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A class of fully nonlinear equations

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Abstract: In this paper we consider a class of fully nonlinear equations which covers the equation introduced by S. Donaldson a decade ago and the equation introduced by Gursky-Streets recently. We solve the equation with uniform weak C^2 estimates, which hold for degenerate case.

Keywords: Fully nonlinear equations, a priori estimates.

1. Introduction

We recall a class of differential operators introduced by S. Donaldson [4] and Gursky-Streets [5]. Consider a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ with the coordinate (t, x). We use the operator $D = (\partial_t, \nabla)$ to denote the first order derivatives. Consider the matrix

$$r = \begin{pmatrix} u_{tt} & \nabla u_t \\ (\nabla u_t)^t & R \end{pmatrix}$$

where $R = \nabla^2 u +$ lower order terms. Given a symmetric matrix P, we use $\sigma_i(P)$ to denote the i-th elementary symmetric function on its eigenvalues $\lambda_1, \cdots, \lambda_n$. The Γ_k^+ cone is given by

$$\Gamma_k^+ = \{P : \sigma_i(P) > 0, 1 \leq i \leq k\}.$$

Assume $u_{tt} > 0$ and $R \in \Gamma_k^+$, consider the operator

$$F_k(r) = u_{tt}\sigma_k(R) - (T_{k-1}(R), \nabla u_t \otimes \nabla u_t), \tag{1.1}$$

where T_{k-1} is the $(k-1)$-th Newton transformation which takes the form of

$$T_{k-1}(R)_{ij} = \sigma_k(R) \frac{\partial}{\partial R^{ij}} \log \sigma_k(R).$$

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This operator appears naturally in two different settings of geodesic equations of certain infinite dimensional Riemannian geometry.

When $k = 1$, the operator was introduced by S. Donaldson [4]

$$F_1(r) = u_{tt}(\Delta u + 1) - |\nabla u_t|^2,$$

when he considered a Weil-Peterson type metric on the space of volume forms (normalized) on a Riemannian manifold (X, g) with fixed total volume. This infinite dimensional space can be parameterized by all smooth functions such that

$$\{\phi \in C^\infty(X) : 1 + \triangle_g \phi > 0\}.$$

The metric is defined by

$$\|\delta\phi\|_\phi^2 = \int_X (\delta\phi)^2 (1 + \triangle_g \phi) dg.$$

Then the geodesic equation is

$$u_{tt}(1 + \triangle u) - |\nabla u_t|_g^2 = 0. \tag{1.2}$$

For $1 \leq k \leq n$, Gursky-Streets [5] introduced a family of operators F_k. Consider a conformal class $g_u = e^{-2u} g$ on a Riemannian manifold (M, g). Recall the Schouten tensor

$$A := \frac{1}{n-2}\left(Ric - \frac{1}{2(n-1)} Rg\right),$$

which plays an important role in conformal geometry. Under the conformal change, the Schouten tensor is given by

$$A_u = A(g_u) = A + \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2}|\nabla u|^2 g.$$

When $A_u \in \Gamma_k^+$, Gursky-Streets introduced a family of fully nonlinear elliptic equations of the form

$$u_{tt}\sigma_k(A_u) - (T_{k-1}(A_u), \nabla u_t \otimes \nabla u_t) = 0.$$

When $n = 4$, $k = 2$, this is the geodesic equation of the following metric

$$\langle \psi, \phi \rangle_u = \int_M \phi\psi\sigma_2(g_u^{-1} A_u) dV_u,$$

defined on the space $\mathcal{C}^+ = \{u : A_{g_u} \in \Gamma_2^+, g_u = e^{-2u}g\}$. Gursky and Streets introduced these structures to solve the uniqueness of σ_2 Yamabe problem on a compact Riemannian four manifold. We refer the readers to [5, 8] for more details. When $k = 1$, the Gursky-Streets equation reads

$$u_{tt}(\Delta u - (n/2-1)|\nabla u|^2 + A(x)) - |\nabla u_t|^2 = 0.$$

The Donaldson equation and the Gursky-Streets equation are closely related in this case. In this paper we discuss a class of equations of the following form,

$$u_{tt}\left(\Delta u - b|\nabla u|^2 + a(x)\right) - |\nabla u_t|^2 = f, \tag{1.3}$$

with boundary condition

$$u(\cdot, 0) = u_0, u(\cdot, 1) = u_1,$$

where $a(x) : M \to \mathbb{R}$ is a positive smooth function and b is a nonnegative constant. We define the function space

$$\mathcal{H} = \{\phi \in C^\infty(M), \Delta\phi - b|\nabla\phi|^2 + a(x) > 0\}$$

and $u_0, u_1 \in \mathcal{H}$. Note that the sign $-b|\nabla u|^2$ makes the space $\mathcal{H}$ convex, meaning that if $u_0, u_1 \in \mathcal{H}$, then $(1-t)u_0 + tu_1 \in \mathcal{H}$ for any $t \in [0, 1]$.

