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Skrepa morphisms

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Abstract: For a complete algebraic space with a nef quasipolarization and its closed subspace which includes the exceptional locus of the quasipolarization, the paper establishes the existence of an infinitesimal neighborhood of the subspace such that for any quasipolarized morphism of the neighborhood into a polarized scheme there exists a quasipolarized modification of the space into a polarized scheme. This is a quasipolarized version of the well-known Artin modification. As applications nonprojective versions of the semiampleness criterion of Birkar in general and of Keel in positive characteristics are obtained.

Introduction

Constructions in mathematics and, in particular, in geometry play an important role both in a search of required objects and in proofs of expected results. Starting from Diophant from Alexandria projections are among the most favourite tools in mathematics. Projective structure is also useful for construction of many important classes of morphisms, e.g., morphisms into a projective space, blowups of ideals, modifications (Artin, Moishezon) and flips in the LMMP. However, some modifications can bring to nonprojective varieties. In this paper we consider a construction of a morphism of an algebraic space into a projective space. As usually the morphisms is given by sections of an invertible sheaf but we assume the base freeness only on some sufficiently small closed subspace (cf. Corollary 18) and treat the required morphism as a morphism of a special quasipolarized diagram – *skrepa* [Sh15, §2] – into polarized pair. A required subspace is actually topologically proper subspace if the sheaf is big. In positive characteristic there exists such a canonical (smallest) subspace which determines the semiampleness and stable free multiplicities.

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Skrepa is only a new word for an old concept. Let $\mathcal{C}$ be a category in which closed inclusions (or closed immersions) are defined. A skrepa in $\mathcal{C}$ is a diagram

$$\begin{array}{ccc} X & \supseteq & E \\ & & \downarrow \\ & & Q \end{array}$$

in $\mathcal{C}$ with a closed inclusion $E \subseteq X$. Usually we write this diagram in a line: $X \supseteq E \to Q$. A *morphism* of the skrepa is a morphism of the diagram into an object in $\mathcal{C}$, that is, a commutative diagram

$$\begin{array}{ccc} X & \supseteq & E \\ \downarrow & \swarrow & \downarrow \\ P & \leftarrow & Q \end{array} \quad \text{or} \quad \begin{array}{ccc} X & \supseteq & E \\ \downarrow & & \downarrow \\ P & \leftarrow & Q \end{array}.$$

Such a morphism is *universal* if it is a colimit of the skrepa. This colimit is also known as an amalgam(ated sum) or a pushout. Usually, in the paper, skrepas and their morphisms are considered in the category quasi-PSp of quasipolarized spaces.

Theorem 1 (Existence of a modification)**.** *Let X be a complete algebraic space and M be a nef invertible sheaf on X. Then, for any closed subspace E of X, there exists a sufficiently small closed subspace $F \subseteq X$ such that $F \supseteq E$ and, for any nef invertible sheaf L on X, any skrepa*

$$(X, M \otimes L) \supseteq (F, M \otimes L_{|F}) \xrightarrow{\varphi} (Q, H_Q), (Q, H_Q) \in \text{PSch},$$

has a morphism in quasi-PSp *into a polarized scheme. Moreover, there exists such a universal morphism (for the skrepa), it is fibered over the colimit and is a modification of the skrepa.*

More precisely, we can take F such that

$$\text{Supp}\, F = \mathbb{E}(M) \cup \text{Supp}\, E, \tag{1}$$

where $\mathbb{E}(M)$ denotes the exceptional locus of M. In particular, if X is irreducible, E has the codimension ≥ 1 in X and M is big then F has also the codimension ≥ 1 in X.

The universal morphism of the theorem depend on L and φ.

The existence of a morphism for the skrepa in the theorem implies the existence of a quasipolarized morphism $\psi\colon (X, M \otimes L) \to (P, H)$ into a polarized pair (P, H), that is, into a pair with a projective scheme P and its ample divisor H such that $\psi^*H \simeq M \otimes L$. This has important applications, e.g., the stable freeness and semiampleness of $M \otimes L$. This also implies (induces) existence of a morphism $\varphi\colon (F, M \otimes L)_{|F}) \to (Q, H_Q)$ into a polarized scheme (Q, H_Q) for any closed subspace $F \subseteq X$. Thus appearance of a skrepa with φ is natural and necessary. The theorem states the existence of such a skrepa which is sufficient for the existence of ψ. A subspace in this case should be not too small, e.g., not empty, because otherwise ψ may not exist. For the sufficiency, F should at least include $\mathbb{E}(M)$. If we would like to add extra assumptions on ψ (pinching) then we can add additional E to F. On the other hand, a large F, e.g., $F = X$ is not useful. Optimal F should be sufficiently small and this means (1). In other words, F is an infinitesimal neighborhood of $\mathbb{E}(M) \cup \operatorname{Supp} E$. Fortunately, we can find such a neighborhood of finite type. This is the main achievement of the theorem and of the paper. Unfortunately, there are no canonical and/or smallest with respect to inclusion F.

Corollary 1 (Cf. [Bir, Theorem 1.5])**.** *Let X be a complete algebraic space and M be a nef invertible sheaf on X. Then there exists a closed subspace $F \subseteq X$ such that*

$\operatorname{Supp} F = \mathbb{E}(M)$*; and,*
for any nef invertible sheaf L on X, $M \otimes L$ is semiample if and only if $M \otimes L_{|F}$ is semiample.

Moreover, $M \otimes L$ is stably free if and only if $M \otimes L_{|F}$ is stably free for all $m \gg 0$.

In general, the last statement does not hold for the freeness instead of the stable freeness. The stable freeness is weaker than the freeness but stronger than the semiampleness.

Recall that an invertible sheaf L (a Cartier divisor D) is *stably free* if $L^{\otimes n}$ (resp. nD) is base point free for every $n \gg 0$ [Sh15, §1.3]. E.g. ample L (resp. D) is always stably free but is not necessarily free. This implies the following categorical meaning of the stable freeness. Let (X, L) be a complete space with an invertible sheaf L. Then L is stably free if and only if there exists a morphism $\varphi\colon (X, L) \to (Y, M)$ such that M is ample and $L \simeq \varphi^*M$ (see Example 2, (3)). Actually there exists such a universal morphism and it is a contraction. To prove use a Stein factorization of any morphism given by free $L^{\otimes n}$ and Proposition 2 below. The freeness implies the stable freeness but not conversely. However, the stable freeness is more important for us. (See also for Semiample in Section 1 below.)

Proof. The last statement of the corollary is immediate by Theorem 1 with $E = \emptyset$ and the last paragraph about the stable freeness.

The main statement of the corollary follows from this for $L := M^{\otimes(m-1)} \otimes L^{\otimes m}$ with a sufficiently large positive integer m such that $(M \otimes L)^{\otimes m}$ and/or $(M \otimes L)^{\otimes m}{}_{|F}$ are stably free. (Cf. Example 2, (3) and Corollary 19.) ☐

If $\mathbb{E}(M) = \emptyset$ then M is stably free by the corollary. The corresponding contraction of X is an isomorphism and M is ample. This also follows by the Nakai-Moishezon criterion (see for Basic property of exceptional loci, (5) in Section 1) which we tacitly use in the paper. However it can be established by finite surjective reduction and dimensional induction (cf. the proof in [K90, Theorem 3.11]). Using a normalization the criterion can be reduced to normal, irreducible spaces. Moreover, using Chow's lemma the normal case can be reduced to the semiampleness of certain big and nef sheaves on normal projective varieties. In this situation we use Proposition 1 below with $L = \mathcal{O}_X$. A required morphism φ exists by dimensional induction. Notice that the criterion is stated for Cartier divisors in [H, Theorem 5.1] and over an algebraically closed field. However it holds for invertible sheaves or line bundles over arbitrary fields [K90, Theorem 3.11]. In its turn, Theorem 1 gives a polarized Artin modification if $\mathbb{E}(M \otimes L) = \emptyset$, or equivalently, φ is finite (see Example 4).

As we will see in Section 4 devoted to the proof of Theorem 1 the main piece of the theorem is its following projective case.

Proposition 1. *Let X be a projective scheme, E be an effective Cartier divisor, H, M be two invertible sheaves on X, and m be a positive integer such that*

$$\mathcal{O}_X(E) \simeq M^{\otimes m} \otimes H^{\vee};$$

H is ample; and
M is nef (and big).

Then there exists a positive integer e_0 such that, for any integer $e \geq e_0$ and any nef sheaf L on X, every skrepa

$$(X, M \otimes L) \supset (eE, M \otimes L_{|eE}) \xrightarrow{\varphi} (P, H_P), (P, H_P) \in \mathrm{PSch}, \varphi^* H_P \simeq M \otimes L_{|eE},$$

in quasi-PSp *has a morphism in* quasi-PSp *into a polarized scheme (Y, H_Y).*

Actually, there exists a universal morphism and it is a skrepa modification.

In the statement, $H^{\vee}$ denotes the dual sheaf to H (see for Category of quasipolarizations in Section 1). A proof will be given in Section 4.

Remarks 1. (1) In the proof a universal (stable) model (Y, H_Y) is constructed under the fixed restriction on a subscheme eE.

(2) Constructed stable Y is actually universal and a modification of X [Ar]. However, some other possible morphisms of the diagram in the statement are not necessary of this kind. Our proof combines two constructions of [Ar, Theorem (6.1) and Corollary (6.10)]. This is a very special projective and more precise (quasipolarized) case.

We verify also that $\mathbb{E}(M) \subseteq E$. So, the diagram of Proposition 1 can be treated as a fine one with only ample images of morphisms [Sh15, §5.5], that is, for any required morphism to (Y, H_Y) in quasi-PSp, H_Y is ample on the image of X. By [Sh15, Interpretation of ample epimorphisms, §5] the stable model in our proof is a colimit, an amalgam of the diagram.

(3) By Noetherian induction and according to the proof of Theorem 1, Corollary 1 and Proposition 1 there exists respectively minimal F and smallest e satisfying the statements and expected morphisms exist respectively for some larger F and every larger e. But smallest F may not exist (cf. with a conductor in Proposition 9 and Corollary 19).

This recalls [Bir, Theorem 1.4] and possibly its proof applies to our situation. Notice that our proof of Corollary 1 in the projective case is different from that of [Bir, Theorem 1.4]. We use a reduction to primary (projective) schemes.

In the positive characteristic, using Kollar's result [K97, Proposition 6.6], we can simplify F. However a payment for this is a divisibility of φ and only a sufficiency in the statement.

Corollary 2. *Under the assumptions of Theorem 1 suppose that the base field k has a positive characteristic p. Then there exists a positive power $q = p^r$ of p such that, for any nef invertible sheaf L on X, any skrepa*

$$(X, (M \otimes L)^{\otimes q}) \supseteq (E, (M \otimes L)^{\otimes q}{}_{|E}) \overset{\varphi^{\otimes q}}{\to} (P, H_P^{\otimes q}), \varphi^* H_P \simeq M \otimes L_{|E},$$

in quasi-PSp*, with a^q multiplication for $a \in k$ and with a quasipolarized space (P, H_P), has a morphism in* quasi-PSp *into a polarized scheme (Y, H_Y) if there exists a morphism of*

$$(F, M \otimes L_{|F}) \supseteq (E, M \otimes L_{|E}) \overset{\varphi}{\to} (P, H_P)$$

in quasi-PSp *into a polarized scheme (Q, H_Q), where F is the smallest closed subspace of X satisfying the equation (1) and containing E. Under the same assumption, if E is reduced then $F = \mathbb{E}(M) \cup E$ is also reduced.*

If k is perfect, the morphism $\varphi^{\otimes q}$ of skrepa and its morphism can be considered also over original k, that is, with multiplication by a.

Proof. We can suppose that E already includes $\mathbb{E}(M)$ and $F = E$ *in the corollary*. We can assume also that $(P, H_P) = (Q', H_{Q'})$ is polarized, that is, φ is the restriction of $(F, N_{|F}) \to (Q', H_{Q'})$ on E, where N denotes the tensor product $M \otimes L$ (cf. For simplicity in Section 4). Hence $\operatorname{Supp} F = \operatorname{Supp} E$ and $E \subseteq F$ is a universal homeomorphism (cf. [St, Lemma 55.51.6]) where F is *as in Theorem* 1 or in Corollary 19. Therefore [K97, Proposition 6.6] gives a commutative diagram

$$\begin{array}{ccccccc}
(E, N^{\otimes q}{}_{|E}) & \subseteq & (F, N^{\otimes q}{}_{|F}) & \stackrel{\psi}{\to} & (E^{(q)}, N^{(q)}_{|E}) & & \\
\varphi^{\otimes q} \downarrow & & & \searrow & \varphi^{(q)} \downarrow & & \\
(P, H_P^{\otimes q}) & & \to & & (P^{(q)}, H_P^{(q)}) & = & (Q, H_Q) \\
\downarrow & & & & \downarrow & & \\
\operatorname{Spec} k & & \to & & \operatorname{Spec} k & &
\end{array},$$

where q depends only on F (see Example 1). Its subdiagram

$$\begin{array}{ccccc}
(E, N^{\otimes q}{}_{|E}) & \subseteq & (F, N^{\otimes q}{}_{|F}) & & \\
\varphi^{\otimes q} \downarrow & & \downarrow & & \\
(P, H_P^{\otimes q}) & \to & (P^{(q)}, H_P^{(q)}) & = & (Q, H_Q) \\
& & \downarrow & & \\
& & \operatorname{Spec} k & &
\end{array}$$

is a morphism from $(F, N^{\otimes q}{}_{|F}) \supseteq (E, N^{\otimes q}{}_{|E}) \stackrel{\varphi^{\otimes q}}{\to} (P, H_P^{\otimes q})$ into a polarized scheme (Q, H_Q). So, by Theorem 1 or Corollary 19, a required morphism into (Y, H_Y) exists. Indeed,

$$N^{\otimes q} = M \otimes M^{\otimes (q-1)} \otimes L^{\otimes q}$$

holds and $M^{\otimes (q-1)} \otimes L^{\otimes q}$ is nef.

To verify that $\psi^* N^{(q)}_{|E} = N^{\otimes q}{}_{|F}$ we apply the universal property of [K97, Proposition 6.6] and Example 1 to the quasipolarized square:

$$\begin{array}{ccc}
(E, N^{\otimes q}{}_{|E}) & \subseteq & (F, N^{\otimes q}{}_{|F}) \\
\cap| & & \cap| \\
(X, N^{\otimes q}) & = & (X, N^{\otimes q}).
\end{array}$$

Note that $k = k^q$ if k is perfect. □

Corollary 3 (Cf. [K, Theorem 0.2])**.** *Let X be a complete algebraic space over a field of positive characteristic p and M be a nef invertible sheaf on X. Then, for any nef invertible sheaf L on X, $M \otimes L$ is semiample if and only if $M \otimes L_{|\mathbb{E}(M)}$ is semiample. Moreover, there exists a positive power $q = p^r$ of p such that $(M \otimes L)^{\otimes q}$ is stably free if $M \otimes L_{|\mathbb{E}(M)}$ is stably free.*

There exists a minimal power q for given L, M, and (larger) minimal q for all nef L.

Proof. Immediate by Corollary 2. □

Improved and special versions of Theorem 1 and of its corollaries will be given in Section 5.

Most of our results can be interpreted in terms of graded algebras of global sections (cf. [Sh15, Interpretation of ample epimorphisms, §5]). For instance, the universal property of Corollary 13 means that the maximal graded subalgebra in

$$\oplus_{i\ge 0} H^0(X, N^{\otimes i})$$

of extensions of global section $\varphi^*(s), s \in H^0(P, iH_P)$, over E is finitely generated and extensions globally generate sheaves $N^{\otimes i}$ for every $i \gg 0$. In its proof we use that some, possibly, nonmaximal algebra exists. However the most important results of the paper, e.g., Theorem 1 are related to the existence of such an algebra. The existence needs vanishing or injectivity theorems. This works well for projective schemes (cf. Proposition 1). However, for algebraic spaces, this is not applicable directly. We use a reduction to the projective case based on Chow's lemma and a descent in terms of colimits (conductors). Actually, for this, it would be sufficient a few results from the theory of colimits for algebraic spaces. In absence of required standard references on the subject, especially, in the category of quasipolarized spaces, we add a quite general and extensive introduction to such a theory in Sections 1–3. He hope that it has an independent interest. After that the proof of Theorem 1 is quite short, only 8 pages in Section 4.

Agreements All schemes, algebraic spaces and morphisms are over k, separated and of finite type over a field k if it is not stated opposite. The local properties for spaces are considered in their étale topology.

1. Quasipolarized morphisms

Quasipolarized spaces quasi-PSp(k) or simply quasi-PSp denote the category of quasipolarized spaces over a field k [Sh15, §1]. The objects of quasi-PSp are pairs $(X, L)/k$ where X is algebraic space and L is an invertible sheaf on X – a *quasipolarization* of X. A morphism of quasipolarized spaces $\varphi\colon (X_1, L_1) \to (X_2, L_2)$ is also a pair $(\varphi, \overset{\varphi}{\simeq})$ where $\varphi\colon X_1 \to X_2$ is a morphism of spaces and $\varphi^* L_2 \overset{\varphi}{\simeq} L_1$ is an isomorphism of sheaves. However we denote the morphism simply by φ.

Let $\psi\colon (X_2, L_2) \to (X_3, L_3)$ be another morphism in quasi-PSp with $\psi^* L_3 \overset{\psi}{\simeq} L_2$. Recall that, for the composition $\psi \circ \varphi$, the isomorphism $(\psi \circ \varphi)^* L_3 \overset{\psi\circ\varphi}{\simeq} L_1$ is the composition

$$(\psi \circ \varphi)^* L_3 = \varphi^* \psi^* L_3 \overset{\varphi^*\psi}{\simeq} \varphi^* L_2 \overset{\varphi}{\simeq} L_1.$$

This is important for the proof of Lemma 1 below.

Notice that every algebraic space X has a canonical quasipolarization $\mathcal{O}_X$, the structure sheaf of X. Moreover, it is compatible with morphisms of spaces: $\varphi^* \mathcal{O}_{X_2} = \mathcal{O}_{X_1}$. (Cf. Natural morphisms below.) This gives a canonical inclusion the category of algebraic spaces into quasi-PSp. The inclusion is not full.

Another possible canonical quasipolarization is $\omega_X = \wedge^n \Omega_{X/k}$, the canonical sheaf of X, if it is invertible. Every étale morphism respects this quasipolarization. A dualizing sheaf ω_X° is not pretended to be canonical even if it is invertible [H, Definition p. 241]. $\mathbb{Q}$-versions of the canonical quasipolarization, e.g., $\omega_X^{\otimes m \vee\vee}$, are also canonical but not good.

The category quasi-PSp contains some important relative subcategories (e.g., see for Finite diagram of spaces in Section 3). The objects of such a relative category quasi-PSp $/(X, L)$ are quasipolarized morphisms $(X_1, L_1) \to (X, L)$ and the morphisms are relative over (X, L). The category is canonically equivalent to its naturally quasipolarized subcategory with only natural morphisms by the proof of Lemma 1 below. Note that all algebraic spaces are relative over k or $T = \operatorname{Spec} k$ but only without quasipolarization. Indeed, every quasipolarization on T is isomorphic to k, trivial. By Lemma 1 the only possible relative category over (T, k) is equivalent to the subcategory of algebraic spaces with trivial quasipolarization $\mathcal{O}_X$ on every space X.

Category of quasipolarizations Let X be an algebraic space. Then its quasipolarizations form the full subcategory quasi-PSp(X). Its objects are quasipolarizations L on X with the isomorphisms $L \simeq M$ as morphisms. Tensor products and dualizing preserve this subcategory. This operations are not canonical (unique) but defined up to a canonical isomorphism as inverse image φ^* on sheaves. The latter can be considered as a functor $\varphi^*\colon$ quasi-PSp$(X_2) \to$

quasi-PSp$(X_1), \varphi\colon X_1 \to X_2$. The functor preserves tensor products, dualizing and the structure sheaf.

Notice also the following exact sequence

$$1 \to \operatorname{Aut} L \to \operatorname{Aut}(X, L) \to \operatorname{Aut} X \to 1.$$

In addition, $\operatorname{Aut} L = H^0(X, \mathcal{O}_X^\times)$, the global unites.

Example 1 ([SGA5, Expose XV, §1, n° 1,2]). Fix $q = p^r$ where $p = \operatorname{char} k$. The qth Frobenius functor

$$\mathrm{Sp}(k) \to \mathrm{Sp}(k), X \xrightarrow{\varphi} Y \mapsto X^{(q)} \xrightarrow{\varphi^{(q)}} Y^{(q)}$$

has a natural extension to quasipolarised spaces over k

$$\text{quasi-PSp}(k) \to \text{quasi-PSp}(k), (X, L) \xrightarrow{\varphi} (Y, M) \mapsto (X^{(q)}, L^{(q)}) \xrightarrow{\varphi^{(q)}} (Y^{(q)}, M^{(q)})$$

of algebraic spaces over k. Indeed, the quasipolarization $L^{(q)}$ is given by a sheaf with transition functions $f^{(q)}$ where f are transition functions for L. Moreover, there exists a natural transformation F^q of the qth tensor power functor of quasi-PSp into the Frobenius functor:

$$F^q\colon (X, L^{\otimes q}) \to (X^{(q)}, L^{(q)}).$$

Usually, to consider the last morphism as a (k-linear) morphism over k, multiplication of structure sheaf $\mathcal{O}_X$ by $a, a \in k$, has to be changed on multiplication a^q. For a perfect field k we can do opposite and change multiplication of $\mathcal{O}_{X^{(q)}}$ by $a^{1/q}$ because the Frobenius $F^q\colon \operatorname{Spec} k \to \operatorname{Spec} k, F^{q*}a = a^q$, is an isomorphism.

Recall that, for every fixed positive integer n, there exists the nth tensor power functor

$$\text{quasi-PSp}(k) \to \text{quasi-PSp}(k), (X, L) \xrightarrow{\varphi} (Y, M) \mapsto (X, L^{\otimes n}) \xrightarrow{\varphi^{\otimes n}} (Y, M^{\otimes n}).$$

Nef and big The nef property of an invertible sheaf L on a space X means its semipositivity [Sh15, Remark 1]: f^*L has nonnegative degree for every complete curve $f\colon C \to X$.

The *big* property of L on X means the big property on an irreducible component of X [Sh15, Remark 1, flavor (3)].

Polarized spaces A quasipolarization L on a space X is a *polarization* if L is ample. A polarized space is a quasipolarized space (X, L) with such a polarization. The polarized spaces form a full subcategory PSp in quasi-PSp. If X is complete and has a polarization then X is projective, and a scheme. Hence complete polarized spaces form the full subcategory PSch in quasi-PSp, PSp.

Semiample An invertible sheaf L on space X is *semiample* if there exists a proper morphism $\varphi\colon X \to Y$ and an ample invertible sheaf M on Y such that $L \simeq_{\mathbb{Q}} \varphi^* M$. The $\mathbb{Q}$-isomorphism $\simeq_{\mathbb{Q}}$ means that there exist a positive integer m and an isomorphism $L^{\otimes m} \simeq \varphi^* M^{\otimes m}$. Thus we can convert the morphism φ into a morphism of quasipolarized spaces $(X, L^{\otimes m}) \to (Y, M^{\otimes m})$ where $(Y, M^{\otimes m})$ belongs to PSp. The existence of such a morphism for $m = 1$ implies a slightly stronger property that $L^{\otimes n}$ is generated by global sections for every $n \gg 0$ [H, Theorem 7.6, Chapter II], which is well-known is a *stable* (base point) *freeness*. E.g., every polarization is stably free and this implies the same property for some power $L^{\otimes n}$ of semiample L. In particular, some power $L^{\otimes n}$ is generated by global section – (base point) *free*. The converse holds for complete X by [H, Theorem 7.1 and Corollary 4.8 (e), Chapter II]. Without the completeness of X we get a separated morphism $\varphi\colon X \to Y$ and an invertible sheaf M on Y such that $L \simeq_{\mathbb{Q}} \varphi^* M$. So, for complete X the stable freeness implies semiampleness. But not every semiample sheaf is stably free, e.g., sheaves corresponding to torsions in the Picard group.

Natural morphisms A quasipolarized morphism $\varphi\colon (X, L) \to (Y, M)$ is *natural* or *naturally* quasipolarized if the structure isomorphism of quasipolarizations is identical: $L = \varphi^* M$. A diagram in quasi-PSp is *natural* or *naturally* quasipolarized if so does every arrow of the diagram. A morphism φ is *divisorially* quasipolarized if the quasipolarizations $L = \mathcal{O}_X(E), M = \mathcal{O}_Y(F)$ are divisorial with Cartier divisors E, F respectively on X, Y, and $E = \varphi^* F$ holds. So, $\varphi^* M = \varphi^* \mathcal{O}_Y(F) = \mathcal{O}_X(\varphi^* F) = \mathcal{O}_X(E) = L$ hold. A diagram is divisorially quasipolarized if so does every its arrow. The next subtlety is important in view of Examples 1 below.

Lemma 1. *Let D be a quasipolarized diagram such that it has a morphism into a quasipolarized space in* quasi-PSp*. Then D is isomorphic to a natural diagram.*

Moreover, if the morphism goes to a quasiprojective space then D is isomorphic to a divisorially polarized diagram.

Recall that a morphism of a diagram into an object in a category is a morphism of the diagram into the constant diagram of the same type for the object (see for Colimit of a diagram in Section 3 below). The morphism can be treated also a diagram of the same type in the relative category over the object.

Proof. Let $\delta\colon D \to (T,O)$ be a morphism in quasi-PSp. Denote by $D_=$ the diagram D with a natural quasipolarization induced from (T,O): a vertex $(X,L) \in D$ goes to the vertex $(X, \delta_X^* O)$ where $\delta_X\colon (X,L) \to (T,O)$ is a component of δ on X. This gives also an isomorphism $(X,L) \simeq (X, \delta_X^* O)$ which is identical on X and determined by δ_X on sheaves: $L \overset{\delta_X}{\simeq} \delta_X^* O$. Note that by construction $D_=$ is a diagram over (T,O) with a natural morphism $\delta_=\colon D \to (T,O)$, which coincides with δ on spaces, and the isomorphism $D \simeq D_=$ is actually is an isomorphism of diagrams over (T,O).

We consider the isomorphism as a natural transformation of $D/(T,O)$ into $D_=/(T,O)$. An arrow of D

$$\begin{array}{ccc} (X,L) & \overset{\varphi}{\to} & (Y,M) \\ \delta_X \searrow & & \delta_Y \swarrow \\ & (T,O) & \end{array}$$

goes to the arrow of $D_=$

$$\begin{array}{ccc} (X,\delta_X^* O) & \overset{\varphi_=}{\to} & (Y,\delta_Y^* O) \\ \delta_{=X} \searrow & & \delta_{=Y} \swarrow \\ & (T,O) & \end{array}.$$

The morphism $\varphi_=$ is the same on spaces but is natural (identical) on sheaves. Note for this that $\delta_Y \circ \varphi = \delta_X$. Hence $\varphi^* \delta_Y^* O = \delta_X^* O$. The transformation is natural: the following square is commutative

$$\begin{array}{ccc} (X,L) & \overset{\varphi}{\to} & (Y,M) \\ S| & & S| \\ (X,\delta_X^* O) & \overset{\varphi_=}{\to} & (Y,\delta_Y^* O) \end{array}$$

in quasi-PSp. Immediate on spaces by construction. Immediate for sheaves by construction and definition: the structure isomorphism $L \simeq \varphi^* \delta_Y^* O$ for the composition $(X,L) \overset{\varphi}{\to} (Y,M) \to (Y,\delta_Y^* O)$ is the composition $L \overset{\varphi}{\simeq} \varphi^* M \overset{\varphi^*\delta_Y}{\simeq} \varphi^* \delta_Y^* O$ which is $L \overset{\delta_X}{\simeq} \delta_X^* O$, the structure isomorphism because $\delta_X = \delta_Y \circ \varphi$ in quasi-PSp. Similarly, $L \simeq \varphi^* \delta_Y^* O$ for $(X,L) \to (X,\delta_X^* O) \overset{\varphi_=}{\to} (Y,\delta_Y^* O)$ is the composition $L \overset{\delta_X}{\simeq} \delta_X^* O = \varphi^* \delta_Y^* O$ which is also $L \overset{\delta_X}{\simeq} \delta_X^* O$.

Any identical morphism of D goes to that of in $D_=$. The composition of arrows $\varphi\colon (X,L)\to(Y,M)$ and $\psi\colon (Y,M)\to(Z,N)$ from D goes to the composition $\psi_=\circ\varphi_=$ again by construction.

If T is quasiprojective then $O\overset{\varepsilon}{\simeq}\mathcal{O}_T(E)$ for a rather general Cartier divisor E. The latter implies that every δ^*E is a Cartier divisor and $\delta_X^*O\overset{\delta^*\varepsilon}{\simeq}\mathcal{O}_X(\delta_X^*E)$. This gives an isomorphism of the natural diagram $D_=$ with the divisorial D_E: the vertices of D_E are $(X,\mathcal{O}_X(\delta_X^*E))$ and the arrows are

$$\varphi_E\colon (X,\mathcal{O}_X(\delta_X^*E))\to(Y,\mathcal{O}_Y(\delta_Y^*E)), \varphi^*\delta_Y^*E=\delta_X^*E. \qquad \square$$

There are diagrams in quasi-PSp without a natural quasipolarization up to an isomorphism. In particular, they do not have colimits in quasi-PSp. On the other hand, a natural quasipolarization does not guarantee the existence of a colimit.

