

Recent progress and problems on the Bartnik quasi-local mass*

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*This work is dedicated with pleasure to Robert Bartnik in honor
of his 60th birthday*

Abstract: This paper surveys recent progress on issues related to the Bartnik quasi-local mass m_B . In addition we formulate a number of new problems and conjectures regarding foundational properties of the mass m_B .

1. Introduction

In this article, we survey recent progress on the Bartnik quasi-local mass m_B , discuss several problems with the various definitions and properties of m_B and present new conjectures worthy of further study.

We begin with a discussion of quasi-local mass in the time-symmetric or Riemannian setting of (space-like) 3-manifolds, which has been studied in much more detail. The constraint equations for (g, K) then take the form $K = 0$ and $R_g \geq 0$ (dominant energy condition). In §5, we discuss the generalization to the space-time setting. (Of course any suitable definition of quasi-local mass should extend to the 4d space-time setting to be physically meaningful).

Let Ω be a compact, connected 3-manifold with connected boundary; typically Ω will be a closed 3-ball \bar{B} with $\partial\Omega = S^2$. Let g_Ω be a smooth metric of non-negative scalar curvature $R_{g_\Omega} \geq 0$ on Ω . Let $\gamma = g_\Omega|_{\partial\Omega}$ be the induced metric on $\partial\Omega$ and let A be the second fundamental form of $\partial\Omega$ in (Ω, g_Ω) , with respect to the outward unit normal, so that $A = \gamma$ for $S^2(1) \subset \mathbb{R}^3$.

It is generally recognized that the basic properties desired of a quasi-local mass $m_{QL}(\Omega)$ of the domain Ω are the following:

- QL0: (Existence) $m_{QL}(\Omega)$ is well-defined on the class of domains Ω above.

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- QL1: (Positivity) $m_{QL}(\Omega) \geq 0$ with equality if and only if Ω is flat, i.e. there is an isometric embedding $F : (\Omega, g_\Omega) \rightarrow (\mathbb{R}^3, g_{Euc})$.
- QL2: (Monotonicity) If $\Omega_1 \subset \Omega_2$, then $m(\Omega_1) \leq m(\Omega_2)$.
- QL3: (Asymptotic behavior) If Ω_i is an exhaustion of a complete asymptotically flat manifold (M, g) with $R_g \geq 0$, then $\lim m_{QL}(\Omega_i) = m_{ADM}$.
- QL4: (Quasi-local) $m(\Omega)$ depends only on the first order geometry of $\partial\Omega$, i.e. (γ, A) .

One could of course add further properties, such as the mass of a domain in the Schwarzschild metric surrounding the horizon be equal to the ADM mass m of the Schwarzschild metric, small sphere limit behavior, etc.

Unfortunately, to the author's knowledge, there is no known definition of $m_{QL}(\Omega)$ which satisfies all of QL0-QL4, and perhaps there is no such quantity; we refer to [29] for a detailed discussion.

The Bartnik mass is a very natural and direct localization of the global ADM mass meant to capture the properties QL0-QL4. The original definition of Bartnik is the following, cf. [9], [10]. Let $\widehat{\mathcal{P}}^0$ denote the collection of smooth complete Riemannian 3-manifolds $(\widehat{M}, \widehat{g})$ such that \widehat{g} is asymptotically flat (AF), of non-negative and integrable scalar curvature $R_{\widehat{g}} \geq 0$, and $(\widehat{M}, \widehat{g})$ has no horizons, i.e. no (stable) minimal surfaces. The Bartnik mass of a smooth bounded domain is then defined by

$$(1.1) \quad m_B(\Omega) = \inf\{m_{ADM}(\widehat{M}, \widehat{g}) : (\Omega, g_\Omega) \subset (\widehat{M}, \widehat{g}) \in \widehat{\mathcal{P}}^0\},$$

where the infimum is taken over all $(\widehat{M}, \widehat{g}) \in \widehat{\mathcal{P}}^0$ for which (Ω, g_Ω) is isometrically embedded in $(\widehat{M}, \widehat{g})$.

Many of the properties QL0-QL4 hold for m_B . Namely, Huisken-Ilmanen [19] proved that $m_B \geq 0$ with equality if and only if Ω is locally flat, i.e. there is an isometric immersion of Ω (or its universal cover) into \mathbb{R}^3 . (This cannot be improved to an embedding $\Omega \subset \mathbb{R}^3$ by the work in [7]). The property QL2 follows immediately from the definition and QL3 also follows from the work in [19].

The main drawback of the definition (1.1) is that, in contrast to all other definitions of m_{QL} , it does not satisfy the quasi-local criterion QL4. This is because the no-horizon condition is global and depends in particular on the bounding domain Ω , not just the geometry at $\partial\Omega$. Note that QL0 is also not fully satisfied, since Ω is not allowed to have any minimal surfaces.

Bartnik observed in [9], [10] that an AF extension (M, g) of $\partial\Omega$ with $\widehat{M} = \Omega \cup M$ which realizes the infimum in (1.1) will in general only be Lipschitz along the "seam" $\partial\Omega = \partial M$. By a simple and elegant argument

using the 2nd variational formula for area, he showed that a minimizer should obey the boundary conditions

$$(1.2) \quad \gamma_{\partial\Omega} = \gamma_{\partial M}, \quad H_{\partial\Omega} = H_{\partial M},$$

where $H_{\partial M}$ is the mean curvature of ∂M with respect to the unit normal pointing into M . The relation (1.2) implies that the scalar curvature is defined as a non-negative distribution across the seam. (Actually this holds for $H_{\partial\Omega} \geq H_{\partial M}$ and this condition is sometimes used instead of (1.2)). Standard minimal surface arguments show that if $H_{\partial M} \leq 0$ then any extension (M, g) has a horizon, so that it is common practice to assume

$$H = H_{\partial M} > 0.$$

The quasi-local data (γ, H) on ∂M are now called Bartnik boundary data. Let $\mathcal{P}_{(\gamma, H)}$ be the space of AF Riemannian manifolds (M, g) with non-negative, integrable scalar curvature, inducing the data (γ, H) at ∂M . Let $\mathcal{P}_{(\gamma, H)}^0 \subset \mathcal{P}_{(\gamma, H)}$ be the subset of such (M, g) with no horizons, i.e. immersed minimal 2-spheres, surrounding ∂M . A natural modification of the definition (1.1) which restores QL4 is then given by

$$(1.3) \quad m_B(\Omega) := m_B(\gamma, H) = \inf\{m_{ADM}(M, g) : (M, g) \in \mathcal{P}_{(\gamma, H)}^0\}.$$

The definition (1.3) now satisfies QL4 and in addition QL0 holds for a larger class of domains Ω . However, it remains an interesting open problem whether QL0 holds in general; given boundary data (γ, H) (perhaps arising as boundary data of (Ω, g_Ω) with $R_{g_\Omega} \geq 0$), is it always true that

$$(1.4) \quad \mathcal{P}_{(\gamma, H)} \neq \emptyset?$$

This open problem is raised in [10], cf. also the recent results and survey given in [7]. The same question holds with respect to $\mathcal{P}_{(\gamma, H)}^0$, but here much less is known and the problem appears much harder. In addition, it is not clear or known whether the monotonicity property QL2 holds for the mass (1.3), again because of the no-horizon condition.