A main result of the paper is the following,

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold and $f \in C^k(M \times [0, 1])$ with $k \geq 2$ is a positive function. The Dirichlet problem (1.3) has a unique solution $u(x, t) \in C^{k+1,\beta}(M \times [0, 1])$ for any $\beta \in [0, 1)$. The uniform C^1 estimates and estimates of $u_{tt}, |u_{tk}|, \Delta u$ do not depend on* $\inf f$, *but on (M, g), boundary datum u_0, u_1 and*

$$\max\left\{\sup f, \sup|Df^{1/2}|, \sup|f_{tt}|, \sup|\Delta f|\right\}$$

for any $t \in [0, 1]$.

Remark 1.2. *This generalizes the results in [2], where the authors solved the Donaldson equation with righthand side ϵ. Here we consider a class of equations which also covers the Gursky-Streets equation when $k = 1$. Our computations are much more streamlined and simplified.*

As a direct corollary, we solve the homogeneous equation with the weak C^2 bound.

Corollary 1.3. *Let (M, g) be a compact Riemannian manifold. Then there exists a solution to the Dirichlet problem of the homogeneous equation*

$$u_{tt}(\Delta u - b|\nabla u|^2 + a(x)) - |\nabla u_t|^2 = 0$$

such that $u(0,\cdot) = u_0$ and $u(1,\cdot) = u_1$ with the uniform bound,

$$|u|_{C^1} + |u_{tt}| + |\Delta u| + |\nabla u_t| \leq C.$$

2. Solve the equation

For simplicity, we write

$$B_u = \Delta u - b|\nabla u|^2 + a(x).$$

Its linearized operator is given by

$$L_{B_u}(h) = \Delta h - 2b(\nabla u, \nabla h)$$

We write the equation

$$Q(u_{tt}, B_u, \nabla u_t) := u_{tt}B_u - |\nabla u_t|^2 = f, \tag{2.1}$$

where $f \in C^\infty(M \times [0,1])$ is a positive function and $u_0, u_1 \in \mathcal{H}$. When there is no confusion, we also write

$$Q(u) = Q(u_{tt}, B_u, \nabla u_t)$$

We compute the linearized operator, which is given by

$$\begin{aligned} dQ(h) =& u_{tt}[\triangle h - 2b(\nabla u, \nabla h)] + B_u h_{tt} - 2(\nabla h_t, \nabla u_t) \\ =& u_{tt}L_{B_u}(h) + B_u h_{tt} - 2(\nabla h_t, \nabla u_t). \end{aligned}$$

We will use the following notations. At any point $p \in M \times [0,1]$, take local coordinates $(x_1, \cdots, x_n, t)$. We can always diagonalize the metric tensor g as $g_{ij}(p) = \delta_{ij}, \partial_k g_{ij}(p) = 0$. We will use, for any smooth function f on $M \times [0,1]$, the following notations

$$\triangle f_i = \triangle(f_i), \quad \triangle f_{ij} = \triangle(f_{ij}), \quad \triangle f_{,i} = (\triangle f)_{,i} \;\text{ and }\; \triangle f_{,ij} = (\triangle f)_{ij}.$$

For any function f, f_i, f_{ij} etc are covariant derivatives. By Weitzenbock formula, we have

$$\triangle f_i = \triangle f_{,i} + R_{ij}f_j, \tag{2.2}$$

where R_{ij} is the Ricci tensor of the metric g.

The following concavity is important for solving the equation.

Lemma 2.1 (Donaldson [4]). *1. If $A>0$, then $Q(A)>0$ and if $A\geq 0$, $Q(A)\geq 0$.*

2. If A, B are two matrices with $Q(A)=Q(B)>0$, and if the entries A_{00}, B_{00} are positive then for any $s\in[0,1]$,

$$Q(sA+(1-s)B)\geq Q(A), Q(A-B)\leq 0.$$

Moreover, strict inequality holds if the corresponding arguments are not the same.

