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Quantizing Deformation Theory II

Dedicated to my teacher Yuri Ivanovich Manin on the occasion of his eightieth birthday

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Abstract: A quantization of classical deformation theory, based on the Maurer-Cartan Equation $dS + \frac{1}{2}[S, S] = 0$ in dg-Lie algebras, a theory based on the Quantum Master Equation $dS + \hbar\Delta S + \frac{1}{2}\{S, S\} = 0$ in dg-BV-algebras, is proposed. Representability theorems for solutions of the Quantum Master Equation are proven. Examples of "quantum" deformations are presented.

Keywords: Deformation theory, Maurer-Cartan equation, Quantum Master Equation, Differential graded manifold, BV-algebra.

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Introduction

Yuri I. Manin has always been fascinated with the concept of quantization. Observing the chromatic spectrum of his work over the years, I have become more and more convinced that you may quantize more than you expect.

In this paper, I suggest an approach to quantizing deformation theory. Neither the idea, nor the terminology is new: I am referring to John Terilla's paper [40] on *Quantizing Deformation Theory*, preceded by his work [34] with Jae-Suk Park and Thomas Tradler. This partially explains the title of the current paper, which I view as a complement to Terilla's work. I hint on a relation in the last section of this paper. I believe strongly that these works are two tips of one and the same iceberg.

I have been running around disseminating vague ideas of quantum deformation theory for a few years since John hooked me on quantizing deformation theory during the historic Northeast blackout of 2003. Alas, it is a pity it took me so long to get these ideas crystallized, but now I am content with their shining, albeit somewhat superficial.

Conventions on graded algebra and geometry

We will work over a ground field k of characteristic zero. In particular, the symbol $\otimes$ will mean $\otimes_k$. The translation $V[n]$ of a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V^n$ is the same vector space with a redefined degree: $V[n]^p := V^{p+n}$. The dual V^* of a graded vector space V is understood as the direct sum of the duals of its graded components, graded in such a way that the natural pairing $V^* \otimes V \to k$ is grading-preserving. In particular, $(V[n])^* = V^*[-n]$. One can also write $V^* = \hom_k(V, k)$ with $\hom_k(V, V')$ being the internal Hom in the category of graded vector spaces, as opposed to $\mathrm{Hom}_k(V, V') = \hom_k(V, V')^0$, the vector space of degree-preserving linear maps. By default, the degree of a homogeneous tensor is the sum of the degrees of its factors. Differentials d

are assumed to have degree 1: $|d| = 1$. However, the BV differential Δ will have degree -1. All associative algebras are assumed to be unital and all coassociative coalgebras to be counital.

For the purpose of this paper, we will mostly work with *pointed formal graded manifolds* that are actually pointed formal *graded affine spaces*. These are determined by graded symmetric coalgebras of the type $S(V)$, where V is a graded vector space. The (graded) cocommutative, coassociative comultiplication on $S(V)$ is taken to be the standard shuffle comultiplication. We think of the linear dual algebra $S(V)^*$ as the algebra of functions in a formal neighborhood of 0 in V. The *basepoint*, which corresponds to the origin 0 in the vector space V, is given by the coaugmentation $k = S^0(V) \to S(V)$.

A *morphism* $V \to W$ *of pointed formal graded manifolds* in our restricted, linear category is just a morphism $S(V) \to S(W)$ of coalgebras respecting the coaugmentations. Since the coalgebra $S(W)$ is cofree (in the category of conilpotent cocommutative coalgebras), such a morphism is determined by a degree-zero linear map $S(V) \to W$. Compatibility with coaugmentations forces this linear map to vanish on $k = S^0(V)$.

When we talk about a linear *pointed formal differential graded* (*dg-*) *manifold* V, we assume that it is a linear pointed formal graded manifold V endowed with a *differential*, *i.e.*, the structure coalgebra $S(V)$ is endowed with a *codifferential*, a degree-one, k-linear coderivation D satisfying $D^2 = 0$ and vanishing on $S^0(V)$. A codifferential, like any coderivation on a cofree cocommutative coalgebra, is determined by a degree-one linear map $S^{>0}(V) \to V$, the projection of the coderivation D to the space V of cogenerators of the coalgebra $S(V)$.

Morphisms of pointed formal dg-manifolds have to respect differentials, *i.e.*, the corresponding coalgebra morphisms have to respect the structure codifferentials.

Typically, a deformation functor is defined on the category of local Artin rings. We find it more convenient to work with slightly more general complete local rings (or algebras). Let $(R, \mathfrak{m})$ be a *complete local k-algebra*, that is to say, a local k-algebra R with a maximal ideal $\mathfrak{m}$ such that the canonical ring homomorphisms $k \to R/\mathfrak{m}$ to the residue field and $R \to \varprojlim_n R/\mathfrak{m}^n$ to the completion of R in the $\mathfrak{m}$-adic topology are isomorphisms. Given a dg-vector space V and a complete local algebra $(R, \mathfrak{m})$, we will be considering *completed tensor products*, such as

$$V \mathbin{\widehat{\otimes}} \mathfrak{m} := \varprojlim_n V \otimes \mathfrak{m}/\mathfrak{m}^n.$$

We say that a complete local k-algebra $(R, \mathfrak{m})$ is *of finite type* if all the quotients $R/\mathfrak{m}^n$, $n \geq 1$, are finite-dimensional over k. We will also associate a pointed formal dg-manifold $\operatorname{Spec} R$ to a complete local k-algebra $(R, \mathfrak{m})$ of finite type. This formal pointed manifold is determined by the natural conilpotent coalgebra structure on the continuous linear dual R^* of R, along with the coaugmentation $k \to R^*$, dual to the augmentation $R \to R/\mathfrak{m} = k$. We will set the dg structure on $\operatorname{Spec} R$ to be given by the zero codifferential on R^*. Note that this *pointed formal dg-manifold* is not linear, but rather *affine* in the scheme-theoretic sense.

More general pointed formal dg-manifolds are treated in [5].

Disclaimer

Sometimes people refer to quantum deformations in the context of deformations associated historically with quantum field theory, especially when the deformation parameter is hidden within the variable q. From the point of view of this paper, many such deformations would still be classical. But you never know. For example, a theorem of C. Teleman [39] states that higher-genus Gromov-Witten invariants can be reconstructed from the quantum cup product in the semisimple case. The quantum cup product is a classical deformation of the usual cup product. We argue that deformation theory of algebraic structures associated to higher genera is intrinsically quantum, cf. Section 2.1. Likewise, quantum groups would be classical deformations from our point of view. However, quantum groups are closely related to Lie bialgebras, whose deformation theory should be quantum, cf. Section 3. Deformation quantization [25], given its relation to moduli spaces of algebraic curves, could have an incarnation within Quantum Deformation Theory, but it is still a classical deformation and one does not need to evoke moduli spaces of higher genera to do it.

1. Classical theory: MCE, CME & deformation theory

1.1. The main player of deformation theory

Let me start with the following, hopefully contentious, statement.

Metatheorem 1.1. *Every reasonable deformation problem in mathematics comes from a dg-Lie algebra.*

The proof of this statement can easily be demonstrated by contradiction: If there is a deformation problem that does not come from a dg Lie algebra, the problem is obviously unreasonable. On a more serious note, the metatheorem presents a long and important development stemming from the work of

Gerstenhaber [18], Schlessinger-Stasheff [35], Goldman-Millson [22], Deligne [15], Kontsevich [25] and a few others, who showed that, on the one hand, many major deformation theories, such as those of complex manifolds or associative algebras, come from a corresponding dg Lie algebra and, on the other hand, can be completely described in terms of this dg Lie algebra. For example, the dg Lie algebra governing deformations of a complex manifold M is the Dolbeault complex $(\Omega^{-1,\bullet}(M), \bar{\partial})$ of the holomorphic tangent bundle of M. The bracket, known as the *Nijenhuis bracket*, is given by the commutator of vector fields combined with the wedge product of $(0,q)$-forms. The dg Lie algebra describing deformations of an associative algebra A is its Hochschild complex $C^\bullet(A,A)$ along with the Gerstenhaber bracket. The metatheorem is probably not that much a law, but rather a manifesto: If your favorite deformation theory does not come from a dg Lie algebra, you should make an effort to find one. This might bear fruit for your theory.

