Pure and Applied Mathematics Quarterly
Volume 16, Number 3, 337–369, 2020

Invariants of the Weyl group of type $A_{2l}^{(2)}$

Kenji Iohara and Yoshihisa Saito

Dedicated to Professor Kyoji Saito on the occasion of his 75th Anniversary

Abstract: In this note, we show the polynomiality of the ring of invariants with respect to the Weyl group of type $A_{2l}^{(2)}$.

Keywords: Affile Lie algebra, affine root system of type $A_{2l}^{(2)}$, affine Weyl group, theta function associated with an odd lattice.

Received March 1, 2019.
2010 Mathematics Subject Classification: Primary 17B10, 17B67; secondary 11F50.

Introduction

Let V be a finite dimensional vector space over a field $\mathbb{K}$ of characteristic 0. A *reflection* is a finite order linear transformation on V that fixes a hyperplane. Such a linear transformation is called *real* if it is of order 2 and *complex* if it is of order > 2. A group G of linear transformations is a *finite reflection group* if it is a finite group generated by reflections. In particular, a finite reflection group generated only by real reflections are called *Coxeter group*, otherwise it is called a *complex refelction group*. The G-action on V induces an G-action on its dual V^*, hence on its symmetric algebra $S^\bullet(V^*)$. C. Chevalley (for real case) [Ch] and G. C. Shephard and J. A. Todd (for complex case) [ST] have shown that the ring of invariants $S(V^*)^G$ is a $\mathbb{Z}$-graded algebra generated by $\dim_K V$ homogeneous elements that are algebraically independent. Notice that, for a finite group, there is the so-called Reynold's operator to obtain invariants explicitly.

Now, for an affine Weyl group W, one can consider the W-invariant theta functions defined on a half space Y of a Cartan subalgebra $\mathfrak{h}$ of the affine Lie algebra $\mathfrak{g}$ whose Weyl group is W. A first attempt has been made to determine the structure of the ring of such W-invariant theta functions by E. Looijenga [L] in 1976 whose argument has not been sufficient, as was pointed out by I. N. Bernstein and O. Schwarzman [BS] in 1978. Indeed, the ring $\widetilde{Th}^+$ of W-invariant theta functions is an $\mathcal{O}_{\mathbb{H}}$-module, where $\mathbb{H}$ is the Poincaré upper half plane $\{\tau \in \mathbb{C} | \operatorname{Im}\tau > 0\}$, spanned by the normalized characters $\{\chi_\Lambda\}_{\Lambda\in P_+ \mod \mathbb{C}\delta}$ of simple integrable highest weight modules over $\mathfrak{g}$. We remark that each χ_Λ is holomorphic on Y, as was shown by M. Gorelik and V. Kac [GK].

In 1984, V. G. Kac and D. Peterson [KP] has published a long monumental article on the modular transformations, in particular, the Jacobi transformation, of the characters of simple integrable highest weight modules over affine

Lie algebras. In particular, as an application of their results, they presented a list of the Jacobian of the fundamental characters for affine Lie algebras except for type $F_4^{(1)}, E_6^{(1)}, E_6^{(2)}, E_7^{(1)}$ and $E_8^{(1)}$, and stated that the ring $\widetilde{Th}^+$ of W-invariant theta functions is a polynomial ring over $\mathcal{O}_{\mathbb{H}}$ generated by fundamental characters $\{\chi_{\Lambda_i}\}_{0\leq i\leq l}$. Unfortunately, their proofs have never been published (cf. Ref. [35] in [KP]). In 2006, J. Bernstein and O. Schwarzman in [BS1] and [BS2] presented a detailed version of their announcement [BS], where their final result is a weak form of what V. Kac and D. Peterson has announced. Moreover, J. Bernstein and O. Schwarzman excluded two cases: type $D_l^{(1)}$ and $A_{2l}^{(2)}$.

In this article, we determine the Jacobian of the fundamental characters for the affine Lie algebra $\mathfrak{g}$ of type $A_{2l}^{(2)}$. In particular, we show that the explicit form of Jacobian in Table J of [KP] for type $A_{2l}^{(2)}$ is valid. As a corollary, it follows that the fundamental characters $\{\chi_{\Lambda_i}\}_{0\leq i\leq l}$ of $\mathfrak{g}$ are algebraically independent.

This article is organised as follows. As the practical computation requires many detailed information, Section 1 is devoted to providing root datum, detailed description on the non-degenerate symmetric invariant bilinear form restricted on a Cartan subalgebra $\mathfrak{h}$ and its induced bilinear form on the dual $\mathfrak{h}^*$, and the structure of the affine Weyl group of type $A_{2l}^{(2)}$. In Section 2, we recall the theta functions associated to the Heisenberg subgroup of the affine Weyl group of type $A_{2l}^{(2)}$ and the modular transformation of the normalized characters of irreducible integrable highest weight $\mathfrak{g}$-modules. In Section 3, After some technical preliminary computations, we study the modular transformations of the Jacobian of the fundamental characters. Finally in Section 4, we determine the Jacobian of the fundamental characters of type $A_{2l}^{(2)}$.

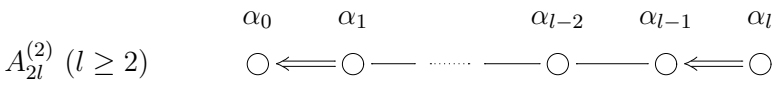
This text contains many well-known facts about the characters of integrable highest weight modules over affine Lie algebras for the sake of reader's convenience.

1. Preliminaries

In this section, we recall the definition of the affine Lie algebra of type $A_{2l}^{(2)}$ and its basic properties. Everything given in this section is completely known, see e.g., [Kac] and/or [MP]. Nevertheless, in order to fix our convention clearly and to avoid unnecessary confusion, we collect several explicit data that would be useful for further computations.

1.1. Basic data

Here, we fix the enumeration of the vertices in the Dynkin diagram of type $A_{2l}^{(2)}$ as follows (cf. [Kac]):



In particular, this implies that the corresponding generalized Cartan matrix $A = (a_{i,j})_{0 \leq i,j \leq l}$ is given by

$$A = \begin{pmatrix} 2 & -2 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 2 & -1 & 0 \\ \vdots & & & \ddots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

for $l \geq 2$, and

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

for $l = 1$. A realization of the generalized Cartan matrix A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that

(i) $\mathfrak{h}$ is an $l + 2$-dimensional $\mathbb{C}$-vector space,
(ii) $\Pi^\vee = \{h_i\}_{0 \leq i \leq l}$ is a linearly independent subset of $\mathfrak{h}$,
(iii) $\Pi = \{\alpha_i\}_{0 \leq i \leq l}$ is a linearly independent subset of $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$, and
(iv) $\langle h_i, \alpha_j \rangle = a_{i,j}$ for any $0 \leq i, j \leq l$.

Here, $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ is the canonical pairing.

The labels $a_0, a_1, \cdots, a_l$ and co-labels $a_0^\vee, a_1^\vee, \cdots, a_l^\vee$ are by definition, relatively prime positive integers satisfying

$$(a_0^\vee, a_1^\vee, \cdots, a_l^\vee)A = (0, 0, \cdots, 0), \qquad A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Explicitly, they are given by the following table:

i	0	1	$\cdots$	$l-1$	l
a_i	2	2	$\cdots$	2	1
$a_i^\vee$	1	2	$\cdots$	2	2

As a corollary, the Coxeter number h and the dual Coxeter number $h^\vee$ are shown to be

$$h := \sum_{i=0}^{l} a_i = 2l + 1, \qquad h^\vee := \sum_{i=0}^{l} a_i^\vee = 2l + 1.$$

Let us introduce some special elements c of $\mathfrak{h}$, and δ of $\mathfrak{h}^*$ by

$$c := \sum_{i=0}^{l} a_i^\vee h_i = h_0 + 2(h_1 + \cdots + h_l) \in \mathfrak{h}, \quad \delta := \sum_{i=0}^{l} a_i \alpha_i = 2(\alpha_0 + \cdots + \alpha_{l-1}) + \alpha_l.$$

By definition, we have

$$\langle c, \alpha_i \rangle = 0, \qquad \langle h_i, \delta \rangle = 0,$$

for any $0 \le i \le l$. Now, define the elements $d \in \mathfrak{h}$ and $\Lambda_0 \in \mathfrak{h}^*$ by

$$\langle d, \alpha_i \rangle = \delta_{i,0}, \qquad \langle h_i, \Lambda_0 \rangle = \delta_{i,0}, \qquad \langle d, \Lambda_0 \rangle = 0.$$

It can be checked that

$$\langle c, \Lambda_0 \rangle = a_0^\vee = 1, \qquad \langle d, \delta \rangle = a_0 = 2.$$

For later purpose, we recall the definition of i-th fundamental weight $\Lambda_i \in \mathfrak{h}^*$, the element of $\mathfrak{h}^*$ satisfying

$$\langle h_i, \Lambda_j \rangle = \delta_{i,j} \qquad (0 \le i, j \le l), \qquad \langle d, \Lambda_i \rangle = 0.$$

Clearly, one has

$$\langle c, \Lambda_i \rangle = a_i^\vee.$$

1.2. Bilinear forms on $\mathfrak{h}$ and $\mathfrak{h}^*$

Here, we recall the bilinear form I on $\mathfrak{h}^*$ and I^* on $\mathfrak{h}$ that are called normalized invariant forms by V. Kac [Kac] in an explicit manner.

On $\mathfrak{h}$:

$$\begin{aligned} &I^*(h_i, h_j) = \frac{a_j}{a_j^\vee} \cdot a_{i,j} \qquad (0 \le i, j \le l), \\ &I^*(h_i, d) = 0 \qquad (0 < i \le l), \\ &I^*(h_0, d) = a_0 = 2, \qquad I^*(d, d) = 0. \end{aligned}$$

In particular, we have

$$I^*(c, d) = 2.$$

On $\mathfrak{h}^*$:

$$\begin{aligned} &I(\alpha_i, \alpha_j) = \frac{a_i^\vee}{a_i} \cdot a_{i,j} \qquad (0 \le i, j \le l), \\ &I(\alpha_i, \Lambda_0) = 0 \qquad (0 < i \le l), \\ &I(\alpha_0, \Lambda_0) = a_0^{-1} = \frac{1}{2}, \qquad I(\Lambda_0, \Lambda_0) = 0. \end{aligned}$$

We also have

$$I(\Lambda_0, \delta) = 1.$$

The above bilinear forms are introduced in such a way that the linear map $\nu : (\mathfrak{h}, I^*) \to (\mathfrak{h}^*, I)$ defined by

$$a_i^\vee \nu(h_i) = a_i \alpha_i, \qquad \nu(c) = \delta, \qquad \nu(d) = a_0 \Lambda_0 = 2\Lambda_0$$

becomes an isometry.

1.3. Affine Lie algebra of type $A_{2l}^{(2)}$

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody Lie algebra attached with the generalized Cartan matrix A which is introduced in Subsection 1.1. Since A is symmetrizable, it is defined to be the Lie algebra over the complex number field $\mathbb{C}$ generated by e_i, f_i $(0 \le i \le l)$ and $\mathfrak{h}$, with the following defining relations:

(i) $[e_i, f_j] = \delta_{i,j} h_i$ for $0 \le i, j \le l$,

(ii) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$,

(iii) $[h, e_i] = \langle h, \alpha_i \rangle e_i$, $[h, f_i] = -\langle h, \alpha_i \rangle f_i$ for $h \in \mathfrak{h}$ and $0 \le i \le l$,

(iv) $(\operatorname{ad} e_i)^{-a_{i,j}+1}(e_j) = 0, \quad (\operatorname{ad} f_i)^{-a_{i,j}+1}(f_j) = 0 \quad \text{for } 0 \le i \ne j \le l.$

The Lie algebra $\mathfrak{g}$ is called the **affine Lie algebra of type** $A_{2l}^{(2)}$. The vector space $\mathfrak{h}$ is regarded as a commutative subalgebra of $\mathfrak{g}$, which is called the Cartan subalgebra of $\mathfrak{g}$.

