Pure and Applied Mathematics Quarterly
Volume 14, Number 3–4, 579–590, 2018

On $J(r,n)$-Jacobsthal quaternions

Dorota Bród and Anetta Szynal-Liana

Abstract: In this paper we introduce the $J(r,n)$-Jacobsthal quaternions and give some of their properties, among others the Binet formula, convolution identity and the generating function.

Keywords: Jacobsthal numbers, quaternions, recurrence relations.

1. Introduction

Let $\mathbb{H}$ be the set of quaternions q of the form

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$.

If $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ are any two quaternions then equality, addition, subtraction and multiplication by scalar are defined.

Equality: $q_1 = q_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$,
subtraction: $q_1 - q_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)j + (d_1 - d_2)k$,
multiplication by scalar $s \in \mathbb{R}$: $sq_1 = sa_1 + sb_1 i + sc_1 j + sd_1 k$.

The quaternion multiplication is defined using the rule

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

Note that (1) implies

$$ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

Received April 17, 2019.
2010 Mathematics Subject Classification: Primary 11B37; secondary 11R52.

The conjugate of a quaternion is defined by

$$\overline{q} = \overline{a + bi + cj + dk} = a - bi - cj - dk.$$

The norm of a quaternion is defined by

$$N(q) = q \cdot \overline{q} = \overline{q} \cdot q = a^2 + b^2 + c^2 + d^2.$$

For the basics on quaternions theory, see [22].

Numbers of the Fibonacci type are defined by the second-order linear recurrence relation of the form $a_n = b_1 a_{n-1} + b_2 a_{n-2}$, where $b_i \in \mathbb{N}$, $i = 1, 2$. For special b_i, $i = 1, 2$, we obtain the recurrence equation which defines the Fibonacci numbers and the like (Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers etc.).

The numbers of the Fibonacci type have many applications in distinct areas of mathematics, also in quaternions theory. In 1963 Horadam [13] introduced nth Fibonacci and Lucas quaternions. Many interesting properties of Fibonacci and Lucas quaternions can be found in [11, 16]. In [14] Horadam mentioned the possibility of introducing Pell quaternions and generalized Pell quaternions. Interesting results of Pell quaternions, Pell-Lucas quaternions obtained recently can be found in [8, 19]. In [18] the Authors investigated Jacobsthal quaternions. There are many generalizations of Fibonacci and Jacobsthal quaternions in the literature, see for example [1, 2, 6, 12, 15, 20]. The another types of generalization of Fibonacci and Jacobsthal quaternions are octonions and quaternion polynomials, see [4, 5, 7].

In this paper we introduce and study the $J(r, n)$-Jacobsthal quaternions.

2. The $J(r, n)$-Jacobsthal numbers

Let $n \geq 0$ be an integer. The nth Jacobsthal number J_n is defined recursively by $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0,\ J_1 = 1$. The first ten terms of the sequence are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171. The direct formula for nth Jacobsthal number has the form $J_n = \frac{2^n - (-1)^n}{3}$, named as the Binet formula for Jacobsthal numbers. Many authors have generalized the second order recurrence of the Jacobsthal sequence, see [9, 10, 17, 21]. In [3] a one-parameter generalization of the Jacobsthal numbers was investigated. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The nth $J(r, n)$-Jacobsthal number $J(r, n)$ is defined as follows

$$J(r, n) = 2^r J(r, n-1) + (2^r + 4^r) J(r, n-2) \text{ for } n \geq 2 \tag{2}$$

with initial conditions $J(r,0)=1,\ J(r,1)=1+2^{r+1}$.

It is easily seen that $J(0,n)=J_{n+2}$. By (2) we obtain

$$\begin{aligned}
&J(r,0)=1\\
&J(r,1)=2\cdot 2^r+1\\
&J(r,2)=3\cdot 4^r+2\cdot 2^r\\
&J(r,3)=5\cdot 8^r+5\cdot 4^r+2^r\\
&J(r,4)=8\cdot 16^r+10\cdot 8^r+3\cdot 4^r\\
&J(r,5)=13\cdot 32^r+20\cdot 16^r+9\cdot 8^r+4^r
\end{aligned}$$

We will now recall some properties of the $J(r,n)$-Jacobsthal numbers.