I may give "plausible reasoning"to convince the reader that the metatheorem should hold for philosophical reasons. I am sure this argument is somewhat of folklore, but I had never thought about it until Jim Stasheff forwarded to me a question of Samir Shah as to *really why* the gods of mathematics had designed the metatheorem to be true. Here is what I think. Deformation theory describes the tangent cone at a point x of the moduli space M of the problem. The tangent cone is $\operatorname{Spec} \operatorname{Gr} \mathcal{O}_{M,x}$, where the associated graded Gr is taken with respect to powers of the maximal ideal of the local ring $\mathcal{O}_{M,x}$, the stalk of the structure sheaf $\mathcal{O}_M$ at x. Usually, this cone is singular. You may resolve this singularity within derived algebraic geometry, for instance, find a free dg-commutative algebra whose cohomology is $\operatorname{Gr} \mathcal{O}_{M,x}$. A free dg-commutative algebra is equivalent to an L_∞-algebra, *cf.* a remark before Theorem 1.3 below. Then you take a quasi-isomorphic dg-Lie algebra, and you are done.

1.2. The Maurer-Cartan Equation & deformation functor

Given a dg Lie algebra $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^n$ with a differential d of degree $|d| = 1$, the *Maurer-Cartan set*

$$\mathrm{MC}_{\mathfrak{g}} := \{S \in \mathfrak{g}^1 \mid dS + \frac{1}{2}[S,S] = 0\}$$

is the set of degree-one solutions S, called *Maurer-Cartan elements*, of the *Maurer-Cartan Equation* (*MCE*)

$$dS + \frac{1}{2}[S,S] = 0, \tag{1}$$

also known as the *CME*, the *Classical Master Equation*.

A dg Lie algebra $\mathfrak{g}$ defines a much richer object, called a *deformation functor*:

$$\begin{aligned} \mathcal{CLAlg} &\to \mathcal{S}et, \\ (R, \mathfrak{m}) &\mapsto \mathrm{MC}_{\mathfrak{g}}(R), \end{aligned}$$

where $\mathcal{CLAlg}$ is the *category of complete local k-algebras of finite type* and $(R, \mathfrak{m})$ is an object of it, $\mathcal{S}et$ is the category of sets, and

$$\mathrm{MC}_{\mathfrak{g}}(R) := \{S \in (\mathfrak{g} \mathbin{\widehat{\otimes}} \mathfrak{m})^1 \mid dS + \frac{1}{2}[S, S] = 0\}. \tag{2}$$

This set is interpreted as the set of deformations over $\operatorname{Spec} R$ of the mathematical object whose deformation theory is governed by $\mathfrak{g}$. For example, when $\mathfrak{g}$ is the Hochschild complex of an associative algebra A, the set $\mathrm{MC}_g(R)$ is the set of associative R-linear multiplications on $A \mathbin{\widehat{\otimes}} R$ extending the original multiplication on A.

Example 1.2. Let $\mathfrak{h}$ be a Lie algebra. Then the dg Lie algebra of based, graded coderivations

$$\mathfrak{g} := \mathrm{Coder}_*(S(\mathfrak{h}[1])) = \mathrm{hom}_k(S^{>0}(\mathfrak{h}[1]), \mathfrak{h}[1])$$

of the cofree conilpotent cocommutative coalgebra $S(\mathfrak{h}[1])$ describes the deformation theory of the Lie algebra $\mathfrak{h}$. The differential on $\mathfrak{g}$ is defined as follows:

$$d := \begin{cases} -[-,-] : S^2(\mathfrak{h}[1]) \to \mathfrak{h}[1], \text{ the Lie bracket on } \mathfrak{h}, & \text{for } n = 2, \\ 0 & \text{for all other } n. \end{cases}$$

The funny sign is a matter of convention, which becomes useful when one generalizes this theory to the case when $\mathfrak{h}$ is an L_∞-algebra. It is a straightforward checkup that the condition that d is a codifferential, $d^2 = 0$, is equivalent to the Jacobi identity for the Lie bracket $[-,-]$. A *deformation of* $\mathfrak{h}$ *over a complete local algebra* $(R, \mathfrak{m})$ is, by definition, a new bracket $[-,-]'$ on $\mathfrak{h} \mathbin{\widehat{\otimes}} R$ which reduces to the original bracket $[-,-]$ on $\mathfrak{h}$ modulo $\mathfrak{m}$. Thus, we have

$$[x, y]' = [x, y] + S(x, y) \qquad \text{for each } x, y \in \mathfrak{h}$$

and some $S(x, y) \in \mathrm{Hom}_k(\Lambda^2(\mathfrak{h}), \mathfrak{h}) \mathbin{\widehat{\otimes}} \mathfrak{m} = \mathrm{hom}_k(S^{>0}(\mathfrak{h}[1]), \mathfrak{h}[1])^1 \mathbin{\widehat{\otimes}} \mathfrak{m}$ such that $[-,-]'$ satisfies the Jacobi identity. As above, this new bracket produces a new, deformed codifferential on $\mathfrak{g} \mathbin{\widehat{\otimes}} R$:

$$d' = d + [S, -],$$

for which the equation $(d')^2 = 0$ is equivalent to the Jacobi identity for $[-,-]'$. Now observe that $(d')^2 = dS + \frac{1}{2}[S,S]$. This implies that a deformation of $\mathfrak{h}$ over R is equivalent to the choice of a Maurer-Cartan element $S \in \mathrm{MC}_{\mathfrak{g}}(R)$.

1.3. Representability theorems

Considering the opposite *category* $\mathcal{FLA}\mathit{ff} := \mathcal{CLA}\mathit{lg}^{\mathrm{op}}$ *of formal local affine k-schemes of finite type* whose object corresponding to an algebra R is denoted by $\operatorname{Spec} R$, we may turn the deformation functor into a contravariant one:

$$\begin{aligned} \mathcal{FLA}\mathit{ff}^{\,\mathrm{op}} &\to \mathcal{S}\mathit{et}, \\ \operatorname{Spec} R &\mapsto \mathrm{MC}_{\mathfrak{g}}(R), \end{aligned}$$

and speak of its representability, possibly in a larger category.

Note that a dg Lie algebra $\mathfrak{g}$ defines a pointed formal dg-manifold $\mathfrak{g}[1]$ determined by the symmetric coalgebra $S(\mathfrak{g}[1])$ with the codifferential induced by the linear map $l : S(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ whose restriction l_n to $S^n(\mathfrak{g}[1])$ is defined by the following formula:

$$l_n := \begin{cases} d : \mathfrak{g}[1] \to \mathfrak{g}[1], \text{ the differential on } \mathfrak{g}, & \text{for } n = 1, \\ \pm[-,-] : S^2(\mathfrak{g}[1]) \to \mathfrak{g}[1], \text{ the Lie bracket on } \mathfrak{g}, & \text{for } n = 2, \\ 0 & \text{for all other } n. \end{cases}$$

The sign for $n = 2$ is given by $l_2(x,y) = (-1)^{|x|}[x,y]$ for $x, y \in \mathfrak{g}[1]$.

Remark. It is also useful to recall at this point that the structure of a pointed formal dg-manifold on the pointed formal graded manifold $\mathfrak{g}[1]$ associated to a graded vector space $\mathfrak{g}$ is equivalent to the structure of an L_∞*-algebra* on $\mathfrak{g}$. Moreover, an L_∞*-morphism* $\mathfrak{g}' \to \mathfrak{g}$ between two L_∞-algebras is by definition a morphism of pointed formal manifolds $\mathfrak{g}'[1] \to \mathfrak{g}[1]$, which is, by definition, nothing but a morphism of coaugmented dg-coalgebras $S(\mathfrak{g}'[1]) \to S(\mathfrak{g}[1])$. Since $S(\mathfrak{g}[1])$ is cofree, every such morphism is determined by a linear map $S^{>0}(\mathfrak{g}'[1]) \to \mathfrak{g}[1]$, projection of the morphism to the cogenerating space $\mathfrak{g}[1]$. Thus, the *category* L_∞-$\mathcal{A}\mathit{lg}$ *of* L_∞*-algebras with* L_∞*-morphisms* becomes a full subcategory of the *category* $\mathcal{PFDGM}\mathit{an}$ *of pointed formal dg-manifolds*, namely the full subcategory of *linear* pointed formal dg-manifolds.

