Pure and Applied Mathematics Quarterly
Volume 16, Number 2, 229–312, 2020

Notes on the universal elliptic KZB connection*

Richard Hain

To Eduard Looijenga, friend and colleague

Abstract: In this paper, we give an exposition of the elliptic KZB connection over the universal elliptic curve and use it to compute the limit mixed Hodge structure on the unipotent fundamental group of the first order Tate curve. We also give an explicit algebraic formula for the restriction of the elliptic KZB connection to the moduli space of non-zero abelian differentials on an elliptic curve.
Keywords: Eisenstein series, moduli of elliptic curves, mixed elliptic motives, Hodge theory, algebraic de Rham theory of fundamental groups, elliptic KZB connection.

Received June 14, 2016. Accepted March 12, 2017.
*Supported in part by the NSF through grants DMS-0706955 and DMS-1005675.

Introduction

The universal elliptic KZB[1] connections generalize the connections defined by the physicists Knizhnik and Zamolodchikov [22] in genus 0 and Bernard [1] in genus 1. For each $n \geq 1$, the universal elliptic KZB connection is an integrable connection on a bundle of pronilpotent Lie algebras over $\mathcal{M}_{1,1+n}$, the moduli space of $(n+1)$-pointed smooth projective curves of genus 1, regarded as a stack over $\mathbb{C}$. The fiber of the connection over the point corresponding to

[1]For Knizhnik–Zamolodchikov–Bernard.

the $(n+1)$-pointed genus 1 curve $(E;0,x_1,\dots,x_n)$ is the Lie algebra of the unipotent completion of $\pi_1^{\mathrm{un}}(C_n(E',(x_1,\dots,x_n)))$, where $E' := E - \{0\}$ and

$$C_n(E') = (E')^n - \text{fat diagonal}$$

is the configuration space of n points in E'. Explicit constructions of the universal elliptic KZB connection were given by Calaque, Enriquez and Etingof (for all $n \geq 1$) in [3] and, independently, by Levin and Racinet (for $n = 1$ only) in [24]. We will generally drop the adjectives "universal" and "elliptic". Since we consider only universal elliptic KZB connections, there should be no confusion.

In mathematics, KZB connections play a role in representation theory [3] and in the study of periods of mixed elliptic motives [9, 18]. In this paper, we focus on the KZB connection over $\mathcal{M}_{1,2}$, the $n = 1$ case. This is the most important in the theory of mixed elliptic motives.

In this paper, we give a complete exposition of the construction of the KZB connection in the $n = 1$ case. We use it to compute the limit mixed Hodge structure (MHS) on the Lie algebra of the unipotent fundamental group of the first order Tate curve $E_{\partial/\partial q}$ (i.e., the restriction of the universal elliptic curve over the q-disk to the tangent vector $\partial/\partial q$ at $q = 0$) with its identity removed and with a canonical tangential base point $\partial/\partial w$ at its identity.[2] In particular, we show that its periods are multiple zeta values. We also use it to derive certain formulas which relate this limit MHS to the MHS on the unipotent fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}$, which we regard as the nodal cubic with its singular point and identity element removed. We also show that, when restricted to $\mathcal{M}_{1,\vec{1}}$, the moduli space of elliptic curves with a non-zero abelian differential (equivalently, a non-zero tangent vector at the identity), the elliptic KZB connection is defined over $\mathbb{Q}$ and we give an explicit formula for this connection in terms of the coordinates on $\mathcal{M}_{1,\vec{1}/\mathbb{Q}}$.

This paper grew out of notes from a seminar at Duke University during the summer of 2007 in which we read the paper of Levin and Racinet [24]. Because this paper is derived from lecture notes, the style is sometimes a little expansive and background which might otherwise be omitted is included.

The paper is in four parts. The first contains some background material. The second part is a complete exposition of the elliptic KZB equation. This exposition follows the approach of Levin and Racinet, which expresses the elliptic KZB connection in terms of Kronecker's Jacobi form, $F(\xi,\eta,\tau)$, [21]

[2] Limit mixed Hodge structures are reviewed in Section 15. Tangential base points and their relationship to limit MHSs are explained in Section 16.

and Eisenstein series. This function F was rediscovered by Zagier in [33] and can be expressed in terms of classical theta functions. Zagier [33] showed that this Jacobi form is a generating function for the periods of modular forms of level 1, a fact whose relevance is still not completely understood in the context of mixed elliptic motives.

During the seminar, we were unable to verify some of the computations in the Levin-Racinet paper without modifying several factors of automorphy. Such differences may have arisen because of differing conventions. This paper uses the modified factors of automorphy. Because of this, and because it is not likely that the paper of Levin and Racinet will be published, complete proofs of the modular behaviour and integrability of the elliptic KZB connection are given in Part 2.

Levin and Racinet define Hodge and weight filtrations on the fibers of the elliptic KZB connection. In Part 3, we prove that with these filtrations, the KZB connection is an admissible variation of MHS isomorphic to the canonical variation of MHS whose fiber over $[E,x]$ is the Lie algebra of $\pi_1^{\mathrm{un}}(E-\{0\},x)$ with its canonical MHS. This allows us to explicitly compute the limit MHS on the fiber associated to a tangent vector at the identity of the nodal cubic. In particular, we prove that its periods are multiple zeta values. The explicit formula for the KZB connection allows us to compute a formula for the canonical map of Lie algebras induced by the homomorphism

$$\pi_1^{\mathrm{un}}(\mathbb{P}^1-\{0,1,\infty\},\partial/\partial w) \to \pi_1^{\mathrm{un}}(E'_{\partial/\partial q},\partial/\partial w)$$

and also for the logarithm of the monodromy action

$$\pi_1^{\mathrm{un}}(E'_{\partial/\partial q},\partial/\partial w) \to \pi_1^{\mathrm{un}}(E'_{\partial/\partial q},\partial/\partial w).$$

Here w is the parameter in $\mathbb{P}^1-\{0,1,\infty\}$ and q is the coordinate $\exp(2\pi i\tau)$ in the q-disk.

For applications to elliptic and modular motives, it is important to know that the elliptic KZB connection is defined over $\mathbb{Q}$. Levin and Racinet [24] state this as a result and sketch a proof of it. In Part 4 we elaborate on their computations and give an explicit formula for the restriction of the KZB connection to $\mathcal{M}_{1,\vec{1}}$ and show that its canonical extension to $\overline{\mathcal{M}}_{1,\vec{1}}$ is also defined over $\mathbb{Q}$. The story for $\overline{\mathcal{M}}_{1,2}$ is more complicated and has been verified by Ma Luo. It will appear in his Duke PhD thesis.

Background material on the topology of moduli spaces of elliptic curves (viewed as orbifolds) and their associated mapping class groups is not included. It can be found, for example, in [15]. The books of Serre [27] and Silverman [29] are excellent references for background material on modular forms.

0.1. Some conventions

We use the topologist's convention for path multiplication: if $\alpha, \beta : [0,1] \to X$ are paths in a topological space with $\alpha(1) = \beta(0)$, then $\alpha\beta : [0,1] \to X$ is the path obtained by first traversing α and then β.

The adjoint action of an element u of the enveloping algebra of a Lie algebra $\mathfrak{g}$ on an element x of $\mathfrak{g}$ will often be denoted by $u \cdot x$. This will be extended to power series u of elements of $\mathfrak{g}$ when it makes sense. For example if $\mathbf{t} \in \mathfrak{g}$, then

$$e^{\mathbf{t}} \cdot x = \sum_{n=0}^{\infty} \mathrm{ad}_{\mathbf{t}}^n(x)/n!$$

If δ is a derivation of $\mathfrak{g}$, then $\delta(f \cdot u) = \delta(f) \cdot u + f \cdot \delta(u)$.

We will be sloppy and denote the generic element of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So, unless otherwise mentioned, the entries of γ are a, b, c and d.

We will use the terms "local system" and "locally constant sheaf" interchangeably. Local systems of vector spaces over a smooth manifold correspond to vector bundles with a flat (i.e., integrable) connection. Sometimes we will abuse terminology and refer to such a local system as a "flat bundle".

Part 1. Background

In this part, we present the background needed to understand the universal elliptic KZB connection. In parts 3 and 4, the reader will also need to be familiar with the basics of Deligne's theory of mixed Hodge structures. Introductory references are listed in Section 15.

1. The universal elliptic curve

The material in this section is standard. We will assume that the reader is familiar with the construction of $\mathcal{M}_{1,1}$ as the orbifold quotient of the upper half plane

$$\mathfrak{h} := \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$$

by $\mathrm{SL}_2(\mathbb{Z})$, the construction of its Deligne-Mumford compactification $\overline{\mathcal{M}}_{1,1}$ (as an orbifold), the construction of the standard line bundle $\mathcal{L}$ over $\mathcal{M}_{1,1}$, and its extension $\overline{\mathcal{L}}$ to $\overline{\mathcal{M}}_{1,1}$. Denote their kth powers by $\mathcal{L}_k$ and $\overline{\mathcal{L}}_k$, respectively. In particular, their inverses will be denoted by $\mathcal{L}_{-1}$ and $\overline{\mathcal{L}}_{-1}$. This material is classical and can be found, for example, in the first four sections of [15].

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ by right multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} m & n \end{pmatrix} \mapsto \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Denote the corresponding semi-direct product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ by Γ. This is the set $\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$ with multiplication:

$$(\gamma_1, v_1)(\gamma_2, v_2) = (\gamma_1\gamma_2, v_1\gamma_2 + v_2)$$

where $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ and $v_1, v_2 \in \mathbb{Z}^2$.

The group Γ acts on $X := \mathbb{C} \times \mathfrak{h}$ on the left:

$$(m, n) : (\xi, \tau) \mapsto \left(\xi + \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \tau\right)$$

and

$$\gamma : (\xi, \tau) \mapsto \left((c\tau + d)^{-1}\xi, \gamma\tau\right)$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

The quotient $\Gamma\backslash X$ is the universal elliptic curve $\mathcal{E}$; the map $\Gamma\backslash X \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ induced by the projection $X \to \mathfrak{h}$ is the projection $\mathcal{E} \to \mathcal{M}_{1,1}$.

The universal elliptic curve can be compactified using the Tate curve to obtain a proper orbifold map $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$ whose fiber over $q = 0$ is the nodal cubic. Its pullback to the q-disk $\mathbb{D}$, with the double point removed, is the quotient of $\mathbb{C}^* \times \mathbb{D}$ by the group action $\mathbb{Z} \times \mathbb{C}^* \times \mathbb{D} \to \mathbb{C}^* \times \mathbb{D}$ defined by

$$n : (w, q) \mapsto \begin{cases} (q^n w, q) & q \neq 0, \\ (w, q) & q = 0. \end{cases}$$

Note that, although this group action is not continuous, the quotient (endowed with the quotient topology) is Hausdorff and is a complex manifold. The fiber over $q = 0$ is the group $\mathbb{C}^*$. The zero section (aka, the identity section) passes through it at $w = 1$.

Proposition 1.1. *The normal bundle of the zero section of* $\overline{\mathcal{E}}$ *is* $\overline{\mathcal{L}}_{-1}$.

Proof. The line bundle $\mathcal{L}_{-1}$ is the quotient of $\mathbb{C} \times \mathfrak{h}$ by the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\xi, \tau) \mapsto \big((c\tau + d)^{-1}\xi, \gamma\tau\big).$$

The identity $\mathbb{C}\times\mathfrak{h} \to \mathbb{C}\times\mathfrak{h}$ is equivariant with respect to the natural inclusion $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and thus induces a quotient mapping $\mathcal{L}_{-1} \to \mathcal{E}$ that commutes with the projections to $\mathcal{M}_{1,1}$. This projection extends over $q = 0$. This is well known and follows from the result of Exercise 47 in [15, §5.2]. □

Corollary 1.2. *A neighbourhood of the zero section of* $\overline{\mathcal{L}}_{-1}$ *is biholomorphic with a neighbourhood of the identity section of* $\overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. □

Denote by L' the complex manifold obtained by removing the 0-section from a holomorphic line bundle L.

Corollary 1.3. *The moduli space* $\mathcal{M}'_{1,\vec{1}}$ *of pairs* $(E, \vec{v})$*, where* E *is a stable elliptic curve and* $\vec{v}$ *is a (possible vanishing) tangent vector at the identity is naturally isomorphic with* $\overline{\mathcal{L}}_{-1}$*. In particular, the moduli space of smooth elliptic curves and a non-zero tangent vector at the identity* $\mathcal{M}_{1,\vec{1}}$ *is isomorphic to* $\mathcal{L}'_{-1}$*.* □

1.1. Fundamental groups

A non-zero point x of an elliptic curve E determines (and is determined by) an orbifold map $[E, x] : \mathbb{C} \to \mathcal{E}'$.

Proposition 1.4. *The fundamental group of* $\mathcal{E}'$ *with respect to the base point* $[E, x]$ *is an extension*

$$1 \to \pi_1(E', x) \to \pi_1(\mathcal{E}', [E, x]) \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

In particular, it is isomorphic to an extension of $\mathrm{SL}_2(\mathbb{Z})$ *by a free group of rank* 2*.*

Proof. The function $\mathbb{R}^2 \times \mathfrak{h} \to \mathbb{C} \times \mathfrak{h}$ defined by $(u, v, \tau) \mapsto (u + v\tau, \tau)$ is a homeomorphism. It induces a homeomorphism $(\mathbb{R}/\mathbb{Z})^2 \times \mathfrak{h} \to \mathcal{E}_\mathfrak{h}$ which restricts to give a homeomorphism

$$\big((\mathbb{R}/\mathbb{Z})^2 - \{0\}\big) \times \mathfrak{h} \to \mathcal{E}'_\mathfrak{h},$$

where $\mathcal{E}_\mathfrak{h}$ denotes the universal elliptic curve $\mathbb{Z}^2\backslash(\mathbb{C}\times\mathfrak{h})$ over $\mathfrak{h}$ and $\mathcal{E}'_\mathfrak{h}$ denotes $\mathcal{E}_\mathfrak{h}$ with the 0-section removed. It follows that $\mathcal{E}'_\mathfrak{h}$ is homotopy equivalent to each of its fibers E'_τ. In particular, the inclusion $(E', x) \to (\mathcal{E}'_{\mathfrak{h}}, (E, x))$ induces an isomorphism on fundamental groups.

The result follows from covering space theory as the covering $\mathcal{E}'_{\mathfrak{h}} \to \mathcal{E}'$ is Galois with Galois group $\mathrm{SL}_2(\mathbb{Z})$. □

Corollary 1.5. *For each point $[E, x]$ of $\mathcal{E}'$, there is a natural action of $\pi_1(\mathcal{E}', [E, x])$ on $\pi_1(E', x)$.*

Proof. Since $\pi_1(E', x)$ is a normal subgroup of $\pi_1(\mathcal{E}', [E, x])$, one has the conjugation action $g : \gamma \mapsto g\gamma g^{-1}$ of $\pi_1(\mathcal{E}', [E, x])$ on $\pi_1(E', x)$. □

Denote the $\mathbb{C}^*$ bundle obtained from $\mathcal{L}_k$ by removing the 0-section by $\mathcal{L}'_k$. Its (orbifold) fundamental group is a central extension

$$0 \to \mathbb{Z} \to \pi_1(\mathcal{L}'_k, *) \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

Remark 1.6. It is well-known that $\pi_1(\mathcal{L}'_{-1})$ is naturally isomorphic to each of the following groups:

- (i) the braid group B_3 on 3-strings;
- (ii) the fundamental group of $\mathbb{C}^2$ with the cusp $x^2 = y^3$ removed;
- (iii) the fundamental group of the complement of the trefoil knot;
- (iv) the inverse image $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ in the universal covering group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$.

Details can be found, for example, in [15].

Proposition 1.1 implies that if E is an elliptic curve and $\vec{v}$ is a non-zero tangent vector at $0 \in E$, there is a natural homomorphism

$$\pi_1(\mathcal{L}'_{-1}, [E, \vec{v}]) \to \pi_1(\mathcal{E}', [E, \vec{v}]).$$

Composing this with the action above we obtain an action

$$\pi_1(\mathcal{L}'_{-1}, [E, \vec{v}]) \to \mathrm{Aut}\, \pi_1(E', \vec{v}).$$

Denote the element of $\pi_1(E', \vec{v})$ that corresponds to moving once around the identity in the positive direction by c_o. Denote by z_o the image in $\pi_1(\mathcal{L}'_{-1}, [E, \vec{v}]) \cong \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of the *positive* generator of the fundamental group of the fiber $\mathcal{L}'_{-1,E} \cong \mathbb{C}^*$ over $[E]$ of the projection $\mathcal{L}'_{-1} \to \mathcal{M}_{1,1}$.

Proposition 1.7. *This action of $\pi_1(\mathcal{L}'_{-1}, [E, \vec{v}])$ on $\pi_1(E', \vec{v})$ fixes c_o.*

Proof. Observe that c_o is the image of z_o under the continuous mapping $\mathcal{L}'_{-1} \to \mathcal{E}'$. The result follows as z_o is central in $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$. □

Since $\pi_1(\mathcal{L}'_{-1},[E,\vec{v}])$ acts on $\pi_1(E',\vec{v})$, we can form the semi-direct product

$$\pi_1(\mathcal{L}'_{-1},[E,\vec{v}]) \ltimes \pi_1(E',\vec{v}).$$

Lemma 1.8. *The element* $c_o^{-1}z_o$ *is central in* $\pi_1(\mathcal{L}'_{-1},[E,\vec{v}]) \ltimes \pi_1(E',\vec{v})$.

Proof. Note that z_o acts on $\pi_1(E',\vec{v})$ by conjugation by c_o. Since z_o is central in $\pi_1(\mathcal{L}'_{-1},[E,\vec{v}])$ and since each element of $\pi_1(\mathcal{L}'_{-1},[E,\vec{v}])$ fixes c_o, we see that $c_o^{-1}z_o$ commutes with each element of $\pi_1(\mathcal{L}'_{-1},[E,\vec{v}])$.

If $g \in \pi_1(E',\vec{v})$, then

$$gc_o^{-1}z_og^{-1} = gc_o^{-1}(z_og^{-1}z_o^{-1})z_o = gc_o^{-1}(c_og^{-1}c_o^{-1})z_o = c_o^{-1}z_o.$$

□

This semi-direct product can be realized as the fundamental group of the pullback $\mathcal{E}'_{\mathcal{L}}$ of $\mathcal{E}'$ to $\mathcal{L}'_{-1}$. This has a (continuous) section. Since $\mathcal{E}'_{\mathcal{L}}$ is a $\mathbb{C}^*$ covering of $\mathcal{E}'$, we obtain:

Proposition 1.9. *The kernel of the natural homomorphism*

$$\pi_1(\mathcal{E}'_{\mathcal{L}},[E,\vec{v}]) \cong \pi_1(\mathcal{L}'_{-1},[E,\vec{v}]) \ltimes \pi_1(E',\vec{v}) \to \pi_1(\mathcal{E}',[E,\vec{v}])$$

is the infinite cyclic subgroup generated by $c_o^{-1}z_o$. *This homomorphism induces an isomorphism*

$$\big(\pi_1(\mathcal{L}'_{-1},[E,\vec{v}]) \ltimes \pi_1(E',\vec{v})\big)/\langle c_o^{-1}z_o\rangle \to \pi_1(\mathcal{E}',[E,\vec{v}]).$$

□

In mapping class group notation, this result says that there is a natural isomorphism

$$\Gamma_{1,2} \cong \big(\Gamma_{1,\vec{1}} \ltimes \pi_1(E',\vec{v})\big)/\mathbb{Z}.$$

In the Hodge and Galois worlds, the copy of $\mathbb{Z}$ is a copy of $\mathbb{Z}(1)$.

1.2. The local system $\mathbb{H}$

This is the local system (i.e., locally constant sheaf) over $\mathcal{M}_{1,1}$ whose fiber over $[E] \in \mathcal{M}_{1,1}$ is $H_1(E;\mathbb{C})$. We identify it, via Poincaré duality $H_1(E) \to H^1(E)$, with the local system $R^1\pi_*\mathbb{C}$ over $\mathcal{M}_{1,1}$ associated to the universal elliptic curve $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$. This has fiber $H^1(E;\mathbb{C})$ over $[E] \in \mathcal{M}_{1,1}$.

We consider two ways of framing (i.e., trivializing) the pullback of $\mathbb{H}$ to $\mathfrak{h}$. Denote the universal elliptic curve over $\mathfrak{h}$ by $\mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$. It is the quotient of $\mathbb{C} \times \mathfrak{h}$ by the standard action of $\mathbb{Z}^2$ given above. The first homology of $E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ is naturally isomorphic to $\Lambda_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$. Let $\mathbf{a}, \mathbf{b}$ be the basis of $H_1(E_\tau; \mathbb{Z})$ that corresponds to the basis $1, \tau$ of Λ_τ.

Denote the dual basis of $H^1(E_\tau; \mathbb{C}) \cong \mathrm{Hom}(H_1(E_\tau), \mathbb{C})$ by $\check{\mathbf{a}}, \check{\mathbf{b}}$. Then, under Poincaré duality,

$$\check{\mathbf{a}} = -\mathbf{b} \text{ and } \check{\mathbf{b}} = \mathbf{a}.$$

Denote the element $d\xi$ of $H^1(E_\tau, \mathbb{C})$ by w_τ. Then

$$w_\tau = \check{\mathbf{a}} + \tau\check{\mathbf{b}} = \tau\mathbf{a} - \mathbf{b}.$$

The two framings $\mathbf{a}, \mathbf{b}$ and $2\pi i\check{\mathbf{b}}, \omega_\tau$ of $\mathbb{H}$ over $\mathfrak{h}$ are related by

$$\begin{pmatrix} 2\pi i\check{\mathbf{b}} & w_\tau \end{pmatrix} = \begin{pmatrix} \check{\mathbf{b}} & \check{\mathbf{a}} \end{pmatrix} \begin{pmatrix} 2\pi i & \tau \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 2\pi i & \tau \\ 0 & -1 \end{pmatrix}.$$

Remark 1.10. The local system $\mathbb{H}$ underlies a polarized variation of Hodge structure over $\mathfrak{h}$ of weight -1. The Hodge subbundle $F^0\mathcal{H}$ of the corresponding flat bundle $\mathcal{H} = \mathbb{H} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathfrak{h}}$ is $\mathcal{O}(\mathfrak{h})\omega$.

2. Unipotent completion

Suppose that π is a discrete group and that R is a commutative ring. Denote the group algebra of π over R by $R\pi$. This is an R-algebra. The *augmentation* is the homomorphism $\epsilon : R\pi \to R$ that takes each $\gamma \in \pi$ to 1. Its kernel, denoted J, is called the *augmentation ideal.* The powers of J define a topology on $R\pi$. A base of neighbourhoods of 0 consist of the powers of J:

$$R\pi \supseteq J \supseteq J^2 \supseteq J^3 \supseteq \cdots$$

The completion of $R\pi$ in this topology is called the J-adic completion of π and is denoted by $R\pi^\wedge$. In concrete terms:

$$R\pi^\wedge = \varinjlim_n R\pi/J^n.$$

Denote its augmentation ideal by $J^\wedge$.

The group algebra also has a "coproduct"

$$\Delta : R\pi \to R\pi \otimes R\pi.$$

This is an augmentation preserving algebra homomorphism, which is continuous in the J-adic topology. It thus induces a ring homomorphism

$$\Delta : R\pi^\wedge \to R\pi^\wedge \hat{\otimes} R\pi^\wedge.$$

Now suppose that R is a field F of characteristic zero. Note that each element of $1 + J^\wedge$ is a unit. Define

$$\mathcal{P}(F) = \{x \in F\pi^\wedge : \epsilon(x) = 1 \text{ and } \Delta x = x \otimes x\}$$

and

$$\mathfrak{p} = \{x \in F\pi^\wedge : \Delta x = x \otimes 1 + 1 \otimes x\}.$$

Elements of $\mathfrak{p}$ are said to be *primitive*; elements of $\mathcal{P}$ are said to be *group-like.*

Proposition 2.1. *(i) $\mathcal{P}(F)$ is a subgroup of the group $1 + J^\wedge$;*
(ii) $\mathfrak{p}$ is a Lie algebra, with bracket $[u, v] = uv - vu$, which lies in $J^\wedge$;
(iii) The logarithm and exponential mappings

$$J^\wedge \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + J^\wedge$$

are continuous bijections, which induce continuous bijections

$$\mathfrak{p} \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathcal{P}(F) .$$

The third part implies that the exponential map

$$\exp : (\mathfrak{p}, \mathrm{BCH}) \to \mathcal{P}$$

is a group isomorphism, where the multiplication on $\mathfrak{p}$ is defined using the Baker-Campbell-Hausdorff formula [28]:

$$\mathrm{BCH}(u, v) := \log(e^u e^v) = u + v + \frac{1}{2}[u, v] + \cdots$$

Proof. The first two assertions are easily verified, as is the first part of the third assertion. To prove the last assertion, note that since exp is continuous, $\exp \Delta(x) = \Delta \exp(x)$ for all $x \in J^\wedge$. Now, $x \in J^\wedge$ is primitive if and only if

$$\Delta x = x \otimes 1 + 1 \otimes x.$$

Since $x \otimes 1$ and $1 \otimes x$ commute, this holds if and only if

$$\Delta \exp(x) = \exp(\Delta(x)) = \exp(x \otimes 1) \exp(1 \otimes x) = \exp(x) \otimes \exp(x).$$

That is, $x \in J^\wedge$ is primitive if and only if $\exp x$ is group-like. □

Since $\epsilon(\gamma) = 1$ for all $\gamma \in \pi$, there is a homomorphism $\pi \to 1 + J^\wedge$. By the definition of the coproduct Δ, the image of this homomorphism lands in $\mathcal{P}(F)$. Thus, the inclusion $\pi \to F\pi$ induces a natural homomorphism $\pi \to \mathcal{P}(F)$

Definition 2.2. Suppose that $H_1(\pi; F)$ is finite dimensional (e.g., π is finitely generated). The homomorphism $\pi \to \mathcal{P}(F)$ is called the *unipotent (or Malcev) completion of* π *over* F. The prounipotent group $\mathcal{P}$ is denoted π^{un}. The Lie algebra of the unipotent completion is the Lie algebra $\mathfrak{p}$. It is also called the Malcev Lie algebra associated to π.

Unipotent completion can be viewed as a functor from the category of groups to the category of prounipotent groups over F:

$$\pi \rightsquigarrow \mathcal{P}(F)$$

There is also the functor $\pi \rightsquigarrow \mathfrak{p}$ that assigns to a group, the Lie algebra of its unipotent completion over F. There are therefore natural homomorphism

$$\operatorname{Aut}\pi \to \operatorname{Aut}\mathcal{P} \text{ and } \operatorname{Aut}\pi \to \operatorname{Aut}\mathfrak{p}.$$

Remark 2.3. When π is the fundamental group of an algebraic variety, $\mathfrak{p}$ carries additional structure: If π is the fundamental group of a complex algebraic variety and $F = \mathbb{Q}$, then $\mathfrak{p}$ has a natural mixed Hodge structure; if π is the fundamental group of a smooth algebraic variety defined over $\mathbb{Q}$ with $\mathbb{Q}$-rational base point, then the absolute Galois group $G_\mathbb{Q}$ acts on $\mathfrak{p} \otimes \mathbb{Q}_\ell$.

2.1. The unipotent completion of a free group

Suppose that π is the free group $\langle x_1, \dots, x_n\rangle$ generated by the set $\{x_1, \dots, x_n\}$.

Consider the ring

$$F\langle\langle X_1, \dots, X_n\rangle\rangle$$

of formal power series in the non-commuting indeterminants X_j. Define an augmentation

$$\epsilon : F\langle\langle X_1, \dots, X_n\rangle\rangle \to F$$

by sending a power series to its constant term. The augmentation ideal $\ker\epsilon$ is the maximal ideal $I = (X_1, \dots, X_n)$.

Define a coproduct

$$\Delta : F\langle\langle X_1, \dots, X_n\rangle\rangle \to F\langle\langle X_1, \dots, X_n\rangle\rangle \hat{\otimes} F\langle\langle X_1, \dots, X_n\rangle\rangle$$

by defining each X_j to be primitive:

$$\Delta X_j := X_j \otimes 1 + 1 \otimes X_j.$$

There is a unique group homomorphism

$$\pi \to F\langle\langle X_1, \dots, X_n \rangle\rangle$$

that takes x_j to $\exp(X_j)$. This extends to a ring homomorphism

$$\theta : F\pi \to F\langle\langle X_1, \dots, X_n \rangle\rangle.$$

Since $\epsilon(x_j) = 1 = \epsilon(\exp(X_j))$, θ is augmentation preserving, and therefore extends to a continuous homomorphism

$$\hat{\theta} : F\pi^\wedge \to F\langle\langle X_1, \dots, X_n \rangle\rangle$$

As in the case of completed group algebras, one can define primitive and group-like elements of $F\langle\langle X_1, \dots, X_n \rangle\rangle$. As there, an element of $1+I$ is group-like if and only if it is the exponential of a primitive element. Since $\exp(X_j)$ is group-like, it is easy to check that $\hat{\theta}$ preserves both the product and the coproduct. (One says that it is a homomorphism of complete Hopf algebras.)

It is easy to use universal mapping properties to prove:

Proposition 2.4. *The homomorphism $\hat{\theta}$ is an isomorphism of complete Hopf algebras.* □

Corollary 2.5. *The restriction of $\hat{\theta}$ induces a natural isomorphism*

$$d\theta : \mathfrak{p} \to \mathbb{L}(X_1, \dots, X_n)^\wedge$$

of topological Lie algebras.