Theorem 1 ([3], Binet formula)**.** *For $n\geq 0$ the nth $J(r,n)$-Jacobsthal number is given by*

$$\begin{aligned}J(r,n)=&\frac{\sqrt{4\cdot 2^r+5\cdot 4^r}+3\cdot 2^r+2}{2\sqrt{4\cdot 2^r+5\cdot 4^r}}{\lambda_1}^n\\
&+\frac{\sqrt{4\cdot 2^r+5\cdot 4^r}-3\cdot 2^r-2}{2\sqrt{4\cdot 2^r+5\cdot 4^r}}{\lambda_2}^n,\end{aligned}$$

where

$$\lambda_1=2^{r-1}+\frac{1}{2}\sqrt{4\cdot 2^r+5\cdot 4^r},\quad \lambda_2=2^{r-1}-\frac{1}{2}\sqrt{4\cdot 2^r+5\cdot 4^r}. \tag{3}$$

Theorem 2 ([3])**.** *The generating function of the sequence $\{J(r,n)\}$ has the following form*

$$f(x)=\frac{1+(1+2^r)x}{1-2^rx-(2^r+4^r)x^2}.$$

Proposition 3 ([3])**.** *Let $n\geq 4$, $r\geq 0$ be integers. Then*

$$J(r,n)=(3\cdot 8^r+2\cdot 4^r)J(r,n-3)+(2\cdot 16^r+3\cdot 8^r+4^r)J(r,n-4).$$

Theorem 4 ([3])**.** *Let $n\geq 1$, $r\geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1}J(r,l)=\frac{J(r,n)+(2^r+4^r)J(r,n-1)-2-2^r}{4^r+2^{r+1}-1}.$$

Theorem 5 ([3], Convolution identity)**.** *Let n,m,r be integers such that $m\geq 2, n\geq 1$, $r\geq 0$. Then*

$$J(r,m+n)=2^rJ(r,m-1)J(r,n)+(4^r+8^r)J(r,m-2)J(r,n-1).$$

3. The $J(r,n)$-Jacobsthal quaternions

For a nonnegative integer n, the nth $J(r,n)$-Jacobsthal quaternion JQ_n^r is defined as

$$JQ_n^r = J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3), \tag{4}$$

where $J(r,n)$ is given by (2).

Theorem 6. *Let $n \geq 0$, $r \geq 0$ be integers. Then*

(i) $2^r JQ_{n+1}^r + (2^r + 4^r) JQ_n^r = JQ_{n+2}^r$,
(ii) $JQ_n^r + \overline{JQ_n^r} = 2J(r,n)$,
(iii) $(JQ_n^r)^2 = 2J(r,n) JQ_n^r - N(JQ_n^r)$.

Proof. (i)

$$\begin{aligned}
&2^r JQ_{n+1}^r + (2^r + 4^r) JQ_n^r \\
&= 2^r (J(r,n+1) + iJ(r,n+2) + jJ(r,n+3) + kJ(r,n+4)) \\
&\quad + (2^r + 4^r)(J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3)) \\
&= J(r,n+2) + iJ(r,n+3) + jJ(r,n+4) + kJ(r,n+5) = JQ_{n+2}^r.
\end{aligned}$$

(ii) By the definition of the conjugate of the quaternion we have

$$\begin{aligned}
JQ_n^r + \overline{JQ_n^r} &= J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3) \\
&\quad + J(r,n) - iJ(r,n+1) - jJ(r,n+2) - kJ(r,n+3) \\
&= 2J(r,n).
\end{aligned}$$

(iii) By simple calculations we get

$$\begin{aligned}
(JQ_n^r)^2 &= J^2(r,n) - J^2(r,n+1) - J^2(r,n+2) - J^2(r,n+3) \\
&\quad + 2iJ(r,n)J(r,n+1) + 2jJ(r,n)J(r,n+2) + 2kJ(r,n)J(r,n+3) \\
&\quad + (ij + ji)J(r,n+1)J(r,n+2) + (ik + ki)J(r,n+1)J(r,n+3) \\
&\quad + (jk + kj)J(r,n+2)J(r,n+3) \\
&= J^2(r,n) - J^2(r,n+1) - J^2(r,n+2) - J^2(r,n+3) \\
&\quad + 2(iJ(r,n)J(r,n+1) + jJ(r,n)J(r,n+2) + kJ(r,n)J(r,n+3)) \\
&= 2J(r,n)(J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3)) \\
&\quad - (J^2(r,n) + J^2(r,n+1) + J^2(r,n+2) + J^2(r,n+3)) \\
&= 2J(r,n) JQ_n^r - N(JQ_n^r).
\end{aligned}$$

