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On the total mass of asymptotically hyperbolic manifolds*

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It is a pleasure to dedicate this work to Robert Bartnik on the occasion of his 60th birthday

Abstract: Generalising a proof by Bartnik in the asymptotically Euclidean case, we give an elementary proof of positivity of the hyperbolic mass near the hyperbolic space.

1. Introduction

The question of positivity of total energy in general relativity has turned out to be a particularly challenging problem (cf. [14] and references therein), with several open questions remaining. It therefore appears of interest to provide simple proofs when available.

In his well-known paper on the mass of asymptotically Euclidean manifolds [2], Robert Bartnik gave an elementary proof of positivity of the ADM mass near the Euclidean metric. Inspired by his work, we establish a similar result for the hyperbolic mass near the hyperbolic metric. The argument turns out to be somewhat more involved and calculation-intensive.

Indeed, we provide an elementary proof of positivity of the hyperbolic mass, near the hyperbolic space, for metrics with scalar curvature bounded below by that of the hyperbolic space. Namely, ignoring an overall dimension-dependent constant, consider the usual definition (cf., e.g. [7]) of the mass m of a metric g asymptotic to a metric $\overline{g}$ with a static KID V (see below for terminology):

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$$m(V) = \lim_{R\to\infty} \int_{r=R} [V g^{mj} g^{i\ell} \left(\overline{D}_m g_{j\ell} - \overline{D}_\ell g_{jm}\right) + (g^{mj} g^{ki} - g^{ij} g^{km})(g_{jm} - \overline{g}_{jm})\overline{D}_k V] d\sigma_i \,. \tag{1.1}$$

We prove the following:

Theorem 1.1. *For $n \geq 3$, let $(M, \overline{g})$ be $\mathbb{R}^n$ equipped with the hyperbolic metric,*

$$\overline{g} = \frac{dr^2}{1+r^2} + r^2 d\Omega^2 \,, \tag{1.2}$$

where $d\Omega^2$ is the canonical metric on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$. Let $(A^0, \vec{A}) \in \mathbb{R}^{n+1}$ satisfy $|\vec{A}| := \sqrt{(A^1)^2 + \ldots + (A^n)^2} \leq A^0$ and set

$$V = A^0 \sqrt{1+r^2} + \sum_i A^i x^i \,. \tag{1.3}$$

Let g be a metric on M asymptotic to $\overline{g}$ with well-defined total mass m. There exists $\delta > 0$ such that if

$$\|g - \overline{g}\|_{L^\infty} + \|\overline{D} g\|_{L^\infty} < \delta \,,$$

where $\overline{D}$ is the covariant derivative operator of $\overline{g}$, then g can be put into the gauge

$$\check{\psi}^j := \overline{D}_i g^{ij} - \frac{1}{2} g^{jk} \overline{g}_{\ell m} \overline{D}_k g^{\ell m} = 0 \tag{1.4}$$

in which we have

$$m(V) \;\geq\; \int_M \left[R - \overline{R} + \frac{1}{8n} |\overline{D} g|^2_{\overline{g}}\right] V d\mu_{\overline{g}} \tag{1.5}$$

where, in local coordinates, $d\mu_{\overline{g}} = \sqrt{\det \overline{g}}\, d^n x$.

It follows clearly from (1.5) that $m(V) \geq 0$ if

$$R \geq \overline{R} \,. \tag{1.6}$$

Equivalently, if we set

$$m_0 := m(V = \sqrt{1+r^2}) \,, \quad m_i := m(V = x^i) \,, \tag{1.7}$$

then, under (1.6), the vector (m_μ) is timelike future-pointing with respect to the Lorentzian quadratic form $m_0^2 - m_1^2 \dots - m_n^2$. The inequality (1.6) holds of course for general relativistic initial data sets with vanishing trace of extrinsic curvature and with matter fields satisfying the dominant energy condition. Note that in vacuum, or in the presence of matter fields satisfying well behaved equations, under suitable further smallness assumptions on the extrinsic curvature of the initial data surface and on the matter fields, the condition of vanishing of the trace of the extrinsic curvature can be enforced by moving slightly the initial data hypersurface in space-time, after invoking the implicit-function theorem.

Theorem 1.1 is, essentially, a special case of Theorem 3.1 below, with the constants coming from (3.15). At the heart of its proof lies the identity, which we derive below and which holds for any asymptotically hyperbolic background $(M, \overline{g})$ with a static KID V, under the usual conditions for existence of the mass:

$$
\begin{aligned}
(1.8) \qquad m \;=\; & \int_M \Big[(R-\overline{R})V + \Big(\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}} + \frac{1}{4}|\overline{D}\hat{h}|^2_{\overline{g}} \\
& -\frac{1}{2}\hat{h}^{i\ell}\hat{h}^{jm}\overline{R}_{\ell mij} - \frac{n+2}{2n}\phi\hat{h}^{ij}\overline{R}_{ij} - \frac{n^2-4}{8n^2}\lambda\phi^2 \\
& -\frac{1}{2}(|\check{\psi}|^2_{\overline{g}} - \check{\psi}^i\overline{D}_i\phi)\Big)V + \Big(h^k{}_i\check{\psi}^i + \frac{1}{2}\phi\check{\psi}^k\Big)\overline{D}_k V \\
& +\Big(O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big)V \\
& +O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\Big]d\mu_{\overline{g}}\,;
\end{aligned}
$$

see (3.1)-(3.4) for notation. Throughout this work, the reader can assume that indices are raised and lowered using the background metric $\overline{g}$. We then use a weighted Poincaré inequality to control the non-obviously-positive terms in (1.8).

The calculations leading to (1.8), presented in Section 4, are vaguely reminiscent of those in [1], but the relation of the formulae presented there to the hyperbolic mass is not clear.

We made an attempt to use similar ideas for perturbations of the Horowitz-Myers instantons [8, 11], with only partial results so far [3].

Remark 1.2. The Birmingham-Kottler [12, 4] metrics with zero mass,

$$
(1.9) \qquad g = -\left(\frac{r^2}{\ell^2} + \kappa\right)dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + \kappa} + r^2 h_\kappa\,,
$$

with constants $\ell > 0$ and $\kappa \in \{0, \pm 1\}$, where $({}^{n-1}N, h_\kappa)$ is an $(n-1)$-dimensional space form with Ricci scalar equal to $(n-1)(n-2)\kappa$, are space forms. Therefore all the *calculations* here apply verbatim to the case of toroidal and hyperbolic conformal boundary at infinity for such metrics. There are, however, issues with the gauge, boundaries, and the weighted Poincaré inequality which would need to be addressed to be able to obtain a positivity result:

1. In the $\kappa = 0$ case the associated manifold $(0, \infty) \times {}^{n-1}N$ is complete with one locally asymptotically hyperbolic end, where $r \to \infty$, and one cuspidal end, where $r \to 0$. Since the manifold is complete without boundary, the proof of existence of the gauge should go through for perturbations which vanish in the cuspidal end, but requires checking. We note that positivity of the mass in the spin case has been established in whole generality by Wang [15], using a variation of Witten's proof, and in [6] in dimension $n \leq 7$, but the non-spin higher dimensional case remains open.
2. In the case $\kappa = -1$ the manifold of interest is $[\ell, \infty) \times {}^{n-1}N$, where the boundary $\{\ell\} \times {}^{n-1}N$ satisfies a mean-curvature inequality. If the perturbations are *not* supported away from the boundary there will be terms arising from integration by parts which are likely to destroy positivity, since in this case there exist well behaved solutions with negative mass.

2. Static KIDs

Let $(M, \overline{g})$ be a smooth n-dimensional Riemannian manifold, $n \geq 2$ and let V be a *static KID* on $(M, \overline{g})$, i.e. V is a solution to

$$\overline{D}_i \overline{D}_j V = V \left(\overline{R}_{ij} - \frac{\overline{R}}{n-1} \overline{g}_{ij} \right) . \tag{2.1}$$

When $\overline{g}$ has constant scalar curvature, an equivalent form is

$$\Delta_{\overline{g}} V + \lambda V = 0 \,, \quad \overline{D}_i \overline{D}_j V = V(\overline{R}_{ij} - \lambda \overline{g}_{ij}) \,, \tag{2.2}$$

for some constant $\lambda \in \mathbb{R}$. Here $\overline{R}_{ij}$ denotes the Ricci tensor of the metric $\overline{g}$, $\overline{D}$ the Levi-Civita connection of $\overline{g}$, and $\Delta_{\overline{g}} = \overline{D}^k \overline{D}_k$ is the Laplacian of $\overline{g}$.