Theorem 1.3 (Quillen, as per [27])**.** *The deformation functor* $\mathrm{MC}_{\mathfrak{g}}$ *is represented by the pointed formal dg-manifold* $\mathfrak{g}[1]$, i.e.*, there is a natural isomorphism*

$$\begin{aligned}\mathrm{MC}_{\mathfrak{g}}(R) &\xrightarrow{\sim} \mathrm{Mor}_{\mathcal{PFDGM}an}(\operatorname{Spec} R, \mathfrak{g}[1]) \\ &:= \mathrm{Hom}_{\mathcal{CDGC}oalg}((R^*, 0), (S(\mathfrak{g}[1]), D)).\end{aligned}$$

Here $\operatorname{Spec} R$ is regarded as a pointed formal dg-manifold with a zero differential, as in the Conventions section of the introduction, $\mathrm{Mor}_{\mathcal{PFDGM}an}$ stands for the set of morphisms of pointed formal dg-manifolds, and $\mathrm{Hom}_{\mathcal{CDGC}oalg}$ for the set of homomorphisms of *coaugmented differential graded coalgebras.*

Remark. We do not consider solutions of the Maurer-Cartan equation *up to homotopy*, or *gauge equivalence* classes of solutions, here and in the sequel (for the Quantum Master Equation) for a reason. We can always extend the scalars and tensor the given dg-Lie or L_∞-algebra $\mathfrak{g}$ with the dg-algebra of polynomial differential forms on the n-simplex Δ^n: $\mathfrak{g} \otimes \Omega^\bullet(\Delta^n)$. If we do this for each $n \geq 0$, we will obtain a simplicial dg-Lie algebra. Solutions of the Maurer-Cartan equation in this simplicial dg-Lie algebra will form a deformation functor with values in simplicial sets, whose topology will reflect homotopy-theoretic properties of the deformation functor. For example, the set π_0 of its path components will be the set of gauge equivalence classes of solutions. Thus, a mere extension of the deformation functor $\mathrm{MC}_{\mathfrak{g}}$ to dg-commutative algebras will recover necessary homotopy-theoretic information.

Proof. Ignore the differentials for the time being. Since $S(\mathfrak{g}[1])$ is cofree, homomorphisms $R^* \to S(\mathfrak{g}[1])$ of conilpotent coaugmented coalgebras are in natural bijection with homogeneous k-linear maps $\mathfrak{m}^* \to \mathfrak{g}[1]$, which are in bijection with the space $\mathfrak{g}^1 \mathbin{\widehat{\otimes}} \mathfrak{m} = (\mathfrak{g} \mathbin{\widehat{\otimes}} \mathfrak{m})^1$, because of our finiteness assumption for $(R, \mathfrak{m})$. Explicitly, a coalgebra homomorphism corresponding to an element $S \in (\mathfrak{g} \mathbin{\widehat{\otimes}} \mathfrak{m})^1 = (\mathfrak{g}[1] \mathbin{\widehat{\otimes}} \mathfrak{m})^0 = \mathrm{Hom}_k(\mathfrak{m}^*, \mathfrak{g}[1])$ is $\exp(S) \in \mathrm{Hom}_k(R^*, S(\mathfrak{g}[1]))$, where the exponential is taken in the sense of the convolution product on the space of linear maps from a cocommutative coalgebra to a commutative algebra.

Now recall that the homomorphism $\exp(S)$ must respect the differentials. In this case, this means $D \circ \exp(S) = 0$. However, $D \circ \exp(S)$, being a coderivation of R^* with values in the cofree conilpotent coalgebra $S(\mathfrak{g}[1])$ over the homomorphism $\exp(S)$, is determined by its projection

$$l_1(S) + \frac{1}{2!} l_2(S, S) = dS + \frac{1}{2}[S, S]$$

to the space $\mathfrak{g}[1]$ of cogenerators. $\square$

The unsettling discrepancy between the category on which the Maurer-Cartan functor $\mathrm{MC}_{\mathfrak{g}}$ is defined and the category in which it is "represented" may nicely be resolved by the following tune-up.

Theorem 1.4 (Chuang-Lazarev [11]). *For any dg Lie algebra* $\mathfrak{g}$*, the contravariant functor* $\mathrm{MC}_{\mathfrak{g}}$*, as in 2, extended from* $\mathcal{FLA}\mathit{ff}$ *to the category* $L_\infty-\mathcal{A}lg$ *of* L_∞*-algebras:*

$$\mathrm{MC}_{\mathfrak{g}} : \mathrm{L}_\infty\text{-}\mathcal{A}lg^{\mathrm{op}} \to \mathcal{S}et,$$
$$\mathrm{MC}_{\mathfrak{g}}(\mathfrak{g}') := \{S \in \hom_k^1(S^{>0}(\mathfrak{g}'[1]), \mathfrak{g}) \mid DS + \frac{1}{2}[S,S] = 0\},$$

where D *is the standard differential on* $\hom_k$ *combining the differentials on* $\mathfrak{g}$ *and* $S(\mathfrak{g}'[1])$ *and the bracket combines the bracket on* $\mathfrak{g}$ *with the coproduct on* $S(\mathfrak{g}'[1])$*, is represented by the dg-Lie algebra* $\mathfrak{g}$ *itself, regarded as an* L_∞*-algebra. In other words, there is a natural isomorphism*

$$\begin{aligned}\mathrm{MC}_{\mathfrak{g}}(\mathfrak{g}') &\xrightarrow{\sim} \mathrm{Mor}_{\mathrm{L}_\infty\text{-}\mathcal{A}lg}(\mathfrak{g}', \mathfrak{g})\\ &:= \mathrm{Mor}_{\mathcal{PFDGM}an}(\mathfrak{g}'[1], \mathfrak{g}[1]) := \mathrm{Hom}_{\mathcal{CDGC}oalg}(S(\mathfrak{g}'[1]), S(\mathfrak{g}[1])).\end{aligned}$$

Proof. The proof of Theorem 1.3 works verbatim in this case. ☐

Remark. This theorem admits even finer tuning, in which the dg-Lie algebra $\mathfrak{g}$ is replaced with an L_∞-algebra and the MCE 1 is replaced with an *Extended Maurer-Cartan Equation* (*EMCE*):

$$DS + \frac{1}{2!}[S,S] + \frac{1}{3!}[S,S,S] + \cdots = 0,$$

which may equivalently be written as

$$\sum_{n=1}^{\infty} \frac{1}{n!} l_n(S, \dots, S) = 0,$$

where $l_1 := D$ is the differential and $l_n := [-, \dots, -]$, $n \geq 2$, are the higher, L_∞ brackets on $\mathfrak{g}$ (with "scalars extended" to $\hom_k(S(\mathfrak{g}'[1]), -)$). Everything else in the wording of the theorem remains intact.

2. Quantum theory: QME & quantum deformation theory

2.1. Overture

Let me start this section with a probably more contentious metatheorem than the previous one.

Metatheorem 2.1. *Every reasonable quantum deformation problem comes from a* BV_∞*-algebra.*

The classical version, Metatheorem 1.1, can be stated equivalently in a similar form, as follows.

Metatheorem 1.1′. *Every reasonable classical deformation problem comes from an* L_∞*-algebra.*

Indeed, on the one hand, any dg-Lie algebra is an L_∞-algebra, and on the other hand, if we have managed to construct an L_∞-algebra governing our deformation problem, then a quasi-isomorphic dg-Lie algebra will describe this deformation problem equally well. A similar statement about quasi-isomorphic dg-BV-algebras and solutions of the Quantum Master Equation 4 below is proven by K. Costello, [14, Section 5].

In the rest of the paper, I would like to provide evidence for the quantum metatheorem. Before doing that, I need to define a few things. Roughly speaking, under a *quantum deformation problem* I understand a deformation problem for a structure based on graphs rather than trees or engaging higher genera rather than genus zero. Those include structures of algebras over PROPs and modular operads, rather than over operads and dioperads. Examples of such could be Frobenius algebras, any types of bialgebras and their homotopy versions, such as L_∞-bialgebras and IBL_∞-algebras. Perhaps, deformations of stable maps in Gromov-Witten theory may also be classified as quantum deformations. I will return to examples later, after discussing the appropriate setup. So, what is a BV_∞-algebra?

2.2. Differential graded BV- and BV_∞-algebras

As in the classical case, typical quantum deformation problems will be coming from dg-BV-algebras, rather than BV_∞ ones. Thus, let us first discuss dg-BV-algebras.

Definition 2.2. A *dg-BV-algebra* is a dg-commutative associative algebra (V, d) with a second-order differential Δ. By a *second-order differential* on a dg-commutative algebra (V, d) we mean a linear operator $\Delta : V \to V$ of degree -1, called a *BV operator*, (graded) commuting with the differential d: $[\Delta, d] = 0$, annihilating the constants: $\Delta(1) = 0$, squaring to zero: $\Delta^2 = 0$, and being a *differential operator of second order*, which is a shortcut for *order* ≤ 2:

$$[[[\Delta, L_a], L_b], L_c] = 0 \qquad \text{for any } a, b, c \in V,$$

where $L_a : x \mapsto ax$ is the operator of left multiplication by a on V. For the purpose of this note, we will also assume a rather nonstandard piece of structure, that of a conilpotent graded cocommutative coalgebra on V. We will impose minimal compatibility between the two structures, namely, that the unit of the algebra structure is a coaugmentation of the coalgebra structure and that the counit of the coalgebra structure is an augmentation of the algebra structure. We will also assume that the differentials are compatible with the augmentation homomorphism $V \to k$, where both d and Δ act trivially on k.