There have been a number of further modifications or variations of the definition of the Bartnik mass m_B ; we refer to the recent paper by Jauregui [21] and further references therein for a careful and detailed discussion. Further discussion of the no-horizon condition is given in §2 and §3.

It is to be expected that an extension (M, g) realizing the infimum in (1.3) or (1.1) satisfies strong conditions. In [9], [10], Bartnik presented a

natural physical argument that extensions (M, g) realizing the infimum in (1.1) (or (1.3)) should be solutions of the static vacuum Einstein equations. Namely, any dynamical gravitational field carries energy (contributing to the Bondi mass at null infinity) and so an extension of minimal mass should have no gravitational dynamics, i.e. be time-independent. For similar reasons, a minimal-mass extension should have no mass coming from matter sources, and so be vacuum. A time-independent vacuum solution which is time-symmetric ($K = 0$) is static vacuum. Bartnik then conjectured that such heuristic physical reasoning could be made mathematically rigorous.

The static vacuum Einstein equations are the equations for a pair (g, u) on M where u is a potential function (forming the lapse function of the space-time $\mathcal{M} = I \times M$) given by

$$(1.5) \quad u\text{Ric} = D^2u, \quad \Delta u = 0.$$

These equations are equivalent to the statement that the space-time metric $g_{\mathcal{M}} = -u^2 dt^2 + g$ is Ricci-flat, i.e. a vacuum solution of the Einstein equations. One usually adds the physical requirements that $u > 0$ and $u \rightarrow 1$ at infinity. A natural and physically well-motivated approach to proving the conjecture above was developed in [11], based on the Regge-Teitelboim Hamiltonian, cf. also [12]. This is discussed further in §2. A full proof along these lines is given in [7]. A completely different proof of part of this conjecture was given by Corvino in [17] and more recently in [18].

There is a useful analogy of the Bartnik mass with the gravitational capacity of a body $\Omega \subset \mathbb{R}^3$ in Newtonian gravity, or a charged body in electrostatics. Here one minimizes the Dirichlet energy,

$$(1.6) \quad E(v) = \int_M |dv|^2,$$

over $M = \mathbb{R}^3 \setminus \Omega$ with boundary conditions $v = 1$ at $\partial\Omega$ and $v \rightarrow 0$ at infinity. (One could also set $v' = 1 - v$ with $v' \rightarrow 1$ at infinity). Classical results show that the infimum of (1.6) is realized by a unique harmonic function u , $\Delta u = 0$ on M ; u represents the gravitational potential of the single layer $\partial\Omega$. The capacity of Ω , equal to the total mass or charge up to a constant, is then given by

$$(1.7) \quad E(u) = \inf E(v) = - \int_{\partial M} N(u),$$

where N is the unit normal into M at ∂M .

In [9], [10], Bartnik made the bold conjecture that a minimizer of (1.1) (or (1.3)) also always exists and is unique; this is now called the minimization conjecture;

Bartnik Minimization Conjecture. Any given boundary data (γ, H) are realized by a unique AF minimizer (M, g, u) , $u > 0$, which is a solution of the static vacuum Einstein equations (1.5) with boundary data (γ, H) .

If true, this suggests the following conjecture [9], [10], which thus serves as a test of the minimization conjecture but is also of independent interest in geometric PDE theory.

Bartnik Static Extension Conjecture. Given boundary data (γ, H) on ∂M , there exists a unique AF solution (M, g, u) , $u > 0$, of the static vacuum Einstein equations (1.5) which induces the data (γ, H) at ∂M .

Unfortunately, as discussed in later sections, the minimization conjecture has recently been proved to be false in general and both conjectures are very likely to be false in various regimes of boundary data (γ, H) . It thus becomes of basic interest to determine the realm of (γ, H) for which these conjectures might be valid.

In the following sections of the paper, we present a discussion of various aspects of the Bartnik mass m_B and some of its further modifications.

2. On the no-horizon condition

The no-horizon condition(s), i.e. non-existence of minimal surfaces or minimal 2-spheres discussed above, are a global requirement on the class of extensions over which one minimizes. Thus $m_B(\Omega)$ or $m_B(\gamma, H)$ depends on a large-scale, global property of the boundary data (γ, H) . The mass m_B is undefined if $\mathcal{P}_{(\gamma, H)}^0 = \emptyset$, but it is not clear why. Moreover, it is very difficult to determine if a given AF extension with $R_g \geq 0$ has a minimal surface surrounding ∂M or no such surfaces. The same remarks hold for the global configuration $(\widehat{M}, \widehat{g})$. This places a severe restriction on developing the foundations for a theory underlying the Bartnik mass or on understanding its behavior as a function of the boundary data (γ, H) .

The physical reasoning for a no-horizon condition is that mass behind a horizon is hidden (causally) from infinity and so is not captured by the ADM mass. One can always add enough matter sources (positive scalar curvature) to produce a horizon, so that the original body and such sources are hidden from infinity, where only an arbitrarily small mass may be detectable. Such configurations are the “bag of gold” initial data sets coined by Wheeler [31],

cf. [26]. However, while the existence of a horizon allows for this behavior, the absence of horizons is not stable under time evolution. An initial time-symmetric slice M (or \widehat{M}) may have no horizons, and so be in $\mathcal{P}_{(\gamma,H)}^0$ (or $\widehat{\mathcal{P}}^0$), but during the course of time evolution of the physical space-time horizons may develop.

This leads more naturally to the space-time Bartnik mass (cf. §5). However, it is possible for instance that there exists a time-symmetric slice \widehat{M}' at a later time which has a horizon, so that such a slice is no longer in $\widehat{\mathcal{P}}^0$, even though the ADM mass remains conserved under such time evolution. This may occur even if the initial data for the Einstein evolution is vacuum, so $R_g = 0$. Of course restricting to the vacuum case, it is no longer so clear if the mass at infinity can be made arbitrarily small.