It is well-known that $\mathfrak{g}$ admits the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha\Big),$$

where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \,|\, [h, X] = \langle h, \alpha\rangle X \text{ for every } h \in \mathfrak{h}\}$ is the root space associated to $\alpha \in \mathfrak{h}^*$, and $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \,|\, \mathfrak{g}_\alpha \ne \{0\}\}$ is the set of all roots. An element $\alpha \in \Delta$ is called a real root if $I(\alpha, \alpha) > 0$, and the set of all real roots is denoted by Δ^{re}. Set $\Delta^{im} := \Delta \setminus \Delta^{re}$. An element $\alpha \in \Delta^{im}$ is called an imaginary root. The explicit forms of real and imaginary roots will be given in the next subsection.

For $\alpha \in \Delta^{re}$, define a reflection $r_\alpha \in GL(\mathfrak{h}^*)$ by

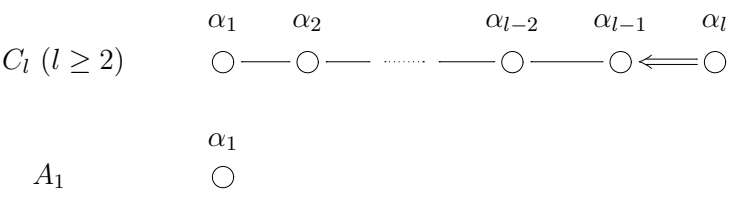
$$r_\alpha(\lambda) := \lambda - \left\langle \frac{2}{I(\alpha, \alpha)} \nu^{-1}(\alpha), \lambda \right\rangle \alpha$$

for $\lambda \in \mathfrak{h}^*$. Let W be the Weyl group of $\mathfrak{g}$, that is, W is a subgroup of $GL(\mathfrak{h}^*)$ generated by r_α ($\alpha \in \Delta^{re}$). It is known that the action of W on $\mathfrak{h}^*$ preserves the set of roots Δ. The detailed structure of the Weyl group W will be given in the next subsection also.

1.4. Affine Weyl group

Here, we describe the structure of the Weyl group of type $A_{2l}^{(2)}$, paying attention to the fact that the lattice of the translation part is **not** an even lattice.

To describe its structure, we regard the 0-th node as the special vertex, namely, the complementary subdiagrams



will be called the finite part of the Dynkin diagram of type $A_{2l}^{(2)}$.

Let $\mathfrak{h}_f$ be the subspace of $\mathfrak{h}$ defined by

$$\mathfrak{h}_f := \bigoplus_{i=1}^{l} \mathbb{C}h_i.$$

This can be viewed as the Cartan subalgebra of the finite part. In particular, we fix the splitting of $\mathfrak{h}$ as follows:

$$\mathfrak{h} = \mathfrak{h}_f \oplus (\mathbb{C}c \oplus \mathbb{C}d). \tag{1}$$

This is an orthogonal decomposition with respect to the normalized invariant form introduced in the previous subsection. We denote the canonical projection to the first component by $\bar{\cdot} : \mathfrak{h} \twoheadrightarrow \mathfrak{h}_f$. Similarly, we set

$$\mathfrak{h}_f^* := \bigoplus_{i=1}^{l} \mathbb{C}\alpha_i$$

and the canonical projection to the first component of the decomposition

$$\mathfrak{h}^* = \mathfrak{h}_f^* \oplus (\mathbb{C}\delta \oplus \mathbb{C}\Lambda_0) \tag{2}$$

will be denoted by the same symbol $\bar{\cdot} : \mathfrak{h}^* \twoheadrightarrow \mathfrak{h}_f^*$. It should be noticed that the image of the restriction of $\nu : \mathfrak{h} \to \mathfrak{h}^*$ to $\mathfrak{h}_f$ is $\mathfrak{h}_f^*$.

Now, we recall the root system of the finite part. Although most of the informations can be found in [Bour] for example, we describe this to fix the convention.

We identify $\mathfrak{h}_f$ and $\mathfrak{h}_f^*$ via the isometry ν. An orthonormal basis $\{\varepsilon_i\}_{1\le i\le l}$ of $\mathfrak{h}_f^*$ with respect to I should be so chosen that the root system Δ_f of the finite part and the set of simple roots $\Pi_f = \{\alpha_i\}_{1\le i\le l}$ has the next description:

$$\Delta_f = \{\pm\varepsilon_i \pm \varepsilon_j\}_{1\le i,j\le l} \cup \{\pm 2\varepsilon_i\}_{1\le i\le l}, \qquad \alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & i < l, \\ 2\varepsilon_i & i = l. \end{cases}$$

N.B. For $l = 1$, $\Delta_f = \{\pm 2\varepsilon_1\}$. □

In particular, the sets Δ_f^+, $\Delta_{f,l}^+$ and $\Delta_{f,s}^+$ of positive roots, positive long roots and short roots, respectively are given by

$$\Delta_f^+ = \{\varepsilon_i \pm \varepsilon_j\}_{1\le i<j\le l} \cup \{2\varepsilon_i\}_{1\le i\le l},$$
$$\Delta_{f,l}^+ = \{2\varepsilon_i\}_{1\le i\le l}, \qquad \Delta_{f,s}^+ = \{\varepsilon_i \pm \varepsilon_j\}_{1\le i<j\le l}.$$

We remark that $\varpi_i := \overline{\Lambda_i} = \sum_{j=1}^{i} \varepsilon_j$ for $1 \leq i \leq l$ is the i-th fundamental weight of $\mathfrak{h}_f$. The highest root $\theta \in \mathfrak{h}_f^*$ can be written as

$$\theta = \delta - a_0\alpha_0 = \delta - 2\alpha_0 = 2\varepsilon_1,$$

and its coroot $\theta^\vee \in \mathfrak{h}_f$ as

$$\theta^\vee = \frac{2}{I(\theta,\theta)}\nu^{-1}(\theta) = \sum_{i=1}^{l} h_i, \qquad \nu(\theta^\vee) = \frac{1}{2}\theta = \varepsilon_1.$$

Let W_f be a subgroup generated by r_{α_i} for $\alpha_i \in \Pi_f$. This group is isomorphic to the Weyl group of type C_l-type. That is, $W_f \cong \mathfrak{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z})^l$.

The set of real roots Δ^{re} of $\mathfrak{g}$ is described as follows: for $l \geq 2$,

$$\Delta^{re} = \Delta_s^{re} \cup \Delta_m^{re} \cup \Delta_l^{re},$$
$$\text{where} \quad \begin{cases} \Delta_s^{re} := \left\{ \frac{1}{2}(\alpha + (2r-1)\delta) \,|\, \alpha \in \Delta_{f,l},\, r \in \mathbb{Z} \right\}, \\ \Delta_m^{re} := \{\alpha + r\delta \,|\, \alpha \in \Delta_{f,s},\, r \in \mathbb{Z}\}, \\ \Delta_l^{re} := \{\alpha + 2r\delta \,|\, \alpha \in \Delta_{f,l},\, r \in \mathbb{Z}\}, \end{cases}$$

and for $l = 1$,

$$\Delta^{re} = \Delta_s^{re} \cup \Delta_l^{re},$$
$$\text{where} \quad \begin{cases} \Delta_s^{re} := \left\{ \frac{1}{2}(\alpha + (2r-1)\delta) \,|\, \alpha \in \Delta_f,\, r \in \mathbb{Z} \right\}, \\ \Delta_l^{re} := \{\alpha + 2r\delta \,|\, \alpha \in \Delta_f,\, r \in \mathbb{Z}\}. \end{cases}$$

The set of imaginary roots can be described as

$$\Delta^{im} = \mathbb{Z}\delta \setminus \{0\}.$$

For detail, see [Kac].

Now, we can describe the Weyl group W. As $c = h_0 + 2\theta^\vee$, one has

$$\begin{aligned} r_{\alpha_0} r_\theta(\lambda) &= r_{\alpha_0}(\lambda - \langle\theta^\vee, \lambda\rangle\theta) = \lambda - \lambda(\theta^\vee)\theta - \langle h_0, \lambda - \langle\theta^\vee,\lambda\rangle\theta\rangle\alpha_0 \\ &= \lambda + \frac{1}{2}\langle c,\lambda\rangle\theta - \left(\langle\theta^\vee,\lambda\rangle + \frac{1}{2}\langle c,\lambda\rangle\right)\delta \\ &= \lambda + \langle c,\lambda\rangle\nu(\theta^\vee) - \left(I(\lambda,\nu(\theta^\vee)) + \frac{1}{2}\langle c,\lambda\rangle I(\nu(\theta^\vee),\nu(\theta^\vee))\right)\delta, \end{aligned}$$

for $\lambda \in \mathfrak{h}^*$. Set

$$M := \mathbb{Z}\nu(W_f \cdot \theta^\vee) = \bigoplus_{i=1}^{l} \mathbb{Z}\varepsilon_i,$$

and, for $\alpha \in \mathfrak{h}^*_{f,\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, we define $t_\alpha \in \mathrm{End}_{\mathbb{C}}(\mathfrak{h}^*)$ by

$$t_\alpha(\lambda) := \lambda + \langle c, \lambda\rangle\alpha - \left(I(\lambda,\alpha) + \frac{1}{2}\langle c,\lambda\rangle I(\alpha,\alpha)\right)\delta.$$

It is known that the assignment of the lattice M to $\mathrm{End}_{\mathbb{C}}(\mathfrak{h}^*)$ defined by α to t_α gives an injective group homomorphism $M \hookrightarrow W$. Furthermore, one has

$$W \cong W_f \ltimes M \cong (\mathfrak{S}_l \ltimes (\mathbb{Z}/2/\mathbb{Z})^l) \ltimes M.$$

We note that the lattice M is an **odd** lattice. This fact is essential in the following discussion.

2. Main theorem

Let $\mathfrak{g}$ be the affine Lie algebra of type $A^{(2)}_{2l}$ recalled in the previous section.

2.1. Formal characters

Let $\mathfrak{n}_\pm$ be the subalgebra of $\mathfrak{g}$ generated by $\{e_i\}_{0\le i\le l}$ (resp. $\{f_i\}_{0\le i\le l}$). We have the so-called triangular decomposition: $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Let V be a $\mathfrak{h}$-diagonalizable module, i.e., $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ where $V_\lambda := \{v \in V \,|\, h.v = \langle h,\lambda\rangle v \quad h \in \mathfrak{h}\}$. Set $\mathcal{P}(V) := \{\lambda \in \mathfrak{h}^* | V_\lambda \neq \{0\}\}$. Let $\mathcal{O}$ be the BGG category of $\mathfrak{g}$-modules, that is, it is the subcategory of $\mathfrak{g}$-modules whose objects are $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ satisfying

(i) $\dim V_\lambda < \infty$ for any $\lambda \in \mathcal{P}(V)$,
(ii) there exists $\lambda_1, \lambda_2, \cdots, \lambda_r \in \mathfrak{h}^*$ such that $\mathcal{P}(V) \subset \bigcup_{i=1}^r (\lambda_i - \mathbb{Z}_{\ge 0}\Pi)$.

A typical object of this category is a so-called highest weight module defined as follows. We say that a $\mathfrak{g}$-module V is a highest weight module with highest weight $\Lambda \in \mathfrak{h}^*$ if

(i) $\dim V_\Lambda = 1$
(ii) $\mathfrak{n}_+.V_\Lambda = \{0\}$ and $V = U(\mathfrak{g}).V_\Lambda$.

