain $\deg \circ f^{\gamma}_0 = f^{\gamma}_0 \circ \deg$. Hence we have the following:

Corollary 5.22.

$$\nabla^{\mathrm{E}}_{\frac{d}{dz}} \circ f^{\gamma}_0 - f^{\gamma}_0 \circ \nabla^{\mathrm{E}}_{\frac{d}{dz}} = \frac{1}{z}\left(d^{\gamma} \circ F^{\epsilon,\gamma}_{\nabla^{\mathrm{E}}_E\gamma} - F^{\epsilon,\gamma}_{\nabla^{\mathrm{E}}_E\gamma} \circ d^{\gamma}\right).$$

Proposition 5.21 and Corollary 5.22 imply that f^{γ}_0 intertwines the Euler connections on cohomology.

Remark 5.23 (Continued from Remark 5.19)**.** We consider the situation of the variants of the GGM connection and the Euler connection in Remark 5.19. Instead of assuming that a morphism $\{f_k\}_{0\leq k}$ of normed CH modules from $\widetilde{M}$ to $\widetilde{N}$ preserves the CH connections, we assume that $\{f_k\}_{0\leq k}$ satisfies

$$\begin{aligned}&(\widetilde{\nabla}_X \circ f_k - f_k \circ \widetilde{\nabla}_X)(y_1 + \epsilon y'_1, \dots, y_k + \epsilon y'_k | m)\\ =&f_{k+1}(c_1(X), y_1 + \epsilon y'_1, \dots, y_k + \epsilon y'_k | m)\end{aligned}$$

for any $X \in \mathrm{Der}_{\mathbb{K}}(R)$. Here $\widetilde{\nabla}$ in the first term is the connection on $\widetilde{N[1]}$ as in (5.7) and one in the second term is the connection on $\widetilde{L}[1]^{\odot k} \otimes \widetilde{M}[1]$ induced by the connections $\nabla, \widetilde{\nabla}$ on $L, \widetilde{M}$ in (5.6), (5.7) respectively. Then Proposition 5.21 for the variant of the GGM connection ∇^{G} holds and Corollary 5.22 for the variant of the Euler connection ∇^{E} also holds.

6. CH structures on Hochschild invariants

6.1. CH structures on Hochschild invariants

In §6.1, we give an explicit formula of a CH structure on Hochschild invariants of an A_∞ category (cf. [6], [37]). We mainly follow [21] for the definitions of operators on Hochschild invariants. Let R be a graded ring (not normed). Let $\mathscr{A}$ be a set of objects $\mathrm{Ob}\mathscr{A}$ with a set of morphisms $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ for each $X, Y \in \mathrm{Ob}\mathscr{A}$, where $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is a graded R module. We assume $\mathrm{Hom}_{\mathscr{A}}(X, X)$ is equipped with a degree preserving R module morphism η :

$R \to \mathrm{Hom}_{\mathscr{A}}(X,X)$ for each $X \in \mathrm{Ob}\mathscr{A}$. This morphism η is called a unit and $\eta(1)$ is denoted by $\mathbb{1}$. Set

$$\mathscr{A}(X_0,\dots,X_k) := \mathrm{Hom}_{\mathscr{A}}(X_0,X_1)[1] \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{A}}(X_{k-1},X_k)[1].$$

The Hochschild cochain complex and the Hochschild chain complex are defined by

$$\begin{aligned}
CC^\bullet(\mathscr{A})^l &:= \prod_{X_0,\dots,X_k \in \mathrm{Ob}\mathscr{A}} \mathrm{Hom}_R^l(\mathscr{A}(X_0,\dots,X_k), \mathrm{Hom}_{\mathscr{A}}(X_0,X_k)),\\
CC^\bullet(\mathscr{A}) &:= \bigoplus_{l\in\mathbb{Z}} CC^\bullet(\mathscr{A})^l,\\
CC_\bullet(\mathscr{A})^l &:= \bigoplus_{X_0,\dots X_k \in \mathrm{Ob}\mathscr{A}} \left(\mathrm{Hom}_{\mathscr{A}}(X_0,X_1) \otimes \mathscr{A}(X_1,\dots,X_k,X_0)\right)^l,\\
CC_\bullet(\mathscr{A}) &:= \bigoplus_{l\in\mathbb{Z}} CC_\bullet(\mathscr{A})^l.
\end{aligned}$$

