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Null geodesic incompleteness of spacetimes with no CMC Cauchy surfaces*

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The authors are honored to dedicate this work to Robert Bartnik

Abstract: Using an initial data gluing construction, Chruściel, Isenberg, and Pollack constructed a class of vacuum cosmological spacetimes that do not admit Cauchy surfaces with constant mean curvature. We prove that, for sufficiently large values of the gluing parameter, these examples are both future and past null geodesically incomplete.

1. Introduction

It is well-known that constant mean curvature (CMC) Cauchy surfaces play an important role in the mathematical study of solutions to the Einstein field equations. When solving the Einstein constraint equations via the conformal method, the CMC assumption ensures that the resulting equations semi-decouple, hence leading to a far more robust understanding of existence and uniqueness than in the general case. The CMC gauge is also quite useful for studying the Einstein evolution equations, both analytically and numerically. This vital role of CMC Cauchy surfaces raises an interesting open question: When do globally hyperbolic spacetimes admit CMC Cauchy surfaces? It is known that not all globally hyperbolic spacetimes have CMC Cauchy surfaces: In [3], Bartnik found no-CMC spacetimes with dust, and in [8], Chruściel, Isenberg, and Pollack (CIP) found a family of vacuum spacetimes with no CMC Cauchy surfaces using a modified form of *IMP gluing*: a connected sum gluing procedure developed initially by Isenberg, Mazzeo and Pollack (see [6], [9], [13], [14], [15]). We refer to the recent survey [10] of Dilts and Holst for some results and conjectures related to this question.

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In a recent paper [11], Galloway and Ling proved a new existence result for CMC slices: every future timelike geodesically complete cosmological spacetime (recall that a *cosmological spacetime* is a globally hyperbolic spacetime with compact Cauchy surfaces) with everywhere nonpositive timelike sectional curvatures must admit a CMC Cauchy surface. Motivated by the Bartnik splitting conjecture (see Conjecture 2 of [3]), the conjectures of [10], and the conditions of their own result, Galloway and Ling conjecture that a future timelike geodesically complete cosmological spacetime satisfying the strong energy condition must contain a CMC Cauchy surface. In light of this conjecture, geodesic completeness or incompleteness of no-CMC spacetimes becomes relevant. Noting that Bartnik's examples are by construction timelike geodesically incomplete to both the future and the past, we turn our attention toward the CIP examples.

In this note, we prove the following:

Theorem 1. *For sufficiently large gluing parameter, the no-CMC spacetimes constructed in [8] are both future and past null geodesically incomplete.*

We do this by using the symmetry of the construction to show that the central cross-section of the gluing region must be both a marginally outer trapped surface (MOTS) and a marginally inner trapped surface (MITS) with particularly rigid geometry. We then use a covering space argument together with Chruściel and Galloway's generalization of the Penrose singularity theorem (Proposition 1.1 of [7]). In Section 2 we recall the localized IMP gluing construction, and in Section 3 we deduce null incompleteness.

The first and third authors have generalized in [5] the results presented here, showing that this incompleteness is not an artifact of the symmetry, but rather a consequence of the geometry of the underlying IMP gluing construction. Therefore, the maximal globally hyperbolic evolution of any (IMP) glued initial data sets admitting noncompact covers are causal geodesically incomplete for sufficiently large values of the gluing parameter.

The foundational work of Robert Bartnik plays a crucial role in the mathematics discussed in this paper. Indeed, his work in general relativity has had a huge influence over the direction of the field, and many of the currently active branches of research have grown out of the seeds that he planted. On a personal level, the third author is grateful to have known Robert since his time as a graduate student at Stanford in the late 1980s, when Robert visited the department from Australia. It has always been a pleasure to discuss mathematics with him and learn from him. The authors are honored to dedicate this work to him.

