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Positivity of Brown-York mass with quasi-positive boundary data

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We would like to dedicate this paper to Robert Bartnik on the occasion of his sixtieth birthday

Abstract: In this short note, we prove positivity of Brown-York mass under quasi-positive boundary data which generalizes some previous results by the authors. The corresponding rigidity result is obtained.
Keywords: Brown-York mass, quasi-positive, nonnegative scalar metrics.

1. Introduction

Let (Ω^n, g) be a compact manifold with smooth boundary $\partial\Omega$. In this work, we always assume that Ω is connected and orientable. *It is an interesting question to understand the relation between the geometry of Ω in terms of scalar curvature and the intrinsic and extrinsic geometry of $\partial\Omega$ in terms of the mean curvature.* The question is closely related to the notion of quasi-local mass in general relativity. On other hand, given an compact manifold (Σ, γ) without boundary and given a smooth function H on Σ, one basic problem in Riemannian geometry is to study: *under what kind of conditions so that γ is induced by a Riemannian metric g with nonnegative scalar curvature, for example, defined on Ω^n, and H is the mean curvature of Σ in (Ω^n, g) with respect to the outward unit normal vector?* These two problems are closely related and there are no satisfactory answers yet.

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In this kind of study, a result was proved by the authors which implies the positivity of Brown-York quasi-local mass introduced by Brown and York in [2, 3], denoted by $\mathfrak{m}_{BY}(\Sigma;\Omega,g)$. For its definition please see (2.1) below. More specifically, using the quasi-spherical metrics introduced by Bartnik [1], in [13] the authors proved the following:

Theorem 1.1. *Let (Ω^3,g) be a compact, connected Riemannian three manifold with nonnegative scalar curvature with smooth boundary $\partial\Omega$ with positive mean curvature, which consists of spheres with positive Gaussian curvature. Then,*

$$\mathfrak{m}_{BY}(\Sigma_\ell;\Omega,g)\geq 0 \tag{1.1}$$

for each component $\Sigma_\ell\subset\partial\Omega$, $\ell=1,\dots,k$. Moreover, equality holds for some $\ell=1,\dots,k$ if and only if $\partial\Omega$ has only one component and (Ω,g) is isometric to a domain in $\mathbb{R}^3$.

Clearly Theorem 1.1 provides a necessary condition for a boundary data (Σ,γ,H) to be the one induced by a Riemannian metric defined on the ambient manifold and with nonnegative scalar curvature and with positive mean curvature H. Here γ is a metric on Σ with positive Gaussian curvature. The existence of quasi-spherical metric in the proof of the theorem uses the fact that the mean curvature is *positive* at the boundary, see [1, 13, 14]. Otherwise, it is unclear if one can construct such kind of metrics. With these facts in mind, it is natural to ask if Theorem 1.1 is still true in a more general context. In this note, we consider the problem in the situation of quasi-positive boundary data. Here a function defined on a set is said to be *quasi positive* if it is nonnegative and is positive somewhere. The specific results are the following:

Theorem 1.2. *Let (Ω,g) be a compact three manifold with smooth boundary $\partial\Omega$. Let Σ be a component of $\partial\Omega$. Assume the following:*

(a) $\partial\Omega$ has nonnegative mean curvature.
(b) Σ has quasi positive Gaussian curvature.
(c) (Ω,g) has nonnegative scalar curvature.

Then we have:

(i) <u>Positivity:</u> $\mathfrak{m}_{BY}(\Sigma;\Omega,g)\geq 0$.
(ii) <u>Rigidity:</u> Suppose $\mathfrak{m}_{BY}(\Sigma;\Omega,g)=0$, then $\partial\Omega$ is connected, Ω is homeomorphic to the unit ball in $\mathbb{R}^3$ and (Ω,g) is isometric to a domain in $\mathbb{R}^3$.

We first remark that in case $\partial\Omega$ has *quasi positive* Gaussian curvature and has *positive* mean curvature or $\partial\Omega$ has *positive* Gaussian curvature and has *nonnegative* mean curvature, then the positivity part of Theorem 1.2 was proved in [14] and [15] respectively. However, the rigidity part in the first instance was studied in [14] but not solved very satisfactorily. The rigidity part in the second instance was not addressed in [15].

To show Theorem 1.1 we used the method of quasi-spherical metric introduced by Bartnik [1]. However, if the mean curvature is only assumed to be nonnegative, a parabolic equation involved in the quasi-spherical metric may be degenerated. To overcome this difficult, in case $\partial\Omega$ is disconnected, we adopt a careful conformal perturbation on the ambient metric g so that one can use Theorem 1.1 and its generalization for the case that the boundary has positive mean curvature and quasi-positive Gaussian curvature [14]. In case $\partial\Omega = \Sigma$, we use an approximation so that the mean curvature is positive but the scalar curvature may be bounded below by a small negative constant. We then embed the boundary to an hyperbolic space with small negative constant curvature, and use a result in [17] to get nonnegativity of Brown-York mass.

