Pure and Applied Mathematics Quarterly
Volume 14, Number 3–4, 529–561, 2018

# A compactness theorem for rotationally symmetric Riemannian manifolds with positive scalar curvature

Jiewon Park, Wenchuan Tian, Changliang Wang

**Abstract:** Gromov and Sormani have conjectured the following compactness theorem on scalar curvature to hold. Given a sequence of compact Riemannian manifolds with nonnegative scalar curvature and bounded area of minimal surfaces, a subsequence is conjectured to converge in the intrinsic flat sense to a limit space, which has nonnegative generalized scalar curvature and Euclidean tangent cones almost everywhere. In this paper we prove this conjecture for sequences of rotationally symmetric warped product manifolds. We show that the limit space has an $H^1$ warping function which has nonnegative scalar curvature in a weak sense, and has Euclidean tangent cones almost everywhere.

**Keywords:** Scalar curvature compactness, Sormani-Wenger intrinsic flat distance, rotationally symmetric manifolds.



Received January 3, 2019.
**2010 Mathematics Subject Classification:** 53C23.

## 1. Introduction

In [Gro14a] and [Gro14b], Gromov conjectured that the intrinsic flat convergence may preserve a generalized notion of nonnegative scalar curvature. In light of this and examples constructed by Basilio, Dodziuk, and Sormani in [BDS17], Gromov and Sormani proposed the following conjecture in [GS18] (see also [Sor17]).

**Conjecture 1.1.** *Let $\{M_j^3\}$ be a sequence of closed oriented manifolds without boundary satisfying*

$$\mathrm{Vol}(M_j) \le V, \tag{1}$$

$$\mathrm{Diam}(M_j) \le D, \tag{2}$$

$$\mathrm{Scalar}_j \ge 0, \tag{3}$$

*and*

$$\mathrm{MinA}(M_j) \ge A > 0. \tag{4}$$

*Here,* $\mathrm{MinA}(M_j)$ *is defined as the infimum of areas of closed embedded minimal surfaces on $M_j$. Then a subsequence of $\{M_j\}$ converges in the volume preserving intrinsic flat sense to a limit space $M_\infty$,*

$$M_{j_k} \xrightarrow{\mathcal{F}} M_\infty \quad \text{and} \quad \mathbf{M}(M_{j_k}) \to \mathbf{M}(M_\infty). \tag{5}$$

*Moreover, $M_\infty$ has nonnegative generalized scalar curvature, has Euclidean tangent cones almost everywhere, and satisfies the prism inequality.*

The convergence in Conjecture 1.1 is under the Sormani-Wenger intrinsic flat (SWIF) distance between integral current spaces introduced by Sormani and Wenger in [SW11]. In this paper, we will prove the first two parts of

Conjecture 1.1 in the case when $M_j$ are rotationally symmetric Riemannian manifolds. Namely, we prove the convergence to a smooth manifold with $\mathcal{C}^0$ metric which has Euclidean tangent cones almost everywhere and nonnegative scalar curvature in the sense of distributions.

We briefly recall the notion of the intrinsic flat distance following [Sor17]. An integral current space $(X, d, T)$ is a metric space $(X, d)$ with an integral current structure $T$. An oriented Riemannian manifold $(M^m, g)$ of finite volume can be naturally viewed as an integral current space, since it has a natural metric induced by the Riemannian metric $g$, and an integral current structure $T$ acting on differential $m$-forms $\omega$ as

(6) $$T(\omega) = \int_M \omega.$$

The mass of an integral current space $\mathbf{M}(T)$ can be understood as a weighted volume. When the integral current space is an oriented Riemannian manifold, its mass is just its volume, $\mathbf{M}(M) = \mathrm{Vol}(M)$. The boundary of an integral current space was defined by Ambrosio and Kirchheim in [AK00] so that it satisfies Stokes' Theorem. In particular, when the integral current space is a Riemannian manifold $M$, then its boundary is just the usual boundary $\partial M$. We refer to [AK00] for more details about integral current spaces.

Let $Z$ be a metric space and $T_1$ and $T_2$ be two $m$-integral currents on $Z$. Recall the flat distance between integral currents $T_1$ and $T_2$ defined by Federer and Fleming in [FF60] is

(7) $$d_F^Z(T_1, T_2) = \inf\left\{\mathbf{M}(B^{m+1}) + \mathbf{M}(A^m) \mid T_1 - T_2 = A + \partial B\right\}.$$

**Definition 1.2** ([SW11])**.** The SWIF distance between integral current spaces $(X_1, d_1, T_1)$ and $(X_2, d_2, T_2)$ is defined as

(8) $$d_{\mathcal{F}}((X_1, d_1, T_1), (X_2, d_2, T_2)) = \inf\left\{d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2) \mid \varphi_i : X_i \to Z\right\},$$

where the infimum is taken over all common complete metric spaces $Z$ and all isometric embeddings $\varphi_i : X_i \to Z$, where $\varphi_{i\#}$ is the push-forward map on integral currents.