Examples 1. (1) Consider, for example, a diagram

$$\varphi,\psi\colon (X,L)\underset{\to}{\overset{\to}{}}(Y,M)$$

such that $L=\varphi^*M$ but a structure isomorphism $L\simeq\psi^*M$ is not identical. Usually, this diagram is not isomorphic to a diagram with naturally quasipolarized morphisms. Take $Y=X$, $\psi=\varphi$ on spaces and $\psi^*M=\varphi^*M=L\overset{\psi}{\simeq}L=\varphi^*M$ given by a scalar multiplication on $\varepsilon\neq 1\in k^\times$. (Actually, any automorphism of L has a scalar form if X is complete; see for Category of quasipolarizations above.) This diagram is not commutative in quasi-PSp. On the other hand, any isomorphic naturally polarized diagram is commutative on spaces and thus commutative in quasi-PSp. This is not surprising because taking a colimit with quasipolarization should glue sections but the automorphism of multiplication by ε fixes zero – the only invariant section.

(2) The existence of natural (quasi)polarization does not guarantee a divisorial (quasi)polarization. So, by Lemma 1 the diagram below has no quasipolarized morphisms and a colimit.

Suppose that M is a 2-torsion: $M^{\otimes 2}\simeq\mathcal{O}_Y$, $\operatorname{char}k\neq 2$, but $M\not\simeq\mathcal{O}_Y$, $\varphi^*M=L\simeq\mathcal{O}_X$, and $\psi=\chi\circ\varphi$ where $\chi\colon X\to X$ is an endomorphism of degree >1. Then the diagram is not isomorphic to a divisorial one. Indeed, the latter has the form

$$\varphi,\psi\colon (X,E)\underset{\to}{\overset{\to}{}}(Y,F),$$

where E, F are Cartier divisors on X, Y respectively and $\varphi^*F = \psi^*F$. This is impossible because $M \simeq \mathcal{O}_Y(F)$ holds and F is not effective. Hence $\varphi^*F = E$ is not effective too and $E = \psi^*F = \chi^*\varphi^*F = \psi^*E$ but χ has a positive degree. (For this we assume that φ, χ and ψ are surjective.)

E.g., the isomorphism $\varphi^*M \simeq \mathcal{O}_X$ holds, if X, Y are elliptic curves and φ is an isogeny of degree 2. (For χ we can take the isogeny $x \mapsto 2x$.) Then $\chi^*\mathcal{O}_X = \mathcal{O}_X$ and the diagram is natural, except for an almost natural isomorphism $\varphi^*M \simeq \mathcal{O}_X$. In our situation we can take $M = \Omega_Y(p-q)$ where $p, q \in Y$ are two closed points such that $p \ominus q$ is a 2-torsion on Y and $\ominus$ is a subtraction on the elliptic curve Y. Equivalently, M is a required 2-torsion: $M^{\otimes 2} \simeq \mathcal{O}_Y$. Actually, there exists a canonical isomorphism $= \mathcal{O}_X$ according to the following. For the inverse image $L = \varphi^*M$, a canonical isomorphism $\varphi^*\Omega_Y(p-q) = \Omega_X(e_1+e_2-e_3-e_4)$ holds, where $\varphi^*p = e_1+e_2, \varphi^*q = e_3+e_4$ and $e_1, \dots, e_4$ are the 2-torsions on the elliptic curve X, assuming $e_1 = 0$ (cf. Example 2, (4) in Section 3). Note that $\Omega_X(e_1+e_2-e_3-e_4)$ has a global almost canonical generator ω: $\operatorname{res}_{e_1}\omega = 1$. Since $\operatorname{res}_{e_2}\omega = -1$ and e_1, e_2 are permutable, the section ω is canonical up to a sign $+/-$. However, it is enough to verify that

$$\chi^*\Omega_X(e_1+e_2-e_3-e_4) = \Omega_X(e_1+e_2-e_3-e_4).$$

The canonical isomorphism is given by multiplication on the canonically defined rational function $\omega/\varphi^*\omega$.

A tensor product with inverse images of a polarization on a (unquasipolarized) colimit gives a polarized version of the example. It can be constructed also from Examples 2, (1) and (4) below. In both examples the 2nd tensor power of quasipolarizations gives a diagram with a quasipolarized colimit. This is not typical in general according to Examples 7 and 9.

The following results about behaviour of quasipolarizations, Cartier divisors and closed subspaces with respect to a contraction are quite elementary and recall generalities from [H, Section 6, Chapter II] and [EGA, §21]. So, we will be sketchy.

Proposition 2. *Let $\varphi\colon X \twoheadrightarrow Y$ be a contraction of algebraic spaces. Then the invertible sheaves L on X, locally trivial over Y, form a subcategory of* quasi-PSp X *(resp. subgroup of* $\operatorname{Pic} X$*) closed under the tensor product and dualization. The subcategory (resp. subgroup) is equivalent (resp. canonically isomorphic) to the category* quasi-PSp Y *(resp.* $\operatorname{Pic} Y$*) under φ_* and φ^*. The equivalence preserves tensor product and dual of sheaves.*

So, any contraction has a unique canonical quasipolarization on Y coming from L on X if it exists (cf. Stein factorization in Section 2 below). The existence holds if and only if L is locally trivial over Y, that is, $\varphi\colon (X, L) \to Y$ can be quasipolarized locally over Y.

Proof. Note at once that the category of quasipolarizations quasi-PSp on X and its subcategory of locally trivial over Y quasipolarizations are classes, not sets, and operations (tensor product, dualization) are defined up to a canonical isomorphism. If L, M are invertible sheaves on X which are locally trivial over Y then $L \otimes M, L^\vee$ do so. Respectively, the locally trivial property gives a subgroup in $\operatorname{Pic} X$.

The locally trivial property means that L locally over Y is isomorphic to $\mathcal{O}_X$. So, $\varphi_* L$ is invertible on Y. Indeed, since both properties are local over Y and φ_* preserves isomorphisms of sheaves, we can suppose that $L = \mathcal{O}_X$. In this case $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$ holds by definition of a contraction. By the similar reason the locally trivial property holds for $\varphi^* L$ (for every morphism) where L is invertible on Y. Moreover, this implies that φ_*, φ^* give an equivalence (are inverse up to natural isomorphisms).

A complete proof needs the *projection formula*: for any invertible sheaf M on Y and any sheaf L of $\mathcal{O}_X$-modules on X,

$$\varphi_*(L \otimes \varphi^* M) = \varphi_* L \otimes M,$$

and the *kernel computation*: for an invertible sheaf L on X locally trivial over Y,

$$\varphi_* L = \mathcal{O}_Y \Rightarrow L = \mathcal{O}_X.$$

In particular, $M = \varphi_* L$ is a unique invertible sheaf such that $\varphi^* M = L$. The last isomorphisms is induced by the canonical one $\varphi^* \varphi_* L \to L$, or $=$ instead of $\to$. □

Proposition 3. *Let $\varphi\colon X \twoheadrightarrow Y$ be a contraction of algebraic spaces. Then there exists a canonical* 1*-to-*1 *correspondence between the lifting Cartier divisor on Y and the vertical Cartier divisor on X. Those divisors form subgroups respectively in the groups of Cartier divisors of Y and of X. The correspondence is a homomorphism of the subgroups given by φ^* and φ, and agrees with forming fractional ideals (ideal quasipolarizations):*

$$\mathcal{O}_X(\varphi^* D) = \varphi^*(\mathcal{O}_Y(D)), \mathcal{O}_Y(\varphi(D)) = \varphi_* \mathcal{O}_X(D)$$

and even with forming closed subspaces for effective divisors. More precisely, a vertical Cartier divisor D on X is effective if and only if its corresponding *lifting divisor $\varphi(D)$ is effective; an effective or anti-effective Cartier divisor D on X is vertical if and only if $\mathcal{O}_X(D)$ is locally trivial over Y. For a vertical effective Cartier divisor D on X, considered as a closed subspace of X, $\mathcal{I}_D = \mathcal{O}_X(-D)$ is its ideal sheaf, $\varphi(D)$ is a subspace of Y for the Cartier divisor $\varphi(D)$:*

$$\varphi_*\mathcal{I}_D = \mathcal{I}_{\varphi(D)} = \mathcal{O}_Y(-\varphi(D)).$$

Respectively, if D is a lifting effective Cartier divisor on Y, considered as a closed subspace of Y, $\varphi^{-1}D = \varphi^*D$ holds for φ^*D considered as a closed subspace of X:

$$\mathcal{I}_{\varphi^{-1}D} = \mathcal{I}_{\varphi^*D} = \mathcal{O}_X(-\varphi^*D) = \varphi^*\mathcal{O}_Y(-D) = \varphi^*\mathcal{I}_D = (\varphi^{-1}\mathcal{I}_D)\mathcal{O}_X.$$

An effective Cartier divisor D on Y is lifting if and only if the inverse image of its ideal sheaf $\varphi^{-1}(\mathcal{I}_D)\mathcal{O}_X \subseteq \mathcal{O}_X$ is invertible on X, that is, its space preimage $\varphi^{-1}D$ is Cartier.

Proof-Explanation. The Cartier divisors D on X (resp. on Y) can be identified with their fractional ideal invertible sheaves $\mathcal{O}_X(D)$ (resp. $\mathcal{O}_Y(D)$). Effective Cartier divisors correspond to *principal* ideal sheaves locally *generated by nonzero divisors*. A Cartier divisor $D = \{(U_i, f_i)\}$ on Y, for an affine covering $\{U_i\}$ of Y, is *lifting* if φ^*D is well-defined: every $f_i = a_i/b_i$ where a_i, b_i are nonzero divisors in $\Gamma(U_i, \mathcal{O}_Y)$ such that φ^*a_i and φ^*b_i are nonzero divisors in $\Gamma(\varphi^{-1}U_i, \mathcal{O}_X)$ too. So, $\varphi^*D = \{(\varphi^{-1}U_i, \varphi^*f_i)\}$ is well-defined with $\varphi^*f_i = \varphi^*a_i/\varphi^*b_i$.

Warning 1. The covering $\{U_i\}$ of Y should be affine. For a Zariski open subset U in Y and an element $a \in \Gamma(Y, \mathcal{O}_Y)$, to be a nonzero divisor means that a is such a divisor locally for every point of U. The nonzero divisors of $\Gamma(U, \mathcal{O}_Y)$ are among them but not only.

Equivalently, those Cartier divisors correspond to a sheaf of partial rings of fractions $\mathcal{R}\mathrm{at}_\varphi Y$ on Y instead of the total one $\mathcal{R}\mathrm{at}\, Y$. The multiplicative set subsheaf $\mathcal{S}_\varphi \subset \mathcal{O}_Y$ locally is given by the nonzero divisors $a \in \Gamma(U, \mathcal{O}_Y), U \subseteq Y$, such that $\varphi^*a \in \Gamma(\varphi^{-1}U, \mathcal{O}_Y)$ are also nonzero divisors; to be nonzero divisor means that this holds locally in every point of U. For affine U, $\Gamma(U, \mathcal{S}_\varphi)$ is the set of usual nonzero divisors in $\Gamma(U, \mathcal{O}_Y)$. The Cartier divisors corresponding to $\mathcal{S}_\varphi$ form a subgroup $\mathrm{CDiv}_\varphi Y$ in the group of Cartier divisors on Y:

$$\begin{aligned}(\mathcal{R}\mathrm{at}_\varphi Y)^*/\mathcal{O}_Y^* \subseteq (\mathcal{R}\mathrm{at}\, Y)^*/\mathcal{O}_Y^* \text{ and } \mathrm{CDiv}_\varphi Y = \Gamma(Y, (\mathcal{R}\mathrm{at}_\varphi Y)^*/\mathcal{O}_Y^*)\\ \subseteq \mathrm{CDiv}\, Y = \Gamma(Y, (\mathcal{R}\mathrm{at}\, Y)^*/\mathcal{O}_Y^*).\end{aligned}$$

Respectively, the vertical Cartier divisors on X are images $\varphi^* D$ of lifting divisors from Y. The condition is actually local over Y. That is, $D \in \operatorname{CDiv} X$ is *vertical* if D is vertical locally over Y: for every open set (or its covering) U in Y, there exists a lifting Cartier divisor D' on U such that

$$D_{|\varphi^{-1}U} = \varphi^* D'.$$

Actually, D' is unique and

$$D' = \varphi_{|U}(D_{|\varphi^{-1}U})$$

because φ is a contraction (cf. determination of the direct image for fractional ideal sheaves below). This proves that the vertical divisors form a subgroup $\operatorname{CDiv}^\varphi X$ of $\operatorname{CDiv} X$, $\operatorname{CDiv}^\varphi X = \varphi^* \operatorname{CDiv}_\varphi Y \simeq \operatorname{CDiv}_\varphi X$, the isomorphism is given by homomorphisms φ^*, φ and

$$\varphi(D)_{|U} = D' = \varphi_{|U}(D_{|\varphi^{-1}U}).$$

If $D \in \operatorname{CDiv}_\varphi Y$ is a lifting Cartier divisor on Y, then by definition

$$\mathcal{O}_X(\varphi^* D) = \varphi^*(\mathcal{O}_Y(D))$$

If additionally D is effective then $\varphi^* D$ is also effective, $\varphi^{-1} D = \varphi^* D$ holds, where $\varphi^{-1} D$ is the closed space preimage (see for Space preimage in Section 2 below) and $D, \varphi^* D$ are considered also as closed subspaces of Y, X respectively:

$$\mathcal{I}_{\varphi^{-1}D} = \mathcal{I}_{\varphi^* D} = \mathcal{O}_X(-\varphi^* D) = \varphi^* \mathcal{O}_Y(-D) = \varphi^* \mathcal{I}_D = (\varphi^{-1}\mathcal{I}_D)\mathcal{O}_X.$$

Similarly the lifting criterion is immediate by definition (see for Space preimage in Section 2). The statements of this paragraph hold for any morphism φ.

However the contraction property of φ is import for the following. Let D be now a Cartier divisor on X. If D is vertical then by definition $D = \varphi^*\varphi(D)$ holds, $\mathcal{O}_X(D)$ is locally trivial over Y and by Proposition 2

$$\mathcal{O}_Y(\varphi(D)) = \varphi_* \mathcal{O}_X(D).$$

This follows also from local determination of the direct image. Replacing Y by an open affine space of its covering, it is sufficient to find $\Gamma(Y, \varphi_* \mathcal{O}_X(D))$ and to suppose that $\varphi(D)$ is given globally on Y by a section $a \in \Gamma(Y, \mathcal{R}\mathrm{at}_\varphi Y)$. Then

$$D = (X, b), b = \varphi^* a \in \Gamma(X, \mathcal{R}\mathrm{at}\, X).$$

Take any affine covering $\{U_i\}$ of X and any section $s \in \Gamma(X, \mathcal{O}_X(D)) = \Gamma(Y, \varphi_* \mathcal{O}_X(D))$. By construction $s_{|U_i} = f_i(b_{|U_i}), f_i \in \Gamma(U_i, \mathcal{O}_X)$. Since $b_{|U_i \cap U_j}$ is not a zero divisor, then the gluing condition

$$f_{i|U_i\cap U_j} b_{|U_i\cap U_j} = (f_i(b_{|U_i}))_{|U_i\cap U_j} = (f_j(b_{|U_j}))_{|U_i\cap U_j} = f_{j|U_i\cap U_j} b_{|U_i\cap U_j}$$

implies that

$$f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}.$$

Hence there exists a regular function $f \in \Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$ such that $f_{|U_i} = f_i$ and $s = fb$ because φ is a contraction and $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$. Since b, a are not zero divisors, we can canonically identify sections s with the sections $t = fa \in \Gamma(Y, \mathcal{O}_Y(\varphi(D))$. In other words, we established that

$$\Gamma(Y, \varphi_* \mathcal{O}_X(D)) = \Gamma(Y, \mathcal{O}_Y(\varphi(D))) \text{ and } \varphi_* \mathcal{O}_X(D) = \mathcal{O}_Y(\varphi(D)).$$

If $D \in \mathrm{CDiv}^\varphi X$ is effective and considered as a closed subspace then $-D$ is anti-effective and $\mathcal{I}_D = \mathcal{O}_X(-D)$. Hence

$$\mathcal{I}_{\varphi(D)} = \varphi_* \mathcal{I}_D = \varphi_* \mathcal{O}_X(-D) = \mathcal{O}_Y(\varphi(-D)) = \mathcal{O}_Y(-\varphi(D)),$$

where the low script $\varphi(D)$ is the space image of D (see for Space image in Section 2 below), is also an ideal sheaf for the subspace of $\varphi(D)$ and the last subspace is an effective Cartier divisor.

Finally, suppose that D is an effective or anti-effective Cartier divisor on X and $\mathcal{O}_X(D)$ is locally trivial over Y. We contend that D is vertical. It is sufficient to consider the case with anti-effective D. Then $\mathcal{O}_X(D) = \mathcal{I}_{-D}$ is the ideal sheaf of the effective Cartier divisor $-D$. By construction, $\varphi_* I_{-D}$ is an ideal sheaf of Y. By Proposition 2 it is fractional invertible: there exists an anti-effective Cartier divisor D' on Y such that

$$\varphi_* \mathcal{I}_{-D} = \varphi_* \mathcal{O}_X(D) = \mathcal{O}_Y(D') = \mathcal{I}_{-D'}.$$

Moreover, the same proposition implies that $D' = \varphi(D)$ is lifting because the ideal sheaf

$$\varphi^* \mathcal{I}_{-D'} = \varphi^* \mathcal{O}_Y(D') = \varphi^* \varphi_* \mathcal{I}_{-D} = \mathcal{I}_{-D} = \varphi^{-1}(\varphi_* \mathcal{I}_{-D}) \mathcal{O}_X$$

is invertible and corresponds to the Cartier divisor D; the sheaf inverse image of $\mathcal{I}_{-D'}$ is canonically isomorphic to its inverse image as for an ideal sheaf [H, Warning 7.12.2, Chapter II]. The statement of the paragraph does not hold for all Cartier divisors D on X. In general, if $\mathcal{O}_X(D)$ is locally trivial over Y then D is *locally trivial* over Y and vertical *up to a linear equivalence.* □

Corollary 4. *Let $\varphi\colon X \twoheadrightarrow Y$ be a contraction of primary algebraic spaces and E be a(n effective) Cartier divisor on Y. Then E is lifting and $\varphi(\varphi^* E) = E$.*

Proof. Immediate by Proposition 3 because for every (effective) Cartier divisor E, $\varphi^* E$ is a vertical (resp. effective) Cartier divisor by the primary property of X, Y and [Sh15, Proposition 3, 1)]. In other words, $\mathrm{CDiv}_\varphi Y = \mathrm{CDiv}\, Y$ holds. Actually, it is enough for this that φ is dominant, equivalently, an epimorphism of primary spaces. □

Exceptional locus Let (X, L) be a quasipolarized algebraic space. Recall that a closed irreducible subspace Y of X is called *exceptional* with respect to L if $L_{|Y}$ is not big. Respectively, the *exceptional* locus $\mathbb{E}(L)$ of L is the smallest reduced closed subspace of X which includes any integral exceptional with respect to L closed subspace (or subset) of X. For a Cartier divisor D, its exceptional locus $\mathbb{E}(D)$ is the exceptional locus of $\mathcal{O}_X(D)$: $\mathbb{E}(D) = \mathbb{E}(\mathcal{O}_X(D))$. The restriction $D_{|Y}$ is not defined for every closed subspace (cf. lifting divisors in Proposition 3 above) but it is well-defined for every integral $Y \not\subseteq \mathrm{Supp}\, D$. So, we prefer to state all required results for invertible sheaves. Note for this also, that an intersection form for invertible sheaves is well-defined and agrees with those for Cartier divisors if X is complete. In particular, for complete X and its quasipolarization L, the selfintersection

$$L^{\dim X}$$

is well-defined. The selfintersection up to the multiple $1/\dim X!$ is the leading coefficient for the Riemann-Roch-Grothendieck-Hirzebruch polynomial and, actually, for nef and big L, the growth of multisections of L has monomial asymptotic of degree $\dim X$ with the volume

$$\mathrm{Vol}(L) = L^{\dim X} / \dim X!$$

as the leading coefficient.

Basic properties of exceptional loci:

(1) $\mathbb{E}(L) = \emptyset$ or $\dim \mathbb{E}(L) \geq 1$.

(2) For every positive integer l, $\mathbb{E}(L^{\otimes l}) = \mathbb{E}(L)$.

(3) For every closed subspace $Y \subseteq X$, $\mathbb{E}(L_{|Y}) \subseteq \mathbb{E}(L), Y \cap \mathbb{E}(L)$ and, if $Y_{\mathrm{red}} = X_{\mathrm{red}}$, then $\mathbb{E}(L_{|Y}) = \mathbb{E}(L)$.

(4) If $X = \cup X_i$ is a finite decomposition into irreducible closed subspaces X_i (see for Space logic in Section 2 below) then $\mathbb{E}(L) = \cup \mathbb{E}(L_{|X_i})$.

(5) *Nakai-Moishezon criterion*: for complete X, $\mathbb{E}(L) = \emptyset$ if and only if L is ample. See for the scheme case in [H, Theorem 5.1, Appendix A] and the general case in [K90, Theorem 3.11].

(6) For any morphism $(X, L) \to (Y, M)$ in quasi-PSp, $\mathbb{E}(X/Y) \subseteq \mathbb{E}(L)$, where $\mathbb{E}(X/Y)$ is the exceptional locus of X/Y, and $=$ holds if M is ample, equivalently, L is semiample and $X \to Y$ is a *related* morphism, that is, the morphism can be quasipolarized with an ample sheaf M on Y. Recall that the *exceptional* locus $\mathbb{E}(\varphi) = \mathbb{E}(X/Y)$ of a *morphism* $\varphi\colon X \to Y$ is the union of contracted curves.

(7) For any proper dominant on the generic points morphism $\varphi\colon Y \to X$, $(\varphi^{-1}(\mathbb{E}(L)))_{\mathrm{red}} \subseteq \mathbb{E}(\varphi^* L)$ and $\mathbb{E}(L) \subseteq \varphi(\mathbb{E}(\varphi^* L))$; $=$ holds in both cases if φ is finite. The dominant property means that φ is dominant (surjective on generic points) and every generic point of Y goes (on)to a generic point of X.

A proof of the above properties, except for (5), follows from the definition of $\mathbb{E}$. A proof of (7) is also not easy and uses a reduction to a proper birational morphism φ of integral spaces. In the latter case the big property is a birational invariant: L is big if and only if $\varphi^* L$ is big (cf. Step 4 in the proof of next proposition).

Proposition 4. *Let X be an algebraic space and L be an invertible sheaf (or divisor) on it. Then $\mathbb{E}(L)$ (resp. $\mathbb{E}(D)$) is a finite union of integral closed subspaces $E \subseteq X$ with $L_{|E}$ (resp. with $\mathcal{O}_X(D)_{|E}$) being not big. If X is complete and L (resp. D) is nef then $\mathbb{E}(L)$ (resp. $\mathbb{E}(D)$) is the null locus of L (resp. of D) and the irreducible components Y of $\mathbb{E}(L)$ (resp. of $\mathbb{E}(D)$) are numerically exceptional with respect to L (resp. to D):*

$$L_{|Y}^{\dim Y} = 0 \quad (\textit{resp. } D^{\dim Y} Y = 0).$$

Possibly, it is true for $\mathbb{R}$-sheaves (and resp. for Cartier $\mathbb{R}$-divisors; cf. [Sh96, Lemma 6.17]).

Proof. Step 1. By the property (3) above, $\mathbb{E}(L) = \mathbb{E}(L_{|X_{\mathrm{red}}})$ holds and we can suppose that X is reduced.

Step 2. By the property (4) above, we can suppose that X is irreducible

Step 3. We can suppose that L is big. Otherwise $\mathbb{E}(L) = X$.

Step 4. We can suppose that L is quasiprojective. Indeed, by Chow's lemma [Kn, Theorem 3.1, Chapter IV] there exists a proper birational map $f\colon Y \to X$ where Y is quasiprojective, integral. Hence, by the property (7), $\mathbb{E}(M) \subseteq f(\mathbb{E}(f^*M)) \subset X$ and, by Step 3, the last inclusion is proper ($\neq$) if the projective case already established. Here we use also the birational invariance of big property.

Step 5. We can suppose that L is divisorial, that is, there exists a Cartier divisor D such that $L \simeq \mathcal{O}_X(D)$.

After that we can use dimensional induction.

Step 6. Taking a multiple of D instead of D, we can suppose the linear equivalence $D \sim H + E$ where H is an ample Cartier divisor of X and E is an effective Cartier divisor on X. By the property (2), $\mathbb{E}(L) = \mathbb{E}(D) \subseteq E$ and we can use dimensional induction. In other words, if $Y \subseteq X$ is an integral closed subspace and D is not big on Y then $Y \subseteq E$. Otherwise, by construction Y is topologically proper and $D_{|Y} \sim H_{|Y} + E_{|Y}$ where the last restriction gives an effective divisor; two other restrictions are defined up to $\sim$. Then $D_{|Y}$ is big, a contradiction. The bigness follows from the following fact: if $D \sim H + E$ where H is ample and E is effective then D is big. Obvious from the linear point of view but, for nef D, can be proved numerically too. A latter proof can use the numerical criterion of bigness [Sh15, Proposition 1]. It also implies the second half of the proposition when X is complete and L (resp. D) is nef. ☐

2. Space lattice

This section is devoted to some generalities about closed subspaces (see also [Sh15, §3]).

Space logic or the lattice of algebraic subspaces Let X be an algebraic space. Its Zariski *closed subspaces* $Y \subseteq X$ form a *lattice* with *the order* on the subspaces:

for closed subspaces $Y, Z \subseteq X$,

$$Y \geq Z \Leftrightarrow Y \supseteq Z;$$

and with two binary operations on the subspaces:

the joint is the union:

$$Y \vee Z = \sup\{Y, Z\} = Y \cup Z;$$

and

the meet is the intersection:

$$Y \wedge Z = \inf\{Y, Z\} = Y \cap Z.$$

The lattice will be denoted by $\operatorname{Clos} X$.

Lattice of ideal sheaves In the affine case the lattice of subspaces is reverse to the *lattice of ideals*. More precisely, suppose that $X = \operatorname{Spec} A$ where A is a commutative ring. Then there exists a canonical 1-to-1 correspondence between the closed subschemes (the closed subspaces in this situation) $Y \subseteq X$ and the ideals $I \subseteq A$ [H, Corollary 5.10, Chapter II] such that

$$I \mapsto \operatorname{Im}[\operatorname{Spec} A/I \to \operatorname{Spec} A].$$

Let $Y, Z \subseteq X$ be closed affine subschemes and I, J be the corresponding ideals. Then

$Y \supseteq Z \Leftrightarrow I \subseteq J$
$Y \cup Z$ corresponds to $I \cap J = \inf\{I, J\}$; and
$Y \cap Z$ corresponds to $I + J = \sup\{I, J\}$.

More generally, the lattice of subspaces is reverse to the *lattice of quasi-coherent ideal sheaves* of $\mathcal{O}_X$ [H, Proposition 5.9, Chapter II] [Kn, Construction 5.1] (see for Inclusions vs immersions below). If X is Noetherian or of finite type over a field k then the ideal sheaves are coherent.

Under Noetherian or finite type assumptions on X, we can add the infinite intersection operation on $\operatorname{Clos} X$ and the infinite sum operation to the ideal side. Under the same assumptions, we can add the infinite union on $\operatorname{Clos} X$ of coherent type. The corresponding ideal sheaf is the *coherent core*, the largest coherent subsheaf, for intersection of ideal sheaves. An infinite intersection of ideal sheaves usually is not quasicoherent (cf. b-divisors [Sh03, Example 4.12]) and is a global operation on $\operatorname{Clos} X$.

For an ideal or/and an ideal sheaf $\mathcal{I}$ and a natural number n, the power $\mathcal{I}^n$ plays an important role. The corresponding subspace operation will be denoted as the multiple nY of a closed subspace Y in X ($0Y = \emptyset$):

$$\mathcal{I}_{nY} = \mathcal{I}_Y^n \ (\mathcal{I}_Y^0 = \mathcal{O}_X).$$

For any natural numbers $n \geq m$, $nY \supseteq mY$.

An effective Cartier divisors E of X is identified with its closed subspaces: locally the ideal sheaf $\mathcal{I}_E$ is invertible. The identification agrees with multiplicities: for every natural number n, nE is the closed subspace corresponding to the effective Cartier divisor nE.

Usually both lattices are not distributive (for both distributivities): e.g.,

$$(Y \cap T) \cup (Z \cap T) \subseteq (Y \cup Z) \cap T$$

and for the corresponding logic

$$(Y \wedge T) \vee (Z \wedge T) \Rightarrow (Y \vee Z) \wedge T$$

hold but not $=$ and $\Leftrightarrow$ respectively. The distributivities and equivalences hold for reduced closed subspaces.

