

# On the strict convexity of the K-energy

ROBERT J. BERMAN

*Dedicated to Simon Donaldson on the occasion of his 60th birthday*

**Abstract:** Let  $(X, L)$  be a polarized projective complex manifold. We show, by a simple toric one-dimensional example, that Mabuchi’s K-energy functional on the geodesically complete space of bounded positive  $(1, 1)$ -forms in  $c_1(L)$ , endowed with the Mabuchi-Donaldson-Semmes metric, is not strictly convex modulo automorphisms. However, under some further assumptions the strict convexity in question does hold in the toric case. This leads to a uniqueness result saying that a finite energy minimizer of the K-energy (which exists on any toric polarized manifold  $(X, L)$  which is uniformly K-stable) is uniquely determined modulo automorphisms under the assumption that there exists some minimizer with strictly positive curvature current.

## 1. Introduction

Let  $(X, L)$  be a polarized compact complex manifold and denote by  $\mathcal{H}$  the space of all smooth metrics  $\phi$  on the line bundle  $L$  with strictly positive curvature, i.e the curvature two-form  $\omega_\phi$  of  $\phi$  defines a Kähler metric on  $X$ . A leading role in Kähler geometry is played by Mabuchi’s K-energy functional  $\mathcal{M}$  on  $\mathcal{H}$ , which has the property that a metric  $\phi$  in  $\mathcal{H}$  is a critical point for  $\mathcal{M}$  if and only if the Kähler metric  $\omega_\phi$  has constant scalar curvature [25]. From the point of view of Geometric Invariant Theory (GIT) the K-energy can, as shown by Donaldson [18], be identified with the Kempf-Ness “norm-functional” for the natural action of the group of Hamiltonian diffeomorphisms on the space of all complex structures on  $X$ , compatible with a given symplectic form.

As shown by Mabuchi the functional  $\mathcal{M}$  is convex along geodesics in the space  $\mathcal{H}$  endowed with its canonical Riemannian metric (the Mabuchi-Semmes-Donaldson metric). More precisely, as indicated by the GIT interpretation,  $\mathcal{M}$  is *strictly* convex modulo the action of the automorphism group

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$G$  of  $(X, L)$  in the following sense: let  $\phi_t$  be a geodesic in  $\mathcal{H}$  (parametrized so that  $t \in [0, 1]$ ) then

$$(1.1) \quad t \mapsto \mathcal{M}(\phi_t) \text{ is affine iff } \phi(t) = g(t)\phi_0,$$

where  $g(t)\phi_0$  denotes the action on  $\phi_0$  of a one-parameter subgroup in  $G$ , i.e.  $\phi_t$  is equal to the pull-back of  $\phi_0$  under the flow of a holomorphic vector field on  $X$ . However, the boundary value problem for the geodesic equation in  $\mathcal{H}$  does not, in general, admit strong solutions [24]. In order to bypass this complication Chen introduced a natural extension of  $\mathcal{M}$  to the larger space  $\mathcal{H}_{1,1}$  consisting of all (singular) metrics  $\phi$  on  $L$  such that the curvature  $\omega_\phi$  is defined as an  $L^\infty$ -form. The advantage of the latter space is that it is geodesically convex (in the sense of metric spaces; see [15] and references therein). It was conjectured by Chen [13] and confirmed in [3] that  $\mathcal{M}$  is convex on  $\mathcal{H}_{1,1}$ . However, the question whether  $\mathcal{M}$  is strictly convex modulo the action of  $G$  on  $\mathcal{H}_{1,1}$  was left open in [3]. In this short note we give a negative answer to the question already in the simplest case when  $X$  is the Riemann sphere and the metrics are  $S^1$ -invariant.

**Theorem 1.** *Let  $L \rightarrow X$  be the hyperplane line bundle over the Riemann sphere. There exists an  $S^1$ -invariant geodesic  $\phi_t$  in  $\mathcal{H}_{1,1}$  such that  $\mathcal{M}(\phi_t)$  is affine, but  $\phi_t$  is not of the form  $g(t)\phi_0$ .*

In terms of the standard holomorphic coordinate  $z$  on the affine piece  $\mathbb{C}$  of  $X$  the geodesic  $\phi_t$  in the previous theorem can be taken so that  $\phi_t$  is equal to the Fubini-Study metric  $\phi_0$  on the lower-hemisphere, flat on a  $t$ -dependent collar attached to the equator and then glued to a one-parameter curve of the form  $g(t)^*\phi_0$  in the remaining region of the upper hemisphere.

The failure of the strict convexity of  $\mathcal{M}$  on  $\mathcal{H}_{1,1}$  modulo  $G$  appears to be quite surprising in view of the fact that the other canonical functional in Kähler geometry – the Ding functional  $\mathcal{D}$  – is strictly convex on  $\mathcal{H}_{1,1}$  modulo  $G$ . In fact, the Ding functional (which is only defined in the “Fano case”, i.e. when  $L$  is the anti-canonical line bundle over a Fano manifold) is even strictly convex modulo the action of  $G$  on the space of all  $L^\infty$ -metrics on  $L$  with positive curvature current [9].