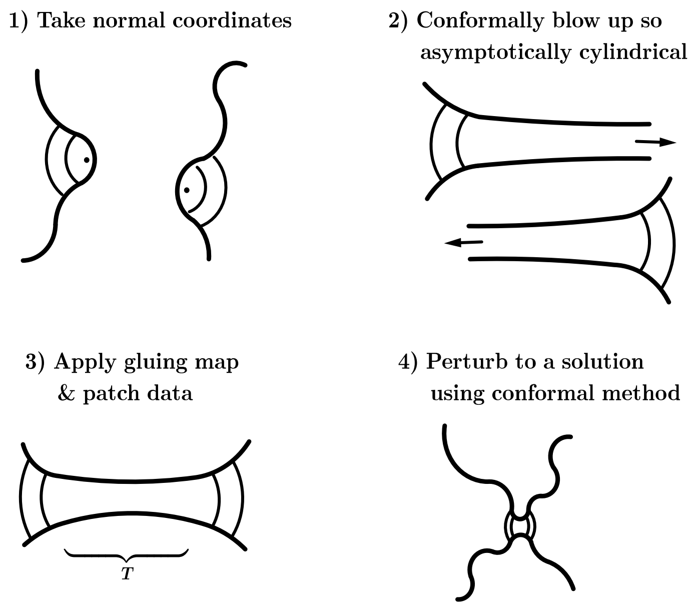

Figure 1: A sketch of IMP gluing.

2. Preliminaries

Here we recall the CIP construction [8], which heavily uses the IMP gluing construction [14], to set up notation and review the known properties of these examples. We begin with a vacuum initial data set $(\mathbb{T}^3, \gamma, K)$ which has no global Killing Initial Data (KIDs) and such that for some $p \in \mathbb{T}^3$, there is a neighborhood of p on which $\tau := \operatorname{tr}_\gamma K \equiv 0$.

To achieve this initial setup, one uses of the work of Beig, Chruściel and Schoen [4] on the generic absence of KIDs in initial data sets, as well as the work of Bartnik [2] on the Plateau problem for prescribed mean curvature spacelike hypersurfaces in a Lorentzian manifold.

The CIP construction then proceeds by applying a localized form of IMP gluing (see [6]) to form a connected sum of $(\mathbb{T}^3, \gamma, K)$ and $(\mathbb{T}^3, \gamma, -K)$ around the points p. The CIP gluing procedure consists of the following steps, summarized in Figure 1 (for more detailed diagrams, see [14]):

- On each copy, we consider the decomposition $\gamma|_{B_{2R}(p)} = dr^2 + r^2 h(r)$ in normal coordinates around p, where r is the geodesic distance from p, and h is a smooth family of metrics on $\mathbb{S}^2$ with $h(0) \equiv \mathring{g}$, the standard round metric on the unit sphere. In these coordinates, consider the conformal factor:

$$\psi_c(p) = \begin{cases} 1 & p \in \mathbb{T}^3 \setminus B_{2R}(p) \\ \text{interpolation} & p \in B_{2R}(p) \setminus B_R(p) \\ r^{1/2} & p \in B_R(p) \end{cases},$$

where by interpolation, we here and henceforth mean interpolation of the explicitly defined functions using radial cutoff functions with bounded derivatives. Now blow up by ψ_c^{-4} so that γ approaches the cylindrical metric as $r \searrow 0$. That is, for $t = -\log r$, the metric $\gamma_c = \psi_c^{-4}\gamma$ decomposes as $\gamma_c|_{B_R(p)\setminus\{p\}} = dt^2 + h(e^{-t})$. We also multiply the transverse-traceless (ie traceless and divergence-free) parts μ and $-\mu$ of K and $-K$, respectively, by ψ_c^2.

- The cylinders are cut off at the parameter $t = T - \log(R)$ for T large, and, distinguishing data on the two $\mathbb{T}^3$'s by the subscripts 1 and 2, we glue by the rule: $(t_1, \theta_1) \sim (t_2, \theta_2)$ if $t_2 = T - 2\log(R) - t_1$ and $\theta_2 = -\theta_1$, where $(t_i, \theta_i) \in [-\log(R), T - \log(R)] \times [0, 2\pi)$. Note also that the identification $\theta_2 \sim -\theta_1$ is due to a reversal of orientation when gluing. On the manifold $M \approx \mathbb{T}^3 \# \mathbb{T}^3$, we define a new coordinate $s \in [-T/2, T/2]$, on the glued cylinder (denoted by C_T) by