For $\varphi,\psi \in CC^\bullet(\mathscr{A})[1]$, we define a composition $\varphi\circ\psi$ by

$$\varphi\circ\psi(x_1,\dots,x_k) := \sum_{0\le i\le j\le k} (-1)^{\#} \varphi(x_1,\dots,x_i,\psi(x_{i+1},\dots,x_j),x_{j+1},\dots,x_k),$$

where $x_i \in \mathrm{Hom}_A(X_{i-1},X_i)[1]$ and the sign $\#$ is $|\psi|'(|x_1|'+\cdots+|x_i|')$. Set

$$[\varphi,\psi] := \varphi\circ\psi - (-1)^{|\varphi|'|\psi|'}\psi\circ\varphi.$$

For $\mathbb{X} = x_0\otimes\cdots\otimes x_k \in \mathscr{A}(X_0,\dots,X_k,X_0) \subset CC_\bullet(\mathscr{A})[1]$, we define $\mathcal{L}_\varphi(\mathbb{X})$ by

$$\begin{aligned}
\mathcal{L}_\varphi(\mathbb{X}) := &\sum_{0\le i\le j\le k} (-1)^{\#_1} x_0\otimes\cdots\otimes x_i \otimes \varphi(x_{i+1},\dots,x_j)\otimes x_{j+1}\cdots\otimes x_k\\
&+ \sum_{0\le i\le j\le k} (-1)^{\#_2} \varphi(x_{j+1},\dots,x_k,x_0,\dots,x_i)\otimes x_{i+1}\otimes\cdots\otimes x_j,
\end{aligned}$$

where

$$\begin{aligned}
\#_1 &:= |\varphi|'(|x_0|'+\cdots+|x_i|'),\\
\#_2 &:= (|x_0|'+\cdots+|x_j|')(|x_{j+1}|'+\cdots+|x_k|').
\end{aligned}$$

Proposition 6.1 (see, e.g., [21], [24, Proposition 3.1.]). *The shifted Hochschild cochain complex $(CC^\bullet(\mathscr{A})[1],[\cdot,\cdot])$ is a graded Lie algebra and $(CC_\bullet(\mathscr{A})[1],\mathcal{L})$ is a graded Lie module over it.*

Following [21, §2], we define a morphism ρ by

$$\begin{aligned}&\rho_{\varphi,\psi}\\&:=\sum_{0\le i\le j\le s\le t\le k}(-1)^{\#}\varphi(x_{j+1},\dots,x_s,\psi(x_{s+1},\dots,x_t),x_{t+1},\dots,x_k,x_0,\dots,x_i)\\&\qquad\qquad\qquad\qquad\qquad\qquad\otimes x_{i+1}\otimes\cdots\otimes x_j,\end{aligned}$$

where $\# = |\psi|'(|x_{j+1}|'+\cdots+|x_s|')+(|x_0|'+\cdots+|x_j|')(|x_{j+1}|'+\cdots+|x_k|')$. The next proposition will be proved in §6.2.

Proposition 6.2.

$$\begin{aligned}&\rho_{[\varphi_1,\varphi_2],\psi}-\rho_{\varphi_1,[\varphi_2,\psi]}+(-1)^{|\varphi_1|'|\varphi_2|'}\rho_{\varphi_2,[\varphi_1,\psi]}\\=&[\mathcal{L}_{\varphi_1},\rho_{\varphi_2,\psi}]-(-1)^{|\varphi_1|'|\varphi_2|'}[\mathcal{L}_{\varphi_2},\rho_{\varphi_1,\psi}].\end{aligned}$$

The reduced Hochschild cochain complex $\overline{CC}^\bullet(\mathscr{A})$ is defined by

$$\overline{CC}^\bullet(\mathscr{A}):=\{\varphi\in CC^\bullet(\mathscr{A})\mid \varphi(\dots,\mathbb{1},\dots)=0\}$$

and the reduced Hochschild chain complex $\overline{CC}_\bullet(\mathscr{A})$ is defined by the relation

$$x_0\otimes x_1\otimes\cdots x_{i-1}\otimes\mathbb{1}\otimes x_{i+1}\otimes\cdots\otimes x_k=0\quad(1\le i\le k).$$