The main issue is to capture the gravitational mass. The mass due to matter sources is localized, in that one has an energy density $\rho : M \rightarrow \mathbb{R}^+$. One can argue that these matter contributions should be subtracted in some sense from the total mass of an extension (M, g) to obtain the gravitational mass of (M, g) . From this perspective, it again makes sense to restrict to vacuum configurations where $R_g = 0$.

Thus we propose the following modification of the Bartnik mass. Let \mathcal{P} be the space of smooth AF Riemannian 3-manifolds (M, g) with $R_g \geq 0$ and compact boundary ∂M , and as above let $\mathcal{P}_{(\gamma,H)} \subset \mathcal{P}$ be the subset such that $(g|_{\partial M}, H_{\partial M}) = (\gamma, H)$; this is the space of extensions of the boundary data (γ, H) with non-negative scalar curvature. Let

$$\mathcal{C}_{(\gamma,H)} = \{(M, g) \in \mathcal{P}_{(\gamma,H)} : R_g = 0\},$$

be the set of solutions of the time-symmetric vacuum constraint equations ($R_g = 0$) with given Bartnik boundary data; (the constraint set with fixed boundary data).

Definition 2.1. *Let*

$$(2.1) \quad m_B^0(\gamma, H) = \inf\{m_{ADM}(M, g) : (M, g) \in \mathcal{C}_{(\gamma,H)}\}.$$

A main point here is the absence of a horizon condition imposed on (M, g) . In particular $m_B^0(\gamma, H)$ satisfies the quasi-local criterion QL4. Some remarks related to this definition were given by Bartnik in the original work [9], but the issue seems to have been neglected since then.

While the positive mass theorem (with corners) implies $m_B^0(\gamma, H) \geq 0$ if (γ, H) are the boundary data of a compact domain Ω with non-negative scalar curvature, the proof of Huisken-Ilmanen [19] that $m_B^0(\Omega) > 0$ unless Ω is locally flat no longer applies. Thus we raise:

Conjecture 2.2. *One has*

$$m_B^0(\Omega) = 0,$$

only if Ω is locally flat.

Of course if the infimum $m_B^0(\Omega)$ in (2.1) is realized by an extension of Ω in $\mathcal{C}_{(g,H)}$, then the conjecture holds, by the positive mass theorem with corners.

Remark 2.3. There is currently little evidence on which to base Conjecture 2.2. As suggested to the author by J. Jauregui, consider a “bag of gold” space (M, g) with boundary data (γ, H) and satisfying the following: $R_g \geq 0$, $\text{supp } R_g$ compact in M , and (M, g) equal to a Schwarzschild metric of arbitrarily small mass outside a compact set, as in the beginning of this section. Using results of Lohkamp [22], the metric g may be deformed within a compact set to a new metric \bar{g} with scalar curvature

$$0 \leq R_{\bar{g}} < \varepsilon$$

and with $|\bar{g} - g|_{C^0} < \varepsilon$, for any prescribed $\varepsilon > 0$. The boundary data (γ, H) and mass m_{ADM} of g remain the same for \bar{g} . Let $\tilde{g} = v^4 \bar{g}$ where v solves the equation

$$-8\tilde{\Delta}v + \tilde{g}v = 0,$$

with $v = 1$ on ∂M and $v \rightarrow 1$ at infinity. The metric \tilde{g} is then scalar-flat $R_{\tilde{g}} = 0$ and satisfies

$$0 \leq \tilde{m} < m, \quad \tilde{\gamma} = \gamma, \quad \tilde{H} \leq H,$$

(cf. [7]). Thus, one can produce a scalar-flat metric with arbitrarily small mass, and with smaller mean curvature at the boundary. This would lead to counterexamples to Conjecture 2.2 if (perhaps by some modified construction) \tilde{H} could be made to equal H (keeping $\tilde{\gamma} = \gamma$) or \tilde{H} at least close to H in a smooth topology. However, in the Lohkamp deformation above, the first and higher derivatives of \bar{g} blow up as $\varepsilon \rightarrow 0$ and this may lead to blow-up of \tilde{H} at ∂M , so that \tilde{H} is far away from H in a smooth topology. It would be of interest to understand this situation better.

The definition (2.1) has a natural variational interpretation closely related to work of Bartnik [12] in the case of complete manifolds without boundary, (cf. discussion at the end of [11] and in [12]). This is based on the Regge-Teitelboim Hamiltonian and an approach to the positive mass theorem suggested by Brill-Deser-Fadeev [15].

To describe this, let $\mathcal{S}(M)$ denote the space of smooth pairs (g, u) on M where g is an asymptotically flat metric and u a function on M with $u \rightarrow 1$ at infinity. Formally, the 4-metric $g_{\mathcal{M}} = -u^2 dt^2 + g$ is a time-independent, i.e. static, metric on the space-time $\mathcal{M} = \mathbb{R} \times M$. Of course $g_{\mathcal{M}}$ is not vacuum in general.

Consider the Regge-Teitelboim Hamiltonian [28] in this setting:

$$(2.2) \quad \begin{aligned} \mathcal{H}_{RT} &: \mathcal{S}(M) \rightarrow \mathbb{R}, \\ \mathcal{H}_{RT}(g, u) &= 16\pi m_{ADM}(g) - \int_M u R_g dv_g. \end{aligned}$$

The second term differs from the Einstein–Hilbert action on the 4-manifold \mathcal{M} by a divergence term. In contrast to the original ADM Hamiltonian, the Regge–Teitelboim Hamiltonian (2.2) (suitably regularized) gives a well-defined variational problem and is a smooth functional on the full manifold $\mathcal{S}(M)$, cf. [28], [11] for details.

The first variation of the RT Hamiltonian is well-known, cf. [28], [10], [7];

$$(2.3) \quad \nabla \mathcal{H} = -(S^*u + \frac{1}{2}uR_g g, R_g, uA - N(u)\gamma, 2u)$$

in the sense that, if (h, u') is any variation of (g, u) inducing the variation (h^T, H'_h) of boundary data (γ, H) , then

$$(2.4) \quad \begin{aligned} -d\mathcal{H}_{(g,u)}(h, u', h^T, H'_h) &= \int_M [\langle S^*u + \frac{1}{2}uR_g g, h \rangle + R_g u'] dv_g + \\ &\int_{\partial M} [\langle uA - N(u)\gamma, h^T \rangle + 2uH'_h] dv_\gamma. \end{aligned}$$

Here N is the unit normal at ∂M pointing into M , A is the 2nd fundamental form of ∂M in M and

$$(2.5) \quad S^*u = D^2u - (\Delta u)g - u \text{ Ric},$$

is the formal L^2 -adjoint of the linearization $R' = DR_g$ of the scalar curvature. The static vacuum Einstein equations (1.5) are equivalent to the system $(g, u) \in \mathcal{S}(M)$ such that

$$(2.6) \quad S^*u = 0, \quad R = 0.$$

(The condition $R = 0$ is equivalent to $\Delta u = 0$ when $S^*u = 0$).