To prove the rigidity part of Theorem 1.2, first we show that if the Brown-York mass is zero, then Ω is homeomorphic to the unit ball in $\mathbb{R}^3$ and g is scalar flat. Then we show that g is Ricci flat. To do this, by suitable approximations, as in [7], one can construct a weak solution of the inverse mean curvature flow (IMCF) in (Ω, g) with a point $p \in \Omega$ as the initial data (see Lemma 3.3 below). We then approximate g by metrics so that Σ has positive Gaussian curvature and positive mean curvature, and so that it also has zero scalar curvature *outside* certain level sets of the IMCF. We can show that the level sets near p have zero Hawking mass. Using the method as in the work of Husiken-Ilmanen [7], one then conclude that g is Ricci flat near p.

It is still an open question whether the Brown-York mass is nonnegative if the mean curvature is *negative* somewhere.

The remaining part of the paper goes as follows: in Section 2, we prove the positivity result of Theorem 1.2; in Section 3, we prove the rigidity result of the theorem.

2. Positivity

Let us first clarify the definition of Brown-York mass. Let (Ω, g) be compact three manifold with smooth boundary $\partial\Omega$. Let Σ be a connected component of $\partial\Omega$ with induced metric γ. Suppose the Gaussian curvature of (Σ, γ) is quasi positive. Then it can be $C^{1,1}$ isometrically embedded in $\mathbb{R}^3$ as a convex surface with mean curvature H_0 which is defined almost everywhere in Σ. Moreover,

$$\int_\Sigma H_0 d\sigma$$

is well-defined and is positive, see [5, 6, 14]. It is well-defined in the sense that it is the same for any $C^{1,1}$ isometric embedding. Here and below mean curvature is computed with respect to the unit outward normal and the mean curvature of the boundary of the unit ball in $\mathbb{R}^3$ is 2. The Brown-York mass [2, 3] of Σ in (Ω, g) is defined as follows:

$$\mathfrak{m}_{BY}(\Sigma; \Omega, g) = \frac{1}{8\pi}\int_\Sigma (H_0 - H) d\sigma. \tag{2.1}$$

Here H is the mean curvature of Σ in (Ω, g). In this section, we want to prove on the positivity of Brown-York mass in Theorem 1.2.

Remark 2.1. We always use the following fact. Suppose the scalar curvature R of (Ω, g) is nonnegative. Let u be the solution of

$$\begin{cases} 8\Delta_g u - Ru = 0 & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

Then u is positive, so that $u^4 g$ has zero scalar curvature and the mean curvature of $\partial\Omega$ with respect to $u^4 g$ is no less than its mean curvature with respect to g.

Lemma 2.1. *Let (Ω, g) and Σ be as in Theorem 1.2. Suppose $\partial\Omega \setminus \Sigma \neq \emptyset$, then*

$$\mathfrak{m}_{BY}(\Sigma; \Omega, g) > 0.$$

Proof. In the following, the area element of $\partial\Omega$ with respect to the metric induced by g will be denoted by $d\sigma_g$, and the mean curvature will be denoted by H_g, etc. Let $\gamma = g|_{T(\Sigma)}$ and let H_0 be the mean curvature when (Σ, γ) is $C^{1,1}$ isometrically embedded in $\mathbb{R}^3$

By Remark 2.1, we may assume that the scalar curvature of (Ω, g) is zero. Moreover, since $\int_\Sigma H_0 d\sigma_g > 0$, we may assume that $H(x_0) > 0$ somewhere in Σ. Let $\Sigma' = \partial\Omega \setminus \Sigma \neq \emptyset$.

First, we want to find a smooth metric g_1 on $\overline{\Omega}$ such that

(i) g_1 has zero scalar curvature;
(ii) the mean curvature H_{g_1} of $\partial\Omega$ is positive; and
(iii) g and g_1 induce the same metric on Σ'.

To construct g_1, let U be a neighborhood of x_0 in Σ such that $H_g \geq c_0 > 0$ in U. Let $0 \leq \phi \leq 1$ be a smooth cutoff function with support in U so that $\phi = 1$ in a neighborhood of x_0. Given $\epsilon > 0$ and let u be the solution of

$$\begin{cases} \Delta_g u = 0 & \text{in } \Omega \\ u = 1 - \epsilon\phi & \text{on } \partial\Omega. \end{cases}$$

For $\epsilon > 0$ small enough, $u > 0$ and $g_1 = u^4 g$ has zero scalar curvature. Moreover,

$$H_{g_1} = \frac{1}{u^2}\left(H_g + \frac{4}{u}\frac{\partial u}{\partial \nu}\right)$$

where ν is the unit outward normal. By the strong maximum principle $H_{g_1} > 0$ outside U. Insider U, $H_g > 0$ and so $H_{g_1} > 0$ provided ϵ is small enough. Fix such an $\epsilon_1 > 0$. Hence $g_1 = u^4 g$ satisfies the conditions mentioned above. In particular, the mean curvature at Σ' with respect to g_1 is bounded below by some positive constant $a > 0$.