Example 2. Let $X = k^2$ be a plane with coordinates x, y, and Y, Z, T be lines given respectively by equations $x = 0, x + y = 0, y = 0$. Then $Y \cap T = Z \cap T$ is the point $(0, 0)$. Respectively $Y \cup Z$ has equation $x(x + y) = 0$ and its intersection with T is a double point with equations $x^2 = 0, y = 0$. So, $(Y \cap T) \cup (Z \cap T) \neq (Y \cup Z) \cap T$.

Inclusions vs immersions When we consider a (e.g., closed) subspace $Y \subseteq X$ we suppose that it is an algebraic space Y *included* into X, that is, the points of Y are points of X and the structure sections, functions, differentials, etc on Y are locally restricted from X. On the other hand, an immersion $\iota \colon Y \hookrightarrow X$ is a composition of a canonical isomorphism onto the image $Y \simeq \iota(Y)$ and of its inclusion $\iota(Y) \subseteq X$ (see for Restrictions and canonical factorization below). However the difference is scarcely perceptible because a subspace, e.g., closed, is actually an equivalence class [H, p. 85]. We mean by a subspace its representative. Such a choice can be arbitrary but usually it is related to constructions which are natural but not unique, that is, defined up to a (unique natural) canonical isomorphism. In our situation, every closed subspace has a unique ideal (sub)sheaf $\mathcal{I}_Y \subseteq \mathcal{O}_X$ such that there exists a canonical isomorphism between the inclusion $Y \subseteq X$ and the immersion $\mathcal{S}\mathrm{pec}\, \mathcal{O}_X/\mathcal{I}_Y \hookrightarrow X$ which gives a commutative triangle. If $=$ denotes the canonical isomorphism then $Y = \mathcal{S}\mathrm{pec}\, \mathcal{O}_X/\mathcal{I}_Y$ is our choice and the immersion became the inclusion. The same applies to any immersion $\iota \colon Y \hookrightarrow X$ instead of the inclusion. But in this case the canonical (for ι) isomorphism $Y \simeq \iota(Y)$ is $\iota_{|Y}$ and not necessarily identical, e.g., a nonidentical automorphism of $\iota(Y)$. Those choices do not matter for natural constructions as restrictions, quotients, $\mathcal{S}$pec, etc. E.g., a quasipolarized lattice $\mathrm{Clos}(X, L)$ with elements $(Y, L_{|Y}), Y \in \mathrm{Clos}\, X$, is well-defined [Sh15, §3.8]. However cf. Example 7 in Section 3.

Space preimage Let $\varphi\colon X \to Y$ be a morphism of spaces and W be a closed subspace of Y. Then the *preimage* $\varphi^{-1}W$ is the largest closed subspace V of X such that $\varphi_{|V}\colon V \to Y$ can be factorize through W:

$$\varphi_{|V}\colon V \to W \subseteq Y.$$

The restriction $\varphi_{|V}$ denotes the composition

$$V \subseteq X \stackrel{\varphi}{\to} Y.$$

The ideal sheaf $\mathcal{I}_V = \varphi^{-1}\mathcal{I}_W$ of V is the image of canonical homomorphism

$$\varphi^*\mathcal{I}_W \to \mathcal{O}_X,$$

that is, generated by the ideal sheaf $\mathcal{I}_W$ of W [H, Caution 7.12.2]. The preimage has also a fiber product description: $\varphi^{-1}W = W \times_Y X$ [H, Exercise 3.11, Chapter II]. The map

$$\varphi^{-1}\colon \operatorname{Clos} Y \to \operatorname{Clos} X$$

preserves the order and intersections but not unions. For instance, if φ denotes the inclusion $T \subset X$ and $Y \in \operatorname{Clos} X$ then $\varphi^{-1}Y = Y \cap T, \varphi^{-1}Z = Z \cap T, \varphi^{-1}(Y \cup Z) = (Y \cup Z) \cap T$ and $\varphi^{-1}(Y \cup Z) \supseteq \varphi^{-1}X \cup \varphi^{-1}Z$ but $\neq$ by Example 2 above; $=$ holds topologically. In general, φ^{-1} preserves finite unions topologically, that is, for reduced closed subspaces. It is also multiplicative: in particular, for every natural number n, $\varphi^{-1}nW = n\varphi^{-1}W$.

If E is an effective Cartier divisor on Y then $\varphi^{-1}E$ has a principal ideal sheaf but it is not necessarily invertible (Cartier) or not lifting. The latter property holds if E is in a rather general position. However, if φ is an epimorphism (see for Epimorphism below) of primary spaces then $\varphi^{-1}E = \varphi^*E$ and is also an effective Cartier divisor (cf. the proof of Corollary 4).

Space image Let $\varphi\colon X \to Y$ be a morphism of spaces and $V \subseteq X$ be a closed subspace of X. Then the *image* $\varphi(V)$ is the smallest closed subspace W of Y such that $V \subseteq \varphi^{-1}W$. The ideal sheaf $\mathcal{I}_{\varphi(V)}$ of $\varphi(V)$ is the preimage of the image ideal sheaf $\operatorname{Im}[\varphi_*\mathcal{I}_V \to \varphi_*\mathcal{O}_X]$ under the canonical homomorphism

$$\varphi^{\#}\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X,$$

where $\mathcal{I}_V$ denotes the ideal sheaf of V on X. The map

$$\varphi\colon \operatorname{Clos} X \to \operatorname{Clos} Y$$

preserves the order and unions (see for Additivity below) but not intersections. Notice that $\varphi^{-1}\colon \operatorname{Clos} Y \to \operatorname{Clos} X$ is not inverse to $\varphi\colon \operatorname{Clos} X \to \operatorname{Clos} Y$. But there are natural inclusions

$$\varphi(\varphi^{-1}Y) \subseteq Y,\ \varphi^{-1}(\varphi(Y)) \supseteq Y$$

and equalities

$$\varphi(\varphi(\varphi^{-1}Y)) = \varphi(Y),\ \varphi^{-1}(\varphi^{-1}(\varphi(Y))) = \varphi^{-1}Y.$$

If $Y \in \operatorname{Clos} X$ is reduced (resp. irreducible, integral, primary) then its image $\varphi(Y)$ is also reduced (resp. irreducible, integral, primary; cf. [Sh15, Proposition 2]). However, if E is an effective Cartier divisor on X then $\varphi(E)$ is not necessarily Cartier (cf. Proposition 3 above).

Additivity Let $\varphi\colon X \to Y$ be a morphisms of algebraic spaces, $E \subseteq X$ be a closed subspace and (E_i) be a collection (not necessarily finite) of closed subspaces such that $E \subseteq \cup E_i$. Then $\varphi(E) \subseteq \cup\varphi(E_i)$. Moreover, = implies =.

Proof. (Cf. the proof of Lemma 4 below.) Actually, it is sufficient to consider the case = where the additivity holds. In this case $\varphi(E) \supseteq \cup\varphi(E_i)$ holds. If $\neq$ then there exists a nonzero ideal sheaf $\mathcal{I}$ on $\varphi(E)$ such that $\cup\varphi(E_i)$ is in a subspace given by the ideal. Then by definition the inverse image ideal $\varphi^{-1}\mathcal{I}$ is a nonzero ideal sheaf on E, because $\varphi^{\#}$ is monomorphic for functions on $\varphi(E)$, and $\cup E_i = E$ is in its proper closed subspace, a contradiction. For a finite collection (E_i), if E_i's are given by $\mathcal{I}_i$'s then $\varphi(E_i)$ is given by

$$\mathcal{I}_{\varphi(E_i)} = \varphi^{\#-1}\operatorname{Im}[\varphi_*\mathcal{I}_i \to \varphi_*\mathcal{O}_X].$$

The ideal sheaf of $\cup\varphi(E_i)$ is

$$\begin{aligned}\cap\mathcal{I}_{\varphi(E_i)} &= \cap\varphi^{\#-1}\operatorname{Im}[\varphi_*\mathcal{I}_i \to \varphi_*\mathcal{O}_X] = \varphi^{\#-1}\operatorname{Im}[\varphi_*(\cap\mathcal{I}_i) \to \varphi_*\mathcal{O}_X]\\ &= \varphi^{\#-1}\operatorname{Im}[\varphi_*\mathcal{I} \to \varphi_*\mathcal{O}_X],\end{aligned}$$

is the ideal sheaf of $\varphi(E)$ and $= 0$ on $\varphi(E)$. ☐

Epimorphism A morphism of algebraic spaces $\varphi\colon X \to Y$ is *epimorphic* or an *epimorphism* if $\operatorname{Im}\varphi = \varphi(X) = Y$. Usually, we use the arrow $\twoheadrightarrow$ only for epimorphisms but not for surjections. Recall that φ is a *categorical epimorphism* means that any diagram

$$X \overset{\varphi}{\to} Y \underset{\overset{\beta}{\to}}{\overset{\alpha}{\to}} Z$$

is commutative if and only if $\alpha = \beta$. A proper epimorphism φ is always surjective by [H, Theorem 4.7, Chapter II]. In other words, the image commutes with reduction: $\varphi(X)_{\mathrm{red}} = \varphi(X_{\mathrm{red}})$ and the latter image is treated as a topological (in particular, set theoretical) one. The converse does not hold even for proper morphisms. Under our separated agreement, the following holds.

Proposition 5. *Every epimorphism is a categorical one. The quasipolarized version holds too.*

Proof. Step 1. Since α, β are separated then $\alpha = \beta$ holds on a largest closed subspace V of Y given by the equation. Indeed, the canonical isomorphism

$$V = \Gamma_\alpha \cap \Gamma_\beta$$

holds under the projection $Y \times Z \to Y$ where $\Gamma_\alpha, \Gamma_\beta$ are respectively graphs of α, β. This means that if $W \subseteq Y$ is a closed subspace such that $\alpha_{|W} = \beta_{|W}$ then $W \subseteq V$. Both graphs $\Gamma_\alpha, \Gamma_\beta$ are closed because morphisms α, β are separated.

Step 2. We can take above W (e.g., Zariski) locally closed. Then $W \subseteq V$ again if $\alpha_{|W} = \beta_{|W}$. Since V is closed, V contains also $\overline{W}$, the smallest closed subspace containing W, equivalently, the image of inclusion $W \subseteq Y$.

Step 3. Suppose now that $W \subseteq Y$ is a primary closed subspace and U is its nonempty Zariski open subset, e.g., affine. Then $W \subseteq V$ if $\alpha_{|U} = \beta_{|U}$. Immediate by Step 2 because $W = \overline{U}$.

Step 4. Reduction to the primary epimorphic case where X, Y, Z are primary and φ, α, β are epimorphic. Since X is Noetherian then $X = \cup X_i$ where the union is finite and every X_i is a primary closed subspace. About the existence of such decomposition see for Primary presentation in Section 3. By definition of restriction the diagram

$$X_i \overset{\varphi_{|X_i}}{\twoheadrightarrow} \varphi(X_i) \subseteq Y \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} Z$$

is commutative and by construction $\varphi_{|X_i}$ is epimorphic. Using canonical factorization and its uniqueness (see Restrictions and canonical factorization below) we can replace Y by $\varphi(X_i)$, Z by $\alpha \circ \varphi(X_i) = \beta \circ \varphi(X_i)$ and α, β respectively by epimorphic restrictions

$$\varphi(X_i) \begin{array}{c} \overset{\alpha_{|\varphi(X_i)}}{\twoheadrightarrow} \\ \overset{\beta_{|\varphi(X_i)}}{\twoheadrightarrow} \end{array} \alpha \circ \varphi(X_i) = \beta \circ \varphi(X_i)$$

[Sh15, §3.5]. So, $\varphi(X_i) \subseteq V$ by the primary and epimorphic assumption because $X_i, \varphi(X_i), \alpha \circ \varphi(X_i) = \beta \circ \varphi(X_i)$ are primary [Sh15, Proposition 2] and by commutativity $\alpha_{|\varphi(X_i)} = \beta_{|\varphi(X_i)}$. Hence by above Additivity $Y = \operatorname{Im}\varphi = \cup\varphi(X_i) \subseteq V$. So, $V = Y$.

Since epimorphism is a local property and by the existence of a dense affine open subscheme in any algebraic space [Kn, Corollary 6.8, Chapter II] we can suppose that all spaces X, Y, Z are affine.

Step 5. (Affine case.) The dual statement for commutative rings: a diagram of commutative rings

$$C \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B \subseteq A$$

is commutative if and only if $\alpha = \beta$. This is the categorical property of inclusions or immersions (cf. Restrictions and canonical factorization).

Step 6. (Quasipolarized case.) Here we use $\operatorname{Aut} L = \Gamma(X, \mathcal{O}_X^\times)$ for (X, L) (see for Category of quasipolarizations in Section 1). Let M, N be quasipolarizations on Y, Z respectively. By definition and since $\alpha = \beta$ on spaces, $\alpha^* N = \beta^* N \overset{\alpha}{\simeq} M$ and $\beta^* N \overset{\beta}{\simeq} M$. Hence, for quasipolarizations, $\alpha = a\beta$ holds for some $a \in \Gamma(Y, \mathcal{O}_Y^\times)$. The relation $\beta \circ \varphi = \alpha \circ \varphi$ for quasipolarizations implies that $\varphi^*\beta = \varphi^*\alpha = \varphi^*(a\beta) = \varphi^\# a \varphi^* \beta$ and $\varphi^\# a = 1$ hold. Therefore $a = 1$ and $\alpha = \beta$ in quasi-PSp because φ is epimorphic and respectively $\varphi^\#$ is monomorphic on functions. $\square$

Notice that the separated property is very important in the proposition.

Example 3. Consider the commutative diagram

$$\mathbb{C}^* \subseteq \mathbb{C} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbb{C} \bigsqcup_{\mathbb{C}^*} \mathbb{C},$$

with the unseparated target $\mathbb{C} \sqcup_{\mathbb{C}^*} \mathbb{C}$ where α, β are two different open immersions: $\alpha \neq \beta$. In this case, $\alpha = \beta$ on $V = \mathbb{C}^*$ which is not closed! Of course, the amalgam $\mathbb{C} \sqcup_{\mathbb{C}^*} \mathbb{C}$ is $\mathbb{C}$ in the separated category, $\alpha = \beta$ and there is no a contradiction.

Stein factorization Every proper morphism $X \to T$ of algebraic spaces has a *Stein* factorization $X \twoheadrightarrow Y \to T$ where $X \twoheadrightarrow Y$ is a contraction and $Y \to T$ is finite ([H, Corollary 11.5, Chapter III] for projective $X \to T$, and [Kn, Theorem 4.1, Chapter IV] in general). The factorization is unique (up to isomorphism) by the epimorphic property of contraction [Kn, Proposition 5.6, Chapter II]. This follows also from the *universal property of Stein factorization*

[Kn, Construction 5.5, Chapter II]: if $X \to Z \to T$ is a factorization of $X \to T$ such that $Z \to T$ is finite then there exists a unique factorization $X \twoheadrightarrow Y \to Z$ of $X \to Z$ through the contraction $X \twoheadrightarrow Y$, the latter factorization is also Stein and $Y \to Z \to T$ is a factorization of $Y \to T$. Use an algebraic (affine) point of view: $f_*\widetilde{N} = \widetilde{{}_AN}$ holds for any quasicoherent sheaf $\widetilde{N}$ on $\operatorname{Spec} B$ under an affine morphism $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ [H, Proposition 5.2(d), Chapter II], in particular, under a finite affine morphism.

By construction the Stein factorization is fibered over T, that is, commutes with restrictions on open sets (of coverings).

Notice also that if $X \to T$ is epimorphic then, by [Sh15, §3.5], the finite morphism $Y \to T$ is epimorphic too.

If X is primary (irreducible) then the Stein factorization if *primary* (resp. *irreducible*): Y is primary by [Sh15, Proposition 2] (resp. irreducible). In many instances we can relax the primary condition to S_1: there are no embedded components (cf. Proposition 6). This condition agrees with the étale and fppt topologies. Consider a (simple) morphism $X \to T$ such that every irreducible component of X goes to an irreducible component of T. In this case, the image also satisfies S_1 if X does so. In particular, this property holds for the contraction Y of the Stein factorization if the factorized morphism has this property and X has S_1. Every finite surjective or birational morphism (on every irreducible component) has this property too. The latter one holds by definition.

By Proposition 2 in Section 1, a Stein factorization

$$(X, L) \overset{\varphi}{\twoheadrightarrow} (Y, \varphi_* L) \overset{\psi}{\to} (T, M)$$

exists for any proper morphism $(X, L) \to (T, M)$ in quasi-PSp where to be proper means that of for $X \to T$. By construction for spaces, on Y there exists a unique canonical quasipolarization $\varphi_* L$ on Y because $L \overset{\psi\circ\varphi}{\simeq} \varphi^*\psi^* M$ is locally trivial over Y. By the same proposition we can push down the latter isomorphism: $\varphi_* L \overset{\psi}{\simeq} \psi^* M$ that gives a required factorization in quasi-PSp. The factorization is unique and universal by Proposition 5 above because any contraction is an epimorphism (cf. [Kn, Proposition 5.6, Chapter II]).

In general, there are no natural push down of a quasipolarization from X to Y (however, cf [EGA, §21.5]).

Restrictions and canonical factorization Let $\varphi\colon X \to Y$ be a morphism of spaces and $W \subseteq X$ be a closed subspace of Y such that $\varphi(X) \subseteq W$. Then φ has a *canonical factorization through* W:

$$\varphi\colon X \to W \subseteq Y.$$

It is unique and $X \to W$ is also denoted as a restriction $\varphi_{|X}$. The existence holds by definition. The uniqueness follows from the *inclusion property*, dual to the categorical epimorphism one: any diagram

$$X \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Y \subseteq Z$$

is commutative if and only if $\alpha = \beta$ where the inclusion $\subseteq$ is locally closed but not necessarily closed. Sometimes, if φ is not proper, we can find a smaller canonical factorization with locally closed W and $\varphi(X)$ but the smallest one does not exist always. The same applies to the statements in this subsection. However in the applications of this paper we need only the closed case. E.g., any morphism $\varphi\colon X \to Y$ has a unique and canonical factorization into

$$X \twoheadrightarrow \operatorname{Im}\varphi \subseteq Y$$

an epimorphism and an inclusion.

The inclusion property can be established in two steps: first, for the closed inclusions and, second, for open inclusions. Both steps are immediate by definition.

More generally, for any closed subspaces $V \subseteq X, W \subseteq Y$ such that $\varphi(V) \subseteq W$ the diagram

$$\begin{array}{ccc} V & \subseteq & X \\ \varphi_{|V}\downarrow & & \varphi\downarrow \\ W & \subseteq & Y \end{array}$$

is well-defined and commutative for a unique morphism $\varphi_{|V}$ which is called a *restriction* of φ *on* V *to* W.

Proof. The uniqueness holds by the inclusion property. The existence can be achieved in two steps:

(1) for $W = Y$; and

(2) for $V = X$.

A construction for (1) follows from definition: $\varphi_{|V}\colon V \to Y$ is the composition of $V \subseteq X$ with φ. A construction for (2) follows from a canonical factorization of φ through W

$$X \twoheadrightarrow W \subseteq Y. \qquad \square$$

Notice also the following two properties of restrictions.

Transitivity: the diagram

$$\begin{array}{ccccc} S & \subseteq & V & \subseteq & X \\ \varphi_{|S} & & \varphi_{|V}\downarrow & & \varphi\downarrow \\ T & \subseteq & W & \subseteq & Y \end{array}$$

is commutative, that is, $\varphi_{|V|S} = \varphi_{|S}$. Immediate by the uniqueness of restriction.

Compliance with images: if the square of morphism of spaces

$$\begin{array}{ccc} X & \stackrel{\varphi}{\to} & Y \\ \alpha\downarrow & & \beta\downarrow \\ V & \stackrel{\psi}{\to} & W \end{array}$$

is commutative then $\beta(\varphi(X)) \subseteq \psi(V)$, and the diagram

$$\begin{array}{ccc} X & \stackrel{\varphi_{|X}}{\to} & \varphi(X) \\ \alpha\downarrow & & \beta_{|\varphi(X)}\downarrow \\ V & \stackrel{\psi_{|V}}{\to} & \psi(V) \end{array}$$

is well-defined and commutative.

Proof. By [Sh15, §3.5]

$$\beta(\varphi(X)) = \psi(\alpha(X)) \subseteq \psi(V).$$

So, the required commutativity follows from the commutativity of the ambient square and of the right square in

$$\begin{array}{ccccc} X & \stackrel{\varphi_{|X}}{\to} & \varphi(X) & \subseteq & Y \\ \alpha\downarrow & & \beta_{|\varphi(X)}\downarrow & & \beta\downarrow \\ V & \stackrel{\psi_{|V}}{\to} & \psi(V) & \subseteq & W \end{array}$$

and by the inclusion property above. The right square is commutative by definition of restriction. □

In all above statements of this subsection and in many statements of this paper the inclusions $\subseteq$ can be replaced by immersions $\hookrightarrow$.

Similar statements hold in quasi-PSp. In the case of immersions, the immersion property in quasi-PSp, instead of the inclusion property, uses an isomorphism with the natural immersion or inclusion property by Lemma 1 in Section 1 (see also [Sh15, §3.14]).

Space image and preimage mappings are not inverse one to another even for contractions (cf. Corollary 4). However there are some approximations to this property.

Lemma 2. *Let $\varphi\colon X \twoheadrightarrow Y$ be a finite epimorphism morphism of algebraic spaces and Z be a closed subspace of Y. Then there exists a natural number c such that, for every $n \geq c$, $\varphi(n(\varphi^{-1}Z)) \supseteq Z$.*

The proper property of a finite morphism is very important for the lemma. E.g., the lemma does not hold for multiples $mP, m \geq 2$, of a point $P \in \mathbb{A}^1$ and the morphism $\varphi\colon (\mathbb{A}^1 \setminus P) \sqcup P \twoheadrightarrow \mathbb{A}^1$. The morphism is epimorphic and surjective but, for every $m, n \geq 1$, $\varphi(n\varphi^{-1}mP) = P$.

Proof. Locally it uses the Artin-Rees lemma. So, we can suppose that $Y = \operatorname{Spec} B$. Then $X = \operatorname{Spec} A$ and $B \subseteq A$ an inclusion corresponding to the surjection φ. By our assumptions A is finite as a module over B. The closed subspace Z is given by an ideal I of B. The subspace $\varphi^{-1}Z$ is given by the ideal IA of A, or as a submodule of A, and its multiple $n\varphi^{-1}Z$ is given by the ideal $(IA)^n = I^nA$ of A, or as a submodule of A, where I^n is an ideal of B. Hence $\varphi(n\varphi^{-1}Z))$ is given by the ideal $I^nA \cap B$ of B. By the Artin-Rees lemma [Mat, Theorem 8.5] there exists a natural number c such that for every $n \geq c+1$,

$$I^nA \cap B = I^{n-c}(I^cA \cap B) \subseteq I^{n-c}B \subseteq I.$$

Equivalently, $\varphi(n(\varphi^{-1}Z)) \supseteq Z$.

By finiteness of Y we can take the maximum of $c+1$ for some covering. □

Corollary 5. *Let $\varphi\colon X \twoheadrightarrow Y$ be a proper epimorphism of (primary) algebraic spaces and E be an effective Cartier divisor on Y. Then, for every natural number $n \gg 0$, $\varphi(n(\varphi^*E)) \supseteq E$. Actually, $\varphi(n(\varphi^{-1}E)) \supseteq E$ for every closed subspace E of Y instead of the divisor E and φ^{-1} instead of φ^*.*

Proof. Here we assume that X, Y are primary and E is an effective Cartier divisor. The general case will be established in Continuation of the proof of Corollary 5 below. Take a Stein factorization

$$X \overset{\chi}{\twoheadrightarrow} Z \overset{\psi}{\twoheadrightarrow} Y$$

of φ. Note that Z is also primary by [Sh15, Proposition 2] and χ, ψ are epimorphic (see for Stein factorization above). Consider E as a closed subspace of X. Let c be as in Lemma 2 for $\varphi = \psi$ and $Z = E$. Then $\varphi^* E, \psi^* E$ are also effective Cartier divisors by Proposition 3 and [Sh15, Proposition 3, 1)] (cf. the proof of Corollary 4). So, by Corollary 4 and Lemma 2, for any $n \geq c$,

$$\varphi(n(\varphi^* E)) = (\psi \circ \chi)(\chi^*(n\psi^* E)) = \psi(n\psi^* E) \supseteq E. \qquad \square$$

Lemma 3. *Let A be a Noetherian ring and I, I_1, I_2 be ideals of A. Then there exists a natural number c such that for any natural number $n \geq c$, we have*

$$(I^n + I_1) \cap (I^n + I_2) \subseteq I^{n-c} + I_1 \cap I_2.$$

Proof. By the Artin-Rees lemma [Mat, Theorem 8.5] there exists a natural number c such that for every $n \geq c$, we have

$$I^n \cap (I_1 + I_2) = I^{n-c}(I^c \cap (I_1 + I_2)).$$

We apply the lemma to $M = A, N = I_1 + I_2$. Notice also that we need only the nontrivial inclusion $\subseteq$.

Consider any $n \geq c$ and $a \in (I^n + I_1) \cap (I^n + I_2)$. That is,

$$a = i_1 + i_{1,1} = i_2 + i_{1,2},\ \ i_1, i_2 \in I^n, i_{1,1} \in I_1, i_{1,2} \in I_2.$$

Since $i_1 \in I^n \subseteq I^{n-c}$ it is sufficient to verify that

$$i_{1,1} \in I^{n-c} + I_1 \cap I_2.$$

By construction

$$i_{1,1} = i + i_{1,2},$$

where $i = i_2 - i_1 \in I^n$, and $i = i_{1,1} - i_{1,2} \in I_1 + I_2$. Hence $i \in I^n \cap (I_1 + I_2)$ and by the Artin-Rees lemma

$$i \in I^{n-c}(I^c \cap (I_1 + I_2)) \subseteq I^{n-c}(I_1 + I_2) = I^{n-c} I_1 + I^{n-c} I_2.$$

So,

$$i_{1,1} - i_{1,2} = i = i_{2,1} - i_{2,2}, i_{2,1} \in I^{n-c} I_1 \subseteq I^{n-c}, I_1; i_{2,2} \in I^{n-c} I_2 \subseteq I^{n-c}, I_2.$$

Therefore

$$i_{1,1} - i_{2,1} = i_{1,2} - i_{2,2} \in I_1 \cap I_2$$

and

$$i_{1,1} = i_{2,1} + (i_{1,1} - i_{2,1}) \in I^{n-c} + I_1 \cap I_2. \qquad \square$$

Theorem 2. *Let A be a Noetherian ring and $I, I_1, \dots, I_n$ be ideals of A. Then there exists a natural number c such that for any natural number $m \geq c$, we have*

$$\cap_{i=1}^n (I^m + I_i) \subseteq I^{m-c} + \cap_{i=1}^n I_i.$$

Proof. Induction for n. For cases $n = 0, 1$, the corollary holds for $c = 0$. For $n = 2$, c exists by Lemma 3.

If $n \geq 3$ then there exists a natural numbers c_1, c_2 such that for any natural number $m \geq c_1$ or c_2, we have respectively

$$\cap_{i=1}^{n-1} (I^m + I_i) \subseteq I^{m-c_1} + \cap_{i=1}^{n-1} I_i$$

and

$$(I^m + \cap_{i=1}^{n-1} I_i) \cap (I^m + I_n) \subseteq I^{m-c_2} + \cap_{i=1}^n I_i.$$

Hence, for any natural number $m \geq c = c_1 + c_2$,

$$\begin{aligned}\cap_{i=1}^n (I^m + I_i) &\subseteq (I^{m-c_1} + \cap_{i=1}^{n-1} I_i) \cap (I^m + I_n) \\ &\subseteq (I^{m-c_1} + \cap_{i=1}^{n-1} I_i) \cap (I^{m-c_1} + I_n) \subseteq I^{m-c_1-c_2} + \cap_{i=1}^n I_i \\ &= I^{m-c} + \cap_{i=1}^n I_i.\end{aligned}$$ □

Corollary 6. *Let X be an algebraic space and $X_1, \dots, X_n$ be its closed subspaces such that $X = \cup_{i=1}^n X_i$. Then, for every closed subspace Y of X, there exists a natural number c such that*

$$(m-c)Y \subseteq \cup_{i=1}^n ((mY) \cap X_i) \text{ for every } m \geq c$$

and

$$Y \subseteq \cup_{i=1}^n ((mY) \cap X_i) \text{ for every } m > c.$$

Notice that, for every closed subspace Y of X there exists a closed subspace Z of X such that

$$Y \subseteq \cup_{i=1}^n (Z \cap X_i).$$

E.g., we can take $Z = X$. This property of Z holds also for any larger closed subspaces V of X: if $V \supseteq Z$ then

$$Y \subseteq \cup_{i=1}^n (Z \cap X_i) \subseteq \cup_{i=1}^n (V \cap X_i).$$

By the Noetherian induction there exists a minimal under inclusion closed subspace Z with the above property. On the other hand such Z includes Y and usually is not Y itself (cf. Example 2). Corollary 6 gives a sufficiently small subspace $Z = mY$ for $m > c$: $(mY)_{\text{red}} = Y_{\text{red}}$, and implies the following.