Remark. Even though the traditional definition of a dg-BV-algebra does not assume any coalgebra structure, imposing it is not unprecedented: it was secretly used in [13, 12] in the study of BV_∞- and IBL_∞-morphisms. Our work [32] with Markl arose from our discovering this secret and attempting to leak this information to the public. The coalgebra requirement is rather mildly restrictive: every augmented dg-commutative algebra carries a *trivial* conilpotent cocommutative comultiplication defined by $\delta(1) := 1 \otimes 1$, $\delta(a) := a \otimes 1 + 1 \otimes a$ for a in the augmentation ideal, see, *e.g.*, [32]. If we do not mention a specific comultiplication in the sequel, we will assume the trivial comultiplication.

Note that the failure of Δ to be a derivation is measured by a Lie bracket of degree -1, often called an *antibracket*:

$$\begin{aligned} \{a, b\} &:= (-1)^{|a|}(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)) \\ &= (-1)^{|a|}[[\Delta, L_a], L_b](1) \qquad \text{for } a, b \in V, \end{aligned} \tag{3}$$

which turns V into a dg-Gerstenhaber algebra.

Example 2.3 (The Chevalley-Eilenberg complex of a dg-Lie algebra)**.** Let $\mathfrak{g}$ be a dg-Lie algebra. Then its *Chevalley-Eilenberg* (*CE*) *complex* $C_\bullet(\mathfrak{g}; k) := S(\mathfrak{g}[-1])$ is a dg-BV-algebra. The differential d is the internal differential on $S(\mathfrak{g}[-1])$, and the BV operator Δ is the following part of the CE differential:

$$\Delta(x_1 \dots x_n) := \sum_{i<j} (-1)^{|x_1|+\dots+|x_i|+\epsilon} x_1 \dots [x_i, x_j] \dots \widehat{x}_j \dots x_n,$$

where $x_1, \dots, x_n \in \mathfrak{g}[-1]$, $|x|$ is the degree of x in $\mathfrak{g}[-1]$, and $(-1)^\epsilon$ is the Koszul sign gotten from commuting $x_1 \dots x_n$ to $x_1 \dots x_i x_j x_{i+1} \dots \widehat{x}_j \dots x_n$ in $S(\mathfrak{g}[-1])$. More generally, if $\mathfrak{g}$ is an L_∞-algebra, then $S(\mathfrak{g}[-1])$ with the CE differential becomes a (commutative) BV_∞-algebra, see [9, 6].

Example 2.4 (The Chevalley-Eilenberg complex of an involutive Lie bialgebra, see [38, 13, 16, 12])**.** Let $\mathfrak{g}$ be an *involutive Lie bialgebra*, that is to say, a Lie bialgebra $(\mathfrak{g}, [-,-], \delta)$, $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ being the cobracket, satisfying an *involutivity condition*: $[-,-] \circ \delta = 0$. Let Δ be the CE differential corresponding to the Lie algebra structure on $\mathfrak{g}$, as in the previous example. Extend the cobracket δ as a degree-one derivation d of $S(\mathfrak{g}[-1])$. Then $(S(\mathfrak{g}[-1]), d, \Delta)$ is a dg-BV-algebra. Note that without the involutivity condition, the BV operator Δ will no longer commute with the differential d, but if we forget Δ, $S(\mathfrak{g}[-1])$ will still be a dg-Gerstenhaber algebra, see [28]. See ibid. for a construction of a different BV operator on the dg-Gerstenhaber algebra $S(\mathfrak{g}[-1])$ in the case when $\dim \mathfrak{g} < \infty$ and the dual $\mathfrak{g}^*$ carries a Lie-bialgebra structure which is triangular, rather than involutive.

Example 2.5 (The Chevalley-Eilenberg complex of a bi-dg-Lie algebra)**.** This example is very important for quantum deformation theory, see Examples 2.15 and 2.16. Let $\mathfrak{g}$ be a graded Lie algebra with two commuting differentials: d of degree 1 and Δ of degree -1. We may call such $\mathfrak{g}$ a *bi-dg-Lie algebra*. Consider the graded symmetric algebra $S(\mathfrak{g}[-1])$ on the shifted graded vector space $\mathfrak{g}[-1]$. Extend the differential d to $S(\mathfrak{g}[-1])$ as an "internal" differential with respect to multiplication. Extend the Lie bracket $[-,-]$ as a degree -1 biderivation to an antibracket $\{-,-\}$ on $S(\mathfrak{g}[-1])$, known as the *Schouten bracket*. Then extend Δ to a second-order differential operator on $S(\mathfrak{g}[-1])$ by the formula

$$\Delta(ab) = (\Delta a)b + (-1)^{|a|}a(\Delta b) + (-1)^{|a|}\{a,b\} \qquad \text{for } a, b \in S(\mathfrak{g}[-1]).$$

The resulting triple $(S(\mathfrak{g}[-1]), d, \Delta)$ is a dg-BV-algebra. By the way, for every dg-BV-algebra V, the shifted space $V[1]$ carries a bi-dg-Lie algebra structure with respect to the antibracket. Moreover, for the dg-BV-algebra $S(\mathfrak{g}[-1])$ of this example, the natural inclusion $\mathfrak{g} = S^1(\mathfrak{g}[-1])[1] \hookrightarrow S(\mathfrak{g}[-1])[1]$ is a morphism of bi-dg-Lie algebras.

Example 2.6 (The bar complex of an associative algebra, see Terilla-Tradler-Wilson [41])**.** Let A be a dg-associative algebra and $T(A[-1])$ be the dg-tensor coalgebra on the shifted dg-vector space $A[-1]$. Then $T(A[-1])$ with the shuffle product and the BV operator

$$\Delta(a_1 \otimes \cdots \otimes a_n) := \sum_i (-1)^{|a_1|+\cdots+|a_i|} a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n$$

for $a_i \in A[-1]$, becomes a dg-BV-algebra. Note that we need to choose a conilpotent cocommutative coproduct on $T(A[-1])$, such as the shuffle coproduct, or the trivial coproduct described in the remark after Definition 2.2, to fit our definition of a dg-BV-algebra. I suspect that $T(A[-1])$, perhaps with the original coassociative coproduct, is responsible for quantum deformation theory over noncommutative, associative parameter rings.

Example 2.7 (The bar complex of an $\mathcal{O}$-algebra)**.** Let $\mathcal{O}$ be a Koszul quadratic operad of vector spaces, $\mathcal{O}^!$ be its Koszul-dual operad, and V be a dg-$\mathcal{O}$-algebra. Then the cofree conilpotent $\mathcal{O}^!$-coalgebra $F^c_{\mathcal{O}^!}(V[1])$ acquires a codifferential $d+\Delta$, where d is the internal differential and Δ is the coderivation corresponding to the $\mathcal{O}$-algebra structure on V. Based on the particular cases of $\mathcal{O}$ being the Lie operad or the associative operad, as in Examples 2.3 and 2.6, respectively, I anticipate that $F^c_{\mathcal{O}^!}(V[-1])$ with the differential d and the BV operator Δ will be a dg-BV algebra and give rise to deformation theory with $\mathcal{O}^!$-algebras as parameter rings. The 2018 Honors thesis [46] of Lucy Yang at the University of Minnesota aims to prove that Δ is indeed a BV operator.

Example 2.8 (Functions on an odd symplectic manifold, see Schwarz [36] and Getzler [19])**.** Let M be an odd symplectic supermanifold of dimension $(n|n)$ with a volume form. Then the algebra $C^\infty(M)$ of smooth functions on M becomes a BV-algebra with the BV operator given by

$$\Delta := \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}$$

in super Darboux coordinates $(x_1, \ldots, x_n |\, \xi_1, \ldots, \xi_n)$. Here we do not assume much of a dg structure, *i.e.*, the grading is actually a $\mathbb{Z}/2\mathbb{Z}$-grading and the differential d is just zero. Also, we do not assume any comultiplication in this example. However, to get one, one can choose a basepoint on M and use the associated augmentation on $C^\infty(M)$ to define a trivial coproduct, as in the Remark after Definition 2.2. Still, in general there will be no compatibility between the augmentation and Δ, which we require of our dg-BV-algebras. Thus, this example, albeit fundamental, is an outlier in our context.

A particular case of this example, which I present in the graded, rather than $\mathbb{Z}/2\mathbb{Z}$-graded version, is the example of $T^*[1]M$, a shifted cotangent bundle to an oriented n-manifold M, see [10, 29]. Functions on this graded manifold are nothing but multivector fields $\Gamma(M, S(T[-1]M))$. A volume form on M gives an isomorphism: $f : \Gamma(M, S^p(T[-1]M)) \to \Omega^{n-p}(M)$. Then $\Delta = f^{-1} \circ d_{\mathrm{dR}} \circ f$ defines the structure of a BV-algebra on $\Gamma(M, S(T[-1]M))$, *i.e.*, a dg-BV-algebra with a zero differential.