$$s = t_1 + \log(R) - T/2 = T/2 - \log(R) - t_2.$$

New data are then constructed by cutoff functions as follows:

$$\gamma_T := \chi_1\gamma_1 + \chi_2\gamma_2, \quad \mu_T := \chi_1\mu_1 + \chi_2\mu_2, \quad \text{and} \quad K_T := \chi_1 K_1 + \chi_2 K_2,$$

where, similar to before, we use the subscripts 1 and 2 to denote restrictions of the conformally transformed data on each gluing region, and where $\{\chi_1, \chi_2\}$ is a partition of unity with respect to an open cover of M whose intersection consists of $\{(s, \theta) \in C_T : s \in (-1, 1)\}$. Note that while it may seem redundant to define both μ_T and K_T on the gluing cylinder since $\operatorname{tr} K \equiv 0$ in the neighborhoods we're considering, we still make some use of K_T to define the glued data at the very end, since $\operatorname{tr} K \not\equiv 0$ outside of the gluing region. We also define a new conformal factor

$$\psi_T = \widetilde{\chi}_1\psi_1 + \widetilde{\chi}_2\psi_2,$$

where $\widetilde{\chi}_1, \widetilde{\chi}_2$ are cutoff functions such that $\widetilde{\chi}_i|_{\mathbb{T}^3_j \setminus B_R(p_j)} = \delta_{ij}$ and on C_T,

$$\widetilde{\chi}_1(s,\theta) = \begin{cases} 1 & s \in [-T/2, T/2-1) \\ \text{interpolation} & s \in [T/2-1, T/2-1/2) \\ 0 & s \in [T/2-1/2, T/2] \end{cases}, \quad \text{and}$$

$$\widetilde{\chi}_2(s,\theta) = \begin{cases} 0 & s \in [-T/2, 1/2-T/2] \\ \text{interpolation} & s \in (1/2-T/2, 1-T/2] \\ 1 & s \in (1-T/2, T/2] \end{cases}.$$

Remark 1. All conformal factors and cutoff functions/partitions of unity must be chosen so that M satisfies the following symmetry:

1. There exists a diffeomorphism $\beta : M \to M$ that takes a point on one $\mathbb{T}^3$ to the corresponding one on the other $\mathbb{T}^3$. In particular, on the gluing neck, $\beta(s,\theta) = (-s,\theta)$, so the cross-section $s = 0$ is fixed by β.
2. The reflection β satisfies: $\beta^*\gamma_T = \gamma_T$ and $\beta^* K_T = -K_T$.

In particular, on the gluing neck, we must have that $\chi_1(s,\theta) = \chi_2(-s,\theta)$ and $\widetilde{\chi}_1(s,\theta) = \widetilde{\chi}_2(-s,\theta)$.

Remark 2. For the purposes of this paper, and in accord with the construction in [14], we call T the *gluing parameter*. We expect that the geometry of the central gluing neck of the resulting initial data set behaves like a small perturbation of a neighborhood of the minimal 2-sphere in a time-symmetric slice of the Schwarzschild spacetime with mass $m_T \sim Ce^{-\alpha T}$ for positive constants C and α which are independent of T. Existence of solutions to the conformally modified momentum and Hamiltonian constraints follows from perturbation arguments as $T \to \infty$, where we see a degeneration in the geometry of the initial data sets.

At this stage, reversing the asymptotically cylindrical blowing-up process, we have data for an approximate solution:

$$(M, \gamma_T, K_T) = \left(\mathbb{T}^3 \# \mathbb{T}^3,\ \psi_T^4 \gamma_T,\ \psi_T^{-2}\mu_T + \frac{\mathrm{tr}_{\gamma_T} K_T}{3} \psi_T^4 \gamma_T \right).$$