□

Theorem 7. *Let $n \geq 0$, $r \geq 0$ be integers. Then*

$$\begin{aligned} &JQ_n^r - iJQ_{n+1}^r - jJQ_{n+2}^r - kJQ_{n+3}^r \\ &= J(r,n) + J(r,n+2) + J(r,n+4) + J(r,n+6). \end{aligned}$$

Proof.

$$\begin{aligned} &JQ_n^r - iJQ_{n+1}^r - jJQ_{n+2}^r - kJQ_{n+3}^r \\ &= J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3) \\ &\quad - i(J(r,n+1) + iJ(r,n+2) + jJ(r,n+3) + kJ(r,n+4)) \\ &\quad - j(J(r,n+2) + iJ(r,n+3) + jJ(r,n+4) + kJ(r,n+5)) \\ &\quad - k(J(r,n+3) + iJ(r,n+4) + jJ(r,n+5) + kJ(r,n+6)). \end{aligned}$$

By simple calculations we get

$$\begin{aligned} &JQ_n^r - iJQ_{n+1}^r - jJQ_{n+2}^r - kJQ_{n+3}^r \\ &= J(,n) + J(r,n+2) + J(r,n+4) + J(r,n+6) \\ &\quad - (ij+ji)J(r,n+3) - (ik+ki)J(r,n+4) - (jk+kj)J(r,n+5) \\ &= J(r,n) + J(r,n+2) + J(r,n+4) + J(r,n+6). \end{aligned}$$

□

Theorem 8. *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$\begin{aligned} \sum_{l=0}^{n-1} JQ_l^r = &\frac{JQ_n^r + (2^r + 4^r)JQ_{n-1}^r - (2+2^r)(1+i+j+k)}{4^r + 2^{r+1} - 1} \\ &- \left(i + j(2+2^{r+1}) + k(2^{r+2} + 3\cdot 4^r + 2)\right). \end{aligned}$$

Proof. By the definition of the $J(r,n)$-Jacobsthal quaternions we have

$$\begin{aligned} \sum_{l=0}^{n-1} JQ_l^r &= JQ_0^r + JQ_1^r + \ldots + JQ_{n-1}^r \\ &= J(r,0) + iJ(r,1) + jJ(r,2) + kJ(r,3) \\ &\quad + J(r,1) + iJ(r,2) + jJ(r,3) + kJ(r,4) + \ldots \\ &\quad + J(r,n-1) + iJ(r,n) + jJ(r,n+1) + kJ(r,n+2) \\ &= J(r,0) + J(r,1) + \ldots + J(r,n-1) \\ &\quad + i\left(J(r,1) + J(r,2) + \ldots + J(r,n) + J(r,0) - J(r,0)\right) \\ &\quad + j\left(J(r,2) + J(r,3) + \ldots + J(r,n+1) + J(r,0) + J(r,1)\right. \end{aligned}$$

$$-J(r,0)-J(r,1))$$
$$+k\left(J(r,3)+J(r,4)+\ldots+J(r,n+2)+J(r,0)+J(r,1)\right.$$
$$\left.+J(r,2)-J(r,0)-J(r,1)-J(r,2)\right).$$