One important motivation for studying strict convexity properties of the Mabuchi functional  $\mathcal{M}$  on suitable completions of  $\mathcal{H}$  comes from the *Yau-Tian-Donaldson conjecture*. In its uniform version the conjecture says that the first Chern class  $c_1(L)$  of  $L$  contains a Kähler metric of constant scalar curvature iff  $(X, L)$  is uniformly K-stable (in the  $L^1$ -sense) relative to a maximal torus of  $G$  [10, 17]. The “only if” direction was established in [8] when

$G$  is trivial (and a similar proof applies in the general case [5]). The proof in [8] uses the convexity of  $\mathcal{M}$  on the finite energy completion  $\mathcal{E}^1$  of  $\mathcal{H}$ .<sup>1</sup> The remaining implication in the Yau-Tian-Donaldson conjecture is still widely open, in general, but a first step would be to establish the existence of a minimizer of  $\mathcal{M}$  in the finite energy space  $\mathcal{E}^1$ , by generalizing the variational approach to the “Fano case” introduced in [4]. Leaving aside the challenging question of the regularity of a minimizer one can still ask if the minimizer is canonical, i.e. uniquely determined modulo  $G$ ? (as conjectured in [16]). The uniqueness in question would follow from the strict convexity of  $\mathcal{M}$  on  $\mathcal{E}^1$  modulo the action of  $G$ . However, by the previous theorem such a strict convexity does not hold, in general. On the other hand, it would be enough to establish the following weaker strict convexity property:

$$t \mapsto \mathcal{M}(\phi_t) \text{ is constant} \implies \phi(t) = g(t)\phi_0,$$

(the converse implication holds if  $(X, L)$  is K-stable). This may still be too optimistic, but here we observe that this approach towards the uniqueness problem can be made to work in the toric case if one assumes some a priori positivity of the curvature current of some minimizer.

**Theorem 2.** *Let  $(X, L)$  be an  $n$ -dimensional polarized toric manifold. Assume that  $(X, L)$  is uniformly K-stable relative to the torus action. Then there exists a finite energy minimizer  $\phi$  of  $\mathcal{M}$  and the minimizer is unique modulo the action of  $\mathbb{C}^{*n}$  under the assumption that there exists some finite energy minimizer  $\phi_0$  whose curvature current is strictly positive on compacts of the dense open orbit of  $\mathbb{C}^{*n}$  in  $X$ .*

### 1.1. Relations to previous results

In view of its simplicity it is somewhat surprising that the counterexample in Theorem 1 does not seem to have been noticed before. The key point of the proof is a generalization of Donaldson formula for the Mabuchi functional  $\mathcal{M}$  in the smooth toric setting to a singular setting (see Lemma 5 and Lemma 7), showing that

$$(1.2) \quad \mathcal{M}(\phi) = \mathcal{F}(u),$$

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<sup>1</sup>The space  $\mathcal{E}^1$  was originally introduced in [22] from a pluripotential point of view and, as shown in [15],  $\mathcal{E}^1$  may be identified with the metric completion of  $\mathcal{H}$  with respect to the  $L^1$ -Finsler version of the Mabuchi-Semmes-Donaldson metric on  $\mathcal{H}$

where the non-linear part of the functional  $\mathcal{F}$  only depends on the non-singular part (in the sense of Alexandrov) of the Hessian of the convex function on the moment polytope of  $(X, L)$ , corresponding to the metric  $\phi$ . This leads, in fact, to a whole class of counter-examples to the strict convexity in question, by taking  $\phi_t$  to be any torus invariant geodesic  $\phi_t$ , emanating from a given  $\phi_0 \in \mathcal{H}$ , whose Legendre transform  $u_t$  is of the form

$$u_t = u_0 + tv,$$

for a convex and piece-wise affine function  $v$ . Such a curve defines a geodesic ray associated to a toric test configuration for  $(X, L)$  and the fact that  $\phi_t \in \mathcal{H}_{1,1}$  then follows from general results (see [14, 27, 28, Section 7]). A by-product of formula 1.2 is a slope formula for the K-energy along toric geodesics of finite energy (see Section 7).

The functional  $\mathcal{F}$  has previously appeared in a series of papers by Zhou-Zhu (see Remark 8). In particular, it was shown in [31, Section 6] (when  $n = 2$ ) that a minimizer  $u$  of  $\mathcal{F}$  is uniquely determined, modulo the complexified torus action, under the stronger assumption that there exists some minimizer  $u_0$  of  $\mathcal{F}$  which is  $C^\infty$ -smooth and strictly convex in the interior (while our assumption is equivalent to merely assuming that  $u$  is  $C^{1,1}$ -smooth in the interior). We also recall that in the toric surface case (i.e. when  $n = 2$ ) it was shown by Donaldson [19] that uniform K-stability is equivalent to K-stability. Moreover, the Yau-Tian-Donaldson in the latter case was settled by Donaldson in a series of papers culminating in [21] (as a consequence, any minimizer of  $\mathcal{M}$  is then smooth and positively curved). Theorem 2 should also be compared with the general uniqueness result for finite energy minimizers of  $\mathcal{M}$  on any Kähler manifold which holds under the assumption that there exists some minimizer which is smooth and strictly positively curved. The proof of the latter result, which generalizes the uniqueness result in [3], exploits a weaker form of strict convexity which holds on the linearized level around a bona fide metric with constant scalar curvature.

## 2. Proofs

We start with some preparations. To keep things as simple as possible we mainly stick to the one-dimensional situation (see [2] for the general convex analytical setup and its relations to polarized toric varieties).