The remaining steps in the CIP gluing construction are to perturb μ_T and ψ_T using the conformal method so that the resulting data

$$(1) \qquad (M, \widetilde{\gamma}_T, \widetilde{K}_T) = \left(\mathbb{T}^3 \# \mathbb{T}^3,\ \widetilde{\psi}_T^4 \gamma_T,\ \widetilde{\psi}_T^{-2}\widetilde{\mu}_T + \frac{\mathrm{tr}_{\gamma_T} K_T}{3} \widetilde{\psi}_T^4 \gamma_T \right)$$

solve the vacuum Einstein equations. The most technical and detailed aspects of finding $\widetilde{\mu}_T$ and $\widetilde{\psi}_T$ are described in [14] Sections 3-6, while Sections 2-4 of [8] sketch the modifications one must make in order to localize the construction: in particular, one must solve boundary value problems with the elliptic operators in question, and then apply a smoothing procedure from [6]. Since we will make use of estimates on these perturbations, we sketch some of the analysis involved:

- The new tensor μ_T is perturbed by solving a boundary value problem with the vector Laplacian so that the resulting tensor, $\widetilde{\mu}_T := \mu_T - \sigma_T$, is transverse-traceless with respect to γ_T in the gluing region (where τ is constant) and $\sigma_T = 0$ on the boundary—since $\widetilde{\mu}_T$ is only different from μ_1 and μ_2 in the very center of the neck where the latter two are interpolated, the perturbation is only needed in that area. Because all the relevant differential operators preserve the symmetry of Remark 1, all analysis can be done in function spaces where the reflection symmetry is preserved.
- Using a contraction mapping argument, the conformal factor ψ_T is perturbed so that the resulting function, $\widetilde{\psi}_T := \psi_T + \eta_T$, satisfies the Lichnerowicz equation (with respect to γ_T with boundary conditions fixing η_T to be 0 on the boundary. Again, all analysis is done in function spaces that preserve the reflection symmetry). In addition, the perturbation term η_T satisfies the following weighted Hölder estimate:

$$||\eta_T||_{k+2,\alpha,\delta} := ||w_T^{-\delta}\eta_T||_{k+2,\alpha} \leq Ce^{-T/4}, \tag{2}$$

  where $k \in \mathbb{Z}_{\geq 0}$, $\delta \in (0,1)$, $w_T|_{C_T} := e^{-T/4}\cosh(s/2)$, the unweighted Hölder norm is defined as in Definition 2 of [14], and $C > 0$ is independent of T.
- Applying the above procedure for small enough R, a compactly supported, smooth deformation procedure per [6] is applied across annuli about the boundaries of the gluing neighborhoods. This perturbation agrees with the IMP construction near the middle of the gluing neck and the original data near the gluing neighborhood boundaries.

We now have that the data $(M, \widetilde{\gamma}_T, \widetilde{K}_T)$ as in Equation 1 satisfies the Einstein vacuum constraint equations as well as the symmetry of Remark 1. When convenient, we suppress the dependence on T and denote the final initial data set by $(M, \widetilde{\gamma}, \widetilde{K})$. However, for the above analysis as well as the bounds we use below, it is necessary that T be sufficiently large.

3. Proof of the Theorem

Let $\widetilde{\Sigma}$ be the cross-section $\{(s, \theta) \in C_T : s = 0\}$ with data induced by $(M, \widetilde{\gamma}, \widetilde{K})$. Using the symmetry of Remark 1, we demonstrate below that $\widetilde{\Sigma}$ is a MOTS and a MITS. First note that

$$\widetilde{K}|_{\widetilde{\Sigma}} = \beta^* \widetilde{K}|_{\widetilde{\Sigma}} = -\widetilde{K}|_{\widetilde{\Sigma}},$$

so $\widetilde{K}|_{\widetilde{\Sigma}} \equiv 0$. Likewise, if we let ν be the unit normal pointing in the positive s direction (since orthogonality properties of the original metric are preserved under conformal transformations, the unit normal to $\widetilde{\Sigma}$ after the final conformal transformation is a rescaling of $\frac{\partial}{\partial s}$), and if we let H_+ and H_- be the mean curvatures of $\widetilde{\Sigma}$ associated to ν and $-\nu$, respectively, we have

$$H_+ = \beta^* H_+ = H_- = -H_+,$$

so $H_{\widetilde{\Sigma}} \equiv 0$. Thus, $\widetilde{\Sigma}$ is a spacetime minimal surface, and in particular satisfies the MOTS and MITS equation

$$\mathrm{tr}_{\widetilde{\Sigma}} \widetilde{K} \pm H_{\widetilde{\Sigma}} = 0.$$