Let $\mathcal{S}_{(\gamma,H)}(M) \subset \mathcal{S}(M)$ be the space of static metrics with fixed Bartnik boundary data (γ, H) . Working in a given function space such as a weighted Hölder space $C_\delta^{m,\alpha}$, it is straightforward (using the implicit function theorem)

to show that $\mathcal{S}_{(\gamma,H)}(M)$ is a smooth, closed Banach submanifold of $\mathcal{S}(M)$, for all choices of (γ, H) .

Thus, (2.4) shows that critical points of the Hamiltonian \mathcal{H}_{RT} on $\mathcal{S}_{(\gamma,H)}(M)$ are given by static vacuum Einstein metrics realizing the given boundary data (γ, H) . Note that although $u \rightarrow 1$ at infinity, it is not assumed at this point that $u > 0$ on M .

Now for the restriction of \mathcal{H}_{RT} to $\mathcal{C}_{(\gamma,H)}$, one clearly has

$$(2.7) \quad \mathcal{H}_{RT}|_{\mathcal{C}_{(\gamma,H)}} = 16\pi m_{ADM}(g) : \mathcal{C}_{(\gamma,H)} \rightarrow \mathbb{R}.$$

It is proved in [7] that, if non-empty, $\mathcal{C}_{(\gamma,H)}$ is a smooth, infinite dimensional Banach manifold in a natural $C_\delta^{m,\alpha}$ Hölder space topology, and that m_{ADM} in (2.7) is a smooth functional on $\mathcal{C}_{(\gamma,H)}$. Moreover, it is shown that critical points of \mathcal{H}_{RT} or m_{ADM} on the constraint manifold $\mathcal{C}_{(\gamma,H)}$ are exactly the static vacuum Einstein metrics (g, u) satisfying the elliptic boundary conditions (γ, H) . In addition, minimizers of the Bartnik mass (1.1), or (1.3), are static vacuum solutions with $u > 0$ and $u \rightarrow 1$ at infinity, cf. [7].

It is worth noting that Miao [24] has proved, based on the black hole uniqueness theorem, that static vacuum solutions (M, g, u) as in (2.6) with $u > 0$ on M have no minimal surfaces surrounding ∂M . On the other hand, it is unknown whether static vacuum solutions (M, g, u) have no compact minimal surfaces at all.

In contrast to the discussion above, there are no critical points of the mass

$$m_{ADM} : \mathcal{S}_{(\gamma,H)}(M) \rightarrow \mathbb{R},$$

so that some constraint, such as $R_g \geq 0$, is necessary to obtain a constrained critical point or minimizer of m_{ADM} . In fact if R_g is not identically zero, i.e. $g \notin \mathcal{C}_{(\gamma,H)}$, then the mass m_{ADM} can be decreased, both infinitesimally and locally, in the space $\mathcal{S}_{(\gamma,H \leq)}^+(M)$, consisting of metrics of non-negative scalar curvature in $\mathcal{S}(M)$ with mean curvature $\leq H$ and induced metric γ at ∂M , cf. [7].

Next we relate the domains of definition of the masses in (1.3) and (2.1).

Conjecture 2.4. *If (M, g) is an extension of the boundary data (γ, H) in \mathcal{P} , then there exists an extension in $\mathcal{C}_{(\gamma,H)}$, i.e.*

$$\mathcal{P}_{(\gamma,H)} \neq \emptyset \Rightarrow \mathcal{C}_{(\gamma,H)} \neq \emptyset.$$

This is of course a very special case of (and much simpler than) the Bartnik minimization conjecture. It is not clear that either $\mathcal{P}_{(\gamma,H)}$ or $\mathcal{C}_{(\gamma,H)}$

are non-empty, for any given (γ, H) arising as boundary data of a compact body Ω with $R_{g_\Omega} \geq 0$; a good testing ground would be the class of immersed spheres in \mathbb{R}^3 discussed in §4.

We also note the following monotonicity property. Suppose $(\Omega_1, g_{\Omega_1}) \subset (\Omega_2, g_{\Omega_2})$ and $R_{g_{\Omega_2}} = 0$, so there are no matter sources in Ω_2 . Then

$$(2.8) \quad m_B^0(\Omega_1) \leq m_B^0(\Omega_2).$$

This follows immediately from the definition (2.1), if one allows metrics in the space $\mathcal{C}_{(\gamma, H)}$ to have Lipschitz seams across smooth surfaces. The condition $R_g = 0$ is then well-defined distributionally. If equality holds in (2.8) and $m_B^0(\Omega_1)$ is realized by a static vacuum extension (M_1, g_1, u_1) , then $(\Omega_2 \setminus \Omega_1, g_2) \subset (M_1, g_1)$ isometrically.

Next we discuss issues related to the Bartnik static extension conjecture. Via the discussion above, these are closely related to the study of m_B itself.

Let \mathbb{E}_{st} be the set of all AF solutions (M, g, u) of the static vacuum Einstein equations (1.5), $u > 0$ on M and let \mathcal{E}_{st} be the associated moduli space, obtained from \mathbb{E}_{st} by quotienting out by the group of diffeomorphisms equal to the identity on ∂M . Given a suitable Hölder space $C_\delta^{m, \alpha}$ topology, it is proved in [4], [8] that \mathcal{E}_{st} is a smooth infinite dimensional Banach manifold. Let \mathcal{B} be the space of Bartnik boundary data (γ, H) ; again this may be given a natural smooth Banach manifold topology. It is proved in [4], [8] that the Bartnik boundary map

$$(2.9) \quad \begin{aligned} \Pi_B : \mathcal{E}_{st} &\rightarrow \mathcal{B}, \\ \Pi_B([g, u]) &= (\gamma, H), \end{aligned}$$

is a smooth Fredholm map, of Fredholm index zero. The Bartnik static extension conjecture from §1 is thus the statement that the smooth map Π_B is a bijection, i.e. both surjective (existence) and injective (uniqueness). The formulation (2.9) gives an effective means to study the static extension conjecture in detail.

Note that the ADM mass

$$(2.10) \quad m_{ADM} : \mathcal{E}_{st} \rightarrow \mathbb{R},$$

is a smooth functional on \mathcal{E}_{st} . It can be “effectively” computed as a usual Komar integral, i.e.

$$m_{ADM} = \int_{\partial M} N(u);$$

compare with (1.7). In regions where Π_B is injective with smooth inverse, the mass $m_B(\gamma, H)$ or $m_B^0(\gamma, H)$ thus becomes a smooth functional of the boundary data (γ, H) , assuming it is realized as in the Minimization Conjecture. In regions where Π_B fails to be injective, this is unlikely to be the case.