Using Theorem 4, we obtain

$$\begin{aligned}
\sum_{l=0}^{n-1} JQ_l^r &= \frac{1}{4^r+2^{r+1}-1}[J(r,n)+(2^r+4^r)J(r,n-1)-2-2^r \\
&\quad + i(J(r,n+1)+(2^r+4^r)J(r,n)-2-2^r) \\
&\quad + j(J(r,n+2)+(2^r+4^r)J(r,n+1)-2-2^r) \\
&\quad + k(J(r,n+3)+(2^r+4^r)J(r,n+2)-2-2^r)] \\
&\quad - i - j(2+2^{r+1}) - k(2^{r+2}+3\cdot 4^r+2) \\
&= \frac{1}{4^r+2^{r+1}-1}\left[J(r,n)+iJ(r,n+1)+jJ(r,n+2)+kJ(r,n+3)\right. \\
&\quad + (2^r+4^r)(J(r,n-1)+iJ(r,n)+jJ(r,n+1)+kJ(r,n+2)) \\
&\quad \left.-(2+2^r)(1+i+j+k)\right] - i - j(2^{r+1}+2) - k(2^{r+2}+3\cdot 4^r+2) \\
&= \frac{JQ_n^r+(2^r+4^r)JQ_{n-1}^r-(2+2^r)(1+i+j+k)}{4^r+2^{r+1}-1} \\
&\quad - \Big(i+j(2+2^{r+1})+k(2^{r+2}+3\cdot 4^r+2)\Big). \qquad \square
\end{aligned}$$

We will give the Binet formula for the $J(r,n)$-Jacobsthal quaternions.

Theorem 9. *Let $n \geq 0$, $r \geq 0$ be integers. Then*

$$JQ_n^r = C_1\lambda_1^n\left(1+i\lambda_1+j\lambda_1^2+k\lambda_1^3\right)+C_2\lambda_2^n\left(1+i\lambda_2+j\lambda_2^2+k\lambda_2^3\right),$$

where

$$C_1=\frac{\sqrt{4\cdot 2^r+5\cdot 4^r}+3\cdot 2^r+2}{2\sqrt{4\cdot 2^r+5\cdot 4^r}},\quad C_2=\frac{\sqrt{4\cdot 2^r+5\cdot 4^r}-3\cdot 2^r-2}{2\sqrt{4\cdot 2^r+5\cdot 4^r}}$$

and λ_1, λ_2 were defined by (3), i.e.

$$\lambda_1=2^{r-1}+\frac{1}{2}\sqrt{4\cdot 2^r+5\cdot 4^r},\quad \lambda_2=2^{r-1}-\frac{1}{2}\sqrt{4\cdot 2^r+5\cdot 4^r}.$$

Proof. By Theorem 1 we get

$$J(r,n) = C_1\lambda_1^n + C_2\lambda_2^n$$

and

$$\begin{aligned} JQ_n^r &= J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3) \\ &= C_1\lambda_1^n + C_2\lambda_2^n + i(C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1}) \\ &\quad + j(C_1\lambda_1^{n+2} + C_2\lambda_2^{n+2}) + k(C_1\lambda_1^{n+3} + C_2\lambda_2^{n+3}) \\ &= C_1\lambda_1^n\left(1 + i\lambda_1 + j\lambda_1^2 + k\lambda_1^3\right) + C_2\lambda_2^n\left(1 + i\lambda_2 + j\lambda_2^2 + k\lambda_2^3\right), \end{aligned}$$

which ends the proof. □

Proposition 10. *Let $n \geq 4$, $r \geq 0$. Then*

$$JQ_n^r = (3\cdot 8^r + 2\cdot 4^r)JQ_{n-3}^r + (2\cdot 16^r + 3\cdot 8^r + 4^r)JQ_{n-4}^r.$$

Proof. Let $A = 3\cdot 8^r + 2\cdot 4^r$, $B = 2\cdot 16^r + 3\cdot 8^r + 4^r$. Using Proposition 3, we obtain

$$\begin{aligned} JQ_n^r &= J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3) \\ &= A\cdot J(r,n-3) + B\cdot J(r,n-4) + i(A\cdot J(r,n-2) + B\cdot J(r,n-3)) \\ &\quad + j(A\cdot J(r,n-1) + B\cdot J(r,n-2)) + k(A\cdot J(r,n) + B\cdot J(r,n-1)) \\ &= A(J(r,n-3) + iJ(r,n-2) + jJ(r,n-1) + kJ(r,n)) \\ &\quad + B(J(r,n-4) + iJ(r,n-3) + jJ(r,n-2) + kJ(r,n-1)). \end{aligned}$$

Hence we have $JQ_n^r = A\cdot JQ_{n-3}^r + B\cdot JQ_{n-4}^r$, which ends the proof. □