When $\lambda < 0$, rescaling $\overline{g}$ by a constant factor if necessary, when the background metric has constant scalar curvature we can without loss of generality assume that $\lambda = -n$ so that

$$\overline{R} := \overline{g}^{ij}\overline{R}_{ij} = \lambda(n-1) = -n(n-1)\,.$$

If $\overline{g}$ is an Einstein metric, namely $\overline{R}_{ij}$ proportional to $\overline{g}_{ij}$, using this last scaling we obtain

$$\overline{R}_{ij} = -(n-1)\overline{g}_{ij}\,, \quad \overline{D}_i\overline{D}_j V = V\overline{g}_{ij}\,. \tag{2.3}$$

This implies

$$\overline{D}_i(|\overline{D}V|^2_{\overline{g}} - V^2) = 0\,, \tag{2.4}$$

where $|\cdot|_{\overline{g}}$ denotes the norm of a tensor with respect to a metric $\overline{g}$. In hyperbolic space, where the sectional curvatures are minus one, and when V takes the form (1.3) in the coordinate system of (1.2), we have

$$|\overline{D}V|^2_{\overline{g}} - V^2 = |\vec{A}|^2 - (A^0)^2\,. \tag{2.5}$$

3. The theorem

It is convenient to introduce some notation:

$$h_{ij} := g_{ij} - \overline{g}_{ij}\,, \tag{3.1}$$

$$\psi^j := \overline{D}_i g^{ij} \quad \Longleftrightarrow \quad g^{ij}\overline{D}_i h_{j\ell} = -g_{\ell j}\psi^j\,, \tag{3.2}$$

$$\phi := g^{ij}h_{ij} \quad \Longrightarrow \quad \overline{\phi} := \overline{g}^{ij}h_{ij} = \phi + O\left(|h|^2_{\overline{g}}\right)\,. \tag{3.3}$$

We will denote by $\check{h}$, respectively by $\hat{h}$, the g-trace-free, respectively the $\overline{g}$-trace-free, part of h:

$$\check{h}_{ij} := h_{ij} - \frac{1}{n}\phi\, g_{ij}\,, \qquad \hat{h}_{ij} := h_{ij} - \frac{1}{n}\overline{\phi}\,\overline{g}_{ij}\,. \tag{3.4}$$

In order to address the question of gauge-freedom, we will apply a diffeomorphism to g so that

$$\check{\psi}^i := \psi^i + \frac{1}{2}g^{ik}\overline{D}_k\phi \tag{3.5}$$

vanishes. Note that the equation $\check{\psi}^i = 0$ reduces to the harmonic-coordinates condition in the case of a flat background, where $\lambda = 0$.

We claim the following:

Theorem 3.1. *Let (M, g) asymptote to an asymptotically hyperbolic space-form $(M, \overline{g})$ and let V be a static KID of $(M, \overline{g})$. Suppose that the usual decay conditions needed for a well-defined mass [7] are satisfied, namely, for large r, in the coordinate system of (1.2),*

$$h_{ij} = o(r^{-n/2}), \quad \overline{D}_k h_{ij} = O(r^{-n/2}), \quad V = O(r), \quad |\overline{D}_k h_{ij}|^2_{\overline{g}} V \in L^1, \quad (R - \overline{R})V \in L^1. \tag{3.6}$$

There exists $\delta > 0$ such that if

$$\|h\|_{L^\infty} + \|\overline{D}h\|_{L^\infty} < \delta$$

and if $|dV|_{\overline{g}} \leq V$, then we have

$$m \geq \int_M \left[R - \overline{R} + \frac{n-2}{8n}|\overline{D}h|^2_{\overline{g}}\right] V d\mu_{\overline{g}} - \frac{1}{2}\int_M \left((|\check{\psi}|^2_{\overline{g}} - \check{\psi}^i \overline{D}_i \phi)V - \left(2h^k{}_i \check{\psi}^i + \phi\check{\psi}^k\right)\overline{D}_k V\right) d\mu_{\overline{g}}. \tag{3.7}$$

A sharper bound can be found in (3.15) below.

It follows from (2.5) that $|dV|_{\overline{g}} \leq V$ holds for static KIDs as in the statement of the theorem. It is well known (cf. e.g., the proof of [5, Theorem 4.5]; compare [10, 13]) that the gauge $\check{\psi}^k = 0$ can always be realised when g is close enough to the hyperbolic metric $\overline{g}$. Hence Theorem 1.1 is indeed a corollary of Theorem 3.1.

Proof. In Section 4 we prove the identity

$$V\left(R - \overline{R}\right) = \overline{\mathcal{D}} + \mathcal{Q} - (h^k{}_i \check{\psi}^i + \frac{1}{2}\phi\check{\psi}^k)\overline{D}_k V, \tag{3.8}$$

where

$$\overline{\mathcal{D}} := \overline{D}_i[V g^{mj} g^{i\ell}\left(\overline{D}_m h_{j\ell} - \overline{D}_\ell h_{jm}\right)] + \overline{D}_i\left[(g^{mj}g^{ki} - g^{ij}g^{km})h_{jm}\overline{D}_k V\right] + \frac{1}{2}\overline{D}_i \underbrace{\left[V\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell} - g^{ik}\overline{D}_j g^{j\ell}\right)\right]}_{(\diamond)} \tag{3.9}$$

$$+\frac{1}{2}\overline{D}_i\left[\left(-3h^{i\ell}h_\ell{}^k+g^{ik}|h|^2_{\overline{g}}\right)\overline{D}_kV\right]$$
$$+\frac{1}{2}\overline{D}_i\left[\left(h^{ki}\phi+\frac{1}{4}\overline{g}^{ki}\phi^2\right)\overline{D}_kV\right]$$

is the sum of all divergence terms and $\mathcal{Q}$ is the sum of all quadratic or higher order terms:

$$\begin{aligned}\mathcal{Q}=\Big(&-\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}}-\frac{1}{4}|\overline{D}\hat{h}|^2_{\overline{g}}\\&+\frac{1}{2}\hat{h}^{i\ell}\hat{h}^{jm}\overline{R}_{\ell mij}+\frac{n+2}{2n}\phi\hat{h}^{ij}\overline{R}_{ij}+\frac{n^2-4}{8n^2}\lambda\phi^2\\&+\frac{1}{2}(|\check{\psi}|^2_{\overline{g}}-\check{\psi}^i\overline{D}_i\phi)+O\left(|h|^3_{\overline{g}}\right)+O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big)V\\&+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\,.\end{aligned}\tag{3.10}$$

Here the Riemann tensor can be replaced by the Weyl tensor, and the Ricci-tensor by its trace-free part.

We note that the term $(\diamond)$ in (3.9) is quadratic in $(h,\overline{D}h)$:

$$\begin{aligned}\overline{g}_{k\ell}\left(g^{jk}\overline{D}_jg^{i\ell}-g^{ik}\overline{D}_jg^{j\ell}\right)&=(g_{k\ell}-h_{k\ell})\left(g^{jk}\overline{D}_jg^{i\ell}-g^{ik}\overline{D}_jg^{j\ell}\right)\\&=-h_{k\ell}\left(g^{jk}\overline{D}_jg^{i\ell}-g^{ik}\overline{D}_jg^{j\ell}\right).\end{aligned}\tag{3.11}$$

It is then easy to see that the integral of the divergence term $\overline{\mathcal{D}}$ gives the total mass when integrated over the whole manifold, after taking into account the fact that the boundary conditions needed for a well-defined mass enforce a vanishing contribution of higher-than-linear terms in the boundary integral. This establishes (1.8).