Proof. This follows immediately from the fact that $\hat{\theta}$ induces an isomorphism on primitive elements and the well-known fact that the set of primitive elements of the power series algebra $F\langle\langle X_1, \dots, X_n \rangle\rangle$ is the completed free Lie algebra $\mathbb{L}(X_1, \dots, X_n)^\wedge$. □

There is a weaker version of the construction of the unipotent completion of a free group, which will be relevant later. Suppose that

$$\theta : \pi \to F\langle\langle X_1, \dots, X_n \rangle\rangle$$

is a homomorphism that satisfies $\theta(x_j) = \exp(U_j)$, where $U_j \in J^\wedge$ and $U_j \equiv X_j \bmod (J^\wedge)^2$. Then it is not difficult to show that θ induces a continuous isomorphism

$$\hat{\theta} : F\pi^{\wedge} \to F\langle\langle X_1, \dots, X_n \rangle\rangle$$

and, by restriction, a Lie algebra isomorphism

$$d\theta : \mathfrak{p} \to \mathbb{L}(X_1, \dots, X_n)^{\wedge}$$

and a group isomorphism

$$\mathcal{P} \to \exp \mathbb{L}(X_1, \dots, X_n)^{\wedge}.$$

3. Factors of automorphy

Suppose that G is a group that acts on a space (or set) X on the left. Suppose that V is a left G-module (or left G-space, etc.). A function $M : G \times X \to \operatorname{Aut} V$ (written $(g, x) \mapsto M_g(x)$) is a *factor of automorphy* if the function

$$V \times X \to V \times X, \qquad g : (v, x) \mapsto (M_g(x)v, gx)$$

is an action. This is equivalent to the condition

$$M_{gh}(x) = M_g(hx)M_h(x) \quad \text{all } g, h \in G,\ x \in X.$$

Note that the projection $V \times X \to X$ is G-equivariant; G-equivariant sections of this projection correspond to functions $f : X \to V$ satisfying $f(gx) = M_g(x)f(x)$ and, by definition, to sections of the "bundle" $G\backslash(X \times V) \to G\backslash X$.[3] Such bundles are flat in the sense that they give rise to a locally constant sheaf. An open set in $G\backslash X$ corresponds to a G-invariant open set U in X. The set of constant sections of the bundle over this set is, by definition, the set of G-invariant locally constant sections of $V \times X \to X$. When V is a real or complex vector space, this bundle has a natural flat connection ∇ which is characterized by the property that a local section s is constant if and only if $\nabla s = 0$. In such cases, we will refer to the bundle $G\backslash(V \times X) \to X$ as being a *flat bundle*.

Three examples that will be generalized and combined to form $\mathcal{P}$ are:

Example 3.1. Fix $k \in \mathbb{Z}$. Let $G = \mathrm{SL}_2(\mathbb{Z})$, $X = \mathfrak{h}$, $V = \mathbb{C}$, and $A_\gamma(\tau) = (c\tau + d)^k$. The (orbifold) quotient of $\mathbb{C} \times \mathfrak{h} \to \mathfrak{h}$ is the line bundle $\mathcal{L}_k \to \mathcal{M}_{1,1}$.

[3] More precisely, G-invariant sections of $V \times X \to X$ correspond to section of the stack bundle $G\backslash\!\backslash(V \times X) \to G\backslash\!\backslash X$.

Note that the fibered product $\mathcal{E} \times_{\mathcal{M}_{1,1}} \mathcal{E} \to \mathcal{M}_{1,1}$ of the universal elliptic curve is the quotient of $\mathbb{C} \times \mathbb{C} \times \mathfrak{h}$ by the $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \oplus \mathbb{Z}^2)$-action

$$\big((m,n),(r,s)\big) : (\xi,\eta,\tau) = \big(\xi + m\tau + n, \eta + r\tau + s, \tau\big)$$

and

$$\gamma : (\xi,\tau) \mapsto \big((c\tau+d)^{-1}\xi, (c\tau+d)^{-1}\eta, \gamma\tau\big)$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Example 3.2. Suppose that $G = \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \oplus \mathbb{Z}^2)$ and that $X = \mathbb{C}\times\mathbb{C}\times\mathfrak{h}$, where the G-action is the one defined above. Let $V = \mathbb{C}$. Define

$$A_\gamma(\xi,\eta,\tau) = \begin{cases} (c\tau+d)e\big(c\xi\eta/(c\tau+d)\big) & \gamma \in \mathrm{SL}_2(\mathbb{Z}), \\ e(\tau)^{-mr}e(\xi)^{-r}e(\eta)^{-m} & \gamma = \big((m,n),(r,s)\big) \end{cases}$$

where $e(u) = \exp(2\pi i u)$. This is a well-defined factor of automorphy. The quotient

$$G\backslash(\mathbb{C} \times X) \to G\backslash X$$

is a line bundle

$$\mathcal{N} \to \mathcal{E} \times_{\mathcal{M}_{1,1}} \mathcal{E}$$

over the self product over $\mathcal{M}_{1,1}$ of the universal elliptic curve. The restriction of $\mathcal{N}$ to the zero section $\mathcal{M}_{1,1}$ is the line bundle $\mathcal{L} = \mathcal{L}_1$. (Just look at the factor of automorphy when $\xi = \eta = 0$.)

Remark 3.3. Later (Prop. 8.1) we will see that the restriction of $\mathcal{N}$ to the fiber E^2 over $[E]$ is the pullback of the Poincaré line bundle over $E \times E$ to $E \times E$ along the map $(\xi,\eta) \mapsto (\xi,-\eta)$.

The next example gives an alternative description of the local system $\mathbb{H}$.

Example 3.4. Let $G = \mathrm{SL}_2(\mathbb{Z})$, $X = \mathfrak{h}$ and $V = \mathbb{C}^2$. Then

$$M_\gamma(\tau) = \begin{pmatrix} (c\tau+d)^{-1} & 0 \\ 2\pi i c & c\tau+d \end{pmatrix}$$

is a factor of automorphy. The resulting bundle is the vector bundle associated to the local system $\mathbb{H} \to \mathcal{M}_{1,1}$ defined in Section 1.2. To see this, we set

$$\mathbf{t} = \omega_\tau/2\pi i \in H^1(E_\tau,\mathbb{C}).$$

Then $\mathbf{a}$ and $\mathbf{t}$ comprise a framing of the pullback $\mathbb{H}_{\mathfrak{h}}$ of $\mathbb{H}$ to $\mathfrak{h}$, which gives an isomorphism $\mathbb{C}^2 \times \mathfrak{h} \to \mathbb{H}_{\mathfrak{h}}$ via

$$(u, v, \tau) \mapsto \begin{pmatrix} (\mathbf{a}, \tau) & (\mathbf{t}, \tau) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.1}$$

Here, $(\mathbf{a}, \tau)$ denotes $\mathbf{a}$ viewed as an element of $H_1(E_\tau)$. Likewise, $(\mathbf{t}, \tau)$ denotes the element $\omega_\tau/2\pi i$ of $H^1(E_\tau)$.

Since $\Lambda_{\gamma\tau} = (c\tau+d)\Lambda_\tau$, multiplication by $(c\tau+d)$ induces an isomorphism

$$E_\tau \to E_{\gamma\tau}.$$

This induces the identification of the fibers of $\mathbb{H}_{\mathfrak{h}}$ over τ and $\gamma\tau$. For convenience, set $\mathbf{a} = (\mathbf{a}, \tau) \in H_1(E_\tau)$ and $\mathbf{a}' = (\mathbf{a}, \gamma\tau) \in H_1(E_{\gamma\tau})$. Similarly with $\mathbf{b}$ and $\mathbf{b}'$, and with $\mathbf{t}$ and $\mathbf{t}'$. Then

$$2\pi i \mathbf{t}' = \omega_{\gamma\tau} = (c\tau + d)^{-1}\omega_\tau = 2\pi i(c\tau + d)^{-1}\mathbf{t}$$

and

$$\begin{aligned}
\begin{pmatrix} \mathbf{a}' & \omega_{\gamma\tau} \end{pmatrix} &= (c\tau + d)^{-1} \begin{pmatrix} \mathbf{a}' & \mathbf{b}' \end{pmatrix} \begin{pmatrix} c\tau + d & a\tau + b \\ 0 & -(c\tau + d) \end{pmatrix} \\
&= (c\tau + d)^{-1} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} \begin{pmatrix} c\tau + d & a\tau + b \\ 0 & -(c\tau + d) \end{pmatrix} \\
&= (c\tau + d)^{-1} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} (c\tau + d)d & \tau \\ (c\tau + d)c & -1 \end{pmatrix} \\
&= (c\tau + d)^{-1} \begin{pmatrix} \mathbf{a} & \omega_\tau \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix} \begin{pmatrix} (c\tau + d)d & \tau \\ (c\tau + d)c & -1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a} & \omega_\tau \end{pmatrix} \begin{pmatrix} c\tau + d & 0 \\ -c & (c\tau + d)^{-1} \end{pmatrix}
\end{aligned}$$

from which we conclude that

$$\begin{pmatrix} \mathbf{a} & \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{a}' & \mathbf{t}' \end{pmatrix} M_\gamma(\tau).$$

Equation (3.1) now implies that the bundle with factor of automorphy $M_\gamma(\tau)$ is isomorphic to $\mathbb{H}$ as the following points correspond:

$$(\begin{pmatrix} u \\ v \end{pmatrix}, \tau) \leftrightarrow \begin{pmatrix} \mathbf{a} & \mathbf{t} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathbf{a}' & \mathbf{t}' \end{pmatrix} M_\gamma(\tau) \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow (M_\gamma(\tau) \begin{pmatrix} u \\ v \end{pmatrix}, \gamma\tau)$$

Note that $\mathbf{t}$ and $\mathbf{a}$ are both invariant under $\tau \mapsto \tau + 1$. It follows that $\mathbb{H}$ is trivial over the q-disk.

Since the bundle $\mathbb{H}$ exists over $\mathcal{M}_{1,1}$, this computation gives a conceptual proof that $M_\gamma(\tau)$ is a factor of automorphy.

Remark 3.5. It is useful to keep in mind that

$$\mathbf{a} \in H_1(E_\tau, \mathbb{Z}) \text{ and } \langle \mathbf{a}, \mathbf{t} \rangle = -(2\pi i)^{-1} \in \mathbb{Z}(-1).$$

Note that $\mathbf{t}$ spans a line sub-bundle of $\mathcal{H} := \mathbb{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}_{1,1}}$. This line bundle is the *Hodge bundle* $F^1\mathcal{H}$ and is isomorphic to $\mathcal{L}$. The factor of automorphy of $\mathbb{H}$ implies that the quotient of $\mathcal{H}$ by F^1 is isomorphic to $\mathcal{L}_{-1}$, so that we have an exact sequence

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{L}_{-1} \to 0.$$

Later we will see that this splits, even over $\overline{\mathcal{M}}_{1,1}$. (Cf. Remark 19.2 and the last paragraph of Section 19.2.)

4. Some Lie theory

Let $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle$ be the completion of the free associative algebra generated by the indeterminants $\mathbf{t}$ and $\mathbf{a}$. It is a topological algebra. Denote the closure of the free Lie algebra $\mathbb{L}(\mathbf{t}, \mathbf{a})$ in $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle$ by $\mathfrak{p}$. It is a topological Lie algebra.

Define a continuous action $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle \times \mathfrak{p} \to \mathfrak{p}$ of $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle$ on $\mathfrak{p}$ by

$$f(\mathbf{t}, \mathbf{a}) : x \mapsto f(\mathbf{t}, \mathbf{a}) \cdot x := f(\mathrm{ad}_{\mathbf{t}}, \mathrm{ad}_{\mathbf{a}})(x)$$

for all $x \in \mathfrak{p}$.

For later use, we record the following fact:

Proposition 4.1. *Suppose that $A : [a, b] \to \mathbb{L}(X_1, \dots, X_n)^\wedge$ is smooth.*[4] *If $X : [a, b] \to \mathbb{C}\langle\langle X_1, \dots, X_n \rangle\rangle$ satisfies the initial value problem*

$$X' = AX, \quad X(0) = 1,$$

then $X(t)$ is group-like for all $t \in [a, b]$.

Proof. This follows from standard Lie theory. It can also be proved directly as follows. Since the diagonal Δ is linear, since Δ is an algebra homomorphism,

[4]That is, each coefficient of the power series $A(t)$ is a smooth function of $t \in [a, b]$.

and since A is primitive, we have

$$(\Delta X)' = \Delta(X') = \Delta(AX) = (\Delta A)(\Delta X) = (A \otimes 1 + 1 \otimes A)\Delta X.$$

On the other hand,

$$\begin{aligned}(X \otimes X)' = X' \otimes X + X \otimes X' &= (AX) \otimes X + X \otimes (AX)\\ &= (A \otimes 1 + 1 \otimes A)(X \otimes X).\end{aligned}$$

Thus both ΔX and $X \otimes X$ satisfy the IVP

$$Y' = (A \otimes 1 + 1 \otimes A)Y, \quad Y(0) = 1 \otimes 1,$$

where $Y : [a, b] \to \mathbb{C}\langle\langle X_1, \dots, X_n\rangle\rangle \otimes \mathbb{C}\langle\langle X_1, \dots, X_n\rangle\rangle$. It follows that $\Delta X = X \otimes X$ for all t. □

4.1. Two identities

For later use we recall two standard identities. To avoid confusion, we shall denote composition of endomorphisms ϕ and ψ of $\mathfrak{p}$ by $\phi \circ \psi$.

Recall that if V is a vector space and $u, \phi \in \operatorname{End} V$, then in $\operatorname{End} V$ we have

$$\exp(\operatorname{ad}\phi)(u) = e^{\phi} \circ u \circ e^{-\phi}.$$

Applying this in the case where $V = \mathfrak{p}$, we see that for all $\delta \in \operatorname{Der}\mathfrak{p}$ and $\phi \in \mathbb{C}\langle\langle \mathbf{t}, \mathbf{a}\rangle\rangle$,

$$\exp(\phi) \cdot \delta = e^{\phi} \circ \delta \circ e^{-\phi}.$$

In particular, if ω is a 1-form on a manifold that takes values in $\mathfrak{p}$, then

$$e(-m\mathbf{t}) \circ \omega \circ e(m\mathbf{t}) = e(-m\mathbf{t}) \cdot \omega, \tag{4.1}$$

where $e(u) := \exp(2\pi i u)$.

Lemma 4.2. *Suppose that $u \in \mathbb{C}\langle\langle \mathbf{t}, \mathbf{a}\rangle\rangle$. If δ is a continuous derivation of $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a}\rangle\rangle$, then*

$$e^{-u}\delta(e^u) = \frac{1 - \exp(-\operatorname{ad}_u)}{\operatorname{ad}_u}\delta(u) \text{ and } \delta(e^u)e^{-u} = \frac{\exp(\operatorname{ad}_u) - 1}{\operatorname{ad}_u}\delta(u).$$

Proof. The functions $e^{-su}\delta(e^{su})$ and $\frac{1-\exp(-s\,\mathrm{ad}_u)}{\mathrm{ad}_u}\delta(u)$ both satisfy the differential equation

$$X'(s) = \delta(u) - \mathrm{ad}_u(X).$$

Since both functions vanish when $s = 0$, they are equal for all $s \in \mathbb{C}$. In particular, they are equal when $s = 1$. This proves the first identity. The second is proved similarly using the differential equation $Y' = \delta(u) + \mathrm{ad}_u(Y)$. $\square$

5. Connections and monodromy

Suppose that Γ is a discrete group, G is a Lie (or proalgebraic) group and that X is a topological space. Suppose that Γ acts on X on the left. (Think of this action as being discontinuous and fixed point free, but it does not have to be.) Suppose that the action of Γ lifts to the trivial right principal G-bundle $G \times X \to X$:

$$\gamma : (g, x) \mapsto \big(M_\gamma(x)g, \gamma x\big)$$

where $M_\gamma : X \to G$ is a factor of automorphy.

5.1. Connections

Denote the Lie algebra of G by $\mathfrak{g}$. Sections of the bundle $G \times X \to X$ will be identified with functions $X \to G$ in the obvious way. A Lie algebra valued 1-form

$$\omega \in E^1(X) \otimes \mathfrak{g}$$

defines a connection on the trivial bundle $G \times X \to X$ by the formula

$$\nabla f = df + \omega f$$

where f is a locally defined function $X \to G$.

Proposition 5.1. *The connection ∇ is Γ-invariant if and only if for all $\gamma \in \Gamma$,*

$$\gamma^*\omega = \mathrm{Ad}(M_\gamma)\omega - dM_\gamma M_\gamma^{-1}.$$

The connection ∇ is flat if and only if ω satisfies

$$d\omega + \frac{1}{2}[\omega, \omega] = 0. \qquad \square$$

Example 5.2. The sections $\mathbf{a}$ and $\mathbf{b}$ of the Hodge bundle $\mathbb{H}_{\mathfrak{h}}$ over $\mathfrak{h}$ are flat. Since they give local framings of the associated vector bundle $\mathcal{H} := \mathbb{H} \otimes_{\mathbb{C}} \mathcal{O}$, there is a flat connection on $\mathcal{H}$, which is characterized by the property that $\nabla \mathbf{a} = \nabla \mathbf{b} = 0$. Since $\mathbf{t} = \omega_\tau / 2\pi i = (\tau \mathbf{a} - \mathbf{b})/2\pi i$, we have

$$2\pi i \nabla \mathbf{t} = \nabla(\tau \mathbf{a} - \mathbf{b}) = \mathbf{a} d\tau.$$

It follows that, in terms of the framing $\mathbf{a}$, $\mathbf{t}$ of $\mathcal{H}$, the connection is given by

$$\nabla = d + (2\pi i)^{-1} \mathbf{a} \frac{\partial}{\partial \mathbf{t}} \otimes d\tau.$$

5.2. Parallel transport

Every path $\alpha : [0,1] \to X$ has a horizontal lift $\tilde{\alpha} : [0,1] \to G$ that starts at $1 \in G$. In other words, the section

$$t \mapsto \big(\tilde{\alpha}(t), \alpha(t)\big) \in G \times X$$

is a flat section of the bundle that projects to α and begins at $(1, \alpha(0))$.[5]

The function $\tilde{\alpha}$ is the unique solution of the ODE

$$d\tilde{\alpha} = -(\alpha^* \omega)\tilde{\alpha}, \quad \tilde{\alpha}(0) = 1.$$

Note that the uniqueness of solutions of ODEs implies that the horizontal lift of α that begins at $g \in G$ is $t \mapsto \tilde{\alpha}(t) g$.

Denote the value of the lift $\tilde{\alpha}$ at $t = 1$ by $T(\alpha)$. The function

$$T : \alpha \mapsto T(\alpha)$$

is called the (parallel) transport function associated to ∇. When ∇ is flat, $T(\alpha)$ depends only on the homotopy class of α relative to its endpoints.

An immediate consequence of the uniqueness of solutions to IVPs:

Lemma 5.3. *If α and β are composable paths, then $T(\alpha\beta) = T(\beta)T(\alpha)$.* □

To make the transport multiplicative, we will work with $T(\alpha)^{-1}$. A formula for the transport and the inverse transport can be given using Chen's iterated integrals. First a basic fact from ODE.

[5]Note that this does not require the connection to be flat.

Lemma 5.4. *Suppose that R is a topological algebra (such as $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a}\rangle\rangle$ or $\mathfrak{gl}_n(\mathbb{C})$) and that $A : [a,b] \to R$ is a smooth function. A function $X : [a,b] \to R^\times$ is a solution of the IVP*

$$X' = -AX, \quad X(0) = 1$$

if and only $Y = X^{-1}(t)$ is a solution of the IVP

$$Y' = YA, \quad Y(0) = 1.$$

Proof. Suppose that X satisfies $X' = -AX$ and $X(0) = I$. Then

$$0 = X^{-1}(XX^{-1})' = X^{-1}\big(X(X^{-1})'\big) - X^{-1}\big(AXX^{-1}\big) = (X^{-1})' - X^{-1}A.$$

The opposite direction is proved similarly. □

Recall that if $\omega_1, \dots, \omega_r$ are 1-forms on a manifold X taking values in an associative algebra A, and if γ is a piecewise smooth path in X then one defines the iterated integral

$$\int_\gamma \omega_1\omega_2\dots\omega_r = \int_{0\le t_1\le\cdots\le t_r\le 1} f_1(t_1)f_2(t_2)\dots f_r(t_r)dt_1dt_2\dots dt_r \tag{5.1}$$

where $\gamma^*\omega_j = f_j(t)dt$. See [5, 11] for more background.

Corollary 5.5 (Transport Formula)**.** *The inverse transport is given by*

$$T(\alpha)^{-1} = 1 + \int_\alpha \omega + \int_\alpha \omega\omega + \int_\alpha \omega\omega\omega + \cdots$$

Proof. This follows from Chen's transport formula (cf. [5, 11]) and the previous lemma. □

For future use, we record the following standard fact.

Proposition 5.6. *If the connection ∇ is Γ-invariant, then for all paths $\alpha : [0,1] \to X$ and all $\gamma \in \Gamma$*

$$T(\gamma\circ\alpha) = M_\gamma\big(\alpha(1)\big)T(\alpha)M_\gamma\big(\alpha(0)\big)^{-1}.$$

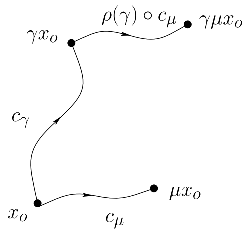

Figure 1: The cocycle relation $c_{\gamma\mu} = c_\gamma \cdot \big(\rho(\gamma) \circ c_\mu\big)$.

5.3. Monodromy

Suppose that the connection $d+\omega$ is Γ-invariant and flat. Our task in this section is to explain how to compute the associated monodromy representation from the transport function T of ω and the factor of automorphy M.

By covering space theory, the choice of a point $x_o \in X$ determines a surjective homomorphism

$$\rho : \pi_1(\Gamma\backslash X, \bar{x}_o) \to \Gamma$$

whose kernel is $\pi_1(X, x_o)$, where $\bar{x}_o$ denotes the image of x_o in $\Gamma\backslash X$.

To each $\gamma \in \pi_1(\Gamma\backslash X)$, let c_γ be its lift to a path in X that begins at x_o. Note that its end point is $\rho(\gamma)\cdot x_o$ and that the homotopy class of c_γ depends only upon γ.

Lemma 5.7. *If $\gamma, \mu \in \pi_1(\Gamma\backslash X, \bar{x}_o)$, then $c_{\gamma\mu} = c_\gamma \cdot \big(\rho(\gamma) \circ c_\mu\big)$.*

Here $\cdot$ denotes path multiplication and $\circ$ denotes composition. The proof is best given by the picture Figure 1.

To obtain a homomorphism (instead of an anti-homomorphism), we need to take inverses. Define

$$\Theta_{x_o} : \pi_1(\Gamma\backslash X, \bar{x}_o) \to G$$

by

$$\Theta_{x_o}(\gamma) = T(c_\gamma)^{-1} M_{\rho(\gamma)}(x_o).$$

Note that $\Theta_{x_o}(\gamma)^{-1}$ is the element of the fiber G over x_o that is identified with the point $T(c_\gamma)$ in the fiber G over $\rho(\gamma)\cdot x_o$. It is thus the result of parallel transporting $1 \in G$ about the loop γ.

Proposition 5.8. *The monodromy representation* $\pi_1(\Gamma\backslash X, \bar{x}_o) \to G$ *of the flat bundle* $\Gamma\backslash(G \times X) \to \Gamma\backslash X$ *with respect to the identification above is*

$$\Theta_{x_o} : \pi_1(\Gamma\backslash X, \bar{x}_o) \to G.$$

Proof. Just trace through the identifications. But to reassure the reader, we show that Θ_{x_o} is a group homomorphism. (We'll drop ρ and the x_o below.) If $\gamma, \mu \in \pi_1(\Gamma\backslash X, \bar{x}_o)$, then

$$\begin{aligned}\Theta(\gamma\mu) &= T(c_{\gamma\mu})^{-1} M_{\gamma\mu}(x_o) \\ &= T(c_\gamma \cdot (\gamma \circ c_\mu))^{-1} M_{\gamma\mu}(x_o) \\ &= T(c_\gamma)^{-1} T(\gamma \circ c_\mu)^{-1} M_\gamma(\mu \cdot x_o) M_\mu(x_o) \\ &= T(c_\gamma)^{-1} M_\gamma(x_o) T(c_\mu)^{-1} M_\gamma(\mu \cdot x_o)^{-1} M_\gamma(\mu \cdot x_o) M_\mu(x_o) \\ &= \Theta(\gamma)\Theta(\mu).\end{aligned}$$

□

Combining this with the transport formula above, we obtain a formula for the monodromy in terms of ω and the factor of automorphy.

Corollary 5.9. *For all* $x \in X$ *and* $\gamma \in \pi_1(\Gamma\backslash X, \bar{x})$,

$$\Theta_x(\gamma) = \Big(1 + \int_{c_\gamma} \omega + \int_{c_\gamma} \omega\omega + \int_{c_\gamma} \omega\omega\omega + \cdots\Big) M_\gamma(x). \qquad \square$$

Part 2. The universal elliptic KZB connection

6. The bundle $\boldsymbol{\mathcal{P}}$ over $\mathcal{E}'$

Before we define the universal elliptic KZB connection, we need to define the bundle $\boldsymbol{\mathcal{P}}$ over $\mathcal{E}'$ on which it lives.

6.1. The flat bundle $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$

The bundle $\boldsymbol{\mathcal{P}}$ with the KZB connection will be the de Rham realization of a topological local system $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$. To provide context, we first construct it.

Denote by Y the universal covering space of $\mathcal{E}'$. This is also the universal covering space of $\mathcal{E}'_{\mathfrak{h}} = (\mathbb{C} \times \mathfrak{h}) - \Lambda_{\mathfrak{h}}$. Choose a base point $[E_o, x_o]$ of $\mathcal{E}'$ and a lift y_o of it to Y. This determines an isomorphism of $\mathrm{Aut}(Y/\mathcal{E}')$ with $\pi_1(\mathcal{E}', [E_o, x_o])$.

Denote the unipotent completion of $\pi_1(E'_o, x_o)$ over $\mathbb{C}$ by $\mathcal{P}_o$. The natural action

$$\pi_1(\mathcal{E}', [E_o, x_o]) \times \pi_1(E'_o, x_o) \to \pi_1(E'_o, x_o), \quad (g, \gamma) \mapsto g\gamma g^{-1}$$

determines a left action of $\pi_1(\mathcal{E}', [E_o, x_o])$ on $\mathcal{P}_o$. We can therefore form the quotient

$$\pi_1(\mathcal{E}', [E_o, x_o]) \backslash (\mathcal{P}_o \times Y)$$

by the diagonal $\pi_1(\mathcal{E}', [E_o, x_o])$-action. This is a flat right principal $\mathcal{P}_o$-bundle which we shall denote by $\boldsymbol{\mathcal{P}}^{\mathrm{top}} \to \mathcal{E}'$. Its fiber over $[E, x]$ is naturally isomorphic to the unipotent completion of $\pi_1(E', x)$.

Since the Lie algebra $\mathfrak{p}_o$ of $\mathcal{P}_o$ can be viewed as a group with multiplication defined by the Baker-Campbell-Hausdorff formula, we can (and will) view $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$ as a local system of Lie algebras. (Cf. the comment following Prop. 2.1.)

6.2. The bundle $\boldsymbol{\mathcal{P}}$

Here we construct a bundle $\boldsymbol{\mathcal{P}}$ over $\mathcal{E}$ on which the universal elliptic KZB connection lives. Its fiber over each point of $\mathcal{E}'$ is the Lie algebra

$$\mathfrak{p} := \mathbb{L}(\mathbf{t}, \mathbf{a})^\wedge.$$

Denote the corresponding group $\exp \mathfrak{p}$ by $\mathcal{P}$. It is prounipotent. The universal elliptic KZB connection on it is constructed in Section 9. It is flat. In Section 14 we will prove that it is isomorphic to the flat bundle $\boldsymbol{\mathcal{P}}$.

The bundle $\boldsymbol{\mathcal{P}}$ will be constructed as the quotient of $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$ by a lift of the action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C} \times \mathfrak{h}$ to $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$.

The (completed) universal enveloping algebra of $\mathfrak{p}$ is the power series algebra $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle$. The adjoint action defines the ring homomorphism

$$\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle \to \operatorname{End} \mathfrak{p}$$

that takes $f(\mathbf{t}, \mathbf{a})$ to $f(\mathrm{ad}_{\mathbf{t}}, \mathrm{ad}_{\mathbf{a}}) \in \operatorname{End} \mathfrak{p}$. This restricts to a homomorphism $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle^\times \to \operatorname{Aut} \mathfrak{p}$.

We use the notation of Section 3. Take $G = \Gamma := \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, $X = \mathbb{C} \times \mathfrak{h}$, and $V = \mathfrak{p}$. Note that Γ acts on $\mathfrak{p} \times X$ via the projection $\Gamma \to \mathrm{SL}_2(\mathbb{Z})$ using the factor of automorphy $M_\gamma(\tau)$ defined in (6.1).