Theorem 11. *Let n, m, r be integers such that $m \geq 2, n \geq 1$, $r \geq 0$. Then*

$$\begin{aligned} &2JQ_{m+n}^r = 2^r JQ_{m-1}^r JQ_n^r + (4^r + 8^r)JQ_{m-2}^r JQ_{n-1}^r \\ &+J(r,m+n) + J(r,m+n+2) + J(r,m+n+4) + J(r,m+n+6). \end{aligned}$$

Proof. By (4) we have

$$\begin{aligned} &2^r JQ_{m-1}^r JQ_n^r + (4^r + 8^r)JQ_{m-2}^r JQ_{n-1}^r \\ &= 2^r(J(r,m-1) + iJ(r,m) + jJ(r,m+1) + kJ(r,m+2)) \\ &\quad \cdot (J(r,n) + iJ(r,n+1) + jJ(r,n+2) + kJ(r,n+3)) \end{aligned}$$

$$+ (4^r + 8^r)(J(r, m-2) + iJ(r, m-1) + jJ(r, m) + kJ(r, m+1)) \\ \cdot (J(r, n-1) + iJ(r, n) + jJ(r, n+1) + kJ(r, n+2)).$$

By simple calculations and using Theorem 5 we get

$$\begin{aligned}
& 2^r JQ^r_{m-1} JQ^r_n + (4^r + 8^r) JQ^r_{m-2} JQ^r_{n-1} \\
= {} & 2^r (J(r, m-1)J(r, n) + iJ(r, m-1)J(r, n+1) \\
& + jJ(r, m-1)J(r, n+2) + kJ(r, m-1)J(r, n+3)) \\
& + iJ(r, m)J(r, n) - J(r, m)J(r, n+1) + kJ(r, m)J(r, n+2) \\
& - jJ(r, m)J(r, n+3) + jJ(r, m+1)J(r, n) - kJ(r, m+1)J(r, n+1) \\
& - J(r, m+1)J(r, n+2) + iJ(r, m+1)J(r, n+3) + kJ(r, m+2)J(r, n) \\
& + jJ(r, m+2)J(r, n+1) - iJ(r, m+2)J(r, n+2) \\
& - J(r, m+2)J(r, n+3)) \\
& + (4^r + 8^r)(J(r, m-2)J(r, n-1) + iJ(r, m-2)J(r, n) \\
& + jJ(r, m-2)J(r, n+1) + kJ(r, m-2)J(r, n+2) \\
& + iJ(r, m-1)J(r, n-1) \\
& - J(r, m-1)J(r, n) + kJ(r, m-1)J(r, n+1) - jJ(r, m-1)J(r, n+2) \\
& + jJ(r, m)J(r, n-1) - kJ(r, m)J(r, n) - J(r, m)J(r, n+1) \\
& + iJ(r, m)J(r, n+2) + kJ(r, m+1)J(r, n-1) + jJ(r, m+1)J(r, n) \\
& - iJ(r, m+1)J(r, n+1) - J(r, m+1)J(r, n+2)) \\
= {} & 2^r J(r, m-1)J(r, n) + (4^r + 8^r)(J(r, m-2)J(r, n-1) \\
& + i(2^r J(r, m-1)J(r, n+1) + (4^r + 8^r)J(r, m-2)J(r, n)) \\
& + j(2^r J(r, m-1)J(r, n+2) + (4^r + 8^r)J(r, m-2)J(r, n+1)) \\
& + k(2^r J(r, m-1)J(r, n+3) + (4^r + 8^r)J(r, m-2)J(r, n+2)) \\
& + i(2^r J(r, m)J(r, n) + (4^r + 8^r)J(r, m-1)J(r, n-1)) \\
& + j(2^r J(r, m+1)J(r, n) + (4^r + 8^r)J(r, m)J(r, n-1)) \\
& + k(2^r J(r, m)J(r, n+2) + (4^r + 8^r)J(r, m-1)J(r, n+1)) \\
& - 2^r J(r, m)J(r, n+1) - (4^r + 8^r)J(r, m-1)J(r, n) \\
& - 2^r J(r, m+1)J(r, n+2) - (4^r + 8^r)J(r, m)J(r, n+1)
\end{aligned}$$