We specialise now to the space-form version (3.10) of $\mathcal{Q}$, which reads

$$\begin{aligned}\mathcal{Q}=\Big[&-\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}}-\frac{1}{4}|\overline{D}\hat{h}|^2_{\overline{g}}+\frac{1}{2}|\hat{h}|^2_{\overline{g}}-\frac{n^2-4}{8n}\phi^2\\&+O\left(|h|^3_{\overline{g}}\right)+O(|h|_{\overline{g}}|\overline{D}h|^2_{\overline{g}})\Big]V\\&+\frac{1}{2}(|\check{\psi}|^2_{\overline{g}}-\check{\psi}^i\overline{D}_i\phi)V+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\,.\end{aligned}\tag{3.12}$$

In order to absorb the undifferentiated terms we use the weighted Poincaré inequality (A.8) below, namely

$$\int|\hat{h}|^2_{\overline{g}}V\,d\mu_{\overline{g}}\;\leq\;\frac{1}{n}\int\Big[(|\overline{D}\hat{h}|^2_{\overline{g}}-|\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}}-|\overline{\operatorname{div}}\,\hat{h}-\hat{h}_{dV}|^2_{\overline{g}})V\tag{3.13}$$

$$+\overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk})\Big]d\mu_{\overline{g}}\,.$$

with $\mathscr{D}$ defined in (A.1). This leads to

$$\begin{aligned}(3.14)\quad \int \mathcal{Q}d\mu_{\overline{g}} \leq \int \Big(-\Big[&\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}} + \frac{n-2}{4n}|\overline{D}\hat{h}|^2_{\overline{g}} + \frac{n^2-4}{8n}\phi^2\\ &+\frac{1}{2n}(|\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} + |\overline{\mathrm{div}}\,\hat{h} - \hat{h}_{dV}|^2_{\overline{g}})\Big]V\\ &+\frac{1}{2n}\overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk}) + \frac{1}{2}(|\check{\psi}|^2_{\overline{g}} - \check{\psi}^i\overline{D}_i\phi)V\\ &+O\left(|h|^3_{\overline{g}}\right)V + O(|h|_{\overline{g}}|\overline{D}h|^2_{\overline{g}}V) + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\Big)d\mu_{\overline{g}}\,.\end{aligned}$$

Hence

$$\begin{aligned}(3.15)\quad m \quad \geq \quad &\int_M \Big[R - \overline{R} + \frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}} + \frac{n-2}{4n}|\overline{D}\hat{h}|^2_{\overline{g}} + \frac{n^2-4}{8n}\phi^2\\ &+\frac{1}{2n}(|\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} + |\overline{\mathrm{div}}\,\hat{h} - \hat{h}_{dV}|^2_{\overline{g}})\Big]Vd\mu_{\overline{g}}\\ &-\int_M\left(\frac{1}{2}(|\check{\psi}|^2_{\overline{g}} - \check{\psi}^i\overline{D}_i\phi)V - \left(h^k{}_i\check{\psi}^i + \frac{1}{2}\phi\check{\psi}^k\right)\overline{D}_kV\right.\\ &\left.+O\left(|h|^3_{\overline{g}}\right)V + O(|h|_{\overline{g}}|\overline{D}h|^2_{\overline{g}}V) + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\right)d\mu_{\overline{g}}\,.\end{aligned}$$

It is now clear that we can choose $|h|_{\overline{g}} + |\overline{D}h|_{\overline{g}}$ small enough so that (3.7) holds. □

4. The Ricci scalar of asymptotically anti de-Sitter spacetimes

The aim of this section is to derive the curvature identities (3.8) and (3.12) needed in the proof of Theorem 3.1.

We consider the following metric

$$(4.1)\qquad g_{ij} = \overline{g}_{ij} + h_{ij}\,,$$

where $\overline{g}_{ki}$ is an anti de-Sitter metric. If we denote the connection of the background metric by $\overline{D}$, we have the relation

$$(4.2)\qquad D_k \equiv \overline{D}_k + \delta\Gamma_k\,,$$

where $\delta\Gamma_k$ is a $(1,2)$ tensor equal to $\delta\Gamma^{\cdot}_{\cdot k} := \Gamma^{\cdot}_{\cdot k} - \overline{\Gamma}^{\cdot}_{\cdot k}$. For example, applying D_k on a vector component v^i we get

$$D_k v^i = \overline{D}_k v^i + \delta\Gamma^i{}_{jk} v^j \,, \tag{4.3}$$

where

$$\begin{aligned} \delta\Gamma^i{}_{jk} &= \frac{1}{2} g^{i\ell} \left(\overline{D}_j g_{k\ell} + \overline{D}_k g_{\ell j} - \overline{D}_\ell g_{jk} \right) \\ &= \frac{1}{2} g^{i\ell} \left(\overline{D}_j h_{k\ell} + \overline{D}_k h_{\ell j} - \overline{D}_\ell h_{jk} \right) . \end{aligned} \tag{4.4}$$

The Riemann tensors of the metrics g_{ki} and $\overline{g}_{ki}$ are related to each other via the following equation

$$R^k{}_{imj} = \overline{R}^k{}_{imj} + \overline{D}_m \delta\Gamma^k{}_{ij} - \overline{D}_j \delta\Gamma^k{}_{im} + \delta\Gamma^k{}_{m\ell} \delta\Gamma^\ell{}_{ij} - \delta\Gamma^k{}_{j\ell} \delta\Gamma^\ell{}_{im} \,. \tag{4.5}$$

Contracting the first and third indices, one obtains

$$R_{ij} = \overline{R}_{ij} + \overline{D}_k \delta\Gamma^k{}_{ij} - \overline{D}_j \delta\Gamma^k{}_{ik} + \delta\Gamma^k{}_{k\ell} \delta\Gamma^\ell{}_{ij} - \delta\Gamma^k{}_{j\ell} \delta\Gamma^\ell{}_{ik} \,. \tag{4.6}$$

Inserting

$$\delta\Gamma^k{}_{ik} = \frac{1}{2} g^{k\ell} \left(\overline{D}_i h_{k\ell} + \overline{D}_k h_{i\ell} - \overline{D}_\ell h_{ki} \right) = \frac{1}{2} g^{k\ell} \overline{D}_i h_{k\ell} \tag{4.7}$$

into (4.6), we obtain

$$\begin{aligned} R_{ij} \;=\; & \overline{R}_{ij} + \frac{1}{2} \Bigg[\overline{D}_k g^{k\ell} \left(\overline{D}_i h_{j\ell} + \overline{D}_j h_{\ell i} - \overline{D}_\ell h_{ji} \right) \\ & + g^{k\ell} \left(\overline{D}_k \overline{D}_i h_{j\ell} + \overline{D}_k \overline{D}_j h_{\ell i} - \overline{D}_k \overline{D}_\ell h_{ji} \right) - \overline{D}_j g^{k\ell} \overline{D}_i h_{k\ell} \\ & - g^{k\ell} \overline{D}_j \overline{D}_i h_{k\ell} + \frac{1}{2} g^{kp} g^{\ell q} \overline{D}_\ell h_{pk} \left(\overline{D}_i h_{jq} + \overline{D}_j h_{iq} - \overline{D}_q h_{ij} \right) \\ & - \frac{1}{2} g^{kp} g^{\ell q} \left(\overline{D}_j h_{\ell p} + \overline{D}_\ell h_{pj} - \overline{D}_p h_{j\ell} \right) \left(\overline{D}_i h_{kq} + \overline{D}_k h_{iq} - \overline{D}_q h_{ki} \right) \Bigg] . \end{aligned} \tag{4.8}$$

And the Ricci scalar reads

(4.9)

$$\begin{aligned} R &= g^{ij}R_{ij} \\ &= \overline{R}_{ij}g^{ij} + \frac{1}{2}g^{ij}\Big[\overline{D}_k g^{k\ell}\left(2\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right) \\ &\quad + 2g^{k\ell}\left(\overline{D}_k\overline{D}_i h_{j\ell} - \overline{D}_k\overline{D}_\ell h_{ji}\right) \\ &\quad - \overline{D}_j g^{k\ell}\overline{D}_i h_{k\ell} + \frac{1}{2}g^{kp}g^{\ell q}\overline{D}_\ell h_{pk}\left(2\overline{D}_i h_{jq} - \overline{D}_q h_{ij}\right) \\ &\quad - \frac{1}{2}g^{kp}g^{\ell q}\left(\overline{D}_j h_{\ell p} + \overline{D}_\ell h_{pj} - \overline{D}_p h_{j\ell}\right)\left(\overline{D}_i h_{kq} + \overline{D}_k h_{iq} - \overline{D}_q h_{ki}\right)\Big]. \end{aligned}$$