Then $\overline{CC}^\bullet(\mathscr{A})[1]$ is a sub graded Lie algebra and $\mathcal{L}$ naturally induces a graded Lie module structure on $\overline{CC}_\bullet(\mathscr{A})[1]$, which is also denoted by $\mathcal{L}$. We define the following morphisms:

$$\begin{aligned}B(\mathbb{X})&:=\sum_{0\le i\le k}(-1)^{\#_1}\mathbb{1}\otimes x_{i+1}\otimes\cdots\otimes x_k\otimes x_0\otimes\cdots\otimes x_i,\\B^1_\varphi(\mathbb{X})&:=\sum_{0\le i\le j\le s\le k}(-1)^{\#_1+\#_2}\mathbb{1}\otimes x_{i+1}\otimes\cdots\otimes x_j\otimes\varphi(x_{j+1},\dots,x_s)\otimes\cdots\\&\qquad\qquad\qquad\cdots\otimes x_k\otimes x_0\otimes\cdots\otimes x_i,\end{aligned}$$

where $\#_1=(|x_0|'+\cdots+|x_i|')(|x_{i+1}|'+\cdots+|x_k|')$ and $\#_2=|\varphi|'(|x_{i+1}|'+\cdots+|x_j|')$. By the definition, we can check the following (e.g., [21, Definition 2.1 and Theorem 2.2]):

$$[B,B]=0,\tag{6.1}$$

$$[B,\mathcal{L}_\varphi]=0,\tag{6.2}$$

$$[B,B^1_\varphi]=0.\tag{6.3}$$

The next proposition will be proved in §6.2.

Proposition 6.3. $[B, \rho_{\varphi,\psi}] + B^1_{[\varphi,\psi]} - (-1)^{|\varphi|'}[\mathcal{L}_\varphi, B^1_\psi] = 0$

Now we recall the definition of A_∞ structures.

Definition 6.4. A *strictly unital curved A_∞ structure* on $\mathscr{A}$ is an element $m \in CC^\bullet(\mathscr{A})[1]^1$ with

$$[m, m] = 0, \quad m_2(\mathbb{1}, x) = (-1)^{|x|} m_2(x, \mathbb{1}), \quad m_k(\dots, \mathbb{1}, \dots) = 0 \quad (k \neq 2).$$

Here the length k part of m is denoted by m_k. In case $m_0 = 0$, m is called a *strictly unital A_∞ structure* on $\mathscr{A}$.

Let m be a strictly unital curved A_∞ structure on $\mathscr{A}$ and set

$$b := \mathcal{L}_m, \quad b^1_\varphi := \rho_{m,\varphi}, \quad \delta := [m, \cdot\,].$$

Then these morphisms naturally induce morphisms on reduced Hochschild complexes, which are denoted by the same symbols. Note that $(\delta, [\cdot,\cdot])$ makes $\overline{CC}^\bullet(\mathscr{A})[1]$ into a DGLA and $(b, \mathcal{L})$ makes $\overline{CC}_\bullet(\mathscr{A})[1]$ into a DGLA module over the DGLA $\overline{CC}^\bullet(\mathscr{A})[1]$. By Proposition 6.2 and $[m, m] = 0$, we have

$$\rho_{\delta\varphi,\psi} - b^1_{[\varphi,\psi]} + (-1)^{|\varphi|'}\rho_{\varphi,\delta\psi} = [b, \rho_{\varphi,\psi}] - (-1)^{|\varphi|'}[\mathcal{L}_\varphi, b^1_\psi], \tag{6.4}$$

$$b^1_{\delta\psi} + [b, b^1_\psi] = 0. \tag{6.5}$$

We use the following identity (see, e.g., [21, Definition 2.1 and Theorem 2.2])

$$[B, b^1_\varphi] + [b, B^1_\varphi] + B^1_{\delta\varphi} + \mathcal{L}_\varphi = 0. \tag{6.6}$$

Now we set

$$L := \overline{CC}^\bullet(\mathscr{A})[1], \quad \widetilde{M} := \overline{CC}_\bullet(\mathscr{A})[1][[z]].$$