The bundle $\boldsymbol{\mathcal{P}}$ is defined using factors of automorphy $\widetilde{M}_\gamma(\xi, \tau)$ which live in the group $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle^\times$, where $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle$ via its left action on the generators via the factor of automorphy $M_\gamma(\tau)$ defined in Example 3.4. Specifically, $M_\gamma(\tau)$ is defined by

$$M_\gamma(\tau) : \begin{cases} \mathbf{a} & \mapsto (c\tau+d)^{-1}\mathbf{a} + 2\pi i c\mathbf{t} \\ \mathbf{t} & \mapsto (c\tau+d)\mathbf{t}. \end{cases} \tag{6.1}$$

The general factor of automorphy is defined by

$$\widetilde{M}_\gamma(\xi,\tau) = \begin{cases} M_\gamma(\tau)\circ e\big(\frac{c\xi\mathbf{t}}{c\tau+d}\big) & \gamma\in \mathrm{SL}_2(\mathbb{Z}); \\ e(-m\mathbf{t}) & (m,n)\in\mathbb{Z}^2. \end{cases} \tag{6.2}$$

Here $e(u) := \exp(2\pi i u)$. This is a factor of automorphy for Γ as

$$e(c\xi\mathbf{t} - m\mathbf{t})\circ M_\gamma(\tau) = \widetilde{M}_\gamma(\xi + (m,n)\gamma(\tau,1)^T,\tau)\circ e(-m\mathbf{t}),$$

where $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ and $(m,n)\in\mathbb{Z}^2$.[6]

Proposition 6.1. *This is a well-defined factor of automorphy.*

Proof. The first task is to show that $\widetilde{M}$ is well-defined on $\Gamma\times\mathbb{C}\times\mathfrak{h}$ — that is, it is compatible with the relation in $\Gamma = \mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$. This relation is

$$(m,n)\circ\gamma = \gamma\circ(m,n)(\cdot\gamma),$$

where $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, $\circ$ denotes composition in Γ and $\cdot$ denotes the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then we have to show that

$$e(-m\mathbf{t})\circ\widetilde{M}_\gamma(\xi,\tau) = \widetilde{M}_\gamma(\xi + (m,n)\gamma(\tau,1)^T,\tau)\circ e\big(-(ma+nc)\mathbf{t}\big).$$

Since

$$M_\gamma(\tau)\circ e(\phi) = e(M_\gamma(\tau)\cdot\phi)\circ M_\gamma(\tau)$$

for all $\phi\in\mathbb{C}\langle\langle\mathbf{t},\mathbf{a}\rangle\rangle$, and since $M_\gamma(\tau) : \mathbf{t}\mapsto(c\tau+d)\mathbf{t}$ (see above), we have

$$e(-m\mathbf{t})\circ M_\gamma(\tau) = M_\gamma(\tau)\circ e\bigg(\frac{-m\mathbf{t}}{c\tau+d}\bigg).$$

[6] Denote left multiplication by $\phi\in\mathbb{C}\langle\langle\mathbf{t},\mathbf{a}\rangle\rangle$ by L_ϕ. For $M\in\mathrm{Aut}\,H$, we have $M\circ L_{M^{-1}\phi} = L_\phi\circ M$. In particular, $M_\gamma(\tau)\circ L_{\mathbf{t}/(c\tau+d)} = L_{\mathbf{t}}\circ M_\gamma(\tau)$.

Thus, the left-hand side expands to

$$\begin{aligned}e(-m\mathbf{t})\circ \widetilde{M}_\gamma(\xi,\tau) &= e(-m\mathbf{t})\circ M_\gamma(\tau)\circ e\Big(\frac{c\xi\mathbf{t}}{c\tau+d}\Big)\\ &= M_\gamma(\tau)\circ e\Big(\frac{c\xi\mathbf{t}-m\mathbf{t}}{c\tau+d}\Big)\end{aligned}$$

The right-hand side expands to

$$\begin{aligned}&\widetilde{M}_\gamma(\xi+(ma+nc)\tau+(mb+nd),\tau)\circ e\big(-(ma+nc)\mathbf{t}\big)\\ &= M_\gamma(\tau)\circ e\Big(\frac{c\big(\xi+(ma+nc)\tau+(mb+nd)\big)\mathbf{t}}{c\tau+d}\Big)\circ e\big(-(ma+nc)\mathbf{t}\big)\\ &= M_\gamma(\tau)\circ e\Big(\frac{c\xi\mathbf{t}-m\mathbf{t}}{c\tau+d}\Big),\end{aligned}$$

which equals the left-hand side. It follows that $\widetilde{M}$ is a well-defined function on $\Gamma\times\mathbb{C}\times\mathfrak{h}$.

Since $\widetilde{M}$ defines a homomorphism $\mathbb{Z}^2\to\mathbb{Q}\langle\langle\mathbf{t}\rangle\rangle^\times$, to complete the proof we need only check that the restriction of $\widetilde{M}$ to $\mathrm{SL}_2(\mathbb{Z})\times\mathbb{C}\times\mathfrak{h}$ is a factor of automorphy. We will use the fact that $M_\gamma(\tau)$ is a factor of automorphy.

Let

$$\gamma_1=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad \gamma_2=\begin{pmatrix}p & q\\ r & s\end{pmatrix}\text{ and }\gamma_1\gamma_2=\begin{pmatrix}e & f\\ g & h\end{pmatrix}.$$

Set $(\xi',\tau')=\gamma_2(\xi,\tau)=(\xi/(r\tau+s),\gamma_2\tau)$. Then

$$\begin{aligned}&\widetilde{M}_{\gamma_1}(\xi',\tau')\widetilde{M}_{\gamma_2}(\xi,\tau)\\ &= M_{\gamma_1}(\gamma_2\tau)\circ e\Big(\frac{c\xi'\mathbf{t}}{c\tau'+d}\Big)\circ M_{\gamma_2}(\tau)\circ e\Big(\frac{r\xi\mathbf{t}}{r\tau+s}\Big)\\ &= M_{\gamma_1}(\gamma_2\tau)M_{\gamma_2}(\tau)\circ e\Big(\frac{c\xi'\mathbf{t}}{(c\tau'+d)(r\tau+s)}\Big)\circ e\Big(\frac{r\xi\mathbf{t}}{r\tau+s}\Big)\\ &= M_{\gamma_1\gamma_2}(\tau)\circ e\Big(\frac{c\xi\mathbf{t}}{(r\tau+s)\big(c(p\tau+q)+d(r\tau+s)\big)}+\frac{r\xi\mathbf{t}}{r\tau+s}\Big)\\ &= M_{\gamma_1\gamma_2}(\tau)\circ e\Big(\frac{\big(c+r(g\tau+h)\big)\xi\mathbf{t}}{(r\tau+s)(g\tau+h)}\Big)\\ &= M_{\gamma_1\gamma_2}(\tau)\circ e\Big(\frac{g\xi\mathbf{t}}{g\tau+h}\Big)\\ &= \widetilde{M}_{\gamma_1\gamma_2}(\xi,\tau).\end{aligned}$$

□

Remark 6.2. The bundle $\boldsymbol{\mathcal{P}}$ is a bundle of free Lie algebras. Its quotient by the commutator subalgebra of each fiber is the bundle over $\mathfrak{h}$ with framing $\mathbf{t}, \mathbf{a}$ and factor of automorphy $M_\gamma(\tau)$. So it is isomorphic to $\mathbb{H}$ by Example 3.4, as it should be.

7. Eisenstein series and Bernoulli numbers

Define the Bernoulli numbers B_n by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Recall that $B_0 = 1$, $B_1 = -1/2$ and that $B_{2k+1} = 0$ when $k > 0$.

There are several ways to normalize Eisenstein series $G_{2k} : \mathfrak{h} \to \mathbb{C}$. We will use the normalization used by Zagier [33]:

$$G_{2k}(\tau) = \frac{1}{2}\frac{(2k-1)!}{(2\pi i)^{2k}} \sum_{\substack{\lambda \in \mathbb{Z}\oplus\mathbb{Z}\tau \\ \lambda \neq 0}} \frac{1}{\lambda^{2k}} = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

(properly summed when $k = 1$), where $q = e(\tau)$ and $\sigma_k(n) = \sum_{d|n} d^k$. In particular

$$G_{2k}|_{q=0} = -\frac{B_{2k}}{4k} = \frac{(2k-1)!}{(2\pi i)^{2k}}\, \zeta(2k).$$

When $k > 1$, G_{2k} is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2k$:

$$G_k(\gamma\tau) = (c\tau + d)^k G_k(\tau), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

And G_2 satisfies

$$G_2(\gamma\tau) = (c\tau + d)^2 G_2(\tau) + ic(c\tau + d)/4\pi.$$

(Cf. [33, p. 457], bottom of page, and [33, p. 459], near bottom of page.)

The role of $G_2(\tau)$ in this work should be clarified by the following result, which follows from the transformation law for G_2 above.

Lemma 7.1. *If* $\mathrm{SL}_2(\mathbb{Z})$ *acts on* $\mathbb{C} \times \mathfrak{h}$ *by* $\gamma : (\xi, \tau) \mapsto \big(\xi/(c\tau + d), \gamma\tau\big)$, *then the form*

$$\frac{d\xi}{\xi} - 2 \cdot 2\pi i\, G_2(\tau)\, d\tau$$

is $\mathrm{SL}_2(\mathbb{Z})$*-invariant.*

This 1-form represents a generator of $H^1(\mathcal{L}'_{-1}, \mathbb{Z}(1)) \cong \mathbb{Z}$.

7.1. Some useful identities

The following well-known identities are used later in the paper. Since

$$\frac{1}{2}\coth(u/2) = \frac{1}{2}\frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}} = \frac{1}{2}\frac{e^u+1}{e^u-1} = \frac{1}{2}+\frac{1}{u}\frac{u}{e^u-1} = \sum_{m=0}^{\infty}\frac{B_{2m}}{(2m)!}u^{2m-1},$$

we have

$$\text{(7.1)}\qquad \frac{1}{u}-\frac{u/4}{\sinh^2(u/2)} = \frac{1}{u}+\frac{u}{2}\frac{d}{du}\coth(u/2) = \sum_{m=1}^{\infty}(2m-1)\frac{B_{2m}}{(2m)!}u^{2m-1}.$$

Rearranging gives the useful alternative form

$$\text{(7.2)}\qquad \sum_{m=0}^{\infty}(2m-1)\frac{B_{2m}}{(2m)!}\,u^{2m-1} = -\frac{u/4}{\sinh^2(u/2)}.$$

8. The Jacobi form $F(\xi,\eta,\tau)$

There are two versions of the function $F(u,v,\tau)$, one used by Levin-Racinet [24], the other by Zagier [33].[7] Denote them by $F(\xi,\eta,\tau)$ and $F^{\mathrm{Zag}}(u,v,\tau)$, respectively. Zagier's function is defined by

$$F^{\mathrm{Zag}}(u,v,\tau) := \frac{\theta'(0,\tau)\theta(u+v,\tau)}{\theta(u,\tau)\theta(v,\tau)},$$

where θ is the classical theta function

$$\theta(u,\tau) := \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{(n+\frac{1}{2})u},\quad q=e(\tau)$$

and θ' is its derivative with respect to u.

Their periodicity properties imply that $u=2\pi i\xi$, $v=2\pi i\eta$. Since

$$F(\xi,\eta,\tau) = \frac{1}{\xi}+\frac{1}{\eta} \bmod \text{ holomorphic functions}$$

and

$$F^{\mathrm{Zag}}(u,v,\tau) = \frac{1}{u}+\frac{1}{v} \bmod \text{ holomorphic functions}$$

[7] Calaque-Enriquez-Etingof [3] do not explicitly use the Jacobi form F. However, their connection is expressed in terms of the function $k(z,x|\tau)$, which is $F^{\mathrm{Zag}}(z,x,\tau)-1/x$. See their Section 1.2.

near the origin, it follows that

$$F(\xi,\eta,\tau) = 2\pi i F^{\mathrm{Zag}}(2\pi i\xi, 2\pi i\eta, \tau).$$

It satisfies the *symmetry condition*

$$F(\xi,\eta,\tau) = F(\eta,\xi,\tau) = -F(-\xi,-\eta,\tau).$$

8.1. Expansions

We use the formulas in [33], but write them using F in place of F^{Zag}.[8]

Set $q = \exp(2\pi i\tau)$. Then

(8.1)
$$F(\xi,\eta,\tau) = \pi i\big[\coth(\pi i\xi) + \coth(\pi i\eta)\big] + 4\pi \sum_{n=1}^{\infty}\Big(\sum_{d|n} \sin\big[2\pi(\frac{n}{d}\xi + d\eta)\big]\Big) q^n.$$

(8.2)
$$F(\xi,\eta,\tau) = \frac{1}{\xi} + \frac{1}{\eta} - 2\sum_{r,s=0}^{\infty} (2\pi i)^{1+\max\{r,s\}} \Big(\frac{\partial}{\partial\tau}\Big)^{\min\{r,s\}} G_{|r-s|+1}(\tau)\,\frac{\xi^r}{r!}\frac{\eta^s}{s!}.$$

8.2. Derivatives

Differentiating these with respect to η yields:

$$\begin{aligned}\frac{1}{\eta} + \eta\frac{\partial F}{\partial \eta}(\xi,\eta,\tau) &= -2\sum_{\substack{r\geq 0\\ s\geq 1}} (2\pi i)^{1+\max\{r,s\}} \Big(\frac{\partial}{\partial\tau}\Big)^{\min\{r,s\}} G_{|r-s|+1}(\tau)\,\frac{\xi^r}{r!}\frac{\eta^s}{(s-1)!} \\ &= \Big(\frac{1}{\eta} - \frac{(\pi i)^2\eta}{\sinh^2(\pi i\eta)}\Big) \\ &\quad + 8\pi^2\eta\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\cos\big[2\pi(\frac{n}{d}\xi + d\eta)\big]\Big) q^n. \qquad (8.3)\end{aligned}$$

Comparing the result of differentiating this with respect to ξ and (8.2) with respect to τ, we obtain the *heat equation*:

$$2\pi i\frac{\partial F}{\partial \tau}(\xi,\eta,\tau) = \frac{\partial^2 F}{\partial\xi\partial\eta}(\xi,\eta,\tau).$$

[8] Note the conflict in notation: Zagier sets $\xi = \exp u$ and $\eta = \exp v$, which conflicts with the variables (ξ,η) used by Levin-Racinet: $2\pi i(\xi,\eta) = (u,v)$. See [33, p. 455].

8.3. Elliptic and modularity properties

The *elliptic property,* [33, p. 456] is:

$$F(\xi + m\tau + n, \eta, \tau) = e(-m\eta)F(\xi, \eta, \tau) \qquad (m, n) \in \mathbb{Z}^2. \tag{8.4}$$

Here, as previously, $e(x) = \exp(2\pi i x)$. Zagier states a more general form of this, which follows from this one using the symmetry property of $F(\xi, \eta, \tau)$.

The *modularity property* is:

$$F(\xi/(c\tau + d), \eta/(c\tau + d), \gamma\tau) = (c\tau + d)e\big(c\xi\eta/(c\tau + d)\big)F(\xi, \eta, \tau). \tag{8.5}$$

In particular

$$F(\xi, \eta, \tau + 1) = F(\xi, \eta, \tau) = F(\xi + 1, \eta, \tau).$$

Proposition 8.1. *The function F induces a meromorphic section of the line bundle $\mathcal{N} \to \overline{\mathcal{E}} \times_{\overline{\mathcal{M}}_{1,1}} \overline{\mathcal{E}}$ that was constructed in Example 3.2. The divisor of the section is $[\Gamma_\iota] - [0_1] - [0_2]$, where Γ_ι is the graph in $E \times E$ of the involution ι that takes a point of E to its inverse.*

Proof. Since $F(\xi, \eta, \tau)$ is meromorphic on $\mathbb{C}^2$ for each $\tau \in \mathfrak{h}$, $\operatorname{div} F$ has no vertical components over $\mathcal{M}_{1,1}$. The polar locus of F over $\mathcal{M}_{1,1}$ contains $[0_1] + [0_2]$ with multiplicity one on every fiber over $\mathcal{M}_{1,1}$. The zero divisor of F over $\mathcal{M}_{1,1}$ contains $[\Gamma_\iota]$ with multiplicity one on the generic fiber over $\mathcal{M}_{1,1}$. Since the class of $\operatorname{div} F$ in $H^2(E_\tau^2)$ is constant and since $\operatorname{div} F$ has no vertical components, it suffices to show that the class of $\operatorname{div} F$ is exactly $[\Gamma_\iota] - [0_1] - [0_2]$ on an open set of fibers and also over $q = 0$.

Identity (8.1) implies that

$$\frac{1}{\pi i} F(\xi, \eta)|_{q=0} = \frac{w + 1}{w - 1} + \frac{u + 1}{u - 1}$$

where $w = \exp(2\pi i \xi)$ and $u = \exp(2\pi i \eta)$. The coordinates on the normalization of $E_0 \times E_0$ are (w, u). The identity sections are $w = 1$ and $u = 1$. The involution is given by $w = 1/w$. It is easily checked that

$$\frac{w + 1}{w - 1} + \frac{u + 1}{u - 1} = 0$$

implies that $wu = 1$. It follows that the restriction of $\operatorname{div} F$ to $E_0 \times E_0$ is $[\Gamma_\iota] - [0_1] - [0_2]$. But this implies that the divisor of F is $[\Gamma_\iota] - [0_1] - [0_2]$ on all nearby fibers. The result follows. ∎

This result can also be proved using the formula for F in terms of theta functions.

8.4. The Weierstrass $\wp$ function

Recall that

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\mathbb{Z}\oplus\mathbb{Z}\tau \\ \lambda\neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right].$$

The next result follows from the standard identity

$$\begin{aligned}\wp(z,\tau) &= \frac{1}{z^2} + \sum_{m=2}^{\infty}(2m-1)\Big(\sum_{\substack{\lambda\in\mathbb{Z}\oplus\mathbb{Z}\tau \\ \lambda\neq 0}}\frac{1}{\lambda^{2m}}\Big)z^{2m-2} \\ &= \frac{1}{z^2} + \sum_{m=2}^{\infty}\frac{2(2\pi i)^{2m}}{(2m-2)!}G_{2m}(\tau)z^{2m-2} \\ &= \frac{1}{z^2} + \sum_{m=1}^{\infty}\frac{2(2\pi i)^{2m+2}}{(2m)!}G_{2m+2}(\tau)z^{2m}.\end{aligned}$$

Lemma 8.2. *Suppose that x, y are commuting indeterminants. Then*

$$\begin{aligned}\frac{1}{2}\frac{xy}{x+y}\Big(\big(\wp(x,\tau) - \frac{1}{x^2}\big) - \big(\wp(y,\tau) - \frac{1}{y^2}\big)\Big) \\ = \sum_{m\geq 1}\frac{(2\pi i)^{2m+2}}{(2m)!}G_{2m+2}(\tau)\sum_{\substack{j+k=2m+1 \\ j,k>0}}(-1)^j x^j y^k\end{aligned}$$

in the ring $\mathcal{O}(\mathfrak{h})[[x,y]]$ of formal power series with coefficients in $\mathcal{O}(\mathfrak{h})$. □

8.5. The addition formula

The following identity is used in the proof of the integrability of the elliptic KZB connection.

Proposition 8.3 (Addition Formula)**.**

$$\begin{aligned}F(\xi,\eta_1,\tau)\frac{\partial F}{\partial\eta}(\xi,\eta_2,\tau) - F(\xi,\eta_2,\tau)\frac{\partial F}{\partial\eta}(\xi,\eta_1,\tau) \\ = F(\xi,\eta_1+\eta_2,\tau)\big(\wp(\eta_1,\tau) - \wp(\eta_2,\tau)\big).\end{aligned}$$

9. The universal elliptic KZB connection

This is a Γ-invariant flat connection constructed by Calaque, Enriquez and Etingof [3] and by Levin and Racinet [24] on the bundle

$$\mathfrak{p} \times \mathbb{C} \times \mathfrak{h} \to \mathbb{C} \times \mathfrak{h}. \tag{9.1}$$

So it descends to a flat connection on the bundle $\boldsymbol{\mathcal{P}} \to \mathcal{E}'$. It has regular singularities along the universal lattice:

$$\Lambda_{\mathfrak{h}} := \{(m\tau + n, \tau) \in \mathbb{C} \times \mathfrak{h}\}.$$

It therefore descends to a meromorphic connection on the bundle $\boldsymbol{\mathcal{P}} \to \mathcal{E}$ with regular singularities along the zero-section. In Section 12 we show that the natural extension of this connection to the q-disk has regular singularities along the nodal cubic.

In this section we will follow Levin-Racinet (with modifications).

9.1. Derivations

We have already explained the algebra homomorphism

$$\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a} \rangle\rangle \to \operatorname{End} \mathfrak{p}, \qquad f(\mathbf{t}, \mathbf{a}) \mapsto \{x \mapsto f(\mathbf{t}, \mathbf{a}) \cdot x\},$$

where $f(\mathbf{t}, \mathbf{a}) \cdot x := f(\operatorname{ad}_{\mathbf{t}}, \operatorname{ad}_{\mathbf{a}})(x)$. We will view $\mathfrak{p}$ as a Lie subalgebra of $\operatorname{Der} \mathfrak{p}$ via the adjoint action $\operatorname{ad} : \mathfrak{p} \to \operatorname{Der} \mathfrak{p}$, which is an inclusion as $\mathfrak{p}$ has trivial center. Every derivation δ can be written uniquely in the form

$$\delta = \delta(\mathbf{a}) \frac{\partial}{\partial \mathbf{a}} + \delta(\mathbf{t}) \frac{\partial}{\partial \mathbf{t}}.$$

Consequently, there is a linear isomorphism

$$\mathfrak{p} \frac{\partial}{\partial \mathbf{t}} \oplus \mathfrak{p} \frac{\partial}{\partial \mathbf{a}} \xrightarrow{\simeq} \operatorname{Der} \mathfrak{p}.$$

9.2. The formula

The connection is defined by a 1-form

$$\omega \in \Omega^1(\mathbb{C} \times \mathfrak{h}, \log \Lambda) \hat{\otimes} \operatorname{End} \mathfrak{p}$$

via the formula

$$\nabla f = df + \omega f$$

where $f : \mathbb{C} \times \mathfrak{h} \to \mathfrak{p}$ is a (locally defined) section of (9.1). Specifically,

$$\omega = \frac{1}{2\pi i} d\tau \otimes \mathbf{a}\frac{\partial}{\partial \mathbf{t}} + \psi + \nu$$

where

$$\psi = \sum_{m\geq 1}\left(\frac{(2\pi i)^{2m+1}}{(2m)!}G_{2m+2}(\tau)d\tau \otimes \sum_{\substack{j+k=2m+1\\ j,k>0}} (-1)^j [\mathrm{ad}_{\mathbf{t}}^j(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^k(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}}\right)$$

and

$$\nu = \mathbf{t}F(\xi, \mathbf{t}, \tau)\cdot \mathbf{a}\, d\xi + \frac{1}{2\pi i}\left(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi, \mathbf{t}, \tau)\right)\cdot \mathbf{a}\, d\tau.$$

Note that each term takes values in $\mathrm{Der}\,\mathfrak{p}$. Later we will show that its restriction to a punctured first order neighbourhood of the identity section takes values in a smaller subalgebra.

Remark 9.1. Each term of the lower central series of $\boldsymbol{\mathcal{P}}$ is preserved by the connection. The connection thus induces a connection on the bundle of abelianizations, which is isomorphic to $\mathbb{H}$ (cf. Remark 6.2). Example 5.2 implies that this induced connection on $\mathbb{H}$ is the natural connection.

9.3. Modularity

Recall that $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. In this section we shall prove:

Proposition 9.2. *The universal elliptic KZB connection is* $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$*-invariant. That is,*

$$\gamma^*\omega = \mathrm{Ad}\left(\widetilde{M}_\gamma\right)\cdot \omega - d\widetilde{M}_\gamma \widetilde{M}_\gamma^{-1}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

It suffices to check that the connection is invariant under $\mathbb{Z}^2$ and $\mathrm{SL}_2(\mathbb{Z})$. These are proved in the two following subsections.

9.3.1. Ellipticity: invariance under $\mathbb{Z}^2$

Lemma 9.3. *For all* $\delta \in \mathrm{Der}\,\mathfrak{p}$ *we have*

$$e(-m\mathbf{t})\cdot \delta = \delta + \frac{1 - e(-m\,\mathrm{ad}_{\mathbf{t}})}{\mathrm{ad}_{\mathbf{t}}}\delta(\mathbf{t}).$$

Proof. For all $x \in \mathfrak{p}$

$$\begin{aligned}
\big(e(-m\mathbf{t})\cdot\delta\big)(x) &= e(-m\mathbf{t})\delta\big(e(m\mathbf{t})(x)\big) && \text{(Equation 4.1)}\\
&= e(-m\mathbf{t})\delta(e(m\mathbf{t}))(x) + e(-m\mathbf{t})e(m\mathbf{t})\delta(x)\\
&= \delta(x) + \frac{1-e(-m\,\mathrm{ad}_{\mathbf{t}})}{2\pi i m\,\mathrm{ad}_{\mathbf{t}}}\delta(2\pi i m\mathbf{t})\cdot x && \text{(Lemma 4.2)}\\
&= \delta(x) + \frac{1-e(-m\,\mathrm{ad}_{\mathbf{t}})}{\mathrm{ad}_{\mathbf{t}}}\delta(\mathbf{t})\cdot x\\
&= \Big(\delta + \frac{1-e(-m\,\mathrm{ad}_{\mathbf{t}})}{\mathrm{ad}_{\mathbf{t}}}\delta(\mathbf{t})\Big)(x).
\end{aligned}$$

□

Corollary 9.4. *If $(m,n)\in\mathbb{Z}^2$, then*

$$(m,n)^*\Big(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial\mathbf{t}}d\tau\Big) - e(-m\mathbf{t})\cdot\Big(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial\mathbf{t}}d\tau\Big) = -\frac{1}{2\pi i}\frac{1-e(-m\mathbf{t})}{\mathbf{t}}(\mathbf{a})d\tau.$$

Proof. Apply the previous lemma with $\delta = \mathbf{a}\frac{\partial}{\partial\mathbf{t}}$. □

Corollary 9.5. *If $a,b\in\mathbb{N}$, then*

$$e(-m\mathbf{t})\cdot[\mathbf{t}^a\cdot\mathbf{a},\mathbf{t}^b\cdot\mathbf{a}]\frac{\partial}{\partial\mathbf{a}} = [\mathbf{t}^a\cdot\mathbf{a},\mathbf{t}^b\cdot\mathbf{a}]\frac{\partial}{\partial\mathbf{a}}.$$

Proof. This follows directly from the previous lemma as the derivation

$$[\mathbf{t}^a\cdot\mathbf{a},\mathbf{t}^b\cdot\mathbf{a}]\frac{\partial}{\partial\mathbf{a}}$$

annihilates $\mathbf{t}$. □

Corollary 9.6. *If $(m,n)\in\mathbb{Z}^2$, then $(m,n)^*\psi = e(-m\mathbf{t})\cdot\psi = \psi$.* □

Lemma 9.7. *For all $(m,n)\in\mathbb{Z}^2$*

$$(m,n)^*\nu - e(-m\mathbf{t})\cdot\nu = \frac{1}{2\pi i}\frac{1-e(-m\,\mathrm{ad}_{\mathbf{t}})}{\mathrm{ad}_{\mathbf{t}}}(\mathbf{a})d\tau.$$

Proof. Write $\nu = \nu_1 + \nu_2$, where

$$\nu_1 = \mathbf{t}F(\xi,\mathbf{t},\tau)\cdot\mathbf{a}\,d\xi \text{ and } \nu_2 = \frac{1}{2\pi i}\Big(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial\mathbf{t}}(\xi,\mathbf{t},\tau)\Big)\cdot\mathbf{a}\,d\tau.$$

Then

$$\begin{aligned}
&(m,n)^*\nu_1 - e(-m\mathbf{t})\cdot\nu_1 \\
&= \mathbf{t}F(\xi+m\tau+n,\mathbf{t})\cdot\mathbf{a}\,d(\xi+m\tau+n) - \mathbf{t}e(-mt)F(\xi,\mathbf{t})\cdot\mathbf{a}\,d\xi \\
&= \mathbf{t}e(-m\mathbf{t})F(\xi,\mathbf{t})\cdot\mathbf{a}\,(d\xi+md\tau) - \mathbf{t}e(-m\mathbf{t})F(\xi,\mathbf{t})\cdot\mathbf{a}\,d\xi \\
&= m\mathbf{t}e(-m\mathbf{t})F(\xi,\mathbf{t})\cdot\mathbf{a}\,d\tau.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial F}{\partial \mathbf{t}}(\xi+m\tau+n,\mathbf{t}) &= \frac{\partial}{\partial \mathbf{t}}(e(-m\mathbf{t})F(\xi,\mathbf{t})) \\
&= e(-m\mathbf{t})\frac{\partial F}{\partial \mathbf{t}}(\xi,\mathbf{t}) - 2\pi i m e(-m\mathbf{t})F(\xi,\mathbf{t}).
\end{aligned}$$

Thus

$$\begin{aligned}
&2\pi i\big((m,n)^*\nu_2 - e(-m\mathbf{t})\cdot\nu_2\big) \\
&= \left(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi+m\tau+n,\mathbf{t},\tau)\right)\cdot\mathbf{a}\,d\tau - e(-m\mathbf{t})\left(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi,\mathbf{t},\tau)\right)\cdot\mathbf{a}\,d\tau \\
&= -2\pi i m\mathbf{t}e(-m\mathbf{t})F(\xi,\mathbf{t})\cdot\mathbf{a}\,d\tau + \frac{1}{\mathbf{t}}\big(1-e(-m\mathbf{t})\big)\cdot\mathbf{a}\,d\tau.
\end{aligned}$$

□

If $(m,n)\in\mathbb{Z}^2$, then the results above imply that

$$(m,n)^*\omega = e(-m\mathbf{t})\cdot\omega(\xi,\tau).$$

Since $e(-m\mathbf{t})$ does not depend on (ξ,τ), $de(-m\mathbf{t}) = 0$ and ω is invariant under $\mathbb{Z}^2$.