Using

$$\begin{aligned} \frac{1}{2}g^{ij}g^{kp}g^{\ell q}\left(\overline{D}_j h_{\ell p} + \overline{D}_\ell h_{pj} - \overline{D}_p h_{j\ell}\right)\left(\overline{D}_i h_{kq} + \overline{D}_k h_{iq} - \overline{D}_q h_{ki}\right) \\ = \frac{1}{2}g^{ij}g^{kp}g^{\ell q}\overline{D}_p h_{j\ell}\left(2\overline{D}_q h_{ki} - \overline{D}_k h_{iq}\right), \end{aligned} \tag{4.10}$$

this can be rewritten as

$$\begin{aligned} R &= \overline{R}_{ij}g^{ij} + g^{ij}g^{k\ell}\left(\overline{D}_k\overline{D}_i h_{j\ell} - \overline{D}_k\overline{D}_\ell h_{ji}\right) \\ &\quad + \frac{1}{2}g^{ij}\Big[\overline{D}_k g^{k\ell}\left(2\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right) \\ &\quad - \overline{D}_j g^{k\ell}\overline{D}_i h_{k\ell} + \frac{1}{2}g^{kp}g^{\ell q}\Big[\overline{D}_\ell h_{pk}\left(2\overline{D}_i h_{jq} - \overline{D}_q h_{ij}\right) \\ &\quad - \overline{D}_p h_{j\ell}\left(2\overline{D}_q h_{ki} - \overline{D}_k h_{iq}\right)\Big]\Big]. \end{aligned} \tag{4.11}$$

In order to isolate the contribution of the mass we group all second-derivative terms in (4.9) in a divergence *with respect to the background metric* (similar to [7], except that there the divergence was taken with respect to the physical metric):

(4.12)

$$\begin{aligned} R &= \overline{R}_{ij}g^{ij} + \overline{D}_k[g^{ij}g^{k\ell}\left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right)] \\ &\quad - \overline{D}_k\left(g^{ij}g^{k\ell}\right)\left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right) \\ &\quad + \frac{1}{2}g^{ij}\Big[\overline{D}_k g^{k\ell}\left(2\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right) - \overline{D}_j g^{k\ell}\overline{D}_i h_{k\ell} \\ &\quad + \frac{1}{2}g^{kp}g^{\ell q}\left[\overline{D}_\ell h_{pk}\left(2\overline{D}_i h_{jq} - \overline{D}_q h_{ij}\right) - \overline{D}_p h_{j\ell}\left(2\overline{D}_q h_{ki} - \overline{D}_k h_{iq}\right)\right]\Big]. \end{aligned}$$

Note that

$$0 = \overline{D}_j \delta_i^k = \overline{D}_j (g^{kp} g_{pi}) = g_{pi} \overline{D}_j g^{kp} + g^{kp} \overline{D}_j g_{pi} \,,$$

equivalently

$$\overline{D}_j g^{kp} = -g^{\ell k} g^{ip} \overline{D}_j g_{i\ell} = -g^{\ell k} g^{ip} \overline{D}_j h_{i\ell} \,. \tag{4.13}$$

This allows us to rewrite (4.12) as

(4.14)

$$\begin{aligned} R &= \overline{R}_{ij} g^{ij} + \overline{D}_k [g^{ij} g^{k\ell} \left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right)] - \overline{D}_k \left(g^{ij} g^{k\ell} \right) \left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right) \\ &\quad + \frac{1}{2} g^{ij} \Big[- g^{kp} g^{\ell q} \overline{D}_k h_{pq} \left(2 \overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right) + g^{kp} g^{\ell q} \overline{D}_j h_{pq} \overline{D}_i h_{k\ell} \\ &\quad + \frac{1}{2} g^{kp} g^{\ell q} \left[\overline{D}_\ell h_{pk} \left(2 \overline{D}_i h_{jq} - \overline{D}_q h_{ij} \right) - \overline{D}_p h_{j\ell} \left(2 \overline{D}_q h_{ki} - \overline{D}_k h_{iq} \right) \right] \Big] \\ &= \overline{R}_{ij} g^{ij} + \overline{D}_k [g^{ij} g^{k\ell} \left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right)] \\ &\quad + \left(g^{k\ell} g^{ip} g^{jq} \overline{D}_k h_{pq} + g^{ij} g^{kp} g^{\ell q} \overline{D}_k h_{pq} \right) \left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right) \\ &\quad + \frac{1}{2} g^{ij} g^{kp} g^{\ell q} \Big[- \overline{D}_k h_{pq} \left(2 \overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right) + \overline{D}_j h_{pq} \overline{D}_i h_{k\ell} \\ &\quad + \frac{1}{2} \left[\overline{D}_\ell h_{pk} \left(2 \overline{D}_i h_{jq} - \overline{D}_q h_{ij} \right) - \overline{D}_p h_{j\ell} \left(2 \overline{D}_q h_{ki} - \overline{D}_k h_{iq} \right) \right] \Big] \,. \end{aligned}$$

After some simplifications one gets

$$R = \overline{R}_{ij} g^{ij} + \overline{D}_k [g^{ij} g^{k\ell} \left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji} \right)] + Q \,, \tag{4.15}$$

where

$$Q := \underbrace{\frac{1}{4} g^{ij} g^{kp} g^{\ell q} \left(2 \overline{D}_p h_{j\ell} \overline{D}_q h_{ki} \right.}_{=: Q_1} \left. - \overline{D}_\ell h_{kp} \overline{D}_q h_{ij} - \overline{D}_i h_{pq} \overline{D}_j h_{k\ell} \right) . \tag{4.16}$$

We note that

$$g^{ij} = \overline{g}^{ij} - h^{ij} + \chi^{ij} \,, \tag{4.17}$$

where

$$h^i{}_\ell = \overline{g}^{ik} h_{k\ell} \,, \quad h^{ij} = \overline{g}^{ik} \overline{g}^{j\ell} h_{k\ell} \,, \tag{4.18}$$

and

$$\chi^{ij} := \overline{g}^{ik}\overline{g}^{j\ell}\overline{g}^{mn}h_{km}h_{n\ell} + O(|h|^3_{\overline{g}}) = O(|h|^2_{\overline{g}})\,. \tag{4.19}$$

In the notation of (3.2)-(3.5), the identity (4.15) becomes

$$-\frac{1}{2}\overline{D}_k\left(g^{kl}D_l\phi\right) = R - \overline{R} + \overline{R}_{ij}h^{ij} - \underbrace{\overline{D}_k\left(g^{k\ell}h_{ji}\overline{D}_\ell g^{ij} - \check{\psi}^k\right) \underbrace{-Q}_{O(|h|^2+|\overline{D}h|^2)}}_{\text{“higher order terms”}}\,. \tag{4.20}$$

If both g and $\overline{g}$ satisfy the vacuum scalar constraint equation, so that $R = \overline{R}$, and in the gauge $\check{\psi}^i = 0$, (4.20) takes the form

$$\begin{aligned} -\frac{1}{2}\overline{D}_k\left(g^{kl}D_l\phi\right) - \frac{\overline{R}}{n}\phi \\ = \quad \overline{R}_{ij}\hat{h}^{ij} \underbrace{-\frac{\overline{R}}{n}(\phi-\overline{\phi}) - \overline{D}_k\left(g^{k\ell}h_{ji}\overline{D}_\ell g^{ij}\right) + O(|h|^2+|\overline{D}h|^2)}_{\text{“higher order terms”}}\,, \end{aligned} \tag{4.21}$$

which becomes an elliptic equation for ϕ when all “higher order terms” are thought to be negligible. Note that when $\overline{g}$ is a space-form the linear term at the right-hand side vanishes, which implies that ϕ itself is higher order. However, this is not true in general, in particular one cannot assume that $\phi = 0$ for general perturbations of e.g. the Horowitz-Myers metrics.