We note that the natural z-linear extensions of morphisms $\mathcal{L}, \rho$, etc. are denoted by the same symbols. For $\varphi \in L[[z]]$ we put

$$d := b + zB, \quad I_\varphi := b^1_\varphi + zB^1_\varphi.$$

Then combining Propositions 6.1, 6.2, 6.3, and the equalities (6.1), (6.2), (6.3), (6.4), (6.5), (6.6) we obtain the following theorem:

Theorem 6.5. *The morphisms $\mathcal{L}, I, \rho$ give a CH structure on $\widetilde{M}$ over the DGLA L.*

Proof. Using Proposition 6.1, the equalities (6.1), (6.2), and $[m, m] = 0$, we see that $(\widetilde{M}, d, \mathcal{L})$ is a DGLA module over the DGLA $L[[z]]$, where the DGLA structure on L is z-linearly extended to $L[[z]]$. By Proposition 3.8, it is sufficient to show the equalities (3.8)), (3.9), (3.10). We note that the equality (3.10) is a direct consequence of Proposition 6.2. We check (3.8) and (3.9). Since

$$\begin{aligned}&[d, I_\varphi] + I_{\delta\varphi} + z\mathcal{L}_\varphi \\ =&[b, b^1_\varphi] + b^1_{\delta\varphi} + z([b, B^1_\varphi] + [B, b^1_\varphi] + B^1_{\delta\varphi} + \mathcal{L}_\varphi) + z^2[B, B^1_\varphi],\end{aligned}$$

the equality (3.8) follows from (6.3), (6.5), (6.6). Since

$$\begin{aligned}&I_{[\varphi,\psi]} - (-1)^{|\varphi|'}[\mathcal{L}_\varphi, I_\psi] + [d, \rho_{\varphi,\psi}] - \rho_{\delta\varphi,\psi} - (-1)^{|\varphi|'}\rho_{\varphi,\delta\psi} \\ =&b^1_{[\varphi,\psi]} - (-1)^{|\varphi|'}[\mathcal{L}_\varphi, b^1_\psi] + [b, \rho_{\varphi,\psi}] - \rho_{\delta\varphi,\psi} - (-1)^{|\varphi|'}\rho_{\varphi,\delta\psi} \\ &+ z(B^1_{[\varphi,\psi]} - (-1)^{|\varphi|'}[\mathcal{L}_\varphi, B^1_\psi] + [B, \rho_{\varphi,\psi}]),\end{aligned}$$

the equality (3.9) follows from (6.4) and Proposition 6.3. □

Definition 6.6. The cohomology of $(\overline{CC}_\bullet(\mathscr{A})[[z]], b + zB)$ is called a *negative cyclic homology* and the cohomology of $(\overline{CC}_\bullet(\mathscr{A})((z)), b + zB)$ is called a *periodic cyclic homology.*

Remark 6.7. Let $\mathbb{K}$ be an ungraded field and R be a graded algebra over $\mathbb{K}^e = \mathbb{K}((e))$. Note that $\mathbb{K}$ and $\mathbb{K}^e$ are equipped with the trivial norms. Assume that R is equipped with a norm $\|\cdot\|$ such that R is a graded complete normed algebra over $\mathbb{K}^e$, i.e., $\|\cdot\|$ is a ring norm and a $\mathbb{K}^e$ module norm. We also assume $\|R\| \le 1$. Let m be a strictly unital A_∞ structure on $\mathscr{A}$ over $\mathbb{K}$. Then m gives a DGLA structure on $\overline{CC}^\bullet(\mathscr{A})[1]$ and this DGLA structure naturally extends to $\overline{CC}^\bullet(\mathscr{A})[1]\widehat{\otimes}_{\mathbb{K}} R$, where $\overline{CC}^\bullet(\mathscr{A})[1]$ is considered as a graded $\mathbb{K}$ module with the trivial norm. In this note, we define a *gapped filtered graded A_∞ structure on $\mathscr{A}$ by a Maurer-Cartan element $m_+ \in \overline{CC}^\bullet(\mathscr{A})[1]\hat{\otimes}_{\mathbb{K}} R$ with $\|m_+\| < 1$ and* we call such a triple $(\mathscr{A}, m, m_+)$ a *gapped filtered graded A_∞ category over R.* See [14, Definition 3.2.26] for the definition of gappedness of a filtered A_∞ algebra over the universal Novikov ring, and see also [14, Definition 3.7.5].