Next, for any $\epsilon > 0$ let v be the harmonic function in Ω so that $v = 1$ on Σ and $v = 1 - \epsilon$ on Σ'. Then for ϵ small enough, $v^4 g$ is a smooth metric on $\overline{\Omega}$ such that the mean curvature of Σ with respect to $v^4 g$ is larger than the mean curvature with respect to g. Moreover, the mean curvature of Σ' with respect to $v^4 g$ is bounded in absolute value by $\frac{a}{2}$, provided ϵ is small enough. Choose such an $\epsilon_2 > 0$. Let $g_2 = v^4 g$. Then g_2, g induce the same metric on Σ and $(1 - \epsilon_2)^4 g_1$ and g_2 induce the same metric on Σ'.

Let $M_1 = \Omega$ with metric $(1 - \epsilon_2)^4 g_1$ and $M_2 = \Omega$ with metric g_2. We can glue the M_1 and M_2 along Σ'. Denote the resulting manifold by M_3 and the resulting metric by g_3. Then the boundary of M_3 consists of two copies of Σ denoted by Σ_1 and Σ_2. Moreover the following are true:

(i) g_3 is smooth except along Σ'. Moreover, g_3 is Lipschitz and is smooth on each side of Σ'.

(ii) The scalar curvature of g_3 is zero away from Σ'.

(iii) The mean curvature of Σ_1 and Σ_2 are positive.

(iv) The mean curvature jump at Σ' is positive. Namely, if we choose the unit normal pointing outside Σ' in M_1, then the mean curvature jump is at least $a - \frac{a}{2} = \frac{a}{2} > 0$.

(v) g and g_3 induce the same metric on Σ which corresponds to Σ_2.

(vi) The mean curvature of Σ_2 with respect to g_3 is larger than the mean curvature of Σ with respect to g.

We claim that

$$\int_{\Sigma_2}(H_0 - H_{g_3})d\sigma_{g_3} \geq 0. \tag{2.2}$$

If the claim is true, then by (v) and (vi) above, we conclude the lemma is true.

To prove the claim we further glue two copies of M_3 along Σ_1. Denote the resulting manifold by M_4 and the resulting metric by g_4. The boundary of M_4 consists of two copies of Σ_2, denoted by $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2$. The following are true:

(i) g_4 is smooth except along those parts coming from Σ' or from Σ_1. Moreover, g_4 is Lipschitz and is smooth on each side of these surfaces.
(ii) The scalar curvature of g_4 is zero away from those parts coming from Σ' or from Σ_1.
(iii) The mean curvature of $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ with respect to g_4 are positive. In fact they are equal the mean curvature of Σ_2 with respect to g_3.
(iv) The mean curvature jump at those parts coming Σ' or Σ_1 are positive, because the mean curvature of Σ_1 with respect to g_3 is positive.
(v) $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2$ with respect to the induced metric from g_4 is isometric to $(\Sigma, g|_{T(\Sigma)})$.

By [9, Theorem 3.3], there exists a smooth metric h on M_4 with nonnegative scalar curvature so that h, and g_4 induce the same metric on ∂M_4 and

$$\int_{\partial M_4} H_h d\sigma_h > \int_{\partial M_4} H_{g_4} d\sigma_{g_4} = 2\int_{\Sigma} H_{g_3} d\sigma_{g_3}.$$

Moreover, $H_h > 0$ on ∂M_4. Since each component of ∂M_4 with metric induced by h is isometric to Σ with metric induced by g, it has quasi positive Gaussian curvature. By [14, Theorem 0.2], we conclude that

$$2\int_{\Sigma} H_0 d\sigma \geq \int_{\partial M_4} H_h d\sigma \geq 2\int_{\Sigma} H_{g_3} d\sigma_{g_3}.$$

Hence the claim is true. This completes the proof of the lemma. □

Lemma 2.2. *Let* (Ω, g) *and* Σ *be as in Theorem 1.2. Suppose* $\partial\Omega = \Sigma$*, then*

$$\mathfrak{m}_{BY}(\Sigma; \Omega, g) \geq 0.$$

Proof. By Remark 2.1, we may assume that g is scalar flat. Note that $\partial\Omega = \Sigma$ is a sphere because its Gaussian curvature is quasi positive. Moreover, we may assume that the mean curvature H of Σ is quasi positive. Let $x_0 \in \Sigma$ with

$H(x_0) > 0$. Let U be an neighborhood of x_0 in Σ such that $H_g \geq c_0 > 0$ in U. Let $0 \leq \phi \leq 1$ be a smooth cutoff function with support in U so that $\phi = 1$ in a neighborhood of x_0. Given $\epsilon > 0$ and let $u = u(\epsilon)$ be the solution of

$$\begin{cases} \Delta_g u &= 0 \quad \text{in } \Omega \\ u &= 1 - \epsilon\phi \quad \text{on } \partial\Omega. \end{cases}$$