In particular, the last condition implies that $V = U(\mathfrak{n}_-).V_\Lambda$ as a vector space and

$$\mathcal{P}(V) := \{\lambda \in \mathfrak{h}^* | V_\lambda \neq \{0\}\} \subset \Lambda - \mathbb{Z}_{\ge 0}\Pi.$$

A typical example is given as follows. For $\Lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\Lambda = \mathbb{C}v_\Lambda$ be the one dimensional module over $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$ defined by

$$h.v_\Lambda = \langle h, \Lambda \rangle v_\Lambda \quad (h \in \mathfrak{h}), \qquad \mathfrak{n}_+.v_\Lambda = 0.$$

The induced $\mathfrak{g}$-module $M(\Lambda) := \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}_\Lambda$ is called the Verma module with highest weight Λ. It can be shown that for any highest weight $\mathfrak{g}$-module V with highest weight $\Lambda \in \mathfrak{h}^*$, there exists a surjective $\mathfrak{g}$-module map $M(\Lambda) \twoheadrightarrow V$. The smallest among such V can be obtained by taking the quotient of $M(\Lambda)$ by its maximal proper submodule and the resulting $\mathfrak{g}$-module is the irreducible highest $\mathfrak{g}$-module with highest weight Λ, denoted by $L(\Lambda)$.

Let $\mathcal{E}$ be the formal linear combination of e^λ ($\lambda \in \mathfrak{h}^*$) with the next condition: $\sum_\lambda c_\lambda e^\lambda \in \mathcal{E} \Rightarrow \exists \lambda_1, \lambda_2, \cdots, ; \lambda_r \in \mathfrak{h}^*$ such that

$$\{\lambda | c_\lambda \neq 0\} \subset \bigcup_{i=1}^{r} (\lambda_i - \mathbb{Z}_{\geq 0}\Pi).$$

We introduce the ring structure on $\mathcal{E}$ by $e^\lambda \cdot e^\mu := e^{\lambda+\mu}$.

The formal character of $V \in \mathcal{O}$ is, by definition, the element $\mathrm{ch}V \in \mathcal{E}$ defined by

$$\mathrm{ch}V = \sum_{\lambda \in \mathcal{P}(V)} (\dim V_\lambda) e^\lambda.$$

$\mathrm{ch}(\cdot)$ can be viewed as an additive function defined on $\mathcal{O}$ with values in $\mathcal{E}$. For example, The formal character of the Verma module $M(\Lambda)$ is given by

$$\mathrm{ch}M(\Lambda) = e^\Lambda \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\mathrm{mult}(\alpha)},$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$ and $\mathrm{mult}(\alpha) = \dim \mathfrak{g}_\alpha$ is the multiplicity of the root α. Set $\varepsilon(w) = (-1)^{l(w)}$ where $l(w)$ is the length of an element $w \in W$. For $\Lambda \in P_+ := \{\Lambda \in \mathfrak{h}^* | \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0} | 0 \leq i \leq l\}$, the character of $L(\Lambda)$ is known as Weyl-Kac character formula and is given by

$$\mathrm{ch}L(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}},$$

where $\rho \in \mathfrak{h}^*$ is the so-called Weyl vector, i.e., it satisfies $\langle h_i, \rho \rangle = 1$ for any $0 \leq i \leq l$ and $\langle d, \rho \rangle = 0$. In particular, for $\Lambda = 0$, as $L(0) = \mathbb{C}$ is the trivial representation, one obtains the so-called denominator identity:

$$\sum_{w \in W} \varepsilon(w) e^{w(\rho)} = e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}. \tag{3}$$

This implies that the Weyl-Kac character formula can be rephrased as follows:

$$\mathrm{ch}L(\Lambda) = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}}{\sum_{w\in W}\varepsilon(w)e^{w(\rho)}}. \tag{4}$$

2.2. Normalized characters of type $A_{2l}^{(2)}$

For $\Lambda \in P_+$ with $\langle c, \Lambda\rangle = k \in \mathbb{Z}_{\geq 0}$, the normalized character χ_Λ of the irreducible highest weight $\mathfrak{g}$- module $L(\Lambda)$ with highest weight Λ is defined by

$$\chi_\Lambda := e^{-m_\Lambda \delta}\mathrm{ch}L(\Lambda),$$

where the number m_Λ, called the conformal anomaly, is defined by

$$m_\Lambda := \frac{I(\Lambda+\rho, \Lambda+\rho)}{2(k+h^\vee)} - \frac{I(\rho,\rho)}{2h^\vee},$$

with $h^\vee = 2l+1$ being the dual Coxeter number of $A_{2l}^{(2)}$. Here, the bilinear form is normalized as in §1.2 (cf. Chapter 6 of [Kac]). For $\Lambda \in P_+$ with $\langle c, \Lambda\rangle = k$, set

$$A_{\Lambda+\rho} := \sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)-\frac{I(\Lambda+\rho,\Lambda+\rho)}{2(k+h^\vee)}\delta}.$$

This is a W anti-invariant. By (4), one has

$$\chi_\Lambda = \frac{A_{\Lambda+\rho}}{A_\rho}. \tag{5}$$

We regard this normalized character χ_Λ as a function defined on a certain subset of $\mathfrak{h}$ as follows. For $\lambda \in \mathfrak{h}^*$, e^λ can be viewed as a function defined on $\mathfrak{h}$: $e^\lambda(h) := e^{\langle h,\lambda\rangle}$. It is shown in [GK] that the normalized character χ_Λ for $\Lambda \in P_+$ can be viewed as a holomorphic function on a complex domain

$$Y := \{h \in \mathfrak{h} | \mathrm{Re}\,\langle h, \delta\rangle > 0\,\}.$$

We introduce a coordinate system on $\mathfrak{h}$ as follows. Set $\epsilon_i := \nu^{-1}(\varepsilon_i)$ for $1 \leq i \leq l$, where $\nu : \mathfrak{h} \to \mathfrak{h}^*$ is the isometry defined in §1.2. Then, $\{\epsilon_i\}_{1\leq i\leq l}$ is an orthonormal basis of $\mathfrak{h}_f^*$ with respect to $I^*|_{\mathfrak{h}_f^*\times\mathfrak{h}_f^*}$. For $h \in Y$, define $z \in \mathfrak{h}_f, \tau, t \in \mathbb{C}$ as in [Kac]:

$$h = 2\pi\sqrt{-1}\left(z - \tau\nu^{-1}(\Lambda_0) + t\nu^{-1}(\delta)\right) = 2\pi\sqrt{-1}\left(z - \frac{1}{2}\tau d + tc\right), \tag{6}$$

and write $z = \sum_{i=1}^{l} z_i \epsilon_i$. With this coordinate system, we see that

$$(7) \qquad \begin{aligned} & Y \xrightarrow{\sim} \mathbb{H} \times \mathfrak{h}_f \times \mathbb{C} \xrightarrow{\sim} \mathbb{H} \times \mathbb{C}^l \times \mathbb{C}; \\ & h \longmapsto (\tau, z, t) \mapsto (\tau, (z_1, z_2, \cdots, z_l), t), \end{aligned}$$

where $\mathbb{H}$ is the upper half plane $\{\, \tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0 \,\}$. We write $\chi_\Lambda(\tau, z, t) := \chi_\Lambda(h)$.

For $k \in \mathbb{Z}_{>0}$, set

$$P^k := \{\lambda \in \mathfrak{h}^* | \, \langle c, \lambda \rangle = k, \ \overline{\lambda} \in M\},$$

and for $\lambda \in P^k$, set

$$\Theta_\lambda := e^{-\frac{I(\lambda,\lambda)}{2k}\delta} \sum_{\alpha \in M} e^{t_\alpha(\lambda)}.$$

This is the **classical theta function of degree** k. Its value on $h \in \mathfrak{h}$ is given by

$$(8) \qquad \Theta_\lambda(\tau, z, t) = e^{2\pi\sqrt{-1}kt} \sum_{\gamma \in M + k^{-1}\overline{\lambda}} e^{\pi\sqrt{-1}k\tau I(\gamma,\gamma) + 2\pi\sqrt{-1}kI(\gamma,\nu(z))}.$$

Now, we recall the modular transformations of the classical theta functions. Recall that $SL(2, \mathbb{Z})$ is the group generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

As $SL(2, \mathbb{Z})$ acts on the complex domain Y by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.(\tau, z, t) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t - \frac{cI^*(z, z)}{2(c\tau + d)} \right),$$

This action induces a right action of $SL(2, \mathbb{Z})$ on $\mathcal{O}_Y$ as follows: for $g \in SL(2, \mathbb{Z})$ and $F \in \mathcal{O}_Y$,

$$F|_g(\tau, z, t) := F(g.(\tau, z, t)).$$

The next formula is a simple application of the Poisson resummation formula:

Proposition 2.1 (cf. Theorem 13.5 in [Kac]). *Let $\lambda \in P^k$. One has*

$$\Theta_\lambda \left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{I^*(z, z)}{2\tau} \right)$$

$$= \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}l} k^{-\frac{1}{2}l} \sum_{\mu \in P^k \bmod \mathbb{C}\delta + kM} e^{-\frac{2\pi\sqrt{-1}I(\overline{\lambda},\overline{\mu})}{k}} \Theta_\mu(\tau, z, t),$$

$$\Theta_\lambda(\tau + 2, z, t) = e^{\frac{2\pi\sqrt{-1}}{k} I(\overline{\lambda},\overline{\lambda})} \Theta_\lambda(\tau, z, t).$$

In particular, when k is even, one also has

$$\Theta_\lambda(\tau + 1, z, t) = e^{\frac{\pi\sqrt{-1}}{k} I(\overline{\lambda},\overline{\lambda})} \Theta_\lambda(\tau, z, t).$$

In fact, the lattice M is not an even lattice.

For $k \in \mathbb{Z}_{\geq 0}$, let $P_+^k := P_+ \cap P^k$ be the set of dominant integral weights of level k. As an application of the above proposition, V. Kac and D. Peterson [KP] proved that the $\mathbb{C}$-span of $\{A_{\Lambda+\rho}\}_{\Lambda \in P_+^k \bmod \mathbb{C}\delta}$ admits an action of a certain subgroup of $SL(2, \mathbb{Z})$, so does the $\mathbb{C}$-span of the normalized characters $\{\chi_\lambda(\tau, z, t)\}_{\lambda \in P_k^+ \bmod \mathbb{C}\delta}$.

Theorem 2.1 ([KP]). *Let $\mathfrak{g}$ be the affine Lie algebra of type $A_{2l}^{(2)}$. For $\lambda \in P_+^k$ $(k \in \mathbb{Z}_{>0})$, one has*

$$\chi_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{I^*(z,z)}{2\tau}\right) = \sum_{\mu \in P_+^k \bmod \mathbb{C}\delta} a(\lambda, \mu) \chi_\mu(\tau, z, t),$$

where the matrix $(a(\lambda, \mu))_{\lambda,\mu \in P_+^k \bmod \mathbb{C}\delta}$ is given by

$$a(\lambda, \mu) = (\sqrt{-1})^{l^2} (k + 2l + 1)^{-\frac{1}{2}l} \times \sum_{w \in \mathfrak{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z})^l} \varepsilon(w) \exp\left(-\frac{2\pi\sqrt{-1} I(\overline{\lambda + \rho}, w(\overline{\mu + \rho}))}{k + 2l + 1}\right).$$

One also has

$$\chi_\lambda(\tau + 2, z, t) = e^{4\pi\sqrt{-1} m_\lambda} \chi_\lambda(\tau, z, t).$$

Remark 2.1. *For later purpose, we recall the modular transformations of the denominator:*

$$A_\rho\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{I^*(z,z)}{2\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}l} (\sqrt{-1})^{-l^2} A_\rho(\tau, z, t),$$

$$A_\rho(\tau + 2, z, t) = \exp\left(\frac{l(l+1)}{3}\pi\sqrt{-1}\right) A_\rho(\tau, z, t).$$

Remark 2.2. *It can be checked that, for* $w = \sigma\eta \in W_f$ ($\sigma \in \mathfrak{S}_l, \eta = (\eta_1, \cdots, \eta_l) \in \{\pm 1\}^l$), *one has* $\det {}_{\mathfrak{h}_f^*}(w) = \mathrm{sgn}(\sigma) \prod_{i=1}^{l} \eta_i$.