Corollary 7. *Let X be an algebraic space and $X_1, \ldots, X_n$ be its closed subspaces such that $X = \cup_{i=1}^n X_i$. Then, for every closed subspace Y of X, there exists a minimal under inclusion closed subspace Z of X such that*

$$Y \subseteq \cup_{i=1}^n (Z \cap X_i) \subseteq Z \text{ and } Z_{\text{red}} = Y_{\text{red}}.$$

Proof. Immediate by Corollary 6. □

Proof of Corollary 6. By the finite type properties we can suppose that $X = \operatorname{Spec} A$ is affine and the ring is Noetherian. Then every closed subspace X_i of X is a subscheme given by an ideal I_i of A. Respectively denote by $I \subseteq A$ the ideal of Y. The assumption $X = \cup_{i=1}^n X_i$ means that $\cap_{i=1}^n I_i = 0$. The first required inclusion in terms of ideals means that

$$I^{m-c} \supseteq \cap_{i=1}^n (I^m + I_i).$$

The latter holds by Theorem 2.

For $m > c$, we have $Y \subseteq (m-c)Y$ or $\mathcal{I} \supseteq \mathcal{I}^{m-c}$ for the ideal sheaf $\mathcal{I}$ of Y. This implies the second inclusion of the corollary. □

Continuation of the proof of Corollary 5. Step 1. (Reduction to the primary case.) Indeed, $X = \cup X_i$ where every X_i is primary (see for Primary presentation in Section 3 below). So, $Y = \cup \varphi(X_i)$ by Additivity above, every $Y_i = \varphi(X_i)$ is primary by [Sh15, Proposition 2] and every restriction $\varphi_i = \varphi_{|X_i} \colon X_i \twoheadrightarrow Y_i$ is epimorphic by construction. By Corollary 6,

$$E \subseteq \cup((mE) \cap Y_i)$$

for every natural number $m \gg 0$. Therefore it is sufficient to establish the corollary for primary X, Y. Indeed, if, for every Y_i and every $n \gg 0$,

$$(mE) \cap Y_i \subseteq \varphi_i n \varphi_i^{-1}((mE) \cap Y_i)) \subseteq \varphi_i \varphi_i^{-1} n((mE) \cap Y_i)),$$

where the last n-multiple is taken in Y_i, then

$$(mE) \cap Y_i \subseteq \varphi_i \varphi_i^{-1} n((mE) \cap Y_i)) \subseteq \varphi_i \varphi_i^{-1}((nmE) \cap Y_i) \subseteq \varphi \varphi^{-1} nmE.$$

(We use here certain monotonicities for multiples and for restrictions.) Hence $E \subseteq \varphi \varphi^{-1} nmE$, that is, required result for every sufficiently divisible $n \gg 0$. By the monotonicity $mE \subseteq nE$ for every $m \leq n$ we can omit the divisibility.

We assume now that both X and Y are primary.

Step 2. (Dimensional induction.) If $E_{\text{red}} = Y_{\text{red}}$ then $nE = Y$ for every $n \gg 0$ and $E \subseteq Y = \varphi\varphi^{-1}Y = \varphi n\varphi^{-1}E$ for those n. Otherwise there exists a Cartier divisor C in Y such that $E \subseteq C$. By already established the Cartier case: for some $n > 0$,

$$\varphi_{|n\varphi^*C}\colon n\varphi^*C \twoheadrightarrow \varphi(n\varphi^*C)$$

is proper epimorphic and $E \subseteq \varphi(n\varphi^*C)$. By construction $\dim \varphi(n\varphi^*C) < \dim Y$ and we can use the dimensional induction. □

Proposition 6. *Let $(\varphi_i\colon Y_i \to X), i \in I$, be a finite collection of proper morphisms of algebraic spaces such that*

every Y_i is primary;

(2) $$X = \cup_{i\in I} \operatorname{Im}\varphi_i;$$

where

every reduced image $\varphi_i Y_{i\text{red}}$ is an irreducible component of X_{red}.

Then, for any effective Cartier divisor E of X, every $\varphi_i^{-1}E = \varphi_i^*E$ is Cartier (E is lifting) and there exists a natural number c such that, for every natural number $n \geq c$,

$$E \subseteq \cup_{i\in I}\varphi_i n\varphi_i^*E.$$

Actually, only under one assumption (2), for any closed subspace E of X, there exists a natural number c such that, for every natural number $n \geq c$,

$$E \subseteq \cup_{i\in I}\varphi_i n\varphi_i^{-1}E.$$

Proof. We can suppose that every φ_i is a proper epimorphism onto $X_i = \operatorname{Im}\varphi_i$. Corollary 6 gives the inclusion

$$E \subseteq \cup_{i\in I} nE_i,\ E_i = E \cap X_i,$$

for every $n \gg 0$, where the multiple nE_i is taken in X_i. For Cartier E, every E_i is an effective Cartier divisor on X_i by our assumptions, Proposition 3 above and [Sh15, Proposition 3]; and $nE_i = nE \cap X_i$ is its n-th multiple on X_i. On the other hand, by Corollary 5, for every $i \in I$ and every $m \gg 0$,

$$nE_i \subseteq \varphi_i m\varphi_i^*nE_i = \varphi_i mn\varphi_i^*E_i.$$

This implies the existence of required c (cf. Step 1 in Continuation of the proof of Corollary 5).

We can apply also Corollary 5 to a proper epimorphism $\varphi = \sqcup\varphi_i\colon \sqcup Y_i \twoheadrightarrow X$. This works in general for a closed subspace E. $\square$

3. Colimits

Diagrams By definition a diagram of a category is a certain mapping of an oriented graph into the category. In particular, the vertices and arrows of a diagram form collections but not sets. However if a category is rather large and has a lot of isomorphic objects we can suppose up to isomorphism of diagrams that the diagram mapping is injective at least on the vertices and they form a set (cf. diagrams in [Sh15]). To avoid difficult notation, in the paper we treat diagrams in the latter naive sense. In particular, for a diagram D, $X \in D$ means that X is a vertex of D. For morphisms of a diagram and its colimit we can remove redundant (copies of) arrows and also suppose that the arrows form a set.

Colimit of a diagram For a diagram D of a category, its *colimit* or *inductive limit* is a universal morphism δ of D into an object E in the category. A *morphism* $\delta\colon D \to E$ is a collection of morphisms $\delta_X\colon X \to E$ for every vertex X of D such that, for any arrow $X_1 \to X_2$ of D, the diagram

$$\begin{array}{ccccc} X_1 & & \to & & X_2 \\ & \delta_1 \searrow & & \delta_2 \swarrow & \\ & & E & & \end{array} \quad , \delta_i = \delta_{X_i}, i = 1, 2,$$

is a commutative. A morphism $\delta_{\mathrm{colim}}\colon D \to \operatorname{colim} D$ is *universal* if any other morphism δ is a composition of δ_{colim} with a unique morphism $\varepsilon\colon \operatorname{colim} D \to E$, that is, $\delta_X = \varepsilon \circ \delta_{\mathrm{colim},X}$ for every vertex X of D.

The main purpose of the section and of the whole paper is to present as a colimit certain birational constructions, e.g., a modification of skrepa.

Some basic properties of colimits are given in [Sh15, Functorial properties of colimits, §5]. Here we treat in details only the following. Another important notion of a fibered colimit is explained after Lemma 5.

Epimorphic property of colimits Suppose that the category has the usual properties of images, e.g, this is true for the category of algebraic spaces with morphisms over k (see for Space image and Restrictions and canonical factorization in Section 2). Let $\delta\colon D \to E$ be a morphism of a diagram D into (a space) E. In the category of spaces, the image of δ is the smallest closed subspace $\operatorname{Im}\delta \subseteq E$ with respect to inclusion such that, for every vertex X_i of D, $\operatorname{Im}\delta_i \subseteq \operatorname{Im}\delta, \delta_i = \delta_{X_i}$. Equivalently, $\operatorname{Im}\delta = \cup \operatorname{Im}\delta_i$ holds.

Let $\delta\colon D \to E$ be a morphism of a diagram D into (a space) E and $E_1 \subseteq E$ be (e.g., a closed subspace) such that $\operatorname{Im}\delta \subseteq E_1$. Then δ goes through E_1. More precisely, there exists a unique factorization of δ through the inclusion $\varepsilon\colon E_1 \overset{\subseteq}{\to} E$: $\delta = \varepsilon \circ \delta_1$ or, for every X_i,

$$\delta_i = \varepsilon \circ \delta_{1,i}\colon X_i \to E.$$

In particular, δ goes through $\operatorname{Im}\delta$. For algebraic spaces, δ_1 and ε are proper if δ is proper: every δ_i is proper, or D is finite, complete: every X_i is complete. Under the latter assumptions, for finite D, any colimit δ is surjective: $\operatorname{Im}(\delta_{\mathrm{red}}) = (\operatorname{colim} D)_{\mathrm{red}}$ holds topologically. This follows from the following epimorphic property and the surjective property of proper epimorphisms [H, Theorem 4.7, Chapter II]. Actually, the same holds always for closed points.

Epimorphic property. Let $\delta\colon D \to \operatorname{colim} D$ be a colimit of algebraic spaces. Then it is *epimorphic*: $\operatorname{Im}\delta = \operatorname{colim} D$; δ is also surjective on closed points and surjective on all points for finite D.

In general δ_i and ε are neither epimorphic, nor surjective. For infinite D, δ is not necessarily surjective on some nonclosed points. Quasipolarized versions of above statements hold in quasi-PSp. See for a proof of the epimorphic property below.

Remarks 2. (1) The epimorphic property is typical for colimits in many categories even it does not mean usually surjectivity. For instance, if $G + H$ is a sum of two groups then it is not a union of their images but it is generated by them.

(2) The *monomorphic* property is dual and typical for limits. If $\delta\colon \lim D \to D$ is a limit and it goes through an epimorphism $\varphi\colon \lim D \twoheadrightarrow Y$ then φ is an isomorphism. E.g., in the category of modules over a ring R,

$$\ker\delta = \cap_{X\in D} \ker\delta_X = 0.$$

The property is dual to the epimorphic property for colimits of affine spaces.

Quasipolarized colimits They are colimits in quasi-PSp, that is, diagrams and their morphisms are quasipolarized. By Lemma 1 every fixed quasipolarized morphism $\delta\colon D \to (T, O)$ of a quasipolarized diagram, up to a quasipolarized isomorphism, can be assumed to be *natural*: for every morphism $\varphi\colon (X_1, L_1) \to (X_2, L_2)$ of D, $\varphi^* L_2 = L_1$ and $\delta_i^* O = L_i$.

Algebraic spaces fit better colimits that schemes. However existence of colimits even in the category of spaces do not have an easy answer. The existence of colimits can be reduced to the existence of a morphism (see Propositions 8, 11 and Corollaries 22, 23). This explains our interest to morphisms of skrepas, especially, in quasi-PSp. Colimits usually do not exist for algebraic spaces of finite type or for quasipolarized spaces but even, if they exist, they can be quite weird, i.g., not fibered (see Examples 8 and 9). E.g., colimits of finite diagrams of complete algebraic spaces always exits (see Corollary 23) but they are typically not fibered.

Examples 2. (1) A coproduct of algebraic spaces is their lattice disjoint union. In the affine case the corresponding ring is a product of component rings. In general, a finite coproduct has finite type, is Noetherian and a usual disjoint union, in particular, topological (cf. ultraproduct [Shaf, Example 11, §4]); and has a canonical quasipolarization if every its component is quasipolarized.

(2) ([K97, Example 8.5].) Consider a skrepa

$$\begin{array}{ccc} \operatorname{Spec} k[x,y]/(y^2) & \supseteq & \operatorname{Spec} k[x] \\ & & \varphi \downarrow \\ & & \operatorname{Spec} k \end{array} .$$

Then it has a nonNoetherian colimit $X_\infty = \operatorname{Spec} k[y, xy, x^2y, \dots]/(y^2)$. It has approximations $X_n = \operatorname{Spec} k[y, xy, x^2y, \dots, x^n y]/(y^2)$, that is, there exists a morphism of the skrepa into X_n and $X_\infty \cdots \twoheadrightarrow X_n \twoheadrightarrow X_{n-1} \cdots \twoheadrightarrow X_0$ is a limit. Notice that φ is neither finite, nor proper but all X_n are finite over X_0 (and over $\operatorname{Spec} k$; cf. [Sh15, Lemma 5]).

By the general reduction of colimit to a coequalizer of coproducts of vertices and by (1) there exists a coequlizer without a colimit of finite type. In quasi-PSp the existence of a colimit for spaces does not guarantee it existence with a quasipolarization (see Example 1, (1)). By Proposition 8 below, a finite diagram of spaces (in quasi-PSp) has a colimit if it has a finite morphism (resp. in quasi-PSp). (For finite polarized diagrams it is also necessary.)

(3) (See also [Sh15, §2].) A *skrepa* in quasi-PSp is a diagram

$$(X, M) \supseteq (E, M_{|E}) \stackrel{\varphi}{\to} (P, H_P), \varphi^* H_P \simeq M_{|E},$$

in quasi-PSp with closed inclusion $\supseteq$ or immersion. Its morphism is a commutative square

$$\begin{array}{ccc} (X, M) & \supseteq & (E, M_{|E}) \\ \delta_X \downarrow & & \varphi \downarrow \\ (Y, H_Y) & \stackrel{\delta_P}{\leftarrow} & (P, H_P) \end{array} .$$

Skrepas play an important role in geometry. E.g., if $E = \emptyset$, then the skrepa has a morphism in quasi-PSp into a polarized scheme (Y, H_Y) if and only if M is stably free.

On the other hand, if X is complete, $(P, H_P) \in$ PSch and $\mathbb{E}(M) \subseteq E$ hold then any skrepa morphism has a polarized image $\mathrm{Im}(\delta)$, in particular, its colimit belongs to PSch. This follows from Basic properties of exceptional locus (5-6) in Section 1 (cf. [Sh15, §5.5]).

In many situations, it is easier to construct a *modification*: a commutative square

$$\begin{array}{ccc} (X, M) & \supseteq & (E, M_{|E}) \\ \delta_X \downarrow & & \varphi \downarrow \\ (Y, H_Y) & \supseteq & (P, H_P), \end{array}$$

where

(1) δ_P is a closed inclusion or immersion;
(2) $\delta_X^{-1} P = E$ and

$$\delta_{X|X\setminus E} \colon X \setminus E \simeq Y \setminus P$$

is an isomorphism.

In particular, φ is the restriction $\delta_{X|E}$. Un(quasi)polarized modifications were introduced by Artin [Ar]. A modification is always surjective but possibly neither epimorphic and (epimorphic) nor universal or a colimit (amalgam). However if a skrepa has a modification and a colimit

$$\begin{array}{ccc} (X, M) & \supseteq & (E, M_{|E}) \\ \psi \downarrow & & \varphi \downarrow \\ (Z, H_Z) & \stackrel{\theta}{\leftarrow} & (P, H_P), \end{array}$$

then the latter one is a modification too. This follows from the following property of isomorphisms: if $X \twoheadrightarrow Z \to Y$ is a factorization of an isomorphism starting from an epimorphism then both factors are isomorphisms. By the universal property of colimit there exists a canonical morphism $\varepsilon \colon Z \to Y$ such that the composition

$$P \stackrel{\theta_{|P}}{\twoheadrightarrow} \theta(P) \stackrel{\varepsilon_{|\theta(P)}}{\to} P$$

is identical. Hence θ is a closed immersion or inclusion up to isomorphism. So, if θ is already the inclusion then $\varepsilon_{|P} = \mathrm{Id}_P$ and $\psi^{-1} P \subseteq \delta_X^{-1} P = E$ and actually $= E$. Moreover, the composition

$$X \setminus E \overset{\psi|_{X\setminus E}}{\twoheadrightarrow} \psi(X \setminus E) \overset{\varepsilon|_{\psi(X\setminus E)}}{\rightarrow} Y \setminus P$$

is an isomorphism. Hence $\psi_{|X\setminus E}$ is an isomorphism and the colimit is a modification.

Actually, if a colimit exists and it is fibered over Y then we can relax the property (2) up to

(2)' $\delta_X^{-1} P \subseteq E$.

The corresponding morphism will be called a *premodification*. Indeed, since the colimit is fibered, we can compute it over $Y \setminus P$, that is, assuming $E = \emptyset$ by (2)'. But then a canonical colimit over $Y \setminus P$ is $X \setminus E$.

Notice that for certain morphisms of skrepas we can find a colimit according Theorem 1 above and to Lemma 7 below. Skrepa is a quite subtle concept as shows the following example (cf. also Example 4).

(4) Usually a diagram D of the form $\circ \leftarrow \circ \rightarrow \circ$ does not have a colimit but if exists it is weird. Consider, a diagram

$$Y \overset{\mathbb{Z}/n\mathbb{Z}}{\twoheadleftarrow} X \overset{\mathbb{Z}/mZ}{\twoheadrightarrow} Z,$$

where arrows are quotients by cyclic groups of automorphisms of X. If they generate an infinite group of automorphisms of X with a dense orbit then a colimit is pt.. E.g., this holds for the affine line $X = \mathbb{A}^1 = k, \operatorname{char} k = 0$, with subgroups generated by involutions: $x \mapsto -x, x \mapsto 1 - x$.

The following example with quasipolarizations is more interesting. Let $X = E$ be an elliptic curve with addition $\oplus$ and negation $\ominus$ and $e \neq 0$ be its 2-torsion, $\operatorname{char} k \neq 2$. Consider two involutions $x \mapsto x \oplus e, x \mapsto \ominus x$ and their quotients

$$X \overset{\psi}{\twoheadrightarrow} Y, X \overset{\varphi}{\twoheadrightarrow} Z,$$

respectively. The diagram

$$Y \overset{\psi}{\twoheadleftarrow} X \overset{\varphi}{\twoheadrightarrow} Z, \tag{3}$$

has a colimit V which is the quotient by the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by the involutions. Indeed, the involutions commute and the group has order 4:

$$\ominus(x \oplus e) = (\ominus x) \oplus (\ominus e) = (\ominus x) \oplus e.$$

The quotient V is isomorphic to $\mathbb{P}^1$.

On the other hand, Y is also an elliptic curve isogenous to X. Take on Y the log sheaf $\Omega_Y(p-q)$ (see Example 1, (2) in Section 1) where $p=\psi(e_1)=\psi(e_2), q=\psi(e_3)=\psi(e_4)$ and $e_1=0, e_2=e, e_3, e_4$ are 4 2-torsion points on X. Then $\psi^*\Omega_Y(p-q)=\Omega_X(e_1+e_2-e_3-e_4)$. Since the involution $\ominus$ on X has 4 fixed points $e_1,\dots,e_4$, Z is isomorphic to $\mathbb{P}^1$ and φ is ramified in $e_1,\dots,e_4$ with multiplicities 2. Hence

$$\varphi^*\Omega_Z(s+t)=\Omega_X(e_1+e_2-e_3-e_4),$$

where $s=\varphi(e_1)$ and $t=\varphi(e_2)$. This gives a natural quasipolarization on the diagram (3):

$$(Y,\Omega_Y(p-q))\overset{\psi}{\twoheadleftarrow}(X,\Omega_X(e_1+e_2-e_3-e_4))\overset{\varphi}{\twoheadrightarrow}(Z,\Omega_Z(s+t)).$$

The quasipolarized diagram does not have a quasipolarized colimit. We can suppose that the colimit is normal and is isomorphic to pt. or $\mathbb{P}^1$. By construction the polarization in both cases has degree 0. So, it is isomorphic to a trivial one. Hence $\Omega_Y(p-q)\simeq\mathcal{O}_Y$, a contradiction.

A tensor product with a polarization H on V gives a polarized version. The diagram

$$\begin{aligned}(Y,\Omega_Y(p-q)\otimes\alpha^*H)&\overset{\psi}{\twoheadleftarrow}(X,\Omega_X(e_1+e_2-e_3-e_4)\otimes\psi^*\alpha^*H)\\&\overset{\varphi}{\twoheadrightarrow}(Z,\Omega_Z(s+t)\otimes\beta^*H)\end{aligned}$$

is finite, naturally polarized but without a (quasi)polarized colimit, where $\alpha\colon Y\to V, \beta\colon Z\to V$ are canonical morphism to the (unquasipolarized) colimit V.

Notice that a diagram $Y\leftarrow X\rightarrow Z$ with finite morphisms of algebraic spaces and a finite colimit has a quasipolarized colimit for any quasipolarization of the diagram if locally the unites of $\mathcal{O}_X$ are products unites coming from Y and Z (cf. arguments of Example 4). This is usually failed according to the last example.

Lemma 4. *Let $\varphi,\psi\colon X\to Y$ be two morphism of algebraic spaces and $X=\cup X_i$ be a covering by closed subspaces such that, for every X_i,*

$$\varphi_{|X_i}=\psi_{|X_i}.$$

Then $\varphi=\psi$.

If φ,ψ are quasipolarized, the quasipolarized version holds too.

The covering is not necessarily finite. The union is the smallest closed subspace containing every X_i (see for Lattice of ideal sheaves in Section 2). The existence of a morphism $\varphi\colon X \to Y$ with given restriction $\varphi_{|X_i}$ holds by definition if $X = \operatorname{colim} X_i$ (see Theorem 3 below).

Proof. We can suppose that X, Y are affine. Using coordinate functions or all regular functions on Y we can suppose that $\varphi, \psi\colon X \to k$ are regular functions on X. Then $\varphi - \psi$ is vanishing on X and $= 0$.

If L, M are quasipolarizations of X, Y respectively. Then, for quasipolarized φ, ψ,

$$L \overset{\varphi}{\simeq} \varphi^* M \text{ and } L \overset{\psi}{\simeq} \psi^* M = \varphi^* M.$$

Hence, for isomorphisms of sheaves, $\psi = a\varphi$ holds for some $a \in \Gamma(X, \mathcal{O}_X^\times)$ (see for Category of quasipolarizations in Section 1). By our assumptions, for every X_i, $a_{|X_i} = 1$ holds. So, $a = 1$ and $\varphi = \psi$ in quasi-PSp.

Alternative proof for spaces. By our assumptions

$$\sqcup X_i \overset{\varepsilon}{\twoheadrightarrow} X$$

is an epimorphism. On the other hand the disjoint union is a colimit (see Example 2, (1)). So, by our assumptions $\varphi \circ \varepsilon = \psi \circ \varepsilon$ holds. Hence by Proposition 5 $\varphi = \psi$. Of course, this works directly only for finite covers. In general, we need to use the largest closed subspace where $\varphi = \psi$ (see for Step 1 in the proof of Proposition 5). All finite unions are inside of it and it is $X = \cup X_i$. □

The uniqueness of a morphism from a colimit can be replaced by the epimorphicity of a universal morphism.

Corollary 8. *Let D be a diagram of algebraic spaces and $\delta\colon D \to X$ be an epimorphism onto a space X such that for any other morphism $\varphi\colon D \to Y$ there exists a morphism of $\psi\colon X \to Y$ of δ into φ. Then δ is a colimit of D.*

Proof. It is sufficient the uniqueness of ψ. So, let $\vartheta\colon X \to Y$ be another morphism of δ into φ. Since $\delta\colon D \to X$ is surjective and by Lemma 4, it is sufficient to verify that, for every vertex V of D, $\psi_{|\delta_V V} = \vartheta_{|\delta_V V}$ holds. On the other hand,

$$\psi_{|\delta_V V} \circ [V \twoheadrightarrow \delta_V V] = \psi \circ \delta_V = \varphi_V = \vartheta \circ \delta_V = \vartheta_{|\delta_V V} \circ [V \twoheadrightarrow \delta_V V]$$

by the commutativity of

$$
\begin{array}{ccc}
 & & X \\
 & \delta_V \nearrow & \\
V & & \psi \text{ or } \vartheta \downarrow \\
 & \varphi_V \searrow & \\
 & & Y
\end{array}
$$

according to the definition of restriction for a morphism and by the canonical factorization. Hence by the epimorphicity of $V \twoheadrightarrow \delta_V V$,

$$\psi_{|\delta_V V} = \vartheta_{|\delta_V V}$$

holds. $\square$

Proof of Epimorphic property for colimits. By definition, for every $X \in D$, $\operatorname{Im}\delta_X \subseteq \operatorname{Im}\delta$. Hence, by the canonical factorization, for every $\delta_X \colon X \to \operatorname{colim} D$, there exists a canonical factorization

$$X \overset{\delta_{\operatorname{Im}\delta,X}}{\to} \operatorname{Im}\delta \subseteq \operatorname{colim} D.$$

Moreover, by Restrictions and canonical factorization in Section 2, $\delta_{\operatorname{Im}\delta,X}$ form a morphism $\delta_{\operatorname{Im}\delta} \colon D \to \operatorname{Im}\delta$. We contend that $\delta_{\operatorname{Im}\delta}$ is also a colimit and the inclusion $\operatorname{Im}\delta \subseteq \operatorname{colim} D$ is a canonical morphism of colimits. Hence, by the universal property of colimits it is actually an identity.

Indeed, by Restrictions and canonical factorization, for every arrow $X \to Y$ of D, the commutative triangle

$$
\begin{array}{ccc}
X & & \\
 & \delta_X \searrow & \\
\downarrow & & \operatorname{colim} D \\
 & \delta_Y \nearrow & \\
Y & &
\end{array}
$$

gives a commutative diagram

$$
\begin{array}{ccccc}
X & & & & \\
 & \delta_{\operatorname{Im}\delta,X} \searrow & & \delta_X \searrow & \\
\downarrow & & \operatorname{Im}\delta & \subseteq & \operatorname{colim} D \\
 & \delta_{\operatorname{Im}\delta,Y} \nearrow & & \delta_Y \nearrow & \\
Y & & & &
\end{array}
.
$$

This gives the canonical factorization of $\delta_{\operatorname{Im}\delta}$ through δ:

$$D \overset{\delta_{\operatorname{Im}\delta}}{\twoheadrightarrow} \operatorname{Im}\delta \subseteq \operatorname{colim} D.$$

By Corollary 8 and since $\delta_{\operatorname{Im}\delta}$ is epimorphic, it is sufficient to verify the existence of the universal property for $\delta_{\operatorname{Im}\delta}$.

Let $\gamma\colon D \to E$ be a morphism of D into an algebraic space E and $\varepsilon\colon \operatorname{colim} D \to E$ be a morphism of the universal property for δ. Then $\varepsilon_{|\operatorname{Im}\delta}\colon \operatorname{Im}\delta \to E$ is a morphism of the universal property for $\delta_{\operatorname{Im}\delta}$. Equivalently, the diagram

$$\begin{array}{ccccc} & & \operatorname{Im}\delta & \subseteq & \operatorname{colim} D \\ & \delta_{\operatorname{Im}\delta}\nearrow & & & \\ D & & \varepsilon_{|\operatorname{Im}\delta}\downarrow & \varepsilon\swarrow & \\ & \gamma\searrow & & & \\ & & E & & \end{array}$$

is commutative. The diagram is commutative because, for every $X \in D$, the diagram

$$\begin{array}{ccccc} & & \operatorname{Im}\delta & \subseteq & \operatorname{colim} D \\ & \delta_{\operatorname{Im}\delta,X}\nearrow & & & \\ X & & \varepsilon_{|\operatorname{Im}\delta}\downarrow & \varepsilon\swarrow & \\ & \gamma_X\searrow & & & \\ & & E & & \end{array}$$

is commutative by construction and Restrictions and canonical factorization.

The surjectivity for closed points can be established similarly where we take $\operatorname{Im}\delta$ without a missing point (see for Restrictions and canonical factorization in Section 2). For finite D, this gives a surjection on closed points for the morphism of coproduct of vertices of D to $\operatorname{colim} D$. In its turn, this implies surjection for all points by the Noetherian property of Zariski topology.

If the colimit δ is quasipolarized then the image $\operatorname{Im}\delta = \operatorname{colim} D$ has the restricted (natural) quasipolarization and the identity with colimit become the identity in quasi-PSp. □

Finite diagram of spaces We say that a diagram D of algebraic spaces is *finite* if it has finitely many vertices and arrows. Similarly we can define a finite diagram in any category. E.g., the diagram D is *finite over* X if D is finite in the category of spaces finite over X, that is, every vertex of D is a finite morphism $Y \to X$. Equivalently, D can be treated as a diagram D

of spaces Y such that every vertex $Y \in D$ has a unique structure morphism $Y \to X$, it is finite and every arrow of D is a morphism over X, that is, the structure morphisms form a morphism $D \to X$.

Those finite diagrams form a much larger class than the finite diagrams as functors from a finite category. However, if we suppose that the category is finitely generated then we get essentially the same class.