These remarks lead naturally to consideration of a further choice m_B^{st} for the definition of Bartnik mass. Namely $m_B^{st}(\gamma, H)$ is defined only for

$$(\gamma, H) \in \text{Im } \Pi_B$$

and for such (γ, H) ,

$$(2.11) \quad m_B^{st}(\gamma, H) = \inf\{m_{ADM}(M, g, u) : \Pi_B(M, g, u) = (\gamma, H)\}.$$

In many situations, one would expect that $(\Pi_B)^{-1}(\gamma, H)$ is compact, and generically a finite number of points; this will be the case in regions $\mathcal{U} \subset \mathcal{E}_{st}$ where Π_B is proper. In such cases, the infimum in (2.11) is then realized as the smallest value of a continuous function on a compact set. Thus m_B^{st} is, in principle, effectively computable.

Note that the mass functional (2.10) does not map into \mathbb{R}^+ in general, i.e. it is not assumed in (2.10) or (2.9) that the boundary data (γ, H) bound a body Ω with $R_g \geq 0$. This is closely related to the *fill-in problem*, dual to the extension problem (1.4) discussed above: given data (γ, H) on $\partial\Omega$, when does there exist a metric g_Ω on Ω with $R_{g_\Omega} \geq 0$ and with Bartnik boundary data (γ, H) ?

It is proved by Jauregui in [20] that for any (γ, H) , with Gauss curvature $K_\gamma > 0$ and $H > 0$, one has a basic trichotomy: there exists $\lambda_0 > 0$ so that $(\gamma, \lambda H)$ has a fill-in with $R > 0$ for all $0 < \lambda < \lambda_0$ and no fill-in with $R \geq 0$ for $\lambda > \lambda_0$. It is conjectured in [20] that the data $(\gamma, \lambda_0 H)$ have a fill-in which is a solution to the static vacuum equations (1.5); thus static vacuum solutions provide a division between these two behaviors. We refer to [5] for an analysis of the behavior of static vacuum solutions on bounded domains Ω .

Regarding the dual extension problem, it is proved in [6] that given any fixed (γ, H) as above (with γ now arbitrary), there is a $\lambda_0 > 0$ such that $(\gamma, \lambda_0 H)$ are boundary data of a complete exterior AF static vacuum solution (M, g, u) , $M = \mathbb{R}^3 \setminus B$. Moreover, for $\lambda > \lambda_0$, the data $(\gamma, \lambda H)$ have an AF extension with non-negative scalar curvature, i.e. $\mathcal{P}_{(\gamma, \lambda H)} \neq \emptyset$, cf. [7]. On the other hand, the trichotomy above breaks down for $\lambda < \lambda_0$: Mantoulidis-Schoen [23] have constructed large families of data $(\gamma, \lambda H)$ with $\lambda > 0$ arbitrarily small such that $\mathcal{P}_{(\gamma, \lambda H)} \neq \emptyset$.

Unfortunately, it is proved in [8] that Π_B in (2.9) is not a proper map and so is not a homeomorphism, for a rather large domain within \mathcal{E}_{st} or \mathcal{B} . At this

time, it seems very unlikely that Π_B is either globally surjective or globally injective, and so the domain in \mathcal{E}_{st} on which Π_B is a smooth diffeomorphism onto its image becomes of basic importance; these issues are discussed further in §4.

3. The outer-minimizing modification

An interesting modification of the Bartnik mass m_B has been suggested by Bray [14], based on considerations from the Riemannian Penrose inequality [19], [13]. Given $(M, g) \in \mathcal{P}$, ∂M is said to be outer-minimizing in (M, g) if

$$area(\partial M) < area(\Sigma)$$

for any surface $\Sigma \subset M$ surrounding ∂M , (so that the cycle $\partial M - \Sigma$ is null-homologous). Let $\mathcal{P}^{out} \subset \mathcal{P}$ be the subset for which ∂M is outer-minimizing, so that \mathcal{P}^{out} is an open domain in \mathcal{P} . Given Bartnik boundary data (γ, H) , let $\mathcal{P}_{(\gamma, H)}^{out} = \mathcal{P}_{(\gamma, H)} \cap \mathcal{P}^{out}$ be the space of outer-minimizing extensions of (γ, H) . Define then

$$(3.1) \quad m_B^{out}(\gamma, H) = \inf\{m_{ADM}(M, g) : (M, g) \in \mathcal{P}_{(\gamma, H)}^{out}\}.$$

The mass m_B^{out} has a number of advantageous properties which follow from the fundamental work of Huisken-Ilmanen [19]. Namely the Hawking mass m_H is a lower bound for m_B^{out} :

$$m_H \leq m_B^{out}.$$

For domains Ω in the Schwarzschild metric (M, g_{Sch}) containing the horizon $r = 2m$, one has

$$(3.2) \quad m_B^{out}(\Omega) = m_{Sch}.$$

The relation (3.2) is unknown for any other version (1.1), (1.3) or (2.1) of the Bartnik mass. Directly from the definition, one sees that a restricted version of the monotonicity property QL2 also holds; if $\Omega_1 \subset \Omega_2$ and Ω_1 is outer-minimizing in Ω_2 then $m_B^{out}(\Omega_1) \leq m_B^{out}(\Omega_2)$. Further, QL3 holds. However, as with (1.1), the quasi-local property QL4 does not hold.

Perhaps the main issue in extending the understanding of m_B^{out} is the domain of existence property QL0. Namely, just as with the no-horizon condition, the definition m_B^{out} again depends on a large-scale, global property of the boundary data (γ, H) . It is very difficult to effectively determine whether

Bartnik boundary data (γ, H) admit an outer-minimizing extension in \mathcal{P} , i.e. for which (γ, H) is

$$\mathcal{P}_{(\gamma, H)}^{out} \neq \emptyset.$$

The mass m_B^{out} is not defined when $\mathcal{P}_{(\gamma, H)}^{out} = \emptyset$ but there is no solid physical reason for this.