We return to the calculation of the mass. Let V be a static KID as in Section 2. Multiplying (4.15) by V we obtain

$$\begin{aligned} VR &= V\Big(\overline{R}_{ij}g^{ij} + \overline{D}_k[g^{ij}g^{k\ell}\left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right)] + Q\Big) \\ &= V\Big(\overline{R}_{ij}g^{ij} + Q\Big) + \sigma\,, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \sigma &:= V\overline{D}_k[g^{ij}g^{k\ell}\left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right)] \\ &= \overline{D}_k[Vg^{ij}g^{k\ell}\left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right)] \underbrace{-g^{ij}g^{k\ell}\left(\overline{D}_i h_{j\ell} - \overline{D}_\ell h_{ji}\right)\overline{D}_k V}_{=:*}\,. \end{aligned} \tag{4.23}$$

Then

$$* = \Big(-\overline{D}_i(g^{ij}g^{k\ell}h_{j\ell}\overline{D}_k V) + h_{j\ell}\overline{D}_i(g^{ij}g^{k\ell}\overline{D}_k V) \tag{4.24}$$

$$
\begin{aligned}
&+ \overline{D}_\ell(g^{ij}g^{k\ell}h_{ji}\overline{D}_k V) - h_{ji}\overline{D}_\ell(g^{ij}g^{k\ell}\overline{D}_k V)\Big)\\
= \;& \Big(\overline{D}_\ell((g^{ij}g^{k\ell} - g^{\ell j}g^{ki})h_{ji}\overline{D}_k V) + \underbrace{h_{j\ell}(g^{ij}g^{k\ell} - g^{\ell j}g^{ki})\overline{D}_i\overline{D}_k V}_{\text{used in (4.36)}}\\
&+ \underbrace{h_{j\ell}\overline{D}_i(g^{ij}g^{k\ell} - g^{\ell j}g^{ki})\overline{D}_k V}_{\text{first term in (4.29)}}\Big)\,.
\end{aligned}
$$

The last two terms in (4.16) are manifestly negative, which is the desired sign for our purposes. The part Q_1 of Q requires further manipulations, as follows:

$$
\begin{aligned}
(4.25)\; VQ_1 &= \frac{1}{2}Vg^{ij}g^{kp}g^{\ell q}\overline{D}_p h_{j\ell}\overline{D}_q h_{ki}\\
&= \frac{1}{2}V\overline{g}^{ij}\overline{D}_k h_j{}^{\ell}\overline{D}_\ell h^k{}_i + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)V\\
&= \frac{1}{2}V\overline{g}_{k\ell}\overline{D}_i g^{jk}\overline{D}_j g^{i\ell} + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)V\\
&= \frac{1}{2}V\Big\{\overline{D}_i\left[\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell} - g^{ik}\overline{D}_j g^{j\ell}\right)\right] + \overline{g}_{k\ell}\overline{D}_i g^{ik}\overline{D}_j g^{j\ell}\\
&\quad -\overline{g}_{k\ell}g^{ik}\left(\overline{R}_{mi}g^{m\ell} - \overline{R}^{\ell}{}_{mij}g^{jm}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big\}\,.
\end{aligned}
$$

In the notation of (4.17), Equation (4.25) becomes

$$
\begin{aligned}
(4.26)\; VQ_1 &= \frac{1}{2}V\Big\{\overline{D}_i\left[\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell} - g^{ik}\overline{D}_j g^{j\ell}\right)\right] + \overline{g}_{k\ell}\overline{D}_i g^{ik}\overline{D}_j g^{j\ell}\\
&\quad -\chi^{ij}\overline{R}_{ij} + h^i{}_\ell h^{jm}\overline{R}^{\ell}{}_{mij} + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big\}\\
&= \frac{1}{2}\Big\{\overline{D}_i\left[V\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell} - g^{ik}\overline{D}_j g^{j\ell}\right)\right]\\
&\quad \underbrace{-\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell} - g^{ik}\overline{D}_j g^{j\ell}\right)\overline{D}_i V}_{\text{second term in (4.29)}}\\
&\quad + \left(|\psi|^2_{\overline{g}} - \chi^{ij}\overline{R}_{ij} + h^i{}_\ell h^{jm}\overline{R}^{\ell}{}_{mij}\right)V\Big\}\\
&\quad + O\left(|h|^3_{\overline{g}}\right)V + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)V\,.
\end{aligned}
$$

In the special case where $\overline{g}$ is a (suitably normalised) hyperbolic space-form we have

$$
(4.27)\qquad \overline{R}^i{}_{jk\ell} = \frac{\overline{R}}{n(n-1)}\left(\delta^i_k\overline{g}_{j\ell} - \delta^i_\ell\overline{g}_{jk}\right) = -\left(\delta^i_k\overline{g}_{j\ell} - \delta^i_\ell\overline{g}_{jk}\right)\,,
$$

and the relations in (2.3) are satisfied. In this case (4.26) becomes

(4.28)
$$\begin{aligned} VQ_1 &= \frac{1}{2}\Big\{\overline{D}_i\left[V\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell}-g^{ik}\overline{D}_j g^{j\ell}\right)\right] \\ &\quad -\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell}-g^{ik}\overline{D}_j g^{j\ell}\right)\overline{D}_i V \\ &\quad +\left(|\psi|^2_{\overline{g}}-\phi^2+n|h|^2_g\right)V\Big\}+O\left(|h|^3_{\overline{g}}\right)V+O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)V. \end{aligned}$$

In order to simplify the expressions derived so far we consider similar terms separately:

1. We wish to add the second term of (4.26) and the third term of (4.24):

(4.29)
$$\begin{aligned} &h_{j\ell}\overline{D}_i(g^{ij}g^{k\ell}-g^{\ell j}g^{ki})\overline{D}_k V-\frac{1}{2}\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell}-g^{ik}\overline{D}_j g^{j\ell}\right)\overline{D}_i V \\ &\quad = h_{j\ell}\left(\psi^j g^{k\ell}+g^{ij}\overline{D}_i g^{k\ell}-\overline{D}_i g^{\ell j}g^{ki}-g^{\ell j}\psi^k\right)\overline{D}_k V \\ &\qquad +\frac{1}{2}\left(h_{k\ell}g^{jk}\overline{D}_j g^{i\ell}-h_{k\ell}g^{ik}\psi^\ell\right)\overline{D}_i V \\ &\quad = \frac{1}{2}\Big[h^k{}_j\psi^j+3h^i{}_\ell\overline{D}_i g^{k\ell}+g^{ki}\overline{D}_i|h|^2_{\overline{g}}-2\phi\psi^k+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)\Big]\overline{D}_k V \\ &\quad = \underbrace{\frac{1}{2}\Big[-2h^k{}_i\psi^i-2\phi\psi^k}_{=:\mathcal{A}^k\text{, taken care of in (4.34)}}\ \underbrace{-3\overline{D}_i\left(h^{i\ell}h_\ell{}^k\right)+g^{ki}\overline{D}_i|h|^2_{\overline{g}}}_{=:\mathcal{P}^k\text{, taken care of in (4.31)}} \\ &\qquad +O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)\Big]\overline{D}_k V\,, \end{aligned}$$

where we used

(4.30)
$$\begin{aligned} 3h^i{}_\ell\overline{D}_i g^{k\ell} &= -3h^i{}_\ell\overline{D}_i h^{k\ell}+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right) \\ &= -3\overline{D}_i\left(h^{i\ell}h_\ell{}^k\right)+3\overline{D}_i h^{i\ell}h_\ell{}^k+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right) \\ &= -3\overline{D}_i\left(h^{i\ell}h_\ell{}^k\right)-3h^k{}_i\psi^i+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right). \end{aligned}$$

We may rewrite the terms indicated by $\mathcal{P}$ in terms of total divergences as follows

(4.31)
$$\begin{aligned} \frac{1}{2}\mathcal{P}^k\overline{D}_k V &= \frac{1}{2}\overline{D}_i\left[\left(-3h^{i\ell}h_\ell{}^k+g^{ik}|h|^2_{\overline{g}}\right)\overline{D}_k V\right] \\ &\quad +\frac{1}{2}(3h^{i\ell}h_\ell{}^k-g^{ik}|h|^2_{\overline{g}})V\left(\overline{R}_{ik}-\lambda\overline{g}_{ik}\right)+O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V| \end{aligned}$$

$$= \frac{1}{2}\overline{D}_i\left[\left(-3h^{i\ell}h_\ell{}^k + g^{ik}|h|^2_{\overline{g}}\right)\overline{D}_k V\right] + \frac{1}{2}\Big\{3h^{i\ell}h_\ell{}^k\overline{R}_{ik} - \left[\overline{R} + \lambda(3-n)\right]|h|^2_{\overline{g}} + O\left(|h|^3_{\overline{g}}\right)\Big\}V + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|\,.$$

When $\overline{g}$ is Einstein, the result simplifies to:

$$\frac{1}{2}\mathcal{P}^k\overline{D}_k V = \frac{1}{2}\overline{D}_i\left[\left(-3h^{i\ell}h_\ell{}^k + g^{ik}|h|^2_{\overline{g}}\right)\overline{D}_k V\right] + \frac{1}{2}\left[(3-n)|h|^2_{\overline{g}} + O\left(|h|^3_{\overline{g}}\right)\right]V + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|\,. \tag{4.32}$$