Definition 2.9. A (*commutative*) BV$_\infty$-*algebra* is a graded commutative associative algebra V with a sequence of differential operators Δ_n of order $\leq n$

with $n \geq 1$. A *differential operator of order* $\leq n$ on a graded commutative algebra V is a linear operator $\Delta_n : V \to V$ satisfying

$$[\ldots[[\Delta_n, L_{v_0}], L_{v_1}], \ldots, L_{v_n}] = 0 \qquad \text{for any } v_0, v_1, \ldots, v_n \in V.$$

We also require that the $k[[\hbar]]$-linear operator $\widehat{d} := \sum_{n \geq 1} \hbar^{n-1} \Delta_n : V[[\hbar]] \to V[[\hbar]]$, which we call a BV_∞ *operator*, where $\hbar$ is a formal variable with $|\hbar| = 2$, be of total degree 1, kill constants: $\widehat{d}(1) = 0$, and square to zero: $\widehat{d}^2 = 0$. Another requirement, specific to this paper, is that V have a conilpotent graded cocommutative coalgebra structure, mildly compatible with the algebra structure, as in Definition 2.2: the unit of the algebra structure is a coaugmentation of the coalgebra structure, and the counit of the coalgebra structure is an augmentation of the algebra structure. And we again assume that $\widehat{d}$ is compatible with the augmentation homomorphism $V[[\hbar]] \to k[[\hbar]]$, where $\widehat{d}$ acts trivially on $k[[\hbar]]$.

Example 2.10. Every dg-BV-algebra (V, d, Δ) is a BV_∞-algebra with $\widehat{d} = d + \hbar\Delta$ or $\Delta_1 = d$, $\Delta_2 = \Delta$, and $\Delta_n = 0$ for all other n.

The BV_∞ operator $\widehat{d}$ generates a whole family of "derived" antibrackets

$$\begin{aligned}\{v_1, v_2, \ldots, v_n\} &:= \frac{1}{\hbar^{n-1}}[\ldots[[\widehat{d}, L_{v_1}], L_{v_2}], \ldots, L_{v_n}](1)\\ &= [\ldots[[\Delta_n + \hbar\Delta_{n+1} + \hbar^2\Delta_{n+2} + \ldots, L_{v_1}], L_{v_2}], \ldots, L_{v_n}](1)\end{aligned}$$

for $v_1, v_2, \ldots, v_n \in V$. Note a conventional sign difference with Equation 3 for $n = 2$. These antibrackets (after being multiplied back by $\hbar^{n-1}$, to be precise) define the structure of an L_∞-algebra on V, compatible with the product on V in the way that the failure of the nth antibracket to be a multiderivation is measured by the $(n+1)$st antibracket. These statements are an original result of J. Alfaro, I. A. Batalin, K. Bering, P. H. Damgaard and R. Marnelius [3, 7, 8], see also F. Akman [1, 2], Th. Th. Voronov [43, 44], and D. Bashkirov and the author [6].

Commutative BV_∞-algebras appeared in [13] in the study of symplectic field theory. We will be dropping the adjective "commutative," despite the fact that our commutative BV_∞-algebras do not fit the definition of an $\mathcal{O}_\infty$-algebra in the sense of being an algebra over a cofibrant model $\mathcal{O}_\infty$ of an operad $\mathcal{O}$. The correct, C_∞ version of the notion of a BV_∞-algebra and its relation to the notion of a commutative BV_∞-algebra is described in [17].

2.3. QME & quantum deformation functor

Again, let $\hbar$ be a formal variable of degree 2:

$$|\hbar| = 2.$$

First of all, a dg-BV-algebra V will be related to quantum deformations through the corresponding *quantum deformation functor*

$$\begin{gathered}\mathcal{CLA}lg \to \mathcal{S}et,\\ (R, \mathfrak{m}) \mapsto \mathrm{QM}_V(R),\end{gathered}$$

which associates to a complete local algebra $(R, \mathfrak{m})$ the set

$$\mathrm{QM}_V(R) := \{S \in V[[\hbar]]^2 \mathbin{\widehat{\otimes}} \mathfrak{m} \mid dS + \hbar\Delta S + \frac{1}{2}\{S, S\} = 0\}$$

of solutions of the *Quantum Master Equation* (*QME*):

$$dS + \hbar\Delta S + \frac{1}{2}\{S, S\} = 0, \tag{4}$$

which is equivalent to

$$\widehat{d}\, e^{S/\hbar} = 0,$$

where

$$\widehat{d} := d + \hbar\Delta,$$

in the space $V((\hbar)) \mathbin{\widehat{\otimes}} R$ of formal Laurent series, because of the following remarkable formula

$$e^{-S/\hbar} \circ \widehat{d} \circ e^{S/\hbar} = \widehat{d} + \{S, -\} + \frac{1}{\hbar}\left(\widehat{d}S + \frac{1}{2}\{S, S\}\right), \tag{5}$$

for operators on $V((\hbar)) \mathbin{\widehat{\otimes}} R$, where we abuse notation by writing elements, such as $e^{S/\hbar}$, in lieu of the operators, such as $L_{e^{S/\hbar}}$, of left multiplication by these elements. This formula follows from a celebrated, and much more compact, identity

$$\mathrm{Ad}_{e^A} = e^{\mathrm{ad}_A}$$

for linear operators on a Lie algebra, in our case evaluated on $\widehat{d}$ with $A = L_{-S/\hbar}$. If we apply the operators on both sides of Equation 5 to the unit element 1, we obtain an equation

$$e^{-S/\hbar}\widehat{d}\,(e^{S/\hbar}) = \frac{1}{\hbar}\left(\widehat{d}S + \frac{1}{2}\{S, S\}\right)$$

on elements in $V((\hbar)) \mathbin{\widehat{\otimes}} R$, which yields the equivalence of the two forms of the QME above.

The quantum deformation functor admits a generalization to a BV_∞-algebra $(V, \widehat{d})$. Here are the adjustments one needs to make in this case. The quantum deformation functor $\mathrm{QM}_V : \mathcal{CLA}lg \to \mathcal{S}et$ associates to a complete local algebra $(R, \mathfrak{m})$ the set

$$\mathrm{QM}_V(R) := \{S \in V[[\hbar]]^2 \mathbin{\widehat{\otimes}} \mathfrak{m} \mid \widehat{d}\, e^{S/\hbar} = 0\} \tag{6}$$

of solutions of the *Quantum Master Equation* (*QME*):

$$\widehat{d}\, e^{S/\hbar} = 0, \tag{7}$$

which is equivalent to

$$\widehat{d}S + \frac{1}{2!}\{S, S\} + \frac{1}{3!}\{S, S, S\} + \dots = 0.$$

Again, the equivalence follows from a generalization of Equation 5:

$$e^{-S/\hbar} \circ \widehat{d} \circ e^{S/\hbar} = \widehat{d} + \{S, -\} + \frac{1}{2!}\{S, S, -\} + \dots \\ + \frac{1}{\hbar}\left(\widehat{d}S + \frac{1}{2!}\{S, S\} + \frac{1}{3!}\{S, S, S\} + \dots\right),$$

proven exactly in the same way as in the dg-BV case.

Solutions to the QME in a dg-BV- or BV_∞-algebra, in general, may be considered as distinguished deformations of the BV or BV_∞ operator. In particular cases, they may describe interesting structures. For example, for the dg-BV-algebra $\Gamma(M, S(T[-1]M))$ of multivector fields on an oriented manifold M, see the end of Example 2.8, a solution of the QME is a linear polynomial $S_0 + \hbar S_1$, where S_0 is a bivector field and S_1 is just a function satisfying the relations $[S_0, S_0] = 0$ and $\Delta(S_0) + [S_1, S_0] = 0$. As Ricardo Campos has pointed out to me, these data define a unimodular Poisson structure on M, see [45].

2.4. Quantum representability theorem

To talk about representable functors in the QME context, we need to switch to different categories, those of BV_∞-spaces and BV_∞-algebras.

Definition 2.11. Let $(V, \widehat{d})$ and $(V', \widehat{d'})$ be two BV_∞-algebras. A BV_∞-*morphism* $V \to V'$ is a k-linear map $\varphi : V \to V'[[\hbar]]$ of degree two such that

1. $\varphi(1) = 0$;
2. $\widehat{d'} \circ \exp(\varphi/\hbar) = \exp(\varphi/\hbar) \circ \widehat{d}$, where the exponential exp is taken with respect to the convolution product on $\operatorname{Hom}_k(V, V'((\hbar)))$;
3. $\varphi = \varphi_0 + \hbar\varphi_1 + \hbar^2\varphi_2 + \dots$, where $\varphi_n : V \to V'$ is a *differential operator of order* $\leq n+1$ *over the trivial algebra homomorphism* $V \to V'$, which takes the augmentation ideal $\mathfrak{m}$ of V to zero, *i.e.*, $\varphi_n(\mathfrak{m}^{n+2}) = 0$.