Consider the simplest case of flat boundary data, i.e. boundary data (γ, H) of a flat bounded domain in \mathbb{R}^3 for which $m_B = 0$. At least when $\partial\Omega = S^2$, flat boundary data (γ, H) uniquely determine the bounding domain Ω (up to rigid motion). However there are very few general results on characterizing the data (γ, H) for which the boundary is outer-minimizing in \mathbb{R}^3 . The wonderful solution of the Weyl embedding problem by Nirenberg [25] and Pogorelov [27] gives an affirmative answer when the Gauss curvature $K_\gamma > 0$; convex boundaries are outer-minimizing in \mathbb{R}^3 . However, there do not appear to be any sound physical reasons why the mass of a non-convex body in \mathbb{R}^3 should not be well-defined; one would expect it is defined and the mass is zero. Thus it is of basic interest to understand more general conditions on (γ, H) for which $\partial\Omega$ is outer-minimizing in \mathbb{R}^3 . Moreover, there are no known analogs of the Nirenberg-Pogorelov result for general solutions (M, g, u) of the static vacuum Einstein equations.

A further issue with m_B^{out} is the failure of the Static Extension and Minimization conjectures in this context, i.e. the non-existence of static vacuum extensions realizing m_B^{out} . This behavior occurs for a very natural and simple class of boundary data (γ, H) . The following result is essentially given in [7]:

Proposition 3.1. *Let γ be a smooth metric on S^2 distinct from the round metric γ_{2m} of radius $2m$. Then there is a constant $H_m > 0$, depending only on m and a lower bound on the C^0 -distance between γ and $\{\gamma_{2m'}, m' \in \mathbb{R}^+\}$, such that, for any smooth function H satisfying*

$$0 < H < H_m,$$

pointwise, the Bartnik boundary data (γ, H) are not the boundary data of a solution (M, g, u) of the static vacuum Einstein (2.6) with ∂M outer-minimizing in (M, g) .

Proof. The proof is by contradiction. Thus, let (γ_i, H_i) , $H_i > 0$, be a sequence of smooth Bartnik boundary data converging smoothly to $(\gamma, 0)$. Suppose for each i , (γ_i, H_i) are boundary data of static vacuum solutions (M, g_i, u_i) , $u_i > 0$, with ∂M outer-minimizing in (M, g_i) . By the basic compactness theorem in [8], a subsequence of (M, g_i, u_i) converges, smoothly up to ∂M ,

to a limit solution (M, g, u) , $u > 0$, of the static vacuum Einstein equations with Bartnik boundary data

$$\lim_{i \rightarrow \infty} (\gamma_i, H_i) = (\gamma, 0).$$

This compactness theorem strongly uses (a weak form of) the outer-minimizing property.

By the black hole uniqueness theorem and its extension by Miao in [24], the only static vacuum solution (M, g, u) with $u > 0$, $u \rightarrow 1$ at infinity and horizon $H = 0$ boundary is the Schwarzschild metric with boundary data $(\gamma_{2m}, 0)$ for some m . It follows that if $\gamma \neq \gamma_{2m'}$ for some m' , i.e. γ is not a round metric, then such a sequence cannot exist. Thus, there is a neighborhood of the boundary data $(\gamma, 0)$ where there are no outer-minimizing solutions of the static vacuum Einstein equations. \square

In sum, the black hole uniqueness theorem causes the non-existence of static vacuum solutions (M, g, u) with sufficiently small mean curvature and non-round boundary metric. It follows in particular that the mass m_B^{out} cannot be realized by boundary data (γ, H) with H sufficiently small and γ not a round metric.

We believe that Proposition 3.1 holds for all static vacuum solutions, without the outer-minimizing assumption. It follows from the work in [8] that if (M, g, u) is a static vacuum solution with γ not a round metric and H sufficiently small, then the curvature of (M, g) becomes either very large near ∂M or the injectivity radius normal to ∂M in (M, g) becomes very small (or both). There does not seem to be any particular reason which would cause such behavior with such controlled boundary data; however this remains an open problem.

4. Embeddings and immersions

It is proved in [7] that there exists a large family of smooth immersions $F : S^2 \rightarrow \mathbb{R}^3$, which extend to smooth embeddings of the interior 3-ball $F : B^3 \rightarrow \mathbb{R}^3$, for which the corresponding Bartnik data (γ, H) induced from F have

$$m_B(\gamma, H) = 0,$$

but (γ, H) are not realizable as boundary data of mass-minimizing extensions. In particular, for such data $\mathcal{P}_{(\gamma, H)}^0 \neq \emptyset$ so that m_B is defined but is not achieved by a mass-minimizer. This gives the first counterexamples to the

Bartnik minimization conjecture; the discussion at the end of §3 suggests there are more.

Within this family of boundary data (γ, H) , the non-existence of mass-minimizers is due to the breakdown of the manifold-with-boundary structure near the boundary, for all minimizing sequences for the Bartnik mass of (γ, H) . This occurs even with smooth control of the boundary data (γ, H) .

We conjecture that a similar principle holds for general immersed 2-spheres. Let $\text{Imm}_A(S^2, \mathbb{R}^3)$ be the space of smooth Alexandrov immersed 2-spheres in \mathbb{R}^3 , i.e. immersions of spheres which extend to immersions of the 3-ball B^3 . The inverse image $F^{-1}(\mathbb{R}^3)$ is thus a locally flat 3-ball (Ω, g_Ω) and F is the developing map giving an isometric immersion of the abstract body $\Omega \simeq B^3$ into \mathbb{R}^3 . Let $\mathcal{I} = \text{Imm}_A(S^2, \mathbb{R}^3) \setminus \text{Emb}(S^2, \mathbb{R}^3)$ be the complement of the space of embeddings $S^2 \rightarrow \mathbb{R}^3$. The constructions in [7] take place at the boundary $\partial\mathcal{I}$. For $F \in \mathcal{I}$, let (γ, H) be the induced metric and mean curvature from the Euclidean metric.

Conjecture 4.1. *For $F \in \mathcal{I}$ as above, either*

$$\mathcal{P}_{(\gamma, H)} = \emptyset,$$

(so that $m_B(\gamma, H)$ is not defined), or $m_B(\gamma, H)$ is not achieved by a static vacuum solution with boundary data (γ, H) .

This behavior is not restricted to flat \mathbb{R}^3 ; one may conjecture the same behavior for immersions of spheres into any given static vacuum solution (M, g, u) .

For $F \in \mathcal{I}$ as above, the abstract body (Ω, g_Ω) becomes “twisted” and has self-intersections when viewed as a body in the static vacuum solution $(\mathbb{R}^3, g_{Eucl}, 1)$ (or more generally (M, g, u)). However, note that (Ω, g_Ω) may be smoothly approximated by embedded space-like 3-balls (Ω_i, g_i, K_i) in Minkowski space time $\mathbb{R}^{1,3}$, i.e. the 4-dimensional (vacuum) space-time development of the initial Cauchy hypersurface $(\mathbb{R}^3, g_{Eucl}, K = 0)$. Thus there are always space-like AF extensions (M, g, K) of $(\Omega, g_\Omega, 0)$ with approximately (and possibly even the same) boundary data in the 4-dimensional space-time; cf. the discussion in §5. This passage from a fixed 3-dimensional space-like slice to the the full 4-dimensional space-time bears resemblance to the passage from the Brown-York mass [16] to the more recent Wang-Yau mass [30].