### 2.1. Convex preparations

Let  $\phi(x)$  be a lower semi-continuous (lsc) convex function on  $\mathbb{R}$  (taking values in  $] - \infty, \infty]$ ). Its point-wise derivative  $\phi'(x)$  exists a.e. on  $\mathbb{R}$  and defines an element in  $L^{\infty}_{loc}(\mathbb{R})$ . We will denote by  $\partial\phi$  the subgradient of  $\phi$ , which is a set-valued map on  $\mathbb{R}$  with the property that  $(\partial\phi)(x)$  is a singleton iff  $\phi'(x)$  exists at  $x$ . Similarly, we will denote by  $\partial^2\phi$  the measure on  $\mathbb{R}$  defined by the second order distributional derivative of  $\phi$ . By Lebesgue's theorem we can decompose

$$\partial^2\phi = \partial_s^2\phi + \phi'',$$

where  $\partial_s^2\phi$  denotes the singular part of the measure  $\partial^2\phi$  and  $\phi'' \in L^1_{loc}(\mathbb{R})$  denotes the regular part (wrt Lebesgue measure  $dx$ ), which coincides with the second order derivative of  $\phi$  almost everywhere on  $x$ . We set

$$(2.1) \quad \phi_0(x) := \log(1 + e^x)$$

and

$$\mathcal{P}_+(\mathbb{R}) := \{\phi : \phi \text{ convex on } \mathbb{R}, \phi - \phi_0 \in L^{\infty}(\mathbb{R})\}$$

( $\partial^2\phi$  is a probability measure for any such  $\phi$ ). Given a function  $\phi$  in  $\mathcal{P}_+(\mathbb{R})$  we will denote by  $u$  its *Legendre transform* which defines a finite lsc convex function  $u$  on  $[0, 1]$  (which is equal to  $\infty$  on  $]0, 1[^c$ )

$$u(y) := (\phi^*)(y) := \sup_{x \in \mathbb{R}} xy - \phi(x)$$

Since  $\phi = u^*$  the map  $\phi \mapsto u$  gives a bijection

$$\mathcal{P}_+(\mathbb{R}) \longleftrightarrow \{u : u \text{ convex on } [0, 1]\} \cap L^{\infty}[0, 1]$$

which is an isometry wrt the  $L^{\infty}$ -norms. Moreover,  $\phi$  is smooth and strictly convex on  $\mathbb{R}$  iff  $u$  is smooth and strictly convex on  $]0, 1[$ , as follows from the formula

$$(2.2) \quad \phi(x) = xy - u'(y),$$

if  $x = u'(y)$  and  $u$  is differentiable at  $y$  (and vice versa if  $\phi$  is replaced by  $u$ ). Moreover, if  $u$  is two times differentiable at  $y = 0$  and  $u''(y) > 0$  then  $\phi''$  is differentiable at  $x$  and

$$(2.3) \quad \phi''(x) = 1/u''(y)$$

### 2.2. Complex preparations

Let  $(X, L) := (\mathbb{P}^1, \mathcal{O}(1))$  be the complex projective line  $\mathbb{P}^1$  endowed with the hyperplane line bundle  $\mathcal{O}(1)$ . Realizing  $\mathbb{P}^1$  as the Riemann sphere (i.e. the one-point compactification of the complex line  $\mathbb{C}$ ) a locally bounded metric  $\Phi$  on  $\mathcal{O}(1)$  may, in the standard way, be identified with a convex function  $\Phi(z)$  on  $\mathbb{C}$  such that

$$\Phi - \Phi_0 \in L^\infty(\mathbb{C}), \quad \Phi_0(z) := \log(1 + |z|^2),$$

where  $\Phi_0$  corresponds to the Fubini-Study metric on  $\mathcal{O}(1)$ , which defines a smooth metric with strictly positively curvature  $\omega_0$  on  $\mathbb{P}^1$  (coinciding with the standard  $SU(2)$ -invariant two-form on  $\mathbb{P}^1$ ). Moreover, the metric  $\Phi$  on  $\mathcal{O}(1)$  has semi-positive curvature  $\omega_\Phi$  on  $\mathbb{P}^1$  iff  $\Phi(z)$  is subharmonic on  $\mathbb{C}$ . More precisely,

$$(2.4) \quad \omega_{\Phi|\mathbb{C}} = \frac{i}{2\pi} \partial\bar{\partial}\Phi := \frac{i}{2\pi} \frac{\partial^2\Phi}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$$

We will denote by  $\mathcal{H}_b^{S^1}$  the space of all bounded (i.e  $L^\infty$ ) metrics  $\Phi$  on  $\mathcal{O}(1) \rightarrow \mathbb{P}^1$  with semi-positive curvature which are  $S^1$ -invariant (wrt the standard action of  $S^1$ ). Setting

$$x := \log |z|^2$$

gives a correspondence

$$\mathcal{H}_b^{S^1} \longleftrightarrow \mathcal{P}_+(\mathbb{R}), \quad \Phi(z) \mapsto \phi(x) := \Phi(e^{2x})$$

As is well-known, under the Legendre transform the Mabuchi-Donaldson-Semmes metric on  $\mathcal{H}^T$  corresponds the standard flat metric induced from  $L^2[0, 1]$ . It follows that a geodesic  $\Phi_t$  in  $\mathcal{H}_b^{S^1}$  corresponds to a curve  $\phi_t$  in  $\mathcal{P}_+(\mathbb{R})$  with the property that the corresponding curve

$$u_t := \phi_t^*$$

of bounded convex functions on  $[0, 1]$  is affine wrt  $t$ .