This definition is somewhat more general than the original one by Cieliebak and Latschev [13] (or Cieliebak, Fukaya, and Latschev [12]): if we require our φ_n to be a differential operator of order $\leq n$ over the trivial algebra homomorphism for each $n \geq 0$, then $\varphi/\hbar$ will be a BV_∞-morphism in their sense. The exponential makes sense, because of Condition 1 and the conilpotency of the coproduct on V. Composition of BV_∞-morphisms φ and ψ is done by composing $\exp(\varphi/\hbar)$ and $\exp(\psi/\hbar)$. The fact that composition of exponentials is the exponential of a BV_∞-morphism follows from the existence of the logarithm and its extension to Laurent series in $\hbar$, see [6, 32], and checking that the series $\hbar \log(\exp(\varphi/\hbar) \circ \exp(\psi/\hbar))$ satisfies Properties 1–3 of Definition 2.11. For example, to see that the series does not contain any negative powers of $\hbar$, one verifies that $\lim_{\hbar\to 0} \hbar \log(\exp(\varphi/\hbar) \circ \exp(\psi/\hbar))$ is finite.

Let $\mathcal{BV}_\infty$-$\mathcal{A}lg$ denote the category of BV_∞-algebras and $\mathcal{BV}_\infty$-$\mathcal{S}p$ the same category, interpreted geometrically: if $(V, \widehat{d})$ is a BV_∞-algebra, $\operatorname{Spec}^* V$ will denote the corresponding geometric object, which we call a BV_∞-*space*. The idea is that this is a geometric object, generalized functions, or distributions, on which form the BV_∞-algebra V.

Observe that the opposite category of complete local algebras forms a subcategory of the category of BV_∞-algebras:

$$\mathcal{CLA}lg \subseteq \mathcal{BV}_\infty\text{-}\mathcal{A}lg^{\mathrm{op}} .$$

To see this, observe that if $(R, \mathfrak{m})$ is a complete local k-algebra, then its k-linear dual R^* with the BV_∞ operator $\widehat{d} = 0$ and multiplication defined to be zero on $\mathfrak{m}^*$ is a BV_∞-algebra. A homomorphism $f : (R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ of complete local algebras induces a dual morphism $f^* : S^* \to R^*$ of coalgebras. Then $\varphi := \hbar \log f^*$ is a BV_∞-morphism $S^* \to R^*$. Indeed, let e be the unit of $\operatorname{Hom}_k(S^*, R^*)$ under the convolution product. It is given by composing the unit morphism $k \to R^*$ with the counit morphism $S^* \to k$. Then $\varphi = \hbar \log f^* = \hbar(f^* - \mathsf{e})$, because $\operatorname{Im}(f^* - \mathsf{e}) \subseteq \mathfrak{m}_R^*$ and $(\mathfrak{m}_R^*)^2 = 0$. Therefore,

$\varphi(1) = \hbar(1-1) = 0$. Also $\log f^* = f^* - \mathsf{e}$ automatically vanishes on $(\mathfrak{m}_S^*)^3 = 0$. Thus, it makes sense to talk about a functor $\mathcal{CLA}lg \to \mathcal{S}et$, such as QM_V, being represented by an object of the category $\mathcal{BV}_\infty$-$\mathcal{A}lg$ or $\mathcal{BV}_\infty$-$\mathcal{S}p$.

Theorem 2.12. *The quantum deformation functor* QM_V *associated to a dg-BV- or a* BV_∞*-algebra* V *is represented by the* BV_∞*-space* $\operatorname{Spec}^* V$, i.e., *there is a natural isomorphism*

$$\begin{aligned} \mathrm{QM}_V(R) &\xrightarrow{\sim} \mathrm{Mor}_{\mathcal{BV}_\infty\text{-}\mathcal{S}p}(\operatorname{Spec}^* R^*, \operatorname{Spec}^* V) \\ &:= \mathrm{Hom}_{\mathcal{BV}_\infty\text{-}\mathcal{A}lg}((R^*, 0), (V, \widehat{d})). \end{aligned}$$

Park, Terilla, and Tradler in [34] prove a representability theorem of a rather different flavor for the quantum deformation functor up to gauge equivalence. Our result is closer to but does not directly follow from Münster-Sachs [33, Section 4.3] or Markl-V [32, Corollary 41]. However, the proof, which we repeat here for completeness, is similar.

Proof. A solution $S \in V[[\hbar]]^2 \mathbin{\widehat{\otimes}} \mathfrak{m}$ of the QME 7 is by definition equivalent to a degree-two k-linear map $S : \mathfrak{m}^* \to V[[\hbar]]$ satisfying $\widehat{d} \exp(S/\hbar) = 0$ or a degree-two k-linear map $S : R^* \to V[[\hbar]]$ such that $S(1) = 0$. Each such S automatically satisfies Property 3 of Definition 2.11, because $(\mathfrak{m}^*)^{n+2} = 0$ for all $n \geq 0$. □

A quantum analogue of Chuang-Lazarev's Theorem 1.4 has a more natural wording and, naturally, a totally trivial proof. The quantum deformation functor associated to a BV_∞-algebra $(V, \widehat{d})$ may be extended to a functor

$$\mathrm{QM}_V : \mathcal{BV}_\infty\text{-}\mathcal{A}lg^{\mathrm{op}} \to \mathcal{S}et$$

which takes a BV_∞-algebra $(V', \widehat{d'})$ to the set of BV_∞-morphisms $V' \to V$. One may view the equation $\widehat{d} \exp(\varphi/\hbar) = \exp(\varphi/\hbar)\widehat{d'}$ as a QME on φ. Then, tautologically, the functor QM_V is represented by the BV_∞-algebra $(V, \widehat{d})$ or the BV_∞-space $\operatorname{Spec}^* V$.

On the other hand, the following less general representability theorem may be more interesting.

Before wording the theorem, note that the category L_∞-$\mathcal{A}lg$ of L_∞-algebras (and thereby the equivalent category $\mathcal{PFDGM}an$ of pointed formal dg-manifolds) is a subcategory of the category $\mathcal{BV}_\infty$-$\mathcal{A}lg$ of BV_∞-algebras. Indeed, if $\mathfrak{g}$ is an L_∞-algebra, then $S(\mathfrak{g}[-1])$ is a BV_∞-algebra, see Example 2.3. An L_∞-morphism $\mathfrak{g} \to \mathfrak{h}$ is, by definition, a morphism of coaugmented dg-coalgebras $S(\mathfrak{g}[1]) \to S(\mathfrak{h}[1])$, which is determined by its components

$\varphi_n : S^n(\mathfrak{g}[1]) \to \mathfrak{h}[1]$, $n \geq 1$, and $\varphi = \sum_{n\geq 1} \hbar^n \varphi_n$ defines a BV_∞-morphism $S(\mathfrak{g}[-1]) \to S(\mathfrak{h}[-1])$, see [6, Theorem 4.8].

Now define a version of the quantum deformation functor on the opposite category $\mathrm{L}_\infty\text{-}\mathcal{A}lg^{\mathrm{op}}$ of the category of L_∞-algebras. Let $(V, \widehat{d}_V)$ be a BV_∞-algebra and $\mathfrak{g}$ an L_∞-algebra. Then $S(\mathfrak{g}[1])$ is a coaugmented conilpotent cocommutative dg-coalgebra with the codifferential $D_1 + D_2 + \dots$ defining the L_∞ structure on $\mathfrak{g}$: D_n extends the nth bracket $l_n : S^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ to a degree-one coderivation of $S(\mathfrak{g}[1])$. Likewise, $S(\mathfrak{g}[-1])$ is a coaugmented conilpotent cocommutative graded coalgebra with the codifferential $\widehat{d}_{\mathfrak{g}} := D_1 + \hbar D_2 + \hbar^2 D_3 + \dots$ on $S(\mathfrak{g}[-1])[[\hbar]]$. Hence, the graded vector space $\hom_k(S(\mathfrak{g}[-1]), V)$ becomes a BV_∞-algebra with respect to the convolution product and the BV_∞ operator $\widehat{D}(\Phi) := \widehat{d}_V \circ \Phi - (-1)^{|\Phi|} \Phi \circ \widehat{d}_{\mathfrak{g}}$.[1] Thus, we can define the value of the quantum deformation functor associated to V on the L_∞-algebra $\mathfrak{g}$ as the set