9.3.2. Modularity: invariance under $\mathbf{SL_2}(\mathbb{Z})$ Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Recall that $M_\gamma(\tau)$ is defined by

$$\begin{aligned}
\mathbf{a} &\mapsto (c\tau+d)^{-1}\mathbf{a} + 2\pi i c\mathbf{t} \\
\mathbf{t} &\mapsto (c\tau+d)\mathbf{t}.
\end{aligned} \tag{9.2}$$

Its inverse is the linear map

$$\begin{aligned} \mathbf{a} &\mapsto (c\tau + d)\mathbf{a} - 2\pi i c\mathbf{t} \\ \mathbf{t} &\mapsto (c\tau + d)^{-1}\mathbf{t}. \end{aligned} \tag{9.3}$$

Lemma 9.8. *If $\gamma \in SL_2(\mathbb{Z})$ and $a, b \in \mathbb{N}$, then*

$$e(c\xi\mathbf{t}/(c\tau+d)) \cdot [\mathrm{ad}_{\mathbf{t}}^a(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}} = [\mathrm{ad}_{\mathbf{t}}^a(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}}.$$

Proof. This follows from (4.1) as the derivation $\delta = [\mathrm{ad}_{\mathbf{t}}^a(A), \mathrm{ad}_{\mathbf{t}}^b(A)]\frac{\partial}{\partial \mathbf{a}}$ vanishes on $\mathbf{t}$. □

Lemma 9.9. *If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $a, b \in \mathbb{N}$, then*

$$\mathrm{Ad}(M_\gamma(\tau))[\mathrm{ad}_{\mathbf{t}}^a(A), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}} = (c\tau+d)^{a+b-1}[\mathrm{ad}_{\mathbf{t}}^a(A), \mathrm{ad}_{\mathbf{t}}^b(A)]\frac{\partial}{\partial \mathbf{a}}.$$

Proof. Set $\delta = [\mathrm{ad}_{\mathbf{t}}^a(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}}$. Since $M_\gamma(\tau)^{-1}(\mathbf{t}) = (c\tau + d)^{-1}\mathbf{t}$, $\delta \circ M_\gamma^{-1}(\mathbf{t}) = 0$. Consequently, $\mathrm{Ad}(M_\gamma(\tau))\delta$ is of the form $f(\mathbf{t}, \mathbf{a})\frac{\partial}{\partial \mathbf{a}}$. The coefficient $f(\mathbf{t}, \mathbf{a})$ is computed as follows:

$$\begin{aligned} &\mathrm{Ad}(M_\gamma(\tau))\delta(\mathbf{a}) \\ &= M_\gamma(\tau) \circ \delta \circ M_\gamma(\tau)^{-1}(\mathbf{a}) \\ &= M_\gamma(\tau) \circ \delta\big((c\tau+d)\mathbf{a} - 2\pi i c\mathbf{t}\big) \\ &= (c\tau+d)M_\gamma(\tau)\big([\mathrm{ad}_{\mathbf{t}}^a(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]\big) \\ &= (c\tau+d)^{a+b+1}[\mathrm{ad}_{\mathbf{t}}^a\left((c\tau+d)^{-1}\mathbf{a} + 2\pi i c\mathbf{t}\right), \mathrm{ad}_{\mathbf{t}}^b\left((c\tau+d)^{-1}\mathbf{a} + 2\pi i c\mathbf{t}\right)] \\ &= (c\tau+d)^{a+b-1}[\mathrm{ad}_{\mathbf{t}}^a(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]. \end{aligned}$$

□

Corollary 9.10. *If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma^*\psi = \mathrm{Ad}(\widetilde{M}_\gamma)\psi$.*

Proof. This follows as, for each $k \geq 1$, the expression

$$G_{2k+2}(\tau)d\tau \otimes \sum_{\substack{a+b=2k+1 \\ a,b>0}} [\mathrm{ad}_{\mathbf{t}}^a(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}^b(\mathbf{a})]\frac{\partial}{\partial \mathbf{a}}$$

is multiplied by $(c\tau+d)^{2k}$ by both γ^* and $\widetilde{M}_\gamma(\xi, \tau)$. □

Lemma 9.11. *Set* $\nu_1 = \mathbf{t}F(\xi, \mathbf{t}, \tau) \cdot \mathbf{a}d\xi$. *Then*

$$\gamma^*\nu_1 - \widetilde{M}_\gamma(\xi,\tau)\nu_1 = -2\pi i c\mathbf{t}\, d\xi - \frac{c\xi\mathbf{t}}{c\tau+d} e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot \mathbf{a}\, d\tau.$$

Proof. First,

$$\begin{aligned}\widetilde{M}_\gamma(\xi,\tau)\nu_1 &= \widetilde{M}_\gamma(\xi,\tau)\big[\mathbf{t}F(\xi,\mathbf{t},\tau)\cdot\mathbf{a}\big]d\xi \\ &= e(c\xi\mathbf{t})(c\tau+d)\mathbf{t}F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot\big((c\tau+d)^{-1}\mathbf{a}+2\pi i c\mathbf{t}\big)d\xi \\ &= e(c\xi\mathbf{t})\mathbf{t}F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot\mathbf{a}\, d\xi + 2\pi i c\mathbf{t}\, d\xi\end{aligned}$$

as the value of $\mathbf{t}F(\xi,(c\tau+d)\mathbf{t},\tau)$ at $\mathbf{t}=0$ is $(c\tau+d)^{-1}$. This and the modular property of $F(\xi,\mathbf{t},\tau)$ then yield:

$$\begin{aligned}\gamma^*\nu_1 &= \mathbf{t}F(\xi/(c\tau+d),\mathbf{t},\gamma\tau)\cdot\mathbf{a}\gamma^* d\xi \\ &= (c\tau+d)\mathbf{t}e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot\mathbf{a}\left(\frac{d\xi}{c\tau+d} - \frac{c\xi d\tau}{(c\tau+d)^2}\right) \\ &= \mathbf{t}e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot\mathbf{a}\left(d\xi - \frac{c\xi d\tau}{c\tau+d}\right) \\ &= \widetilde{M}_\gamma(\xi,\tau)\nu_1 - 2\pi i c\mathbf{t}\, d\xi - \frac{c\xi\mathbf{t}}{c\tau+d}e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot\mathbf{a}\, d\tau.\end{aligned}$$

□

As a special case of the general formula, we have:

Lemma 9.12. *In* $\operatorname{Der}\mathfrak{p}$ *we have:*

$$e(c\xi\,\mathrm{ad}_\mathbf{t})\left(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial\mathbf{t}}\right) = \frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial\mathbf{t}} + \frac{1}{2\pi i}\frac{1-e(c\xi\mathbf{t})}{\mathbf{t}}\cdot\mathbf{a}.$$

Lemma 9.13. *Set*

$$\nu_2 = \frac{1}{2\pi i}\left(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial\mathbf{t}}(\xi,\mathbf{t},\tau)\right)\cdot\mathbf{a}\, d\tau.$$

Then

$$\begin{aligned}\gamma^*\nu_2 - \widetilde{M}_\gamma(\xi,\tau)\nu_2 = \frac{1}{2\pi i}&\frac{1-e(c\xi\mathbf{t})}{\mathbf{t}}\cdot\mathbf{a}\frac{d\tau}{(c\tau+d)^2} \\ &+ \frac{c\xi\mathbf{t}}{c\tau+d}e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\cdot\mathbf{a}\, d\tau.\end{aligned}$$

Proof. First note that the modularity property of $F(\xi,\mathbf{t},\tau)$ implies that

$$\begin{aligned}&\frac{\partial F}{\partial \mathbf{t}}(\xi/(c\tau+d),\mathbf{t},\gamma\tau)\\&=(c\tau+d)\frac{\partial}{\partial \mathbf{t}}\big[e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\big]\\&=(c\tau+d)e(c\xi\mathbf{t})\big[c\xi F(\xi,(c\tau+d)\mathbf{t},\tau)+(c\tau+d)\frac{\partial F}{\partial \mathbf{t}}(\xi,(c\tau+d)\mathbf{t},\tau)\big].\end{aligned}$$

Thus

$$\begin{aligned}2\pi i\gamma^*\nu_2&=\Big(\frac{1}{\mathbf{t}}+\mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi/(c\tau+d),\mathbf{t},\gamma\tau)\Big)\cdot\mathbf{a}\,d\Big(\frac{a\tau+b}{c\tau+d}\Big)\\&=\Big(\frac{1}{\mathbf{t}}+(c\tau+d)^2e(c\xi\mathbf{t})\mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi,(c\tau+d)\mathbf{t},\tau)\\&\qquad+c\xi\mathbf{t}(c\tau+d)e(c\xi\mathbf{t})F(\xi,(c\tau+d)\mathbf{t},\tau)\Big)\cdot\mathbf{a}\,\frac{d\tau}{(c\tau+d)^2}.\end{aligned}$$

Since

$$\widetilde{M}_\gamma(\xi,\tau)(\mathbf{t})=(c\tau+d)\mathbf{t}\text{ and }\widetilde{M}_\gamma(\xi,\tau)(\mathbf{a})=e(c\xi\mathbf{t})\cdot\mathbf{a}/(c\tau+d)+2\pi ic\mathbf{t}$$

we have

$$\begin{aligned}&2\pi i\widetilde{M}_\gamma(\xi,\tau)\nu_2\\&=\Big(\frac{1}{(c\tau+d)\mathbf{t}}+(c\tau+d)\mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi,(c\tau+d)\mathbf{t},\tau)\Big)\\&\quad\cdot\big(e(c\xi\mathbf{t})\cdot\mathbf{a}/(c\tau+d)+2\pi ic\mathbf{t}\big)\,d\tau\\&=e(c\xi\mathbf{t})\Big(\frac{1}{\mathbf{t}}+(c\tau+d)^2\mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(\xi,(c\tau+d)\mathbf{t},\tau)\Big)\cdot\mathbf{a}\frac{d\tau}{(c\tau+d)^2}\end{aligned}$$

as $\frac{1}{\eta}+\eta\frac{\partial F}{\partial\eta}(\xi,\eta,\tau)$ is holomorphic in η and vanishes at $\eta=0$ by (8.3).

The previous lemma implies that

$$\big(e(c\xi\,\mathrm{ad}_{\mathbf{t}})-1\big)\Big(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial\mathbf{t}}\Big)=\frac{1}{2\pi i}\Big(\frac{1-e(c\xi\mathbf{t})}{\mathbf{t}}\Big)\cdot\mathbf{a}.$$

Now assemble the pieces to obtain the result. □

Combining the last two computations, we obtain:

Corollary 9.14. *For all* $\gamma\in\mathrm{SL}_2(\mathbb{Z})$,

$$\gamma^*\nu-\widetilde{M}_\gamma\nu=\frac{1-e(c\xi\mathbf{t})}{2\pi i\mathbf{t}}\cdot\mathbf{a}\frac{d\tau}{(c\tau+d)^2}-2\pi ic\mathbf{t}\,d\xi.$$

Lemma 9.15. *For all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$d\widetilde{M}_\gamma \widetilde{M}_\gamma^{-1} = e(c\xi\mathbf{t}) \cdot \big(dM_\gamma M_\gamma^{-1}\big) + 2\pi i c\mathbf{t}\, d\xi.$$

Proof. Since $\widetilde{M}_\gamma(\xi,\tau) = e(c\xi\mathbf{t})M_\gamma(\tau)$, we have

$$\begin{aligned} d\widetilde{M}_\gamma \widetilde{M}_\gamma^{-1} &= d\big(e(c\xi\mathbf{t})M_\gamma\big)M_\gamma^{-1}e(-c\xi\mathbf{t}) \\ &= \big(e(c\xi\mathbf{t})dM_\gamma + 2\pi i c\mathbf{t}e(c\xi\mathbf{t})M_\gamma d\xi\big)M_\gamma^{-1}e(-c\xi\mathbf{t}) \\ &= e(c\xi\mathbf{t}) \cdot \big(dM_\gamma M_\gamma^{-1}\big) + 2\pi i c\mathbf{t}\, d\xi. \end{aligned}$$

□

Lemma 9.16. *For all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, *we have*

$$\gamma^*\left(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau\right) - M_\gamma\left(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau\right) + dM_\gamma M_\gamma^{-1} = 0.$$

Proof. This is best done using matrices with respect to the basis $\{\mathbf{a},\mathbf{t}\}$ of H. We have

$$\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau = \frac{1}{2\pi i}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\tau$$

and

$$M_\gamma(\tau) = \begin{pmatrix} (c\tau+d)^{-1} & 0 \\ 2\pi i c & c\tau+d \end{pmatrix}, \qquad M_\gamma(\tau)^{-1} = \begin{pmatrix} c\tau+d & 0 \\ -2\pi i c & (c\tau+d)^{-1} \end{pmatrix}.$$

So

$$\begin{aligned} dM_\gamma M_\gamma^{-1} &= \begin{pmatrix} -c(c\tau+d)^{-2} & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} c\tau+d & 0 \\ -2\pi i c & (c\tau+d)^{-1} \end{pmatrix} d\tau \\ &= \begin{pmatrix} -c & 0 \\ -2\pi i c^2(c\tau+d) & c \end{pmatrix} \frac{d\tau}{c\tau+d} \end{aligned}$$

and

$$\begin{aligned} &M_\gamma\left(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau\right) \\ &= \frac{1}{2\pi i}\begin{pmatrix} (c\tau+d)^{-1} & 0 \\ 2\pi i c & c\tau+d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c\tau+d & 0 \\ -2\pi i c & (c\tau+d)^{-1} \end{pmatrix} d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\begin{pmatrix} -2\pi i c & (c\tau+d)^{-1} \\ -(2\pi i c)^2(c\tau+d) & 2\pi i c \end{pmatrix}\frac{d\tau}{c\tau+d} \\
&= \frac{1}{2\pi i}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\frac{d\tau}{(c\tau+d)^2} + \begin{pmatrix} -c & 0 \\ -2\pi i c^2(c\tau+d) & c \end{pmatrix}\frac{d\tau}{c\tau+d} \\
&= \gamma^*\Big(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) + dM_\gamma M_\gamma^{-1}.
\end{aligned}
$$

□

Final computation: For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have:

$$
\begin{aligned}
&\gamma^*\omega - \widetilde{M}_\gamma\omega + d\widetilde{M}_\gamma\widetilde{M}_\gamma^{-1} \\
&= \gamma^*\Big(\nu + \frac{\mathbf{a}}{2\pi i}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) - \widetilde{M}_\gamma\Big(\nu + \frac{\mathbf{a}}{2\pi i}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) + e(c\xi\mathbf{t})\cdot\big(dM_\gamma M_\gamma^{-1}\big) + 2\pi i c\mathbf{t}\, d\xi \\
&= \gamma^*\Big(\frac{\mathbf{a}}{2\pi i}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) + \frac{1-e(c\xi\mathbf{t})}{2\pi i\mathbf{t}}\cdot\mathbf{a}\frac{d\tau}{(c\tau+d)^2} - e(c\xi\mathbf{t})\cdot M_\gamma\Big(\frac{\mathbf{a}}{2\pi i}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) \\
&\qquad\qquad + e(c\xi\mathbf{t})\cdot\big(dM_\gamma M_\gamma^{-1}\big) \\
&= e(c\xi\mathbf{t})\cdot\Big(\gamma^*\Big(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) - M_\gamma\Big(\frac{1}{2\pi i}\mathbf{a}\frac{\partial}{\partial \mathbf{t}}d\tau\Big) + dM_\gamma M_\gamma^{-1}\Big) \\
&= 0.
\end{aligned}
$$

9.4. Integrability

Proposition 9.17. *The 1-form ω is closed.*

Proof. It is clear that

$$
d\Big(\frac{1}{2\pi i}d\tau \otimes \mathbf{a}\frac{\partial}{\partial \mathbf{t}} + \psi\Big) = 0
$$

as these terms do not depend upon ξ. The heat equation implies that

$$
\begin{aligned}
2\pi i\, d\nu &= 2\pi i\ \mathrm{ad}_\mathbf{t}\, dF(\xi,\mathrm{ad}_\mathbf{t},\tau)(\mathbf{a})\wedge d\xi + d\Big(\frac{1}{\mathrm{ad}_\mathbf{t}} + \mathrm{ad}_\mathbf{t}\,\frac{\partial F}{\partial \mathbf{t}}(\xi,\mathrm{ad}_\mathbf{t},\tau)\Big)(\mathbf{a})\wedge d\tau \\
&= \mathrm{ad}_\mathbf{t}\,\Big(2\pi i\frac{\partial F}{\partial \tau}(\xi,\mathrm{ad}_\mathbf{t},\tau) - \frac{\partial^2 F}{\partial\xi\partial \mathbf{t}}(\xi,\mathrm{ad}_\mathbf{t},\tau)\Big)(\mathbf{a})d\tau\wedge d\xi \\
&= 0,
\end{aligned}
$$

so that $d\omega = 0$. □

The proof of the vanishing of $[\omega, \omega]$ is quite involved. For this we employ the elegant calculus developed by Levin and Racinet in [24, §3.1].

9.4.1. The Levin-Racinet calculus For $U, V \in \mathbb{L}(\mathbf{a}, \mathbf{t})^\wedge$, define

$$x^r y^s \circ (U, V) = [\mathbf{t}^r \cdot U, \mathbf{t}^s \cdot V].$$

This extends linearly to an action $f(x, y) \circ (U, V)$ of polynomials and power series $f(x, y)$ in commuting indeterminants on ordered pairs of elements of $\mathbb{L}(\mathbf{t}, \mathbf{a})$. When U and V are equal, one has the identity $f(x, y) \circ (U, U) = -f(y, x) \circ (U, U)$, so that

$$f(x, y) \circ (U, U) = \frac{1}{2}\big(f(x, y) - f(y, x)\big) \circ (U, U). \tag{9.4}$$

As an example of how this notation is used, note that Lemma 8.2 implies that

$$2\pi i \psi = \frac{1}{2} \frac{xy}{x+y}\Big(\wp(x) - \frac{1}{x^2} - \wp(y) + \frac{1}{y^2}\Big) \circ (\mathbf{a}, \mathbf{a}) \frac{\partial}{\partial \mathbf{a}} \otimes d\tau. \tag{9.5}$$

Two more identities will be needed in the proof of the vanishing of $[\omega, \omega]$.

Lemma 9.18. *Suppose that $U, V \in \mathbb{L}(\mathbf{t}, \mathbf{a})^\wedge$.*

(i) (Jacobi identity) If $f(x, y) \in \mathbb{C}[[x, y]]$, then

$$\operatorname{ad}_{\mathbf{t}} \big(f(x, y) \circ (U, V)\big) = (x + y) f(x, y) \circ (U, V).$$

(ii) If δ is a continuous derivation of $\mathbb{C}\langle\langle \mathbf{t}, \mathbf{a}\rangle\rangle$ and $g(x) \in \mathbb{C}[[x]]$, then

$$\delta(g(\operatorname{ad}_{\mathbf{t}})V) = g(\operatorname{ad}_{\mathbf{t}})\delta(V) + \left(\frac{g(x+y) - g(y)}{x}\right) \circ (\delta(\mathbf{t}), V).$$

Proof. The first identity encodes the Jacobi identity and is left as an easy exercise. To prove the second, note that both sides are linear in g, so that, by continuity, it suffices to prove the result when g is a monomial x^n. This holds trivially when $n \leq 1$. The general case follows by induction using the Jacobi identity. □

9.5. Integrability

The following computation completes the proof of integrability.

Lemma 9.19. *The 2-form $[\omega, \omega]$ vanishes, so that ω is integrable.*

Proof. Note that

$$\pi i\,[\omega,\omega] = \Big[d\tau\otimes \mathbf{a}\frac{\partial}{\partial \mathbf{t}} + 2\pi i\,\psi + d\tau\otimes\Big(\frac{1}{\mathrm{ad}_{\mathbf{t}}} + \mathrm{ad}_{\mathbf{t}}\,\frac{\partial F}{\partial \mathbf{t}}(\xi,\mathrm{ad}_{\mathbf{t}},\tau)\Big)(\mathbf{a}), \mathrm{ad}_{\mathbf{t}}\,F(\xi,\mathrm{ad}_{\mathbf{t}},\tau)(\mathbf{a})\otimes d\xi\Big].$$

The expression (9.5) implies that the coefficient of $d\tau\wedge d\xi$ is

$$[\mathbf{a}\frac{\partial}{\partial \mathbf{t}}, \mathbf{t}F(\mathbf{t})(\mathbf{a})] + \frac{1}{2}\Big[\frac{xy}{x+y}(\wp(x) - \frac{1}{x^2} - \wp(y) + \frac{1}{y^2})\circ(\mathbf{a},\mathbf{a})\frac{\partial}{\partial \mathbf{a}}, \mathbf{t}F(\mathbf{t})(\mathbf{a})\Big] + \Big[\frac{1}{\mathbf{t}} + \mathbf{t}F'(\mathbf{t})(\mathbf{a}), \mathbf{t}F(\mathbf{t})(\mathbf{a})\Big]$$

where $F(z)$ denotes $F(\xi,z,\tau)$ and $F'(z)$ denotes $\partial F/\partial z(\xi,z,\tau)$. We'll compute these three terms, one at a time.

Since $[\delta,\mathrm{ad}_v] = \mathrm{ad}_{\delta(v)}$, Lemma 9.18 and equation (9.4) imply that the first term is

$$\begin{aligned}[\mathbf{a}\frac{\partial}{\partial \mathbf{t}}, \mathbf{t}F(\mathbf{t})\cdot\mathbf{a}] &= \Big(\frac{(x+y)F(x+y) - yF(y)}{x}\Big)\circ(\mathbf{a},\mathbf{a})\\ &= \frac{1}{2}\Big(\Big(\frac{y^2-x^2}{xy}\Big)\,F(x+y) - \frac{y}{x}\,F(y) + \frac{x}{y}\,F(x)\Big)\circ(\mathbf{a},\mathbf{a}).\end{aligned}$$

The identity $[\delta,\mathrm{ad}_v] = \mathrm{ad}_{\delta(v)}$ and the Jacobi identity (Lemma 9.18) imply that the second term is

$$\begin{aligned}&\frac{1}{2}\,\mathrm{ad}_{\mathbf{t}F(\mathbf{t})}\Big(\frac{xy}{x+y}\Big(\wp(x) - \frac{1}{x^2} - \wp(y) + \frac{1}{y^2}\Big)\circ(\mathbf{a},\mathbf{a})\Big)\\ &= \frac{1}{2}xyF(x+y)\Big(\wp(x) - \frac{1}{x^2} - \wp(y) + \frac{1}{y^2}\Big)\circ(\mathbf{a},\mathbf{a})\\ &= \frac{1}{2}\Big(\Big(\frac{x^2-y^2}{xy}\Big)\,F(x+y) + xy(\wp(x)-\wp(y))F(x+y)\Big)\circ(\mathbf{a},\mathbf{a})\end{aligned}$$

The addition formula, Proposition 8.3, implies that the third term is

$$\begin{aligned}&\Big(\Big(\frac{1}{x} + xF'(x)\Big)yF(y)\Big)\circ(\mathbf{a},\mathbf{a})\\ &= \Big(\frac{y}{x}\,F(y) + xyF'(x)F(y)\Big)\circ(\mathbf{a},\mathbf{a})\end{aligned}$$

$$= \frac{1}{2}\Big(\frac{y}{x}\,F(y) - \frac{x}{y}\,F(x) + xy\big(F'(x)F(y) - F'(y)F(x)\big)\Big) \circ (\mathbf{a},\mathbf{a})$$
$$= \frac{1}{2}\Big(\frac{y}{x}\,F(y) - \frac{x}{y}\,F(x) - xy\big(\wp(x) - \wp(y)\big)F(x+y)\Big) \circ (\mathbf{a},\mathbf{a}).$$

These three terms clearly sum to 0. □

Part 3. Hodge theory and applications

The main goal of this section is to show that the elliptic KZB connection underlies an admissible variation of mixed Hodge structure (MHS) over $\mathcal{E}' = \mathcal{M}_{1,2}$ and to show that this variation is isomorphic to the variation of MHS whose fiber over $[E, x]$ is the Lie algebra of the unipotent fundamental group of (E', x). We use this to show that the periods of the limit MHS on the fiber of $\boldsymbol{\mathcal{P}}$ over $[E_{\partial/\partial q}, \partial/\partial w]$ are multizeta values and to derive an explicit formula for the natural morphism of MHS

$$\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0,1,\infty\}, \partial/\partial w) \to \pi_1^{\mathrm{un}}(E'_{\partial/\partial q}, \partial/\partial w).$$

As preparation, we show that the KZB connection extends to a meromorphic connection over $\overline{\mathcal{M}}_{1,2}$ with regular singularities and pronilpotent monodromy about the boundary divisors. To fill a gap in the literature, we prove, in Section 14, that the local system associated to the universal elliptic KZB connection on $\boldsymbol{\mathcal{P}}$ is the local system $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$. It can be used to prove the analogous result for the KZB connection over $\mathcal{M}_{1,1+n}$. This complements results in [24, §4] and [3, §4.3].

Throughout, $\mathfrak{p} = \mathbb{L}(\mathbf{t}, \mathbf{a})^\wedge$ and $\mathcal{P}$ is the corresponding prounipotent group. Set

$$\mathrm{Der}^0\,\mathfrak{p} = \{\delta \in \mathrm{Der}\,\mathfrak{p} : \delta([\mathbf{t}, \mathbf{a}]) = 0\}.$$

This is the infinitesimal analogue of the mapping class group $\Gamma_{1,\vec{1}}$.

The reader is assumed to be familiar with the basics of Deligne's theory of mixed Hodge structures [7]. A good introductory reference is the book [30] by Steenbrink and Peters. Another good introductory reference is Carlson's paper [4]. An exposition of the construction of the mixed Hodge structure on the unipotent fundamental group of a smooth variety can be found in [11].

10. Extending $\boldsymbol{\mathcal{P}}$ to $\overline{\mathcal{M}}_{1,2}$

The punctured universal elliptic curve $\mathcal{E}'$ is isomorphic to $\mathcal{M}_{1,2}$ and is the complement of a normal crossing divisor in $\overline{\mathcal{M}}_{1,2}$. This divisor has two components: the zero section and the nodal cubic. The flat bundle $\mathcal{P}$ over $\mathcal{E}'$ has prounipotent monodromy about each, and thus extends naturally to a bundle over $\overline{\mathcal{M}}_{1,2}$ with regular singularities and (pro) nilpotent residues along each component.

This extension is easily described. First, the complement of the Tate curve in $\overline{\mathcal{M}}_{1,2}$ is the universal elliptic curve $\mathcal{E}$, which is a quotient of $\mathbb{C} \times \mathfrak{h}$. The bundle $\boldsymbol{\mathcal{P}}$ defined in Section 6.2 is defined over $\mathcal{E}$, not just over $\mathcal{E}'$. We take this to be the extension across the zero section.

To extend $\boldsymbol{\mathcal{P}}$ across the Tate curve, recall from Example 3.4 that the holomorphic vector bundle $\mathcal{H} := \mathbb{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{D}^*}$ associated to $\mathbb{H}$ is trivial on the punctured q-disk $\mathbb{D}^*$. The framing $\mathbf{t}$, $\mathbf{a}$ of $\mathcal{H}$ over $\mathbb{D}^*$ determines an extension $\overline{\mathcal{H}}$ of $\mathbb{H}$ to the entire q-disk $\mathbb{D}$ that is framed by $\mathbf{t}$ and $\mathbf{a}$. The formula for the natural connection on $\mathcal{H}$ in Example 5.2 implies it extends to a meromorphic connection on $\overline{\mathcal{H}}$ with a regular singular point at the cusp $q = 0$ and with nilpotent residue. This implies that $\overline{\mathcal{H}}$ is Deligne's canonical extension of $\mathcal{H}$ to $\overline{\mathcal{M}}_{1,1}$. (Cf. [6].)

Since $\mathfrak{p} = \mathbb{L}(\mathbf{t}, \mathbf{a})^\wedge$, this determines an extension of $\mathfrak{p} \times \mathbb{C} \times \mathbb{D}^* \to \mathbb{C} \times \mathbb{D}^*$ to $\mathbb{C} \times \mathbb{D}$; its fiber over $q \in \mathbb{D}$ is the free Lie algebra generated by the fiber of $\overline{\mathcal{H}}$ over q, which is naturally isomorphic to $\mathbb{L}(\mathbf{t}, \mathbf{a})^\wedge$. The pullback of the universal elliptic curve over $\overline{\mathcal{M}}_{1,1}$ to $\mathbb{D}$ minus the double point P of the nodal cubic is the quotient of $\mathbb{C} \times \mathfrak{h}$ by the subgroup

$$\Gamma := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \ltimes \mathbb{Z}^2$$

of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

The action of Γ on $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$ induces an action of Γ on $\mathfrak{p} \times \mathbb{C} \times \mathbb{D}$. The pullback of $\boldsymbol{\mathcal{P}}$ to $\mathcal{E}_{\mathbb{D}^*}$ thus extends to a bundle $\overline{\boldsymbol{\mathcal{P}}}$ over $\mathcal{E}_{\mathbb{D}}$ (minus the double point P of the nodal cubic) as the quotient of this action.