For $\epsilon > 0$ small enough, $g(\epsilon) = u^4 g$ has zero scalar curvature so that $\partial\Omega$ has positive mean curvature. Let $\gamma(\epsilon)$ be the metric on Σ induced by $g(\epsilon)$ and let $K(\epsilon)$ be the Gaussian curvature of Σ with respect to $\gamma(\epsilon)$. Then

$$K(\epsilon) > -\kappa^2(\epsilon) \tag{2.3}$$

where $\kappa(\epsilon) > 0$, $\kappa(\epsilon) \to 0$ as $\epsilon \to 0$. By [12], we can isometrically embed $(\Sigma, \gamma(\epsilon))$ in $\mathbb{H}_{-\kappa^2(\epsilon)}$ as a strictly convex surface in the ball model defined in the ball

$$\{|x| < \kappa^{-2}(\epsilon)\}.$$

Moreover, we may assume that the origin is inside the embedded surface. Let $H(\epsilon)$ be the mean curvature of Σ with respect to $g(\epsilon)$ and let $H_{\kappa(\epsilon)}$ be the mean curvature when $(\Sigma, \gamma(\epsilon))$ is isometrically embedded in the hyperbolic space $\mathbb{H}_{-\kappa^2(\epsilon)}$ with constant curvature $-\kappa(\epsilon)$. By [17], we have

$$\int_\Sigma (H_{\kappa(\epsilon)} - H(\epsilon)) \cosh(\kappa(\epsilon) r) d\sigma_{g(\epsilon)} \geq 0 \tag{2.4}$$

where r is the distance from the origin in $\mathbb{H}_\kappa(\epsilon)$.

Observe that we can find $\epsilon_i \to 0$ such that $g(\epsilon_i) \to g$ in C^∞ norm on $\overline{\Omega}$. Hence the intrinsic diameter of $(\Sigma, \gamma(\epsilon_i))$ is bounded by a constant independent of i, we conclude that r is bounded by a constant independent of i. By [8, p.7152-7154], one can choose $\epsilon_i \to 0$ such that:

- $H_{\kappa(\epsilon_i)}$ are uniformly bounded from above. (Note that $H_{\kappa(\epsilon_i)} > 0$).
- If $\mathbf{X}_i = (x^1, x^2, x^3)$ is the isometric embedding of $(\Sigma, \gamma(\epsilon_i))$, then the C^2 norm with respect to the fixed metric σ are uniformly bounded.

Together with (2.4), we conclude that

$$\liminf_{i \to \infty} \int_\Sigma (H_{\kappa(\epsilon_i)} - H_g) d\sigma \geq 0.$$

Moreover, $\mathbf{X}_i$ converge to a $C^{1,1}$ embedding of (Σ, σ) in $\mathbb{R}^3$ as a convex surface. As in [14], one can conclude that

$$\lim_{i\to\infty} \int_\Sigma H_{\kappa(\epsilon_i)} d\sigma = \int_\Sigma H_0 d\sigma,$$

where H_0 is the mean curvature of Σ when (Σ, γ) is isometrically $C^{1,1}$ embedded in $\mathbb{R}^3$. Here $\gamma = g|_{T(\Sigma)}$. From this the lemma follows. □

Proof of Theorem 1.2 (i) Positivity. Let (Ω, g), Σ be as in Theorem 1.2. Then by Lemmas 2.1 and 2.2, we have

$$\mathfrak{m}_{BY}(\Sigma; \Omega, g) \geq 0.$$

□

3. Rigidity

In the section, we will prove the rigidity part in Theorem 1.2. First we have the following:

Lemma 3.1. *Let $(\Omega, g), \Sigma$ be as in Theorem 1.2 so that $\partial\Omega = \Sigma$. Suppose Ω is not homeomorphic to the unit ball in $\mathbb{R}^3$, then*

$$\mathfrak{m}_{BY}(\Sigma; \Omega, g) > 0.$$

Proof. Since the Gaussian curvature of Σ is quasi positive, Σ is a topological sphere. If Ω is a handle body, then it is homeomorphic to the unit ball. Suppose this is not the case, then Ω is not a handle body. By [10, Theorem 1' and Proposition 1] there is an embedded minimal surface S which is either a sphere or a minimal projective space inside Ω.

Case 1: Suppose S is a sphere. Since S is orientable, there is a smooth unit normal vector field on S and there is an embedding $F : S \times (-1, 1) \to \Omega$ so that $F(\cdot, 0) = S$ and the image of F is a tubular neighborhood N of S in Ω. Then $N \setminus S$ is a manifold so that part of its boundary are two copies of S with two components. Hence $\Omega \setminus S$ is a manifold with boundary consisting of $\partial\Omega$ and two copies of S. Let $\widetilde{\Omega}$ be the connected component of $\Omega \setminus S$ containing $\partial\Omega = \Sigma$. Then $(\widetilde{\Omega}, g)$ has nonnegative scalar curvature so that $\partial\widetilde{\Omega}$ is disconnected, and $\mathfrak{m}_{BY}(\Sigma, \Omega, g) = \mathfrak{m}_{BY}(\Sigma, \widetilde{\Omega}, g)$, which is positive by Lemma 2.1.