$$- 2^r J(r,m+2)J(r,n+3) - (4^r + 8^r)J(r,m+1)J(r,n+2).$$

Using Theorem 5 again, we obtain

$$\begin{aligned}
&2^r JQ^r_{m-1} JQ^r_n + (4^r + 8^r) JQ^r_{m-2} JQ^r_{n-1} \\
&= 2(J(r,m+n) + iJ(r,m+n+1) + jJ(r,m+n+2) \\
&\quad + kJ(r,m+n+3)) - (J(r,m+n) + J(r,m+n+2) \\
&\quad + J(r,m+n+4) + J(r,m+n+6)) \\
&= 2JQ^r_{m+n} - (J(r,m+n) + J(r,m+n+2) \\
&\quad + J(r,m+n+4) + J(r,m+n+6)),
\end{aligned}$$

which ends the proof. □

At the end we shall give the ordinary generating functions for the $J(r,n)$-Jacobsthal quaternions.

Theorem 12. *The generating function for the $J(r,n)$-Jacobsthal quaternion sequence $\{JQ^r_n\}$ is*

$$G(x) = \frac{JQ^r_0 + (JQ^r_1 - 2^r JQ^r_0)x}{1 - 2^r x - (2^r + 4^r)x^2}.$$

Proof. Assuming that the generating function of the $J(r,n)$-Jacobsthal quaternion sequence $\{JQ^r_n\}$ has the form $G(x) = \sum_{n=0}^{\infty} JQ^r_n x^n$, we obtain

$$\begin{aligned}
&(1 - 2^r x - (2^r + 4^r)x^2)G(x) \\
&= (1 - 2^r x - (2^r + 4^r)x^2) \cdot (JQ^r_0 + JQ^r_1 x + JQ^r_2 x^2 + \ldots) \\
&= JQ^r_0 + JQ^r_1 x + JQ^r_2 x^2 + \ldots \\
&\quad - 2^r JQ^r_0 x - 2^r JQ^r_1 x^2 - 2^r JQ^r_2 x^3 - \ldots \\
&\quad - (2^r + 4^r) JQ^r_0 x^2 - (2^r + 4^r) JQ^r_1 x^3 - (2^r + 4^r) JQ^r_2 x^4 - \ldots \\
&= JQ^r_0 + (JQ^r_1 - 2^r JQ^r_0)x,
\end{aligned}$$

since $JQ^r_n = 2^r JQ^r_{n-1} + (2^r + 4^r) JQ^r_{n-2}$ and the coefficients of x^n for $n \geq 2$ are equal to zero.

Moreover,

$$JQ^r_0 = 1 + i(2 \cdot 2^r + 1) + j(3 \cdot 4^r + 2 \cdot 2^r) + k(5 \cdot 8^r + 5 \cdot 4^r + 2^r)$$

and

$$JQ_1^r - 2^r JQ_0^r = 2^r + 1 + i(4^r + 2^r) + j(2\cdot 8^r + 3\cdot 4^r + 2^r) + k(3\cdot 16^r + 5\cdot 8^r + 2\cdot 4^r). \quad \square$$

Acknowledgments

The authors would like to thank the referees for helpful valuable suggestions which resulted in improvements to this paper.

References

[1] G. Anatriello and G. Vincenzi, On H-Jacobsthal and H-Jacobsthal-Lucas sequences, and related quaternions, *Analele Stiintifice ale Universitatii Ovidius Constanta* **27(3)** (2019).

[2] F. T. Aydin and S. Yüce, A New Approach to Jacobsthal Quaternions, *Filomat* **31(18)** (2017), 5567–5579. DOI: https://doi.org/10.2298/FIL1718567T. MR3744161

[3] D. Bród, On a new Jacobsthal-type sequence, *Ars Combinatoria*, in press.