We have its equivalent form.

Lemma 2.2 ([2]). *Consider the function*

$$f(x,y,z_1,\cdots,z_n)=\log\left(xy-\sum z_i^2\right).$$

Then f is concave when $x>0, y>0, xy-\sum z_i^2>0$.

First we assume u solves the Dirichlet problem (1.3) and derive the *a priori* estimates. With these estimates, it is standard to use the method of continuity to solve the equation.

2.1. C^0 estimates and uniqueness

Denote $U_c=ct(1-t)+(1-t)u_0+tu_1$ for any number c.

Lemma 2.3. *For some $c>0$ big enough,*

$$U_{-c}\leq u\leq (1-t)u_0+tu_1. \tag{2.3}$$

Moreover, the solution u is unique.

Proof. First we have

$$u_{tt}>0.$$

It follows that

$$\frac{u(\cdot,t)-u(\cdot,0)}{t-0}<\frac{u(\cdot,1)-u(\cdot,t)}{1-t}.$$

Namely

$$u(t)<(1-t)u_0+tu_1.$$

Note that $u = U_{-c}$ on the boundary. If $u < U_{-c}$ for some point, then $v = u - U_{-c}$ obtains its minimum in the interior, say at p. Then, $\nabla v = 0, D^2 v \geq 0$ at p. By the concavity of $\log Q$, we have

$$Q^{-1}dQ(v) \leq \log Q(u) - \log Q(U_{-c}), \tag{2.4}$$

where $Q^{-1}dQ$ takes value at u. Clearly $Q(U_{-c}) = 2cB_{U_{-c}} - |\nabla u_0 - \nabla u_1|^2$. Note that $B_{U_{-c}} \geq (1-t)B_{u_0} + tB_{u_1}$ is strictly positive. If we choose c sufficiently large, the righthand side of (2.4) is negative. However at p, $\nabla v = 0, D^2 v \geq 0$, we claim $dQ(v) \geq 0$. Contradiction. To see the claim, we choose a vector $(x_0, Y) = (x_0, y_1, \cdots y_n)$, then by $D^2 v(p) \geq 0$ we have,

$$v_{tt}x_0^2 - 2x_0(\nabla v_t, Y) + Y\nabla^2 v Y^t \geq 0$$

Choose $x_0 = B_u, Y = \nabla u_t$ and note $Y\nabla^2 v Y^t \leq \Delta v |\nabla u_t|^2$. It follows

$$2(\nabla u_t, \nabla v_t) \leq v_{tt}B_u + B_u^{-1}\Delta v |\nabla u_t|^2$$

We compute

$$\begin{aligned} dQ(v) =& v_{tt}B_u + u_{tt}(\Delta v - 2b(\nabla u, \nabla v)) - 2(\nabla u_t, \nabla v_t) \\ \geq& (u_{tt} - B_u^{-1}|\nabla u_t|^2)\Delta v \geq 0. \end{aligned}$$

This contradicts (2.4) if c is sufficiently large.

Now we prove the uniqueness. Suppose u and $\tilde{u}$ are two solutions with the same boundary condition, then for any $\varepsilon \in (0, 1)$,

$$u \leq \tilde{u} + \varepsilon t(1-t).$$

This follows the same maximum principle argument above. Let $\varepsilon \to 0$, we obtain $u \leq \tilde{u}$. Interchanging u and $\tilde{u}$ this shows that $\tilde{u} = u$. ☐

2.2. C^1 estimates

Proposition 2.4. *We have the following,*

$$-c + u_1 - u_0 \leq u_t(0, \cdot) \leq u_1 - u_0 \leq u_t(1, \cdot) \leq u_1 - u_0 + c.$$

Proof. By Lemma 2.3,

$$-ct(1-t) + (1-t)u_0 + tu_1 \leq u \leq (1-t)u_0 + tu_1.$$

Since $u_{tt} > 0$, u_t obtains its maximum on the boundary. It is then easy to verify that the estimate holds. ☐

Remark 2.5. *Since $u + At + B$ still solves the equation for any constants A, B. The boundary data changes as, $u_0 \to u_0 + B$, $u_1 \to u_1 + A + B$ and $u_t \to u_t + A$. (Note that ∇u remains the same.) Since we have uniform bound on $|u|_{C^0}$ and $|u_t|$, we can choose A, B accordingly such that $1 \le |u_t| \le C$, and $1 \le -u \le C$. We assume this normalization in the following.*

We need some preparations. We have the following straightforward computations.