Case 2: Suppose S is a projective space. $f : \mathbb{RP}^2 \to \Omega$ is an embedding with $S = f(\mathbb{RP}^2)$. We want to construct a double cover $p : \hat{\Omega} \to \Omega$ so that $p^{-1}(f(\mathbb{RP}^2)) \cong \mathbb{S}^2$.

Let V be the normal bundle of the embedding f. Note that $\mathbb{RP}^2$ has only two non-isomorphic real line bundles, namely the tautological line bundle and the trivial one. Since Ω is orientable, V is isomorphic the tautological line bundle $((\mathbb{S}^2 \times \mathbb{R})/\sim) \to (\mathbb{S}^2/\sim) \cong \mathbb{RP}^2$ with $(x,k) \sim (-x,-k)$ on $\mathbb{S}^2 \times \mathbb{R}$.

By the tubular neighborhood theorem, there exists an open embedding $G : ((\mathbb{S}^2 \times \mathbb{R})/\sim) \cong V \to \Omega$ whose restriction on the zero section is equal to f. Let $\Omega' = G((\mathbb{S}^2 \times [-1,1])/\sim)$ and $\Omega'' = \Omega \setminus G((\mathbb{S}^2 \times (-1,1))/\sim)$. Then $\Omega = \Omega' \cup \Omega''$ with $\Omega' \cap \Omega'' = \partial\Omega' \cong \mathbb{S}^2$.

Let Ω_+, Ω_- be two identical copies of Ω''. Define $\phi : \mathbb{S}^2 \times \{-1,1\} \to \Omega_+ \sqcup \Omega_-$ by $\phi(x,1) = g([(x,1)]) \in \Omega_+$ and $\phi(x,-1) = g([(x,-1)]) \in \Omega_-$ for $x \in \mathbb{S}^2$. Let $\hat{\Omega} = \mathbb{S}^2 \times [-1,1] \cup_\phi (\Omega_+ \sqcup \Omega_-)$. Then the obvious map $p : \hat{\Omega} \to \Omega$ has the desired properties. By the construction, we see that $(\hat{\Omega}, \hat{g})$ has nonnegative scalar curvature and $\partial\hat{\Omega}$ two components, each of them has quasi-positive mean curvature with respect to outward unit norm vector and quasi-positive Gauss curvature. In fact, near each component, $(\hat{\Omega}, \hat{g})$ is isometric to neighborhood of Σ in (Ω, g). On the other hand, $2\mathfrak{m}_{BY}(\Sigma, \Omega, g) = \mathfrak{m}_{BY}(\partial\hat{\Omega}, \hat{\Omega}, g)$, which is positive by Lemma 2.1. This completes the proof of the lemma. ☐

Let (Ω, g) and Σ be as in Theorem 1.2. Suppose $\mathfrak{m}_{BY}(\Sigma; \Omega, g) = 0$. By Lemmas 2.1 and 3.1, we conclude that $\partial\Omega = \Sigma$ and Ω is homeomorphic to the unit ball. By Remark 2.1, we conclude that g is scalar flat. Moreover, since Σ has quasi positive Gaussian curvature, we conclude that Σ has quasi positive mean curvature, otherwise $\mathfrak{m}_{BY}(\Sigma; \Omega, g) > 0$. In the rest of this section, we always assume the above facts. In remains to prove that g is Ricci flat.

We need the following two lemmas.

Lemma 3.2. *Let (Ω, g) and Σ be as above. For any p in Ω and for any $\rho > 0$ small enough, there is a sequence of smooth metrics g_i on $\overline{\Omega}$ with the following properties:*

- *(i) $g_i \to g$ in C^∞ norm in $\overline{\Omega}$ as $i \to \infty$.*
- *(ii) Σ has positive mean curvature H_i with respect to g_i.*
- *(iii) Let γ_i be the induced metric of g_i on Σ. Then the Gaussian curvature of (Σ, γ_i) has positive Gaussian curvature.*
- *(iv) The scalar curvature of g_i is zero outside $B(p, 2\rho)$.*
- *(v) The mean curvature of $\partial B_g(p,s)$ with respect to g_i is positive for all $s < 2\rho$ for all i.*
- *(vi) $\mathfrak{m}_{BY}(\Sigma; \Omega, g_i) \to 0$ as $i \to \infty$.*

Proof. Let $\rho > 0$ be small enough so that $\partial B_g(p, s)$ is diffeomorphic to the sphere and so that its mean curvature is larger than $1/s$ for all $0 < s < 2\rho$. Fix a smooth cutoff function $\phi \geq 0$ so that $\phi = 1$ in $B(p, \rho)$ and $\phi = 0$ outside $B(p, 2\rho)$. Let v be the solution of $\Delta_g v = \epsilon\phi$ in Ω and $v = 1$ on Σ. Then for $\epsilon > 0$ small enough, $v > 0$. Let $g_\epsilon = v^4 g$. For ϵ small enough, g_ϵ satisfies:

- $g_\epsilon \to g$ in C^∞ norm in $\overline{\Omega}$ as $\epsilon \to 0$.
- The scalar curvature of g_ϵ is zero outside $B(p, 2\rho)$.
- The mean curvature of Σ with respect to g_ϵ is positive. This follows from the strong maximum principle that $\frac{\partial v}{\partial \nu} > 0$ where ν is the unit outward normal of Σ with respect to g.