Returning to the general case, in the notation of (4.29) we find

(4.33)

$$\begin{aligned}
\mathcal{A}^k\overline{D}_k V &\equiv -\left(h^k{}_i\psi^i + \phi\psi^k\right)\overline{D}_k V \\
&= -\left(h^k{}_i\check{\psi}^i + \phi\check{\psi}^k\right)\overline{D}_k V + \frac{1}{2}\left(h^k{}_i g^{i\ell}\overline{D}_\ell\phi + \phi g^{k\ell}\overline{D}_\ell\phi\right)\overline{D}_k V \\
&= -\left(h^k{}_i\check{\psi}^i + \phi\check{\psi}^k\right)\overline{D}_k V \\
&\quad + \frac{1}{2}\left[h^{k\ell}\overline{D}_\ell\phi + \frac{1}{2}\overline{g}^{k\ell}\overline{D}_\ell\phi^2 + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)\right]\overline{D}_k V \\
&= -\left(h^k{}_i\check{\psi}^i + \phi\check{\psi}^k\right)\overline{D}_k V + \frac{1}{2}\Bigg\{\overline{D}_\ell\left[\left(h^{k\ell}\phi + \frac{1}{2}\overline{g}^{k\ell}\phi^2\right)\overline{D}_k V\right] \\
&\quad + \underbrace{(\psi^k - \check{\psi}^k)\phi\overline{D}_k V}_{=-\frac{1}{4}\left[\overline{D}_\ell\left(\overline{g}^{k\ell}\phi^2\overline{D}_k V\right) - \overline{g}^{k\ell}\phi^2\overline{D}_\ell\overline{D}_k V + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\right]} + \check{\psi}^k\phi\overline{D}_k V \\
&\quad - (h^{k\ell}\phi + \frac{1}{2}\overline{g}^{k\ell}\phi^2)\overline{D}_k\overline{D}_\ell V + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\Bigg\}\,.
\end{aligned}$$

Using (2.2), we thus obtain

$$\begin{aligned}
\mathcal{A}^k\overline{D}_k V &= \underbrace{-\left(h^k{}_i\check{\psi}^i + \frac{1}{2}\phi\check{\psi}^k\right)\overline{D}_k V}_{=:\mathcal{G}} + \frac{1}{2}\overline{D}_\ell\left[\left(h^{k\ell}\phi + \frac{1}{4}\overline{g}^{k\ell}\phi^2\right)\overline{D}_k V\right] \\
&\quad + \frac{1}{2}\left[\left(\frac{n}{4}+1\right)\lambda\phi^2 - \frac{1}{4}\overline{R}\phi^2 - h^{k\ell}\overline{R}_{k\ell}\phi\right]V \\
&\quad + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}} + O\left(|h|^3_{\overline{g}}\right)V\,.
\end{aligned} \tag{4.34}$$

In the space-form case and using (2.3), (4.34) reads

(4.35)
$$\begin{aligned}\mathcal{A}^k\overline{D}_kV &= \underbrace{-\left(h^k{}_i\check{\psi}^i+\frac{1}{2}\phi\check{\psi}^k\right)\overline{D}_kV}_{=\mathcal{G}}+\frac{1}{2}\overline{D}_\ell\left[\left(h^{k\ell}\phi+\frac{1}{4}\overline{g}^{k\ell}\phi^2\right)\overline{D}_kV\right]\\ &\quad-\frac{1}{2}\left[\left(\frac{n}{4}+1\right)\phi^2V+O\left(|h|^2_{\overline{g}}|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\right]+O\left(|h|^3_{\overline{g}}\right)V.\end{aligned}$$

2. We can add the second term of (4.24) to the first term of (4.22), namely $V\overline{R}_{ij}g^{ij}$, using (2.2). Thus, we have

(4.36)
$$\begin{aligned}&V\overline{R}_{ij}g^{ij}+h_{j\ell}(g^{ij}g^{k\ell}-g^{\ell j}g^{ki})\overline{D}_i\overline{D}_kV\\ &\quad=\ V\overline{R}-h^{ij}\left(\overline{D}_i\overline{D}_jV+\lambda\overline{g}_{ij}V\right)+\overline{R}_{ij}\chi^{ij}\\ &\qquad+h_{j\ell}\left(\overline{g}^{ij}\overline{g}^{k\ell}-\overline{g}^{\ell j}\overline{g}^{ki}+\check{\chi}^{ijkl}\right)\overline{D}_i\overline{D}_kV\\ &\quad=\ V\left[\overline{R}+\left(\chi^{ij}+\hat{\chi}^{ij}\right)\overline{R}_{ij}-\lambda\hat{\chi}^{ij}\overline{g}_{ij}\right]\\ &\quad=\ V\Big[\overline{R}-h^i{}_\ell h^{\ell j}\overline{R}_{ij}+h^{ij}\overline{R}_{ij}\phi+\overline{R}|h|^2_{\overline{g}}\\ &\qquad-\lambda\left(\phi^2+(n-2)|h|^2_{\overline{g}}\right)+O\left(|h|^3_{\overline{g}}\right)\Big],\end{aligned}$$

where

(4.37)
$$\begin{aligned}\check{\chi}^{i\ell jk}\ :=\ &2\Big(-\overline{g}^{j[i}h^{\ell]k}-\overline{g}^{k[\ell}h^{i]j}+h^{j[i}h^{\ell]k}+\overline{g}^{j[i}\chi^{\ell]k}\\ &+\overline{g}^{k[\ell}\chi^{i]j}-h^{j[i}\chi^{\ell]k}-h^{k[\ell}\chi^{i]j}+\chi^{j[i}\chi^{\ell]k}\Big)\\ =\ &2\left(-\overline{g}^{j[i}h^{\ell]k}-\overline{g}^{k[\ell}h^{i]j}\right)+O(|h|^2),\end{aligned}$$

which possesses the algebraic symmetries of the Riemann tensor,

(4.38)
$$\check{\chi}^{i\ell jk}=-\check{\chi}^{\ell ijk}=-\check{\chi}^{i\ell kj}=\check{\chi}^{jki\ell},$$

and $\hat{\chi}^{ik}:=h_{j\ell}\check{\chi}^{i\ell jk}$. Then, we have

(4.39)
$$\begin{aligned}\hat{\chi}^{ik}\overline{g}_{ik}\ &=\ h_{j\ell}\left(\overline{g}^{j\ell}h^{ik}-\overline{g}^{ji}h^{\ell k}-\overline{g}^{k\ell}h^{ij}+\overline{g}^{ki}h^{\ell j}+O\left(|h|^2_{\overline{g}}\right)\right)\overline{g}_{ik}\\ &=\ \phi^2+(n-2)|h|^2_{\overline{g}}+O\left(|h|^3_{\overline{g}}\right).\end{aligned}$$

If $\overline{g}$ is space-form, using (2.3) and keeping in mind that $\lambda = -n$, (4.36) becomes

$$
\begin{aligned}
(4.40) \qquad & V\left[\overline{R} + \left(\chi^{ij} + \hat{\chi}^{ij}\right)\overline{R}_{ij} - \lambda\hat{\chi}^{ij}\overline{g}_{ij}\right] \\
&= V\left[\overline{R} + \chi^{ij}\overline{R}_{ij} - (n-1+\lambda)\hat{\chi}^{ij}\overline{g}_{ij}\right] \\
&= V\overline{R} + V\left[\phi^2 - |h|^2_{\overline{g}} + O\left(|h|^3_{\overline{g}}\right)\right].
\end{aligned}
$$