Notice that, for each $k \in \mathbb{Z}_{\geq 0}$, the $\mathbb{C}$-span of the normalized characters χ_λ ($\lambda \in P_+^k \mod \mathbb{C}\delta$) is Γ_θ-stable, where Γ_θ is the subgroup of $SL(2,\mathbb{Z})$ generated by S and T^2.

In the following, we recall an expansion of χ_Λ in terms of the classical theta functions. An element $\lambda \in \mathcal{P}(L(\Lambda))$ is called maximal if $\lambda + \delta \notin \mathcal{P}(L(\Lambda))$, and let $\max(\Lambda)$ be the set of all maximal elements of $\mathcal{P}(L(\Lambda))$. Hence, we have a decomposition of $\mathcal{P}(L(\Lambda))$:

$$\mathcal{P}(L(\Lambda)) = \bigsqcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \,|\, n \in \mathbb{Z}_{\geq 0}\}.$$

For $\lambda \in \max(\Lambda)$, set

$$c_\lambda^\Lambda := e^{-m_\Lambda(\lambda)\delta} \sum_{n=0}^{\infty} (\dim_{\mathbb{C}} L(\Lambda)_{\lambda - n\delta}) e^{-n\delta}, \quad \text{where } m_\Lambda(\lambda) := m_\Lambda - \frac{I(\lambda,\lambda)}{2k}.$$

Furthermore, we extend the definition of c_λ^Λ to an arbitrary $\lambda \in \mathfrak{h}^*$ as follows. If $(\lambda + \mathbb{C}\delta) \cap \max(\Lambda) = \emptyset$, we set $c_\lambda^\Lambda := 0$. Otherwise, there exists a unique $\mu \in \max(\Lambda)$ such that $\lambda - \mu \in \mathbb{C}\delta$. Hence, we set $c_\lambda^\Lambda := c_\mu^\Lambda$.

Similarly to the case of classical theta functions, we regard the series c_λ^Λ as a (formal) function on Y. By the definition, this function depends only on the variable $\tau \in \mathbb{H}$. The following proposition is well-known in the representation theory of affine Lie algebras. For example, see [Kac], in detail.

Proposition 2.2 ([Kac])**.** *Let* $k > 0$ *be a positive integer and* $\Lambda \in P_+^k$.

(i) *The series* $c_\lambda^\Lambda = c_\lambda^\Lambda(\tau)$ *converges absolutely on the upper half plane* $\mathbb{H}$ *to a holomorphic function.*

(ii) *The normalized character* χ_Λ *has the following expansion in terms of the classical theta functions:*

$$\chi_\Lambda = \sum_{\lambda \in P^k \bmod (kM + \mathbb{C}\delta)} c_\lambda^\Lambda \Theta_\lambda.$$

The holomorphic function $c_\lambda^\Lambda(\tau)$ is called the string function of $\lambda \in \mathfrak{h}^*$.

2.3. Ring of Theta functions

For $\alpha \in \mathfrak{h}_{f,\mathbb{R}}^*$, we define $p_\alpha \in \mathrm{End}_{\mathbb{C}}(\mathfrak{h}^*)$ by

$$p_\alpha(h) = h + 2\pi\sqrt{-1}\nu^{-1}(\alpha).$$

Set

$$N = \mathfrak{h}^*_{f,\mathbb{R}} \times \mathfrak{h}^*_{f,\mathbb{R}} \times \sqrt{-1}\mathbb{R}$$

and define a group structure on N by

$$(\alpha, \beta, u) \cdot (\alpha', \beta', u') = (\alpha + \alpha', \beta + \beta', u + u' + \pi\sqrt{-1}\{I(\alpha, \beta') - I(\alpha', \beta)\}).$$

The group N is called **Heisenberg group**. This group acts on $\mathfrak{h}$ by

$$(\alpha, \beta, u).h = t_\beta(h) + 2\pi\sqrt{-1}\nu^{-1}(\alpha) + \{u - \pi\sqrt{-1}I(\alpha, \beta)\}c.$$

This N action preserves the complex domain Y. We note that $(\alpha, 0, 0).h = p_\alpha(h)$ and $(0, \beta, 0).h = t_\beta(h)$. We consider the subgroup $N_\mathbb{Z}$ of N generated by $(\alpha, 0, 0), (0, \beta, 0)$ with $\alpha, \beta \in M$ and $(0, 0, u)$ with $u \in 2\pi\sqrt{-1}\mathbb{Z}$, i.e.,

$$N_\mathbb{Z} = \{(\alpha, \beta, u) \in N \,|\, \alpha, \beta \in M,\ u + \pi\sqrt{-1}I(\alpha, \beta) \in 2\pi\sqrt{-1}\mathbb{Z}\}.$$

Let $\mathcal{O}_Y$ be the ring of holomorphic functions on Y. We define the right N-action on $\mathcal{O}_Y$ by

$$F|_{(\alpha,\beta,u)}(h) = F((\alpha, \beta, u).h) \qquad F \in \mathcal{O}_Y.$$

Definition 2.1. *For $k \in \mathbb{Z}_{\geq 0}$, define*

$$\widetilde{Th}_k := \left\{ F \in \mathcal{O}_Y \;\middle|\; \begin{array}{ll} (1) & F|_{(\alpha,\beta,u)} = F \quad \forall\, (\alpha, \beta, u) \in N_\mathbb{Z}, \\ (2) & F(h + ac) = e^{ka}F(h) \quad \forall\, h \in Y \text{ and } a \in \mathbb{C}. \end{array} \right\},$$

and set

$$\widetilde{Th} := \bigoplus_{k=0}^{\infty} \widetilde{Th}_k.$$

An element F of $\widetilde{Th}_k$ is called a theta function of degree k.

Remark 2.3. *The ring $\widetilde{Th}$ is a $\mathbb{Z}$-graded algebra over the ring $\widetilde{Th}_0 = \mathcal{O}_\mathbb{H}$, the ring of holomorphic functions on the upper half plane $\mathbb{H}$.*

A typical example of an element of $\widetilde{Th}_k$ for $k \in \mathbb{Z}_{>0}$ is the classical theta function Θ_λ of degree k, i.e., $\lambda \in P^k$.

Remark 2.4. *It follows from (8) that*

$$\Theta_{\lambda + k\alpha + a\delta} = \Theta_\lambda \quad \text{for every } \alpha \in M \text{ and } a \in \mathbb{C}.$$

Thus, a classical theta function of degree k depends only on the finite set $P^k \bmod (kM + \mathbb{C}\delta)$.

Let D be the Laplacian on Y:

$$D = \frac{1}{4\pi^2}\left(2\frac{\partial}{\partial t}\frac{\partial}{\partial \tau} - \sum_{i=1}^{l}\frac{\partial^2}{\partial z_i^2}\right).$$

For $k \in \mathbb{Z}_{\geq 0}$, set

$$Th_k := \begin{cases} \{F \in \widetilde{Th}_k \,|D(F) = 0\} & k > 0, \\ \mathbb{C} & k = 0. \end{cases}$$

V. Kac and D. Peterson showed the following proposition.

Proposition 2.3 ([KP]). *Let k be a positive integer.*

(i) The set $\{\Theta_\lambda | \lambda \in P^k \bmod kM + \mathbb{C}\delta\}$ is a $\mathbb{C}$-basis of Th_k.

(ii) The map: $\mathcal{O}_{\mathbb{H}} \otimes_{\mathbb{C}} Th_k \to \widetilde{Th}_k$ defined by $f \otimes F \mapsto fF$ is an isomorphism of $\mathcal{O}_{\mathbb{H}}$-modules. In other words, $\widetilde{Th}_k$ is a free $\mathcal{O}_{\mathbb{H}}$-module with basis $\{\Theta_\lambda \,|\, \lambda \in P^k \bmod (kM + \mathbb{C}\delta)\}$.

As the finite Weyl group W_f acts on Y, it induces the right action of W_f on $\mathcal{O}_Y$:

$$F|_w(h) := F(w \cdot h) \quad \text{for } F \in \mathcal{O}_Y \text{ and } w \in W_f.$$

For any $k \in \mathbb{Z}_{\geq 0}$, this right W_f-action on $\mathcal{O}_Y$ restricts to a right W_f-action on $\widetilde{Th}_k$. Thus, we set

$$\begin{aligned} \widetilde{Th}_k^+ &:= \{F \in \widetilde{Th}_k \,|\, F|_w = F \quad \forall\, w \in W_f\}, \\ \widetilde{Th}_k^- &:= \{F \in \widetilde{Th}_k \,|\, F|_w = \det{}_{\mathfrak{h}_f^*}(w) F \quad \forall\, w \in W_f\}, \\ Th_k^\pm &:= \widetilde{Th}_k^\pm \cap Th_k. \end{aligned}$$

An element of $\widetilde{Th}^\pm := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \widetilde{Th}_k^\pm$ is called a W_f-invariant (resp. W_f anti-invariant). For $k \in \mathbb{Z}_{>0}$, set

$$P_{++}^k := \{\Lambda \in P^k \,|\, \langle h_i, \Lambda \rangle \in \mathbb{Z}_{>0}\}.$$

Let $\Lambda \in P^k$. Obviously, the element $A_{\Lambda+\rho}$ introduced in §2.2 is a W_f anti-invariant. On the other hand, set

$$S_\Lambda := e^{-\frac{I(\Lambda,\Lambda)}{2k}\delta} \sum_{w \in W} e^{w(\lambda)} = \sum_{w_f \in W_f} \Theta_{w_f(\Lambda)}.$$

Then, it is an element of Th_k^+.

Another important example of W_f-invariants is the normalized character χ_Λ. Indeed, Proposition 2.2 (ii) tells us that χ_Λ is an element of $\widetilde{Th}_k$. In addition, thanks to the description (5) of χ_Λ, it is invariant under the action of W_f. Therefore, one has $\chi_\Lambda \in \widetilde{Th}_k^+$.

Proposition 2.4 ([KP]). *Let k be a non-negative integer.*

(i) The set $\{S_\Lambda | \Lambda \in P_+^k \, mod \mathbb{C}\delta\}$ is a $\mathbb{C}$-basis of Th_k^+.
(ii) For $k \geq 2l+1$, the set $\{A_\lambda | \lambda \in P_{++}^k \, mod \mathbb{C}\delta\}$ is a $\mathbb{C}$-basis of Th_k^-.
(iii) The map Φ_k in Proposition 2.3 is W_f-equivariant. Therefore, the $\mathcal{O}_{\mathbb{H}}$-modules $\widetilde{Th}_k^\pm$ are free over $\{\chi_\Lambda | \Lambda \in P_+^k \, mod \mathbb{C}\delta\}$ (resp. $\{A_\lambda | \lambda \in P_{++}^k \, mod \mathbb{C}\delta\}$).