Since $v = 1$ on Σ, the metrics induced by g, g_ϵ are equal, and will be denoted by γ. In particular, the Gaussian curvature of Σ does not change. If the Gaussian curvature of (Σ, γ) is positive, then $g_i = g_{\epsilon_i}$ with $\epsilon_i \to 0$ are the required metrics. Otherwise, we can find a smooth function η on Σ such that $\eta \leq 0$, $\Delta_\gamma \eta = -1$ in an open set containing $\{K = 0\}$. For fixed $\epsilon > 0$, for $\tau > 0$, and let w be the solution of $\Delta_{g_\epsilon} w = 0$ in Ω so that $w = \exp(\frac{1}{2}\tau\eta)$. Let $h_\tau = w^4 g_\epsilon$. Then

- $h_\tau \to g_\epsilon$ in C^∞ norm in $\overline{\Omega}$ as $\tau \to 0$.
- The scalar curvature of h_τ is zero outside $B(p, 2\rho)$.
- The mean curvature of Σ is positive, provided τ is small enough.
- The Gaussian curvature of Σ with respect to the metric induced by h_τ is positive provided τ is small enough.

From these, it is easy to see the lemma is true. □

The following lemma is basically from [7].

Lemma 3.3. *Let (Ω, g), Σ be as above. For any $p \in \Omega$, there is a weak solution for the inverse mean curvature flow in (Ω, g) with p as the initial data.*

Proof. Let U be a small neighborhood of $\partial\Omega$, then extend $\Omega \cup U$ to be Euclidean near infinity, the resulting metric is denoted by $\hat{g}$.

Let us consider the inverse mean curvature flow (IMCF) in $(M, \hat{g})$ with $\partial B_r(p)$ as the initial data where $r > 0$ is small enough. By Theorem 3.1 in [7], there is a weak solution u_r to this IMCF with $u_r|_{\partial B_r(p)} = 0$ and

$$|\nabla u_r|(x) \leq \sup_{\partial B_r(p) \cap B_\rho(x)} H_+ + \frac{C}{\rho},$$

for any $0 < \rho \le \sigma(x)$, here C is a universal constant independent on ρ and r, $\sigma(x)$ is defined in Definition 3.3 in [7], i.e. for any $x \in \Omega$, let $\tau(x) \in (0, \infty]$ be the supremum of radii r such that $B_r(x) \subset \Omega$, and

$$Rc \ge -\frac{1}{1000r^2} \text{ in } B_r(x),$$

and there is a C^2 function p on $B_r(x)$ such that $p(x) = 0$, $p \ge d^2(, x)$, and $|\nabla p| \le 3d(, x)$, $\nabla^2 p \le 3g$ on $B_r(x)$, define $\sigma(x) = \min\{\tau(x), d(x, \partial\Omega)\}$. Let $\Omega' \subset\subset \Omega$ with $dist(\partial\Omega', \partial\Omega)$ being any fixed small number and $p \in \Omega'$. Without loss of generality, it suffices to consider the case that $x \in \Omega'$, so, we may assume $\sigma(x) \ge \sigma_0$ for any $x \in \Omega'$, here σ_0 is a fixed number that depends only on $dist(\partial\Omega', \partial\Omega)$ and (Ω, g).

Let us choose r small enough so that $\sup_{\partial B_r(p)} H_+ \le \frac{3}{r}$. Now, we claim that for any $x \in \Omega'$

$$|\nabla u_r|(x) \le \frac{C}{d(x,p)}, \tag{3.1}$$

here C is a universal constant independent on r, $d(x, p)$ is the distance function to p with respect to the metric g.

In fact, if $d(x, p) \le 4r$, then we take $\rho = \frac{r}{2}$, here we assume $r \le \frac{\sigma_0}{2}$, we get (3.1); if $d(x, p) > 4r$, let $\rho = \min\{\frac{1}{2} dist(x, p), \frac{\sigma_0}{2}\}$, together with the fact $dist(x, p) \le \Lambda\sigma_0$, where Λ is a universal constant, we still get (3.1).