In the case of \mathbb{R}^3 , the reasoning above suggests that the mass of any body (Ω, g_Ω) arising from \mathcal{I} should still be defined and be zero. As in the discussion in §3, there do not seem to be any physical reasons why the mass of an embedded body in \mathbb{R}^3 should be defined but become undefined in passing

continuously to an immersed body. Moreover, it is very difficult to detect from the boundary data (γ, H) of a flat body Ω whether Ω gives an embedded or immersed sphere in \mathbb{R}^3 .

From this perspective, one may consider the following problem. Define an AF static vacuum solution (M, g, u) , $u > 0$, M open, without boundary, to be *maximal* if it cannot be smoothly extended to any larger domain (M', g', u') , $u' > 0$, with (M, g, u) isometrically embedded in (M', g', u') as a proper subset; thus M is maximal with respect to inclusion. Of course $(\mathbb{R}^3, g_{Eucl}, 1)$ is a maximal solution; it is the only maximal solution which is complete as a metric space.

Problem. Given Bartnik boundary data (γ, H) , is there a maximal static vacuum solution (M, g, u) and an immersion $F : S^2 \rightarrow M$ which realizes (γ, H) , so that the “boundary data” on $F(S^2)$ are (γ, H) .

Note that an affirmative resolution of the problem would then allow a definition of Bartnik mass for any (γ, H) ; namely one may define $\tilde{m}_B(\gamma, H)$ to be the infimum of m_{ADM} over all static vacuum solutions (M, g, u) for which there is an immersion $F : S^2 \rightarrow M$ inducing the data (γ, H) .

From one perspective, this problem may be viewed as an analog of the classical isometric immersion problem for spheres in \mathbb{R}^3 : given a metric γ on S^2 , is there an isometric immersion $F : S^2 \rightarrow \mathbb{R}^3$ realizing γ , so that $F^*(g_{Eucl}) = \gamma$? This is a famous and notoriously hard problem; the main result to date is the solution to the Weyl embedding problem by Nirenberg and Pogorelov. This generalization to the static vacuum case (M, g, u) adds on the prescription of the mean curvature H , corresponding to the extra scalar field u . Note however that a positive solution of the static vacuum Problem above does not imply a resolution of the isometric immersion problem. One significant advantage of the static vacuum problem is that it is given by an elliptic system of equations, in strong contrast to the isometric immersion problem. Thus it might well be more amenable to further progress.

Observe however that the static vacuum Problem above is incompatible with the discussion following Proposition 3.1. Thus, a more reasonable version of the Problem is the following:

Conjecture 4.2. *Given Bartnik boundary data (γ, H) , with (say) $area_\gamma(S^2) = 1$, there is a constant $H_1 > 0$ such that if*

$$H \geq H_1 > 0,$$

then there is a maximal static vacuum solution (M, g, u) and an immersion $F : S^2 \rightarrow M$ which realizes (γ, H) , i.e. the “boundary data” induced on $F(S^2)$ are (γ, H) .

Many of the problems raised here for the Bartnik mass exist also for other QL mass definitions. For instance, the Brown-York mass [16] is well defined only for metrics γ for which there is a unique isometric embedding into \mathbb{R}^3 , e.g. metrics with $K_\gamma > 0$ by the solution to the Weyl problem. Similar convexity requirements are needed so far for the Wang-Yau mass [30].

These and many other definitions of quasi-local mass require solving a complicated system of PDE's to obtain their value; for instance solving the isometric immersion problem or the static vacuum Einstein equations. (The main exception is the Hawking mass). The behavior of solutions of these equations in the large, non-perturbative realm (e.g. far from flat) have not yet been well-understood.

5. Space-time Bartnik mass

The space-time version of the Bartnik mass, introduced in [9] and [10], is considerably more complicated than the the time-symmetric (static) case (1.3). The complications are so severe that until very recently, there had been essentially no progress on this topic.

Let $(\widehat{M}, \widehat{g}, \widehat{K})$ be a complete AF initial data (Cauchy) surface satisfying the Einstein constraint equations

$$(5.1) \quad |K|^2 - H^2 - R_g = \mu, \quad \delta(K - Hg) = J,$$

where (μ, J) are required to satisfy the dominant energy condition $\mu^2 \geq |J|^2$; we have dropped the hat notation for simplicity. The vacuum case $\mu = J = 0$ is of particular interest.

As in §1, let $(\Omega, g_\Omega, K_\Omega)$ be a smooth bounded domain with boundary (say) $\partial\Omega = S^2$ satisfying the constraint equations (5.1), and let $(M, \underline{g}, \underline{K})$ be an AF extension of $\partial\Omega = \partial M$ also satisfying (5.1). We view $\Omega \subset \widehat{M}$, so that $M = \widehat{M} \setminus \Omega$. Let $\widehat{\mathcal{M}}$ be the 4-d space-time generated by the initial data $(\widehat{M}, \widehat{g}, \widehat{K})$ by solving the Einstein evolution equations and similarly $\mathcal{M} \subset \widehat{\mathcal{M}}$.

To begin, one must define suitable boundary data for the quasi-local mass. Here the Bartnik boundary data are given by

$$(5.2) \quad (\gamma, H, tr_{\partial M} K, K(N)^T),$$

where N is the inward unit normal to ∂M in $M \subset \mathcal{M}$ and $K(N)^T$ is the restriction of the 1-form $K(N)$ to ∂M . Note that the boundary conditions $H, K(N)^T$ are “gauge-dependent”, i.e. depend on a choice of the space-like hypersurface M at ∂M or equivalently a choice of time-like unit normal ν

at ∂M . Such gauge-dependence occurs for many other quasi-local masses (e.g. that of Brown-York [16] or Wang-Yau [30]).

Bartnik derived the boundary data (5.2) via two distinct but essentially equivalent arguments given in [10], cf. also [11]. First, the requirement that the energy-momentum terms (μ, J) are distributionally well-defined across the seam or interface $\partial\Omega = \partial M$ of the interior and exterior regions Ω and M leads by straightforward analysis to the equality of (5.2) at $\partial\Omega$ and ∂M . The data (γ, H) arise from the Hamiltonian constraint, just as in the static case discussed in §2, while the data $(tr_{\partial M}K, K(N)^T)$ arise from the divergence or momentum constraint in (5.1).