As the map $P_+^k \to P_{++}^{k+2l+1}; \lambda \mapsto \lambda + \rho$ is bijective, (5) and the above proposition implies

Corollary 2.1. *The space of W_f anti-invariants $\widetilde{Th}^-$ is a free $\widetilde{Th}^+$-module over A_ρ.*

As $P_+ \bmod \mathbb{C}\delta$ is the monoid generated by $\{\Lambda_i\}_{0 \leq i \leq l}$ and $\langle c, \Lambda_i \rangle = a_i^\vee$, it can be shown that the Poincaré series $P_{\widetilde{Th}^+}(T)$ of the graded $\mathcal{O}_{\mathbb{H}}$-algebra $\widetilde{Th}^+$ is given by

$$P_{\widetilde{Th}^+}(T) := \sum_{k \in \mathbb{Z}_{\geq 0}} \mathrm{rank}_{\mathcal{O}_{\mathbb{H}}}(\widetilde{Th}_k^+) T^k = \prod_{i=0}^{l} (1 - T^{a_i^\vee})^{-1}.$$

By the same computation, this implies

$$\sum_{k \in \mathbb{Z}_{\geq 0}} \dim_{\mathbb{C}}(Th_k^+) T^k = \prod_{i=0}^{l} (1 - T^{a_i^\vee})^{-1}. \tag{9}$$

For each $\tau \in \mathbb{H}$, let Y_τ be the subset of Y that corresponds to $\{\tau\} \times \mathfrak{h}_f \times \mathbb{C}$ via the isomorphism $Y \cong \mathbb{H} \times \mathfrak{h}_f \times \mathbb{C}$ discussed in (7), $\iota_{Y_\tau} : Y_\tau \hookrightarrow Y$ the corresponding embedding. The above computation leads to the next conjecture:

Conjecture 2.1 (cf. [KP]). *Let $\mathfrak{g}$ be the affine Lie algebra of type $A_{2l}^{(2)}$.*

(i) For each $\tau \in \mathbb{H}$, $\widetilde{Th}^+|_{Y_\tau}$ is a polynomial ring generated by $\{\iota_{Y_\tau}^*(\chi_{\Lambda_i})\}_{0 \leq i \leq l}$ over $\mathbb{C}$.
(ii) The graded ring $\widetilde{Th}^+$ is a polynomial algebra over $\mathcal{O}_{\mathbb{H}}$ generated by $\{\chi_{\Lambda_i}\}_{0 \leq i \leq l}$.

2.4. Jacobian of fundamental characters

Let $\Lambda \in P_+$ be a dominant integral weight. For an integer $0 \le i \le l$, we define the directional derivative $\partial_i \chi_\Lambda$ by

$$(\partial_i \chi_\Lambda)(\tau, z, t) := \lim_{s \to 0} \frac{\chi_\Lambda(h + 2\pi\sqrt{-1} s h_j) - \chi_\Lambda(h)}{s}.$$

As $c = h_0 + 2\sum_{i=1}^{l} h_i$, it follows that

$$(\partial_0 \chi_\Lambda)(\tau, z, t) = 2\pi\sqrt{-1}\langle c, \Lambda\rangle \chi_\Lambda(\tau, z, t) - 2\sum_{i=1}^{l} \partial_i \chi_\Lambda(\tau, z, t).$$

Hence, the Jacobian of the fundamental characters is, by definition,

$$J(\tau, z, t) := \det(\partial_j \chi_{\Lambda_i})_{i,j=0,1,\cdots,l} = 2\pi\sqrt{-1} \begin{vmatrix} \chi_0 & \partial_1\chi_0 & \cdots & \partial_l\chi_0 \\ 2\chi_1 & \partial_1\chi_1 & \cdots & \partial_l\chi_1 \\ \vdots & \vdots & & \vdots \\ 2\chi_l & \partial_1\chi_l & \cdots & \partial_l\chi_l \end{vmatrix}.$$

It can be easily seen that this determinant is an element of $\widetilde{Th}^{-}_{2l+1}$, i.e., there exists a holomorphic function $F \in \mathcal{O}_{\mathbb{H}}$ such that $J(\tau, z, t) = F(\tau) A_\rho(\tau, z, t)$.

In the rest of this article, we will determine this function F and prove Conjecture 2.1.

Remark 2.5. *A weaker statement is proved by I. Bernstein and O. Schwarzmann [BS1] and [BS2] for any affine root systems except for $D_l^{(1)}$ and $A_{2l}^{(2)}$.*

3. Preliminary computations

In this section, we study the modular transformations and leading terms of the Jacobian of certains characters.

3.1. Explicit formulas on $a(\lambda, \mu)$

As $a_0^\vee = 1$ and $a_i^\vee = 2$ for $0 < i \le l$, one sees that

(i) $P_+^1 \mod \mathbb{C}\delta = \{\Lambda_0\}$ and
(ii) $P_+^2 \mod \mathbb{C}\delta = \{2\Lambda_0\} \cup \{\Lambda_i\}_{1 \le i \le l}$.

First, we compute $a(\Lambda_0, \Lambda_0)$. By Theorem 2.1 and the denominator identity, we have

$$\begin{aligned} a(\Lambda_0, \Lambda_0) &= (\sqrt{-1})^{l^2}(2(l+1))^{-\frac{1}{2}l} \sum_{w \in W_f} \varepsilon(w) \exp\left(-\frac{\pi\sqrt{-1}I(w(\overline{\rho}), \overline{\rho})}{l+1}\right) \\ &= (\sqrt{-1})^{l^2}(2(l+1))^{-\frac{1}{2}l} \prod_{\alpha \in \Delta_f^+} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}) \left(-\frac{\pi\sqrt{-1}}{l+1}\overline{\rho}\right) \\ &= 2^{l^2-\frac{1}{2}l}(l+1)^{-\frac{1}{2}l} \prod_{\alpha \in \Delta_f^+} \sin\left(\frac{I(\overline{\rho}, \alpha)}{2(l+1)}\pi\right). \end{aligned}$$

Hence, by §1.4 and the well-known formula

$$\prod_{k=1}^{n-1} \sin\left(\frac{k}{n}\pi\right) = \frac{n}{2^{n-1}} \tag{10}$$

for $n \in \mathbb{Z}_{>1}$, it follows that

$$\begin{aligned} &\prod_{\alpha \in \Delta_f^+} \sin\left(\frac{I(\overline{\rho}, \alpha)}{2(l+1)}\pi\right) \\ &= \prod_{1 \le i < j \le l} \sin\left(\frac{j-i}{2(l+1)}\pi\right) \sin\left(\frac{j+i}{2(l+1)}\pi\right) \prod_{i=1}^{l} \sin\left(\frac{i}{l+1}\pi\right) \\ &= 2^{-l^2+\frac{1}{2}l} \cdot (l+1)^{\frac{1}{2}l}, \end{aligned}$$

thus we obtain

$$a(\Lambda_0, \Lambda_0) = 1. \tag{11}$$

Next, for level 2 case, we set

$$\lambda_i = \begin{cases} 2\Lambda_0 & i = 0, \\ \Lambda_i & 0 < i \le l. \end{cases}$$

By Remark 2.2, for $\overline{\lambda + \rho} = \sum_{i=1}^{l} m_i \varepsilon_i$ and $\overline{\mu + \rho} = \sum_{i=1}^{l} n_i \varepsilon_i$, we have

$$\sum_{w \in \mathfrak{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z})^l} \varepsilon(w) \exp\left(-\frac{2\pi\sqrt{-1}I(\overline{\lambda + \rho}, w(\overline{\mu + \rho}))}{2l+3}\right)$$

$$= \sum_{\sigma\in\mathfrak{S}_l} \varepsilon(\sigma) \prod_{r=1}^{l} \left(\exp\left(-\frac{2\pi\sqrt{-1}}{2l+3} m_i n_{\sigma^{-1}(r)} \right) - \exp\left(\frac{2\pi\sqrt{-1}}{2l+3} m_r n_{\sigma^{-1}(r)} \right) \right)$$

$$= (-2\sqrt{-1})^l \sum_{\sigma\in\mathfrak{S}_l} \varepsilon(\sigma) \prod_{r=1}^{l} \left(2\sin\left(\frac{2m_r n_{\sigma^{-1}(r)}}{2l+3}\pi \right) \right)$$

$$= (-2\sqrt{-1})^l \det\left(\sin\left(\frac{2m_r n_s}{2l+3}\pi \right) \right)_{1\le r,s\le l}.$$

As $\overline{\Lambda_i+\rho} = \sum_{k\le i}(l+2-k)\varepsilon_k + \sum_{k>i}(l+1-k)\varepsilon_k$ and

$$\det\left(\sin\left(\frac{2m_r n_s}{2l+3}\pi \right) \right)_{1\le r,s\le l} = \det\left(\sin\left(\frac{2m_{l+1-r} n_{l+1-s}}{2l+3}\pi \right) \right)_{1\le r,s\le l},$$

to compute $a(\lambda_i,\lambda_j)$, it suffices to compute the $(-1)^{i+j}$ times $(l+1-i, l+1-j)$-entry of the cofactor matrix of the matrix

$$M_2 := \left(\sin\left(\frac{2rs}{2l+3}\pi \right) \right)_{1\le r,s\le l+1}.$$

With the aid of C_l-type denominator identity (cf. [Kr]), i.e., the identity

$$\det(X_i^j - X_i^{-j})_{1\le i,j\le l} = (\prod_{i=1}^{l} X_i)^{-l} \prod_{1\le i<j\le l} (1-X_iX_j)(X_i-X_j) \prod_{i=1}^{l} (X_i^2-1).$$

in $\mathbb{C}[X_1^{\pm1},\cdots,X_l^{\pm1}]$, one can show that

$$\det M_2 = (-1)^{\frac{1}{2}l(l+1)} 2^{-(l+1)} (2l+3)^{\frac{1}{2}(l+1)}.$$

Moreover, it can be shown that

$$M_2^2 = \frac{1}{4}(2l+3) I_{l+1}.$$

Combining these facts, we obtain

$$a(\lambda_i,\lambda_j) = \frac{2}{\sqrt{2l+3}} \cos\left(\frac{(2i+1)(2j+1)}{2(2l+3)}\pi \right) \qquad (0\le i,j\le l). \tag{12}$$

3.2. Modular transformation of Jacobians

For $0 \leq i \leq l$, set

$$J^i(\tau,z,t) := 2\pi\sqrt{-1}\begin{vmatrix} \chi_{\Lambda_0} & \partial_1\chi_{\Lambda_0} & \cdots & \partial_l\chi_{\Lambda_0} \\ 2\chi_{\lambda_{i_1}} & \partial_1\chi_{\lambda_{i_1}} & \cdots & \partial_l\chi_{\lambda_{i_1}} \\ \vdots & \vdots & & \vdots \\ 2\chi_{\lambda_{i_l}} & \partial_1\chi_{\lambda_{i_l}} & \cdots & \partial_l\chi_{\lambda_{i_l}} \end{vmatrix} \qquad (\tau,z,t) \in \mathbb{H}\times\mathfrak{h}_f\times\mathbb{C},$$

where $i_1, i_2, \cdots i_l$ are integers such that $i_1 < i_2 < \cdots < i_l$ and $\{i_1, i_2, \cdots, i_l\} = \{0, 1, \cdots, l\} \setminus \{i\}$. Note that

$$J^0(\tau,z,t) = J(\tau,z,t).$$

Since these determinants are W anti-invariant and their degrees are $\sum_{i=0}^{l} a_i^\vee = 2l+1$, it follows that $J^i(\tau,z,t) \in \widetilde{Th}^-_{2l+1} = \mathcal{O}_{\mathbb{H}}A_\rho$, i.e., there exist holomorphic functions $F^i \in \mathcal{O}_{\mathbb{H}}$ such that

$$J^i(\tau,z,t) = F^i(\tau)A_\rho(\tau,z,t). \tag{13}$$

Below, we determine these holomorphic functions $\{F^i(\tau)\}_{0\leq i\leq l}$ explicitly. For this purpose, we compute the modular transformations of $J^i(\tau,z,t)$.