Now consider the following covering space of M: given one of the tori at the beginning of the gluing construction, take a universal cover and on each copy of the gluing neighborhood, identically glue in the other torus (using pullback data on the universal cover of the first torus) as described above. Call the resulting space $\mathcal{N}$. Fixing a single copy of $\widetilde{\Sigma}$ in this covering space, we are in the situation of Proposition 1.1 of [7]:

- i Because the CIP construction is vacuum, it trivially satisfies the null energy condition, and $\mathcal{N}$ is a noncompact Cauchy surface for its spacetime evolution.
- ii The hypersurface $\widetilde{\Sigma}$ is a closed, connected MOTS, and its complement in $\mathcal{N}$ consists of two disjoint open sets, one of which has noncompact closure (without loss of generality, let ν point toward this end).
- iii We must show that either the null second fundamental form χ of $\widetilde{\Sigma}$ is not identically zero, that $\widetilde{\Sigma}$ is strictly stable, or that there exists a null geodesic along which a certain genericity condition holds.

We now show that the last item is satisfied; in particular, we will show that either the first or second condition of the (iii) holds.

Suppose that $\chi \equiv 0$. It suffices to show that there exists a function $\phi \in C^\infty(\widetilde{\Sigma})$ such that $L\phi > 0$, where $L : C^\infty(\widetilde{\Sigma}) \to C^\infty(\widetilde{\Sigma})$ is the MOTS stability operator (see [1]):

$$\begin{aligned} L\phi :&= -\Delta\phi + 2\langle X, \nabla\phi\rangle \\ &\quad + \left(\frac{1}{2}R_{\widetilde{\Sigma}} - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2 + \operatorname{div} X - |X|^2\right)\phi, \end{aligned}$$

and where all differential operators and inner products are taken with respect to the induced metric on $\widetilde{\Sigma}$, $R_{\widetilde{\Sigma}}$ is the scalar curvature of $\widetilde{\Sigma}$, μ and J are the respective energy and momentum densities, and $X := \left(\widetilde{K}(\nu, \cdot)|_{T\widetilde{\Sigma}}\right)^\sharp$. Now in our case, μ and J are both zero because $\mathcal{N}$ is vacuum, and χ disappears by assumption. In addition, since $\widetilde{K}|_{\widetilde{\Sigma}} \equiv 0$ and all derivatives are taken with respect to $\widetilde{\Sigma}$, all X terms disappear, whence the stability operator simplifies to

$$L\phi = -\Delta\phi + \frac{1}{2}R_{\widetilde{\Sigma}}\phi.$$

Let $\phi \equiv 1$, so we are left to show that $R_{\widetilde{\Sigma}} > 0$. But now recall that before the final conformal transformation in the IMP gluing construction, the spherical cross sections close to the middle of the neck have metrics that are arbitrarily close to the standard spherical metric for T large. Thus, we may choose T large enough so that the scalar curvature of the $s = 0$ slice is positive. Denote the $s = 0$ slice prior to the final conformal transformation by (Σ, h)—that is, the data on Σ is induced by (M, γ_T, K_T). Then using the formula for scalar curvature after a conformal transformation, we see the scalar curvature of $\widetilde{\Sigma}$ is given by:

$$\begin{aligned} R_{\widetilde{\Sigma}} &= (\widetilde{\psi}_T)^{-4}\left(R_\Sigma - 4\Delta(\log(\widetilde{\psi}_T))\right) \\ &= (\widetilde{\psi}_T)^{-4}\left(R_\Sigma + 4\left(\frac{||\nabla\eta_T||^2}{\widetilde{\psi}_T^2} - \frac{\Delta\eta_T}{\widetilde{\psi}_T}\right)\right), \end{aligned}$$

where all derivatives and inner products are taken with respect to the induced metric on Σ. Note that the second equality follows because $\psi_T|_{C_T}$ is a function of s, so it is constant on Σ. We must show that the last two terms can be made arbitrarily small for T large. From our definitions of ψ_T and w_T in Section 2, we see that

$$\psi_T|_\Sigma = \psi_T(0) = 2e^{\log(R)/2}e^{-T/4} \quad \text{and} \quad w_T|_\Sigma = e^{-T/4}.$$