Proposition 2.6. *We have*

$$dQ(t) = 0, dQ(t^2) = 2B_u.$$

Proposition 2.7. *We have*

$$dQ(u) = 2f - (a + b|\nabla u|^2)u_{tt}$$

Proof. We compute

$$dQ(u) = u_{tt}(\Delta u - 2b|\nabla u|^2) + B_u u_{tt} - 2|\nabla u_t|^2$$

Using the equation this completes the proof. □

Proposition 2.8. *Given ϕ, ψ, we have*

$$dQ(\phi\psi) = \psi dQ(\phi) + \phi dQ(\psi) + 2q_u(D\phi, D\psi), \tag{2.5}$$

where the quadratic form is given by

$$q_u(D\phi, D\psi) = u_{tt}(\nabla\phi, \nabla\psi) + B_u(\phi_t, \psi_t) - (\nabla u_t, \phi_t\nabla\psi + \psi_t\nabla\phi)$$

Note that $q_u(D\phi, D\phi) \ge 0$.

Proposition 2.9. *We compute*

$$dQ(|\nabla u|^2) = 2u_{tt}(R_{ij}u_iu_j - a_iu_i) + 2f_iu_i + 2q_u(Du_i, Du_i) \tag{2.6}$$

Proof. We compute,

$$dQ(u_i) = u_{tt}(\Delta u_i - 2b(\nabla u, \nabla u_i)) + B_u u_{tti} - 2u_{tk}u_{tki}$$

Taking derivative of the equation, we get

$$u_{tt}((\Delta u)_i - 2b(\nabla u, \nabla u_i) + a_i) + B_u u_{tti} - 2u_{tk}u_{tki} = f_i.$$

It follows that

$$dQ(u_i) = u_{tt}(R_{ij}u_j - a_i) + f_i. \tag{2.7}$$

Applying (2.5) to $\phi = u_i$, we get (2.6). □

Lemma 2.10. *There exists a uniform constant*

$$C_2 = C_2(g, |u_0|_{C^1}, |u_1|_{C^1}, \sup f, |\nabla f^{1/2}|)$$

such that

$$|\nabla u| \leq C_2.$$

Proof. To bound ∇u, take

$$h = \frac{1}{2}\left(|\nabla u|^2 + \lambda u^2 + At^2\right),$$

where λ is a constant determined later. We want to show that h is bounded. Namely, there exists a constant C_1 depending only on $\sup f, |\nabla f^{1/2}|$ and the boundary data such that

$$\max h \leq C_2.$$

Since h is uniformly bounded on the boundary, we assume h takes its maximum at $(p, t_0) \in M \times (0, 1)$. We compute, using (2.5),

$$dQ(u^2) = -2u(a + b|\nabla u|^2) + 4fu + 2q_u(Du, Du)$$

It follows that, using (2.6),

$$\begin{aligned} dQ(h) =& u_{tt}(R_{ij}u_iu_j - a_iu_i) + f_iu_i + q_u(Du_i, Du_i) \\ &- \lambda u(a + b|\nabla u|^2) + 2\lambda fu + \lambda q_u(Du, Du) + AB_u \\ \geq& - C_0u_{tt}(|\nabla u|^2 + |\nabla u|) - |\nabla u||\nabla f| \\ &- \lambda u(a + b|\nabla u|^2) + 2\lambda fu + \lambda q_u(Du, Du) + AB_u, \end{aligned} \tag{2.8}$$

where C_0 depends on $\max |Ric|$ and $|\nabla a|$. We compute

$$\begin{aligned} q_u(Du, Du) =& u_{tt}|\nabla u|^2 + B_uu_t^2 - 2u_{tk}u_tu_k \\ \geq& \frac{1}{2}u_{tt}|\nabla u|^2 - B_uu_t^2, \end{aligned}$$

where we use the fact that

$$\frac{1}{2}u_{tt}|\nabla u|^2 + 2B_u u_t^2 \geq 2\sqrt{u_{tt}B_u}|\nabla u||u_t| \geq 2|\nabla u_t||\nabla u||u_t|$$