Remark 6.8. We consider the same setting as in Remark 6.7. Let $(\mathscr{A}, m, m_+)$ be a gapped filtered graded A_∞ category over R as above. Then the trivial connection ∇ on the normed DGLA $\overline{CC}^\bullet(\mathscr{A})[1]\widehat{\otimes}_{\mathbb{K}} R$ is an L_∞ connection and the trivial "connection" $\widetilde{\nabla}$ on the normed CH module

$$(\overline{CC}_\bullet(\mathscr{A})[1]\widehat{\otimes}_{\mathbb{K}} R)[[z]]$$

is a CH connection. Here the normed DGLA structure and the normed CH structure are determined by m (not twisted by the Maurer-Cartan element m_+). Since m_+ is a Maurer-Cartan element of $\overline{CC}^\bullet(\mathscr{A})[1]\widehat{\otimes}_{\mathbb{K}}R$, we can consider the Euler connection on the cohomology of the (m_+-twisted) normed CH module $(\overline{CC}_\bullet(\mathscr{A})[1]\widehat{\otimes}_{\mathbb{K}}R)[[z]]^{m_+}$. See Subsection 5.2 for this notation. In a way similar to [21], we can prove flatness of ∇^{G} on cohomology and also flatness of ∇^{E} by Proposition 5.17.

6.2. Proofs of Propositions 6.2 and 6.3

Propositions 6.2 and 6.3 follow by direct calculation. To write down the proofs, we use operadic notation. In [26], they introduce a 2-colored operad called the Kontsevich-Soibelman operad and show that this operad naturally acts on Hochschild invariants (see also [7], [38]). In this article, we will follow [38, §3]. They use "trees" to describe the operadic structure. We use parentheses notation to describe the operadic structure. We note that there is a one to one correspondence between trees and parentheses. To simplify notation, the special vertex *out* is omitted, i.e., $\{out\{\cdots\}\}$ is denoted by $\{\cdots\}$ (see also Example 6.10 below).

Remark 6.9. Precisely, we consider the degree one shift of the Kontsevich-Soibelman operad which acts on the pair $(\overline{CC}^\bullet(\mathscr{A})[1], \overline{CC}_\bullet(\mathscr{A})[1])$. The $\overline{CC}^\bullet(\mathscr{A})[1]$ part of this structure is closely related to the brace algebra [20].

Example 6.10.

$$\begin{aligned}&\varphi\circ\psi=\varphi\{\psi\},\ \mathcal{L}_\varphi=\{in,\varphi\}+\{\varphi\{in\}\},\ \rho_{\varphi,\psi}=\{\varphi\{\psi,in\}\},\\&B=\{\mathbb{1},in\},\ B^1_\varphi=\{\mathbb{1},\varphi,in\}.\end{aligned}$$

Proof of Proposition 6.2. Proposition 6.2 easily follows from the next lemma.

Lemma 6.11.

$$\begin{aligned}&\rho_{\varphi_1\circ\varphi_2,\psi}-\rho_{\varphi_1,[\varphi_2,\psi]}+(-1)^{|\varphi_1|'|\varphi_2|'}[\mathcal{L}_{\varphi_2},\rho_{\varphi_1,\psi}]\\=&\{\varphi_1\{\varphi_2\{\psi,in\}\}\}+(-1)^{|\varphi_1|'|\varphi_2|'}\{\varphi_2\{\varphi_1\{\psi,in\}\}\}.\end{aligned}$$

Proof. By the definition, we have

$$\rho_{\varphi_1,\varphi_2\circ\psi}=\{\varphi_1\{\varphi_2\{\psi\},in\}\},\quad \rho_{\varphi_1,\psi\circ\varphi_2}=\{\varphi_1\{\psi\{\varphi_2\},in\}\}.$$