Summarizing we obtain, quite generally,

$$
(4.41) \qquad V\left(R - \overline{R}\right) \;=\; \overline{\mathcal{D}} + \mathcal{Q} + \mathcal{G}\,,
$$

where

(4.42)

$$
\begin{aligned}
\overline{\mathcal{D}} \;:=\; & \overline{D}_i[Vg^{mj}g^{i\ell}\left(\overline{D}_m h_{j\ell} - \overline{D}_\ell h_{jm}\right) + (g^{mj}g^{ki} - g^{ij}g^{km})h_{jm}\overline{D}_k V] \\
& + \frac{1}{2}\overline{D}_i\left[V\overline{g}_{k\ell}\left(g^{jk}\overline{D}_j g^{i\ell} - g^{ik}\overline{D}_j g^{j\ell}\right)\right] \\
& + \frac{1}{2}\overline{D}_i\left[\left(-3h^{i\ell}h_\ell{}^k + g^{ik}|h|^2_{\overline{g}}\right)\overline{D}_k V\right] \\
& + \frac{1}{2}\overline{D}_i\left[\left(h^{ki}\phi + \frac{1}{4}\overline{g}^{ki}\phi^2\right)\overline{D}_k V\right]
\end{aligned}
$$

is the sum of all divergence terms, and where $\mathcal{G}$ is the gauge-dependent term defined in (4.34), which has no obvious sign but which can be made to vanish by a gauge transformation. Finally, $\mathcal{Q}$ is the sum of quadratic terms and error terms, in the general case given by

$$
\begin{aligned}
(4.43) \quad \mathcal{Q} \;:=\; & \Big\{ -\frac{1}{4}g^{ij}g^{kp}g^{\ell q}\left(\overline{D}_\ell h_{kp}\overline{D}_q h_{ij} + \overline{D}_i h_{pq}\overline{D}_j h_{k\ell}\right) \\
& + \frac{1}{2}\left(|\psi|^2_{\overline{g}} - \chi^{ij}\overline{R}_{ij} + h^i{}_\ell h^{jm}\overline{R}^\ell{}_{mij}\right) \\
& + \frac{3}{2}h^{i\ell}h_\ell{}^k\overline{R}_{ik} - \frac{1}{2}\left[\overline{R} + \lambda\,(3-n)\right]|h|^2_{\overline{g}} \\
& + \frac{1}{2}\left[\left(\frac{n}{4}+1\right)\lambda\phi^2 - \frac{1}{4}\overline{R}\phi^2 - h^{k\ell}\overline{R}_{k\ell}\phi\right] \\
& - h^i{}_\ell h^{\ell j}\overline{R}_{ij} + h^{ij}\overline{R}_{ij}\phi + \overline{R}|h|^2_{\overline{g}} \\
& - \lambda\left[\phi^2 + (n-2)|h|^2_{\overline{g}}\right] + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big\}V \\
& + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\{ -\frac{1}{4}|\overline{D}\phi|^2_{\overline{g}} - \frac{1}{4}|\overline{D}h|^2_{\overline{g}} + \frac{1}{2}|\psi|^2_{\overline{g}} + \frac{1}{2}h^{i\ell}h^{jm}\overline{R}_{\ell mij} \\
&\quad + \frac{1}{2}h^{ij}\overline{R}_{ij}\phi + \frac{1}{2}\left[\overline{R} - \lambda(n-1)\right]|h|^2_{\overline{g}} \\
&\quad + \frac{1}{8}\left[(n-4)\,\lambda - \overline{R}\right]\phi^2 + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big\}V \\
&\quad + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\,.
\end{aligned}
$$

For space-forms this becomes

$$
\begin{aligned}
\mathcal{Q} &= \Big[-\frac{1}{4}g^{ij}g^{kp}g^{\ell q}\left(\overline{D}_\ell h_{kp}\overline{D}_q h_{ij} + \overline{D}_i h_{pq}\overline{D}_j h_{k\ell}\right) \\
&\quad + \frac{1}{2}\left(|\psi|^2_{\overline{g}} - \phi^2 + n|h|^2_g + (3-n)|h|^2_{\overline{g}}\right) + \phi^2 - |h|^2_{\overline{g}} \\
&\quad - \frac{1}{2}\left(\frac{n}{4}+1\right)\phi^2 + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big]V \\
&= \Big[-\frac{1}{4}|\overline{D}\phi|^2_{\overline{g}} - \frac{1}{4}|\overline{D}h|^2_{\overline{g}} + \frac{1}{2}|\psi|^2_{\overline{g}} - \frac{n}{8}\phi^2 + \frac{1}{2}|h|^2_{\overline{g}} \\
&\quad + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big]V\,.
\end{aligned} \tag{4.44}
$$

Using (3.4) and

$$
|h|^2_{\overline{g}} = |\hat{h}|^2_{\overline{g}} + \frac{1}{n}\overline{\phi}^2\,, \qquad |\overline{D}h|^2_{\overline{g}} = |\overline{D}\hat{h}|^2_{\overline{g}} + \frac{1}{n}|\overline{D}\,\overline{\phi}|^2_{\overline{g}}\,, \tag{4.45}
$$

we can rewrite $\mathcal{Q}$ of (4.43) in terms of the trace-free part of h and of $\check{\psi}$:

$$
\begin{aligned}
\mathcal{Q} &= \Big(-\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}} - \frac{1}{4}|\overline{D}\hat{h}|^2_{\overline{g}} + \frac{1}{2}\hat{h}^{i\ell}\hat{h}^{jm}\overline{R}_{\ell mij} + \frac{n+2}{2n}\phi\hat{h}^{ij}\overline{R}_{ij} \\
&\quad + \frac{n^2-4}{8n^2}\lambda\phi^2 + \frac{1}{2}(|\check{\psi}|^2_{\overline{g}} - \check{\psi}^i\overline{D}_i\phi) + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big)V \\
&\quad + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\,,
\end{aligned} \tag{4.46}
$$

where we used (2.2). Putting this into (4.41) we obtain (3.8). Also, when $\overline{g}$ is a space-form metric, this gives

$$
\begin{aligned}
\mathcal{Q} &= \Big[-\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}} - \frac{1}{4}|\overline{D}\hat{h}|^2_{\overline{g}} + \frac{1}{2}|\hat{h}|^2_{\overline{g}} - \frac{n^2-4}{8n}\phi^2 \\
&\quad + O\left(|h|^3_{\overline{g}}\right) + O(|h|_{\overline{g}}|\overline{D}h|^2_{\overline{g}})\Big]V \\
&\quad + \frac{1}{2}(|\check{\psi}|^2_{\overline{g}} - \check{\psi}^i\overline{D}_i\phi)V + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}}\,,
\end{aligned} \tag{4.47}
$$

which is precisely (3.12).

Appendix A. A weighted Poincaré inequality

When $\check{\psi} = 0$ all terms in (3.12) have the desired negative sign except for those involving undifferentiated occurrences of $\hat{h}$. To address this, some integral identities will be needed. Set

$$(\overline{\mathscr{D}}\hat{h})_{ijk} := \tfrac{1}{\sqrt{2}}\left(\overline{D}_i\hat{h}_{jk} - \overline{D}_j\hat{h}_{ik}\right), \quad (\overline{\mathscr{L}}v)_{ij} := \tfrac{1}{2}\left(\overline{D}_i v_j + \overline{D}_j v_i\right), \tag{A.1}$$
$$(\overline{\operatorname{div}}\,\hat{h})_j := -\overline{D}_i\hat{h}^i{}_j, \quad (\hat{h}_{dV})_i := V^{-1}\hat{h}_{ij}\overline{D}^j V,$$

and note that $\overline{\mathscr{L}}^* = \overline{\operatorname{div}}$. For any symmetric tensor $\check{h}$ we have (cf., e.g., [9, Section 3])

$$(\overline{\mathscr{D}}^*\overline{\mathscr{D}} + \overline{\mathscr{L}}\,\overline{\mathscr{L}}^*)\check{h} = (\overline{D}^*\overline{D} + \overline{\operatorname{Ric}} - \overline{\operatorname{Riem}})\check{h}, \tag{A.2}$$

where

$$[(\overline{\operatorname{Ric}} - \overline{\operatorname{Riem}})\hat{h}]_{ij} = \frac{1}{2}(\overline{R}_{ik}\hat{h}^k{}_j + \overline{R}_{jk}\hat{h}^k{}_i - 2\overline{R}_{ikj\ell}\hat{h}^{k\ell}). \tag{A.3}$$

Assume, first, that $\overline{g}$ is a space-form. Multiplying (A.2) by $V\hat{h}$ and integrating by parts, after some simple manipulations one obtains

$$\begin{aligned}\int |\hat{h}|^2_{\overline{g}} V d\mu_{\overline{g}} &= \frac{1}{n+1}\int \Big[(|\overline{D}\hat{h}|^2_{\overline{g}} - |\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} - |\overline{\operatorname{div}}\,\hat{h}|^2_{\overline{g}})V \\ &\qquad + \overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk}) - 2\hat{h}^{ik}\overline{D}_i V\overline{D}^j\hat{h}_{jk}\Big]d\mu_{\overline{g}} \\ &= \frac{1}{n+1}\int \Big[(|\overline{D}\hat{h}|^2_{\overline{g}} - |\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} - |\overline{\operatorname{div}}\,\hat{h} - \hat{h}_{dV}|^2_{\overline{g}})V \\ &\qquad + \overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk}) + |\hat{h}_{dV}|^2_{\overline{g}} V\Big]d\mu_{\overline{g}}.\end{aligned} \tag{A.4}$$