In particular, taking  $\phi_0$  to be defined by 2.1, the following curve defines a geodesic in  $\mathcal{H}_b^{S^1}$ :

$$\phi_t(x) := (u_0 + tv)^*, \quad u_0(y) := \phi_0^*(y) = y \log y + (1 - y) \log(1 - y),$$

where  $v$  is the following convex piece-wise affine function on  $[0, 1]$ :

$$v(y) := 0, \quad y \in [0, \frac{1}{2}], \quad v(y) = -\frac{1}{2} + y, \quad y \in [\frac{1}{2}, 1]$$

**Lemma 3.** *Assume that  $t \in ]0, 1[$ . Then  $\phi_t$  defines an  $S^1$ -invariant metric on  $\mathcal{O}(1) \rightarrow \mathbb{P}^1$  which is  $C^{1,1}$ -smooth on  $\mathbb{P}^1$  with semi-positive curvature and  $C^\infty$ -smooth and strictly positively curved on the complement of a  $t$ -dependent neighborhood of the equator. More precisely, in the logarithmic coordinate  $x \in \mathbb{R}$  we have  $\phi_t(x) = \phi_0(x)$  when  $x \leq 0$  and  $\phi_t(x) = \phi_0(0) + x/2$  when  $x \in [0, t]$  and  $\phi_t(x) = \phi_0(x - t) + t/2$  when  $x \geq t$ .*

*Proof.* *Step 1:*  $\phi_t$  is in  $C^{1,1}(\mathbb{R})$ .

This step only uses that  $v$  is convex and bounded on  $[0, 1]$ . Let  $v^{(j)}$  be a sequence of smooth strictly convex function on  $[0, 1]$  such that  $\|v^{(j)} - v\|_{L^\infty[0,1]} \rightarrow 0$ . Set  $u^{(j)}(t) := u_0 + tv^{(j)}$  and  $\phi^{(j)} := u^{(j)*}$ . Since the Legendre transform preserves the  $L^\infty$ -norm we have  $\|\phi^{(j)} - \phi\|_{L^\infty(\mathbb{R})} \rightarrow 0$ . By construction,  $u_t^{(j)}$  is smooth and satisfies  $(u_t^{(j)})'' \geq (u_0)'' \geq 1/C > 0$ . Hence, by 2.3,  $(\phi_t^{(j)})'' \leq C$  and letting  $j \rightarrow \infty$  thus implies that  $\partial^2\phi = \phi'' \leq C$ , showing that  $\phi_t$  is in  $C^{1,1}(\mathbb{R})$ .

*Step 2:* Explicit description of  $\phi_t$

We fix  $t > 0$  and observe that the map  $y \mapsto u'_t(y)$  induces two diffeomorphisms (with inverses  $x \mapsto \phi'_t(x)$ )

$$(2.5) \quad u'_t : Y_- := ]0, \frac{1}{2}[ \mapsto X_- := ]-\infty, 0[, \quad Y_+ := ]\frac{1}{2}, 1[ \mapsto X_+ := ]t, \infty[$$

This follows directly from the fact that  $u'_t$  is strictly positive on  $Y_\pm$  and converges to 0 and  $t$  when  $y \rightarrow 1/2$  from the left and right, respectively. We claim that this implies, by general principles, that the restriction of  $\phi_t$  to  $X_\pm$  only depends on the restriction of  $u_t$  to  $Y_\pm$ . Indeed, if  $x = u'_t(y)$  for  $y \in Y_\pm$  then formula 2.2 shows that  $\phi_t|_{X_\pm}$  only depends on the restriction of  $u$  to  $X_-$ . Hence, the restriction of  $\phi_t$  to  $X_-$  is given by  $u_0^* = \phi_0$  and the restriction of  $\phi_t$  to  $X_+$  is given by the Legendre transform of  $u_0(y) + t(y - 1/2)$ , which is equal to  $\phi_0(x - t) + t/2$ . Finally, it follows from the diffeomorphism 2.5 (using, for example, that  $\phi'_t$  is increasing) that  $\phi'_t = 1/2$  on  $[0, t]$ .  $\square$

*Remark 4.* A more symmetric form of the geodesic  $\phi_t$  may be obtained by setting  $\tilde{\phi}_t(x) := 2\phi_t(x) - x$ , which has the property that  $\tilde{\phi}_t(x) := \tilde{\phi}_0(x) := \log(e^{-x} + e^x)$  when  $x \leq 0$  and  $\tilde{\phi}_t(x) = \tilde{\phi}_0(0)$  when  $x \in [0, t]$  and  $\tilde{\phi}_t(x) = \tilde{\phi}_0(x - t)$  when  $x \geq t$ . Geometrically,  $\tilde{\phi}_t$  defines a geodesic ray of metrics on