Since $-u \geq 1$ and $|u_t| \leq C_1$, we obtain,

$$dQ(h) > u_{tt}(\frac{\lambda}{2}|\nabla u|^2 - C_0|\nabla u|^2 - C_0|\nabla u|) - |\nabla u||\nabla f| + 2\lambda f u + (A - C_0\lambda)B_u$$

Choose $A = C_0\lambda + \lambda$. Note that

$$\lambda u_{tt}|\nabla u|^2/4 + \lambda B_u \geq \lambda\sqrt{u_{tt}B_u}|\nabla u||u_t| \geq \lambda\sqrt{f}|\nabla u|$$

We compute that

$$dQ(h) > u_{tt}(\lambda|\nabla u|^2/4 - C_0|\nabla u|^2 - C_0|\nabla u|) + \lambda\sqrt{f}|\nabla u| - |\nabla f||\nabla u| + 2\lambda f u.$$

Since at p, $dQ(h) \leq 0$. It follows that, at p,

$$|\nabla u|(p) \leq C_2,$$

where C_2 depends on $|\nabla f^{1/2}|$ in addition. This completes the proof. □

2.3. C^2 estimates

First we derive the boundary estimates. Due to the flatness of the boundary (in t direction), the estimates of "normal-normal" direction u_{tt} can be obtained from the equation that

$$u_{tt} \leq B_u^{-1}(|\nabla u_t|^2 + f),$$

once the boundary estimates hold for $|\nabla u_t|$. To bound the mixed term $|\nabla u_t|$ in the boundary estimates, we construct barrier functions using similar ideas in [7, 6]. The argument is purely local.

Lemma 2.11. *There exists a uniform constant C_2, such that at $t = 0$ and $t = 1$,*

$$u_{tt}, |\nabla u_t| \leq C_2$$

where $C_2 = C_2(g, |u_0|_{C^2}, |u_1|_{C^2}, |\nabla f^{1/2}|, \sup f)$

Proof. We only argue for $t = 0$. First we compute

$$
\begin{aligned}
dQ(u - u_0) &= 2f - u_{tt}(\Delta u_0 - b|\nabla u_0|^2 + a) \\
&\quad - bu_{tt}(|\nabla u|^2 + |\nabla u_0|^2 - 2(\nabla u, \nabla u_0)) \\
&\leq 2f - u_{tt} B_{u_0}
\end{aligned}
$$

For a fix point $p \in M$, take a geodesic ball $B_r(p) \subset M$ around p such that r is less than injectivity radius. Consider the region

$$U = \{(x, t) \in B_r(p) \times [0, 1] : d^2(x, p) + t^2 \leq r^2\}$$

Denote, for positive constants A, B, c which will be specified below,

$$h = A(u_0 - u - 3ct + ct^2) - B(t^2 + d^2(x)) + (\nabla u - \nabla u_0)_i,$$

where $i = 1, 2, \cdots, n$ and $d(x) = d(p, x)$ is the distance function. Note that h is local function define on $\bar{U}$. We choose c, B sufficiently large such that $h \leq 0$ on ∂U. We compute, using (2.7),

$$|dQ((\nabla u - \nabla u_0)_i)| \leq C_0 u_{tt} + |\nabla f|.$$

Note that for $x \in B_r(p)$ for r sufficiently small,

$$|dQ(d^2)| = u_{tt}|(\Delta d^2 - 2b(\nabla u, \nabla d^2))| \leq C_0 u_{tt}.$$

It then follows that

$$dQ(h) \geq A(u_{tt} B_{u_0} - 2f + 2cB_u - 2f) - 2BB_u - (B + 1)C_0 u_{tt} - |\nabla f|$$

Choose A sufficiently large, $c \geq 1$ such that $AB_{u_0} \geq 2$, $AB_{u_0} - 2(B+1)C_0 \geq 0$ and $Ac \geq 2B$, we get that

$$dQ(h) \geq u_{tt} + AcB_u - 2fA - |\nabla f| \geq 2\sqrt{u_{tt} B_u Ac} - 2fA - |\nabla f|$$