By direct calculation, we see that

$$
\begin{aligned}
&\rho_{\varphi_1\circ\varphi_2,\psi}\\
=&\{\varphi_1\{\varphi_2,\psi,in\}\}\\
&+(-1)^{|\varphi_2|'|\psi|'}(\{\varphi_1\{\psi,\varphi_2,in\}\}+\{\varphi_1\{\psi,in,\varphi_2\}\}+\{\varphi_1\{\psi,\varphi_2\{in\}\}\})\\
&+\{\varphi_1\{\varphi_2\{\psi\},in\}\}+\{\varphi_1\{\varphi_2\{\psi,in\}\}\}.\\
&\rho_{\varphi_1,\psi}\circ\mathcal{L}_{\varphi_2}\\
=&\{\varphi_1\{\psi,in\},\varphi_2\}+\{\varphi_1\{\psi\{\varphi_2\},in\}\}+\{\varphi_1\{\psi,\varphi_2,in\}\}+\{\varphi_1\{\psi,in,\varphi_2\}\}\\
&+(-1)^{|\varphi_2|'|\psi|'}\{\varphi_1\{\varphi_2,\psi,in\}\}+\{\varphi_1\{\psi,\varphi_2\{in\}\}\}.\\
&\mathcal{L}_{\varphi_2}\circ\rho_{\varphi_1,\psi}\\
=&(-1)^{|\varphi_1|'|\varphi_2|'+|\varphi_2|'|\psi|'}\{\varphi_1\{\psi,in\},\varphi_2\}+\{\varphi_2\{\varphi_1\{\psi,in\}\}\}.
\end{aligned}
$$

This lemma follows from these equations. Note that $\mathcal{L}_{\varphi_2}$ is degree $|\varphi_2|'$ and $\rho_{\varphi_1,\psi}$ is degree $|\varphi_1|'+|\psi|'$. We also note that in this calculation we do not need to assume φ_1,φ_2,ψ are reduced cochains. $\square$

We next prove Proposition 6.3.

Proof of Proposition 6.3. By the definition, we have

$$
\begin{aligned}
[B,\rho_{\varphi,\psi}]=B\circ\rho_{\varphi,\psi}&=\{\mathbb{1},\varphi\{\psi,in\}\},\\
B^1_{\varphi\circ\psi}&=\{\mathbb{1},\varphi\{\psi\},in\},\\
B^1_{\psi\circ\varphi}&=\{\mathbb{1},\psi\{\varphi\},in\}.
\end{aligned}
$$

By direct calculation, we have

$$
\begin{aligned}
B^1_\psi\circ\mathcal{L}_\varphi=&\{\mathbb{1},\psi,\varphi\{in\}\}+\{\mathbb{1},\psi,in,\varphi\}+\{\mathbb{1},\psi,\varphi,in\}\\
&+(-1)^{|\varphi|'|\psi|'}\{\mathbb{1},\varphi,\psi,in\}+\{\mathbb{1},\psi\{\varphi\},in\},\\
\mathcal{L}_\varphi\circ B^1_\psi=&(-1)^{|\varphi|'}(\{\mathbb{1},\varphi,\psi,in\}+\{\mathbb{1},\varphi\{\psi,in\}\}+\{\mathbb{1},\varphi\{\psi\},in\})\\
&+(-1)^{|\varphi|'+|\varphi|'|\psi|'}(\{\mathbb{1},\psi,\varphi\{in\}\}+\{\mathbb{1},\psi,in,\varphi\}+\{\mathbb{1},\psi,\varphi,in\}).
\end{aligned}
$$

Since B^1_ψ is degree $|\psi|'+1$, we have

$$
(-1)^{|\varphi|'}[\mathcal{L}_\varphi,B^1_\psi]=\{\mathbb{1},\varphi\{\psi,in\}\}+\{\mathbb{1},\varphi\{\psi\},in\}-(-1)^{|\varphi|'|\psi|'}\{\mathbb{1},\psi\{\varphi\},in\}.
$$

The proposition follows from these equations. $\square$

7. Primitive forms without higher residue pairings

In this section, we briefly recall a part of the definition of the primitive forms [30]. Let $\mathbb{K}$ be a field (ungraded) and R be a graded algebra over the graded ring $\mathbb{K}^e := \mathbb{K}((e))$. Let $\widetilde{M}$ be a finitely generated free graded $R[[z]]$ module equipped with an integrable (flat) meromorphic connection

$$\nabla : \mathrm{Der}_{\mathbb{K}}(R((z))) \to \mathrm{End}_{\mathbb{K}}\left(\widetilde{M}((z))\right),$$