$\mathcal{O}(2) \rightarrow \mathbb{P}^1$ , expressed in terms of the trivialization of  $\mathcal{O}(2)$  over  $\mathbb{C}^* \subset \mathbb{P}^1$  induced from the embedding  $\mathbb{C}^* \rightarrow \mathbb{C}^2 \rightarrow \mathbb{P}^2$  defined by  $F(z) := (z^{-1}, z) \in \mathbb{C}^2$ , where  $X$  is identified with the closure  $\overline{F(\mathbb{C})}$  of  $F(\mathbb{C})$  in  $\mathbb{P}^2$  and  $\mathcal{O}(2)$  with the restriction of  $\mathcal{O}_{\mathbb{P}^2}(1)$  to  $\overline{F(\mathbb{C})}$ . A direct calculation reveals that  $\tilde{\phi}_t(x)$  (and hence also  $\phi_t(x)$ ) is in fact  $C^{1,1}$ -smooth when viewed as a function on  $\mathbb{R} \times \mathbb{R}$ . This implies that the Laplacian of the corresponding local potentials over  $\mathbb{P}^1 \times D^*$  (where  $D^*$  denotes the punctured unit disc with holomorphic coordinate  $\tau$  such that  $t := -\log |\tau|^2$ ) is locally bounded, i.e. the geodesic has Chen’s regularity [12] in the space-time variables. It should also be pointed out that  $\tilde{\phi}_t$  can be realized as the geodesic ray, emanating from the Fubini-Study metric, associated to the toric test configuration of  $(X, L) := (\mathbb{P}^1, \mathcal{O}(2))$  determined (in the sense of [19, 14, 28]) by the piece-wise affine function  $\tilde{v}(y) = \max\{0, y\}$  on the moment polytope  $[-1, 1]$  of  $(\mathbb{P}^1, \mathcal{O}(2))$  (as in [14]). Using this realization the  $C^{1,1}$ -regularity also follows from the general results in [27, 14] which show that the Laplacian (or equivalently complex Hessian) of the corresponding potential is locally bounded over  $X \times D^*$ . Indeed, in the toric setting boundedness of the complex Hessian is equivalent to boundedness of the real Hessian, i.e. to  $C^{1,1}$ -regularity.

**2.2.1. The K-energy** Let  $(X, L)$  be a polarized compact complex manifold. We recall that the K-energy functional was originally defined by Mabuchi [25] on the space  $\mathcal{H}$  of all smooth metrics  $\Phi$  on  $L$  with strictly positive curvature by specifying its differential (more precisely, this determines  $\mathcal{M}$  up to an additive constant). Chen extended  $\mathcal{M}$  to the space  $\mathcal{H}_{1,1}$  consisting of all (singular) metrics  $\phi$  on  $L$  such that the curvature  $\omega_\phi$  of  $\Phi$  is defined as an  $L^\infty$ -form [13]. The extension is based on the Chen-Tian formula for  $\mathcal{M}$  on  $\mathcal{H}$  which may be expressed as follows in terms of a fixed Kähler form  $\omega_0$  on  $X$ :

$$(2.6) \quad \mathcal{M}(u) = \left( \frac{\bar{R}}{n+1} \mathcal{E}(\Phi) - \mathcal{E}^{\text{Ric } \omega_0}(\Phi) \right) + H_{\omega_0^n}(\omega_\Phi^n), \quad \bar{R} := \frac{nc_1(X) \cdot [\omega_0]^{n-1}}{[\omega_0]^n},$$

where

$$(2.7) \quad H_{\mu_0}(\mu) := \int_X \log \left( \frac{\mu}{\mu_0} \right) \mu$$

and  $\mathcal{E}$  and  $\mathcal{E}^{\text{Ric } \omega_0}$  are defined, up to an additive constant, by their differentials on  $\mathcal{H}$ :

$$(2.8) \quad d\mathcal{E}|_\Phi = (n+1)\omega_\Phi^n, \quad d\mathcal{E}^{\text{Ric } \omega_0}|_\Phi = n\omega_\Phi^{n-1} \wedge \text{Ric } \omega_0$$



with  $\text{Ric } \omega_0$  denoting the two-form defined by the Ricci curvature of  $\omega_0$  (see [3] for a simple direct proof of the Chen-Tian formula). The extension of  $\mathcal{M}$  to  $\mathcal{H}_{1,1}$  is obtained by observing that both terms appearing in the rhs of formula 2.6 are well-defined (and finite) when  $\Phi \in \mathcal{H}_{1,1}$ . We note that the functional appearing in the first bracket of the formula is continuous wrt the  $L^\infty$ -norm on  $\mathcal{H}_{1,1}$ . Indeed, it follows readily from the definitions that both  $\mathcal{E}$  and  $\mathcal{E}^{\text{Ric } \omega_0}$  are even Lip continuous wrt the  $L^\infty$ -norm.

In the present setting where  $X = \mathbb{P}^1$  we can, for concreteness, take  $\omega_0 = \omega_{\Phi_0}$ , whose restriction to  $\mathbb{C}$  is equal to a constant times  $e^{-2\Phi_0} dz \wedge d\bar{z}$ .

### 2.3. Conclusion of proof of Theorem 1

The proof will follow from the following extension to  $\mathcal{H}_{1,1}^{S^1}$  of a formula due to Donaldson when  $\Phi \in \mathcal{H}^{S^1}$  [19, Prop 3.2.8].

**Lemma 5.** *Assume that  $\Phi$  is in  $\mathcal{H}_{1,1}^{S^1}$ . Then*

$$\mathcal{M}(\Phi) = \mathcal{L}(u) - \int_{[0,1]} \log(u''(y)) dy, \quad \mathcal{L}(u) = \frac{1}{2}(u(1) + u(0)) - \int_0^1 u(y) dy$$

where  $u'' \in L^1_{loc}$  denotes the non-singular part of  $\partial^2 u$ .