Now choose c sufficiently large (depending on $\sup f$ and $|\nabla f^{1/2}|$) such that

$$dQ(h) \geq 2\sqrt{fAc} - 2Af - |\nabla f| > 0.$$

By the maximum principle, it follows that $h \leq 0$ in U. Since $h(p, 0) = 0$, it follows that $\partial_t h(p, 0) \leq 0$. Since i and p are arbitrary, this implies that $|\nabla u_t|(p, 0) \leq C_2$ at $t = 0$, where C_2 depends on $|\nabla f^{1/2}|$ in particular. □

Now we derive the interior C^2 estimates. We need some preparations to simply the computations. We write $r = (r_i)$ and

$$Q(r) = r_0 r_1 - \sum_{i\geq 2} r_i^2, G(r) = \log Q(r),$$

where $r = (u_{tt}, B_u, \nabla_i u_t)$. Then the equation $Q(r) = f$ can be written as $G(r) = \log f$. Denote, for $0 \leq i \leq n+1$,

$$Q^i = \frac{\partial Q}{\partial r_i}, G^i = \frac{\partial G}{\partial r_i} Q^{-1} A^i, Q^{i,j} = \frac{\partial^2 Q}{\partial r_i \partial r_j}, G^{i,j} = \frac{\partial^2 G}{\partial r_i \partial r_j}.$$

With this notation, we also record the linearization of $Q(r)$. We have

$$dQ(\psi) = u_{tt}(\Delta\psi - 2b(\nabla u, \nabla\psi)) + B_u \psi_{tt} - 2u_{tk}\psi_{tk} \tag{2.9}$$

If we write $(R_i) = (\psi_{tt}, L_{B_u}\psi, \nabla\psi_t)$, then

$$dQ(\psi) = \sum_i Q^i R_i.$$

First we have the following interior estimates.

Lemma 2.12. *There is a uniform positive constants C_2 such that*

$$u_{tt} \leq C_2,$$

where $C_2 = C_2(g, |u_0|_{C^2}, |u_1|_{C^2}, \sup f^{-1}|f_t|^2, \sup -f_{tt}, \sup f)$.

Proof. We can compute by $G = \log Q = \log f$

$$G^i \partial_t r_i = Q^{-1} dQ(u_t) = f^{-1} f_t. \tag{2.10}$$

Taking derivative again, we have

$$G^{i,j}\partial_t r_i \partial_t r_j + G^i \partial_t^2 r_i = f^{-1} f_{tt} - f^{-2} f_t^2.$$

By concavity of G, we have

$$G^i \partial_t^2 r_i \geq f^{-1} f_{tt} - f^{-2} f_t^2.$$

Note that

$$\partial_t^2 B_u = L_{B_u}(u_{tt}) - 2b|\nabla u_t|^2.$$

It follows that we have

$$G^i\partial_t^2 r_i = Q^{-1}\left(dQ(u_{tt}) - 2bu_{tt}|\nabla u_t|^2\right)$$

Hence we have

$$dQ(u_{tt}) \geq f_{tt} - f^{-1}f_t^2$$

We compute

$$dQ(u_{tt}-u) \geq (a+b|\nabla u|^2)u_{tt} + f_{tt} - f^{-1}f_t^2 - 2f$$

If $u_{tt}-u$ takes the maximum at the boundary, then by the boundary estimate this is done. If the maximum appears interior, at the maximum point of $u_{tt}-u$, we have

$$u_{tt} \leq C_3,$$

where $C_3 = C_3(\sup f, \sup -f_{tt}, \sup f^{-1}f_t^2)$. This completes the proof. □

Next we want to bound Δu. We use the similar computation relying on the concavity of $G = \log Q$.

Lemma 2.13. *There exists a uniform constant C_4 such that*

$$\Delta u \leq C_4,$$

where $C_4 = C_4(g, |u_0|_{C^2}, |u_1|_{C^2}, \sup f, \sup -\Delta f, \sup Df^{1/2})$

Proof. We only need to control the interior maximum. We compute

$$\begin{aligned}\nabla G &= f^{-1}\nabla f = G^i\nabla r_i,\\ G^{i,j}\nabla r_i\nabla r_j + G^i\Delta r_i &= f^{-1}\Delta f - f^{-2}|\nabla f|^2.\end{aligned} \tag{2.11}$$