Let $\lambda \in P_+$. Recall that, by Theorem 2.1, there exists a unitary matrix $(a(\lambda,\mu))_{\lambda,\mu\in P_+^{\langle c,\lambda\rangle} \bmod \mathbb{C}\delta}$ such that

$$\chi_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{I^*(z,z)}{2\tau}\right) = \sum_{\mu\in P_+^{\langle c,\lambda\rangle}} a(\lambda,\mu)\chi_\mu(\tau,z,t).$$

Differentiating both sides of this formula, we obtain

$$\begin{aligned} &(\partial_i\chi_\lambda)\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{I^*(z,z)}{2\tau}\right) \\ &= \tau \sum_{\mu\in P_+^{\langle c,\lambda\rangle}} a(\lambda,\mu)(\partial_i\chi_\mu)(\tau,z,t) \\ &\quad + 2\pi\sqrt{-1}I^*(z,\alpha_i^\vee)\langle c,\lambda\rangle \sum_{\mu\in P_+^{\langle c,\lambda\rangle}} a(\lambda,\mu)\chi_\mu(\tau,z,t), \end{aligned}$$

for any $0 < i \leq l$.

Set $M_S := (a(\lambda_i, \lambda_j))_{0\le i,j\le l}$. Let $\tilde{M}_S = \left((\tilde{M}_S)_{i,j}\right)_{0\le i,j\le l}$ be the cofactor matrix of M_S. For $0 \le i \le l$, let $0 \le i_1 < i_2 < \cdots < i_l$ be non-negative integers such that $\{i_1, i_2, \cdots, i_l\} = \{0, 1, \cdots, l\} \setminus \{i\}$. By direct calculation, one obtains

$$\begin{aligned}
&J^i\left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{I^*(z,z)}{2\tau}\right)\\
&= 2\pi\sqrt{-1}a(\Lambda_0, \Lambda_0)\tau^l\\
&\times \begin{vmatrix}
\chi_{\Lambda_0} & \partial_1\chi_{\Lambda_0} & \cdots & \partial_l\chi_{\Lambda_0}\\
2\sum_j a(\lambda_{i_1}, \lambda_j)\chi_{\lambda_j} & \sum_j a(\lambda_{i_1}, \lambda_j)\partial_1\chi_{\lambda_j} & \cdots & \sum_j a(\lambda_{i_1}, \lambda_j)\partial_l\chi_{\lambda_j}\\
\vdots & \vdots & & \vdots\\
2\sum_j a(\lambda_{i_l}, \lambda_j)\chi_{\lambda_j} & \sum_j a(\lambda_{i_l}, \lambda_j)\partial_1\chi_{\lambda_j} & \cdots & \sum_j a(\lambda_{i_l}, \lambda_j)\partial_l\chi_{\lambda_j}
\end{vmatrix}\\
&= \tau^l \sum_{j=0}^{l} (-1)^{i+j}(\tilde{M}_S)_{i,j} J^j(\tau, z, t).
\end{aligned}$$

On the other hand, it can be checked that $M_S^2 = I_{l+1}$. Recall the super denominator identity of type $B(0,n)$, i.e., the next identity in $\mathbb{C}[X_1^{\pm\frac{1}{2}}, \cdots X_n^{\pm\frac{1}{2}}]$:

$$\begin{aligned}
&\det(X_i^{j-\frac{1}{2}} + X_i^{-j+\frac{1}{2}})_{1\le i,j\le n}\\
&= (\prod_{i=1}^{n} X_i)^{-n+\frac{1}{2}} \prod_{1\le i<j\le n} (1 - X_iX_j)(X_i - X_j) \prod_{i=1}^{n}(1 + X_i).
\end{aligned} \tag{14}$$

With the aid of this formula, we can show

Lemma 3.1. $\det M_S = (-1)^{\binom{l+1}{2}}$.

Proof. Setting $\omega = \exp\left(\frac{2\pi\sqrt{-1}}{2l+3}\right)$, $n = l+1$ and $X_i = \omega^{i-\frac{1}{2}}$ $(1 \le i \le l+1)$ in (14), we have

$$\begin{aligned}
&(2l+3)^{\frac{1}{2}(l+1)} \det(M_s)\\
&= \det\left(\omega^{(i-\frac{1}{2})(j-\frac{1}{2})} + \omega^{-(i-\frac{1}{2})(j-\frac{1}{2})}\right)_{1\le i,j\le l+1}\\
&= \left(\prod_{i=1}^{l+1} \omega^{i-\frac{1}{2}}\right)^{-(l+\frac{1}{2})} \prod_{1\le <i<j\le l+1} (1 - \omega^{i+j-1})(\omega^{i-\frac{1}{2}} - \omega^{j-\frac{1}{2}}) \prod_{i=1}^{l+1}(1 + \omega^{i-\frac{1}{2}}).
\end{aligned}$$

Now, by direct computation, it follows that

$$\begin{aligned}
&\left(\prod_{i=1}^{l+1}\omega^{i-\frac{1}{2}}\right)^{-(l+\frac{1}{2})}\prod_{1\leq<i<j\leq l+1}(1-\omega^{i+j-1})(\omega^{i-\frac{1}{2}}-\omega^{j-\frac{1}{2}})\prod_{i=1}^{l+1}(1+\omega^{i-\frac{1}{2}})\\
=&(-1)^{\binom{l+1}{2}}\cdot 2^{(l+1)^2}\\
&\times\prod_{1\leq i<j\leq l+1}\sin\left(\frac{i+j-1}{2l+3}\pi\right)\sin\left(\frac{j-i}{2l+3}\pi\right)\prod_{i=1}^{l+1}\cos\left(\frac{2i-1}{2l+3}\cdot\frac{\pi}{2}\right)\\
=&(-1)^{\binom{l+1}{2}}\cdot 2^{(l+1)^2}\left(\prod_{i=1}^{l+1}\sin\left(\frac{i}{2l+3}\pi\right)\right)^{l+1}.
\end{aligned}$$

By (10), we obtain

$$\begin{aligned}
\det(M_S)&=(2l+3)^{-\frac{1}{2}(l+1)}\cdot(-1)^{\binom{l+1}{2}}\cdot 2^{(l+1)^2}\left(\prod_{i=1}^{2l+2}\sin\left(\frac{i}{2l+3}\pi\right)\right)^{\frac{1}{2}(l+1)}\\
&=(-1)^{\binom{l+1}{2}}.
\end{aligned}$$

□

Hence, we obtain

$$\begin{aligned}
(\tilde{M}_S)_{i,j}&=(-1)^{\binom{l+1}{2}}(M_S)_{i,j}\\
&=(-1)^{\binom{l+1}{2}}a(\lambda_i,\lambda_j)\\
&=(-1)^{\binom{l+1}{2}}\frac{2}{\sqrt{2l+3}}\cos\left(\frac{(2i+1)(2j+1)}{2(2l+3)}\pi\right).
\end{aligned}$$

Thus, we obtain the next formula:

$$\begin{aligned}
(15)\quad&(-1)^iJ^i|_S(\tau,z,t)\\
&=(-1)^iJ^i\left(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{I^*(z,z)}{2\tau}\right)\\
&=\tau^l(-1)^{\binom{l+1}{2}}\frac{2}{\sqrt{2l+3}}\sum_{j=0}^{l}\cos\left(\frac{(2i+1)(2j+1)}{2(2l+3)}\pi\right)(-1)^jJ^j(\tau,z,t).
\end{aligned}$$

By definition, the conformal anomaly of the weights $\Lambda_0,\lambda_i(0\leq i\leq l)$ are given by

$$m_{\Lambda_0}=-\frac{1}{24}l,\qquad m_{\lambda_i}=\frac{1}{2(2l+3)}(2(l+1)i-i^2)-\frac{l(l+1)}{6(2l+3)},$$

which implies

$$m_{\Lambda_0} + \sum_{i=0}^{l} m_{\lambda_i} = \frac{1}{24} l(2l+1).$$

Thus, we obtain the next formula:

(16)
$$\begin{aligned}(-1)^i J^i|_{T^2}(\tau, z, t) &= (-1)^i J^i(\tau+2, z, t) \\ &= \exp\left(4\pi\sqrt{-1}\left(\frac{1}{24} l(2l+1) - m_{\lambda_i}\right)\right)(-1)^i J^i(\tau, z, t).\end{aligned}$$

3.3. Leading terms of Jacobians

In this subsection, we compute the leading degree of each Jacobian $J^i(\tau, z, t)$ $(0 \le i \le l)$ as a q-series, where $q := e^{\pi\sqrt{-1}\tau}$. Let

$$\begin{aligned}P^f &= \{\, \lambda \in \mathfrak{h}_f^* \,|\, \langle h_i, \lambda\rangle \in \mathbb{Z} \quad 1 \le \forall\, i \le l \,\} \\ P_+^f &= \{\, \lambda \in \mathfrak{h}_f^* \,|\, \langle h_i, \lambda\rangle \in \mathbb{Z}_{\ge 0} \quad 1 \le \forall\, i \le l \,\}\end{aligned}$$

be the set of integral (resp. dominant integral) weights of $\mathfrak{h}_f$. The Weyl group acts on P^f, hence on its group algebra

$$\mathbb{C}[P^f] = \left\{ \sum_{\lambda \in P^f} c_\lambda e^\lambda \;\middle|\; \begin{array}{ll} \text{i)} & c_\lambda \in \mathbb{C}, \\ \text{ii)} & \sharp\{\lambda \,|\, c_\lambda \ne 0\} < \infty. \end{array} \right\}$$

For $\lambda \in P^f$, set $a_\lambda := \sum_{w \in W_f} \varepsilon(w) e^{w(\lambda)}$. The set $\{a_{\lambda+\overline{\rho}}\}_{\lambda \in P_+^f}$ spans the set of set of W_f anti-invariants $\mathbb{C}[P^f]^{-W_f}$ and it satisfies $a_{w(\lambda)} = \varepsilon(w) a_\lambda$. For $\lambda \in P^f$, set $\chi_\lambda^f := \dfrac{a_{\lambda+\overline{\rho}}}{a_{\overline{\rho}}}$. It follows that $\chi_\lambda^f \in \mathbb{C}[P^f]^{W_f}$ is the Weyl character formula for the finite root system Δ_f when $\lambda \in P_+^f$.

Now, regard e^λ $(\lambda \in P^f)$ as functions on $\mathfrak{h}_f$ defined by $h \mapsto e^{\langle h, \lambda\rangle}$. We rewrite the anti-invariant $A_{\Lambda+\rho}$ $(\Lambda \in P_k^+)$ of the affine Weyl group W:

$$\begin{aligned}&e^{-2(k+2l+1)\pi\sqrt{-1}t} A_{\Lambda+\rho}(\tau, z, t) \\ &= \sum_{w \in W_f} \varepsilon(w) \sum_{\alpha \in M} q^{\frac{1}{2(k+2l+1)}\left|(k+2l+1)\alpha + w(\overline{\Lambda}+\overline{\rho})\right|^2} e^{2\pi\sqrt{-1}\langle z, (k+2l+1)\alpha + w(\overline{\Lambda}+\overline{\rho})\rangle} \\ &= \sum_{\alpha \in M} q^{\frac{1}{2(k+2l+1)}\left|(k+2l+1)\alpha + \overline{\Lambda}+\overline{\rho}\right|^2} \sum_{w \in W_f} \varepsilon(w) e^{2\pi\sqrt{-1}\langle z, w((k+2l+1)\alpha + \overline{\Lambda}+\overline{\rho})\rangle}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\gamma\in M+\frac{\overline{\Lambda}+\overline{\rho}}{k+2l+1}} q^{\frac{1}{2}(k+2l+1)I(\gamma,\gamma)} \sum_{w\in W_f} \varepsilon(w) e^{2\pi\sqrt{-1}\langle z, w((k+2l+1)\gamma)\rangle} \\
&= \sum_{\gamma\in M+\frac{\overline{\Lambda}+\overline{\rho}}{k+2l+1}} a_{(k+2l+1)\gamma}(2\pi\sqrt{-1}z) q^{\frac{1}{2}(k+2l+1)I(\gamma,\gamma)},
\end{aligned}
$$

where $|\beta|^2 := I(\beta,\beta)$ for $\beta \in \mathfrak{h}_f^*$. In particular, for $\Lambda = 0, \Lambda_0, \lambda_i$ $(0 \le i \le l)$, the first few terms of this formula are given as follows.