Formulas (8.1), (8.2) and (8.3) imply that this extension has regular singularities along the identity section and along the nodal cubic $q = 0$.

Proposition 10.1. *The meromorphic extension of the elliptic KZB connection defined above has regular singularities along the two boundary components of $\overline{\mathcal{M}}_{1,2}$: the nodal cubic E_0 and the identity section. It has pronilpotent residue at each codimension 1 boundary point.* □

11. Restriction to E'_τ

Fix $\tau \in \mathfrak{h}$. The first task in proving that the universal elliptic KZB connection has the expected monodromy is to check that its restriction to the fiber E'_τ of $\mathcal{E}' \to \mathcal{M}_{1,1}$ induces an isomorphism of $\pi_1(E'_\tau, x)^{\mathrm{un}}$ with $\mathcal{P}$.

The restriction of the universal elliptic KZB connection to E'_τ is

$$\nabla = d + \nu_1 = d + \mathbf{t}F(\xi, \mathbf{t}, \tau) \cdot \mathbf{a}\, d\xi.$$

Identify $\mathfrak{p}$ with the image of the adjoint action $\mathrm{ad} : \mathfrak{p} \to \operatorname{Der} \mathfrak{p}$ which is injective as $\mathfrak{p}$ has trivial center. With this identification, ∇ takes values in $\mathfrak{p}$. Fix $x \in \mathbb{C} - \Lambda_\tau$. The associated monodromy representation $\rho_x : \pi_1(E'_\tau, x) \to \mathcal{P}$ is given by (cf. Cor. 5.9)

$$\rho_x(\gamma) = \Big(1 + \int_{c_\gamma} \nu_1 + \int_{c_\gamma} \nu_1\nu_1 + \int_{c_\gamma} \nu_1\nu_1\nu_1 + \cdots\Big) e(-m(\gamma)\mathbf{t})$$

where $\rho(\gamma) = \big(m(\gamma), n(\gamma)\big) \in \mathbb{Z}^2$. (That is, the class of γ in $H_1(E'_\tau)$ is $n(\gamma)\mathbf{a} + m(\gamma)\mathbf{b}$.)

Proposition 11.1. *If $[\gamma] = n\mathbf{a} + m\mathbf{b} \in H_1(E'_\tau)$, then*

$$\Theta_x(\gamma) \equiv 1 + (m\tau + n)\mathbf{a} - 2\pi i m \mathbf{t} \bmod (\mathbf{t}, \mathbf{a})^2.$$

Proof. Observe that

$$\begin{aligned} \nu_1 &= \mathbf{t}F(\xi, \mathbf{t}, \tau) \cdot \mathbf{a}\, d\xi \\ &= \mathbf{t}\Big(\frac{1}{\mathbf{t}} + \frac{1}{\xi} + \text{ holomorphic in } \xi\Big) \cdot \mathbf{a}\, d\xi \\ &\equiv \mathbf{a}\, d\xi \bmod (\mathbf{t}, \mathbf{a})^2 \end{aligned}$$

and that $\operatorname{Res}_{\xi=0} \nu_1 \equiv [\mathbf{t}, \mathbf{a}] \bmod (\mathbf{t}, \mathbf{a})^3$. It follows that $\Theta_x(\mathbf{a}) \equiv 1 + \mathbf{a} \bmod (\mathbf{t}, \mathbf{a})^2$ and that

$$\Theta_x(\mathbf{b}) \equiv (1 + \tau\mathbf{a})e(-\mathbf{t}) \equiv 1 + \tau\mathbf{a} - 2\pi i \mathbf{t} \bmod (\mathbf{t}, \mathbf{a})^2.$$

□

Corollary 11.2. *The universal elliptic KZB connection induces the identification*

$$\mathbb{C}\mathbf{a} \oplus \mathbb{C}\mathbf{t} \to H_1(E_\tau; \mathbb{C})$$

that takes $\mathbf{a}$ to $\mathbf{a}$ and $2\pi i\mathbf{t}$ to $\tau\mathbf{a} - \mathbf{b}$, the Poincaré dual of ω_τ.

This corresponds to the framing of the bundle $\mathbb{H}$ given in Example 3.4. (This statement can also be deduced from Remark 9.1.)

The universal connection induces an isomorphism of the unipotent completion of $\pi_1(E'_\tau, x)$ with $\mathcal{P}$ for all $(x, \tau) \in \mathcal{E}_\mathfrak{h}$. This is a special case of [3, Prop. 2.2].

Corollary 11.3. *The monodromy of the restriction of the universal elliptic KZB connection to the fiber E'_τ of $\mathcal{E}'$ over $[E_\tau] \in \mathcal{M}_{1,1}$ is a homomorphism $\pi_1(E'_\tau, x) \to \mathcal{P}$ that induces an isomorphism $\pi_1(E'_\tau, x)^{\mathrm{un}} \to \mathcal{P}$.*

11.1. A better framing of $\mathbb{H}$

To get rid of the powers of $2\pi i$ in the formulas, we replace $2\pi i\, d\tau$ by dq/q and set

$$T = 2\pi i\, \mathbf{t} \text{ and } A = (2\pi i)^{-1}\mathbf{a}. \tag{11.1}$$

Remark 11.4. There is a conceptual reason the basis A, T is a good choice. Denote the fiber of the universal elliptic curve $\mathcal{E}_\mathbb{D} \to \mathbb{D}$ over $q \in \mathbb{D}$ by E_q. For each nonzero tangent vector $\vec{v}$ of $0 \in \mathbb{D}$, there is a limit MHS on H_1 of the fiber, which we denote by $H_1(E_{\vec{v}})$ and think of as the homology of the fiber over $\vec{v}$. This MHS is an extension

$$0 \to \mathbb{Q}(1) \to H_1(E_{\vec{v}}) \to \mathbb{Q}(0) \to 0$$

which splits when $\vec{v} = \partial/\partial q$. In this case, the copy of $\mathbb{Q}(1)$ is spanned by A and the copy of $\mathbb{Q}(1)$ is spanned by T. As will become apparent in Part 4, this limit MHS has a $\mathbb{Q}$-DR form. The basis A, T is a $\mathbb{Q}$-DR basis of $H_1(E_{\partial/\partial q})$. The basis above is the extension of this basis to a framing of the bundle $\overline{\mathcal{H}}_\mathbb{D}$ in which $F^0\overline{\mathcal{H}}$ is trivialized by T. The basis $\mathbf{a}$, $\mathbf{b}$ is a $\mathbb{Q}$-Betti basis of $H_1(E_{\vec{v}})tro$.

In this frame $[\mathbf{t}, \mathbf{a}] = [T, A]$,

$$\frac{1}{2\pi i} d\tau \otimes \mathbf{a}\frac{\partial}{\partial \mathbf{t}} = \frac{dq}{q} \otimes A\frac{\partial}{\partial T}$$

and the terms in the KZB connection become:

$$\psi = \sum_{m\geq 1}\left(\frac{G_{2m+2}(\tau)}{(2m)!}\frac{dq}{q} \otimes \sum_{\substack{j+k=2m+1\\ j,k>0}} (-1)^j [\mathrm{ad}_T^j(A), \mathrm{ad}_T^k(A)]\frac{\partial}{\partial A}\right),$$

and

$$\nu = TF^{\mathrm{Zag}}(2\pi i\xi, T, \tau)\cdot A\frac{dq}{q} + \left(\frac{1}{T} + T\frac{\partial}{\partial T}F^{\mathrm{Zag}}(2\pi i\xi, T, \tau)\right)\cdot A\frac{dq}{q}$$

Remark 11.5. The periodicity properties (8.4) and (8.5) of F and the formulas for the factors of automorphy (6.1) and (6.2) imply that the connection is pulled back from a connection on the trivial bundle $\mathbb{L}(A,T)^\wedge \times \mathbb{C}^* \times \mathbb{D} \to \mathbb{C}^* \times \mathbb{D}$ along the map $\mathbb{C}\times\mathfrak{h} \to \mathbb{C}^*\times\mathbb{D}$ defined by $(\xi,\tau) \mapsto (w,q) := (e(\xi), e(\tau))$.

12. Restriction to the first-order Tate curve

In this section we compute the restriction of the universal elliptic KZB connection to the first order Tate curve. This allows us to see that the connection has regular singularities along the nodal cubic. Restricting further to the the regular locus $\mathbb{P}^1 - \{0, 1, \infty\}$ of the nodal cubic minus its identity is the first step in computing the image of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v})^{\mathrm{un}}$ in the limit mixed Hodge structure on the unipotent fundamental group of the first order smoothing of the Tate curve.

The restriction of the natural extension of the KZB connection to the boundary divisor $q = 0$ is the image of ω under the restriction mapping

$$\Omega^1_{\overline{\mathcal{E}}}(\log(\overline{\mathcal{M}}_{1,1} \cup \overline{E}_0)) \to \Omega^1_{\overline{\mathcal{E}}}(\log(\overline{\mathcal{M}}_{1,1} \cup \overline{E}_0)) \otimes_{\mathcal{O}(\overline{E})} \mathcal{O}_{E_0},$$

where $E_0 \cong \mathbb{G}_m$ is the fiber $\overline{E}_0$ of $\overline{\mathcal{E}}$ over $q = 0$ with the double point removed. The identity section of $\overline{\mathcal{E}}$ is identified with $\overline{\mathcal{M}}_{1,1}$. In concrete terms the restriction mapping is given by

$$G(\xi, q)\frac{d\xi}{\xi} + H(\xi, q)\frac{dq}{q} \mapsto G(\xi, 0)\frac{d\xi}{\xi} + H(\xi, 0)\frac{dq}{q}$$

where G and H are holomorphic functions of (ξ, q), and then setting $w = e(\xi)$.

Formula (8.1) implies that

$$F(\xi, \eta)|_{q=0} = \pi i\left(\frac{e(\xi)+1}{e(\xi)-1} + \frac{e(\eta)+1}{e(\eta)-1}\right) = \pi i\left(\frac{w+1}{w-1} + \coth(\pi i\eta)\right)$$

where $w := e(\xi)$ is the parameter in the normalization $\mathbb{P}^1$ of the nodal cubic E_0. From this and the identity (8.3), it follows that when $q = 0$

$$\frac{1}{\eta} + \eta\frac{\partial F}{\partial \eta}(\xi, \eta)|_{q=0} = \frac{1}{\eta} - \frac{(\pi i)^2\eta}{\sinh^2(\pi i\eta)} = \frac{1}{\eta} - \frac{\pi^2\eta}{\sin^2(\pi\eta)},$$

which is holomorphic at $\eta = 0$.

The restriction of the connection to a first order neighbourhood of E_0 is given by the 1-form:

$$\omega_0 = \frac{dq}{q} \otimes A\frac{\partial}{\partial T} + \psi_0 + \nu_0,$$

where

$$\psi_0 = \sum_{m\geq 1} \left(\frac{1}{(2m)!} G_{2m+2}|_{q=0} \frac{dq}{q} \otimes \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\mathrm{ad}_T^j(A), \mathrm{ad}_T^k(A)] \right) \frac{\partial}{\partial A}$$

$$= -\sum_{m\geq 1} \left(\frac{(2m+1)B_{2m+2}}{(2m+2)!} \frac{dq}{q} \otimes \sum_{\substack{j+k=2m+1 \\ j>k>0}} (-1)^j [\mathrm{ad}_T^j(A), \mathrm{ad}_T^k(A)] \right) \frac{\partial}{\partial A}$$

and, using the identity (7.1),

$$\nu_0 = \frac{T}{2}\left(\frac{w+1}{w-1} + \frac{e^T+1}{e^T-1}\right) \cdot A \, \frac{dw}{w} + \frac{T/4}{\sinh^2(T/2)} \cdot A \, \frac{dq}{q}$$

$$= [T, A] \, \frac{dw}{w-1} + \left(\frac{T}{e^T-1}\right) \cdot A \, \frac{dw}{w} + \left(\frac{1}{T} - \frac{T/4}{\sinh^2(T/2)}\right) \cdot A \, \frac{dq}{q}$$

$$= [T, A] \, \frac{dw}{w-1} + \left(\frac{T}{e^T-1}\right) \cdot A \, \frac{dw}{w} + \sum_{m=1}^{\infty} (2m-1) \frac{B_{2m}}{(2m)!} \, \mathrm{ad}_T^{2m-1} A \frac{dq}{q}$$

At this stage, it is convenient to define $\epsilon_{2m} \in \mathrm{Der}_{2m} \mathfrak{p}$ (the derivations of $\mathfrak{p}$ of degree $2m$) by[9]

(12.1)
$$\epsilon_{2m} = \begin{cases} -A \frac{\partial}{\partial T} & m = 0; \\ \mathrm{ad}_T^{2m-1}(A) - \sum_{\substack{j+k=2m-1 \\ j>k>0}} (-1)^j [\mathrm{ad}_T^j(A), \mathrm{ad}_T^k(A)] \frac{\partial}{\partial A} & m > 0. \end{cases}$$

Assembling the pieces, we see that the restriction of the KZB connection form to a first order neighbourhood of the Tate curve is

(12.2)
$$\omega_0 = [T, A] \, \frac{dw}{w-1} + \left(\frac{T}{e^T-1}\right) \cdot A \, \frac{dw}{w} + \sum_{m=0}^{\infty} (2m-1) \frac{B_{2m}}{(2m)!} \epsilon_{2m} \frac{dq}{q}.$$

12.1. Monodromy logarithms

The residue of the connection at (w, q) acts on the fiber $\mathbb{L}(A, T)^\wedge$ of $\mathfrak{p}$ over it as a derivation. At each point $(w, 0)$, where $w \neq 0, 1$, the residue of the

[9] These derivations occur in the work [32] of Tsunogai on the action of the absolute Galois group on the fundamental group of a once punctured elliptic curve. They also occur in the paper of Calaque et al. [3, §3.1].

connection (12.2) is

$$(12.3)\qquad N_q = \sum_{m\geq 0}(2m-1)\frac{B_{2m}}{(2m)!}\,\epsilon_{2m}.$$

The residue N_w of the connection of each point along the identity section $w=1$ is

$$N_w = \mathrm{ad}_{[T,A]}\,.$$

Since the KZB connection is flat and since the two boundary components intersect transversely at $(w,q)=(1,1)$, $[N_q,N_w]=0$. This implies that each ϵ_{2n} annihilates $[T,A]$, and therefore lies in

$$\mathrm{Der}^0\,\mathfrak{p} := \{\delta\in \mathrm{Der}\,\mathfrak{p} : \delta([T,A])=0\}.$$

Proposition 12.1 (Tsunogai [32])**.** *For all $m\geq 0$, $\epsilon_{2m}\in \mathrm{Der}^0\,\mathfrak{p}$. When $m>1$, ϵ_{2m} is a highest weight vector of weight $2m-2$ for the natural $\mathfrak{sl}_2$-action; that is, it is annihilated by $A\partial/\partial T$.* □

It can be shown that ϵ_{2m} is a highest weight vector of the *unique* copy of $S^{2m}H$ in $\mathrm{Gr}^W_{-2m}\,\mathrm{Der}^0\,\mathfrak{p}$.

The following computation follows from identity (7.2) and the fact that

$$\epsilon_{2m}(T) = (\mathrm{ad}_T^{2m-1}A)\cdot T = -\,\mathrm{ad}_T^{2m}(A) = -T^{2m}\cdot A.$$

Proposition 12.2. *The value of N_q on T is*

$$N_q(T) = \frac{1}{4}\left(\frac{T^2}{\sinh^2(T/2)}\right)\cdot A.$$ □

12.2. Pullback to $\mathbb{P}^1 - \{0,1,\infty\}$

We can pullback the connection to $\mathbb{P}^1$ along the "map" $E_0' = \mathbb{P}^1-\{0,1,\infty\}\to E_{\partial/\partial q}$ to the fiber of $\mathcal{E}$ over the tangent vector $\partial/\partial q$. Just set dq to zero to get:

$$(12.4)\qquad \omega_{E_0'} = [T,A]\,\frac{dw}{w-1} + \left(\frac{T}{e^T-1}\right)\cdot A\,\frac{dw}{w}.$$

Since

$$\frac{T}{e^T-1}+\frac{T}{e^{-T}-1}+T=0,$$

it follows that the residues R_0, R_1 and R_∞ of $\omega_{E_0'}$ at $0, 1, \infty$ are:

$$R_0 = \left(\frac{T}{e^T - 1}\right)\cdot A, \quad R_1 = [T, A], \quad R_\infty = \left(\frac{T}{e^{-T}-1}\right)\cdot A \tag{12.5}$$

13. Restriction to $\mathcal{M}_{1,\vec{1}}$

In this section, we compute the restriction of the universal elliptic KZB connection to the first order neighbourhood $\overline{\mathcal{M}}_{1,\vec{1}}$ of the 0-section of $\overline{\mathcal{E}}$. In algebraic terms, this restriction map is induced by the $\mathcal{O}_{\overline{\mathcal{E}}}$-module homomorphism

$$\Omega^1_{\overline{\mathcal{E}}}(\log \overline{\mathcal{M}}_{1,1}) \to \Omega^1_{\overline{\mathcal{E}}}(\log \overline{\mathcal{M}}_{1,1}) \otimes_{\mathcal{O}_{\overline{\mathcal{E}}}} \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}$$

Here we are identifying the zero-section of $\overline{\mathcal{E}}$ with $\overline{\mathcal{M}}_{1,1}$. This computation will allow us to see that the restricted connection takes values in $\operatorname{Der}^0 \mathfrak{p}$.

In concrete terms the restriction mapping is given by

$$G(\xi,\tau)\frac{d\xi}{\xi} + H(\xi,\tau)d\tau \mapsto G(0,\tau)\frac{d\xi}{\xi} + H(0,\tau)d\tau$$

where G and H are holomorphic functions of (ξ,τ). The restricted connection is thus given by the 1-form

$$\omega' = \frac{dq}{q}\otimes A\frac{\partial}{\partial T} + \psi + \nu'$$

where

$$\begin{aligned}\nu' &= [\mathbf{t},\mathbf{a}]\frac{d\xi}{\xi} + \frac{1}{2\pi i}\left(\frac{1}{\mathbf{t}} + \mathbf{t}\frac{\partial F}{\partial \mathbf{t}}(0,\mathbf{t},\tau)\right)\cdot \mathbf{a}\, d\tau \\ &= [T,A]\left(\frac{d\xi}{\xi} - 2G_2(\tau)\frac{dq}{q}\right) - \sum_{m\geq 1}\frac{2}{(2m)!}G_{2m+2}(\tau)\,\frac{dq}{q}\otimes \operatorname{ad}_T^{2m+1}(A).\end{aligned}$$

Note that both terms in this last expression are $\mathrm{SL}_2(\mathbb{Z})$-invariant and that the term ψ remains unchanged as it does not depend on ξ:

$$\psi = \sum_{m\geq 1}\left(\frac{2}{(2m)!}G_{2m+2}(\tau)\frac{dq}{q}\otimes \sum_{\substack{j+k=2m+1\\ j>k>0}}(-1)^j[\operatorname{ad}_T^j(A), \operatorname{ad}_T^k(A)]\frac{\partial}{\partial A}\right).$$

Proposition 13.1. *The restriction of the KZB connection to* $\mathcal{M}_{1,\vec{1}}$ *is given by the* $\mathrm{SL}_2(\mathbb{Z})$*-invariant 1-form*

$$\omega' = -\frac{dq}{q}\otimes\epsilon_0-\Big(2G_2(\tau)\frac{dq}{q}-\frac{d\xi}{\xi}\Big)\otimes\epsilon_2-\sum_{m=2}^{\infty}\frac{2}{(2m-2)!}G_{2m}(\tau)\frac{dq}{q}\otimes\epsilon_{2m} \tag{13.1}$$

on $\operatorname{Der}^0\mathbb{L}(A,T)^\wedge\times\mathbb{C}\times\mathfrak{h}\to\mathbb{C}\times\mathfrak{h}$.

This gives an alternative computation of N_q:

$$N_q=\operatorname{Res}_{q=0}\omega'=\sum_{m=0}^{\infty}(2m-1)\frac{B_{2m}}{(2m)!}\epsilon_{2m}.$$

14. Rigidity

We have not yet proved that $\boldsymbol{\mathcal{P}}$ with the KZB connection is isomorphic to the flat bundle $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$ defined in Section 6.1. This will be resolved in this section by proving that both have the same monodromy representation.

The punctured universal elliptic curve $\mathcal{E}'$ is the moduli space $\mathcal{M}_{1,2}$. Choose a base point $[E_o,x_o]$ of $\mathcal{M}_{1,2}$, where $x_o\neq 0$. There is a natural isomorphism

$$\pi_1(\mathcal{M}_{1,2},[E_o,x_o])\cong\pi_0\operatorname{Diff}^+(E_o,x_o,0)\cong\Gamma_{1,2},$$

where $\Gamma_{1,2}$ is the mapping class group of a genus 1 curve with 2 marked points.

The restriction of the universal elliptic KZB connection to E_o defines a homomorphism $\pi_1(E_o',x_o)\to\operatorname{Aut}\mathfrak{p}$ whose image lies in the subgroup $\mathcal{P}=\exp\mathfrak{p}$ which acts on $\mathfrak{p}$ via the adjoint action. Corollary 11.3 implies that it induces an isomorphism $\pi_1^{\mathrm{un}}(E_o',x_o)\to\mathcal{P}$.

Identify $\mathcal{P}$ with $\pi_1^{\mathrm{un}}(E_o',x_o)$ via this isomorphism. Then one has the monodromy representations

$$\rho^{\mathrm{KZB}}:\Gamma_{1,2}\to\operatorname{Aut}\mathcal{P}\text{ and }\rho^{\mathrm{top}}:\Gamma_{1,2}\to\operatorname{Aut}\mathcal{P}$$

of $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$. To prove that $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$ and $\boldsymbol{\mathcal{P}}$ are isomorphic, we have to prove that $\rho^{\mathrm{KZB}}=\rho^{\mathrm{top}}$. Observe that if $\gamma\in\pi_1(E_o',x_o)$, then $\rho^{\mathrm{top}}(\gamma)$ and $\rho^{\mathrm{KZB}}(\gamma)$ are both conjugation by the image of γ in $\mathcal{P}$ as the restriction of $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$ to E_o' are isomorphic.

To prove that $\rho^{\mathrm{KZB}}=\rho^{\mathrm{top}}$ it is useful to consider a more abstract situation. Suppose that N is a normal subgroup of a discrete group Γ. Denote the unipotent completion of N by $\mathcal{N}$.[10] The homomorphism $\Gamma\to\operatorname{Aut}N$ that

[10] One can use any field over char 0, but we will take $\mathbb{C}$.

takes $g \in \Gamma$ to $n \mapsto gng^{-1}$ induces a homomorphism $\phi : G \to \operatorname{Aut}\mathcal{N}$. The restriction of ϕ to N takes $n \in N$ to $\iota_{\theta(n)}$, where $\theta : N \to \mathcal{N}$ is the natural homomorphism and ι_u denotes conjugation by $u \in \mathcal{N}$.

Lemma 14.1. *If $\mathcal{N}$ has trivial center, ϕ is the unique homomorphism $\Gamma \to \operatorname{Aut}\mathcal{N}$ whose restriction to N is $n \mapsto \iota_{\theta(n)}$.*

Proof. The condition that $\mathcal{N}$ have trivial center implies that the centralizer of $\operatorname{im}\theta$ in $\operatorname{Aut}\mathcal{N}$ is trivial: if $\sigma \in \operatorname{Aut}\mathcal{N}$, then

$$\sigma\iota_{\theta(n)}\sigma^{-1} = \iota_{\sigma(\theta(n))}.$$

So σ centralizers $\operatorname{im}\theta$ if and only if $\iota_{\theta(n)} = \iota_{\sigma(\theta(n))}$ for all $n \in N$. Since $\operatorname{im}\theta$ is Zariski dense in $\mathcal{N}$ and since $\mathcal{N}$ has trivial center, this implies that $\sigma = \mathrm{id}$.

Suppose now that $\alpha : \Gamma \to \operatorname{Aut}\mathcal{N}$ is a homomorphism whose restriction to N is $n \mapsto \iota_{\theta(n)}$. If $g \in \Gamma$, then

$$\begin{aligned}\alpha(g)\iota_{\theta(n)}\alpha(g)^{-1} = \alpha(g)\alpha(n)\alpha(g)^{-1} = \alpha(gng^{-1})\\ = \iota_{gng^{-1}} = \cdots = \phi(g)\iota_{\theta(n)}\phi(g)^{-1}\end{aligned}$$

for all $n \in N$. So $\alpha(g)^{-1}\phi(g)$ centralizes $\operatorname{im}\theta$ and is therefore trivial. □

Applying the lemma with $\Gamma = \Gamma_{1,2}$, $N = \pi_1(E_o', x_o)$, $\mathcal{N} = \mathcal{P}$ and $\phi = \rho^{\mathrm{top}}$ establishes the equality of ρ^{KZB} and ρ^{top}.

Theorem 14.2. *The exponential mapping induces an isomorphism of the locally constant sheaf over $\mathcal{E}'$ of flat sections of the universal elliptic KZB connection on $\boldsymbol{\mathcal{P}}$ with the locally constant sheaf $\boldsymbol{\mathcal{P}}^{\mathrm{top}}$ over $\mathcal{E}'$. Equivalently, the diagram*

$$\begin{array}{ccc}\pi_1(\mathcal{E}', [E, x_o]) & \xrightarrow{\rho^{\mathrm{KZB}}} & \operatorname{Aut}\mathfrak{p} \\ \| & & \downarrow \cong \\ \pi_1(\mathcal{E}', [E, x_o]) & \xrightarrow[\rho^{\mathrm{top}}]{} & \operatorname{Aut}\mathcal{P}\end{array}$$

commutes.

Remark 14.3. A similar argument can be used to prove that the local system associated to the KZB connection over $\mathcal{M}_{1,1+n}$ constructed in [3] is the canonical local system whose fiber over $[E, 0, x_1, \ldots, x_n]$ is the unipotent completion of the fundamental group of the configuration space of n points on E' with base point $(x_1, \ldots, x_n)$.

Any other connection on the extension of $\boldsymbol{\mathcal{P}}$ to $\overline{\mathcal{M}}_{1,2}$ with regular singularities with pronilpotent residue and conjugate monodromy representation differs from the KZB connection by a holomorphic map $\overline{\mathcal{M}}_{1,2} \to \operatorname{Aut}\mathfrak{p}$. Since $\operatorname{Aut}\mathfrak{p}$ is an affine group and since $\overline{\mathcal{M}}_{1,2}$ is complete, the change of gauge must be constant.

Corollary 14.4. *The universal elliptic KZB connection is the unique meromorphic connection on* $\mathfrak{p}\times\mathbb{C}\times\mathfrak{h}$ *(up to a constant change of gauge) satisfying:*

(i) it is $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ *invariant with the same factors of automorphy as the elliptic KZB connection,*
(ii) it has poles along $\{(z,\tau) \in \mathbb{C}\times\mathfrak{h} : z \in \Lambda_\tau\}$ *and is holomorphic elsewhere,*
(iii) its natural extension to $\boldsymbol{\mathcal{P}}$ *over* $\overline{\mathcal{M}}_{1,2}$ *has regular singularities with pronilpotent residues,*
(iv) for any one $[E,x] \in \mathcal{M}_{1,2}$*, the restriction of the connection to* $\mathcal{E}'$ *induces an isomorphism* $\pi_1(E',x)^{\mathrm{un}} \to \mathcal{P}$*.*

One consequence of this rigidity result is that the canonical extension of the connection constructed by Calaque, Enriquez and Etingof [3] over $\mathcal{M}_{1,2}$ equals the connection constructed by Levin and Racinet in [24].

15. Hodge theory

In this section we show that, with appropriate filtrations, $\boldsymbol{\mathcal{P}}$ is an admissible variation of mixed Hodge structure (MHS) over $\mathcal{E}'$. We assume that the reader is familiar with the definition of mixed Hodge structures. We begin by recalling the definition of an admissible variation of MHS over a smooth variety. Further details can be found in [31] and [20].

15.1. Admissible variations of mixed Hodge structure

Suppose that X is a smooth projective variety (or orbifold) and that $U = X - D$ is the complement of a normal crossing divisor D in X.

(i) Suppose that $\mathbb{V}$ is a local system of finite dimensional $\mathbb{Q}$-vector spaces over U. For simplicity, we assume that the local monodromy operator at each smooth point P of D is unipotent. This holds for each finite dimensional quotient of $\boldsymbol{\mathcal{P}}$.
(ii) Let $\mathcal{V}$ be Deligne's canonical extension of the flat vector bundle $\mathbb{V}\otimes_{\mathbb{Q}}\mathcal{O}_U$ to X. It is a holomorphic vector bundle with a connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X(\log D)$$

with regular singularities along D. It is characterized by the property that the residue of the connection at each smooth point of D is nilpotent.