By the concavity of G, we have

$$Q^i\Delta r_i \geq \Delta f - f^{-1}|\nabla f|^2. \tag{2.12}$$

There exists a difference between $Q^i\Delta r_i$ and $dQ(\Delta u)$ coming from communication of covariant derivatives and the nonlinear term $-b|\nabla u|^2$. We compute

$$(\Delta r_i) = (\Delta u_{tt}, \Delta B_u, \Delta u_{tk})$$

The Bochner-Weitzenbock identity gives

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 + 2(\nabla\Delta u, \nabla u) + 2Ric(\nabla u, \nabla u)$$

Hence we have

$$\begin{aligned}\Delta B_u =&\Delta(\Delta u - b|\nabla u|^2 + a)\\ =&L_{B_u}(\Delta u) - 2b|\nabla^2 u|^2 - 2bRic(\nabla u, \nabla u) + \Delta a\end{aligned}$$

We also have

$$\Delta u_{tk} = Ric_{kj}u_{tj} + (\Delta u)_{tk}$$

It follows that

$$\begin{aligned}Q^i\Delta r_i =&dQ(\Delta u) - 2bu_{tt}\left(|\nabla^2 u|^2 + Ric(\nabla u, \nabla u)\right)\\ &+ u_{tt}\Delta a - 2Ric(\nabla u_t, \nabla u_t)\end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$dQ(\Delta u) \geq 2bu_{tt}|\nabla^2 u|^2 + 2Ric(\nabla u_t, \nabla u_t) - C_2 + \Delta f - f^{-1}|\nabla f|^2 \tag{2.14}$$

Since $b \geq 0$, the nonlinear term $-b|\nabla u|^2$ results in a good term $2bu_{tt}|\nabla^2 u|^2$. Now we denote $v = \Delta u + \lambda t^2$. Then we have

$$dQ(\Delta u + \lambda t^2) \geq 2\lambda B_u - C_1|\nabla u_t|^2 - C_2 + \Delta f - f^{-1}|\nabla f|^2.$$

Since $|\nabla u_t|^2 \leq u_{tt}B_u \leq CB_u$, we can choose λ sufficiently large such that

$$dQ(\Delta u + \lambda t^2) \geq B_u - C_2 + \Delta f - f^{-1}|\nabla f|^2.$$

This is sufficiently to bound Δu from above. □

To get higher regularity, we assume that f is strictly positive. The Hölder estimate of D^2u follows from Evans-Krylov theory using the concavity of $\log Q$. Once we get the Hölder estimates of D^2u, the standard boot-strapping argument gives all higher order derivatives of u.

2.4. Solve the equation

To solve (1.3) for a general positive f, we consider the following continuity family for $s \in [0, 1]$

$$Q(u) = (1 - s)Q(U_{-c}) + sf, \tag{2.15}$$

with the boundary condition

$$u(\cdot, 0, s) = u_0, u(\cdot, 1, s) = u_1.$$

When c is big enough, $Q(U_{-c})$ is positive and bounded away from 0. We shall now prove that if $f \in C^k(X \times [0,1])$ with $k \geq 2$ then we can find of solution of (1.3) such that $u \in C^{k+1,\beta}(X \times [0,1])$ for any $0 \leq \beta < 1$. Consider the set

$$S = \left\{ s \in [0,1] : \text{ the equation (2.15) has a solution in } \ C^{k-1,\beta}(X \times [0,1]) \right\}$$

Obviously $0 \in S$. Hence we only need to show that S is both open and close. It is clear that $Q : C^{k+1,\beta} \to C^{k-1,\beta}$ is open if

$$B_u > 0 \ \text{ and } \ Q(u) > 0.$$

In this case dQ is an invertible elliptic operator and openness follows. The closeness of S follows from the a prior estimates derived in Section 2. Hence Theorem 1.1 holds.

Since our estimates on $|u|_{C^1}, u_{tt}, \Delta u, |\nabla u_t|$ does not depend on $\inf f$, we can solve the equation

$$Q(u) = sf$$

for $s \in (0,1]$ and $f > 0$. Taking $s \to 0$, this gives a strong solution of the homogeneous equation

$$Q(u) = u_{tt} B_u - |\nabla u_t|^2 = 0,$$

which has the uniform bound on $|u|_{C^1}, u_{tt}, \Delta u, |\nabla u_t|$. This proves Corollary 1.3.