$(\Lambda = 0)$:

$$
\begin{aligned}
&e^{-2(2l+1)\pi\sqrt{-1}t} q^{-\frac{1}{12}l(l+1)} A_\rho(\tau,z,t) \\
&\quad = a_{\overline{\rho}}(2\pi\sqrt{-1}z) - a_{\overline{\Lambda}_1+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{1}{2}} + a_{\overline{\Lambda}_1+\overline{\Lambda}_2+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{3}{2}} \\
&\qquad\qquad - a_{2\overline{\Lambda}_2+\overline{\rho}}(2\pi\sqrt{-1}z) q^2 + O(q^{\frac{5}{2}}),
\end{aligned}
$$

$(\Lambda = \Lambda_0)$:

$$
\begin{aligned}
&e^{-4(l+1)\pi\sqrt{-1}t} q^{-\frac{1}{24}l(2l+1)} A_{\Lambda_0+\rho}(\tau,z,t) \\
&\quad = a_{\overline{\rho}}(2\pi\sqrt{-1}z) - a_{2\overline{\Lambda}_1+\overline{\rho}}(2\pi\sqrt{-1}z) q + a_{2\overline{\Lambda}_1+\overline{\Lambda}_2+\overline{\rho}}(2\pi\sqrt{-1}z) q^2 + O(q^3),
\end{aligned}
$$

$(\Lambda = 2\Lambda_0)$:

$$
\begin{aligned}
&e^{-2(2l+3)\pi\sqrt{-1}t} q^{-\frac{l(l+1)(2l+1)}{12(2l+3)}} A_{2\Lambda_0+\rho}(\tau,z,t) \\
&\quad = a_{\overline{\rho}}(2\pi\sqrt{-1}z) - a_{3\overline{\Lambda}_1+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{3}{2}} + a_{3\overline{\Lambda}_1+\overline{\Lambda}_2+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{5}{2}} + O(q^{\frac{7}{2}}),
\end{aligned}
$$

$(\Lambda = \lambda_i, i \ne 0)$:

$$
\begin{aligned}
&e^{-2(2l+3)\pi\sqrt{-1}t} q^{-\frac{l(l+1)(2l+1)}{12(2l+3)} - \frac{i(2(l+1)-i)}{2(2l+3)}} A_{\Lambda_i+\rho}(\tau,z,t) \\
&\quad = a_{\overline{\Lambda}_i+\overline{\rho}}(2\pi\sqrt{-1}z) - a_{\overline{\Lambda}_1+\overline{\Lambda}_i+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{1}{2}} \\
&\qquad + \begin{cases} a_{2(\overline{\Lambda}_1+\overline{\Lambda}_2)+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{5}{2}} + O(q^3) & i = 1 \\ a_{2\overline{\Lambda}_1+\overline{\Lambda}_2+\overline{\Lambda}_i+\overline{\rho}}(2\pi\sqrt{-1}z) q^{\frac{3}{2}} + O(q^2) & i > 1. \end{cases}
\end{aligned}
$$

Thus, the normalized characters have the next expansions:

$$
\begin{aligned}
&e^{-4\pi\sqrt{-1}t} q^{\frac{l(l+1)}{6(2l+3)}} \chi_{2\Lambda_0}(\tau,z,t) \\
&\quad = 1 + \chi^f_{\overline{\Lambda}_1}(2\pi\sqrt{-1}z) q^{\frac{1}{2}} + (\chi^f_{\overline{\Lambda}_1})^2(2\pi\sqrt{-1}z) q
\end{aligned}
$$

$$+\left((\chi^f_{\overline{\Lambda_1}})^3-\chi^f_{\overline{3\Lambda_1}}-\chi^f_{\overline{\Lambda_1+\Lambda_2}}\right)(2\pi\sqrt{-1}z)q^{\frac{3}{2}}+O(q^2)$$

and, for $0<i\le l$,

$$\begin{aligned}e^{-4\pi\sqrt{-1}t}&q^{\frac{3i(i-2(l+1))+l(l+1)}{6(2l+3)}}\chi_{\lambda_i}(\tau,z,t)\\&=\chi^f_{\overline{\Lambda_i}}(2\pi\sqrt{-1}z)+(\chi^f_{\overline{\Lambda_1}}\chi^f_{\overline{\Lambda_i}}-\chi^f_{\overline{\Lambda_1+\Lambda_i}})(2\pi\sqrt{-1}z)q^{\frac{1}{2}}\\&\qquad+\left(\chi^f_{\overline{\Lambda_1}}(\chi^f_{\overline{\Lambda_1}}\chi^f_{\overline{\Lambda_i}}-\chi^f_{\overline{\Lambda_1+\Lambda_i}})\right)(2\pi\sqrt{-1}z)q+O(q^{\frac{3}{2}}).\end{aligned}$$

Notice that $\chi^f_{\overline{\Lambda_1}}\chi^f_{\overline{\Lambda_i}}-\chi^f_{\overline{\Lambda_1+\Lambda_i}}=\chi^f_{\overline{\Lambda_{i-1}}}+\chi^f_{\overline{\Lambda_{i+1}}}$ where we set $\chi^f_{\overline{\Lambda_{l+1}}}=0$.

Now, we analyze the first few terms of χ_{Λ_0}. Recall that the character of the fundamental representation $L(\Lambda_0)$ has the special expression (cf. [KP]):

$$\mathrm{ch}L(\Lambda_0)=\frac{\sum_{\alpha\in M}e^{t_\alpha(\Lambda_0)}}{\prod_{r=1}^{\infty}(1-e^{-r\delta})^l}.$$

Hence, it follows from the Jacobi triple product identity that

$$\begin{aligned}&\chi_{\Lambda_0}(\tau,z,t)\\&=e^{2\pi\sqrt{-1}t}q^{-\frac{1}{24}l}\prod_{i=1}^{l}\left(\prod_{r>0}(1+q^{r-\frac{1}{2}}e^{2\pi\sqrt{-1}\langle z,\varepsilon_i\rangle})(1+q^{r-\frac{1}{2}}e^{-2\pi\sqrt{-1}\langle z,\varepsilon_i\rangle})\right).\end{aligned}$$

With this expression, one can derive the leading terms of the normalized character $\chi_{\Lambda_0}(\tau,z,t)$ as follows. For $r\in\mathbb{Z}_{>0}$, thanks to the W_f-invariance, it can be shown that

$$\begin{aligned}&\prod_{i=1}^{l}(1+q^{r-\frac{1}{2}}e^{2\pi\sqrt{-1}\langle z,\varepsilon_i\rangle})(1+q^{r-\frac{1}{2}}e^{-2\pi\sqrt{-1}\langle z,\varepsilon_i\rangle})\\&=q^{l(r-\frac{1}{2})}\left(\sum_{j=0}^{[\frac{l}{2}]}\chi^f_{\overline{\Lambda_{l-2j}}}(2\pi\sqrt{-1}z)+\sum_{i=1}^{l}(q^{i(r-\frac{1}{2})}+q^{-i(r-\frac{1}{2})})\sum_{j=0}^{[\frac{l-i}{2}]}\chi^f_{\overline{\Lambda_{l-i-2j}}}(2\pi\sqrt{-1}z)\right)\\&=\sum_{i=0}^{2l}\left(\sum_{j=0}^{[\frac{\min\{i,2l-i\}}{2}]}\chi^f_{\overline{\Lambda_{\min\{i,2l-i\}-2j}}}(2\pi\sqrt{-1}z)\right)q^{i(r-\frac{1}{2})},\end{aligned}$$

where we set $\chi_{\overline{\Lambda_0}=1}$ and $[x]$ for $x\in\mathbb{R}$ signifies the maximal integer $\le x$. Thus, the leading terms of

$$e^{-2(2l+1)\pi\sqrt{-1}t}q^{-(m_{\Lambda_0}-m_{\lambda_i}+\sum_{j=0}^{l}m_{\lambda_j})}J^i(\tau,z,t)$$

is given by the leading terms of

$$\begin{vmatrix}
1+O(q^{\frac{1}{2}}) & \partial_1\chi^f_{\Lambda_i}(2\pi\sqrt{-1}z)q^{\frac{i}{2}}+O(q^{\frac{i+1}{2}}) & \cdots & \partial_l\chi^f_{\Lambda_i}(2\pi\sqrt{-1}z)q^{\frac{i}{2}}+O(q^{\frac{i+1}{2}}) \\
2+O(q^{\frac{1}{2}}) & \partial_1\chi^f_{\Lambda_1}(2\pi\sqrt{-1}z)q+O(q^{\frac{3}{2}}) & \cdots & \partial_l\chi^f_{\Lambda_1}(2\pi\sqrt{-1}z)q+O(q^{\frac{3}{2}}) \\
2\chi^f_{\Lambda_1}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \partial_1\chi^f_{\Lambda_1}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \cdots & \partial_l\chi^f_{\Lambda_1}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) \\
\vdots & \vdots & & \vdots \\
2\chi^f_{\Lambda_{i-1}}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \partial_1\chi^f_{\Lambda_{i-1}}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \cdots & \partial_l\chi^f_{\Lambda_{i-1}}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) \\
2\chi^f_{\Lambda_{i+1}}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \partial_1\chi^f_{\Lambda_{i+1}}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \cdots & \partial_l\chi^f_{\Lambda_{i+1}}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) \\
\vdots & \vdots & & \vdots \\
2\chi^f_{\Lambda_l}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \partial_1\chi^f_{\Lambda_l}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}}) & \cdots & \partial_l\chi^f_{\Lambda_l}(2\pi\sqrt{-1}z)+O(q^{\frac{1}{2}})
\end{vmatrix}.$$

In particular, the leading degree with respect to q of this determinant is $\frac{1}{2}i$. Therefore, since

$$m_{\Lambda_0}-m_{\lambda_i}+\sum_{j=0}^{l}m_{\lambda_j}=\frac{(2i+1)^2}{8(2l+3)}+\frac{1}{24}(l-1)+\frac{1}{12}l(l+1)-\frac{1}{2}i,$$

we have

$$\begin{aligned}J^i(\tau,z,t) \;&\propto e^{2(2l+1)\pi\sqrt{-1}t}q^{\frac{(2i+1)^2}{8(2l+3)}+\frac{1}{24}(l-1)+\frac{1}{12}l(l+1)} \\ &\quad\times\det(\partial_i\chi^f_{\Lambda_j}(2\pi\sqrt{-1}z))_{1\le i,j\le l}(1+O(q^{\frac{1}{2}})).\end{aligned}\tag{17}$$

3.4. Functional equations on $\{F^i(\tau)\}_{0\le i\le l}$

By (13) and Remark 2.1, the equations (15), (16) and (17) imply the next equations:

$$\begin{aligned}&(-1)^iF^i\left(-\frac{1}{\tau}\right) \\ &\quad=\left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}l}\frac{2}{\sqrt{2l+3}}\sum_{j=0}^{l}\cos\left(\frac{(2i+1)(2j+1)}{2(2l+3)}\pi\right)(-1)^jF^j(\tau), \\ &(-1)^iF^i(\tau+2,z,t)=\exp\left(-4\pi\sqrt{-1}\left(m_{\lambda_i}+\frac{1}{24}l\right)\right)(-1)^iF^i(\tau),\end{aligned}\tag{18}$$

and also with the computation of the leading term of A_ρ in Subsection 3.3, one obtain

$$(-1)^iF^i(\tau)\propto q^{\frac{(2i+1)^2}{8(2l+3)}+\frac{1}{24}(l-1)}(1+O(q^{\frac{1}{2}})),\tag{19}$$

where $\propto$ means the left hand side is proportional to the right hand side up to a non-zero scalar factor.