Recall that any finite space Y over X has the form $Y = \mathcal{S}\mathrm{pec}\, \varphi_* \mathcal{O}_Y$ where $\varphi\colon Y \to X$ is the structure morphism and $\mathcal{O}_Y$ is the structure sheaf of Y. The direct image $\varphi_* \mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-modules. Respectively, any morphism $\psi\colon X_1 \to X_2/X$ of two spaces over X with structure morphisms φ_1, φ_2 is given by a unique homomorphism $\psi^*\colon \varphi_{2*}\mathcal{O}_{X_2} \to \varphi_{1*}\mathcal{O}_{X_1}$ of $\mathcal{O}_X$-algebras: $\psi = \mathcal{S}\mathrm{pec}\, \psi^*$. Note that ψ is automatically finite. For a diagram D of algebraic spaces finite over X, denote by O the corresponding diagram of $\mathcal{O}_X$-algebras $\varphi_* \mathcal{O}_Y$. Then a canonical isomorphism $D = \mathcal{S}\mathrm{pec}\, O$ holds where D is considered as a *sheaf of diagrams* of spaces over X. In other words, for every opens set $U \subseteq X$ in an appropriate topology (Zariski for schemes, étale or fppf for spaces) isomorphisms of restricted sheaves $D_{|U} = (\mathcal{S}\mathrm{pec}\, O)_{|U} = \mathcal{S}\mathrm{pec}(O_{|U})$ hold. The diagram isomorphism is *fibered* and/or has a (quasi)coherent nature: if $U = \mathrm{Spec}\, A$ is affine then $Y_{|U} = \mathrm{Spec}\, B, Y \in D$, as a space or even scheme where $B = \Gamma(U, \varphi_* \mathcal{O}_Y)$. For spaces U is an affine covering. The diagram D is finite over X if and only if O is finite over $\mathcal{O}_X$. For a separated scheme X we can use [H, Proposition 5.2(d), Chapter II] but for separated spaces [Kn, Proposition 7.2, Chapter II; Theorem 4.1, Chapter IV].

General reduction to the relative case is as follows.

Lemma 5. *Let D be a diagram over T. Then*

$$\mathrm{colim}\, D = \mathrm{colim}_{/T}\, D,$$

where $\mathrm{colim}_{/T}$ *is a colimit in the category of spaces over T. Moreover, the colimit of D exists if and only if it exists over T.*

If additionally D is finite over T, then the colimit is relative in the category of finite spaces over T.

In the relative situation we say that $\mathrm{colim}\, D$ is *fibered* over T if, for any open set U (in a covering for spaces) of T,

$$(\mathrm{colim}\, D)_{|U} = \mathrm{colim}(D_{|U}).$$

For relative finite diagrams it is equivalent to the above coherent nature.

Proof. If $X = \operatorname{colim} D$ exists then, by the universal property of a colimit, X and the morphism $\delta\colon D \to X$ are canonically defined over T. Moreover, the colimit is universal over T too.

Conversely, suppose that $X = \operatorname{colim}_{/T} D$ exists. Denote by $\psi\colon D \to T$ the morphism which gives the structure of D over T. Take any morphism $\varphi\colon D \to Y$ (not necessarily over T). Then there exists a canonical morphism

$$\varphi \times \psi\colon D \to Y \times T,$$

where the morphism $V \to T$ of a vertex V of D is the product

$$\varphi_V \times \psi_V\colon V \to Y \times T$$

of morphisms $\varphi_V\colon V \to Y, \psi_V\colon V \to T$. Indeed, if $\alpha\colon V \to W$ is an arrow of D then

$$(\varphi_W \times \psi_W) \circ \alpha = (\varphi_W \circ \alpha) \times (\psi_W \circ \alpha) = \varphi_V \times \psi_V.$$

By construction $Y \times T$ has a canonical morphism pr_T into T, and the morphism $D \to Y \times T$ is defined over T:

$$\psi_V = pr_T \circ (\varphi_V \times \psi_V)\colon V \to Y \times T \to T.$$

By the universal property of δ there exists a unique morphism $\chi\colon X \to Y \times T$ such that for every vertex V of D the diagram

$$\begin{array}{ccc} & & X \\ & \delta_V \nearrow & \\ V & & \chi \downarrow \\ & \varphi_V \times \psi_V \searrow & \\ & & Y \times T \end{array}$$

is commutative. This gives a commutative diagram

$$\begin{array}{ccc} & & X \\ & \delta_V \nearrow & \\ V & & \chi_Y \downarrow \\ & \varphi_V \searrow & \\ & & Y \end{array}$$

with the vertical morphism

$$\chi_Y = \mathrm{pr}_Y \circ \chi\colon X \to Y \times T \to Y.$$

Immediate by the commutativity of

$$\begin{array}{ccc} & & Y\times T \\ & \varphi_V\times\psi_V \nearrow & \\ V & & \mathrm{pr}_Y \downarrow \\ & \varphi_V \searrow & \\ & & Y \end{array}.$$

Hence there exists a morphism $\chi_Y\colon X\to Y$ from the universal morphism $D\to X$ into $D\to Y$. This morphism is unique, by Epimorphic property for δ and by Corollary 8, and X is a colimit of D.

Suppose now that D is finite over T. Then we can verify that the colimit X is relative in the category of finite spaces over T. Indeed, let $\varphi\colon D\to Y/T$ be a relative morphism of D into an algebraic space Y over T, possibly not finite over T. Then the image $\varphi(D)\subseteq Y$ is finite over T by [H, Exercise 4.4, Chapter II] [Kn, Proposition 7.2, Chapter II]. We use here also the property that a finite disjoint union of finite morphisms is finite by Example 2, (1). This give a canonical factorization of φ into the composition $D\twoheadrightarrow\varphi(D)\subseteq Y$. By the universal property in the category of finite spaces over T, there exists a morphism $\varepsilon\colon X\to\varphi(D)$ with the commutative diagram

$$\begin{array}{cccc} & & X & \\ & \nearrow & & \\ D & & \varepsilon\downarrow & \\ & \searrow & & \\ & & \varphi(D) & \subseteq\ Y \end{array}.$$

Since $D\to X$ is epimorphic, the morphism ε is unique again by Corollary 8. ☐

Proposition 7. *Let D be a quasipolarized diagram such that*

D has a colimit $\lambda\colon D\to\operatorname{colim}D$ without the quasipolarization;
the colimit is fibered, that is, for every open set U (in a covering) of $\operatorname{colim}D$,

$$U=(\operatorname{colim}D)_{|U}=\operatorname{colim}(D_{|U});$$

and
there exists a quasipolarized morphism of D.

Then the colimit can be quasipolarized. Moreover, the category of locally trivial natural quasipolarizations of D over $\operatorname{colim}D$ is equivalent to $\text{quasi-PSp}\operatorname{colim}D$ *under λ^* and* $\lim\lambda_*$.

So, the colimit λ behaves as a contraction (cf. Proposition 2). Notice that the *locally trivial* property of D means that locally over $\operatorname{colim} D$ the diagram is isomorphic to a trivially quasipolarized one. By the proposition it is not true in general (see Examples 1, (2) and 2, (4)). However the trivial property holds always for certain (finite) diagrams, e.g., for skrepas with a finite morphism (see Example 4).

Proof. Below we suppose that D actually denotes the diagram only of spaces and M is its quasipolarization. Let $\delta\colon (D, M) \to (T, O)$ be a morphism in quasi-PSp and $\lambda\colon D \to Z = \operatorname{colim} D$ be a colimit of D as of a diagram of (separated) algebraic spaces. By the universal property of colimits there exists and unique morphism $\varepsilon\colon Z \to T$ such that for every $X \in D$ the diagram

$$\begin{array}{ccccc} & & & & Z \\ & & \lambda_X \nearrow & & \\ X & & & & \varepsilon \downarrow \\ & & \delta_X \searrow & & \\ & & & & T \end{array}$$

is commutative. By construction δ is quasipolarised. By Lemma 1 we can replace (D, M) on an isomorphic diagram such that δ became naturally quasipolarized: for every $X \in D$, $M_{|X} = \delta_X^* O$, and, for every arrow $\varphi\colon X \to Y$ in D, $\varphi^*(M_{|Y}) = M_{|X}$ hold.

In its turn the canonical morphism ε induces a natural quasipolariztion on λ. Indeed, we can take a natural quasipolariztion $\varepsilon^* O$ on Z. Then the natural quasipolarization on D, induced from $(Z, \varepsilon^* O)$ under λ, is M and λ is quasipolarized because, for every $X \in D$, $\lambda_X^* \varepsilon^* O = \delta_X^* O = M_{|X}$ holds. We contend that $\varepsilon^* O$ is a required quasipolarization of the colimit. By construction M is locally trivial over Z.

It looks that the quasipolarization on Z depends on a morphism δ. However it is not true and there exists a universal quasipolarization on Z (cf. Examples 2). For this it is sufficient to verify that

$$\varepsilon^* O = \lim \lambda_* M, \ \text{(canonical isomorphism)}$$

where $\lambda_* M$ the following diagram of $\mathcal{O}_Z$-modules on Z. (If D is finite or proper over T then the diagram of modules is finite over Z. In general the modules are only quasicoherent over $\mathcal{O}_Z$.) The vertices of $\lambda_* M$ are $\mathcal{O}_Z$-modules $\lambda_{X*} M_{|X}$ for every vertex $X \in D$. Every arrow $\varphi\colon X \to Y$ of D gives the arrow

$$\varphi^*\colon \lambda_{Y*} M_{|Y} \to \lambda_{X*} M_{|X},$$

a homomorphism of $\mathcal{O}_Z$-modules. Additionally, there exists a canonical homomorphism $\varepsilon^*O \to \lambda_* M$ of a $\mathcal{O}_Z$-module to the diagram of $\mathcal{O}_Z$-modules. For every vertex $X \in D$:

$$\lambda_{X*} \circ \lambda_X^* \colon \varepsilon^* O \to \lambda_X^* \varepsilon^* O = M_{|X} \to \lambda_{X*} M_{|X}.$$

The composition $\lambda_{X*} \circ \lambda_X^*$ is considered on sections of the sheaves over Z: if U is an open subset or set in a covering of X in étale (or other topology) and $s \in \Gamma(U, \varepsilon^* O)$ is such a section then it goes as follows

$$\begin{aligned} s \mapsto \lambda_X^* s &\in \Gamma(\lambda_X^{-1} U, \lambda_X^* \varepsilon^* O) \\ &= \Gamma(\lambda_X^{-1} U, M_{|\lambda_X^{-1} U}) \mapsto \lambda_{X*}(\lambda_X^* s) \in \Gamma(U, \lambda_{X*} M_{|X}). \end{aligned}$$

This is a homomorphism into the diagram $\lambda_* M$: for every arrow $\varphi \colon X \to Y$, and any section s as above,

$$\varphi^* \lambda_{Y*} \lambda_Y^* s = \varphi^* \lambda_Y^* s = \lambda_X^* s = \lambda_{X*} \lambda_X^* s,$$

because $\lambda_X^* s = \varphi^* \lambda_Y^* s$ holds.

By the universal property there exists a canonical homomorphism $\varepsilon^* O \to \lim \lambda_* M$. The last limit exists because the category of sheaves on Z is Abelian. Thus we need to verify that $\varepsilon^* O \to \lim \lambda_* M$ is an isomorphism. For this we can use affine covering and trivializations on affine open sets. In other words, we can suppose that $\varepsilon^* O \simeq \mathcal{O}_Z$. Then the diagram (D, M) is isomorphic to the trivial natural diagram: for every $X \in D$, $M_{|X} = \mathcal{O}_X$ and $\varepsilon^* O = \mathcal{O}_Z$ hold. So, we need to verify the canonical isomorphism

$$\mathcal{O}_Z = \lim \lambda_* \mathcal{O}_D,\ \mathcal{O}_{D|X} = \mathcal{O}_X,$$

$\mathcal{O}_D$ is the structure sheaf of D. By the fibered assumption $Z = \operatorname{colim} D$ still holds. On the other hand, the required identity $=$ is given by canonical homomorphism $\mu \colon \mathcal{O}_Z \to \lim \lambda_* \mathcal{O}_D$. The last homomorphism is also a homomorphism of $\mathcal{O}_Z$-algebras, not only of $\mathcal{O}_Z$-modules. This can be translated into the commutative diagram (morphism $\nu = \mathcal{S}\mathrm{pec}\,\mu$ of $\mathcal{S}\mathrm{pec}\,\lambda_*$ into λ)

$$D \ \to \ \mathcal{S}\mathrm{pec}\,\lambda_* \mathcal{O}_D \quad \begin{array}{c} \mathcal{S}\mathrm{pec}\,\lambda_* \nearrow \\ \\ \lambda \searrow \end{array} \quad \begin{array}{c} \mathcal{S}\mathrm{pec} \lim \lambda_* \mathcal{O}_D \\ \nu = \mathcal{S}\mathrm{pec}\,\mu \downarrow \\ Z = \operatorname{colim} D \end{array} \ , \lambda_* \colon \lim \lambda_* \mathcal{O}_D \to \lambda_* \mathcal{O}_D,$$

of spaces. We need to establish that ν is an isomorphism of spaces. By Corollary 8 and the universal property of Z, it is sufficient to establish the epimorphic property of $\mathcal{S}\mathrm{pec}\,\lambda_*$, equivalently, λ_* is injective or a monomorphism. Note that by construction $D \to \mathcal{S}\mathrm{pec}\,\lambda_*\mathcal{O}_D$ is epimorphic: for every $X \in D$, $X \twoheadrightarrow \mathcal{S}\mathrm{pec}\,\lambda_{X*}\mathcal{O}_X$ is epimorphic. (For spaces and schemes $\mathrm{Im}[X \twoheadrightarrow \mathcal{S}\mathrm{pec}\,\lambda_{X*}\mathcal{O}_X] = \mathcal{S}\mathrm{pec}\,\lambda_{X*}\mathcal{O}_X$ in the space and scheme sense respectively. If λ_X is finite or affine then the last epimorphism is an isomorphism.) The injectivity of λ_* is the monomorphic property of limits in an Abelian category (see Remark 2, (2)):

$$\lim \lambda_*\mathcal{O}_D \hookrightarrow \lambda_*\mathcal{O}_D.$$

Actually, we proved the existence of a canonical isomorphism between ε^*O and $\lim \lambda_* M$. The latter should be a quasipolarization on Z (a descent) and the isomorphism gives required quasipolarized $\varepsilon\colon (Z, \lim \lambda_* M) \to (T, O)$ for the universal property in quasi-PSp. So, for every quasipolarization M locally trivial over Z, $\lim \lambda_* M$ is a quasipolarization on Z which gives the required equivalence. □

Proposition 8. *Let X be a Noetherian algebraic space and D be a finite diagram over X. Then* $\mathrm{colim}\, D$ *exists and is Noetherian and finite over X. If O is the corresponding diagram of $\mathcal{O}_X$-algebras then a canonical isomorphisms of spaces*

$$\mathrm{colim}\, D = \mathcal{S}\mathrm{pec}\lim O$$

holds where the colimit, $\mathcal{S}\mathrm{pec}$ *and limit are considered as sheaves or fibered over X.*

If D is quasipolarized then the colimit is quasipolarized too.

If X is of finite type over k then $\mathrm{colim}\, D$ *is of finite type over k too.*

Conversely, if D is a finite diagram and every structure morphism $X \to \mathrm{colim}\, D$, $X \in D$, is finite then D is finite over $\mathrm{colim}\, D$.

Proof. By Lemma 5 it is sufficient to find a relative colimit over X. Under the above notations, $D = \mathcal{S}\mathrm{pec}\, O$ and a $\mathrm{colim}\, D$ has the structure sheaf $\lim O$, that is, $\mathrm{colim}\, D = \mathrm{Spec}\lim O$. Since D is finite over X, by the same lemma, we can take the colimit in the category of finite spaces over X. The $\mathcal{S}\mathrm{pec}$ functor transforms the limits of $\mathcal{O}_X$-algebras into colimits in the category of finite spaces over X. The functor agrees with restrictions on open sets [H, Exercise 5.17, Chapter II].

The $\lim O$ is coherent as the $\mathcal{O}_X$-module because can be presented as an equalizer of two homomorphisms $h_1, h_2 : \mathcal{F} \underset{\to}{\overset{\to}{}} \mathcal{G}$ of $\mathcal{O}_X$-algebras which are coherent as $\mathcal{O}_X$-modules. Since D is finite we can suppose that both $\mathcal{O}_X$-algebras $\mathcal{F}, \mathcal{G}$ are coherent as $\mathcal{O}_X$-modules. (We can construct $\mathcal{F}, \mathcal{G}$ as finite products of copies of vertices of O.) The equalizer is $\ker(h_1 - h_2)$ as an $\mathcal{O}_X$-module and is a coherent $\mathcal{O}_X$-module. This implies also the fibered property by the exact property of localizations [Mat, Theorem 4.5].

If X is of finite type over k then so does $\operatorname{colim} D$ because it is finite over X.

If D is quasipolarized then $\operatorname{colim} D$ has a natural quasipolarization which makes the colimit quasipolarized. Notice that D is relative and its quasipolarization is supposed to be over X, that is, there exists a quasipolarization M on D and an invertible sheaf M such that, for every vertex $\varphi\colon Y \to X$ in D, $\varphi\colon (Y, M_{|Y}) \to (X, M_{|X})$ is naturally quasipolarized. Hence by Proposition 7 the colimit is quasipolarized as an absolute one in quasi-PSp. □

Remarks 3. (1) By Lemma 5 the colimit in Proposition 8 can be treated relatively that we use in the proof. The same holds for the limit of algebras.

(2) For any diagram D of algebraic spaces finite or affine over X, $\operatorname{colim} D$ exists and has the same form $\operatorname{colim} D = \operatorname{Spec} \lim O$ but it is possible not of finite type. In the quasipolarized case $\operatorname{colim} D$ has a natural quasipolarization as in the proof above.

(3) A diagram D is over a space X if and only if D has a morphism into X. The same holds in quasi-PSp over quasipolarized X. By Lemma 1 up to isomorphism any relative quasipolarization is natural.

For certain diagrams we can induce a canonical quasipolarization on their colimits.

Example 4 ((Quasi)polarized Artin modification [Ar, Theorem 6.1]; cf. Examples 2 above and 7 below.). Let $(X, H) \supseteq (E, H_{|E}) \xrightarrow{\varphi} (P, H_P), (P, H_P) \in$ quasi-PSp, be a *finite* skrepa, that is, φ is finite. Then it always has a colimit (Y, H_Y) in quasi-PSp and it is fibered. Actually, the colimit is a modification. The existence of Y and modification property, for spaces without quasipolariztion, established in [Ar, Theorem (6.1)]. This implies that the skrepa is finite over Y and by Proposition 8 it is fibered. So, by Proposition 7, it is sufficient to verify that any quasipolarization of the skrepa is locally trivial over Y. Taking an affine covering, we can suppose that spaces X, E, P are affine and $H, H_{|E}, H_P$ are trivial: $H = \mathcal{O}_X, H_{|E} = \mathcal{O}_E, H_P = \mathcal{O}_P$. However, the isomorphisms of skrepa

$$\mathcal{O}_{X|E} \overset{\supseteq}{\simeq} \mathcal{O}_E \overset{\varphi}{\simeq} \varphi^* \mathcal{O}_P$$

are, possibly, unnatural, that is, they are isomorphisms of modules but not of rings. We need to find an isomorphic skrepa with trivial (natural) isomorphisms, possibly, after taking a smaller space X. Since any automorphism of $\mathcal{O}_E$ is a multiplication by a unite of $\Gamma(E, \mathcal{O}_E)$ (see for Category of quasipolarizations in Section 1), we can suppose that $\mathcal{O}_E \overset{\varphi}{=} \varphi^* \mathcal{O}_P$ is already natural. On the other hand, $1 \in \Gamma(E, \mathcal{O}_E)$ corresponds to some element $e \in \Gamma(X, \mathcal{O}_X)$ such that $e_{|E} = 1$. If e is a unite of $\Gamma(X, \mathcal{O}_X)$, we can replace $(X, \mathcal{O}_X)$ by itself using the quasipolarized automorphism of $(X, \mathcal{O}_X)$ which is identical on X and such that $e \mapsto 1 \in \Gamma(X, \mathcal{O}_X)$; this is the multiplication by e^{-1} in $\Gamma(X, \mathcal{O}_X)$. If e is not a unite in $\Gamma(X, \mathcal{O}_X)$, it does so for a smaller open neighborhood $U \subset X = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ of $E \subseteq U$, e.g., take $U = X \setminus (e)$, where (e) is the principal closed subspace given by the ideal (e) of $\Gamma(X, \mathcal{O}_X)$. Note that, if $E \neq \emptyset$, then e is not nilpotent, U is also affine and $\neq \emptyset$. If $E = \emptyset$, the colimit is trivial.

So, every polarized skrepa $(X, H) \supseteq (E, H_{|E}) \xrightarrow{\varphi} (P, H_P), (X, H), (P, H_P) \in \mathrm{PSch}$, with proper φ has a colimit, in particular a morphism into a polarized pair. Note that φ is finite in this situation automatically (cf. [Sh15, Corollary 2]). If X, P are complete, then the skrepa has a morphism into a polarized projective space because some rather high tensor powers $H^{\otimes n}, H_P^{\otimes n}$ of the diagram are generated by global sections. This can be done as in the proof of Proposition 1. (The Serre vanishing [H, Theorem 5.2, Chapter III] is enough for extension of sections from E.) As in the proof, for $n \gg 0$ and for all allowed sections, the morphism on the image is a modification and an amalgam of the skrepa.

Proposition 9. *Let $B \subseteq A$ be an inclusion of rings and I be an ideal of B. The following statements are equivalent:*

I is a conductor of the inclusion $B \subseteq A$;
the diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \supseteq & \varphi^{-1} Z \\ \varphi \downarrow & & \varphi_{|\varphi^{-1}Z} \downarrow \\ \operatorname{Spec} B & \supseteq & Z \end{array}$$

is a modification and an amalgam, $\varphi^{-1} Z \twoheadrightarrow Z$ is an epimorphism, where Z is a subscheme given by the ideal I.
If $J \subseteq I$ are ideals of A and I is a conductor then J is also a conductor.

For similar diagrams of algebraic spaces we can use the second statement as a definition of a conductor subspace (cf. Proposition 10).

Example 5. Let X be a complete nonsingular curve and $\mathfrak{m} = \sum n_P P$ be an effective divisor (module) on X. Then there exists a unique proper birational epimorphism $X \twoheadrightarrow X_\mathfrak{m}$ on a curve $X_\mathfrak{m}$ such that the effective Cartier divisor $\mathfrak{m}$ goes to a point of $X_\mathfrak{m}$ [Ser, p. 70]. The latter curve is a colimit of the skrepa

$$X \supset \mathfrak{m} \twoheadrightarrow \text{pt.},$$

where $\mathfrak{m}$ is considered as a subscheme given by the ideal sheaf $\mathcal{O}_X(-\mathfrak{m})$.

In general, we replace pt. $= \operatorname{Spec} k$ by an Artinian scheme. For instance, let Y be a plane curve with a simple triple point P. Taking a general hyperplane section we can construct an effective Cartier divisor E on Y such that

Supp $E = P$; and

$\varphi^* E = P_1 + P_2 + P_3$ on a normalization $\varphi\colon X \to Y$.

Note that the divisor $\varphi^* E$ gives a different curve $X_{\varphi^* E}$, a seminormalization of Y. This implies that the image $\varphi(\varphi^* E)$ is not E but its reduction $E_{\text{red}} = \operatorname{Supp} E = P$. This implies also that there are rational functions f on X such that f is regular in every point $P_i, i = 1, 2, 3$ with $f(P_1) = f(P_2) = f(P_3) = 0$, but f as a rational function on Y is not regular at P. However $f = g/s$, where g is regular at P and s is an equation of E. By construction $g(P) = 0$ but we can't divide g by s on Y. However we can construct Y as a direct limit of

$$X \subset n\varphi^* E \twoheadrightarrow \varphi(n\varphi^* E)$$

for every $n \geq 2$.

Proof of Proposition 9. Recall that I is a *conductor* of $B \subseteq A$ if I is an ideal of both rings. See also for Step 3 in the proof of Proposition 10. □

Lemma 6. *Let $\varphi\colon X \twoheadrightarrow Y$ be a finite birational morphism of* S_1 *(e.g., primary) algebraic spaces. Then, for any sufficiently large effective Cartier divisor $F \subseteq Y$, the subspace $\varphi\varphi^* F$ is a* conductor *of φ. Moreover, for any closed subspace $G \supseteq \varphi^* F$ of X, the image $\varphi(G)$ is also a conductor.*

The S_1 property can be replaced by the epimorphic property of φ but then F is a closed (principal) subspace topologically proper in every irreducible component of Y and φ^{-1} should be instead of φ^*.

Proof. (Cf. Example 6.) Using affine coverings we reduce the general case to the affine one. (On this way we can loose irreducibility and the primary property.) Let $A \supseteq B$ be inclusion of rings such that its Spec is φ, and s be nonzero divisor of B. It gives an effective Cartier divisor E. Since A, B are S_1 and the inclusion is birational, s is a nonzero divisor of A too; the birational property means an isomorphism of dense open subsets. Suppose also that φ is an isomorphism over $Y \setminus \operatorname{Supp} E$. Then $A_{(s)} = B_{(s)} \supseteq A \supseteq B$ hold. Since A is finite over B as a B-module, A in $B_{(s)}$ is generated over B by finitely many fractions $b_i/s^i, i \geq 1, b_i \in B$. Hence, for some positive integer n, $A \subseteq s^{-n}B$ and $s^n A \subseteq B$ hold. This ideal $s^n A$ of B is a conductor.

If $t \in B$ is a nonzero divisor such that $s^n \mid t$ in B, tA is also a conductor of $A \supseteq B$. This gives required properties for the Cartier divisor F on Y given by t: the subspace $\varphi\varphi^* F$ is given by the ideal tA of B. By construction this holds for any sufficiently large effective Cartier divisor F: for any F containing nE.

The statement about G follows from definition (see Proposition 9). ☐

Example 6. Let $\varphi\colon X \twoheadrightarrow Y$ be a finite birational morphism of primary algebraic spaces, E be an effective Cartier divisor E such that φ is an isomorphism over $Y \setminus \operatorname{Supp} E$ and n be a positive integer such that $\varphi n \varphi^* E \supseteq E$. Notice that if there exists a big divisor on Y then there exists E and by Corollary 5 n with required properties. Moreover, in this situation $\varphi^*\varphi\varphi^* F = \varphi^* F$. In other words, $\varphi\varphi^* F$ is a conductor of φ, that is, so does its ideal sheaf. Equivalently, Y is a pinching of Proposition 10.

By canonicity of constructions, we can suppose that Y is affine. Then we have the following algebraic translation as in Proposition 8 and a proof for the affine case.

Let A, B be commutative rings such that $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, B \subseteq A$, and $F = nE, E$ be Cartier divisors on Y given respectively by nonzero divisors $s = t^n, t$ of B. (They are also nonzero divisors of A because 0 is a primary ideal of A and of B.)

The canonical inclusion $B \subseteq A$ is given by the epimorphism φ. Under our assumptions A is finite as an B-module. Actually, we can weaken this condition (cf. the proof of Lemma 6) and assume that φ is *affine* birational. By construction, F as a subscheme is given by a principal ideal $I = sB$ of B generated by $s \in B$, respectively, E is given by tB [H, Corollary 5.10, Chapter II].

There are canonical isomorphisms $A_{(s)} = A_{(t)} = B_{(s)} = B_{(t)}$ because φ is regular isomorphism of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ over the open subset $\operatorname{Spec} B \setminus \operatorname{Supp} F = \operatorname{Spec} B \setminus \operatorname{Supp} E$. By our assumptions F is sufficiently divisible: $s = t^n$ and

$$sA \cap B \subseteq tB.$$

The last inclusion means that $\varphi\varphi^* F = \varphi(\varphi^{-1} F) \supseteq E$. So, the required equation $\varphi^*\varphi\varphi^* F = \varphi^{-1}\varphi(\varphi^{-1} F) = \varphi^* F = \varphi^{-1} F$ means that $sA \cap B = sA \subseteq B$ is a conductor of the inclusion $B \subseteq A$.

To verify the inclusion $sA \subseteq B$ consider $a \in sA$. The equality $A_{(t)} = B_{(t)}$ means that $a = b/t^m, m \geq 0$, and $b \in B$. So, $at^m = b \in t^m sA \cap B \subseteq sA \cap B \subseteq tB$. Hence $b = ct, c \in B$, and $a = ct/t^m = c/t^{m-1}$ for $m \geq 1$. If we suppose that m is minimal or by induction $m = 0$ and $a = b \in B$.

Lemma 7. *Let D be a diagram*

$$\begin{array}{ccc} X & \supseteq & E \\ & & \psi \downarrow \\ & & Y \end{array}$$

of algebraic spaces over an algebraic space T such that

(1) *X is proper over T;*
(2) *Y is finite over T; and*
(3) *E is a closed subspace which includes the relative exceptional locus of X/T.*

Then a colimit of D exists, is finite over T and canonically isomorphic to a colimit of the following finite diagram over T

$$\begin{array}{ccc} V & \supseteq & \varphi(E) \\ & & \downarrow \\ & & W \end{array}, \tag{4}$$

where V is given by a contraction $\varphi\colon X \twoheadrightarrow V$ of X over T and W is a colimit of a diagram

$$\begin{array}{ccc} \varphi(E) & \twoheadleftarrow & U \\ & & \downarrow \\ & & Y \end{array}$$

given by a contraction $E \twoheadrightarrow U$ of E over T.

Moreover, the colimits, isomorphism, contractions and their factorizations are fibered over T and over the colimits, and if D is quasipolarized over T then they are quasipolarized too.

Remark 1. The colimit W is fibered and canonically isomorphic to a colimit of

$$\begin{array}{ccc} \varphi(E) & \twoheadleftarrow & E \\ & & \downarrow \\ & & Y \end{array}.$$

In particular, the letter exists (see for the end of proof of the lemma).