Remark 2.14. *For the general righthand side $f \geq 0$ (possible degenerate) such that $|Df^{1/2}|$ is uniformly bounded, we can use an approximation argument to get a strong solution, by considering for example the equation*

$$u_{tt}(\Delta u - b|\nabla u|^2 + a(x)) - |\nabla u_t|^2 = f + s$$

for $s \in (0,1]$. Letting $s \to 0$ we get a strong solution. The only technical point is that uniqueness of homogeneous/degenerate equation does not follow directly from the comparison, which requires $f > 0$. On the other hand, we believe that the uniqueness should still hold.

Remark 2.15. *It would be interesting to see whether $|\nabla^2 u|$ is uniformly bounded, independent of $\inf f$. Such a result was proved for complex Monge-Ampere equation recently by [3]. When $n = 2$, the Donaldson equation is one special case of their results and it should work also for (1.3). On the other hand, it would be interesting to see whether such an estimate holds for $n \geq 3$.*

3. Discussions

When $k = 1$, the nonlinear term $-b|\nabla u|^2$ in $B_u = \Delta u - b|\nabla u|^2 + a$ has the "right" sign. Hence we can treat the Donaldson equation and the Gursky-Streets equation together. In [2] only the righthand side $f = \epsilon$ was discussed. Here we give a new argument with more streamlined computations. This also covers the Gursky-Streets equation when $k = 1$.

When $k = n$, the operator

$$F_n(r) = r_{00}\sigma_n(R) - (T_{n-1}(R), r_{0i} \otimes r_{0i}) = \sigma_{n+1}(r),$$

hence it is just the famous Monge-Ampere operator. It is not hard to see that the theory of Monge-Ampere equation can be used directly to solve the equation

$$F_n(r) = f.$$

We shall skip the details.

On the other hand, the Gursky-Streets equation becomes rather subtle when $2 \le k \le n-1$. When $k = 2$, Gursky and Streets obtained a smooth solution with uniform C^1 bound for a perturbed equation [5]. Very recently, the second author solved the Gursky-Streets equation with uniform $C^{1,1}$ bound, for $n \ge 4$. There are several subtle points. First of all, the concavity of the operator $\log F_k(r)$ is rather subtle for $k = 2$, and it is still unknown for $3 \le k \le n-1$; see [8] for the discussion and the conjecture on the concavity. The estimate of second order, in particular Δu appears to be very subtle.

Lastly, we introduce a family of operators, which is the complex companion of F_k. Let $u : \mathbb{R} \times \mathbb{C}^n \to \mathbb{R}$ be a real valued function. Consider the following $(n+1) \times (n+1)$ matrix

$$r = \begin{pmatrix} r_{00} & r_{0i} \\ \bar{r}_{0i} & R \end{pmatrix}$$

where R is a $n \times n$ Hermitian matrix. We take $R = \partial\bar{\partial}u$ and

$$r = \begin{pmatrix} u_{tt} & \partial u_t \\ \bar{\partial} u_t & \partial\bar{\partial}u \end{pmatrix}$$

Denote the operator, $1 \le k \le n$,

$$G_k(r) = u_{tt}\sigma_k(\partial\bar{\partial}u) - (T_{k-1}(\partial\bar{\partial}u), \partial u_t \otimes \bar{\partial}u_t),$$

where $T_{k-1}(R)_{i\bar{j}} = \sigma_k(R)\frac{\partial \log \sigma_k(R)}{\partial R^{i\bar{j}}}$. When $k = 1$, we get that

$$G_1(r) = u_{tt}\Delta u - |\nabla u_t|^2$$

is the Donaldson operator on $\mathbb{R} \times \mathbb{C}^n$. When $k = n$,

$$G_n(r) = u_{tt}\sigma_n(\partial\bar{\partial}u) - (T_{k-1}(\partial\bar{\partial}u), \partial u_t \otimes \bar{\partial}u_t)$$

is a special case of the complex Monge-Ampere operator. Actually this operator is the operator underline the geodesic equation in space of Kähler metrics,

$$\phi_{tt} - |\nabla\phi|^2_{\omega_\phi} = 0,$$

which was studied extensively in the literature. Similar as in [8], we conjecture that,

Conjecture 3.1. *For $2 \le k \le n-1$, the operator $\log G_k(r)$ is concave on r, for $r_{00} > 0$, $\partial\bar{\partial}u$ in Γ_k^+ cone and $G_k(r) > 0$.*

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