$$\mathrm{QM}_V(\mathfrak{g}) := \{S = \sum_{n\geq 0} \hbar^n S_n \mid \widehat{D}\, e^{S/\hbar} = 0\} \tag{8}$$

of solutions to the *QME for the* BV_∞*-algebra* $\hom_k(S(\mathfrak{g}[-1]), V)$, where

$$S_n \in \hom_k^{2-2n}(S^{>0}(\mathfrak{g}[-1]), V) \qquad \text{for each } n \geq 0$$

subject to

$$S_n(S^{>n+1}(\mathfrak{g}[-1])) = 0 \qquad \text{for each } n \geq 0.$$

Theorem 2.13. *Given a* BV_∞*-algebra* $(V, \widehat{d}_V)$*, the associated quantum deformation functor*

$$\mathrm{QM}_V : \mathrm{L}_\infty\text{-}\mathcal{A}lg^{\mathrm{op}} \to \mathcal{S}et$$

is represented by the BV_∞*-algebra* V *in the category of* BV_∞*-algebras or by the* BV_∞*-space* $\mathrm{Spec}^* V$ *in the equivalent category of* BV_∞*-spaces. In other words, there is a natural isomorphism*

$$\begin{aligned}\mathrm{QM}_V(\mathfrak{g}) &\xrightarrow{\sim} \mathrm{Mor}_{\mathcal{BV}_\infty\text{-}\mathcal{S}p}(\mathrm{Spec}^* S(\mathfrak{g}[-1]), \mathrm{Spec}^* V) \\ &:= \mathrm{Hom}_{\mathcal{BV}_\infty\text{-}\mathcal{A}lg}((S(\mathfrak{g}[-1]), \widehat{d}_g), (V, \widehat{d}_V)).\end{aligned}$$

Proof. The proof is almost a tautology: one just needs to observe that the equation $\widehat{d}_V \circ \exp(S/\hbar) = \exp(S/\hbar) \circ \widehat{d}_{\mathfrak{g}}$ defining S as a BV_∞-morphism is equivalent to the QME $\widehat{D} e^{S/\hbar} = 0$ for the BV_∞-algebra $\hom_k(S(\mathfrak{g}[-1]), V)$. $\square$

[1] For $\widehat{D}$ to define a BV_∞ operator, it is essential that $\widehat{d}_{\mathfrak{g}}$ be a coderivation.

2.5. Quantum deformation functor associated to a bi-dg-Lie algebra

Notwithstanding the apparent consistency of the quantum deformation set-up in the previous section, actual examples of quantum deformations, see Section 2.6, require certain modification of the quantum deformation functor. Suppose $S(\mathfrak{g}[-1])$ is the dg-BV algebra arising from a bi-dg-Lie algebra $\mathfrak{g}$, as in Example 2.5. In this case, $\mathfrak{g}$ is a bi-dg-Lie subalgebra of $S(\mathfrak{g}[-1])[1]$, the QME 4 in $S(\mathfrak{g}[-1])$ restricts to an equation in $\mathfrak{g}$, and the following subfunctor of the quantum deformation functor becomes important:

$$\mathrm{QM}_{\mathfrak{g}}(R) := \{S \in \mathfrak{g}[[\hbar]]^1 \mathbin{\widehat{\otimes}} \mathfrak{m} \mid dS + \hbar\Delta S + \frac{1}{2}[S,S] = 0\}. \tag{9}$$

Note that $\mathfrak{g}[[\hbar]]^1 = S^1(\mathfrak{g}[-1])[[\hbar]]^2$ and we have a natural inclusion of functors $\mathrm{QM}_{\mathfrak{g}}(R) \subseteq \mathrm{QM}_{S(\mathfrak{g}[-1])}(R)$. On the other hand, we have a natural identification

$$\mathrm{QM}_{\mathfrak{g}}(R) = \mathrm{MC}_{\mathfrak{g}[[\hbar]]}(R),$$

where $\mathfrak{g}[[\hbar]]$ is considered as a dg-Lie algebra over k with a differential $\widehat{d} = d + \hbar\Delta$. Thus, Theorems 1.3 and 1.4 are applicable and we can state the following easy corollary.

Corollary 2.14. *Given a bi-dg-Lie algebra $\mathfrak{g}$, the quantum deformation functor $\mathrm{QM}_g : \mathcal{FLA}\!f\!f^{\mathrm{op}} \to \mathcal{S}et$ is representable by the pointed formal dg-manifold $\mathfrak{g}[1][[\hbar]]$, and so is the extension of this functor to the category of pointed formal dg-manifolds over $k[[\hbar]]$. In other words, we have natural isomorphisms*

$$\mathrm{QM}_{\mathfrak{g}}(R) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{PFDGM}an}(\operatorname{Spec} R, \mathfrak{g}[[\hbar]][1]),$$
$$\mathrm{QM}_{\mathfrak{g}}(\mathfrak{g}') \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{PFDGM}an/k[[\hbar]]}(\mathfrak{g}'[[\hbar]][1], \mathfrak{g}[[\hbar]][1]),$$

for a complete local algebra R and a bi-dg-Lie algebra $\mathfrak{g}'$ or a more general L_∞-algebra $\mathfrak{g}'[[\hbar]]$ over $k[[\hbar]]$.

2.6. Examples of quantum deformations

Now we can discuss examples of quantum deformations, described by solutions of QME in appropriate bi-dg-Lie and BV_∞-algebras.

Example 2.15. Let $\mathcal{O}$ be a modular dg-operad, V be a dg-vector space with finite-dimensional graded components and an *inner product of degree* -1, *i.e.*, a nondegenerate linear map $S^2(V) \to k[-1]$, and $\mathcal{E}nd_V((g,n)) :=$

$V^{\otimes n}$ be the twisted modular dg-operad of endomorphisms of V. Consider the tensor product $\mathcal{O}\otimes\mathcal{E}nd_V$, which is a twisted modular operad with components $(\mathcal{O}\otimes\mathcal{E}nd_V)((g,n)) := \mathcal{O}((g,n))\otimes\mathcal{E}nd_V((g,n))$. Barannikov in [4], cf. also [23], constructs, in fact, a bi-dg-Lie algebra structure $(\mathfrak{g},[-,-],d,\Delta)$, see Example 2.5, on a shifted direct sum $\mathfrak{g} := \bigoplus_{g,n}(\mathcal{O}\otimes\mathcal{E}nd_V)((g,n))_{S_n}[1] = \bigoplus_{g,n}\mathcal{O}((g,n))\otimes_{S_n}V^{\otimes n}[1]$ of S_n-coinvariants of the components of $\mathcal{O}\otimes\mathcal{E}nd_V$. As we know from Example 2.5, the bi-dg-Lie algebra $\mathfrak{g}$ gives rise to a dg-BV-algebra $S(\mathfrak{g}[-1])$.

According to Barannikov [4], see also [23], solutions $S\in\mathfrak{g}[[\hbar]]^1$ of the Quantum Master Equation

$$dS+\hbar\Delta S+\frac{1}{2}[S,S]=0 \tag{10}$$

are in bijection with $\mathcal{F}(\mathcal{O})$-algebra structures on V, where $\mathcal{F}(\mathcal{O})$ is the Feynman transform [21] of the modular operad $\mathcal{O}$. Thus, we may think of the quantum deformation functor 9 describing deformations of the trivial $\mathcal{F}(\mathcal{O})$-algebra structure on V corresponding to the trivial solution $S_0=0$ of the QME. Deformations of the $\mathcal{F}(\mathcal{O})$-algebra corresponding to a nontrivial solution $S_0\in\mathfrak{g}[[\hbar]]^1$ of Equation 10 may be described by solutions of the QME

$$\widehat{d'}S+\frac{1}{2}[S,S]=0, \qquad S\in\mathfrak{g}[[\hbar]]^1,$$

with $\widehat{d'}=d+\hbar\Delta+[S_0,-]$. As in Example 2.5, the bi-dg-Lie algebra $(\mathfrak{g}[[\hbar]],d+[S_0,-],\Delta,[-,-])$ over $k[[\hbar]]$ gives rise to a BV_∞-algebra $S(\mathfrak{g}[-1])$ with the BV_∞ operator $\widehat{d'}=d+\{S_0,-\}+\hbar\Delta$ being a formal power series in $\hbar$ in which all the terms but those by $\hbar^1$ are derivations.

The simplest example of a modular operad $\mathcal{O}$ is the modular envelope of the commutative operad: $\mathcal{O}((g,n)) := k$ for all (g,n) in the "stable" range. The corresponding notion of an $\mathcal{F}(\mathcal{O})$-algebra was studied by Markl [31], who called it a *loop homotopy Lie algebra*. It is a modular analogue of the (properadic) notion of an IBL_∞-algebra, which we will look at later, in Section 3.