Proof. Step 1. (*Construction of a colimit of* D.) By construction V, U are finite over T and by (2) Y is finite over T. This implies that $\varphi(E)$ and W are finite over T too. Hence by Proposition 8 the diagram (4) has a colimit Z. Moreover the colimit is a modification, in particular, we can replace $W \to Z$ by an inclusion $W \subseteq Z$ (cf. [Ar, Theorem 6.1]). We verify that Z is also a colimit of D. Indeed, we already constructed the commutative diagram

$$\begin{array}{ccccc} X & \supseteq & E & & \\ \downarrow & & \downarrow & \searrow & \\ V & \supseteq & \varphi(E) & \twoheadleftarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ Z & \leftarrow & W & \leftarrow & Y \end{array},$$

where $E \twoheadrightarrow U \to Y$ and $E \twoheadrightarrow U \twoheadrightarrow \varphi(E)$ are Stein factorizations of ψ and $E \twoheadrightarrow \varphi(E)$ respectively. Here we use the universal property of Stein factorization (see for Stein factorization in Section 2). Note for this that E is proper over T by (1) and (3).

The last commutative diagram gives a morphism $\delta\colon D \to Z$. Actually, we can treat δ on D as a canonical extension of δ from the diagram (4). Moreover, this gives a morphism δ of the diagram

$$\begin{array}{ccccc} X & \supseteq & E & & \\ \downarrow & & \downarrow & \searrow & \\ V & \supseteq & \varphi(E) & \twoheadleftarrow & U \\ & & \downarrow & & \downarrow \\ & & W & \leftarrow & Y \end{array} \tag{5}$$

into Z. Any morphism of the diagram (4) can be canonically extended to (5) and, in particular, to D. We would like to reverse the construction.

Step 2. (*Reduction to the existence of a morphism.*) By construction $\delta\colon D \to Z$ is epimorphic:

$$Z = \delta_V V \cup \delta_W W \cup \delta_Y Y = \delta_X X \cup \delta_Y Y,$$

because $X \twoheadrightarrow V$ and $Y \twoheadrightarrow W$ are epimorphic. The former property is by construction [Kn, Proposition 5.6, Chapter II] and the latter one holds because $U \twoheadrightarrow \varphi(E)$ is epimorphic. Indeed, by [Sh15, §3.5]

$$\mathrm{Im}[\varphi(E) \to W] = \mathrm{Im}[U \to W] \subseteq \mathrm{Im}[Y \to W].$$

Hence

$$W = \mathrm{Im}[\varphi(E) \to W] \cup \mathrm{Im}[U \to W] \cup \mathrm{Im}[Y \to W] = \mathrm{Im}[Y \to W].$$

So, by Corollary 8, it is sufficient to verify the existence of a morphism $Z \to S$ from δ into any morphism $\alpha\colon D \to S$. By Lemma 5 we can suppose that S is over T.

Step 3. (*Pushing down* α *to a morphism from the diagram* (4) *to* S.) Equivalently, we extend α to a (unique) morphism from the diagram (5). Note that $\alpha_X X$ is finite over T. (It is possible that S itself is not finite over T, even not proper over T.) By [EGA, Corollaire 18.12.4] (this works for morphism of algebraic spaces too) and since $\alpha_X X$ is proper over T, it is sufficient to establish the quasifinite property of $\alpha_X X$ over T. The image $\alpha_X X$ is proper over T by [H, Exercise 4.4, p.143].

Let $C \subseteq \alpha_X X \subseteq S$ be a complete curve over (a point of) T, that is, in a fibre of $\alpha_X X/T$. Since X is proper over $\alpha_X X$ [H, Corollary 4.8(e), Chapter II] then α_X is surjective on points onto $\alpha_X X$ and there exists a proper curve $C_X \subseteq X$ such that $C = \alpha_X C_X$. (The image is also topological.) So, C_X is (exceptional) over T and by (3) $C_X \subseteq E$. Since Y is finite over T by (2) and by our construction, $\psi(C_X)$ is a point in Y. Hence

$$C = \alpha_X C_X = \alpha_Y \circ \psi(C_X)$$

is a point in S, a contradiction. So, $\alpha_X X$ is finite over T.

By the universal property of contraction φ, there exists a unique (Stein) factorization of α_X

$$\alpha_X\colon X \twoheadrightarrow V \twoheadrightarrow \alpha_X X \subseteq S.$$

The second epimorphism and the inclusion give a required morphism

$$\alpha_V\colon V \twoheadrightarrow \alpha_X X \subseteq S.$$

This gives a partial morphism α

$$\begin{array}{ccccc}
X & \supseteq & E & & \\
\downarrow & & \downarrow & \searrow & \\
V & \supseteq & \varphi(E) & \twoheadleftarrow & U \\
\downarrow & \swarrow & & & \downarrow \\
S & & \leftarrow & & Y
\end{array}$$

of the diagram (5) to S. The square

$$\begin{array}{ccc}
\varphi(E) & \twoheadleftarrow & U \\
\downarrow & & \downarrow \\
S & \leftarrow & Y
\end{array}$$

is commutative because the contraction $E \twoheadrightarrow U$ is epimorphic or by its universal property (Stein). The last square is a morphism of the diagram

$$\begin{array}{ccc} \varphi(E) & \twoheadleftarrow & U \\ & & \downarrow \\ & & Y \end{array}$$

to S. By definition of W this gives a morphism $W \to S$ and the morphism α

$$\begin{array}{ccccc} X & \supseteq & E & & \\ \downarrow & & \downarrow & \searrow & \\ V & \supseteq & \varphi(E) & \twoheadleftarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ S & \leftarrow & W & \leftarrow & Y \end{array}$$

of the whole diagram (5) to S. In particular, this gives a morphism of the diagram (4) to S. By construction there exists a required morphism of Z to S giving a morphism of δ to α. Hence Z is a colimit of D. The properties of the colimit are immediate by construction.

Similarly one can prove that W is a colimit of

$$\begin{array}{ccc} \varphi(E) & \twoheadleftarrow & E \\ & & \downarrow \\ & & Y \end{array}.$$

Finally, the colimits in our constructions are fibered. Indeed, all constructions are fibered over T because the colimits and contracted spaces in Stein factorizations are finite over T. By construction and Proposition 8, for every colimit in the proof, we can replace the base space T by the colimit itself. If D is quasipolarized over T then the diagrams and constructions are quasipolarized too. In this circumstances we use quasipolarized versions of Proposition 8, Corollary 8 and Lemma 5. The last lemma has a quasipolarized version by Proposition 7. Notice that the natural quasipolarization on contraction V, coming from T, is also natural for the Stein factorization (cf. [K, Lemma 1.1]). □

Corollary 9. *Let $\psi\colon X \to Y$ be a proper morphism of algebraic spaces and E be a closed subspace which includes the relative exceptional locus of ψ. Then a colimit of the diagram*

$$\begin{array}{ccc} X & \supseteq & E \\ & & \psi_{|E} \downarrow \\ & & \psi(E) \end{array}$$

exists and is fibered and finite over Y*. The colimit is canonically isomorphic to a colimit of*

$$\begin{array}{ccc} V & \supseteq & \varphi(E) \\ & & \downarrow \\ & & \psi(E), \end{array}$$

where $\varphi\colon X \twoheadrightarrow V, \varphi(E) \twoheadrightarrow \psi(E)$ *are respectively a contraction of* X *over* Y *and a finite surjection in a factorization*

$$E \twoheadrightarrow \varphi(E) \twoheadrightarrow \psi(E)$$

of $\psi_{|E}$.

Moreover, if ψ *is quasipolarized then the colimits, isomorphism and factorization are quasipolarized too.*

Proof. We use Lemma 7 with $T = Y, Y = \psi(E)$, and with $\psi = \psi_{|E}$. Since $\psi_{|E}$ is an epimorphism, $W = \psi(E)$ is a colimit of the diagram

$$\begin{array}{ccc} \varphi(E) & \twoheadleftarrow & E \\ & & \downarrow \\ & & \psi(E) \end{array}.$$

Indeed, there exists a unique factorization of $\psi_{|E}$ through the epimorphism $E \twoheadrightarrow \varphi(E)$ given by the contraction φ. The uniqueness holds because $E \twoheadrightarrow \varphi(E)$ is epimorphic. The existence holds by construction: $\varphi(E) \twoheadrightarrow \psi(E)$ is the restriction of the second morphism $V \to Y$ in the Stein factorization through φ

$$X \twoheadrightarrow V \to Y$$

on $\varphi(E)$ (see for Restrictions and canonical factorization in Section 2).

Hence by Remark 1 and Lemma 7 a colimit of the diagram

$$\begin{array}{ccc} X & \supseteq & E \\ & & \psi_{|E} \downarrow \\ & & \psi(E) \end{array}$$

exists and is canonically isomorphic to a colimit of

$$\begin{array}{ccc} V & \supseteq & \varphi(E) \\ & & \downarrow \\ & & \psi(E) \end{array}.$$

It is finite by construction and Lemma 7.

If ψ is quasipolarized we use the quasipolarized version of Lemma 7 (cf. the proof of Proposition 1). ☐

Proposition 10 (Modification)**.** *Let $\varphi\colon X \twoheadrightarrow Y$ be a proper birational morphism of primary algebraic spaces. Then, for any sufficiently large effective Cartier divisor $F \subset Y$, Y is a colimit of the diagram*

$$\begin{array}{ccc} X & \supset & \varphi^* F \\ & & \varphi_{|\varphi^* F} \downarrow \\ & & \varphi\varphi^* F \end{array} .$$

Such a Cartier divisor F exists if Y has a big invertible sheaf. Moreover, for any closed subspace $G \supseteq \varphi^* F$ in X, Y is a colimit of the diagram

$$\begin{array}{ccc} X & \supset & G \\ & & \varphi_{|G} \downarrow \\ & & \varphi(G) \end{array} .$$

If φ is quasipolarized then the colimits are quasipolarized too.

So, $\varphi\varphi^* F$ is a conductor of φ.

Proof. Step 1. (*Choice of F.*) Consider a Stein factorization

$$X \overset{\psi}{\twoheadrightarrow} V \overset{\chi}{\twoheadrightarrow} Y$$

of φ. Take an effective Cartier divisor F of Y such that $\chi\chi^* F$ is a conductor of χ. Since χ is birational morphism of primary spaces, by Lemma 6 the conductor property holds for any sufficiently large effective Cartier divisor F on Y. If Y has a big invertible sheaf then such F exists [Sh15, §1.2, and Corollary 3].

Step 2. By Corollary 9 and Corollary 4, we can suppose that φ is finite. Indeed, a colimit of

$$\begin{array}{ccc} X & \supset & \varphi^* F \\ & & \downarrow \\ & & \varphi\varphi^* F \end{array}$$

is canonically isomorphic to a finite colimit

$$\begin{array}{ccc} V & \supset & \psi\varphi^* F \\ & & \downarrow \\ & & \varphi\varphi^* F \end{array} .$$

On the other hand, by Corollary 4

$$\psi\varphi^* F = \psi(\chi \circ \psi)^* F = \psi\psi^*\chi^* F = \chi^* F$$

and

$$\varphi\varphi^* F = \chi\psi\varphi^* F = \chi\chi^* F.$$

By construction χ is finite, birational and V is primary (see for Stein factorization in Section 2).

By Lemma 6 $\psi(G) \supseteq \psi\varphi^* F = \chi^* F$ is also a conductor.

Step 3. (*Pinching*. Cf. with Example 6.) So, φ is finite. Then by Proposition 8 is enough to establish that the limit of diagram

$$\begin{array}{ccc} \varphi_*\mathcal{O}_X & \twoheadrightarrow & \varphi_*\mathcal{O}_{\varphi^* F} \\ & & \cup| \\ & & \mathcal{O}_{\varphi\varphi^* F} \end{array}$$

is (canonically) isomorphic to $\mathcal{O}_Y$. (Cf. the affine case of Proposition 9.) The limit is a fiber product. Let $\mathcal{I}$ be the ideal sheaf of $\varphi\varphi^* F$ on Y. Then it is a conductor by Step 1 and $\varphi^*\mathcal{I} = \mathcal{I} = \ker[\varphi_*\mathcal{O}_X \twoheadrightarrow \varphi_*\mathcal{O}_{\varphi^* F}]$ is the ideal sheaf of $\varphi^* F$ in $\varphi_*\mathcal{O}_X$. Hence the diagram has the form

$$\begin{array}{ccc} \varphi_*\mathcal{O}_X & \twoheadrightarrow & \varphi_*\mathcal{O}_X/\mathcal{I} \\ & & \cup| \\ & & \mathcal{O}_Y/\mathcal{I}, \end{array}$$

where the homomorphism $\twoheadrightarrow$ is the quotient by $\mathcal{I}$. Since φ is epimorphic, we get the canonical diagram

$$\begin{array}{ccc} \varphi_*\mathcal{O}_X & \twoheadrightarrow & \varphi_*\mathcal{O}_X/\mathcal{I} \\ \cup| & & \cup| \\ \mathcal{O}_Y & \twoheadrightarrow & \mathcal{O}_Y/\mathcal{I}, \end{array}$$

where $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_Y/\mathcal{I}$ is the quotient by $\mathcal{I}$. The diagram is a required fiber product:

$$\mathcal{O}_Y = \varphi_*\mathcal{O}_X \times_{\varphi_*\mathcal{O}_X/\mathcal{I}} \mathcal{O}_Y/\mathcal{I}.$$

In this case $\psi(G) = G$ is also a conductor and the colimit is the same.

Step 4. If φ is quasipolarized we use quasipolarized versions of Corollary 9 and of Proposition 8. $\square$

The simplest diagrams among finite have only inclusions as arrows but their colimits are not necessarily trivial.

Primary presentation This is a subtle version of a union of closed subspaces. Let X be an algebraic space and $\{X_i \mid i \in I\}$ be family of closed subspaces $X_i \subseteq X$. We say that X is (ind-)*representable* by the family if

$$X_i \to \operatorname{colim}_{i\in I} X_i = \operatorname{colim} D = X,$$

where D denotes the diagram with vertices X_i of the family and the arrow are inclusions $X_i \subseteq X_j$ whenever they hold. The diagram is commutative: the compositions of two subsequent inclusions $X_i \subseteq X_j \subseteq X_h$ is the inclusion $X_i \subseteq X_h$. The representation is *finite, primary* if respectively I is finite, every X_i is primary. By Epimorphic property, $X = \cup_{ß\in I} X_i$ holds if X is representable by D. It is also finite in the sense of diagram (cf. Finite diagram of spaces above). Of course, every space X is representable and finitely, e.g., if some $X_i = X$. However an existence of a primary representation is not obvious.

Theorem 3. *Any Noetherian algebraic space X has a primary finite representation. Moreover, for any finite family $\{X_i \mid i \in I\}$ of primary closed subspaces there exists a larger finite family $\{X_i \mid i \in J\}$ of primary closed spaces which represents X:*

$I \subseteq J$;

for every $i \in I$, X_i is the same for both families; and

$\operatorname{colim}_{i\in J} X_i = X$.

The same holds in quasi-PSp *with (natural) quasipolarizations on subspaces over X.*

To prove the last statement we use the quasipolarization on X (cf. Example 4 above). We put aside the following two questions. Suppose we have a finite commutative diagram D with closed inclusions as above but without an ambient space X:

does a colimit of D exist for spaces?

the same for naturally quasipolarized closed inclusions?

The questions are, possibly, idle but not trivial according to the following.

Example 7. Let $Y, Z \subseteq X$ are closed subspaces of X such that $X = Y \cup Z$. Then the diagram $Y \supseteq Y \cap Z \subseteq Z$ has a canonical colimit X [Sh15, Example 3]. The colimit can be (naturally) quasipolarized for any quasipolarization of X, or even of the diagram itself by Example 4 above. Moreover, the inclusions can be replaced by immersions but the result will be the same up to an isomorphism by Example 4. However there are finite commutative diagrams D of quasipolarized spaces with only closed immersions such that they do not have colimits, even for any tensor power (truncation) of quasipolarizations.

Perhaps, the simplest commutative one has the form of diagram (6) below and the noncommutative one has the form of a coequalizer corresponding to such an example.

Consider a diagram D

$$
\begin{array}{ccc} E_1 & \to & X_1 \\ \downarrow & & \uparrow \\ X_2 & \leftarrow & E_2 \end{array}, \tag{6}
$$

where $X_1 = S_1 \times \mathbb{P}^1, X_2 = S_2 \times \mathbb{P}^1, E_1 = C_1 \times \mathbb{P}^1, E_2 = C_2 \times \mathbb{P}^1$ and S_1, S_2, C_1, C_2 are respectively smooth projective surfaces and curves. All arrows are immersions and are products by $\mathbb{P}^1$ of immersions

$$
S_1 \overset{\iota_1^1}{\hookleftarrow} C_1 \overset{\iota_2^1}{\hookrightarrow} S_2 \text{ and } S_1 \overset{\iota_1^2}{\hookleftarrow} C_2 \overset{\iota_2^2}{\hookrightarrow} S_2.
$$

Suppose that the images of curves have simple intersections in points $s_1 \in S_1, s_2 \in S_2$ respectively:

$$
\iota_1^1 C_1 \cap \iota_1^2 C_2 = s_1 \text{ and } \iota_2^1 C_1 \cap \iota_2^2 C_2 = s_2.
$$

Moreover, we suppose that s_1, s_2 corresponds to the same points $c_1 \in C_1, c_2 \in C_2$:

$$
\iota_1^1 c_1 = \iota_1^2 c_2 = s_1 \text{ and } \iota_2^1 c_1 = \iota_2^2 c_2 = s_2.
$$

Such a diagram exists. E.g., we can take $S_1 \simeq S_2 \simeq \mathbb{P}^2, E_1 \simeq E_2 \simeq \mathbb{P}^1$ and the immersions with the images as lines in $\mathbb{P}^2$ under the isomorphisms. Denote by B the diagram

$$
\begin{array}{ccc} C_1 & \overset{\iota_1^1}{\to} & S_1 \\ \iota_2^1 \downarrow & & \iota_1^2 \uparrow \\ S_2 & \overset{\iota_2^2}{\leftarrow} & C_2 \end{array}.
$$

So, D is the product $B \times \mathbb{P}^1$. A colimit of D exists and is a product $(\operatorname{colim} B) \times \mathbb{P}^1$. On the other hand, $\operatorname{colim} B$ is the gluing of S_1 and S_2 along C_1, C_2. Actually, we can first glue C_1 and C_2 along c_1 and c_2, that is, assuming $c_1 = c_2$. This follows from our assumptions about configuration of immersions. By construction D is a proper diagram over $\operatorname{colim} B$.

Let $\alpha \colon \mathbb{P}^1 \to \mathbb{P}^1$ be a toric automorphism. More precisely, α has two fixed points p, q and is toric with respect to such a log structure. Denote by D^α the following twist of D by α: instead of immersion

$$
\iota_1^1 \times \mathrm{Id}_{\mathbb{P}^1} \colon C_1 \times \mathbb{P}^1 \hookrightarrow S_1 \times \mathbb{P}^1
$$

take immersion

$$\iota_1^1 \times \alpha \colon C_1 \times \mathbb{P}^1 \hookrightarrow S_1 \times \mathbb{P}^1.$$

Then $\operatorname{colim} D^\alpha$ exists but it is fibered over B if and only if α has a finite order.

The required colimit exists by Corollary 21 in Section 5 because D^α is a proper diagram over B. Moreover, it can be computed directly. Note for this that a coequalizer of the diagram

$$\mathbb{P}^1 \underset{\mathrm{Id}_{\mathbb{P}^1}}{\overset{\alpha}{\rightrightarrows}} \mathbb{P}^1$$

is the quotient $\mathbb{P}^1/\alpha$, if α has a finite order, or is the point pt., otherwise. In the former case, $\operatorname{colim} D^\alpha$ is a modification of

$$\operatorname{colim} D \supset \mathbb{P}^1 \times s \twoheadrightarrow \mathbb{P}^1/\alpha.$$

In the latter case, $\operatorname{colim} D^\alpha = B$ and is a premodification of

$$\operatorname{colim} D \supset \mathbb{P}^1 \times s \twoheadrightarrow \mathrm{pt.} = s,$$

where s is the gluing of $s_1 = s_2$. The coequalizer is isomorphic to $\operatorname{colim}(D^\alpha/s)$. Indeed, the diagram D^α/s is

$$\begin{array}{ccc} c_1 \times \mathbb{P}^1 & \xrightarrow{\iota_1^1|_{c_1} \times \alpha} & s_1 \times \mathbb{P}^1 \\ \iota_2^1|_{c_1} \times \mathrm{Id}_{\mathbb{P}^1} \downarrow & & \iota_1^2|_{c_2} \times \mathrm{Id}_{\mathbb{P}^1} \uparrow \\ s_2 \times \mathbb{P}^1 & \xleftarrow{\iota_2^2|_{c_2} \times \mathrm{Id}_{\mathbb{P}^1}} & c_1 \times \mathbb{P}^1 \end{array} \simeq \begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\alpha} & \mathbb{P}^1 \\ \mathrm{Id}_{\mathbb{P}^1} \downarrow & & \mathrm{Id}_{\mathbb{P}^1} \uparrow \\ \mathbb{P}^1 & \xleftarrow{\mathrm{Id}_{\mathbb{P}^1}} & \mathbb{P}^1 \end{array}$$

and its colimit is the above equalizer. Hence if α has an infinite order then $\operatorname{colim} D^\alpha$ maps D^α/s to a point, say, s. In this case, $\operatorname{colim} D^\alpha = B$ and this colimit is not fibered (cf. Example 8 in Section 5). Otherwise, α has a finite order and $\operatorname{colim} D^\alpha$ maps D^α/s to $\mathbb{P}^1/\alpha$.

In both diagrams D and D^α we can induce a natural polarization from $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p+q))$: quasipolarizations on X_1, X_2, E_1, E_2 are respectively

$$\mathcal{O}_{X_1}(S_1 \times p + S_1 \times q), \mathcal{O}_{X_2}(S_2 \times p + S_2 \times q), \mathcal{O}_{E_1}(C_1 \times p + C_1 \times q), \mathcal{O}_{E_2}(C_2 \times p + C_2 \times q).$$

Note that α preserves the polarization $\mathcal{O}_{\mathbb{P}^1}(p+q)$: $\alpha^*(p+q) = p+q$. If α has an infinite order then a quasipolarized colimit (related quasipolarization) does not exists because the relative polarization is positive and not contractible. The same holds, but for another reason, for any $\alpha \neq \mathrm{Id}_{\mathbb{P}^1}$ (including of order 2) and even for any tensor power of quasipolarizations (cf. Examples 1 and 1, (2)). In this situation D^α is not isomorphic to a diagram with inclusions but D is isomorphic to such a diagram.

Actually, D^α can be naturally polarized by tensor products of any polarization from the base B. Notice also that there exists $\alpha \neq \mathrm{Id}_{\mathbb{P}^1}$ if $k \not\simeq \mathbb{F}_2$, the field with two elements.

Proof of Theorem 3. We use a primary decomposition $X = \cup X_i$ into a finite union of primary closed subspaces. For affine schemes this follows from a primary decomposition of the zero ideal in a Noetherian ring. Taking a finite affine covering we get a required decomposition for scheme as the union of images (closures) for elements in decompositions of covering by [Sh15, Proposition 2] because the covering is epimorphic. For spaces we can use étale affine coverings.

We use also Noetherian induction on $\operatorname{Supp} X = X_{\mathrm{red}}$.

Let $\{X_i \mid i \in I\}$ be a given finite family of subspaces on X and Y be a primary maximal component of X. Denote by H the maximal subset in I such that every $X_i, i \in H$, is associated to an integral subspace distinct from that of Y: $X_{i\mathrm{red}} \neq Y_{\mathrm{red}}$.

Take any decomposition $X = Y \cup Z$ such that

(1) Y is a primary maximal component;
(2) $Y \cap Z$ has codimension ≥ 1 in Y; and
(3) for every $i \in H$, $X_i \subseteq Z$.

To construct such a decomposition we can use a primary reduced decomposition of X. Then Y is a maximal component of the decomposition and Z is the union of the rest, and of components $X_i, i \in H$. Note that $\operatorname{Supp} Z$ and $\operatorname{Supp} Y \cap Z$ are proper closed subsets of $\operatorname{Supp} X$. Indeed, every intersection $X_i \cap Y$ is proper and has codimension ≥ 1 in Y. The same holds for primary components $\neq Y$ of the reduced decomposition of X.

By induction Z and $Y \cap Z$ are finitely primary representable. Denote by $\{X_i \mid i \in G\}$ a finite family of closed primary subspaces in $Y \cap Z$ which represents $Y \cap Z$. By induction there exists a finite primary family $\{X_i \mid i \in J\}$ which includes both families $\{X_i \mid i \in H\}, \{X_i \mid i \in G\}$, and represents Z. Then a required family $I \cup J$ can be obtained by adding $\{X_i \mid i \in I \setminus H\}$. We suppose also that Y is in the last family, that is, $Y = X_i$ for some $i \in I \setminus H$.

We need to verify that $X_i \to X, i \in I \cup J$, with canonical inclusions $X_i \subseteq X$ as morphisms is $\operatorname{colim}_{i \in I \cup J} X_i$. Consider any morphism $\varphi_i \colon X_i \to T$

of the diagram $\{X_i \mid i \in I \cup J\}$. The only constrain on the morphisms is

$$\varphi_i{}_{|X_j} = \varphi_j \text{ if } X_j \subseteq X_i.$$

It induces two morphisms

$$X_i \to T, i \in J, \text{ and } X_i \to T, i \in G.$$

By induction the first of them gives

$$\varepsilon_Z \colon Z = \operatorname{colim}_{i \in J} X_i \to T$$

such that $\varepsilon_Z{}_{|X_i} = \varphi_i$ for every $i \in J$. Respectively the second one gives

$$\varepsilon_{Y \cap Z} \colon Y \cap Z = \operatorname{colim}_{i \in G} X_i \to T$$

such that $\varepsilon_{Y \cap Z}{}_{|X_i} = \varphi_i$ for every $i \in G$. By Lemma 4 this implies that

$$\varepsilon_Z{}_{|Y \cap Z} = \varepsilon_{Y \cap Z}.$$

Indeed, $Y \cap Z$ is covered by $X_i, i \in G \subseteq J$:

$$Y \cap Z = \cup_{i \in G} X_i,$$

because the family $X_i \to Y \cap Z, i \in G$, is epimorphic (Epimorphic property above). On the other hand by construction, for every $i \in G$,

$$\varphi_Z{}_{|Y \cap Z}{}_{|X_i} = \varphi_Z{}_{|X_i} = \varphi_i = \varphi_{Y \cap Z}{}_{|X_i}.$$

By construction $Y = X_j$ for some $j \in I \setminus H$. Note that

$$\varepsilon_Y{}_{|Y \cap Z} = \varepsilon_{Y \cap Z}$$

where $\varepsilon_Y = \varphi_j \colon Y = X_j \to T$. For every $i \in G$, $Y = X_j \supset X_i$ and as above

$$\varepsilon_Y{}_{|Y \cap Z}{}_{|X_i} = \varepsilon_Y{}_{|X_i} = \varphi_j{}_{|X_i} = \varphi_i = \varphi_{Y \cap Z}{}_{|X_i}.$$

Therefore by Example 7 there exists a morphism $\varepsilon \colon X \to T$ such that

$$\varepsilon_{|Y} = \varepsilon_Y \text{ and } \varepsilon_{|Z} = \varepsilon_Z.$$

This proves the universal property of X. Indeed, if $i \in I \setminus H$, then by construction $X_{i\mathrm{red}} = Y_{\mathrm{red}}, X_i \subseteq Y = X_j$, and

$$\varepsilon_{|X_i} = \varepsilon_{|Y|X_i} = \varepsilon_Y{}_{|X_i} = \varphi_j{}_{|X_i} = \varphi_i.$$

Otherwise $i \in J \supseteq H$. Then by (3) and construction $X_i \subseteq Z$ and

$$\varepsilon_{|X_i} = \varepsilon_{|Z|X_i} = \varepsilon_Z{}_{|X_i} = \varphi_i.$$

Such a morphism $\varepsilon\colon X \to T$ is unique by Lemma 4 because $X_i, i \in I \cup J$, is a covering family of X by (1) and construction.

The quasipolarized case is immediate by Proposition 8. □

Corollary 10. *Any sufficiently large finite family $\{X_i \mid i \in I\}$ of primary closed subspaces in X has a unique natural closed immersion*

$$s\colon X \hookrightarrow \mathrm{colim}_{i\in I} X_i$$

such that $s_{|X_i} = \delta_i\colon X_i \to \mathrm{colim}_{i\in I} X_i$ for i in some subset of I. Equivalently, s is a section of the canonical morphism $\varepsilon\colon \mathrm{colim}_{i\in I} X_i \to X$ extending the canonical sections $\delta_i\colon X_i \to \mathrm{colim}_{i\in I} X_i$ of ε for those i.

The same holds in quasi-PSp.

A sufficiently large family means that it contains some fixed subfamily. Usually, the immersion s is not surjective if X is not integral or not S_1.