*Proof. Step 1:* Assume that  $\Phi \in \mathcal{H}_{1,1}^{S^1}$ . Then

$$\mathcal{M}(\Phi) = \mathcal{L}(u) + \int_{\mathbb{R}} \phi''(x) \log \phi''(x) dx$$

In the case when  $\Phi \in \mathcal{H}^{S^1}$  (or more generally when  $u$  is continuous on  $[0, 1]$  and smooth and strictly convex in the interior) this follows from Donaldson's formula [19]. To extend the formula to the case when  $\Phi \in \mathcal{H}_{1,1}^{S^1}$  first observe that

$$(2.9) \quad \int \phi_0(x) \phi''(x) dx < \infty,$$

as follows directly from estimating  $\phi'' \leq C\phi_0'' \leq Ae^{-|x|/B}$  and  $\phi_0(x) \leq |x| + C$ . Hence, we can rewrite the Chen-Tian formula 2.6 as

$$(2.10) \quad \mathcal{M}(\Phi) = E_0(\Phi) + \int_{\mathbb{R}} \phi''(x) \log \phi''(x) dx,$$

where

$$E_0(\Phi) = \left( \frac{\bar{R}}{n+1} \mathcal{E}(\Phi) - \mathcal{E}^{\text{Ric } \omega_0}(\Phi) \right) + 2 \int_{\mathbb{R}} \phi_0(x) \phi''(x) dx,$$

Now take a sequence  $\Phi_j \in \mathcal{H}^{S^1}$  such that  $\|\Phi_j - \Phi\|_{L^\infty} \rightarrow 0$  (which equivalently means that  $\|u_j - u\|_{L^\infty[0,1]} \rightarrow 0$ ) and  $\omega_{\Phi_j} \leq C\omega_{\Phi_0}$ , i.e.

$$(2.11) \quad \phi_j''(x) \leq C\phi_0''(x)$$

We claim that

$$(2.12) \quad E_0(\Phi_j) \rightarrow E_0(\Phi).$$

Indeed, as pointed out above the first term appearing in the definition of  $E_0$  is continuous wrt the  $L^\infty$ -norm. To handle the second term first observe that, since  $\|\Phi_j - \Phi\|_{L^\infty(X)} \rightarrow 0$ , the probability measures  $\phi_j''(x)dx$  converge weakly towards  $\phi''(x)dx$  and hence, for any fixed  $R > 0$ ,

$$\lim_{j \rightarrow \infty} \int_{|x| \leq R} \phi_0(x)\phi_j''(x)dx = \int_{|x| \leq R} \phi_0(x)\phi''(x)dx$$

Moreover, the uniform bound 2.11 gives

$$\limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} \int_{|x| \geq R} \phi_0\phi_j''(x)dx \leq C \limsup_{R \rightarrow \infty} \int_{|x| \geq R} \phi_0(x)\phi''(x)dx = 0$$

Hence, letting first  $j \rightarrow \infty$  and then  $R \rightarrow \infty$  proves 2.12.

Now take a sequence  $\Phi_j \in \mathcal{H}^{S^1}$  such that  $\|\Phi_j - \Phi\|_{L^\infty} \rightarrow 0$  which equivalently means that  $\|u_j - u\|_{L^\infty[0,1]} \rightarrow 0$ . By Donaldson's formula

$$E_0(\phi_j) = \mathcal{L}(u_j)$$

and since both sides are continuous wrt the convergence of  $\Phi_j$  towards  $\Phi$  this concludes the proof of Step 1.

*Step 2:* Let  $\phi$  be a convex function on  $\mathbb{R}$  such that  $\partial^2\phi$  is a probability measure which is absolutely continuous wrt  $dx$ . Then

$$(2.13) \quad \int_{\mathbb{R}} \phi''(x) \log \phi''(x)dx = - \int_{[0,1]} \log(u''(y))dy,$$

if the left hand side is finite (and then  $u''(y) > 0$  a.e.).

This formula is a special case of McCann's Monotone change of variables theorem [26, Theorem 4.4]. But it may be illuminating to point out that a simple direct proof can be given in the present setting when  $\phi$  is of the form  $\phi_t$  appearing in Lemma 3. Indeed, then  $\rho := \phi'' = 0$  on a closed interval  $I$  of  $\mathbb{R}$  and  $\phi'$  diffeomorphism of the complement  $I$  onto  $]0, 1[-\{1/2\}$ . Since

$\rho \log \rho = 0$  if  $\rho = 0$  the formula 2.13 then follows directly from making the change of variables  $y = \phi'(x)$  on  $\mathbb{R} - S$ .  $\square$

Now, let  $\Phi_t$  be the geodesic in  $\mathcal{H}_{1,1}^{S^1}$  defined by the curve  $\phi_t$  appearing in Lemma 3. Since  $v$  is piece-wise affine we have  $u_t'' = u_0''$  a.e on  $\mathbb{R}$  and hence the previous lemma gives

$$\mathcal{M}(\Phi_t) = - \int_{[0,1]} \log(u''(y)) dy + t\mathcal{L}(v)$$

which is affine in  $t$ . Moreover,  $\phi_t$  is not induced from the flow of a holomorphic vector field (since this would imply that  $v$  is affine on all of  $[0, 1]$ ). This concludes the proof of Theorem 1.

*Remark 6.* The functional  $E_0$  in formula 2.10 coincides with the (attractive) Newtonian energy of the measure  $\mu = \partial^2\phi$ :

$$E_0(\mu) = \frac{1}{4} \int_{\mathbb{R}^2} |x - y| \mu(x) \otimes \mu(y)$$

and the continuity property of  $E_0$  used in the in Step 1 can be alternatively deduce from the fact that  $E$  is continuous on the space  $\mathcal{P}_1(\mathbb{R})$  of all probability measures with finite first moment (endowed with the  $L^1$ -Wasserstein topology). This point of view is further developed in the higher dimensional toric setting in [1].