(iii) Suppose that $F^\bullet\mathcal{V}$ is a decreasing filtration of $\mathcal{V}$ by holomorphic vector bundles over X and that these satisfy "Griffiths transversality":

$$\nabla : F^p\mathcal{V} \to F^{p-1}\mathcal{V} \otimes \Omega^1_X(\log D) =: F^p\big(\mathcal{V} \otimes \Omega^1_X(\log D)\big).$$

That is, ∇ respects the Hodge filtration.

(iv) There is an increasing filtration $W_\bullet$ of $\mathbb{V}$. It induces a filtration $W_\bullet\mathcal{V}$ of $\mathcal{V}$ by flat sub-bundles.

(v) Suppose that for each $x \in U$ the restriction of $F^\bullet\mathcal{V}$ and $W_\bullet\mathbb{V}$ to the fiber V_x of $\mathbb{V}$ over x define a MHS on V_x.

When X is a curve, these data (a "pre-variation" of MHS) form an *admissible* variation of MHS if for each $P \in D$, there is a *relative weight filtration* $M_\bullet$ of the fiber V_P of $\mathcal{V}$ over P and its nilpotent endomorphism $N = \operatorname{Res}_P \nabla$.[11] This means that:

(i) $M_\bullet$ is an increasing filtration of V_P satisfying $N(M_r V_P) \subseteq M_{r-2}V_P$ and $N(W_m V_P) \subseteq W_m V_P$ for all m and r,

(ii) for each m and each k, N^k induces an isomorphism

$$N^k : \operatorname{Gr}^M_{m+k} \operatorname{Gr}^W_m V_P \to \operatorname{Gr}^M_{m-k} \operatorname{Gr}^W_m V_P.$$

In this case, for each $P \in D$ and each choice of non-zero tangent vector $\vec{v} \in T_P X$, there is a canonical MHS on the fiber V_P of $\mathcal{V}$ over P. This MHS will be denoted $V_{\vec{v}}$. It has weight filtration $M_\bullet$; its Hodge filtration is the restriction of $F^\bullet\mathcal{V}$ to V_P. The $\mathbb{Q}$-structure on V_P is spanned by the elements

$$\lim_{t\to 0}(t/c)^{-N} v(t) \in V_P,$$

where $N = \operatorname{Res}_P \nabla$, t is a local holomorphic parameter on X centered at P, $\vec{v} = c\partial/\partial t$, and $v(t)$ is a flat section of $\mathbb{V}$ defined in an angular sector containing a ray tangent to $\vec{v}$. Each $W_m V_P$ is a sub-MHS.

When X has dimension > 1, the pre-variation is admissible if its restriction to each curve in X is admissible. We will also refer to pro-objects of the category of admissible variations of MHS as admissible variations.

[11] Background material on relative weight filtrations can be found in [31] and [14].

15.2. The variation $\boldsymbol{\mathcal{P}}$

We now show that the KZB equation gives rise to an admissible variation of MHS. Denote the maximal ideal (T, A) of $\mathbb{Q}\langle\langle T, A\rangle\rangle$ by I. As in [24], define Hodge and weight filtrations on $\mathbb{Q}\langle\langle T, A\rangle\rangle$ in the natural way by setting

$$W_{-n}\mathbb{Q}\langle\langle T, A\rangle\rangle = I^n \text{ and } F^{-p}\mathbb{Q}\langle\langle T, A\rangle\rangle = \{x \in \mathbb{Q}\langle\langle T, A\rangle\rangle : \deg_A(x) \leq p\}.$$

We also define a *relative weight filtration* $M_\bullet$ on $\mathbb{Q}\langle\langle T, A\rangle\rangle$ by

$$M_{-2m}\mathbb{Q}\langle\langle T, A\rangle\rangle = \{x \in \mathbb{Q}\langle\langle T, A\rangle\rangle : \deg_A(x) \geq m\}.$$

These filtrations are multiplicative. By restriction, they induce Hodge, weight and relative weight filtrations on $\mathfrak{p}$ and thus on $\operatorname{Der}\mathfrak{p}$. They also induce filtrations on the bundle $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$ over $\mathbb{C} \times \mathfrak{h}$.

Theorem 15.1. *The Hodge and weight filtrations on $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$ descend to Hodge and weight filtrations of the local system $\boldsymbol{\mathcal{P}} \to \mathcal{E}'$. With these filtrations, the local system $\boldsymbol{\mathcal{P}}$ over $\mathcal{E}' = \mathcal{M}_{1,2}$ and its restriction to $\mathcal{M}_{1,\vec{1}}$ are admissible variations of MHS whose weight graded quotients are direct sums of Tate twists of $S^m\mathbb{H}$. The MHS on the fiber over $[E, x]$ is the canonical MHS on the Lie algebra of the unipotent completion of $\pi_1(E', x)$. The relative weight filtration of the limit MHS associated to the tangent vector $\vec{v} = \partial/\partial q + \partial/\partial w$ at the identity of the nodal cubic is $M_\bullet$. In addition, there is a natural homomorphism*

$$\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w)^{\mathrm{un}} \to \pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w)^{\mathrm{un}}$$

whose image is invariant under monodromy and which is a morphism of MHS for all $\lambda \in \mathbb{C}^*$

Proof. It suffices to consider the case where the base is $\mathcal{E}'$. The factor of automorphy $\widetilde{M}_\gamma(\xi, \tau)$ (Cf. (6.2)) preserves both the Hodge and weight filtrations on $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$. They therefore descend to filtrations of the bundle $\boldsymbol{\mathcal{P}}$ over $\mathcal{E}'$. Next observe that each component of the universal elliptic KZB connection is a 1-form that takes values in $F^{-1}W_0 \operatorname{Der}^0 \mathfrak{p}$. This implies that, over $\mathcal{E}'$, the connection satisfies Griffiths transversality and that the weight bundles are sub-local systems of $\boldsymbol{\mathcal{P}}$. The weight filtration is defined over $\mathbb{Q}$ as it is defined in terms of the lower central series filtration of $\mathfrak{p}$.

As explained in Section 10, the natural extension of the universal elliptic KZB connection to the q-disk has regular singular points along the nodal cubic and along the identity section. The extension of the Hodge and weight bundles to the q-disk are the quotients of the bundles

$$(F^p\mathfrak{p}) \times \mathbb{C}^* \times \mathbb{D} \text{ and } (W_m\mathfrak{p}) \times \mathbb{C}^* \times \mathbb{D}$$

over $\mathbb{C}^* \times \mathbb{D}$ (with coordinates (w, q)) by the factor of automorphy. These are well defined as $\widetilde{M}_{(m,n)}(q) = e^{-mT}$, which lies in $F^0W_0M_0 \operatorname{Der}^0 \mathfrak{p}$.

The residue at each point of the identity section is $N_w := \operatorname{ad}_{[T,A]}$. It lies in $F^{-1}W_{-2}M_{-2} \operatorname{Der}^0 \mathfrak{p}$, which implies that the Hodge, weight and (where relevant) the relative weight filtrations extend across the identity section.

According to (12.3) the residue of the connection at each point of the nodal cubic is

$$N_q = \sum_{m \geq 0} (2m-1)\frac{B_{2m}}{(2m)!}\,\epsilon_{2m} \in \operatorname{Der}^0 \mathfrak{p}.$$

It is easy to check that for each $m \geq 0$, $\epsilon_{2m} \in F^{-1}M_{-2}W_{-2m} \operatorname{Der}^0 \mathfrak{p}$ for all $m \geq 0$, so that

$$N_q \in F^{-1}M_{-2}W_0 \operatorname{Der}^0 \mathfrak{p}.$$

Moreover,

$$\operatorname{Gr}^W_\bullet N_q : \operatorname{Gr}^W_\bullet \mathbb{C}\langle\langle T, A\rangle\rangle \to \operatorname{Gr}^W_\bullet \mathbb{C}\langle\langle T, A\rangle\rangle$$

is ϵ_0. Set $H = \mathbb{C}A \oplus \mathbb{C}T$. Since $\operatorname{Gr}^W_{-m} \mathbb{C}\langle\langle T, A\rangle\rangle = S^mH$ placed in weight $-m$ and since $\operatorname{Gr}^W_\bullet N_q = -\epsilon_0 = A\partial/\partial T \in \mathfrak{sl}(H)$, it follows easily from the representation theory of $\mathfrak{sl}(H)$ that N_q^k induces an isomorphism

$$\operatorname{Gr}^W_{-m+k} \mathbb{C}\langle\langle T, A\rangle\rangle \to \operatorname{Gr}^W_{-m-k} \mathbb{C}\langle\langle T, A\rangle\rangle.$$

This implies that $M_\bullet$ is the relative weight filtration of N_q and for $N_q + N_w$, which completes the proof that $\boldsymbol{\mathcal{P}}$ is an admissible variation of MHS over $\mathcal{M}_{1,2}$.

To prove that the MHS on the fiber of $\boldsymbol{\mathcal{P}}$ over $[E] \in \mathcal{M}_{1,1}$ is its canonical MHS (as defined in [12] or [19]) we consider the restriction $\boldsymbol{\mathcal{P}}_E$ of $\boldsymbol{\mathcal{P}}$ to the fiber E'. The above discussion implies that this is an admissible variation of MHS over E'. In fact, it is clearly a unipotent variation of MHS. Fix a base point $x \in E'$. Theorem 14.2 implies that the fiber $\mathfrak{p}(E, x)$ of $\boldsymbol{\mathcal{P}}_E$ over x is naturally isomorphic to the Lie algebra of the unipotent completion of $\pi_1(E', x)$. The monodromy representation $\theta_x : \mathfrak{p}(E, x) \to \operatorname{End} \mathfrak{p}(E, x)$ of $\boldsymbol{\mathcal{P}}_E$ is the adjoint action. Since the center of $\mathfrak{p}(E, x)$ is trivial ($\mathfrak{p}$ is free of rank 2), θ_x is injective. Denote $\mathfrak{p}(E, x)$ with its canonical MHS by $\mathfrak{p}(E, x)^{\mathrm{can}}$ and

$\mathfrak{p}(E,x)$ with the MHS given by the elliptic KZB connection via Theorem 14.2 by $\mathfrak{p}(E,x)^{\mathrm{KZB}}$. The main theorem of [19] implies that

$$\theta_x : \mathfrak{p}(E,x)^{\mathrm{can}} \to \operatorname{End}\mathfrak{p}(E,x)^{\mathrm{KZB}}$$

is a morphism of MHS. On the other hand, since $\mathfrak{p}(E,x)^{\mathrm{KZB}}$ is a Lie algebra in the category of pro-MHS,

$$\theta_x : \mathfrak{p}(E,x)^{\mathrm{KZB}} \to \operatorname{End}\mathfrak{p}(E,x)^{\mathrm{KZB}}$$

is also a morphism of MHS. Since θ_x is injective, this implies that the MHSs $\mathfrak{p}(E,x)^{\mathrm{can}}$ and $\mathfrak{p}(E,x)^{\mathrm{KZB}}$ are equal.

The final statement follows from the construction [13] of limit mixed Hodge structures on homotopy groups. It will be explained in greater detail in [16]. □

We can now prove that N_q annihilates the R_α.

Corollary 15.2. *The derivation N_q annihilates R_0, R_1 and R_∞.*

An elementary proof is given in Appendix A. Here we sketch a more conceptual proof.

Sketch of Proof. Since N_q annihilates $R_1 = [T,A]$, and since $R_0 + R_1 + R_\infty = 0$, it suffices to show that $N_q(R_0) = 0$.

Denote the limit MHS on $\mathbb{Q}\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, w)^\wedge$ associated to the tangent vector $\partial/\partial w \in T_1\mathbb{P}^1$ by $\mathbb{Q}\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \partial/\partial w)^\wedge$ and the limit MHS on $\mathbb{Q}\pi_1(E_q', x)^\wedge$ associated to the tangent vector $\lambda\partial/\partial q + \partial/\partial w$ of $\mathcal{E}$ at the identity of the nodal cubic. One has the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \partial/\partial w)^\wedge & \xrightarrow{\Theta_{KZ}} & \mathbb{C}\langle\langle R_0, R_1\rangle\rangle \\
\downarrow & & \downarrow \\
\mathbb{C}\pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w)^\wedge & \xrightarrow[\Theta_\lambda]{} & \mathbb{C}\langle\langle T, A\rangle\rangle
\end{array}$$

where the left hand vertical mapping is the one given by Theorem 15.1, the right hand vertical map is given by the formulas for R_0 and R_1, and where the top horizontal map is the standard isomorphism given by the KZ-connection. The result follows as the image of Θ_{KZ} is invariant under monodromy and as the logarithm of monodromy acting on $\mathbb{C}\langle\langle T, A\rangle\rangle$ is N_q. □

As noted in Remark 11.4, the limit MHS on $H_1(E_\tau)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$. The basis T, A splits both the Hodge and monodromy weight filtrations. The following statement follows directly from Corollary 11.2.

Lemma 15.3. *The limit MHS on $H_1(E_\tau)$ associated to the tangent vector $\lambda\partial/\partial q$ of the origin of the q-disk has complex basis A and T and integral basis spanned by $\mathbf{a}$ and $-\mathbf{b}$, where*

$$-\mathbf{b} = T - \log\lambda A \text{ and } \mathbf{a} = 2\pi i A,$$

so that the corresponding period matrix is

$$\begin{pmatrix} 1 & -\log\lambda \\ 0 & 2\pi i \end{pmatrix}$$

Denote it by $H_1(E_{\lambda\partial/\partial q})$. □

16. Pause for a picture

Suppose that $f : Y \to X$ is a family of varieties which is locally topologically trivial over the Zariski open subset $X' = X - S$ of X, where S is a normal crossing divisor. Suppose that $P \in S$ and that $\vec{v}$ is a non-zero tangent vector at P that is not tangent to S. Suppose that $\mathbb{V}$ is an admissible variation of MHS over X' whose fiber over $x \in X'$ is a given topological invariant of the fiber Y_x of f over x. It is useful to think of the limit MHS of $\mathbb{V}$ associated to $\vec{v}$ as the MHS on that invariant of the "fiber $Y_{\vec{v}}$ of f over $\vec{v}$." For example, we would like to think of the limit MHS of $\mathbb{H}$ associated to the tangent vector $\vec{v} = \lambda\partial/\partial q$ at the origin of the q-disk as being $H_1(E_{\lambda\partial/\partial q})$, and the limit MHS of $\boldsymbol{\mathcal{P}}$ associated to the tangent vector $\vec{v} = \partial/\partial q + \partial/\partial w$ of the universal elliptic curve at the identity of the nodal cubic as a MHS on the Lie algebra $\mathfrak{p}(E_{\partial/\partial q}, \partial/\partial w)$ of the unipotent fundamental group of $(E'_{\partial/\partial q}, \partial/\partial w)$. The goal of this section is to explain how to make this more precise using real oriented blowups.

16.1. Real oriented blowups and tangential base points

The real oriented blowup of a Riemann surface X at a finite subset S will be denoted by $\mathrm{Bl}_S^o X$. This is a bordered Riemann surface with one boundary circle for each point of S. There is a continuous projection $\pi : \mathrm{Bl}_S^o X \to X$ that induces a biholomorphism

$$\mathrm{Bl}_S^o X - \partial\,\mathrm{Bl}_S^o X \longrightarrow X - S$$

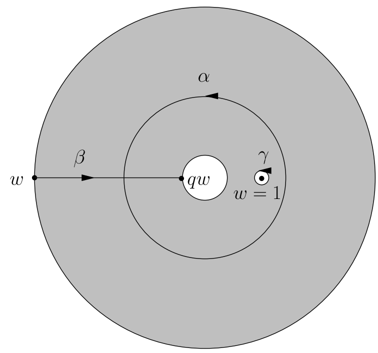

Figure 2: $\mathrm{Bl}_0^o E_q$ as a quotient of $\widehat{A}_{|q|}$.

The fiber of π over $P \in S$ is the quotient of $(T_P X) - \{0\}$ by the multiplicative group of positive real numbers.

Example 16.1. The real oriented blowup of the unit disk at the origin is $[0,1) \times S^1$. The projection to the disk is $(r,\theta) \mapsto re^{i\theta}$. As explained in Appendix B, there is a natural identification of $\mathrm{Bl}_{0,\infty}^o \mathbb{P}^1$ with $[0,1] \times S^1$.

Each non-zero vector $\vec{v} \in T_P X$ determines an element $[\vec{v}] \in \mathrm{Bl}_P^o X$. Set $X' = X - \{P\}$. The fundamental group $\pi_1(X', \vec{v})$ is defined to be $\pi_1(\mathrm{Bl}_P^o X, [\vec{v}])$. (cf. [8].) When $t > 0$, $\pi_1(X', \vec{v})$ and $\pi_1(X', t\vec{v})$ are canonically isomorphic. It is important to note that the MHSs on their unipotent completions will not be isomorphic except when the local monodromy operator associated to $\vec{v}$ acts trivially on $\pi_1^{\mathrm{un}}(X', \vec{v})$.

16.2. The fiber of $\boldsymbol{\mathcal{E}}$ over $\boldsymbol{\partial/\partial q}$

We now sketch the construction of the fiber $E_{\partial/\partial q}$ of $\mathcal{E}$ over $\partial/\partial q$. Full details can be found in Appendix B.

Suppose that $q \in \mathbb{D}^*$. The fiber of the universal elliptic curve over q is $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$. For $0 < r < 1$, set

$$A_r := \{w \in \mathbb{C}^* : r^{1/2} \leq |w| \leq r^{-1/2}\} \text{ and } \widehat{A}_r := \mathrm{Bl}_1^o A_r.$$

Denote their outer and inner boundaries by $\partial_+ A_r$ and $\partial_- A_r$, respectively. The elliptic curve E_q is the quotient of $A_{|q|}$ obtained by identifying $w \in \partial_+ A_{|q|}$ with $qw \in \partial_- A_{|q|}$. Similarly, $\mathrm{Bl}_1^o E_q$ is a quotient of $\widehat{A}_q$. See Figure 2.

The homology class $\mathbf{a}$ corresponds to the class of the positively oriented unit circle α; the homology class $\mathbf{b}$ corresponds to a path β from w to qw,

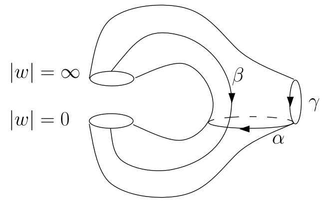

Figure 3: The "nearby fiber" $\mathrm{Bl}_1^o E'_{\partial/\partial q}$.

where $w \in \partial_+ A_{|q|}$. Note that these have intersection number $+1$. From this it is evident that $\mathbf{a}$, the class of α, is the vanishing cycle. The Picard Lefschetz transformation is also evident: as q travels once around $\mathbb{D}^*$ in the positive direction, a remains invariant, but $\mathbf{b}$, the class of β changes to $\mathbf{b}+\mathbf{a}$. This can also be seen from the formula

$$T = \tau\mathbf{a} - \mathbf{b} = \frac{\mathbf{a}}{2\pi i}\log q - \mathbf{b}.$$

Let γ be the boundary circle at $w = 1$ of $\widehat{A}_r$.

As $r \to 0$, A_r "converges to" $\mathrm{Bl}_{0,\infty}^o \mathbb{P}^1$ and $\widehat{A}_r$ "converges to" $\mathrm{Bl}_{0,1,\infty}\mathbb{P}^1$. So $E_{re^{i\theta}}$ and $\mathrm{Bl}_1 E_{re^{i\theta}}$ "converge" to the surfaces obtained by identifying the boundary circles of $\mathrm{Bl}_{0,\infty}^o \mathbb{P}^1$ and $\mathrm{Bl}_{0,1,\infty}\mathbb{P}^1$ at 0 and ∞ by multiplication $e^{i\theta}$. The resulting surface will be denoted by $E_{e^{i\theta}\partial/\partial q}$. See Figure 3. In Appendix B we show that the universal family of elliptic curves over $\mathbb{D}^*$ extends to a family of tori over $\mathrm{Bl}_0^o \mathbb{D}$ whose fiber over $[e^{i\theta}\partial/\partial q]$ is $E_{e^{i\theta}\partial/\partial q}$.

The curve γ represents the commutator of two generators of $\pi_1(\mathrm{Bl}_1(E_{\partial/\partial q}, [\partial/\partial w]))$. This is consistent with the fact that

$$\mathrm{Res}_{w=1}\,\omega_{E'_0} = [T, A] = \frac{1}{2\pi i}[T, \mathbf{a}] = \frac{1}{2\pi i}[\tau\mathbf{a} - \mathbf{b}, \mathbf{a}] = \frac{1}{2\pi i}[\mathbf{a}, \mathbf{b}].$$

17. The KZ-equation and the Drinfeld associator

The quotient map

$$\mathbb{C}\langle\langle X_0, X_1, X_\infty\rangle\rangle/(X_0 + X_1 + X_\infty) \to \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$$

is an isomorphism. We will identify these two rings. Recall that the KZ-connection on $\mathbb{P}^1 - \{0, 1, \infty\}$ is given by

$$\omega_{KZ} = \frac{dw}{w}X_0 + \frac{dw}{w-1}X_1 \in H^0(\Omega^1_{\mathbb{P}^1}(\log\{0, 1, \infty\}))\hat{\otimes}\mathbb{C}\langle\langle X_0, X_1\rangle\rangle.$$

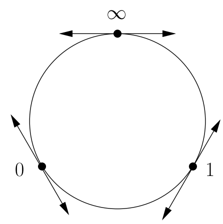

Figure 4: The 6 tangent vectors.

The form ω_{KZ} defines a flat connection on the trivial bundle

$$\mathbb{C}\langle\langle X_0, X_1\rangle\rangle \times \mathbb{P}^1 - \{0, 1, \infty\} \to \mathbb{P}^1 - \{0, 1, \infty\}$$

by the formula

$$\nabla f = df - f\omega_{KZ}.$$

Its transport function induces a transport function

$$\{\text{paths in } \mathbb{P}^1 - \{0, 1, \infty\}\} \to \{\text{group-like elements of } \mathbb{C}\langle\langle X_0, X_1\rangle\rangle\}$$

in $\mathbb{P}^1 - \{0, 1, \infty\}$. It takes the path γ in $\mathbb{P}^1 - \{0, 1, \infty\}$ to

$$T(\gamma) = 1 + \int_\gamma \omega_{KZ} + \int_\gamma \omega_{KZ}\omega_{KZ} + \int_\gamma \omega_{KZ}\omega_{KZ}\omega_{KZ} + \cdots \tag{17.1}$$

Since ω_{KZ} is clearly integrable, the connection is flat and $T(\gamma)$ depends only on the homotopy class of γ relative to its endpoints.

There are six standard tangent vectors of $\mathbb{P}^1 - \{0, 1, \infty\}$. Two are anchored at each of $0, 1, \infty$. They lie in one orbit under the action of the symmetric group S_3 on $\mathbb{P}^1 - \{0, 1, \infty\}$ and are thus determined by the two vectors at $w = 0$, which are $\pm\partial/\partial w$. These have the property that their reduction mod p is non-zero for all prime numbers p.

The (KZ/de Rham) version of the Drinfeld associator is the invertible power series $\Phi(X_0, X_1) \in \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$ obtained by taking the regularized value of the transport (17.1) above on the path $[0, 1]$. It begins:

$$\begin{aligned}\Phi(X_0, X_1) = 1 &- \zeta(2)[X_0, X_1] + \zeta(3)[X_0, [X_0, X_1]] + \zeta(1, 2)[[X_0, X_1], X_1] \\ &- \zeta(4)[X_0, [X_0, [X_0, X_1]]] - \zeta(1, 3)[X_0, [[X_0, X_1], X_1]] \\ &- \zeta(1, 1, 2)[[[X_0, X_1], X_1], X_1] + \frac{1}{2}\zeta(2)^2[X_0, X_1]^2 + \cdots\end{aligned}$$

where, for positive integers $n_1, \dots, n_r$, where $n_r > 1$,

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}}.$$

These are the *multiple zeta numbers*. They generalize the values of the Riemann zeta function at positive integers. An explicit formula for $\Phi(X_0, X_1)$ is given in [23, 10].[12] All coefficients are rational multiples of multiple zeta values.

Several (not all) of its basic properties are summarized in the following result:

Theorem 17.1 (Drinfeld)**.** *The Drinfeld associator* Φ *satisfies:*

(i) $\Phi(X_0, X_1)\Phi(X_1, X_0) = 1$
(ii) In the ring $\mathbb{C}\langle\langle X_0, X_1, X_\infty\rangle\rangle/(X_0 + X_1 + X_\infty)$ *we have*

$$\Phi(X_0, X_1)e^{i\pi X_1}\Phi(X_1, X_\infty)e^{i\pi X_\infty}\Phi(X_\infty, X_0)e^{i\pi X_0} = 1.$$

The normalized value of T on the unique real path from $w = 0$ to $w = 1$ is $\Phi(X_0, X_1)$. View the symmetric group S_3 as $\mathrm{Aut}\{0, 1, \infty\}$. The action of the automorphisms of $\mathbb{P}^1 - \{0, 1, \infty\}$ on the cusps determines an isomorphism

$$\mathrm{Aut}(\mathbb{P}^1, \{0, 1, \infty\}) \to \mathrm{Aut}\{0, 1, \infty\}.$$

Let it act on $\{X_0, X_1, X_\infty\}$ by permuting the indices. Since the connection is invariant under the S_3-action on $\mathbb{P}^1 - \{0, 1, \infty\}$, we have, for example, the following values of the normalized transport on the real paths:

$$T^{\mathrm{norm}}([1, \infty]) = \Phi(X_1, X_\infty), \quad T^{\mathrm{norm}}([0, \infty]) = \Phi(X_0, X_\infty),$$

where $[0, \infty]$ is the path from 0 to ∞ along the negative real axis.

17.1. The fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$

Consider the category whose objects are the 6 tangent vectors of $\mathbb{P}^1$ defined above and whose morphisms are homotopy classes from one tangent vector to

[12] To get this formula for $\Phi(X_0, X_1)$, one has to reverse the order of all monomials — equivalently, replace each bracket $[U, V]$ by its negative $-[U, V]$. This is because Furusho uses the opposite convention for path multiplication.

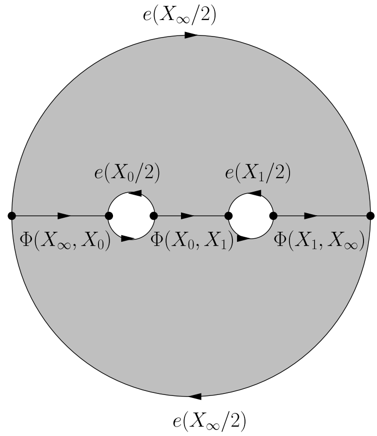

Figure 5: The fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$.

another.[13] Denote it by $\Pi(\mathbb{P}^1, V)$. It is generated by the paths shown in the diagram. As above, a good topological model is to replace $\mathbb{P}^1 - \{0, 1, \infty\}$ by the real oriented blow-up of $\mathbb{P}^1$ at $\{0, 1, \infty\}$, which is a 3-holed sphere, and represent the tangent directions by the corresponding points on the boundary of the blown up sphere.

Define a functor

$$\Theta : \Pi(\mathbb{P}^1, V) \to \{\text{group-like elements of } \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle\}$$

by taking the positively oriented semi-circle about $a \in \{0, 1, \infty\}$ to $e(X_a/2)$ and the real interval from a to b to $\Phi(X_a, X_b)$. Drinfeld's relations imply that Θ is well-defined.

This restricts to a group homomorphism $\Theta_{\vec{v}} : \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}) \to \mathcal{P}$ for each of the 6 distinguished tangent vectors $\vec{v}$.

18. The limit MHS on $\pi_1(E'_{\partial/\partial q}, \partial/\partial w)^{\mathrm{un}}$

The computations of Section 12.2 imply that the restriction $\omega_{E'_0}$ of the KZB connection to $E'_0 = \mathbb{P}^1 - \{0, 1, \infty\}$ is obtained from ω_{KZ} by composing it

[13]That is, the path starts with one tangent vector and ends with the negative of the second. Such a path γ must also satisfy $\gamma(t) \notin \{0, 1, \infty\}$ when $0 < t < 1$. Composition of two such homotopy classes of paths can be defined when the second path begins at the tangent vector where the first ends.

with the ring homomorphism

$$\mathbb{C}\langle\langle X_0, X_1, X_\infty\rangle\rangle/(X_0 + X_1 + X_\infty) \hookrightarrow \mathbb{C}\langle\langle T, A\rangle\rangle$$

defined by

$$\begin{aligned} X_0 &\mapsto R_0 = \left(\frac{T}{e^T - 1}\right) \cdot A, \\ X_1 &\mapsto R_1 = [T, A], \\ X_\infty &\mapsto R_\infty = \left(\frac{T}{e^{-T} - 1}\right) \cdot A, \end{aligned} \tag{18.1}$$

which is well defined as $R_0 + R_1 + R_\infty = 0$.

Since the periods of ω_{KZ} are understood (they are multiple zeta numbers), this formula will allow us to compute the periods of $\omega_{E_0'}$ in terms of multiple zeta numbers.

18.1. The cylinder relation

To construct a well-defined homomorphism

$$\pi_1(\mathcal{E}'_{\partial/\partial q}, \partial/\partial w) \to \mathcal{P},$$

we need to find all solutions $U \in \mathbb{L}(T, A)^\wedge$ of the equation

$$e^{-U} e^{\lambda R_0} e^{U} e^{\lambda R_\infty} = 1 \text{ in } \mathbb{Q}\langle\langle T, A\rangle\rangle \tag{18.2}$$

for all $\lambda \in \mathbb{Q}^\times$. This relation will be called the *cylinder relation.* Note that a solution to the equation with $\lambda = 1$ will be a solution for all λ.