Proof. Let $\{X_i \mid i \in J\}$ be a family representing X. Then any larger family $J \subseteq I$ with same X_i for $i \in J$ satisfies the corollary. Indeed, the canonical morphism s is given by the universal property of colimits

$$X = \mathrm{colim}_{i\in J} X_i \to \mathrm{colim}_{i\in I} X_i.$$

By Lemma 4 s is a unique section of the canonical morphism ε: for every $i \in J$,

$$\varepsilon \circ s_{|X_i} = \varepsilon \circ \delta_i = \mathrm{Id}_{X_i} = \mathrm{Id}_X{}_{|X_i}.$$

Hence s is an isomorphism on the image.

The quasipolarized case is immediate by Proposition 8. □

Corollary 11. *Let $\{X_i \mid i \in I\}$ be a total family of closed primary subspaces of X, that is, every closed primary subspace $Y \subseteq X$ has the form $Y = X_i$ for some $i \in I$. Then*

$$X_i \subseteq X = \mathrm{colim}_{i\in I} X_i.$$

The same holds in quasi-PSp.

A total family is usually not finite.

Proof. It is sufficiently to verify that the inclusions $X_i \subseteq X, i \in I$, give a universal morphism, that is, for any morphism $\varphi_i \colon X_i \to T, \in I$, there exists a unique morphism $\varepsilon \colon X \to T$ such that every $\varphi_i = \varepsilon_{|X_i}$. The uniqueness follows from Lemma 4. On the other hand the existence and uniqueness hold for some sufficiently large finite subfamilies $J \subseteq I$ by Theorem 3. Thus the (uniqueness and) existence hold for I by the same theorem

The quasipolarized case is immediate by Proposition 8. □

4. Construction of a skrepa morphism

This section is the proof of Theorem 1.

For simplicity In this section we use N instead of the product $M \otimes L$. By the numerical part of Proposition 4 $\mathbb{E}(N) \subseteq \mathbb{E}(M)$ holds: big plus nef is big.

Isomorphism in the theorem The isomorphism $\varphi^* H_P \simeq N_{|E}$ is *arbitrary*. A posteriori, by Lemma 1, we can suppose that the isomorphism is natural.

Sufficiently large E We can suppose that E itself is already sufficiently large, in particular, $\mathbb{E}(M) \subseteq E$. Thus we need to find F such that $E \subseteq F$ and $\operatorname{Supp} F = \operatorname{Supp} E$.

Notice that taking larger E we get larger F. So, both can include any (closed) subspace of X.

Morphism in the theorem as a modification After a choice of an appropriate F, we construct a modification

$$\begin{array}{ccc} (X, N) & \supseteq & (F, N_{|F}) \\ \delta_X \downarrow & & \varphi \downarrow \\ (Y, H_Y) & \supseteq & (Q, H_Q) \end{array} .$$

By Basic properties of exceptional loci (5-6) in Section 1, Proposition 4 and [Sh15, Proposition 1], $(Y, H_Y) \in \mathrm{PSch}$ because δ_X is surjective on $Y \setminus Q$. So, N is semiample with a related morphism δ_X, $\mathbb{E}(\delta_X) = \mathbb{E}(N) \subseteq \mathbb{E}(M) \subseteq F$ and $(Q, H_Q) \in \mathrm{PSch}$. This implies the existence of a universal morphism of $(X, N) \supseteq (F, N_{|F}) \stackrel{\varphi}{\to} (Q, H_Q)$ or its colimit by Lemma 7, and the colimit is (finite) over (Y, H_Y). The colimit is fibered over (Y, H_Y) again by Lemma 7 and is a modification (see Example 2, (3)). In construction of a modification below we can omit the isomorphism property (2) (see again for ibid). That is, in this section we construct a premodification. However, the corresponding universal modification or colimit will be a modification (see again for ibid).

So, it sufficient to choose appropriate F and to construct a premodification. For this we use the dimensional and component inductions.

Dimensional induction Suppose that $d = \dim X$. For $d = 0$, the theorem holds for $F = E$ because M, L, N are ample and $\mathbb{E}(M) = \emptyset$ (see for Basic properties of exceptional loci, (1) and (5) in Section 1). So we assume that $d \geq 1$ and the theorem holds for spaces of dimension $< d$. We assume also that $\operatorname{Supp} E$ is a proper subset of X under inclusion: $\operatorname{Supp} E \neq \operatorname{Supp} X$. Otherwise we take $F = X$. Since $\mathbb{E}(M) \subseteq E$ holds, M is big on some irreducible component of X by Proposition 4.

Component induction We can suppose that X is irreducible and moreover it is *primary*. Otherwise we decompose X into a union $X = Z \cup V$ of closed subspaces of X such that

Z is a (maximal) proper under inclusion primary component of X with $\dim Z \leq \dim X$ (possibly, $< \dim X$);

V and $\operatorname{Supp} E$ intersects properly Z under inclusion: $\operatorname{Supp} Z \not\subseteq Z \cap E, V$; but

V contains E as a subspace.

By the component or dimensional induction we can suppose that $(Z, N_{|Z})$ with the closed subspace $Z \cap V$ satisfies the existence of modification for some closed subspace $G \supseteq Z \cap V$. By assumptions and construction $\mathbb{E}(M_{|Z}) \subseteq Z \cap \mathbb{E}(M) \subseteq Z \cap E \subseteq Z \cap V$ (see for Basic property of exceptional loci (3) in Section 1). Hence we can suppose also that $\operatorname{Supp} G = \operatorname{Supp}(Z \cap V) \subseteq V$. In general, $G \not\subseteq V$. However the space $W = G \cup V$ as V satisfies the component or dimensional induction. Indeed, $\dim W = \dim V \leq \dim X$ holds and if $\dim W = \dim X$ then W, V have the same irreducible components and their number is less than for X. Since by Basic property of exceptional loci, (3) in Section 1,

$$\mathbb{E}(M_{|W}) = \mathbb{E}(M_{|V}) \subseteq \mathbb{E}(M) \subseteq E \subseteq V \subseteq W$$

and by either of inductions, there exists a closed subspace F of W such that $F \supseteq E, \operatorname{Supp} F = \operatorname{Supp} E$, and any diagram

$$(W, N_{|W}) \supseteq (F, N_{|F}) \overset{\varphi}{\to} (Q, H_Q), (Q, H_Q) \in \text{PSch},$$

in quasi-PSp has a modification in quasi-PSp into a polarized scheme (T,H_T). The subspace F is also a required subspace for X. By construction $F \supseteq E, \operatorname{Supp} F = \operatorname{Supp} E$ holds. We construct a required (pre)modification of

$$(X,N) \supseteq (F,N_{|F}) \stackrel{\varphi}{\to} (Q,H_Q)$$

into $(Y,H_Y) \in \mathrm{PSch}$ in seven (seals) steps.

Step 1. By construction and Morphism as a modification, there exists a modification

$$\begin{array}{ccc} (W,N_{|W}) & \supseteq & (F,N_{|F}) \\ \psi\downarrow & & \varphi\downarrow \\ (T,H_T) & \supseteq & (Q,H_Q) \end{array}, (T,H_T),(Q,H_Q) \in \mathrm{PSch}\,.$$

Step 2. On the other hand, we have the following commutative diagram with restrictions

$$\begin{array}{ccccccc} (Z\cap V,N_{|Z\cap V}) & \subseteq & (G,N_{|G}) & \subseteq & (Z\cap W,N_{|Z\cap W}) & \subseteq & (W,N_{|W}) \\ & \psi_{|Z\cap V}\searrow & \psi_{|G}\downarrow & & & \psi\swarrow & \\ & & (T,H_T) & & & & \end{array}.$$

Step 3. By the choice of G there exists a modification

$$\begin{array}{ccc} (Z,N_{|Z}) & \supset & (G,N_{|G}) \\ \delta_Z\downarrow & & \psi_{|G}\downarrow \\ (Y,H_Y) & \supset & (T,H_T) \end{array}, (Y,H_Y) \in \mathrm{PSch}\,.$$

Step 4. This gives the commutative diagram

$$\begin{array}{ccccc} (Z,N_{|Z}) & \supset & (Z\cap V,N_{|Z\cap V}) & \subseteq & (V,N_{|V}) \\ \| & & \cap| & & \cap| \\ (Z,N_{|Z}) & \supset & (G,N_{|G}) & \subseteq & (W,N_{|W}) \\ \delta_Z\downarrow & & \psi_{|G}\downarrow & \psi\swarrow & \\ (Y,H_Y) & \supset & (T,H_T) & & \end{array}.$$

Unfortunately, a colimit of $Z \supset G \subseteq W$ is not always X or, possibly, $G \neq Z \cap W$.

Step 5. However, by Example 7, (X,N) is the canonical colimit of $(Z,N_{|Z}) \supset (Z\cap V,N_{|Z\cap V}) \subseteq (V,N_{|V})$.

Step 6. Thus there exists a canonical morphism $\delta_X\colon (X,N)\to(Y,H_Y)$ such that $\delta_{X|_Z}=\delta_Z$ and $\delta_{X|_V}=\delta_V,\delta_{X|_{Z\cap V}}=\delta_{Z\cap V}$ are respectively compositions

$$(V,N_{|V})\overset{\psi|_V}{\to}(T,H_T)\subset(Y,H_Y),(Z\cap V,N_{|Z\cap V})\overset{\psi|_{Z\cap V}}{\to}(T,H_T)\subset(Y,H_Y).$$

Indeed, $\delta_{Z|_{Z\cap V}}=\delta_{Z\cap V}=\delta_{V|_{Z\cap V}}$ by the commutativity of

$$\begin{array}{ccccc}(Z,N_{|Z}) & \supset & (Z\cap V,N_{|Z\cap V}) & \subseteq & (V,N_{|V})\\ \delta_Z\downarrow & & \psi_{|Z\cap V}\downarrow & \psi_{|V}\swarrow & \\ (Y,H_Y) & \supset & (T,H_T) & & \end{array},$$

according to Step 4.

Step 7. Finally, we get a required premodification

$$\begin{array}{ccc}(X,N) & \supset & (F,N_{|F})\\ \delta_X\downarrow & & \varphi\downarrow\\ (Y,H_Y) & \supset & (Q,H_Q)\end{array}.$$

Actually, we need to verify its commutativity: $\delta_F=\delta_{X|_F}$ is the composition

$$(F,N_{|F})\overset{\varphi}{\to}(Q,H_Q)\subseteq(T,H_T)\subset(Y,H_Y).$$

By Step 1 the last composition is equal to

$$(F,N_{|F})\overset{\psi|_F}{\to}(T,H_T)\subset(Y,H_Y).$$

It is sufficient a similar commutativity for W: $\delta_W=\delta_{X|_W}$ is the composition

$$(W,N_{|W})\overset{\psi}{\to}(T,H_T)\subset(Y,H_Y).$$

Indeed, then, by Transitivity in Section 2, $\delta_F=\delta_{W|_F}$ is equal to the required composition.

In its turn, since $W=G\cup V$, by Lemma 4, it is sufficient to verify two other commutativities for G and V: $\delta_G=\delta_{X|_G},\delta_V=\delta_{X|_V}$ are respectively compositions

$$(G,N_{|G})\overset{\psi|_G}{\to}(T,H_T)\subset(Y,H_Y),(V,N_{|V})\overset{\psi|_V}{\to}(T,H_T)\subset(Y,H_Y).$$

The last commutativity by our construction in Step 6. For G, again by Transitivity: $\delta_G = \delta_{X|G} = \delta_{X|Z|G} = \delta_{Z|G}$. However, by Step 3 the last restriction is the required composition.

It is actually a premodification because, for every $x \in X \setminus F$, $\delta_X(x) \notin Q$. If $x \in Z$, then $\delta_X(x) = \delta_Z(x) \notin T$ and $\notin Q$ by Step 3. Otherwise, $x \in V$ and $\delta_X(x) = \delta_V(x) = \psi(x) \notin Q$ by Step 1. If the constructed premodification is not a modification we use Morphism in the theorem as a modification above.

Reduction to the large case (Cf. Sufficiently large E above.) We can suppose that F is some sufficiently large closed subspace of the codimension ≥ 1 in X. (By the last assumption, F is still small and we can use Dimensional induction.) The sufficiently large property means that F can include any fixed subspace of codimension 1 in X, in particular, if F includes a nonzero divisor, the codimension is 1. The subspace F includes E but $\operatorname{Supp} F = \operatorname{Supp} E$ is not required *after the reduction.* To reduce to this case we need to find F such that any skrepa $(X,N) \supset (F, N_{|F}) \to (Q, H_Q)$ has a modification into a polarized pair.

By the dimensional induction there exists a closed subspace D in F such that $D \supseteq E, \operatorname{Supp} D = \operatorname{Supp} E$ hold, and any skrepa $(F, N_{|F}) \supseteq (D, N_{|D}) \to (P, H_P), (P, H_P) \in \mathrm{PSch}$, has a modification

$$\begin{array}{ccc} (F, N_{|F}) & \supseteq & (D, N_{|D}) \\ \downarrow & & \downarrow \\ (Q, H_Q) & \supseteq & (P, H_P) \end{array}, (Q, H_Q) \in \mathrm{PSch}\,.$$

We contend that D satisfies required properties for (X,N) too. That is, we can find a modification of $(X,N) \supset (D, N_{|D}) \to (P, H_P)$ into a polarized pair.

By the reduction there exists a further modification

$$\begin{array}{ccc} (X, N) & \supset & (F, N_{|F}) \\ \psi \downarrow & & \downarrow \\ (Y, H_Y) & \supset & (Q, H_Q) \end{array}, (Y, H_Y) \in \mathrm{PSch}\,.$$

Its composition with the previous modification gives a required premodification and, by Morphism in the theorem as a modification, a modification

$$\begin{array}{ccc} (X, N) & \supseteq & (D, N_{|D}) \\ \downarrow & & \downarrow \\ (Y, H_Y) & \supseteq & (P, H_P) \end{array}.$$

Indeed, for every $x \in X \setminus D$, $\psi(x) \notin Q$ and $\notin P$, if $x \notin F$, otherwise $\psi(x) = \varphi(x) \notin P$.

Now we choose an appropriate large subspace F in X. It depends only on X, E and M.

Projective case Suppose that X is projective. That is, there exists an ample invertible sheaf H on X. Since M is big there exist two positive integers m, h and an effective Cartier divisor G such that

$$\mathcal{O}_X(G) \simeq M^{\otimes m} H^{\vee \otimes h}.$$

Indeed, since X is primary and H is ample, in particular big, there exists an effective Cartier divisor A on X such that

$$\mathcal{O}_X(A) \simeq H^{\otimes h}$$

for some positive integer h. Indeed, by [Sh15, Corollary 3], $A = (s)$ is an effective Cartier divisor where s is a Cartier section of $H^0(X, H^{\otimes h})$. Such a section exists. Similarly, there exists an effective Cartier divisor B on X such that

$$\mathcal{O}_X(B) \simeq M^{\otimes m} \text{ and } B \geq A$$

for some positive integer m. In this case, by the big property of M, there exists a Cartier section $t \in H^0(X, M^{\otimes m})$ for a sufficiently large m such that t is vanishing on A. Take $B = (t)$. Hence

$$\mathcal{O}_X(G) \simeq M^{\otimes m} H^{\vee \otimes h},$$

where $G = B - A$ is an effective Cartier divisor. Moreover, we can suppose that $G > 0$ holds and G contains any given subscheme of X of codimension ≥ 1, in particular, $G \supseteq \mathbb{E}(M)$, e.g., taking $m \gg h$ (cf. with the proof of Proposition 1). Replacing $H^{\otimes h}$ by H, we can suppose that $h = 1$.

Therefore by Proposition 1 we can take $F = eG$ in Theorem 1 or large $G = F$ as a Cartier divisor with a sufficiently (divisible) large multiplicity. Note that this case is independent of inductions. The nonprojective case is more subtle and needs the dimensional induction of Reduction to the large case.

General case If X is not projective we can use Chow's lemma and Proposition 10, presenting a modification as a colimit.

Step 1. There exists a proper birational morphism $\psi : Z \twoheadrightarrow X$, where Z is primary and projective. We suppose that ψ is quasipolarized with ψ^*N on Z and N on X. By Chow's lemma [Kn, Theorem 3.1, Chapter IV] [St, Lemma 63.30.5] there exists such a morphism with projective Z. Since the morphism is an isomorphism over a dense Zariski open subset in X we can replace Z by its maximal primary component dominating X.

Step 2. (*Choice of* G.) Let A be a sufficiently large effective Cartier divisor on X as in Proposition 10. We can suppose also that $\psi\psi^*A \supseteq E$. Indeed, for any sufficiently large $A \subset X$, $E \subseteq A$ holds. Such an effective Cartier divisor A exists and can be constructed as zeros of a section of $H^0(X, M^{\otimes m})$ for sufficiently large m by definition and [Sh15, Corollary 3] because M is big and X is primary. By Corollary 5 $E \subseteq A \subseteq \psi\psi^*nA$ for some positive integer n. (The same holds for any effective Cartier divisor which includes nA.) We take $G = \psi(V)$ where $V = \psi^*nA$ is an effective Cartier divisor on Z. By construction G has codimension ≥ 1 and $= 1$ if $G \neq \emptyset$.

So, by Proposition 10, (X, N) is a canonical colimit of the diagram

$$\begin{array}{ccc} (Z, \psi^*N) & \supset & (V, \psi^*N_{|V}) \\ & & \psi_{|V} \downarrow \\ & & (G, N_{|G}) \end{array} .$$

Step 3. (*Choice of* F.) Let W be a sufficiently large effective Cartier divisor on Z such that

$$\mathcal{O}_Z(W) \simeq (\psi^*M)^{\otimes m} H_Z^{\vee \otimes h},$$

where H_Z is an ample invertible sheaf on Z and m, h are positive integers. Since Z is projective, H_Z exists. Since ψ^*M is big, W exists and we can suppose that $h = 1$ as in Projective case above. We can suppose also that W is sufficiently large, in particular, $V \subseteq W$ and W has already a high multiplicity (e of Proposition 1). By Proposition 1, every skrepa

$$(Z, \psi^*N) \supset (W, \psi^*N_{|W}) \to (Q, H_Q), (Q, H_Q) \in \mathrm{PSch},$$

has a morphism in quasi-PSp into a polarized scheme (Y, H_Y). Indeed, $\psi^*N = \psi^*M \otimes \psi^*L$ holds with nef ψ^*L. Take $F = \psi(W)$.

Step 4. (*Construction of* $(W,\psi^*N_{|W})\to(Q,H_Q)$.) Take the composition

$$(W,\psi^*N_{|W}) \overset{\psi_{|W}}{\twoheadrightarrow} (F,N_{|F}) \overset{\varphi}{\to} (Q,H_Q).$$

By construction we have the commutative diagram

$$\begin{array}{ccccc} & & (Z,\psi^*N) & \supset & (W,\psi^*N_{|W}) \\ & \swarrow & \psi\downarrow & & \psi_{|W}\downarrow \\ & & (X,N) & \supset & (F,N_{|F}) \\ & & & & \varphi\downarrow \\ (Y,H_Y) & & \supset & & (Q,H_Q) \end{array} .$$

It gives a morphism of $(Z,\psi^*N)\supset(W,\psi^*N_{|W})\twoheadrightarrow(F,N_{|F})$ into (Y,H_Y)

Step 5. By Steps 2, 3 and Proposition 10, (X,N) is also a canonical colimit of the diagram

$$\begin{array}{ccc} (Z,\psi^*N) & \supset & (W,\psi^*N_{|W}) \\ & & \psi_{|W}\downarrow \\ & & (F,N_{|F}) \end{array} .$$

Step 6. (*Construction of morphism* $(X,N)\to(Y,H_Y)$.) Hence by the universal property of colimit there exists a required morphism making the diagram

$$\begin{array}{ccccc} & & (Z,\psi^*N) & \supset & (W,\psi^*N_{|W}) \\ & \swarrow & \psi\downarrow & & \psi_{|W}\downarrow \\ & & (X,N) & \supset & (F,N_{|F}) \\ & \swarrow & & & \varphi\downarrow \\ (Y,H_Y) & & \supset & & (Q,H_Q) \end{array}$$

commutative. This gives a required modification

$$\begin{array}{ccc} (X,N) & \supset & (F,N_{|F}) \\ \downarrow & & \varphi\downarrow \\ (Y,H_Y) & \supset & (Q,H_Q) \end{array}$$

because by construction the previous diagram induces isomorphisms

$$Y\setminus Q\simeq Z\setminus W\simeq X\setminus F.$$

To complete the proof of Theorem 1 we need to prove Proposition 1.

Proof of Proposition 1. A required morphism

$$\psi\colon (X, M\otimes L)\to (Y,H_Y)\overset{\beta}{\supset}(P,H_P), (Y,H_Y)\in \mathrm{PSch},$$

will be constructed as a modification of a skrepa $\varphi\colon (eE, M\otimes L_{|eE})\to (P,H_P)\subset (Y,H_Y)$, in particular, $\psi_{|eE}=\varphi$ up to the inclusion β (cf. Remarks 1, (2)). So, the morphism of skrepa has the form

$$\begin{array}{ccccc} (X, M\otimes L) & \supset & (eE, M\otimes L_{|eE}) & \overset{\varphi}{\to} & (P,H_P) \\ & \psi\searrow & \psi_{|eE}\downarrow & \beta\swarrow & \\ & & (Y,H_Y) & & \end{array}.$$

Moreover, $\psi^{-1}(Y\setminus P)=X\setminus E$ holds, and

$$\psi_{|X\setminus E}\colon X\setminus E\to X\setminus P$$

is an isomorphism. Note also that $\mathbb{E}(M)\subseteq E$ under the assumptions of Proposition 1. Indeed, we establish below that M is semiample. Hence, by Basic property of exceptional loci (6) in Section 1, $\mathbb{E}(M)=\mathbb{E}(X/Y)$ is covered by complete curves such that $(M.C)=0$, where $(-.-)$ denotes the intersection number bilinear form between invertible sheaves and curves. Hence $(E.C)=(H^\vee.C)=-(H,C)<0$ and $C\subseteq E$. In this situation the diagram $(X,M)\supset (eE,M_{|eE})\to (P,H_P)$ has a colimit, a universal morphism into a polarized pair, and it is a required modification (cf. [Sh15, Theorem 4]). This is, what we are realizing below.

Picking up e. By the Fujita vanishing [F, Theorem (1)] there exists $e_0>0$ such that

$$h^i(X, H^{\otimes e}\otimes O)=0,\ \text{for every } i>0, e\geq e_0,$$

and every nef invertible sheaf O.

Renormalization. We can suppose that $L=\mathcal{O}_X, M=M\otimes L$ and $H=H\otimes L^{\otimes m}$. Indeed, $M\otimes L, L^{\otimes me}\otimes O$ are also nef, $H\otimes L^{\otimes m}$ is ample, and

$$(M\otimes L)^{\otimes m}\otimes (H\otimes L^{\otimes m})^\vee = M^{\otimes m}\otimes H^\vee,\ (H\otimes L^{\otimes m})^{\otimes e}\otimes O = H^{\otimes e}\otimes L^{\otimes me}\otimes O$$

hold. Thus all assumptions of the proposition and the vanishing also hold after such a renormalization. Equivalently, it is enough to suppose that $L=\mathcal{O}_X$.

The morphism φ can be reconstructed as a morphism into a projective space $\mathbb{P}^N$. Indeed, $(P, H_P^{\otimes n})\subseteq (\mathbb{P}^N, H_{\mathbb{P}})$, where $H_{\mathbb{P}}$ is a hyperplane in $\mathbb{P}^N$. This means that, for a natural number n, the tensor power $H_P^{\otimes n}$ is very ample and this holds for every $n \gg 0$. In this situation the morphism φ is given by $N+1$ sections of $H^0(eE, M^{\otimes n}_{|eE})$ or by the subspace V in it generated by the sections [H, Theorem 7.1, Ch. II]. The sections and the subspace are obtained by the structure isomorphism $\varphi^* H_P^{\otimes n} \simeq M^{\otimes n}{}_{|eE}$.

The morphism ψ can be given by the maximal extension $W \subseteq H^0(X, M^{\otimes n})$ of the sections from V to X. The maximal extension includes all sections vanishing on eE. This can be performed for e under our assumptions and any $n \gg 0$ (in particular, depending on e). Indeed, the restriction short sequence gives the surjection

$$H^0(X, M^{\otimes n}) \twoheadrightarrow H^0(eE, M^{\otimes n}_{|eE})$$

because by the Fujita vanishing

$$h^1(X, M^{\otimes n}(-eE)) = h^1(X, M^{\otimes (n-me)} \otimes H^{\otimes e}) = 0.$$

Note for this that $M^{\otimes (n-me)}$ is nef for every $n \geq me$.

Actually, ψ is a morphism because sections of the extension generate $M^{\otimes n}$. Since it is true near E by construction, it is sufficient to generate on $X \setminus E$. For this we can use presentations $n = am + b$, where a, b are nonnegative integral numbers and $b < m$, and

$$M^{\otimes n} = M^{\otimes (am+b)} \simeq H^{\otimes a} \otimes M^{\otimes b}(aE).$$

Since aE is an effective Cartier divisor, to find generating global sections of $M^{\otimes n}$ on $X \setminus E$, it is sufficient to do so for $H^{\otimes a} \otimes M^{\otimes b}$. But the last sheaf is very ample for every $n \gg 0$ because $a \gg b$ for those n and b belongs to a finite set. So, for some natural number n_0, $M^{\otimes n}$ is generated by global sections (base point free), actually by sections from $W \oplus_V H^0(P, H_P^{\otimes n})$, for any $n \geq n_0$. We suppose also that $H_P^{\otimes n}$ is very ample.

Using sections of $W \oplus_V H^0(P, H_P^{\otimes n})$ we glue Y from the image $\psi(X)$, given by W, and P (cf. [Sh15, Interpretation of ample epimorphisms, §5]). Actually $Y = \psi(X) \cup P \subseteq \mathbb{P}^N$. We use the same space $\mathbb{P}^N$ and the same hyperplane $H_{\mathbb{P}}$ as for (P, H_P); some sections of $W \oplus_V H^0(P, H_P^{\otimes n})$ are vanishing on P or on $\psi(X)$. By construction

$$H_P^{\otimes n} = \mathcal{O}_{\mathbb{P}^N}(H_{\mathbb{P}})_{|P} \text{ and } \psi^*(H_{\mathbb{P}|Y}) \simeq M^{\otimes n}.$$

The last isomorphism extends the isomorphism $\varphi^*(H_{\mathbb{P}|\psi(eE)}) \simeq (M_{|eE})^{\otimes n}$. So, we get a required morphism of the diagram modulo n-truncation, taking n-th tensor power of the quasipolarization.

By construction ψ is a modification for a sufficiently large n because, for those n, ψ give an isomorphism from $X \setminus E$ onto its image $Y \setminus P = \psi(X) \setminus P$. Indeed, $\mathbb{E}(M) \subseteq E$ and $M^{\otimes n}$ has global sections with the ample property on $X \setminus E$.

Finally, to construct a morphism $\psi\colon (X, M) \to (Y, H_Y)$ in quasi-PSp we need to determine a polarization H_Y on Y such that $(H_{\mathbb{P}|Y} = H_Y^{\otimes n})$ $\psi^* H_Y \simeq M$ and the above isomorphism with $M^{\otimes n}$ is the tensor power of the last one. For this it is better to take stabile ψ, that is, for any $n_1 \geq n + n_0$, there is a commutative diagram

$$\begin{array}{ccc} & & (Y_1, H_1) \\ & \psi_1 \nearrow & \\ (X, M^{\otimes n_1}) & & \mathrm{pr} \downarrow \\ & \psi \searrow & \\ & & (Y, H_{\mathbb{P}|Y}) \end{array}$$

with an isomorphism pr, where $\psi_1\colon (X, M^{\otimes n_1}) \to (Y_1, H_1) \subseteq (\mathbb{P}^{N_1}, H_{1,\mathbb{P}})$, $H_1 = H_{1,\mathbb{P}|Y_1}$ in quasi-PSp is the morphism ψ for n_1. However ψ is also quasipolarized if n_1 is replaced by n. The morphism pr is a projection given by an imbedding of sections $H^0(X, M^{\otimes n}) \hookrightarrow H^0(X, M^{\otimes n_1})$. This can be done under the assumption $n_1 - n \geq n_0$ because $M^{\otimes(n_1 - n)}$ is base point free and an imbedding can be given by a product with a general section $s \in H^0(X, M^{\otimes(n_1-n)})$ (with a Cartier divisor (s)). The imbedding and the projection are not unique. The stabile property holds for every rather divisible n by [Sh15, Lemma 5]. A required polarization is

$$H_Y = ((\mathrm{pr}^{-1})^* H_1) \otimes (H_{\mathbb{P}|Y_1})^{\vee\otimes 2}$$

for $n_1 = 2n + 1$. [$M^\vee$ dual to M] Indeed, the structure isomorphisms give a required isomorphism

$$\begin{aligned} \psi^* H_Y = \psi^*((\mathrm{pr}^{-1})^* H_1) \otimes \psi^*(H_{\mathbb{P}|Y_1})^{\vee\otimes 2} &\simeq (\psi_1^* H_1) \otimes M^{\vee\otimes 2n} \\ \simeq M^{\otimes(2n+1)} M^{\vee\otimes 2n} = M & \end{aligned}$$

which converts ψ into a quasipolarized morphism of (X, M). By construction the morphism agrees with quasipolarized φ and β: $\varphi = \psi_{|eE}$ and, under the identification $(P, H_P) = (P_1, H_{P_1})$,

$$\begin{aligned} H_{Y}{}_{|P} &= (((\mathrm{pr}^{-1})^* H_1) \otimes (H_{\mathbb{P}|Y_1})^{\vee\otimes 2})_{|P} = ((\mathrm{pr}^{-1})^* H_1) \otimes {}_{|P} H_P^{\vee\otimes 2n} \\ &= ((\mathrm{pr}^{-1}_{|P_1})^* H_{1|P_1}) \otimes H_P^{\vee\otimes 2n} \\ &= H_P^{\otimes(2n+1)} \otimes H_P^{\vee\otimes 2n} = H_P, P_1 = P, \mathrm{pr}_{P_1} = \mathrm{Id}_P\,. \end{aligned}$$ □

5. Applications

Corollary 12. *Under the assumptions and notation of Theorem 1 suppose that* $\mathbb{E}(M) \subseteq E$. *Let* $(X, N) \supseteq (E, N_{|E}) \to (P, H_P)$ *be a skrepa where* $N = M \otimes L$ *and* $(P, H_P) \in \mathrm{PSch}$. *Then the skrepa has a morphism into a quasipolarized scheme if and only if* N *is stably free on* F *with related contractions* $(V, H_V), (W, H_W)$ *respectively for* $(F, N_{|F}), (E, N_{|E})$ *and the canonical diagram*

$$\begin{array}{ccc} (V, H_V) & \leftarrow & (W, H_W) \\ & & \downarrow \\ & & (P, H_P) \end{array} \tag{7}$$

has a colimit. The diagram is finite over k *and polarized.*

The universal morphism of skrepa is a modification if and only if the canonical morphism of (P, H_P) *into the colimit of (7) is a closed immersion. Or, equivalently, there exists a universal infinitesimal extension* $(Q, H_Q) \supseteq (P, H_P)$ *such that* $(W, H_W) \to (P, H_P) \subseteq (Q, H_Q)$ *goes through* $(W, H_W) \to (V, H_V)$.