### 2.4. Proof of Theorem 2

In this higher dimensional setting we will be rather brief and refer to [1] for more details. Let  $(X, L)$  be an  $n$ -dimensional toric manifold and denote by  $P$  the corresponding moment lattice polytope in  $\mathbb{R}^n$  which contains 0 in its interior. We will denote by  $d\sigma$  the measure on  $\partial P$  induced from the standard integer lattice in  $\mathbb{R}^n$  (which is comparable with the Lebesgue measure on  $\partial P$ ) [19]. The  $n$ -dimensional real torus acting on  $(X, L)$  will be denoted by  $T$ . As above we can then identify a  $T$ -invariant metric  $\Phi$  on  $L$  with positive curvature current with a convex function  $\phi(x)$  on  $\mathbb{R}^n$  (whose sub-gradient maps into  $P$ ) and, via the Legendre transform, with a convex function  $u$  on  $P$ . We will denote by  $\partial^2\phi$  the distributional Hessian of  $\phi$  and by  $(\nabla^2\phi)(x)$  the Alexandrov Hessian of  $\phi$  which is defined for almost all  $x$  (on the subset where  $\phi$  is finite).

Assume that  $(X, L)$  is uniformly K-stable relative to the torus  $T$  (in the  $L^1$ -sense). Concretely, this means (see [23]) that there exists  $\delta > 0$  such that

for any rational piece-wise affine convex function  $u$  on  $P$ ,

$$(2.14) \quad \mathcal{L}(u) := \int_P u dy - c \int_{\partial P} u \geq \delta \inf_{l \in (\mathbb{R}^n)^*} \left( \int_P (u - l) dy - \inf_P (u - l) \right),$$

$$c := \int_P dy / \int_{\partial P} d\sigma,$$

where the inf ranges over all linear functions  $l$  on  $\mathbb{R}^n$  (which, geometrically, may be identified with an element of the real part of the Lie algebra of the complex torus). We note that, by a standard approximation argument, the inequality 2.14 holds for the space  $\mathcal{C}(P)$  of all convex functions  $u$  on  $P$  such that  $u \in L^1(P) \cap L^1(\partial P)$  (where  $u|_{\partial P}(y)$  is defined as the radial boundary limit of  $u$ ).<sup>2</sup> The uniform K-stability implies, by [29, 23], that  $\mathcal{M}$  is coercive relative to  $T$ , i.e. there exist  $C > 0$  such that the following coercivity inequality holds on  $\mathcal{H}^T$ :

$$\mathcal{M}(\Phi) \geq \inf_{g \in \mathcal{C}^{*n}} J(g\Phi) / C - C,$$

where  $J$  denotes Aubin’s  $J$ -functional. The functional  $\mathcal{M}$  admits a canonical extension to the space  $\mathcal{E}^1$  of all (singular) metrics on  $L$  with positive curvature current and finite energy (namely, the greatest lsc extension of  $\mathcal{M}$  from  $\mathcal{H}$  to  $\mathcal{E}^1$ , endowed with the strong topology [6, 7]). The coercivity of  $\mathcal{M}$  combined with the results in [6] (which show that  $\mathcal{M}$  is lsc wrt the weak topology on  $\mathcal{E}^1$ ) implies that there exists a  $T$ -invariant minimizer  $\Phi_0$  of  $\mathcal{M}$  on the space  $\mathcal{E}^1(X, L)$  of all (singular) metrics on  $L$  with positive curvature current and finite energy.

A generalization of the argument used in the proof of Lemma 5 gives the following lemma which extends Donaldson’s formula in [19] to the finite energy setting (the proof is given in [1]):

**Lemma 7.** *Assume that  $\Phi \in \mathcal{E}^1(X, L)^T$  and  $\mathcal{M}(\Phi) < \infty$ . Then  $u \in \mathcal{C}(P)$  and*

$$(2.15) \quad \mathcal{M}(\Phi) = \mathcal{F}(u) := \mathcal{L}(u) - \int_P \log \det(\nabla^2 u(y)) dy,$$

where  $\nabla^2 u$  denotes the Alexandrov Hessian of  $u$  and both terms are finite (in particular,  $\nabla^2 u(y) > 0$  a.e. on  $P$ ).

*Remark 8.* The functional  $\mathcal{F}$  on  $\mathcal{C}(P)$  has previously been studied in a series of papers by Zhou-Zhu (see [30, 29]). In particular it was shown in [30] that  $\mathcal{F}$  admits a minimizer  $u$ . But the point of the previous formula is that it identifies

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<sup>2</sup>In fact, if  $u$  is convex on  $P$  and in  $L^1(\partial P)$ , then automatically  $u \in L^1(P)$ .

$\mathcal{F}$  with the Mabuchi functional on the space  $\mathcal{E}^1(X, L)^T$ . As a byproduct this gives a new proof of the existence of a minimizer  $u$  of  $\mathcal{F}$ .