Combining these with the bound on η_T in (2), we obtain

$$||\eta_T||_{k+2}^{\Sigma} \lesssim e^{-(1+\delta)T/4},$$

which yields

$$||\eta_T||_{k+2}^{\Sigma} \lesssim e^{-\delta T/4}\psi_T|_{\Sigma}$$

for $\delta \in (0,1)$. Thus, for T sufficiently large, we indeed have that $||\nabla\eta_T||^2$ and $|\Delta\eta_T|$ are negligible compared with $\widetilde{\psi}_T^2$ and $\widetilde{\psi}_T$, respectively, and hence $R_{\widetilde{\Sigma}}$ is positive as desired. Thus, if $\chi \equiv 0$, we have $\widetilde{\Sigma}$ is a strictly stable MOTS, so (iii) is satisfied, and we conclude that any spacetime evolution of $\mathcal{N}$ is future null geodesically incomplete.

It remains to show that any spacetime evolution $\widehat{M}$ of M is future null geodesically incomplete. We use the following lemma from [12]:

Lemma 2. *Let $(M,\widetilde{\gamma},\widetilde{K})$ be a smooth spacelike Cauchy surface in a spacetime $(\widehat{M},g)$, and suppose $\pi : \mathcal{N} \to M$ is a Riemannian covering map. Then there exists a Lorentzian covering map $\widehat{\pi} : \widehat{\mathcal{N}} \to \widehat{M}$ extending π such that $(\mathcal{N},\pi^*\widetilde{\gamma},\pi^*\widetilde{K})$ is a Cauchy surface for the spacetime $\widehat{\mathcal{N}}$.*

$$\begin{tikzcd}
(\mathcal{N},\pi^*\widetilde{\gamma},\pi^*\widetilde{K}) \arrow[r, hook] \arrow[d, "\pi"] & (\widehat{\mathcal{N}},\widehat{\pi}^*g) \arrow[d, "\widehat{\pi}"] \\
(M,\widetilde{\gamma},\widetilde{K}) \arrow[r, hook] & (\widehat{M},g)
\end{tikzcd}$$

Now suppose $\widehat{M}$ is future null geodesically complete. Since $\widehat{\mathcal{N}}$ is future null geodesically incomplete, there exists a future inextendible smooth null geodesic $\zeta : [0,\alpha) \to \widehat{\mathcal{N}}$ that terminates at affine parameter $\alpha < \infty$. Consider the smooth null geodesic $\widehat{\pi}(\zeta) \subset \widehat{M}$. Then by future null completeness of $\widehat{M}$, we must be able to find a smooth null geodesic $\widehat{\zeta} : [0,\infty) \to \widehat{M}$ extending $\widehat{\pi}(\zeta)$. Let $\varepsilon > 0$ be small enough so that $\widehat{\zeta}(\alpha-\varepsilon,\alpha+\varepsilon)$ is contained in a single evenly-covered neighborhood $U \subset \widehat{M}$, and pick a smooth local section $\sigma : U \to \widehat{\mathcal{N}}$ of the covering such that $\zeta(t) = \sigma\circ\widehat{\pi}\circ\zeta(t)$ for every $t \in (\alpha-\varepsilon,\alpha)$. Then we see that the smooth null geodesic $\xi : [0,\alpha+\varepsilon) \to \widehat{\mathcal{N}}$ defined by

$$\xi(t) := \begin{cases} \zeta(t) & t \in [0,\alpha) \\ \sigma\circ\widehat{\zeta}(t) & t \in (\alpha-\varepsilon,\alpha+\varepsilon) \end{cases}$$

extends ζ to the future, which contradicts our assumption on ζ. Thus, we must indeed have that any spacetime evolution of M is future null geodesically incomplete.

Lastly, note that since $\widetilde{\Sigma}$ is also marginally inner trapped we may take a time reversal of the above argument—using a covering space that "unwraps" the other torus—to conclude past null geodesic incompleteness. $\square$

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