where $\widetilde{M}((z)) := \widetilde{M} \otimes_{R[[z]]} R((z))$. We assume that ∇ has pole order at most 2 in the z-direction and pole order at most 1 in another direction (i.e., $z\nabla_X m \in \widetilde{M}$ for $X \in \mathrm{Der}_{\mathbb{K}}(R)$ and $m \in \widetilde{M}$). An element $\zeta \in \widetilde{M}^r$ is said to be *primitive* if the morphism

$$[z\nabla\zeta] : \mathrm{Der}_{\mathbb{K}^e}(R) \to (\widetilde{M}/z\widetilde{M})[r+2]$$

is an isomorphism as R modules. If $\zeta \in \widetilde{M}^r$ is primitive, then $z\nabla\zeta : \mathrm{Der}_{\mathbb{K}^e}(R)[[z]] \to \widetilde{M}[r+2]$ is also an isomorphism as $R[[z]]$ modules. By this isomorphism, ∇ gives a meromorphic connection on $\mathrm{Der}_{\mathbb{K}^e}(R)((z))$ which is also denoted by ∇. For the Euler vector field $E \in \mathrm{Der}^0_{\mathbb{K}}(R)$, there exists a unique element $E' \in \mathrm{Der}^0_{\mathbb{K}^e}(R)$ such that $[z\nabla_E\zeta] = [z\nabla_{E'}\zeta]$. By abuse of notation, this vector field E' is also denoted by E and called an Euler vector field.

Definition 7.1 (cf. [31, Definition 6.1, (P1), (P3), (P4), (P5)])**.** An element $\zeta \in \widetilde{M}^r$ is called a *primitive form without higher residue pairing* if

- ζ is primitive.
- $z\nabla_{\frac{d}{dz}}\zeta = -\nabla_E\zeta + \frac{r}{2}\zeta$.
- $z\nabla_X Y \in \mathrm{Der}_{\mathbb{K}^e}(R) \oplus z\mathrm{Der}_{\mathbb{K}^e}(R)$ for all $X, Y \in \mathrm{Der}_{\mathbb{K}^e}(R)$.
- $z^2\nabla_{\frac{d}{dz}} Y \in \mathrm{Der}_{\mathbb{K}^e}(R) \oplus z\mathrm{Der}_{\mathbb{K}^e}(R) \oplus z^2\mathrm{Der}_{\mathbb{K}^e}(R)$ for all $Y \in \mathrm{Der}_{\mathbb{K}^e}(R)$.

Remark 7.2. In this note, there is no specific reason to exclude the parts related to higher residue parings from the definition of the primitive forms.

Remark 7.3 (Continued from Remarks 6.7 and 6.8)**.** Let $(\mathscr{A}, m, m_+)$ be a gapped filtered graded A_∞ category over R as in Remark 6.7. We define a graded $R[[z]]$ module $\widetilde{M}$ by the cohomology of the normed CH module $\overline{CC}_\bullet(\mathscr{A})[1]\widehat{\otimes}_{\mathbb{K}}R)[[z]]^{m_+}$. The graded $R[[z]]$ module $\widetilde{M}$ is equipped with the Euler connection ∇^{E}. If $\widetilde{M}$ is finitely generated and free over $R[[z]]$, then we can consider primitive forms without higher residue pairings on $(\widetilde{M}, \nabla^{\mathrm{E}})$. We note that, in this categorical setting, primitive forms are closely related to (smooth or proper) Calabi-Yau structures (see, e.g., [19]).

Example 7.4 (Continued from Example 5.20)**.** An element $\mathbb{1}$ in A^0 is called a unit (of the hyper commutative algebra A^e) if

$$(\mathbb{1}, x_1, \dots, x_k) = \begin{cases} x_1 & k = 1, \\ 0 & k \geq 2. \end{cases}$$

Suppose that A^e has a unit $\mathbb{1}$, then $\mathbb{1}$ is also a unit of $A^e[[t]]$. Since $z\nabla^{\mathrm{E}}_{\frac{\partial}{\partial t_i}}\mathbb{1} = T_i$, we see that $\mathbb{1}$ is primitive (with respect to the Euler connection) and $\frac{\partial}{\partial t_i}$ is identified with T_i. Hence $\mathbb{1}$ is a primitive form without a higher residue pairing.

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