Let now  $\Phi_0$  and  $\Phi_1$  be two given minimizers of  $\mathcal{M}$  in  $\mathcal{E}^1(X, L)^T$  and denote by  $\Phi_t$  the corresponding geodesic in  $\mathcal{E}^1(X, L)^T$  (which corresponds to  $u_t := u_0 + t(u_1 - u_0)$  under the Legendre transform). By the previous lemma the function  $t \mapsto \mathcal{M}(\Phi_t)$  decomposes in two terms, where the first term is affine in  $t$  and the second one is convex. Since  $\mathcal{M}(\Phi_t)$  is constant (and in particular affine) it follows that the second term,

$$t \mapsto - \int_P \log(\det \nabla^2 u_t(y)) dy$$

is also affine. But this forces, using the arithmetic-geometric means inequality, that

$$(2.16) \quad \nabla^2 u_1 = \nabla^2 u_0 \text{ a.e. on } P.$$

As a consequence the previous function in  $t$  is, in fact, constant. Since  $\mathcal{M}(\Phi_t)$  is also constant in  $t$  formula 2.15 forces  $\mathcal{L}(u_t) = \mathcal{L}(u_0)$  for all  $t$ . Setting  $v := u_1 - u_0$  this means that

$$\mathcal{L}(v) = 0.$$

Now, if  $v$  is convex, then it follows from the assumption of uniform relative K-stability that  $v$  is affine and hence  $\Phi_0$  and  $\Phi_1$  coincide modulo the action of  $\mathbb{C}^{*n}$ . All that remains is thus to show that  $v$  is convex. To this end we invoke the assumption that the distributional Hessian of  $\phi_0$  satisfies

$$\nabla^2 \phi_0 \geq C_K I$$

on any given compact subset  $K$  of  $\mathbb{R}^n$ . We claim that this implies that  $u_0 \in C_{loc}^{1,1}(P)$ . Indeed, since  $\Phi_0$  has finite energy it has full Monge-Ampère mass and hence the closure of the sub-gradient image  $(\partial\Phi_0)(\mathbb{R}^n)$  is equal to  $P$ . It follows (just as in the proof of Lemma 3) that

$$\partial^2 u_0 = \nabla^2 u_0 \leq C_K^{-1} I$$

on the closure of  $(\partial\Phi_0)(K)$  in  $P$ . Since  $K$  was an arbitrary compact subset of  $\mathbb{R}^n$  it follows that  $u_0 \in C_{loc}^{1,1}(P)$ . The proof of the theorem is now concluded by invoking the following lemma (see [26, Lemma 3.2]):

**Lemma 9.** *Let  $u_0$  and  $u_1$  be two finite convex functions on an open convex set  $P \subset \mathbb{R}^n$  such that  $u_0 \in C_{loc}^{1,1}(P)$  and the Alexandrov Hessians satisfy 2.16. Then  $u_1 - u_0$  is convex.*

### 2.5. A generalized slope formula for the K-energy

We conclude the paper by observing that a by-product of Lemma 7 is the following generalization of the slope formula for the K-energy in [11] (which concerns the case when  $\Phi_t$  is defined by a bona fide metric on a test configuration) to the present singular setting:

**Proposition 10.** *(Slope formula) Let  $\Phi_t$  be a geodesic ray in  $\mathcal{E}^1(X, L)^T$  such that  $\Phi_0 \in \mathcal{H}(X, L)^T$  and  $\mathcal{M}(\Phi_t) < \infty$  for any  $t \in [0, \infty[$ . Then*

$$\lim_{t \rightarrow \infty} t^{-1} \mathcal{M}(\Phi_t) = \mathcal{L}(v) < \infty$$

where  $u_t = u + tv$  is the curve of convex functions in  $L^1(\partial P)$  corresponding to  $\Phi_t$  under Legendre transformation.

*Proof.* Since  $\mathcal{M}(\Phi_t) < \infty$  Lemma 7 shows that  $u_t = u_0 + tv \in L^1(\partial P)$  for all  $t \geq 0$ , where  $v := u_1 - u_0$ . Moreover, since  $u_t$  is convex for any  $t \geq 0$  it also follows that  $v$  is convex and  $v \in L^1(\partial P)$ . Now, since  $\partial^2 u_0$  is invertible we can, denoting the inverse by  $A(y)$ , write

$$\int \log(\det \nabla^2(u_0 + tv)(y)) dy = C_0 + \int_P \log(\det(1 + tA(y)\nabla^2 v_0(y))) dy,$$

which is finite for any  $t$  (by Lemma 7). Moreover, since  $\nabla^2 v_0(y) \geq 0$  we have, when  $t \geq 1$ , that

$$\begin{aligned} 0 &\leq \int_P \log(\det(I + tA(y)\nabla^2 v_0(y))) dy \\ &\leq \text{Vol}(P)n \log t + \int_P \log(\det(I + A(y)\nabla^2 v_0(y))) dy, \end{aligned}$$

where all terms are finite. Hence, dividing by  $t$  and letting  $t \rightarrow \infty$  concludes the proof of the proposition. □

In the terminology of [10, 11, 4] this formula shows that the slope of the Mabuchi functional along a finite energy geodesic is equal to the Non-Archimedean Mabuchi functional of the corresponding (singular) Non-Archimedean metric. It would be very interesting to extend this slope formula to the non-toric setting. Indeed, this is the key missing ingredient when trying to extend the variational approach to the (uniform) Yau-Tian-Donaldson conjecture in the ‘‘Fano case’’ in [4] to a general polarized manifold  $(X, L)$ , in order to produce a finite energy minimizer of  $\mathcal{M}$ .

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Robert J. Berman  
Mathematical Sciences  
Chalmers University of Technology and the University of Gothenburg  
SE-412 96 Göteborg  
Sweden  
E-mail: [robertb@chalmers.se](mailto:robertb@chalmers.se)