Lemma 18.1. *For all $\lambda \in \mathbb{C}$, the relation $e^T e^{\lambda R_0} e^{-T} e^{\lambda R_\infty} = 1$ holds in $\mathcal{P}$. That is, $U = -T$ is a solution of the cylinder equation.*

Proof. It suffices to prove the relation $e^T R_0 e^{-T} = -R_\infty$. Since $\phi \exp(u) \phi^{-1} = \exp(\phi u \phi^{-1})$, we have

$$\begin{aligned} e^T R_0 e^{-T} = e^T\left(\left[\frac{T}{e^T - 1}\right] \cdot A\right) e^{-T} &= e^{\mathrm{ad}_T}\left(\left[\frac{T}{e^T - 1}\right] \cdot A\right) \\ &= \left[\frac{Te^T}{e^T - 1}\right] \cdot A = \left[\frac{-T}{e^{-T} - 1}\right] \cdot A = -R_\infty. \end{aligned}$$

□

Proposition 18.2. *Every solution of the cylinder relation (18.2) is of the form*

$$e^U = e^{\lambda R_0} e^{-T}$$

for some $\lambda \in \mathbb{C}$.

Proof. Suppose that U is a solution of the cylinder equation. Set $V = \log(e^U e^T)$. Then $V \in \mathbb{L}(T, A)^\wedge$ and $e^U = e^V e^{-T}$. The cylinder relation implies that

$$e^T e^{-V} e^{R_0} e^V e^{-T} = e^{-U} e^{R_0} e^U = e^{-R_\infty} = e^T e^{R_0} e^{-T}$$

so that $e^{-V} e^{R_0} e^V = e^{R_0}$. The result follows as the centralizer of R_0 in $\mathbb{L}(T, A)^\wedge$ is $\mathbb{C}R_0$. □

18.2. The homomorphisms $\pi_1(E'_{\partial/\partial q}, \partial/\partial w) \to \mathcal{P}$

The positive real axis determines two points v_0 and v_∞ on the real oriented blowup of $\mathbb{P}^1$ at $\{0, 1, \infty\}$ — the point v_0 lies on the circle at 0 and v_∞ lies on the circle at ∞. There is a natural $U(1)$ action on each of these circles.

Suppose that $\lambda \in \mathbb{C}^*$. Write it in the form $re^{i\theta}$. View $E'_{\lambda\partial/\partial q}$ as the quotient of the real oriented blow-up of $\mathbb{P}^1$ at $\{0, 1, \infty\}$ by

$$e^{i\phi} v_\infty \sim e^{i(\theta-\phi)} v_0.$$

Denote the image of the two identified circles in $E_{\lambda\partial/\partial q}$ by C. One can check that as λ moves around the unit circle in the positive direction, the identification changes by a positive Dehn twist about C. Note that for each λ there is a natural inclusion

$$\iota : (\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w) \to (E'_{\lambda\partial/\partial q}, \partial/\partial w)$$

where in both cases $\partial/\partial w$ is in element of the tangent space of $1 \in \mathbb{P}^1$.

To define a homomorphism $\Theta_\lambda : \pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w) \to \mathcal{P}$ such that the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w) & \longrightarrow & \exp \mathbb{L}(X_0, X_1)^\wedge \\ \downarrow{\scriptstyle \iota_*} & & \downarrow \\ \pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w) & \xrightarrow{\Theta_\lambda} & \mathcal{P} \end{array}$$

commutes, where the right-hand vertical map is defined by (18.1). We need to give a "factor of automorphy" for the identification. This is the monodromy

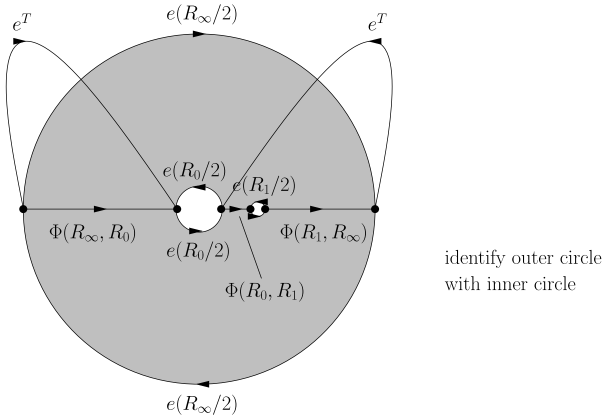

Figure 6: The path torsor of $E'_{\partial/\partial q}$.

along a "path" from $e^{i\phi}v_\infty$ to $e^{i(\theta-\phi)}v_0$. In order that Θ_λ be well defined, this factor of automorphy has to satisfy the cylinder relation.

For $\lambda \in \mathbb{C}^*$, define this factor of automorphy for Θ_λ to be

$$\lambda^{R_0}e^{-T} := e^{\log \lambda R_0}e^{-T}.$$

That is, the (inverse) monodromy in going from the tangent vector λv_0 at $0 \in \mathbb{P}^1$ to the tangent vector v_∞ at ∞ is $\lambda^{R_0}e^{-T}$.

Give $\mathbb{C}\langle\langle A, T\rangle\rangle$ the Hodge, weight and relative weight filtrations defined in Section 15.

Proposition 18.3. *The (complete Hopf algebra) homomorphism*

$$\Theta_\lambda : \mathbb{Q}\pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w)^\wedge \to \mathbb{C}\langle\langle T, A\rangle\rangle$$

is an isomorphism after tensoring the source with $\mathbb{C}$. This and the Hodge and weight filtrations on $\mathbb{C}\langle\langle A, T\rangle\rangle$ defined in Section 15 define a MHS on $\mathbb{Q}\pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w)^\wedge$. This is the canonical limit MHS on the fiber of the universal enveloping algebra of $\boldsymbol{\mathcal{P}}$ corresponding to the tangent vector $\lambda\partial/\partial q + \partial/\partial w$ at the identity of the nodal cubic. Its relative weight filtration is the one defined in Section 15.

Proof. Observe that Θ_λ induces a homomorphism $H_1(E'_{\lambda\partial/\partial q}, \mathbb{Z}) \to \mathbb{C}T \oplus \mathbb{C}A$. It takes $\mathbf{a}$ to $2\pi i R_0 \bmod I^2 = 2\pi i A$ and $\mathbf{b}$ to $\log(\lambda^{R_0}e^{-T}) \bmod I^2 = \log \lambda A - T$. The homomorphism Θ_λ is an isomorphism after complexifying as both its source and target are free and as it induces an isomorphism on I/I^2.

The homomorphism Θ_λ defines a MHS on $\mathbb{Q}\pi_1(E'_{\lambda\partial/\partial q}, \partial/\partial w)^\wedge$ by pulling back the Hodge, weight and relative weight filtrations of $\mathbb{C}\langle\langle T, A\rangle\rangle$. To check that this is the limit MHS associated to $\lambda\partial/\partial q$, it suffices to check that the induced MHS on $H_1(E_{\lambda\partial/\partial q})$ by Θ_λ agrees with the canonical limit MHS. This follows from the discussion above and the fact that the MHS induced on the image of $\mathbb{Q}\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w)^\wedge$ by Θ_λ is its canonical MHS, which follows from the computations in Section 12.2. The point being that the limit MHS on $H_1(E_{\lambda\partial/\partial q})$ corresponding to $\lambda\partial/\partial q$ determines the factor of automorphy. The computations at the beginning of the proof imply that the MHS on $H_1(E_{\lambda\partial/\partial q})$ induced by Θ_λ agrees with the limit MHS that was computed in Lemma 15.3. □

Part 4. The $\mathbb{Q}$-de Rham structure

Levin and Racinet [24, §5] sketch an argument to show that the elliptic KZB connection is defined over $\mathbb{Q}$. This part is an expanded exposition of a special case of their computation. In particular, we explicitly compute (Thm. 20.2) the restriction of the canonical extension of the universal elliptic KZB connection to $\mathcal{M}_{1,\vec{1}}$ in terms of the $\mathbb{Q}$-algebraic coordinates on $\mathcal{M}_{1,\vec{1}}$. When reading [24, §5], it is important to note that that the restriction of $\overline{\boldsymbol{\mathcal{P}}}$ to an elliptic curve E is *not* algebraically trivial. Levin and Racinet trivialize the restriction of $\boldsymbol{\mathcal{P}}$ to E'. The $\mathbb{Q}$-connection they write down is on the corresponding *trivial* extension of $\boldsymbol{\mathcal{P}}|_{E'}$ to E. It does not have a regular singular point at 0. However, it is important in applications, such as those in [18], to know that the canonical extension $\overline{\boldsymbol{\mathcal{P}}}$ and its connection are defined over $\mathbb{Q}$. This has been verified by Ma Luo and will appear in his Duke PhD thesis.

19. The $\mathbb{Q}$-DR structure on $\overline{\boldsymbol{\mathcal{H}}}$ over $\overline{\boldsymbol{\mathcal{M}}}_{1,\vec{1}}$

The first step is to compute the $\mathbb{Q}$-DR structure on $\mathbb{H}$ and its canonical extension $\overline{\mathcal{H}}$. Since $\mathcal{M}_{1,\vec{1}}$ is the moduli space of elliptic curves endowed with a non-zero abelian differential, the Hodge bundle $F^1\mathcal{H}$ is trivialized by its tautological section. We show that the canonical extension of $\mathcal{H}$ over $\overline{\mathcal{M}}_{1,\vec{1}}$ is trivial and that it and its connection are defined over $\mathbb{Q}$. Material in this section must surely be well known and classical (19th C).

19.1. $\boldsymbol{\mathcal{M}}_{1,\vec{1}}$ as a $\mathbb{Q}$-scheme

As explained in Section 1 (also see [15]), $\mathcal{M}_{1,\vec{1}}$ is the quotient $\mathcal{L}'_{-1}$ of $\mathbb{C} \times \mathfrak{h}$ by the action of $\mathrm{SL}_2(\mathbb{Z})$ which acts with factor of automorphy $(c\tau + d)^{-1}$. It is also the complement in $\mathbb{C}^2$ of the discriminant locus $\Delta = 0$, where

$$\Delta = u^3 - 27v^2.$$

The quotient mapping $\mathbb{C} \times \mathfrak{h} \to \mathbb{C}^2 - \Delta^{-1}(0)$ is

$$(\xi, \tau) \mapsto \big(\xi^4 g_2(\tau), \xi^6 g_3(\tau)\big),$$

where

$$g_2(\tau) = 20(2\pi i)^4 G_4(\tau) \text{ and } g_3 = \frac{7}{3}(2\pi i)^6 G_6(\tau).$$

The point (u, v) corresponds to the pair $(E_{u,v}, \omega_{u,v})$ where $E_{u,v}$ is the elliptic curve $y^2 = 4x^3 - ux - v$ and $\omega_{u,v}$ is the abelian differential dx/y. This elliptic curve has discriminant (divided by 16) equal to

$$\Delta := u^3 - 27v^2 = \xi^{12}(g_2^3 - 27g_3^2) = (2\pi i\xi)^{12}\Delta_0,$$

where $\Delta_0 = q\prod_{n\geq 1}(1-q^n)^{24}$ is the Ramanujan τ-function. We will view $\mathcal{M}_{1,\vec{1}}$ as the $\mathbb{Q}$-scheme $\operatorname{Spec}\mathbb{Q}[u, v, \Delta^{-1}]$.

19.2. Trivializing $\mathcal{H}$ over $\mathcal{M}_{1,\vec{1}}$

To trivialize $\mathcal{H}$, we need two linearly independent sections. The first is given by the abelian differential dx/y. The second by xdx/y, a differential of the second kind.

Set

$$\eta_\tau = \wp(z, \tau)dz = \left(1 + 2\sum_{m=1}^{\infty} \frac{G_{2m+2}(\tau)}{(2m)!}(2\pi i z)^{2m+2}\right)\frac{dz}{z^2}.$$

This is a differential of the second kind on E_τ.

Proposition 19.1. *If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\eta_{\gamma\tau} = (c\tau + d)\eta_\tau$ and*

$$\int_{E_\tau} \omega_\tau \smile \eta_\tau = 2\pi i.$$

In particular, $H^1(E_\tau; \mathbb{C}) = \mathbb{C}\omega_\tau \oplus \mathbb{C}\eta_\tau$ for all τ.

Proof. The first assertion follows easily from the definition of η_τ. The second formula follows from a routine residue computation:

Choose a closed disk $D = \{z : |z| \leq R\}$ in E_τ about the origin. Let F be a holomorphic function on D satisfying $F'(z) = \wp(z)$. Let $\varphi : E_\tau \to \mathbb{R}$ be a smooth function that vanishes outside the annulus $A = \{z : R/3 < |z| < R/2\}$ and is identically 1 when $|z| < R/3$. The 1-form

$$\psi := \eta_\tau - d(\varphi F(z))$$

is smooth and closed. Since it agrees with η_τ outside A, it has the same periods as η_τ and thus represents the same cohomology class. Since $\omega_\tau \wedge \psi$ is supported in D, we have

$$\langle \omega_\tau \smile \eta_\tau, E_\tau \rangle = \int_{E_\tau} \omega_\tau \wedge \psi = \int_D dz \wedge \psi = \int_{\partial D} z\psi = \int_{\partial D} z\wp(z)dz = 2\pi i.$$

□

Remark 19.2. This implies that the exact sequence

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{L}_{-1} \to 0$$

over $\mathcal{M}_{1,1}$ splits; the copy of $\mathcal{L}_{-1}$ in $\mathcal{H}$ is spanned locally by η_τ. We will see below that this sequence also splits over $\overline{\mathcal{M}}_{1,1}$. This splitting also follows from the vanishing of $H^1(\overline{\mathcal{M}}_{1,1}, \mathcal{L}_2)$ as there are no modular forms of weight 2 and level 1.

Corollary 19.3. *The sections $\xi^{-1}\omega_\tau$ and $\xi\eta_\tau$ of $\mathcal{H}$ over $\mathbb{C} \times \mathfrak{h}$ are $\mathrm{SL}_2(\mathbb{Z})$-invariant.* □

For a lattice Λ in $\mathbb{C}$, set

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right].$$

The Weierstrass $\wp$-function $\wp(z,\tau)$ defined in Section 8.4 is $\wp_{\Lambda_\tau}(z)$. One checks easily that

$$\wp_{\xi^{-1}\Lambda}(\xi^{-1}z) = \xi^2 \wp_\Lambda(z).$$

Multiplication by ξ^{-1} induces an isomorphism $E_\tau \to \mathbb{C}/\xi^{-1}\Lambda_\tau$ under which dz and $\wp_{\xi^{-1}\Lambda_\tau} dz$ pull back to $\xi^{-1}\omega_\tau$ and $\xi\eta_\tau$, respectively.

Proposition 19.4. *If $(u,v) = (\xi^4 g_2(\tau), \xi^6 g_3(\tau))$, then*

(i) the map

$$z \mapsto [\wp_{\xi^{-1}\Lambda_\tau}(z), \wp'_{\xi^{-1}\Lambda_\tau}(z), 1]$$

from $\mathbb{C}/\xi^{-1}\Lambda_\tau$ to $\mathbb{P}^2$ induces an isomorphism $\mathbb{C}/\xi^{-1}\Lambda_\tau \to E_{u,v}$;

(ii) *under this isomorphism*

$$dx/y = dz = \xi^{-1}\omega_\tau \text{ and } xdx/y = \wp_{\xi^{-1}\Lambda_\tau}(z)dz;$$

(iii) *under the isomorphism* $E_\tau \to \mathbb{C}/\xi^{-1}\Lambda_\tau \to E_{u,v}$, dx/y *and* xdx/y *pull back to* $\xi^{-1}\omega_\tau$ *and* $\xi\eta_\tau$, *respectively.*

□

Corollary 19.5. *For each* $(u,v) \in \mathcal{M}_{1,\vec{1}}$, *the elements* dx/y *and* xdx/y *of the fiber* $H^1(E_{u,v})$ *of* $\mathcal{H}$ *are linearly independent.* □

Denote these sections of $\mathcal{H}$ over $\mathcal{M}_{1,\vec{1}}$ by $\widehat{T}$ and $\widehat{S}$, respectively. They trivialize $\mathcal{H}$ and determine the extension

$$\overline{\mathcal{H}} := \mathcal{O}_{\overline{\mathcal{M}}_{1,\vec{1}}}\widehat{S} \oplus \mathcal{O}_{\overline{\mathcal{M}}_{1,\vec{1}}}\widehat{T}$$

of $\mathcal{H}$ to $\overline{\mathcal{M}}_{1,\vec{1}}$.

Proposition 19.6. *The connection on* $\overline{\mathcal{H}}$ *with respect to this trivialization is*

$$\nabla_0 = d + \Big(-\frac{1}{12}\frac{d\Delta}{\Delta}\otimes\widehat{T} + \frac{3}{2}\frac{\alpha}{\Delta}\otimes\widehat{S}\Big)\frac{\partial}{\partial\widehat{T}} + \Big(-\frac{u}{8}\frac{\alpha}{\Delta}\otimes\widehat{T} + \frac{1}{12}\frac{d\Delta}{\Delta}\otimes\widehat{S}\Big)\frac{\partial}{\partial\widehat{S}}$$

where $\alpha = 2udv - 3vdu$ *and* $\Delta = u^3 - 27v^2$. *This form is logarithmic on* $\overline{\mathcal{M}}_{1,\vec{1}}$ *with nilpotent residue along* $\Delta = 0$ *and is therefore the canonical extension of* $\mathcal{H}$ *over* $\overline{\mathcal{M}}_{1,\vec{1}}$. *It is defined over* $\mathbb{Q}$.

Sketch of Proof. We need to understand how the classes dx/y and xdx/y depend on (u,v). Each of the 1-forms

$$\frac{\partial}{\partial u}\Big(\frac{dx}{y}\Big) = \frac{1}{2}\frac{xdx}{y^3}, \quad \frac{\partial}{\partial v}\Big(\frac{dx}{y}\Big) = \frac{1}{2}\frac{dx}{y^3},$$
$$\frac{\partial}{\partial u}\Big(\frac{xdx}{y}\Big) = \frac{1}{2}\frac{x^2dx}{y^3}, \quad \frac{\partial}{\partial v}\Big(\frac{xdx}{y}\Big) = \frac{1}{2}\frac{xdx}{y^3}$$

is a differential of the second kind on each $E_{u,v}$. So the cohomology class of each is a linear combination of the classes of dx/y and xdx/y.

The differentials $d(1/y)$, $d(x/y)$ and $d(x^2/y)$, the relation $2ydy = (12x^2 - u)dx$, and some linear algebra give

$$\begin{pmatrix} \frac{dx}{y^3} & \frac{xdx}{y^3} & \frac{x^2dx}{y^3}\end{pmatrix} \equiv \frac{3}{\Delta}\begin{pmatrix}\frac{dx}{y} & \frac{xdx}{y}\end{pmatrix}\begin{pmatrix} 3v & -u^2/6 & uv/4 \\ 2u & -3v & u^2/6\end{pmatrix}$$

where $\equiv$ means congruent mod exact forms of the second kind, and thus equal in cohomology.

Now

$$\begin{aligned}\nabla_0\begin{pmatrix}\widehat{T} & \widehat{S}\end{pmatrix} &= \nabla_0\begin{pmatrix}\frac{dx}{y} & \frac{xdx}{y}\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\frac{xdx}{y^3} & \frac{x^2dx}{y^3}\end{pmatrix}du + \frac{1}{2}\begin{pmatrix}\frac{dx}{y^3} & \frac{xdx}{y^3}\end{pmatrix}dv\\ &= \frac{3}{2\Delta}\begin{pmatrix}\frac{dx}{y} & \frac{xdx}{y}\end{pmatrix}\begin{pmatrix}-u^2du/6+3vdv & uvdu/4-u^6dv/6\\ -3vdu+2udv & u^2du/6-3vdv\end{pmatrix}\\ &= \begin{pmatrix}\widehat{T} & \widehat{S}\end{pmatrix}\begin{pmatrix}-\frac{1}{12}\frac{d\Delta}{\Delta} & -\frac{u\alpha}{8\Delta}\\ \frac{3\alpha}{2\Delta} & \frac{1}{12}\frac{d\Delta}{\Delta}\end{pmatrix}\end{aligned}$$

The forms α/Δ and $u\alpha/\Delta$ are logarithmic. One can prove this directly. Alternatively, we can use the fact [6] that a meromorphic form φ on $\mathbb{C}^2$ has logarithmic singularities along $\Delta = 0$ if and only if $\Delta\varphi$ and $\Delta d\varphi$ are both holomorphic along $\Delta = 0$. This holds in our case as

$$\Delta d(\alpha/\Delta) = -du\wedge dv \text{ and } \Delta d(u\alpha/\Delta) = udu\wedge dv.$$

Similarly, one checks that $w\varphi$ and $wd\varphi$ are holomorphic along the line at infinity, where $w = 0$ is a local defining equation of the line at infinity, then φ is logarithmic along the line at infinity. This is easily checked when φ is α/Δ and $u\alpha/\Delta$. □

This implies that the sequence

$$0\to\overline{\mathcal{L}}\to\overline{\mathcal{H}}\to\overline{\mathcal{L}}_{-1}\to 0$$

splits over $\overline{\mathcal{M}}_{1,\vec{1}}$. The lift of $\overline{\mathcal{L}}_{-1}$ in $\overline{\mathcal{H}}$ is $\mathcal{O}_{\overline{\mathcal{M}}_{1,\vec{1}}}\widehat{S}$.

19.3. Transcendental version

We re-derive the formula for the connection in terms of the coordinates $(\xi,\tau)\in\mathbb{C}\times\mathfrak{h}$. This will yield some formulas that are useful in computing the algebraic version of the universal elliptic KZB connection.

As observed above, if $(u,v) = (\xi^4 g_2(\tau), \xi^6 g_3(\tau))$, then $dx/y = \xi^{-1}\omega_\tau$. Since $\widehat{T}$ is the class of dx/y, we have

$$\widehat{T} = \xi^{-1}T.$$

Proposition 19.7. *We have* $\eta_\tau = (2\pi i)^2(A - 2G_2(\tau)T)$ *so that*

$$\widehat{S} = \xi\eta_\tau = (2\pi i)^2\xi(A - 2G_2(\tau)T).$$

Proof. Using the notation of Example 3.4, we have

$$\begin{aligned}\begin{pmatrix}\mathbf{a} & \mathbf{t}\end{pmatrix}\begin{pmatrix}1\\ 8\pi^2G_2(\tau)\end{pmatrix} &= \begin{pmatrix}\mathbf{a}' & \mathbf{t}'\end{pmatrix}\begin{pmatrix}(c\tau+d)^{-1} & 0\\ 2\pi i c & c\tau + d\end{pmatrix}\begin{pmatrix}1\\ 8\pi^2G_2(\tau)\end{pmatrix}\\ &= (c\tau+d)^{-1}\begin{pmatrix}\mathbf{a}' & \mathbf{t}'\end{pmatrix}\begin{pmatrix}1\\ 8\pi^2G_2(\gamma\tau)\end{pmatrix}.\end{aligned}$$

From this it follows that

$$A - 2G_2(\tau)T = (c\tau+d)^{-1}(A' - 2G_2(\gamma\tau)T').$$

Consequently, $A - 2G_2(\tau)T$ is a section of $\mathcal{H}$ that spans a copy of $\mathcal{L}_{-1}$ over $\mathfrak{h}$. But η_τ is another such section. It follows that η_τ is a holomorphic multiple of $A - 2G_2(\tau)T$. This multiple can be determined by pairing with ω_τ. Since

$$\langle T, A - 2G_2(\tau)T\rangle = \langle \omega_\tau, (2\pi i)^{-1}\mathbf{a}\rangle = (2\pi i)^{-1}$$

and

$$\langle T, \eta_\tau\rangle = \int_{E_\tau}\omega_\tau \smile \eta_\tau = 2\pi i,$$

it follows that

$$\eta_\tau = (2\pi i)^2(A - 2G_2(\tau)T).$$

There are several ways to prove the second assertion. One is to observe that

$$\langle \widehat{T}, \widehat{S}\rangle = \langle \frac{dx}{y}, \frac{xdx}{y}\rangle = 2\pi i = \langle T, S\rangle = \langle \xi^{-1}T, \xi S\rangle = \langle \widehat{T}, \xi S\rangle.$$

□

We've already seen in Example 5.2 that the connection ∇_0 on $\mathcal{H}$ over $\mathfrak{h}$ with respect to the framing A, T is given by

$$\nabla_0 = d + 2\pi i A\frac{\partial}{\partial T}\otimes d\tau.$$

Proposition 19.8. *With respect to the framing $\widehat{S}$ and $\widehat{T}$ of $\overline{\mathcal{H}}$ over $\mathcal{M}_{1,\vec{1}}$, the connection on $\mathcal{H}$ is*

$$\nabla_0 = d + \Big((4\pi i\, G_2\, d\tau - \frac{d\xi}{\xi}) \otimes \widehat{T} + \frac{2\pi i}{(2\pi i\xi)^2} d\tau \otimes \widehat{S}\Big)\frac{\partial}{\partial \widehat{T}}$$
$$- \Big((2\pi i\xi)^2(8\pi i\, G_2^2 + 2G_2')\, d\tau \otimes \widehat{T} + (4\pi i\, G_2\, d\tau - \frac{d\xi}{\xi}) \otimes \widehat{S}\Big)\frac{\partial}{\partial \widehat{S}}$$

Proof. Since $S/(2\pi i)^2 = A - 2G_2 T$,

$$\begin{aligned}\nabla_0 \begin{pmatrix} T & S/(2\pi i)^2\end{pmatrix} &= \nabla_0 \begin{pmatrix} T & A\end{pmatrix}\begin{pmatrix} 1 & -2G_2 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} T & A\end{pmatrix}\begin{pmatrix} 0 & -2G_2' \\ 0 & 0\end{pmatrix} d\tau \\ &= \begin{pmatrix} T & A\end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 2\pi i & 0\end{pmatrix}\begin{pmatrix} 1 & -2G_2 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} 0 & -2G_2' \\ 0 & 0\end{pmatrix}\right) d\tau \\ &= \begin{pmatrix} T & S/(2\pi i)^2\end{pmatrix}\begin{pmatrix} 1 & 2G_2 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -2G_2' \\ 2\pi i & -4\pi i G_2\end{pmatrix} d\tau \\ &= \begin{pmatrix} T & S/(2\pi i)^2\end{pmatrix}\begin{pmatrix} 4\pi i G_2 & -(8\pi i G_2^2 + 2G_2') \\ 2\pi i & -4\pi i G_2\end{pmatrix} d\tau\end{aligned}$$

Rescaling, we have

$$\nabla_0 \begin{pmatrix} T & S\end{pmatrix} = \begin{pmatrix} T & S\end{pmatrix}\begin{pmatrix} 4\pi i G_2 & -(2\pi i)^2(8\pi i G_2^2 + 2G_2') \\ (2\pi i)^{-1} & -4\pi i G_2\end{pmatrix} d\tau$$

Denote the 2×2 matrix of 1-forms in this expression by $Bd\tau$. Then

$$\begin{aligned}\nabla_0\begin{pmatrix} \widehat{T} & \widehat{S}\end{pmatrix} &= \nabla_0\begin{pmatrix} T & S\end{pmatrix}\begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi\end{pmatrix} + \begin{pmatrix} T & S\end{pmatrix}\begin{pmatrix} -\xi^{-2} & 0 \\ 0 & 1\end{pmatrix} d\xi \\ &= \begin{pmatrix} \widehat{T} & \widehat{S}\end{pmatrix}\left[\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1}\end{pmatrix} B \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi\end{pmatrix} d\tau + \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1}\end{pmatrix}\begin{pmatrix} -\xi^{-2} & 0 \\ 0 & 1\end{pmatrix} d\xi\right] \\ &= \begin{pmatrix} \widehat{T} & \widehat{S}\end{pmatrix}\begin{pmatrix} -\frac{d\xi}{\xi} + 4\pi i G_2(\tau) d\tau & -2(2\pi i\xi)^2(4\pi i G_2(\tau)^2 + G_2'(\tau)) d\tau \\ \frac{2\pi i}{(2\pi i\xi)^2} d\tau & \frac{d\xi}{\xi} - 4\pi i G_2(\tau) d\tau\end{pmatrix}\end{aligned}$$

□

Comparing this formula with that in Proposition 19.6, we conclude:

$$\frac{d\xi}{\xi} - 4\pi i G_2(\tau) d\tau = \frac{1}{12}\frac{d\Delta}{\Delta} \text{ and } \frac{2\pi i}{(2\pi i\xi)^2} d\tau = \frac{3\alpha}{2\Delta}. \tag{19.1}$$

Taking the quotient of the two off diagonal entries of the connection matrix, we conclude that

$$G_4(\tau) = \frac{6}{5}\big(2G_2(\tau)^2 + G_2'(\tau)/2\pi i\big).$$

This can also be verified by observing that the RHS is a modular form of weight 4 and then computing value of both sides at $q = 0$.