On the other hand, together with Theorem 2.1 in [7] and the remarks following it, we know that by taking a subsequence of $\{u_r\}$, denoted by $\{u_{r_i}\}$, there is a constant C_i so that $\{u_{r_i} - C_i\}$ converges to the weak solution of IMCF $-\infty < u$ in (Ω', g) with p as the initial data. Note that the mean curvature of $\partial B_r(p)$ is positive for all $r \le \delta$, we see that the level set of u in $B_\delta(p) \subset\subset \Omega'$ cannot jump, and

$$|\nabla u|(x) \le \frac{C}{d(x,p)},$$

and $-\infty < u \le t_0$, here t_0 is a universal constant. □

Let us first recall the definition of *minimizing hull* in Ω. A subset E of Ω with locally finite perimeter is said to be a minimizing hull in Ω if $|\partial^* E \cap K| \le |\partial^* F \cap K|$ for any set $F \subset \Omega$ with locally finite perimeter such that $F \supset E$ and $F \setminus E \Subset \Omega$ and for any compact set K with $F \setminus E \subset K \subset \Omega$. Here $\partial^* E, \partial^* F$ are the reduced boundaries of E and F respectively.

By the proof in [16, Theorem 2.5], we see that for t small enough, the slice $N_t = \partial\{u < t\}$ of the weak IMCF in Lemma 3.3 is the boundary of a *minimizing hull* in (Ω, g) which is $C^{1,\alpha}$ smooth and $\int_{N_t} |A|^2 d\sigma < \infty$, and $\mathfrak{m}_H(N_t) \geq 0$.

We are ready to prove the rigidity part of Theorem 1.2.

Proof of Theorem 1.2 (ii) Rigidity. Let $p \in \Omega$. Suppose g is not flat near p. Choose $r > 0$ be small enough with $B(p, 2r) \Subset \Omega$, so that $\partial B(p, s)$ is a sphere with mean curvature at least $1/s$ for all $s < 2r$. Then by Lemma 3.3 and [7], one can find a solution to the IMF given by a locally Lipschitz function u, so that for some a, the following are true: (i) $E_t = \{u < t\}$ is precompact in $B(x, r)$ for $t < a$; (ii) ∂E_t is connected; (iii) E_t is a minimizing hull in (Ω, g); (iv) $\mathfrak{m}_H(\partial E_t, g) > 0$, for $t < a$. Here and below, $\mathfrak{m}_H(\partial U, g)$ is the Hawking mass of the boundary of U with respect to g.

Fix $t_0 < a$ so that $\mathfrak{m}_H(\partial E_{t_0}, g) \geq b$ for some $b > 0$. In the following we denote E_{t_0} by E. For any $\theta > 0$ small enough, we can find $E \subset F \Subset B(x, r)$ such that

$$|\partial E|_g \leq |\partial F|_g \leq |\partial E|_g + \theta; \ \mathfrak{m}_H(\partial F) \geq \mathfrak{m}_H(\partial E) - \theta > 0. \tag{3.2}$$

Moreover ∂F is smooth. Note that F depends on θ.

Since $p \in E_{t_0}$ which is open, we can find $r > \rho > 0$ such that $B(p, 2\rho) \Subset E$.

Next, we want to approximate g. By the Lemma 3.2, for any $\epsilon > 0$ small enough, we can find a smooth metric g_ϵ on $\overline{\Omega}$ so that (i) $||g - g_\epsilon||_{C^4} \leq \epsilon$; (ii) Σ has positive mean curvature H_ϵ with respect to g_ϵ; (iii) the Gaussian curvature of $(\Sigma, g_\epsilon|_{T(\Sigma)})$ has positive Gaussian curvature; (iv) the scalar curvature of g_ϵ is zero outside $B(p, 2\rho)$; (v) the mean curvature of $\partial B(p, s)$ with respect to g_ϵ is positive for all $s < 2r$; (vi) $\mathfrak{m}_{BY}(\Sigma, \Omega, g_\epsilon) \leq \epsilon$; (vii) $|\partial F|_{g_\epsilon} \leq |\partial E|_g + \theta + \epsilon$, $\mathfrak{m}_H(\partial F, g_\epsilon) \geq \mathfrak{m}_H(\partial E, g)) - \theta - \epsilon > 0$.

By (ii), (iii), we can glue Ω to the exterior of the a convex set in $\mathbb{R}^3$ and solve the quasi-spherical metric as in [1, 13] so that the scalar curvature outside the convex set is zero and is asymptotically flat. Denote the manifold by M. We still denote this metric as g_ϵ. Note that g_ϵ has zero scalar curvature outside $B(x, 2\rho)$. However, g_ϵ may have negative scalar curvature inside $B(p, 2\rho)$. By the monotonicity in quasi-spherical metric [13], using the Lemma 3.2 (vi) we may choose g_ϵ so that

$$\mathfrak{m}_{ADM}(g_\epsilon) \leq \epsilon.$$

Fix such an ϵ. Using the method of Miao [11], for $\tau > 0$ small enough, we can find metrics h_τ so that $h_\tau = g_\epsilon$ outside $\{x \in M | d_{g_\epsilon}(x, \Sigma) < \tau\}$ and the scalar curvature inside $\{x \in M | d_{g_\epsilon}(x,\Sigma) < \tau\}$ is uniformly bounded. Let R_τ be the scalar curvature of g_ϵ. One can find a positive solution of