A second and in certain respects more significant argument for the choice of the boundary conditions (5.2) is that they are exactly the boundary terms arising from the first variation of the Regge-Teitelboim Hamiltonian. In more detail, the RT-Hamiltonian is defined by

$$(5.3) \quad \mathcal{H}_{RT}(g, K, \xi) = 16\pi \langle \xi_\infty, \mathbb{P}_{ADM} \rangle - \int_M \langle \xi, \Phi(g, K) \rangle,$$

where \mathbb{P}_{ADM} is the ADM energy-momentum 4-vector, $\xi = (u, X)$ is the lapse-shift, ξ_∞ is their value (u_∞, X_∞) at space-like infinity and Φ is the constraint operator $\Phi(g, K) = (\mu, J)$ as in (5.1). This is the direct generalization of the static Regge-Teitelboim Hamiltonian (2.2).

One may then derive the first variation of \mathcal{H}_{RT} . Just as in the static case, there are bulk integrals over M and boundary integrals at ∂M . A lengthy but well-known calculation shows that the vanishing of the first variation of the bulk integral gives the Euler-Lagrange equations

$$(5.4) \quad Ric_g - u^{-1}D^2u + (trK)K - 2K^2 + u^{-1}\mathcal{L}_XK = 0, \\ \mathcal{L}_Xg = 2uK,$$

in addition to the vacuum constraints (5.1). These are a determined system of 16 equations for the 16 unknowns (g, K, u, X) . As shown by Bartnik in [10], the vanishing of the variation of the boundary terms gives exactly the vanishing of the variation of the boundary data (5.2).

The equations (5.4) are the stationary vacuum Einstein equations, giving time-independent solutions of the vacuum Einstein equations with Killing field $\partial_t = u\nu + X$, where ν is the time-like unit normal to M .

Writing $\mathbb{P}_{ADM} = (E, P)$ in the usual way, the ADM mass m_{ADM} is given by

$$m_{ADM} = \sqrt{|\mathbb{P}_{ADM}|^2} = \sqrt{E^2 - |P|^2}.$$

Each of the definitions of Bartnik mass discussed above in the static case, namely $m_B(\Omega)$ in (1.1), $m_B(\gamma, H)$ in (1.3), m_B^0 in (2.1), m_B^{out} in (3.1) may be formally extended to the space-time setting. For instance, let \mathcal{P}^0 be the space of triples (M, g, K) with $M \simeq \mathbb{R}^3 \setminus B$, with (g, K) asymptotically flat data satisfying the dominant energy constraints (5.1), with (M, g, K) containing no apparent horizons surrounding ∂M . Given Bartnik boundary data $\beta = (\gamma, H, tr_{\partial M} K, K(N)^T)$, let

$$\mathcal{P}_\beta^0 \subset \mathcal{P}^0,$$

be the subset of \mathcal{P}^0 with fixed Bartnik boundary data $\beta = (\gamma, H, tr_{\partial M} K, K(N)^T)$ at ∂M . Then

$$(5.5) \quad m_B(\beta) = \inf\{m_{ADM}(M, g, K) : (M, g, K) \in \mathcal{P}_\beta^0\}.$$

In [10], Bartnik again raised the minimization conjecture that the mass m_B in (5.5) is realized by a unique AF stationary vacuum space-time (M, g, K) with given boundary data (5.2), and the corresponding stationary extension conjecture that all such boundary data are uniquely realized by a stationary vacuum solution. These are of course even more difficult than the static versions. All of the issues, problems and conjectures discussed in previous sections apply here also, both for the mass (5.5) and the analogous space-time modifications of static masses discussed previously.

However, very little has been known in this space-time setting. As a simple example, the analog of the result of Miao [24] that static vacuum solutions (M, g, u) have no horizons surrounding ∂M is unknown for stationary vacuum solutions.

Even the most basic question, raised by Bartnik in [9], of whether the system (5.4) with boundary conditions (5.2) is an elliptic system in a suitable gauge was unresolved. This has recently been proved to be true by An [2]. Given this, one has now at least an effective starting point to study some of these conjectures in more detail.

Thus, analogous to the static case, let \mathbb{E}_{sta} be the space of all AF solutions (g, K) of the stationary vacuum Einstein equations (1.5) on $M \simeq \mathbb{R}^3 \setminus B$ and let \mathcal{E}_{sta} be the corresponding moduli space, obtained by dividing out by a suitable group of diffeomorphisms fixing ∂M . The choice of the correct diffeomorphism or gauge group here is quite subtle, since the group of diffeomorphisms fixing ∂M does not preserve the data (5.2) at ∂M . Let then \mathcal{D}_4 be the group of time-independent diffeomorphisms of the space-time \mathcal{M} fixing ∂M ; any element in \mathcal{D}_4 can be written as a pair (ψ, f) , where $\psi \in \mathcal{D}_3$ is a diffeomorphism of M fixing ∂M and f represents a time translation $f : M \rightarrow \mathbb{R}$

with $f|_{\partial M} = 0$, moving M in \mathcal{M} but fixing ∂M . The correct choice of gauge group $\mathcal{D} \subset \mathcal{D}_4$, as discussed in [2], is

$$\mathcal{D} = \{(\psi, f) \in \mathcal{D}_4 : N(f) = 0 \text{ at } \partial M\},$$

where N is the unit inward normal to ∂M in M . This choice is naturally related to the gauge-dependence of the boundary conditions (5.2) discussed above.

It is then proved in [1], [2] that \mathcal{E}_{sta} is a smooth infinite dimensional Banach manifold (in a suitable $C_\delta^{m,\alpha}$ Hölder topology). Moreover, by [2], the Bartnik boundary map

$$\begin{aligned} \Pi_B : \mathcal{E}_{sta} &\rightarrow \mathcal{B}, \\ \Pi_B(g, K) &= (\gamma, H, tr_{\partial M} K, K(N)^T), \end{aligned}$$

is a smooth Fredholm map. In addition, the map Π_B is a local diffeomorphism in a neighborhood of standard, round flat boundary data $(\gamma_{+1}, 2, 0, 0)$.

It would be very interesting to extend the Regge-Teitelboim-Bartnik program discussed in §2 to the stationary or space-time setting. Thus one would like to study the Hamiltonian (5.3) and its critical points on the constraint space $\mathcal{C}_{(\gamma, H, tr_{\partial M} K, K(N)^T)}$: the space of solutions of the constraint equations (5.1) with fixed boundary data (5.2). The first main task in such a program is to prove that $\mathcal{C}_{(\gamma, H, tr_{\partial M} K, K(N)^T)}$ can be given the structure of a smooth Banach (or Hilbert) manifold. This was proved by Bartnik [11] in the case of empty boundary and in the low-regularity regime. Recent progress on this problem has been obtained by An [3].

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