Another standard example of a modular dg-operad is the homology operad $\mathcal{O}((g,n))=H_\bullet(\overline{\mathcal{M}}_{g,n};k)$ of the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves of genus g with n punctures with respect to attaching. In this case, the notion of an $\mathcal{F}(\mathcal{O})$-algebra on V will be a higher-genus, homotopy extension of the notion of a *gravity algebra*, see [20].

Example 2.16. This is a twisted version of the previous example. Let $\mathcal{O}$ be a twisted modular operad, V be a dg-vector space with finite-dimensional graded components and an inner product of degree 0, and $\mathcal{E}nd_V((g,n)) :=$

$V^{\otimes n}$ be the endomorphism modular operad of V. Consider the tensor product $\mathcal{O} \otimes \mathcal{E}nd_V$, which is a twisted modular operad with components $(\mathcal{O} \otimes \mathcal{E}nd_V)((g,n)) := \mathcal{O}((g,n)) \otimes \mathcal{E}nd_V((g,n))$. Again, Barannikov in [4], cf. [23], constructs, a bi-dg-Lie-algebra structure $(\mathfrak{g}, [-,-], d, \Delta)$, see Example 2.5, on a shifted direct sum $\mathfrak{g} := \bigoplus_{g,n} (\mathcal{O} \otimes \mathcal{E}nd_V)((g,n))_{S_n}[1] = \bigoplus_{g,n} \mathcal{O}((g,n)) \otimes_{S_n} V^{\otimes n}[1]$ of S_n-coinvariants of the components of $\mathcal{O} \otimes \mathcal{E}nd_V$. The bi-dg-Lie algebra $\mathfrak{g}$ is part of the dg-BV-algebra $S(\mathfrak{g}[-1])$ up to shift.

Again, as per [4, 23], solutions of the Quantum Master Equation

$$dS + \hbar \Delta S + \frac{1}{2}[S,S] = 0$$

in $\mathfrak{g}[[\hbar]]^1$ are in bijection with $\mathcal{F}(\mathcal{O})$-algebra structures on V, where $\mathcal{F}(\mathcal{O})$ is the Feynman transform of the twisted modular operad $\mathcal{O}$.

Deformations of the $\mathcal{F}(\mathcal{O})$-algebra corresponding to a nontrivial solution $S_0 \in \mathfrak{g}[[\hbar]]^1$ of the QME are obtained by redefining $\widehat{d'} := \widehat{d} + [S_0, -]$ and considering the QME $\widehat{d'}S + \frac{1}{2}[S,S] = 0$, $S \in \mathfrak{g}[[\hbar]]^1$. As in the previous example, this leads to a BV_∞-algebra of a particular type, with all the BV operators Δ_n, except Δ_2, being derivations.

The model example of a twisted modular operad is the homology operad $\mathcal{O}((g,n)) = H_\bullet(\mathcal{M}_{g,n}; k)$ of the moduli spaces $\mathcal{M}_{g,n}$ of algebraic curves of genus g with n punctures with respect to twist-gluing. In this case, the notion of an $\mathcal{F}(\mathcal{O})$-algebra will be a higher-genus, homotopy extension of the notion of a *hypercommutative*, or *WDVV-algebra*, see [20] and [42]. This structure is equivalent to the structure of a *linear Frobenius manifold* on V and is the genus-zero avatar of *cohomological field theory*, see [26, 30].

There are various versions of the moduli-space example of a twisted modular operad. One, in a different language, appeared in the work of B. Zwiebach [47] and A. Sen and Zwiebach [37]. Translated to the language of our paper, they considered $V = k$ and S^1-equivariant chains on the moduli space of Riemann surfaces with holomorphic disks. This gave a twisted modular dg-operad. Another version of this operad, which uses the real version of the Deligne-Mumford compactification, originated in the paper [24] of T. Kimura, Stasheff and myself. See also Costello [14]. However, for the QME in these cases to be sensible, degree considerations suggested to change grading on chains to grading by codimension, as well as assume that the formal variable $\hbar$ has degree zero. One can view this change of grading as nothing but replacing chains with Poincaré-Lefschetz dual cochains with compact support. A solution to the QME in these cases is regarded as a universal *topological quantum field theory*, the 2d, chain-level, closed-string, nonperturbative flavor to be more precise. Zwiebach also associated this structure with *string-field theory*.

3. Quantizing Deformation Theory I

Recall from Example 2.4 that an involutive Lie bialgebra $\mathfrak{g}$ gives rise to a dg-BV-algebra structure on the symmetric algebra $S(\mathfrak{g}[-1])$. Not surprisingly, a general BV_∞-algebra structure on the symmetric algebra $S(\mathfrak{g}[-1])$ of a suspended graded vector space $\mathfrak{g}$ is equivalent to the structure of a *homotopy involutive Lie bialgebra*, called an IBL_∞-*algebra*, on $\mathfrak{g}$. This is actually the definition thereof, see [12]! Moreover, the notion is equivalent to that of an Ω(coFrob)-algebra as per [16, Theorem 4.10]. Here coFrob is a certain co-Frobenius coproperad and Ω is the cobar construction, producing a dg-properad. The notion of an IBL_∞-algebra generalizes not only that of an involutive Lie bialgebra but also the notion of a bi-dg-Lie algebra, which played an important role in the previous section. Indeed, both the involutive Lie bialgebra and bi-dg-Lie algebra structures on a graded vector space $\mathfrak{g}$ induce dg-BV-structures on the symmetric algebra $S(\mathfrak{g}[-1])$, the Chevalley-Eilenberg complex of $\mathfrak{g}$, see Examples 2.4 and 2.5.

In *Quantizing Deformation Theory* [40], Terilla conjectured the existence of quantized deformation theory, in which commutative k-algebras R would be replaced with (commutative) Frobenius algebras and the Maurer-Cartan equation in an L_∞-algebra would be replaced with a master equation in an IBL_∞-algebra $\mathfrak{g}$. The rationale is that the properad Frob describing Frobenius algebras is a unit in the monoidal category of properads, just like the operad Com describing commutative algebras is a unit in the monoidal category of operads. Equivalently, if V is an algebra over a properad $\mathcal{P}$ and F is a Frobenius algebra, then $V \otimes F$ is again a $\mathcal{P}$-algebra. Moreover, an IBL_∞-algebra is equivalent to an algebra over the dg-properad Ω(coFrob), whereas an L_∞-algebra is equivalent to an algebra over the operadic cobar construction Ω(coCom) for the cocommutative co-operad coCom.

This is an extremely striking analogy, but the current paper falls short of proving Terilla's conjecture. However, I would like to convince the reader that staying within the good old world of deformations over commutative parameters still produces an interesting quantization of deformation theory.

The matter is that there is a subtle difference between extending the structure of an algebra over a properad $\mathcal{P}$ on a dg-vector space V to a tensor product $V \otimes_k R$ and making a complete base change from k to R. The structure of a commutative k-algebra on R is not enough to define the structure of a $\mathcal{P}$-algebra on $V \otimes R$ over k. As mentioned above, endowing R with a Frobenius-algebra structure will suffice. On the other hand, when making a base change, we are rather interested in a $\mathcal{P} \otimes R$-algebra structure on $V \otimes R$ over R, which is always there as long as R is a commutative k-algebra. At the level of operations, not every properadic operation with values, say, in the tensor square $V \otimes V$ extends to an operation with values in $(V \otimes R) \otimes (V \otimes R)$ but it does, if all we want is an operation with values in $(V \otimes R) \otimes_R (V \otimes R)$.

What this means in the case of an IBL_∞-algebra $\mathfrak{g}$ and a complete local algebra R is that $\mathfrak{g} \otimes R$ and $\mathfrak{g} \widehat{\otimes} R$ are also IBL_∞-algebras over R. Likewise, if $\mathfrak{g}'$ is an L_∞-algebra, then $\hom_k(S(\mathfrak{g}'[-1]), \mathfrak{g})$ is an IBL_∞-algebra over the dg-commutative algebra $S(\mathfrak{g}'[-1])^*$. Accordingly, $S(\mathfrak{g}[-1]) \otimes R$, $S(\mathfrak{g}[-1]) \widehat{\otimes} R$ and $\hom_k(S(\mathfrak{g}'[-1]), S(\mathfrak{g}[-1]))$ are BV_∞-algebras over (dg-)commutative algebras R and $S(\mathfrak{g}'[-1])^*$, respectively. Thus, the functors $\mathrm{QM}_V(R)$, see Equation 6, and $\mathrm{QM}_V(\mathfrak{g}')$, see Equation 8, for $V = S(\mathfrak{g}[-1])$ are well-defined and Theorems 2.12 and 2.13 show that these functors are representable.

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