$$\widetilde{R}_\tau u - 8\Delta_{g_\epsilon} u = 0$$

with $u \to 1$ near infinity. Here $\widetilde{R}_\tau = R_\tau$ in $\{x \in M | d_{g_\epsilon}(x,\Sigma) < \tau\}$ and $\widetilde{R}_\tau = 0$ outside this set. Note that $\widetilde{R}_\tau$ is smooth. Hence one can approximate g_ϵ by a smooth metrics $h_\tau = u^4 g_\epsilon$ on the manifold so that, h_τ has zero scalar curvature outside $B(p, 2\rho)$ and

$$\mathfrak{m}_{ADM}(h_\tau) \le 2\epsilon.$$

Moreover, $h_\tau \to g_\epsilon$ uniformly in M, $h_\tau \to g_\epsilon$ in C^∞ norm in any compact set away from Σ.

Note that the mean curvature of Σ with respect to g_ϵ is positive and $\mathfrak{m}_H(\partial F, g_\epsilon) > 0$, one can find F_ϵ which is the strictly minimizing hull of F with respect to g_ϵ inside Ω, see [7]. F_ϵ exists because the mean curvature of $\Sigma = \partial\Omega$ is positive with respect to g_ϵ. Then $F_\epsilon \Subset \Omega$ and is connected because M is homeomorphic to $\mathbb{R}^3$. Using the fact that the scalar curvature of g_ϵ is zero outside ∂F, one can proceed as in the proof [14, Theorem 3.1], to obtain

$$2\epsilon \ge \mathfrak{m}_{ADM}(g_\epsilon) \ge \mathfrak{m}_H(\partial F_\epsilon, g_\epsilon).$$

On the other hand, the mean curvature of ∂F_ϵ is zero on $\partial F_\epsilon \setminus \partial F$ is equal to the mean curvature of ∂F on $\partial F_\epsilon \cap \partial F$, see [7, p.372]. Hence

$$\begin{aligned}
\mathfrak{m}_H(\partial F_\epsilon, g_\epsilon) =& \sqrt{\frac{|\partial F_\epsilon|_{g_\epsilon}}{16\pi}} \left(1 - \frac{1}{16\pi}\int_{\partial F_\epsilon} H^2 d\sigma_{g_\epsilon}\right) \\
\ge& \sqrt{\frac{|\partial F_\epsilon|_{g_\epsilon}}{16\pi}} \left(1 - \frac{1}{16\pi}\int_{\partial F} H^2 d\sigma_{g_\epsilon}\right) \\
=& \sqrt{\frac{|\partial F_\epsilon|_{g_\epsilon}}{|\partial F|_{g_\epsilon}}} \mathfrak{m}_H(\partial F, g_\epsilon) \\
\ge& \sqrt{\frac{|\partial F_\epsilon|_{g_\epsilon}}{|\partial F|_{g_\epsilon}}} (\mathfrak{m}_H(\partial E, g) - \theta - \epsilon).
\end{aligned}$$

Now

$$\begin{aligned}
|\partial F|_{g_\epsilon} \le& (|\partial E|_g + \theta + \epsilon) \\
\le& (|\partial F_\epsilon|_g + \theta + \epsilon) \\
\le& (1+\epsilon)\,(|\partial F_\epsilon|_{g_\epsilon} + \theta + \epsilon)
\end{aligned}$$

and

$$|\partial F|_{g_\epsilon} \geq (1-\epsilon)|\partial F|_g \geq (1-\epsilon)|\partial E|_g$$

here we may assume that $(1+\epsilon)^{-1}g \leq g_\epsilon \leq (1+\epsilon)g$. Hence

$$\begin{aligned}\frac{|\partial F_\epsilon|_{g_\epsilon}}{|\partial F|_{g_\epsilon}} \geq & \frac{1}{1+\epsilon} - (\theta+\epsilon)\cdot\frac{1}{|\partial F_\epsilon|_{g_\epsilon}} \\ \geq & \frac{1}{1+\epsilon} - (\theta+\epsilon)\cdot\frac{1}{(1-\epsilon)|\partial E|_g}\end{aligned}$$

Since $\mathfrak{m}_H(\partial E, g) - \theta - \epsilon > 0$ provided θ, ϵ are small enough, we have

$$2\epsilon \geq \left(\frac{1}{1+\epsilon} - (\theta+\epsilon)\cdot\frac{1}{(1-\epsilon)|\partial E|_g}\right)^{\frac{1}{2}}(\mathfrak{m}_H(\partial E, g) - \theta - \epsilon).$$

Let $\epsilon \to 0$ and then let $\theta \to 0$, we have

$$0 \geq \mathfrak{m}_H(\partial E, g) > 0.$$

This is a contradiction. □

Remark 3.1. It is not difficult to see that by the arguments in the above proof of rigidity, we may also get $\mathfrak{m}_{BY}(\Sigma; \Omega, g) \geq 0$ in case Ω is homeomorphic to a ball.

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