[4] P. Catarino, A note on certain matrices with $h(x)$-Fibonacci quaternion polynomials, *Journal of Difference Equations and Applications* **22(2)** (2016), 343–351. DOI: https://doi.org/10.1080/10236198.2015.1092526. MR3474987

[5] P. Catarino, The $h(x)$-Fibonacci Quaternion Polynomials: Some Combinatorial Properties, *Advances in Applied Clifford Algebras* **26(1)** (2016), 71–79. DOI: https://doi.org/10.1007/s00006-015-0606-1. MR3459990

[6] G. Cerda-Morales, On a Generalization for Tribonacci Quaternions, *Mediterranean Journal of Mathematics* **14** (2017), 239. DOI: https://doi.org/10.1007/s00009-017-1042-3.

[7] C. B. Çimen and A. İpek, On Jacobsthal and Jacobsthal-Lucas Octonions, *Mediterranean Journal of Mathematics* **14** (2017), 37. DOI: https://doi.org/10.1007/s00009-017-0873-2. MR3615053

[8] C. B. Çimen and A. İpek, On Pell Quaternions and Pell-Lucas Quaternions, *Advances in Applied Clifford Algebras* **26** (2016), 39–51. DOI: https://doi.org/10.1007/s00006-015-0571-8. MR3459988

[9] A. Dasdemir, The Representation, Generalized Binet Formula and Sums of The Generalized Jacobsthal p-Sequence, *Hittite Journal of Science and Engineering* **3(2)** (2016), 99–104. DOI: http://dx.doi.org/10.17350/HJSE19030000038.

[10] S. Falcon, On the k-Jacobsthal Numbers, *American Review of Mathematics and Statistics*, **2(1)** (2014), 67–77.

[11] S. Halici, On Fibonacci Quaternions, *Advances in Applied Clifford Algebras* **22** (2012), 321–327. DOI: https://doi.org/10.1007/s00006-011-0317-1. MR2930698

[12] S. Halici and A. Karataş, On a generalization for Fibonacci quaternions, *Chaos, Solitons & Fractals* **98** (2017), 178–182. DOI: https://doi.org/10.1016/j.chaos.2017.03.037. MR3632845

[13] A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, *American Mathematical Monthly* **70** (1963), 289–291. DOI: 10.2307/2313129. https://www.jstor.org/stable/2313129. MR0146137

[14] A. F. Horadam, Quaternion recurrence relations, *Ulam Quarterly* **2** (1993), 23–33. MR1257660

[15] A. İpek, On (p, q)-Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums, *Advances in Applied Clifford Algebras* **27(2)** (2017), 1343–1351. DOI: https://doi.org/10.1007/s00006-016-0704-8. MR3651518

[16] M. R. Iyer, A Note On Fibonacci Quaternions, *The Fibonacci Quarterly* **3** (1969), 225–229. MR0255513

[17] D. Jhala, K. Sisodiya and G. P. S. Rathore, On Some Identities for k-Jacobsthal Numbers, *International Journal of Mathematical Analysis* **7(12)** (2013), 551–556. DOI: http://dx.doi.org/10.12988/ijma.2013.13052. MR3004318

[18] A. Szynal-Liana and I. Włoch, A note on Jacobsthal quaternions, *Advances in Applied Clifford Algebras* **26** (2016), 441–447. DOI: https://doi.org/10.1007/s00006-015-0622-1. MR3460010

[19] A. Szynal-Liana and I. Włoch, The Pell Quaternions and the Pell Octonions, *Advances in Applied Clifford Algebras* **26** (2016), 435–440. DOI: https://doi.org/10.1007/s00006-015-0570-9. MR3460009

[20] D. Tasci, On k-Jacobsthal and k-Jacobsthal Lucas quaternions, *Journal of Science and Arts* **3(40)** (2017), 469–476.

[21] S. Uygun, The (s, t)-Jacobsthal and (s, t)-Jacobsthal Lucas Sequences, *Applied Mathematical Sciences* **9(70)** (2015), 3467–3476. DOI: http://dx.doi.org/10.12988/ams.2015.52166.

[22] J. P.Ward, *Quaternions and Cayley Numbers: Algebra and Applications*, Kluwer Academic Publishers, London (1997). MR1458894

Dorota Bród
Faculty of Mathematics and Applied Physics
Rzeszow University of Technology
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
E-mail: dorotab@prz.edu.pl

Anetta Szynal-Liana
Faculty of Mathematics and Applied Physics
Rzeszow University of Technology
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
E-mail: aszynal@prz.edu.pl