4. Determination of Jaocbians

4.1. Some modular forms

For $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}/2m\mathbb{Z}$, let

$$\theta_{n,m}(\tau) := \sum_{k\in\mathbb{Z}} q^{m\left(k+\frac{n}{2m}\right)^2},$$

be the classical theta functions. Their modular transformations are given by

$$\begin{aligned} &\theta_{n,m}(\tau+1) = e^{\frac{n^2}{2m}\pi\sqrt{-1}}\theta_{n,m}(\tau), \\ &\theta_{n,m}\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{2m\sqrt{-1}}\right)^{\frac{1}{2}} \sum_{n'\in\mathbb{Z}/2m\mathbb{Z}} e^{-\frac{nn'}{m}\pi\sqrt{-1}}\theta_{n',m}(\tau). \end{aligned} \tag{20}$$

Here, we recall the Jacobi triple product identity:

$$\prod_{k=1}^{\infty}(1-p^k)(1-p^{k-1}w)(1-p^kw^{-1}) = \sum_{l\in\mathbb{Z}}(-1)^l p^{\binom{l}{2}} w^l. \tag{21}$$

The Dedekind eta-function is defined as follows:

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = \sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{3}{2}\left(m-\frac{1}{6}\right)^2}.$$

It is a weight $\frac{1}{2}$ modular form:

$$\eta(\tau+1) = e^{\frac{\pi\sqrt{-1}}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}}\eta(\tau).$$

Following [KP], for $M \in \mathbb{Z}_{>0}$ and $0 \le r < M$, set

$$F_r^{(M)}(\tau) := q^{\frac{1}{8M}(M-2r)^2} \prod_{\substack{n\equiv 0 \bmod M \\ n>0}} (1-q^n) \prod_{\substack{n\equiv r \bmod M \\ n>0}} (1-q^n) \prod_{\substack{n\equiv -r \bmod M \\ n>0}} (1-q^n).$$

By definition, $F_0^{(M)}(\tau) = \eta(M\tau)^3$, and for $0 < r < M$, the Jacobi triple product identity (21) gives

$$F_r^{(M)}(\tau) = \theta_{M-2r,2M}(\tau) - \theta_{M+2r,2M}(\tau).$$

In particular, one has

$$F_r^{(M)}(\tau+1) = e^{\frac{(M-2r)^2}{4M}\pi\sqrt{-1}} F_r^{(M)}(\tau).$$

Assume that M is odd (this is the only case we need). Then, its Jacobi transformation $\tau \mapsto -\frac{1}{\tau}$ is given as follows: for $0 < r < M$,

$$\begin{aligned} (22) \qquad & F_r^{(M)}\left(-\frac{1}{\tau}\right) \\ & = 2\left(\frac{\tau}{M\sqrt{-1}}\right)^{\frac{1}{2}} (-1)^{r-\frac{1}{2}(M+1)} \sum_{\substack{0<r'<M \\ r'\equiv 0 \bmod 2}} \sin\left(\frac{rr'}{M}\pi\right) e^{\frac{1}{2}r'\pi\sqrt{-1}} F_{\frac{r'}{2}}^{(M)}(\tau). \end{aligned}$$

Notice that, for $0 < r < M$, one has

$$F_{M-r}^{(M)}(\tau) = F_r^{(M)}(\tau).$$

4.2. Conclusion

It turns out that the functions $\{\eta(\tau)^{l-1} F_{l+1-i}^{(2l+3)}(\tau)\}_{0\le i\le l}$ enjoys the same properties as $\{(-1)^i F^i(\tau)\}_{0\le i\le l}$, i.e., (18). Thus, we see that

$$J^i(\tau,z,t) \propto (-1)^i \eta(\tau)^{l-1} F_{l+1-i}^{(2l+3)}(\tau) A_\rho(\tau,z,t).$$

In particular, since $\eta^{l-1}F_{l+1}^{(2l+3)}$ never vanishes on $\mathbb{H}$, we see that, for any $\tau \in \mathbb{H}$, the fundamental characters $\{\chi_{\Lambda_i}\}_{0\le i\le l}$ are algebraically independent. In particular, we have proved the validity of the first part of the Conjecture 2.1:

Theorem 4.1. *For any $\tau \in \mathbb{H}$, we have*

$$\widetilde{Th}^+\Big|_{Y_\tau} = \mathbb{C}[\iota_{Y_\tau}^*(\chi_{\Lambda_i})\ (0 \le i \le l)],$$

where $\iota_{Y_\tau} : Y_\tau \hookrightarrow Y$ is a natural embedding.

Indeed, both of them are $\mathbb{Z}$-graded $\mathbb{C}$-algebras whose Poincaré series (cf. (9)) are the same.

Remark 4.1. *Let $L(c,h)$ be the irreducible highest weight module over the Virasoro algebra whose highest weight is (c,h). For each integer $0 \le i \le l$, set*

$$c_{2,2l+3} := 1 - \frac{3(2l+1)^2}{2l+3}, \qquad h_i := \frac{(2i+1)^2-(2l+1)^2}{8(2l+3)}.$$

We denote the normalized character $\mathrm{tr}_{L(c_{2,2l+3},h_i)}(q^{L_0-\frac{1}{24}c})$ *by* $\chi_{1,l+1-i}(\tau)$. *It turns out that* (*cf. [IK]*)

$$\chi_{1,l+1-i}(\tau) = \frac{F_{l+1-i}^{(2l+3)}(\tau)}{\eta(\tau)}.$$

In particular, this implies that

$$J^i(\tau,z,t) \propto (-1)^i \eta(\tau)^l \chi_{1,l+1-i}(\tau) A_\rho(\tau,z,t).$$

As for the second part of the Conjecture 2.1, we see that the $\mathcal{O}_{\mathbb{H}}$-algebra $\widetilde{Th}^+$ contains the $\mathcal{O}_{\mathbb{H}}$-algebra generated by $\{\chi_{\Lambda_i}\}_{0\le i\le l}$, that is isomorphic to a polynomial algebra, as a subalgebra. In addition, as a graded $\mathcal{O}_{\mathbb{H}}$-algebra, the Poincaré series of $\widetilde{Th}^+$ and $\mathcal{O}_{\mathbb{H}}[\chi_{\Lambda_i}\ (0\le i\le l)]$ coincide.

Let $\mathcal{K}_{\mathbb{H}}$ be the quotient field of $\mathcal{O}_{\mathbb{H}}$. One has

$$\mathcal{K}_{\mathbb{H}} \otimes_{\mathcal{O}_{\mathbb{H}}} \widetilde{Th}^+ \cong \mathcal{K}_{\mathbb{H}}[\chi_{\Lambda_i}\ (0\le i\le l)]; \qquad f\otimes F \longmapsto fF.$$

On the other hand, by Theorem 4.1, the above isomorphism restricts to

$$\mathcal{O}_{\mathbb{H},\tau} \otimes_{\mathcal{O}_{\mathbb{H}}} \widetilde{Th}^+ \cong \mathcal{O}_{\mathbb{H},\tau}[\chi_{\Lambda_i}\ (0\le i\le l)].$$

Since $\bigcap_{\tau\in\mathbb{H}} \mathcal{O}_{\mathbb{H},\tau} = \mathcal{O}_{\mathbb{H}}$, this implies

Theorem 4.2. $\widetilde{Th}^+ = \mathcal{O}_{\mathbb{H}}[\chi_{\Lambda_i}\ (0\le i\le l)]$.

Hence the second part of the Conjecture 2.1 is also valid.

Acknowledgement

At an early stage of this project, K.I. was partially supported by the program FY2018 JSPS Invitation. He is also partially supported by the French ANR (ANR project ANR-15-CE40-0012). Y.S. is partially supported by JSPS KAKENHI Grant Number 16K05055.

References

[BS] Bernstein I. N. and Schwarzman O., *Chevalley's theorem for complex crystallographic coxeter groups*, Funct. Anal. Appl. **12** (1978), 79–80. MR0515632

[BS1] Bernstein J. and Schwarzman O., *Complex crystallographic Coxeter groups and affine root system*, Jour. Nonlin. Math. Phys. **13** (2), (2006), 163–182. MR2238956

[BS2] Bernstein J. and Schwarzman O., *Chevalley's theorem for the complex crystallographic groups*, Jour. Nonlin. Math. Phys. **13** (3), (2006), 323–351. MR2249806

[Bour] N. Bourbaki, *Groupes et Algèbres de Lie, Chapître 4 à 6*, Hermann, 1968. MR0023803

[Car] Cartier P., *Théorie des caractères II. Détermination des caractères*, Sém. Sophus Lie **1** (1954–1955), Talk no. 19 et 20, 12 pp.

[C] Chandrasekharan K., *Elliptic Functions*, Grund. Math. Wiss. **281**, Springer-Verlag, 1985. MR0808396

[Ch] Chevalley C., *Invariants of finite groups generated by reflections*, Amer. Jour. Math. **77** (1955), 778–782. MR0072877

[GK] Gorelik M. and Kac V. G., *Characters of highest weight modules over affine Lie algebras are meromorphic functions*, Int. Meth. Res. Not. **2007** (2007), ID rnm079, 25 pages. MR2363305

[IK] Iohara K. and Koga Y., *Representation Theory of the Virasoro Algebra*, Springer Monograph in Math., 2011. MR2744610

[Kac] Kac V. G., *Infinite Dimensional Lie Algebras*, third edition, Cambridge Univ. Press, 1990. MR1104219

[KP] Kac V. G. and Peterson D. H., *Infinite dimensional Lie algebras, theta functions and modular forms*, Adv. Math. **53**, (1984), 125–264. MR0750341

[Kr] Krattenthaler C., *Advanced determinant calculus*, The Andrews Fetschrift, Sém. Lothar. Combin. **42**, (1999), Art. B42q, 67 pages. MR1701596

[L] Looijenga E., *Root systems and elliptic curves*, Invent. Math. **38**, (1976), 17–32. MR0466134

[MP] Moody R. V. and Pianzola A., *Lie Algebras with Triangular Decompositions*, Canadian Math. Soc. Ser. Monographs and Adv. Texts, Wiley-InterSci., 1995. MR1323858

[ST] Shephard G. C. and Todd J. A., *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274–304. MR0059914

Kenji Iohara
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 Boulevard du 11 Novembre 1918
F-69622 Villeurbanne cedex
France
E-mail: iohara@math.univ-lyon1.fr

Yoshihisa Saito
Department of Mathematics
Rikkyo University
3-34-1 Nishi-Ikebukuro Toshima-ku
Tokyo 171-8501
Japan
E-mail: yoshihisa@rikkyo.ac.jp; yosihisa@ms.u-tokyo.ac.jp