In a coordinate system in which the (suitably-normalised) anti-de Sitter metric $\overline{\mathbf{g}}$ reads

$$\overline{\mathbf{g}} = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\Omega^2 \tag{A.5}$$

we choose V as in (1.3) with $|\vec{A}| \leq A^0$ so that

$$|dV|_{\overline{g}} < V \quad \Longrightarrow \quad |\hat{h}_{dV}|_{\overline{g}} \leq |\hat{h}|_{\overline{g}}. \tag{A.6}$$

This gives

$$\int |\hat{h}|^2_{\overline{g}} V d\mu_{\overline{g}} \leq \frac{1}{n}\int \Big[(|\overline{D}\hat{h}|^2_{\overline{g}} - |\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} - |\overline{\mathrm{div}}\,\hat{h} - \hat{h}_{dV}|^2_{\overline{g}})V + \overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk})\Big] d\mu_{\overline{g}}\,. \tag{A.7}$$

which provides the desired *weighted Poincaré inequality* for space-forms when the trace-free tensor field $\hat{h}$ decays sufficiently fast so that the divergence term gives no contribution:

$$\int |\hat{h}|^2_{\overline{g}} V d\mu_{\overline{g}} \leq \frac{1}{n}\int |\overline{D}\hat{h}|^2_{\overline{g}} d\mu_{\overline{g}}\,. \tag{A.8}$$

We now indicate how to adapt the above argument to the general case, without assuming that the metric is a space form. For this, multiplying (A.2) by $V\hat{h}$ and integrating by parts we obtain

$$\begin{aligned}\int \left(\overline{R}_{ikj\ell}\hat{h}^{k\ell} - \overline{R}_{ik}\hat{h}^k{}_j\right)\hat{h}^{ij} V d\mu_{\overline{g}} \\ = \int \Big[(|\overline{D}\hat{h}|^2_{\overline{g}} - |\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} - |\overline{\mathrm{div}}\,\hat{h}|^2_{\overline{g}})V + \overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk}) \\ -2\hat{h}^{ik}\overline{D}_i V\overline{D}^j\hat{h}_{jk} - \left(\overline{R}_{ij}\hat{h}^{jk}\hat{h}^i{}_k - \lambda|\hat{h}|^2_{\overline{g}}\right)V\Big] d\mu_{\overline{g}}\,.\end{aligned} \tag{A.9}$$

To continue, it is convenient to introduce

$$\hat{\psi}_i := -\overline{D}_j h^j{}_i + \frac{1}{2}\overline{D}_i\overline{\phi} = -\overline{D}_j\hat{h}^j{}_i + \frac{n+2}{2n}\overline{D}_i\overline{\phi} \tag{A.10}$$

(note that this differs from $\check{\psi}_i$ by higher order terms). In this notation (A.9) can be rewritten as

$$\begin{aligned}\int \left(\overline{R}_{ikj\ell}\hat{h}^{k\ell}\hat{h}^{ij} - \lambda|\hat{h}|^2_{\overline{g}}\right)V d\mu_{\overline{g}} \\ = \int \Big[\Big(|\overline{D}\hat{h}|^2_{\overline{g}} - |\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} - |\hat{\psi} - \frac{n+2}{2n}\overline{D}\,\overline{\phi}|^2_{\overline{g}} \\ +2(\hat{\psi}^k - \frac{n+2}{2n}\overline{D}^k\overline{\phi})(\hat{h}_{dV})_k\Big)V + \overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk})\Big] d\mu_{\overline{g}}\,.\end{aligned} \tag{A.11}$$

One should keep in mind that the divergence term at the right-hand side is irrelevant for many purposes, in that it gives a vanishing contribution for suitably decaying fields when the integral in (A.11) is taken over the whole manifold.

Let $\gamma > 0$ be a constant, which might have to be chosen on a case-by-case basis depending upon the background geometry at hand. The trivial identity

$$-2\overline{D}^k\overline{\phi}(\hat{h}_{dV})_k = -|\gamma^{-1}\overline{D}\,\overline{\phi} + \gamma\hat{h}_{dV}|^2_{\overline{g}} + |\gamma^{-1}\overline{D}\,\overline{\phi}|^2_{\overline{g}} + |\gamma\hat{h}_{dV}|^2_{\overline{g}} \tag{A.12}$$

leads to the following version of (A.11):

$$\begin{aligned}\int \Big(\overline{R}_{ikj\ell}\hat{h}^{k\ell}\hat{h}^{ij} - \lambda|\hat{h}|^2_{\overline{g}} - \frac{n+2}{2n}|\gamma\hat{h}_{dV}|^2_{\overline{g}}\Big)\, V d\mu_{\overline{g}} \\ = \int \Big[\ \Big(|\overline{D}\hat{h}|^2_{\overline{g}} - |\overline{\mathscr{D}}\hat{h}|^2_{\overline{g}} - |\hat{\psi}^k - \frac{n+2}{2n}\overline{D}\,\overline{\phi}|^2_{\overline{g}} + 2\hat{\psi}^k(\hat{h}_{dV})_k \\ -\frac{n+2}{2n}|\gamma^{-1}\overline{D}\phi + \gamma\hat{h}_{dV}|^2_{\overline{g}} + \frac{n+2}{2n}|\gamma^{-1}\overline{D}\,\overline{\phi}|^2_{\overline{g}}\Big)V \\ +\overline{D}_j(\hat{h}_{ik}\overline{D}^i V\hat{h}^{jk})\Big] d\mu_{\overline{g}}\,.\end{aligned} \tag{A.13}$$

Suppose that there exist constants $c \geq 0$ and $\varepsilon > 0$ such that for all $\overline{\phi}$ and $\hat{h}$ we have

$$\begin{aligned}\frac{1}{2}\hat{h}^{i\ell}\hat{h}^{jm}\overline{R}_{\ell mij} + \frac{n+2}{2n}\overline{\phi}\hat{h}^{ij}\overline{R}_{ij} + \frac{n^2-4}{8n^2}\lambda\overline{\phi}^2 \\ \leq c\left(\overline{R}_{ikj\ell}\hat{h}^{k\ell}\hat{h}^{ij} - \lambda|\hat{h}|^2_{\overline{g}} - \frac{n+2}{2n}|\gamma\hat{h}_{dV}|^2_{\overline{g}}\right) - \varepsilon|\hat{h}|^2_{\overline{g}}\,.\end{aligned} \tag{A.14}$$

Integrating (3.10) over the manifold in the gauge $\check{\psi} \equiv 0$, using (A.13)-(A.14) and the decay conditions on h we obtain

$$\begin{aligned}\int \mathcal{Q} d\mu_{\overline{g}} \quad \leq \quad \int \Big\{\Big[-\frac{n+2}{8n}|\overline{D}\,\phi|^2_{\overline{g}} - \frac{1}{4}|\overline{D}\hat{h}|^2_{\overline{g}} - \varepsilon|\hat{h}|^2_{\overline{g}} \\ + O\left(|h|^3_{\overline{g}}\right) + O\left(|h|_{\overline{g}}\,|\overline{D}h|^2_{\overline{g}}\right)\Big]V + O\left(|h|^2_{\overline{g}}\,|\overline{D}h|_{\overline{g}}\right)|\overline{D}V|_{\overline{g}} \\ + c\Big(|\overline{D}\hat{h}|^2_{\overline{g}} + \frac{n+2}{2n}(\gamma^{-2} - \frac{n+2}{2n})|\overline{D}\phi|^2_{\overline{g}}\Big)V\Big\} d\mu_{\overline{g}}\,.\end{aligned} \tag{A.15}$$

The right-hand side will be strictly negative, as desired, for all sufficiently small $\|h\|_{L^\infty}$ and $\|\overline{D}h\|_{L^\infty}$, provided that $V > 0$, that $V^{-1}|\overline{D}V|_{\overline{g}}$ is bounded, and that

$$0 < c < \frac{1}{4}\,, \quad c(\gamma^{-2} - \frac{n+2}{2n}) < \frac{1}{4}\,. \tag{A.16}$$

This reduces the positivity issue to the algebraic inequality (A.14), with γ and c satisfying (A.16). The existence of c, and its value, has to be checked

on a case-by-case basis. We note that this strategy does not allow one to conclude in the case of Horowitz-Myers instantons.

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