By construction $W \to V$ is surjective and the existence of colimit (7) can be divided into two subproblems:

(1) the existence for epimorphic $W \twoheadrightarrow V$, a difficult part;
(2) the existence for a (surjective) immersion $W \hookrightarrow V$ (is known by Example 4 in Section 3 essentially due to Artin).

Note one special case when the colimit (7) exists for any morphism in (1). If $W \hookrightarrow P$ is an emersion, e.g., $E \twoheadrightarrow P$ is a contraction, then the colimit (7) exists again by Example 4.

Proof. Necessity. Suppose that the skrepa has a morphism

$$\begin{array}{ccc} (E, N_{|E}) & \subseteq & (X, N) \\ \downarrow & & \downarrow \\ (P, H_P) & \to & (Y, H_Y) \end{array}, (Y, H_Y) \in \text{quasi-PSp}\,.$$

Replacing (Y, H_Y) by the image of $(X, N) \to (Y, H_Y)$ we can suppose that $(Y, H_Y) \in \mathrm{PSch}$ by Basic properties of exceptional loci (5-6) in Section 1 (cf. Corollary 13 below). Then by definition N is stably free on E and F. On the other hand, restriction on F gives the commutative diagram

$$\begin{array}{ccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) \\ \downarrow & & \downarrow \\ (P, H_P) & \to & (Y, H_Y) \end{array}.$$

Thus the universal property of contractions $(F, N_{|F} \twoheadrightarrow (V, H_V), (E, N_{|F}) \twoheadrightarrow (W, H_W)$ given by the stably free property, splits the last diagram into the commutative diagram

$$\begin{array}{ccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) \\ \downarrow & & \downarrow \\ (W, H_W) & \to & (V, H_V) \\ \downarrow & & \downarrow \\ (P, H_P) & \to & (Y, H_Y) \end{array}.$$

The low square is commutative by the epimorphic property of $E \twoheadrightarrow W$ or by the universal property of the contraction. The morphisms of the low square are finite, polarized and the diagram (7) has a colimit by Proposition 8.

Sufficiency. Conversely, suppose a colimit (Y, H_Y) of (7) exists. Then the canonical commutative diagram

$$\begin{array}{ccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) \\ \downarrow & & \downarrow \\ (W, H_W) & \to & (V, H_V) \\ \downarrow & & \\ (P, H_P) & & \end{array}$$

has a commutative extension as above. This gives a morphism of the skrepa $(F, N_{|F}) \supseteq (E, N_{|E}) \to (P, H_P)$ to (Y, H_Y). By construction $(Y, H_Y) \in \mathrm{PSch}$. The required morphism can be obtained by the composition of the last one with a morphism of $(X, N) \supseteq (F, N_{|F}) \to (Y, H_Y)$ of Theorem 1 (cf. Corollary 19).

The modification case follows from Corollary 13. □

Corollary 13. *Under the assumptions and notation of Theorem 1 suppose that* $\mathbb{E}(M) \subseteq E$*. Let* $(X, N) \supseteq (E, N_{|E}) \stackrel{\varphi}{\to} (P, H_P)$ *be a skrepa where* $N =$

$M \otimes L$ and $(P, H_P) \in$ PSch. If the skrepa has a morphism in quasi-PSp into a polarized scheme then it has a universal morphism or a colimit (Y, H_Y). Moreover, $(Y, H_Y) \in$ PSch and the colimit can be factorize into two colimits with a modification at the right and $(Q, H_Q) \in$ PSch

$$\begin{array}{ccccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) & \subseteq & (X, N) \\ \downarrow & & \downarrow & & \downarrow \\ (P, H_P) & \to & (Q, H_Q) & \subseteq & (Y, H_Y) \end{array}.$$

The colimit and factorization are fibered over Y. The skrepa has a modification and its colimit is a modification if and only if the left square is a (pre)modification.

The existence of a colimit in the corollary is not surprising by Corollary 22 but the fibered property and relation to modifications are not obvious.

Proof. The existence of a colimit is immediate by Lemma 7. The hypotheses (1-2) of the lemma hold by our assumptions. The hypothesis (3) follows from Basic property of exceptional loci (6) in Section 1. The factorization can be constructed by Restrictions and canonical factorization in Section 2. The universal property of colimits, Lemma 7 and our assumptions implies the required properties of the colimits and of the factorization. Notice only that since $\operatorname{Supp} F = \operatorname{Supp} E$ there no difference between modification or premodification because $F \setminus E = \emptyset$. □

The stable free parts of Corollaries 1, 3 have the following diagram versions.

Corollary 14. *Under the assumptions and notation of Corollary 1, $(X, M \otimes L)$ has a morphisms into a polarized scheme if and only if $(F, M \otimes L_{|F})$ does so.*

Proof. Immediate by Corollary 1 and Example 2, (3). □

Corollary 15. *Under the assumptions and notation of Corollary 3 $(X, (M \otimes L)^{\otimes q})$ has a morphisms into a polarized scheme if $(F, M \otimes L_{|\mathbb{E}(M)})$ does so.*

Proof. Immediate by Corollary 3 and Example 2, (3). □

Same F, q of Theorem 1 and Corollaries 1, 2, 3 work for a larger class of quasipolarizations. First, we give the same class with a twist.

Corollary 16. *F, q respectively in Theorem 1 and Corollaries 1, 2, 3 are the same for any sheaf $M' \equiv M$.*

Proof. Indeed, $M'^{\vee} \otimes M \equiv 0$ and corresponding $L' = M'^{\vee} \otimes M \otimes L$ because $M' \otimes L' = M \otimes L$. □

The next statement gives a larger class of quasipolarizations L'.

Corollary 17. *Under the assumptions and notation of in Theorem 1, let L' be a nef invertible sheaf on X such that there exists an invertible sheaf L on X such that $N = M \otimes L$ is semiample and*

(1) *if a closed subspace W of X is exceptional with respect to L' then W is exceptional with respect to N; or,*
(2) *for any closed curve C on X, if $(L'.C) = 0$ then $(N.C) = 0$.*

Then any skrepa

$$(X, L') \supseteq (F, L'_{|F}) \to (Q', H_{Q'}), (Q', H_{Q'}) \in \text{PSch}, \tag{8}$$

has a morphism in quasi-PSp *into a polarized scheme. Moreover, there exists such a universal morphism (for the skrepa), it is fibered over the colimit and is a modification of the skrepa.*

Additionally, in notation of Corollary 1, L' is semiample (stably free) if and only if $L'_{|F}$ is semiample (resp. stably free).

Additionally, in notation of Corollary 2, any skrepa

$$(X, L'^{\otimes q}) \supseteq (E, L'^{\otimes q}{}_{|E}) \xrightarrow{\varphi'^{\otimes q}} (P', H_{P'}^{\otimes q}),$$

with a quasipolarized space $(P', H_{P'})$, has a morphism in quasi-PSp *into a polarized scheme if there exists a morphism of*

$$(F, L'_{|F}) \supseteq (E, L'_{|E}) \xrightarrow{\varphi'} (P', H_{P'})$$

in quasi-PSp *into a polarized scheme.*

Additionally, in notation of Corollary 3, $L'^{\otimes q}$ is stably free if $L'_{|\mathbb{E}(M)}$ is stably free.

Proof. By our assumptions $N_{|F}$ and $L'_{|F}$ are semiample on F. So, $N \otimes L'_{|F}$ is also semiample on F. Moreover, $N^{\otimes n} \otimes L'_{|F}$ is stably free for some positive integer n. This gives a Stein factorization

$$(F, L'_{|F}) \xrightarrow{\varphi} (W, H'_W) \to (Q', H_{Q'}), (W, H'_W) \in \text{PSch},$$

where φ is the contraction related to $L'_{|F}$. On the other hand, the contraction φ is also related to $N \otimes L'_{|F}$, or better to $N^{\otimes n} \otimes L'_{|F}$ by (1-2). So, we have a morphism

$$(F, N^{\otimes n} \otimes L'_{|F}) \overset{\varphi}{\twoheadrightarrow} (W, H_W), (W, H_W) \in \mathrm{PSch}\,.$$

Hence by Theorem 1, there exists a modification

$$\begin{matrix} (X, N^{\otimes n} \otimes L') & \supseteq & (F, N^{\otimes n} \otimes L'_{|F}) \\ \downarrow & & \varphi \downarrow \\ (Y, H_Y) & \supseteq & (W, H_W) \end{matrix}, (Y, H_Y) \in \mathrm{PSch}\,.$$

Since φ is a contraction, $X \twoheadrightarrow Y$ is also a contraction related to $N^{\otimes n} \otimes L'$ by the universal property of contractions. Moreover, we can change n on $2n$. So, $N^{\otimes n}$ and L' are locally trivial over Y and actually the contraction φ is related to L' by (1-2). Moreover, the above modification can by twisted into a modification

$$\begin{matrix} (X, L') & \supseteq & (F, L'_{|F}) \\ \downarrow & & \varphi \downarrow \\ (Y, H'_Y) & \supseteq & (W, H'_W) \end{matrix}, (Y, H'_Y) \in \mathrm{PSch}\,.$$

The composition of the last modification with an Artin quasipolarized modification of

$$(Y, H'_Y) \supseteq (W, H'_W) \to (Q', H_{Q'})$$

(see Example 4) gives a required modification of (8).

The rest follows form Lemma 7 and corresponding corollaries. □

Corollary 18. *Let M be an invertible sheaf on a complete algebraic space X such that M is globally generated, except for, finitely many closed points of X. Then M is stably free and so $M^{\otimes m}$ is globally generated everywhere for any $m \gg 0$.*

Proof. Step 1. (Reduction to the case with $\mathbb{E}(M) = X_{\mathrm{red}}$.) M is globally generated on every connected component of F of the dimension ≥ 1. But, on the connected components of the dimension 0, every sheaf is generated by global sections. Hence M is stably free on F and so does on X by Corollary 1 with $L = \mathcal{O}_X$.

Now we can suppose that $\mathbb{E}(M) = X_{\mathrm{red}}$.

Step 2. M is globally generated on X_{red}. Otherwise, by Nakayama's lemma [Mat, Theorem 2.3, (i)], there exists a closed point $p \in X$ such that every global section $s \in H^0(X, M)$ is vanishing in p: $s(p) = 0$. Let Y be an irreducible component of X_{red} such that $p \in Y$. Since M is globally generated on Y except for finitely many closed points, including P, then there exist n sections $s_1, \dots, s_n$ such that there zeros (s_i) are in general position where $n = \dim Y$. (The linear system $\left|M_{|Y}\right|$ has only finitely many fixed closed points.) The zeros are Cartier divisors on Y and $p \in (s_i)$ for every s_i. Hence M is big on Y:

$$(L_{|Y})^n = (s_1)\dots(s_n) \geq 1,$$

a contradiction.

So, there are global sections of M on X which generate M locally on X_{red}.

Step 3. Again by Nakayama's lemma, M is globally generated on X. □

Corollary 19 (Existence of a skrepa morphism)**.** *Under the assumptions and notation of Theorem 1, for any closed subspace E of X, there exists a sufficiently small closed subspace $F \subseteq X$ such that $F \supseteq E$ and, for any nef invertible sheaf L on X, any skrepa*

$$(X, M \otimes L) \supseteq (E, M \otimes L_{|E}) \xrightarrow{\theta} (P, H_P), \theta^* H_P \simeq M \otimes L_{|E},$$

in quasi-PSp*, with a quasipolarized space (P, H_P), has a morphism in* quasi-PSp *into a polarized scheme (Y, H_Y) if and only if there exists a morphism of*

$$(F, M \otimes L_{|F}) \supseteq (E, M \otimes L_{|E}) \xrightarrow{\theta} (P, H_P)$$

in quasi-PSp *into a polarized scheme (Q, H_Q).*

More precisely, we can take F satisfying (1) of Theorem 1.

In general, the skrepa $(X, N) \supseteq (E, N_{|E}) \xrightarrow{\theta} (P, H_P)$ of the corollary does not have a universal morphism (amalgam) and it is not necessarily its modification if exists (see [K, Theorem 3.0]; cf. Theorem 1 and Corollary 12).

Proof. Necessity. Immediate by definition for any $F \supseteq E$. Suppose that there exists a morphism of $(X, N) \supseteq (E, N_{|E}) \xrightarrow{\theta} (P, H_P)$ into (Y, H_Y), that is, a commutative diagram

$$\begin{array}{ccc} (E, N_{|E}) & \subseteq & (X, N) \\ \theta \downarrow & & \psi \downarrow \\ (P, H_P) & \to & (Y, H_Y) \end{array}, (Y, H_Y) \in \text{PSch},$$

where $N = M \otimes L$ as above. Then there exists a morphism of $(F, N_{|F}) \supseteq (E, N_{|E}) \xrightarrow{\theta} (P, H_P)$ into (Q, H_Q), e.g.,

$$\begin{array}{ccccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) & & \\ \theta\downarrow & & \psi_{|F}\downarrow & & \\ (P, H_P) & \to & (Q, H_Q) & = & (Y, H_Y) \in \mathrm{PSch} \end{array}$$

by definition of the restriction.

Sufficiency. Follows from the commutative diagram

$$\begin{array}{ccccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) & \subseteq & (X, N) \\ \theta\downarrow & & \varphi\downarrow & & \psi\downarrow \\ (P, H_P) & \xrightarrow{\delta} & (Q, H_Q) & \to & (Y, H_Y) \end{array}, \tag{9}$$

where the left square is given by assumptions and the right one is by Theorem 1. □

Corollary 20. *If in Corollary 19* $\mathbb{E}(M) \subseteq E$ *then we can suppose that* $(P, H_P) \in \mathrm{PSch}$ *instead of* (Q, H_Q).

Proof. Instead of the left square of the diagram (9) we can take its restriction

$$\begin{array}{ccc} (E, N_{|E}) & \subseteq & (F, N_{|F}) \\ \theta\downarrow & & \varphi_{|\,\mathrm{Im}\,\varphi\cup\mathrm{Im}\,\delta}\downarrow \\ (P, H_P) & \xrightarrow{\delta_{|\,\mathrm{Im}\,\varphi\cup\mathrm{Im}\,\delta}} & \mathrm{Im}\,\varphi \cup \mathrm{Im}\,\delta \end{array}.$$

Its commutativity follows from the inclusion property of Restrictions and canonical factorization in Section 2. By our assumptions and construction $\delta_{|\,\mathrm{Im}\,\varphi\cup\mathrm{Im}\,\delta}$ is surjective and $\mathrm{Im}\,\varphi \cup \mathrm{Im}\,\delta$ is polarized.

We can use also Lemma 7 and take a colimit of the left square in (9). □

Proper diagram of spaces We say that a diagram D is *proper* over an algebraic space X if D is a finite diagram in the category of spaces proper over X. By the universal property of Stein factorizations, every proper diagram has a finite diagram D^f over X. For a vertex Y of D, Y^f is the contraction in a Stein factorization of the structure morphism $Y \to X$

$$Y \twoheadrightarrow Y^f \to X$$

and is a vertex of D^f with a finite structure morphism $Y^f \to X$. Again by the universal property and an the epimorphic property of contractions, for every arrow $X_1 \to X_2$ of D, we get the commutative diagram

$$\begin{array}{ccc} X_1 & \to & X_2 \\ \downarrow & & \downarrow \\ X_1^f & \to & X_2^f \\ & \searrow \quad \swarrow & \\ & X & \end{array} ;$$

the triangle is commutative because $X_1 \twoheadrightarrow X_1^f$ is epimorphic. If D is quasipolarized then D^f is canonically quasipolarized: for every vertex $(Y, L_{|Y})$ of (D, L), $(Y^f, L_{|Y^f}) \to (X, L_{|X})$ is canonically quasipolarized by Stein factorization in Section 2 and naturally by Lemma 1. Note that D^f depends on the base X. So, we use sometimes D^f/X.

Proposition 11. *Let D be a proper diagram over an algebraic space X. Then* colim D *exists, is proper over X and*

$$\operatorname{colim} D = \operatorname{colim}(D^f / \operatorname{colim} D) \tag{10}$$

holds where $\operatorname{colim}(D^f)$ *and D^f are fibered over* colim D.

If D is quasipolarized then the colimit is quasipolarized too. The colimit has finite type.

In general, colim D is not fibered over X, even over a colimit.

Example 8. Let D be a skrepa $X \supset E \to c$, where $X = \mathbb{P}^1 \times C$, C is a nonsingular curve, $E = \mathbb{P}^1 \times c$, $c \in C$ is a closed point, and $E = \mathbb{P}^1 \times c \to c$ is the projection onto c. Then D is a proper diagram over C with the canonical projection $X \to C$ and $\operatorname{colim} D = \operatorname{colim}_{/C} D = C$. This follows from the semicontinuity of dimension for fibers of the colimit over C. Note also that the colimit is a premodification but not a modification of the skrepa because $X \not\simeq C$ over $C \setminus c$.

On the other hand, the skrepa $D_{|C\setminus c}$ has $E_{|C\setminus c} = \emptyset$ and

$$\operatorname{colim}(D_{|C\setminus c}) = X_{|C\setminus c}$$

is a modification. In particular, colim D is not fibered.

Lemma 8 (Stabilization of proper epimorphisms)**.** *Let $X \twoheadrightarrow \cdots \twoheadrightarrow X_i \twoheadrightarrow \cdots \twoheadrightarrow X_1$ be a sequence of proper epimorphisms $X_i \twoheadrightarrow X_{i-1}$ under a space*

X of finite type and every $X \twoheadrightarrow X_i$ is proper epimorphic. Then the sequence stabilizes: $X_i \twoheadrightarrow X_{i-1}$ is an isomorphism for all $i \gg 0$.

This is a natural generalization of [Sh15, Lemma 5] and actually can be reduced to the latter result.

Proof. It is enough to prove that : $X_i \twoheadrightarrow X_{i-1}$ is finite for all $i \gg 0$ and then to use [Sh15, Lemma 5]. Since the morphisms are proper we verify only the quasifinite property and use [EGA, Corollaire 18.12.4].

Step 1. We can suppose that all X, X_i are reduced and all morphisms $X \twoheadrightarrow X_i \twoheadrightarrow X_{i-1}$ are proper surjective because the quasifinite property is topological.

Step 2. We can suppose that X_i have the same number of irreducible component for all $i \gg 0$ and morphisms $X_i \twoheadrightarrow X_{i-1}$ induce 1-to-1 correspondence between irreducible components for all $i \gg 0$. Indeed, an irreducible component of X_i dominant over some irreducible component of X_{i-1} dominates or goes to such a component. Since $X_i \twoheadrightarrow X_{i-1}$ is surjective, the correspondence is surjective on the irreducible components of X_{i-1}. By the same reason, the number of irreducible components of X_i is not higher than that of X.

Step 3. We can suppose that X, X_i are irreducible. Indeed, the quasifiniteness is enough to verify for morphisms of corresponding irreducible components. Every such a chain is proper surjectively dominated by some irreducible component of X.

Step 4. Morphisms $X_i \twoheadrightarrow X_{i-1}$ are generically finite (alterations) for all $i \gg 0$. Since all morphisms $\varphi_i \colon X \twoheadrightarrow X_{i-1}$ are surjective, $\dim X_i \le \dim X$ for all i. So, $\dim X_i$ are stabilize for all $i \gg 0$.

Removing finitely many X_i we can suppose that all $X_i \twoheadrightarrow X_{i-1}$ are generically finite.

Step 5. There exists a nonempty Zariski open subset $U \subseteq X_1$ that all $X_i \twoheadrightarrow X_{i-1}$ are finite over U. Take a nonempty open subset $U \subseteq X_1$ such that X is flat over U. Actually it is enough that the fibers of X are equidimensional over U. This set is open and if $C \subseteq X_i$ is a curve over a closed point $x \in U$. Then

$$\varphi_1^{-1}x \supseteq \varphi_i^{-1}C$$

and by construction

$$\begin{aligned}\dim \varphi_1^{-1}x = \dim X - \dim X_1, \dim \varphi_i^{-1}C &\ge \dim X - \dim X_i + 1\\ &= \dim X - \dim X_1 + 1,\end{aligned}$$

a contradiction.

Step 6. Finally, we can apply the dimensional induction to the chain $X \twoheadrightarrow \cdots \twoheadrightarrow X_i \twoheadrightarrow \cdots \twoheadrightarrow X_1$ over $X_1 \setminus U$. Morphisms $X_i \twoheadrightarrow X_{i-1}$ are quasifinite over $X_1 \setminus U$ and thus everywhere for all $i \gg 0$. □

Proof of Proposition 11. By Lemma 5 it is enough to find a colimit of D over X. Let $D \twoheadrightarrow Y/X$ be a highest epimorphism. That is, if $D \twoheadrightarrow Z/X$ is another epimorphism with a (epi)morphism $Z \to Y$ then it is an isomorphism. Such a highest epimorphism exists by Lemma 8.

The highest epimorphism is a required colimit again by Lemma 5. Since $D \twoheadrightarrow Y$ is surjective it is enough to verify the existence of a morphism from $D \twoheadrightarrow Y$ to every morphism $\varphi\colon D \to Z/Y$. By construction we get a canonical epimorphism $\varepsilon\colon \operatorname{Im}\varphi \twoheadrightarrow Y$ and it is an isomorphism. A required morphism is the composition

$$Y \overset{\varepsilon^{-1}}{\to} \operatorname{Im}\varphi \subseteq Z.$$

Using the universal property of Stein factorization we can verify the equation (10). The diagram D^f and colimit $Y = \operatorname{colim}(D^f)$ are finite and fibered over Y. This gives the canonical quasipolarization of $\operatorname{colim}(D^f)$ by Proposition 8 in the quasipolarized case. We already know that D^f has a canonical (natural) quasipolarization. □

Corollary 21. *Let D be a finite diagram of algebraic spaces. It has a proper colimit $D \to \operatorname{colim} D$ if and only if it has a proper morphism $D \to X$.*

The same holds in quasi-PSp.

Proof. Immediate by Proposition 11. □

Corollary 22. *A finite diagram of complete quasipolarized algebraic spaces has a colimit if and only if it has a morphism.*

Proof. Immediate by the previous corollary because every morphism of a complete space is proper. □

Some proper quasipolarized diagrams do not have morphisms and colimits. See Examples 1, (1-2) and 2, (4). The nonexistence can hold even for skrepas.

Example 9 (continuation of Example 8). In notation of this example, let $Y \supset E \to c$ be a skrepa, where Y is a blowup of a point on $X = \mathbb{P}^1 \times C$, which does not lie over c, with an exceptional curve F. The skrepa also has a colimit and it is C. However, the quasipolarized skrepa

$$(Y, L) \supseteq (E, L_{|E}) \to (c, k), L = \mathcal{O}_Y(F),$$

does not have a quasipolarized colimit because $L_{|F}$ is not isomorphic to $\mathcal{O}_F$.

Note also a more subtle and relevant to Theorem 1 and to the whole paper example [K, Theorem 3.0] of Keel.

Corollary 23. *Every finite diagram of complete algebraic spaces has a colimit.*

Proof. Immediate by Corollary 21 because such a diagram has a proper morphism to a point. □

Compound skrepa This is a diagram

$$X_1 \supseteq E_1 \to X_2 \supseteq E_2 \to \cdots \to X_n \supseteq E_n \to X_{n+1}$$

of algebraic spaces, where every E_i is a closed subspace of X_i. A *compound preskrepa* is

$$X_1 \supseteq E_1 \to X_2 \supseteq E_2 \to \cdots \to X_n \supseteq E_n$$

the skrepa without the end $\to X_{n+1}$. A quasipolarized compound preskrepa

(11)
$$(X_1, M_1) \supseteq (E_1, {M_1}_{|E_1}) \overset{\varphi_1}{\to} (X_2, M_2) \supseteq (E_2, {M_2}_{|E_2}) \overset{\varphi_2}{\to} \cdots \overset{\varphi_{n-1}}{\to} (X_n, M_n) \supseteq (E_n, {M_n}_{|E_n})$$

is *well-wrought* if every subspace E_i satisfies the properties of F in Theorem 1:

$E_i \supseteq \mathbb{E}(M_i)$; and
for any nef invertible sheaf L_i on X_i, any skrepa

$$(X_i, M_i \otimes L_i) \supseteq (E_i, M_i \otimes {L_i}_{|E_i}) \to (Q_i, H_i), (Q_i, H_i) \in \text{PSch},$$

has a morphism in quasi-PSp into a polarized scheme.

Notice that the sheaves $M_i \otimes L_i$ and $M_i \otimes {L_i}_{|E_i}$ form a quasipolarization on the preskrepa if (and only if) L_i and ${L_i}_{|E_i}$ form a quasipolarization on it.

Corollary 24. *Let D be a well-wrought quasipolarized compound preskrepa* (11)*. Then, for any nef quasipolarization $L_i, i = 1, \ldots, n$, on D, any compound skrepa*

$$(X_1, N_1) \supseteq (E_1, {N_1}_{|E_1}) \to (X_2, N_2) \supseteq (E_2, {N_2}_{|E_2}) \to \cdots \to (X_n, N_n) \supseteq (E_n, {N_n}_{|E_n}) \to (Q, H_Q),$$

where $N_i = M_i \otimes L_i$ *and* $(Q, H_Q) \in$ PSch*, has a morphism in* quasi-PSp *into a polarized scheme. Moreover, the compound skrepa has a colimit, it is a fibered compound modification:*

$$\begin{array}{ccccccccccccc} (X_1,N_1) & \supseteq & (E_1,N_{1|E_1}) & \to & (X_2,N_2) & \supseteq & \dots & \to & (X_n,N_n) & \supseteq & (E_n,N_{n|E_n}) \\ \downarrow & & & \searrow & \downarrow & & \dots & \searrow & \downarrow & & \downarrow \\ (Y_1,H_1) & & \supseteq & & (Y_2,H_2) & \supseteq & \dots & \supseteq & (Y_n,H_n) & \supseteq & (Q,H_Q) \end{array},$$

where all pairs $(Y_1, H_1), \dots (Y_n, H_n), (Q, H_Q)$ *are polarized schemes, the inclusions are natural, that is,* $H_{i|Y_{i+1}} = H_{i+1}, i = 1, \dots, n, H_{n|Q} = H_Q$ *hold additionally, the diagram is commutative, all squares*

$$\begin{array}{ccc} (X_1,N_1)\supseteq(E_1,N_{1|E_1}) \\ \downarrow \quad\quad \downarrow \\ (Y_1,H_1)\supseteq\ (Y_2,H_2) \end{array}, \dots, \begin{array}{c} (X_{n-1},N_{n-1})\supseteq(E_{n-1},N_{n-1|E_{n-1}}) \\ \downarrow \quad\quad \downarrow \\ (Y_{n-1},H_{n-1})\supseteq \quad (Y_n,H_n) \end{array}, \begin{array}{c} (X_n,N_n)\supseteq(E_n,N_{n|E_n}) \\ \downarrow \quad\quad \downarrow \\ (Y_n,H_n)\supseteq\ (Q,H_Q) \end{array}$$

are modifications and amalgams (or colimits) of

$$(X_1,N_1)\supseteq(E_1,N_{1|E_1})\to(Y_2,H_2),\dots,(X_{n-1},N_{n-1})\supseteq(E_{n-1},N_{n-1|E_{n-1}})\to(Y_n,H_n),$$
$$(X_n,N_n)\supseteq(E_n,N_{n|E_n})\to(Q,H_Q)$$

respectively. The diagrams are fibered over Y_1*.*

Proof. Immediate by Theorem 1 and induction on n □

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