20. The $\mathbb{Q}$-DR structure on $\overline{\boldsymbol{\mathcal{P}}}$ over $\overline{\mathcal{M}}_{1,\vec{1}}$

The bundle $\overline{\boldsymbol{\mathcal{P}}}$ over $\overline{\mathcal{M}}_{1,\vec{1}}$ is the trivial bundle whose fiber is $\mathbb{L}(\widehat{S},\widehat{T})^\wedge$. We will define a $\mathbb{Q}$ structure on it — that is, a $\mathbb{Q}$ structure on its truncations by the terms of its lower central series.

Some preliminary observations will be helpful. Since the cup product of the rational differentials dx/y and xdx/y is $2\pi i$, it is natural to multiply their Poincaré duals by $(2\pi i)^{-1}$ to obtain a $\mathbb{Q}$-de Rham basis of the first homology. Motivated by this, we define

$$\widehat{T}_0 = \widehat{T}/2\pi i \text{ and } \widehat{S}_0 = \widehat{S}/2\pi i.$$

Since both basis elements of $\overline{\mathcal{H}}$ are multiplied by the same constant, the formula for the connection on $\overline{\mathcal{H}}$ given in Proposition 19.6 remains valid when we replace $\widehat{S}$ by $\widehat{S}_0$ and $\widehat{T}$ by $\widehat{T}_0$.

Set $\mathfrak{p}_\mathbb{Q} = \mathbb{L}_\mathbb{Q}(\widehat{S}_0,\widehat{T}_0)^\wedge$. Define the $\mathbb{Q}$-structure on $\overline{\boldsymbol{\mathcal{P}}}$ to be $\overline{\mathcal{M}}_{1,\vec{1}/\mathbb{Q}} \times \mathfrak{p}_\mathbb{Q}$.

Define derivations $\hat{\boldsymbol{\epsilon}}_{2m}$ of $\mathfrak{p}_\mathbb{Q}$ by

$$\hat{\boldsymbol{\epsilon}}_{2m} = \begin{cases} -\widehat{S}_0\frac{\partial}{\partial \widehat{T}_0} & m = 0;\\ \widehat{T}_0^{2m-1}\cdot\widehat{S}_0 - \sum_{\substack{j+k=2m-1\\ j>k>0}}(-1)^j[\widehat{T}_0^j\cdot\widehat{S}_0,\widehat{T}_0^k\cdot\widehat{S}_0]\frac{\partial}{\partial\widehat{S}_0} & m>0.\end{cases}$$

Lemma 20.1. *For all $m \geq 0$, we have $(2\pi i\xi)^{2m-2}\hat{\boldsymbol{\epsilon}}_{2m} = \epsilon_{2m}$ in* $\operatorname{Der}\mathfrak{p}$.

Proof. First observe that

$$\epsilon_{2m}(T) = -T^{2m}\cdot A \text{ and } \epsilon_{2m}(A) = \sum_{\substack{j>k\geq 0\\ j+k=2m-1}}(-1)^{j+1}[T^j\cdot A, T^k\cdot A]$$

and

$$\hat{\boldsymbol{\epsilon}}_{2m}(\widehat{T}_0) = -\widehat{T}_0^{2m}\cdot\widehat{S}_0 \text{ and } \hat{\boldsymbol{\epsilon}}_{2m}(\widehat{S}_0) = \sum_{\substack{j>k\geq 0\\ j+k=2m-1}}(-1)^{j+1}[\widehat{T}_0^j\cdot\widehat{S}_0,\widehat{T}_0^k\cdot\widehat{S}_0].$$

One checks easily that, when $m \geq 1$, $\epsilon_{2m}(T) = -T^{2m} \cdot (A - 2G_2T)$ and

$$\epsilon_{2m}(A - 2G_2T) = \sum_{\substack{j>k\geq 0\\ j+k=2m-1}} (-1)^{j+1}[T^j \cdot (A - 2G_2T), T^k \cdot (A - 2G_2T)].$$

The result follows by rescaling as $\widehat{T}_0 = T/(2\pi i\xi)$ and $\widehat{S}_0 = 2\pi i\xi(A - 2G_2T)$. □

The connection ∇_0 on $\mathcal{H}$ defines, and will be viewed as, a $\mathbb{Q}$-rational connection on each graded quotient of $\boldsymbol{\mathcal{P}}$.

Theorem 20.2. *With respect to the framing of $\boldsymbol{\mathcal{P}}$ over $\mathcal{M}_{1,\vec{1}}$ described above, the universal elliptic KZB-connection ∇ is given by*

$$\nabla = \nabla_0 + \frac{1}{12}\frac{d\Delta}{\Delta} \otimes \hat{\boldsymbol{\epsilon}}_2 + \sum_{m\geq 2} \frac{3}{(2m-2)!}\frac{p_{2m}(u,v)(3vdu - 2udv)}{\Delta} \otimes \hat{\boldsymbol{\epsilon}}_{2m}$$

where $\Delta = u^3 - 27v^2$ is the discriminant and where $p_{2m}(u,v) \in \mathbb{Q}[u,v]$ is the polynomial characterized by $(2\pi i\xi)^{2m}G_{2m}(\tau) = p_{2m}(u,v)$. The Hodge bundles $F^p\boldsymbol{\mathcal{P}}$ are all defined over $\mathbb{Q}$.

The polynomial $p_{2m}(u,v)$ is weighted homogeneous of weight $2m$ in u and v, where u is given weight 4 and v is given weight 6. The polynomials of weight up to 24 are:

$$p_4(u,v) = \frac{1}{20}u$$
$$p_6(u,v) = \frac{3}{7}v$$
$$p_8(u,v) = \frac{3}{10}u^2$$
$$p_{10}(u,v) = \frac{108}{11}uv$$
$$p_{12}(u,v) = \frac{756}{65}u^3 + \frac{16200}{91}v^2$$
$$p_{14}(u,v) = 1296\,uv^2$$
$$p_{16}(u,v) = \frac{174636}{85}u^4 + \frac{1166400}{17}uv^2$$
$$p_{18}(u,v) = \frac{9471168}{19}u^3v + \frac{256608000}{133}v^3$$
$$p_{20}(u,v) = \frac{25147584}{25}u^5 + \frac{678844800}{11}u^2v^2$$

$$p_{22}(u,v) = \frac{10671720192}{23}u^4v + \frac{103296384000}{23}uv^3$$
$$p_{24}(u,v) = \frac{73581830784}{65}u^6 + \frac{1410877440000}{13}u^3v^2 + \frac{15547365504000}{91}v^4$$

Proof. With respect to the framing A, T of $\boldsymbol{\mathcal{P}}$, the connection is $\nabla = d + \omega'$ where ω' is the form (13.1). Since the change of frame is homogeneous, the transformed connection is of the form $\nabla = \nabla_0 + \omega'$. We just need to express ω' in the frame given by Lie words in $\widehat{S}_0$, $\widehat{T}_0$. Using the identities (19.1) and the preceding lemma we have

$$\begin{aligned}
\omega' &= -\Big(2G_2(\tau)\frac{dq}{q} - \frac{d\xi}{\xi}\Big)\otimes \epsilon_2 - \sum_{m=2}^{\infty}\frac{2}{(2m-2)!}G_{2m}(\tau)\frac{dq}{q}\otimes\epsilon_{2m}\\
&= \frac{1}{12}\frac{d\Delta}{\Delta}\otimes\hat{\boldsymbol{\epsilon}}_2 - \sum_{m\geq 2}\frac{2}{(2m-2)!}(2\pi i\xi)^{2m}G_{2m}(\tau)\frac{2\pi i}{(2\pi i\xi)^2}\,d\tau\otimes\hat{\boldsymbol{\epsilon}}_{2m}\\
&= \frac{1}{12}\frac{d\Delta}{\Delta}\otimes\hat{\boldsymbol{\epsilon}}_2 - \sum_{m\geq 2}\frac{3}{(2m-2)!}\frac{p_{2m}(u,v)\alpha}{\Delta}\otimes\hat{\boldsymbol{\epsilon}}_{2m}
\end{aligned}$$

The last assertion follows from the fact that $F^p\boldsymbol{\mathcal{P}}$ is trivial and consists of those Lie words whose degree in $\widehat{T}_0$ is $\geq -p$. $\square$

21. The $\mathbb{Q}$-de Rham structure on $F^{2n+1}H^1(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$

Here we compute the $\mathbb{Q}$-structure on $F^{2n+1}H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$. The computation of the $\mathbb{R}$-de Rham structure on all of $H^1(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ can be found in [17, §17.2].

The starting point is the isomorphisms

$$H^0(\overline{\mathcal{M}}_{1,1}, \mathcal{L}_{2n+2}) \to H^0(\Omega^1_{\overline{\mathcal{M}}_{1,1}}(P)\otimes F^{2n}S^{2n}\overline{\mathcal{H}}) \to F^{2n+1}H^1(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}),$$

where P denotes the cusp $q=0$. The second isomorphism takes a 1-form to its cohomology class. The first follows from the isomorphisms $\mathcal{L}\cong F^1\overline{\mathcal{H}}$ and $\Omega^1_{\overline{\mathcal{M}}_{1,1}}(P)\cong\mathcal{L}_2$, which together induce an isomorphism

$$\mathcal{L}_{2n+2} \cong \Omega^1_{\overline{\mathcal{M}}_{1,1}}(P)\otimes F^{2n}S^{2n}\overline{\mathcal{H}}.$$

We now explain the $\mathbb{Q}$-structure. Recall that if f is a modular form of weight $2n+2$ of $\mathrm{SL}_2(\mathbb{Z})$, then

$$\omega_f := f(\tau)\mathbf{w}^{2n}d\tau \in E^1(\mathfrak{h})\otimes S^{2n}H$$

is an $\mathrm{SL}_2(\mathbb{Z})$-invariant 1-form on $\mathfrak{h}$, where $\mathbf{w} := 2\pi i T$ is the section of $H \times \mathfrak{h} \to \mathfrak{h}$ that takes the value $2\pi i \omega_\tau$ at $\tau \in \mathfrak{h}$ and $\mathrm{SL}_2(\mathbb{Z})$ acts on $H = \mathbb{C}A \oplus \mathbb{C}T$ via the factor of automorphy (9.2).[14] It gives a framing of $F^1\mathcal{H}$ over $\mathfrak{h}$. The section $\mathbf{w}$ extends to a framing of $F^1\overline{\mathcal{H}}$ over the q-disk.

Denote the space of modular forms of $\mathrm{SL}_2(\mathbb{Z})$ of weight m whose Fourier coefficients lie in the subfield F of $\mathbb{C}$ by $M_{m,F}$. These form a graded ring $M_{*,F}$ isomorphic to $F[G_4, G_6]$.

Proposition 21.1. *The $\mathbb{Q}$-structure on $F^{2n+1}H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ is*

$$\{f(q)\mathbf{w}^{2n}\frac{dq}{q} : f(q) \in M_{2n+2,\mathbb{Q}}\}.$$

Proof. Embed E_τ into $\mathbb{P}^2$ via the mapping

$$z + \Lambda_\tau \mapsto [\wp_\tau(z)/(2\pi i)^2, \wp'_\tau(z)/(2\pi i)^3, 1].$$

The image is the plane cubic $y^2 = 4x^3 - ux - v$ where

$$u = g_2(\tau)/(2\pi i)^4 = 20G_4(\tau) \text{ and } v = g_3(\tau)/(2\pi i)^6 = \frac{7}{3}G_6(\tau).$$

This curve has discriminant $\Delta_0(\tau)$, where Δ_0 denotes the normalized cusp form of weight 12.

With this normalization $dx/y = 2\pi i\omega_\tau = \mathbf{w}(\tau)$.[15] We choose it because the value of the section $\mathbf{w}$ at $q = 0$ is dw/w, where w is the parameter on the nodal cubic that maps 0 and ∞ to the node and 1 to the identity. (Cf. Exercise 47 in [15].)

We regard $\mathbf{w}$ as the section dx/y of $F^1\mathcal{H}$ over $\mathcal{M}_{1,\vec{1}} = \mathbb{A}^2_{\mathbb{Q}} - \{u^3 - 27v^2 = 0\}$. It is defined over $\mathbb{Q}$. Since x has weight 2 and y weight 3, it has weight -1 under the $\mathbb{G}_m$ action. If $h(u,v) \in \mathbb{Q}[u,v]$ is a polynomial of weight $2n+2$ (where u has weight 4 and v weight 6), then

$$\frac{h(u,v)}{u^3 - 27v^2}(2u\,dv - 3v\,du)\mathbf{w}^{2n}$$

is a $\mathbb{Q}$-rational, $\mathbb{G}_m$-invariant section of $\Omega^1_{\mathcal{M}_{1,\vec{1}/\mathbb{Q}}}(\log D) \otimes F^{2n}S^{2n}\mathcal{H}$ over $\mathcal{M}_{1,\vec{1}/\mathbb{Q}}$, where D denotes the discriminant locus $u^3 - 27v^2 = 0$. Since $\mathcal{M}_{1,1/\mathbb{Q}} = \mathbb{G}_m \backslash\backslash \mathcal{M}_{1,\vec{1}/\mathbb{Q}}$ it descends to a section of $\Omega^1_{\mathcal{M}_{1,1/\mathbb{Q}}}(P) \otimes F^{2n}S^{2n}\mathcal{H}$.

[14] Recall the definitions (11.1).

[15] This would have been a better normalization to use in Part 4.

The identity (19.1) implies that the pullback of this form along the map $\mathfrak{h} \to \mathcal{M}_{1,\vec{1}}$ defined by $\tau \mapsto (20G_4(\tau), 7G_6(\tau)/3)$ is

$$\frac{2}{3}h(20G_4(\tau), 7G_6(\tau)/3)\mathbf{w}^{2n}\frac{dq}{q}.$$

The result follows as $M_{2n+2,\mathbb{Q}}$ is isomorphic to the polynomials in G_4 and G_6 with rational coefficients. □

Appendix A. Vanishing of $N(R_0)$

Here we give an elementary proof of the vanishing of $N_q(R_0)$ established in Corollary 15.2. It does not use limit mixed Hodge structures. Instead we deduce the vanishing from an identity involving Bernoulli numbers.

For non-negative integers, define polynomials

$$\begin{aligned} h_{a,b}(x,y) &= x^{2a-1}y^{2b} - x^{2b}y^{2a-1} + xy(x+y)^{2b-1}(y^{2a-2} - x^{2a-2}) \\ &= xy(y-x)\Big(x^{2a-2}y^{2b-2} + (x+y)^{2b-1} \sum_{\substack{i+j=2a-3 \\ i,j\geq 0}} x^i y^j\Big) \end{aligned}$$

in commuting indeterminants x and y. Note that

$$h_{0,n}(x,y) = -\sum_{\substack{i+j=2n-1 \\ i,j\geq 0}} \Big[\binom{2n}{i+1} - \binom{2n}{j+1}\Big] x^i y^j$$

for all $n \geq 1$.

Theorem A.1. *For all $n \geq 1$,*

$$\begin{aligned} &\sum_{\substack{a+b=n \\ a>0}} (2a-1)\binom{2n}{2a}\frac{B_{2a}B_{2b}}{B_{2n}}\, h_{a,b}(x,y) \\ &= \sum_{\substack{i+j=2n-1 \\ i,j\geq 0}} \Big[\binom{2n}{i+1} - \binom{2n}{j+1}\Big] x^i y^j \in \mathbb{Z}[x,y]. \end{aligned}$$

Equivalently,

$$\sum_{\substack{a+b=n \\ a,b\geq 0}} (2a-1)\frac{B_{2a}}{(2a)!}\frac{B_{2b}}{(2b)!}\, h_{a,b}(x,y) = 0.$$

Proof. It suffices to show that

$$\sum_{n\geq 0}\sum_{\substack{a+b=n\\a,b\geq 0}}(2a-1)\frac{B_{2a}}{(2a)!}\frac{B_{2b}}{(2b)!}\,h_{a,b}(x,y)=0. \tag{A.1}$$

Observe that

$$\begin{aligned}&\sum_{n\geq 0}\sum_{\substack{a+b=n\\a,b\geq 0}}(2a-1)\frac{B_{2a}}{(2a)!}\frac{B_{2b}}{(2b)!}\,u^{2a-1}v^{2b}\\&=\Big(\sum_{a\geq 0}(2a-1)\frac{B_{2a}}{(2a)!}u^{2a-1}\Big)\Big(\sum_{b\geq 0}\frac{B_{2b}}{(2b)!}v^{2b}\Big)\\&=\frac{4u}{\sinh^2(u/2)}\Big(\frac{v}{e^v-1}+\frac{v}{2}\Big)\\&=\frac{ue^u}{(e^u-1)^2}\Big(\frac{v}{e^v-1}+\frac{v}{2}\Big).\end{aligned}$$

Denote this function of (u,v) by $F(u,v)$. The series (A.1) is then

$$F(x,y)-F(y,x)+\frac{x}{x+y}F(y,x+y)-\frac{y}{x+y}F(x,x+y)$$

which is easily to vanish by elementary algebraic manipulations. $\square$

We finish by showing that this identity is equivalent to the vanishing of $N_q(R_0)$. For this we need to relate polynomials to the free Lie algebra $\mathbb{L}(T,A)$. For this we use the Levin-Racinet calculus [24, §3.1]. Recall from Section 9.4.1 that for $U,V\in\mathbb{L}(A,T)$,

$$x^r y^s\circ(U,V)=[T^r\cdot U,T^s\cdot V].$$

This extends linearly to an action $f(x,y)\circ(U,V)$ of polynomials $f(x,y)$ in commuting indeterminants on ordered pairs of elements of $\mathbb{L}(T,A)$. When U and V are equal, one has the identity $f(x,y)\circ(U,U)=-f(y,x)\circ(U,U)$, so that

$$2f(x,y)\circ(U,U)=(f(x,y)-f(y,x))\circ(U,U).$$

In this case we need only consider polynomials $f(x,y)$ satisfying $f(x,y)+f(y,x)=0$.

The significance of the polynomials $h_{a,b}(x,y)$ is given by:

Lemma A.2. *For all $a \geq 0$ and $b \geq 0$ with $a+b>0$,*

$$2\,\epsilon_{2a}(T^{2b}\cdot A) = h_{a,b}(x,y)\circ(A,A).$$

Proof. Observe that $h_{a,b}(x,y) = f_{a,b}(x,y) - f_{a,b}(y,x)$ where

$$f_{a,b}(x,y) = x^a(x+y)^{2b-1}\big(x^a-(-y)^a\big) - x^{2a-1}\big((x+y)^{2b}-y^{2b}\big).$$

The result now follows from the easily verified identity

$$\epsilon_{2a}(T^{2b}\cdot A) = f_{a,b}(x,y)\circ(A,A).$$

□

Theorem A.1 implies the vanishing of $N_q(R_0)$:

$$\begin{aligned} N_q(R_0) &= \sum_{a\geq 0}\sum_{b\geq 0}(2a-1)\frac{B_{2a}}{(2a)!}\frac{B_{2b}}{(2b)!}\,\epsilon_{2a}(T^{2b}\cdot A) \\ &= \frac{1}{2}\Big(\sum_{a\geq 0}\sum_{b\geq 0}(2a-1)\frac{B_{2a}}{(2a)!}\frac{B_{2b}}{(2b)!}\,h_{a,b}(x,y)\Big)\circ(A,A) \\ &= 0. \end{aligned}$$

Appendix B. The universal elliptic curve over $\mathrm{Bl}_{\mathbf{0}}^{+}\mathbb{D}$

Here we justify the claim, made in Section 16, that the fiber of the universal elliptic curve over $e^{i\theta}\partial/\partial q$ is obtained from the real oriented blowup of $\mathbb{P}^1$ at $\{0,\infty\}$ by identifying its two boundary components with a suitable twist.

The map $[0,1)\times S^1 \to \mathbb{D}$ that takes (r,θ) to $e^{i\theta}$ is the real oriented blowup $\mathrm{Bl}_0^o\,\mathbb{D}\to\mathbb{D}$ of the disk. More generally, the map

$$S^1\times[0,1]\to\mathbb{P}^1 \text{ defined by } (\phi,t)\mapsto[te^{i\phi},1-t]$$

is $\mathrm{Bl}_{0,\infty}^o\,\mathbb{P}^1\to\mathbb{P}^1$. With this identification, the inclusion $\mathbb{C}^*\to\mathrm{Bl}_{0,\infty}^o\,\mathbb{P}^1$ takes $se^{i\phi}$ to $\big(\phi, s/(1+s)\big)$.

The fiber of the universal elliptic curve over $q=re^{i\theta}$ is the quotient of

$$A := \{(w,q)\in\mathbb{C}^*\times\mathbb{D}^* : \sqrt{|q|}\leq|w|\leq 1/\sqrt{|q|}\}$$

obtained by glueing w to qw when $|w| = 1/\sqrt{|q|}$. Write $w=se^{i\phi}$ so that we can identify A with

$$\{(s,\phi,re^{i\theta})\in\mathbb{R}\times S^1\times\mathbb{D}^* : \sqrt{r}\leq s\leq 1/\sqrt{r}\}.$$

With this identification, $(1/\sqrt{r}, \phi, re^{i\theta})$ is glued to $(\sqrt{r}, \phi + \theta, re^{i\theta})$.

The function

$$h(r,s) = \Big(\frac{s}{1+s} - \frac{\sqrt{r}}{1+\sqrt{r}}\Big)\Big(\frac{1}{1+\sqrt{r}} - \frac{\sqrt{r}}{1+\sqrt{r}}\Big)^{-1}$$

induces homeomorphisms $h(r,\ \ ) : [\sqrt{r}, 1/\sqrt{r}] \to [0,1]$ for all $r \geq 0$. It has inverse $k(r,\ \ )$, where

$$k(r,t) = \frac{\sqrt{r} - (\sqrt{r}-1)t}{1 + (\sqrt{r}-1)t}$$

Define an equivalence relation on $B := S^1 \times S^1 \times [0,1) \times [0,1)$ by

$$(\phi, \theta, 1, r) \sim (\phi + \theta, \theta, 0, r).$$

Set $\mathcal{B} = B/{\sim}$. The map $(\phi, \theta, t, r) \to (r, \theta)$ defines a projection $\pi : \mathcal{B} \to \mathrm{Bl}_0^o\,\mathbb{D}$. This is a torus bundle over $\mathrm{Bl}_0^o\,\mathbb{D}$. Its fiber B_θ over $(0,\theta) \in \mathrm{Bl}_0^o\,\mathbb{D}$ is the quotient of $\mathrm{Bl}_{0,\infty}^o\,\mathbb{P}^1$ obtained by identifying the two boundary components by a twist by θ. The inclusion $A \hookrightarrow B$ defined by $(s, \phi, re^{i\theta}) \mapsto (\phi, \theta, h(r,s), r)$ induces a map $\mathcal{E}_{\mathbb{D}^*} \to \mathcal{B}$ that commutes with the projections to $\mathbb{D}$ and is a homeomorphism into its image.

The map $B \to \mathbb{C}^* \times \mathbb{D}$ that takes (ϕ, θ, t, r) to $(k(r,t)e^{i\phi}, re^{i\theta})$ induces a map $\mathcal{B} \to \mathcal{E}_{\mathbb{D}}$ such that the diagram

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{E}_{\mathbb{D}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Bl}_0^o\,\mathbb{D} & \longrightarrow & \mathbb{D} \end{array}$$

commutes. For each $\theta \in S^1$, the composite $\mathbb{C}^* \hookrightarrow \mathrm{Bl}_{0,\infty}^o\,\mathbb{P}^1 \to B_\theta \to E_0$ is the natural inclusion of the smooth locus of the nodal cubic E_0 given by the parameter $w = se^{i\phi}$. The map $\mathrm{Bl}_{0,\infty}^o\,\mathbb{P}^1 \to B_\theta \to E_0$ collapses the boundary of $\mathrm{Bl}_{0,\infty}^o\,\mathbb{P}^1$ to the double point of E_0.

Acknowledgments

I am indebted to all participants in the seminar, particularly Aaron Pollack, whose help in checking the factors of automorphy was invaluable. I am grateful to Makoto Matsumoto, Francis Brown and Benjamin Enriquez for their constructive comments, and to Ma Luo and Jin Cao for their numerous corrections. Finally, I would like to thank the referee for his/her thorough reading of the manuscript and for their constructive comments on the exposition.

References

[1] D. Bernard: *On the Wess-Zumino-Witten models on the torus*, Nuclear Phys. B **303** (1988), 77–93. MR0952765

[2] F. Brown: *Mixed Tate motives over* $\mathbb{Z}$, Ann. of Math. **175** (2012), 949–976. MR2993755

[3] D. Calaque, B. Enriquez, P. Etingof: *Universal KZB equations: the elliptic case*, in Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 165–266, Progr. Math., Vol. 269, Birkhäuser, Boston, 2009 [arXiv:math/0702670]. MR2641173

[4] J. Carlson: *Extensions of mixed Hodge structures*, Journées de Géometrie Algébrique d'Angers, Juillet 1979, Sijthoff & Noordhoff, 1980, 107–127. MR0605338

[5] K.-T. Chen: *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), 831–879. MR0454968

[6] P. Deligne: *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, 1970. MR0417174

[7] P. Deligne: *Théorie de Hodge, II*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57. MR0498551

[8] P. Deligne: *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., Vol. 16, Springer, 1989. MR1012168

[9] B. Enriquez: *Elliptic associators*, Selecta Math. (N.S.) **20** (2014), 491–584. MR3177926

[10] H. Furusho: *The multiple zeta value algebra and the stable derivation algebra*, Publ. Res. Inst. Math. Sci. **39** (2003), 695–720 [arXiv:math/0011261]. MR2025460

[11] R. Hain: *The geometry of the mixed Hodge structure on the fundamental group*, Algebraic Geometry, 1985, Proc. Symp. Pure Math., Vol. 46 (1987), 247–282. MR0927984

[12] R. Hain: *The de Rham homotopy theory of complex algebraic varieties, I.* K-Theory **1** (1987), 271–324. MR0908993

[13] R. Hain: *The de Rham homotopy theory of complex algebraic varieties, II.* K-Theory **1** (1987), 481–497. MR0934453

[14] R. Hain: *Relative weight filtrations on completions of mapping class groups*, in Groups of Diffeomorphisms, Advanced Studies in Pure Mathematics, Vol. 52 (May, 2008), 309–368, Mathematical Society of Japan [arXiv:0802.0814]. MR2509715

[15] R. Hain: *Lectures on Moduli Spaces of Elliptic Curves*, in Transformation Groups and Moduli Spaces of Curves, Advanced Lectures in Mathematics, Vol. 16, edited by Lizhen Ji, S.-T. Yau (2010), 95–166, Higher Education Press, Beijing [arXiv:0812.1803]. MR2883686

[16] R. Hain: *Unipotent Path Torsors of Ihara Curves*, in preparation, 2018.

[17] R. Hain: *The Hodge-de Rham Theory of Modular Groups*, in Recent Advances in Hodge Theory Period Domains, Algebraic Cycles, and Arithmetic, Matt Kerr and Greogory Pearlstein, editors, 422–514. Cambridge University Press, 2016 [arXiv:1403.6443]. MR3409885

[18] R. Hain, M. Matsumoto: *Universal mixed elliptic motives*, J. Inst. Math. Jussieu, https://doi.org/10.1017/S1474748018000130 [arXiv:1512.03975].

[19] R. Hain, S. Zucker: *Unipotent variations of mixed Hodge structure*, Invent. Math. **88** (1987), 83–124. MR0877008

[20] M. Kashiwara: *A study of variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22** (1986), no. 5, 991–1024. MR0866665

[21] L. Kronecker: *Zur theorie der elliptischen Funktionen (1881)*. Reproduced in: Leopold Kronecker's Werke Volume IV, pp. 313–318. Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften von K. Hensel, Chelsea, New York, 1968. MR0237286

[22] V. Knizhnik, A. Zamolodchikov: *Current algebra and Wess–Zumino model in two dimensions*, Nuclear Phys. B **247** (1984), 83–103. MR0853258

[23] T. Le, J. Murakami: *Kontsevich's integral for the Kauffman polynomial*, Nagoya Math. J. **142** (1996), 39–65. MR1399467

[24] A. Levin, G. Racinet: *Towards multiple elliptic polylogarithms*, unpublished preprint [arXiv:math/0703237].

[25] A. Pollack: *Relations between derivations arising from modular forms*, undergraduate thesis, Duke University, 2009. Available at: http://dukespace.lib.duke.edu/dspace/handle/10161/1281.

[26] W. Schmid: *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. MR0382272

[27] J.-P. Serre: *A course in arithmetic*, Translated from the French. Graduate Texts in Mathematics, Vol. 7. Springer-Verlag, New York-Heidelberg, 1973. MR0344216

[28] J.-P. Serre: *Lie algebras and Lie groups, Lectures given at Harvard University, 1964*, Benjamin, 1965. (Second edition, Springer Verlag, 1992.) MR0218496

[29] J. Silverman: *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, Vol. 151. Springer-Verlag, New York, 1994. MR1312368

[30] C. Peters, J. Steenbrink: *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 52. Springer-Verlag, 2008. MR2393625

[31] J. Steenbrink, S. Zucker: *Variation of mixed Hodge structure, I*, Invent. Math. **80** (1985), 489–542. MR0791673

[32] H. Tsunogai: *On some derivations of Lie algebras related to Galois representations*, Publ. Res. Inst. Math. Sci. **31** (1995), 113–134. MR1317526

[33] D. Zagier: *Periods of modular forms and Jacobi theta functions*, Invent. Math. **104** (1991), 449–465. MR1106744

Richard Hain
Department of Mathematics
Duke University
Box 90320
Durham, NC 27708
USA
E-mail: hain@math.duke.edu