We refer to [SW11] for properties of the SWIF distance and only mention here Wenger's Compactness Theorem [Wen11], which says that if a sequence of Riemannian manifolds $M_j$ satisfies

(9) $$\mathrm{Diam}(M_j) \leq D,$$

$$\mathrm{Vol}(M_j) \leq V, \tag{10}$$
$$\mathrm{Area}(\partial M_j) \leq A, \tag{11}$$

then there exists a subsequence $M_{j_i}$ such that $M_{j_i} \xrightarrow{\mathcal{F}} M_\infty$, where $M_\infty$ is an integral current space, possibly the 0 space. In Ambrosio and Kirchheim's work [AK00] and Sormani and Wenger's work [SW11], it has been shown that $M_\infty$ can have tangent cones that are normed spaces. Note that there is no hypothesis on MinA or scalar curvature in this compactness theorem. In [AS18], Allen and Sormani constructed rotationally symmetric examples with bounds on MinA, but without bounds on the scalar curvature, which have non-Euclidean tangent cones.

When $M_j$ has nonnegative scalar curvature, Gromov proved in [Gro14b] that if the limit space is smooth and the convergence is $\mathcal{C}^0$, then indeed the scalar curvature is nonnegative on the limit space. In [Bam16], Bamler proved the same result using Ricci flow. In general, the volume is only lower semicontinuous; collapsing, or cancellation, can happen even with a scalar curvature bound, and the mass of the limit space can be 0. Such examples are given in Example 3.1 and by Sormani and Wenger in [SW10]. Also note that the SWIF limit does not always coincide with the Gromov-Hausdorff limit; see Example 3.2, which is an example constructed by Lakzian.

Now we consider rotationally symmetric Riemannian manifolds $(M_j^3, g_j)$, that is, $M_j^3$ are diffeomorphic to $\mathbb{S}^3$ with the metric tensor

$$g_j = ds^2 + f_j(s)^2 g_{\mathbb{S}^2}, \tag{12}$$

where $0 \leq s \leq D_j$, and $f_j(s)$ is a smooth nonnegative function with $f_j(0) = f_j(D_j) = 0$ and $f_j > 0$ everywhere else, and $f_j'(0) = 1$, $f_j'(D_j) = -1$, so that the metric tensor is smooth. Our main result confirms Conjecture 1.1 in this rotationally symmetric setting.

**Theorem 1.3.** *Let $(M_j^3, g_j)$ be a sequence of oriented rotationally symmetric Riemannian manifolds without boundary satisfying*

$$\mathrm{Diam}(M_j) \leq D, \tag{13}$$
$$\mathrm{Scalar}_j \geq 0, \tag{14}$$
$$\mathrm{MinA}(M_j) \geq A > 0, \tag{15}$$

*then a subsequence converges in the volume preserving intrinsic flat sense to a metric space $(M_\infty, g_\infty)$*

$$M_{j_k} \xrightarrow{\mathcal{F}} M_\infty \quad \text{and} \quad \mathbf{M}(M_{j_k}) \to \mathbf{M}(M_\infty). \tag{16}$$

*The metric $g_\infty$ is rotationally symmetric, $\mathcal{C}^0$, $H^1$, and has nonnegative generalized scalar curvature, meaning that the warping function satisfies the inequality in Lemma 2.4 in the sense of distributions.*

**Remark 1.4.** In Theorem 1.3, when $\mathrm{Scalar}_j \geq 0$ is replaced by $\mathrm{Scalar}_j \geq H > 0$, we have the same convergence result and that $M_\infty$ has generalized scalar curvature at least $H$ in the sense of distributions.

**Remark 1.5.** Note that in Theorem 1.3 we do not need to assume a uniform upper bound on volume as in Conjecture 1.1. Actually, with the help of Lemma 2.6, a uniform upper bound on volume follows from the non-negativity of scalar curvature and uniform upper bound on diameter. Lemma 2.6 also implies that the tangent cones are Euclidean almost everywhere on the limit space.

**Remark 1.6.** In Section 3, we will illustrate that if $\mathrm{MinA}(M_j)$ has no positive lower bound then the sequence $M_j$ could collapse to the zero current. We will also recall an interesting example obtained by Lakzian in [Lak16] to illustrate the difference between SWIF limit and Gromov-Hausdorff limit of sequences of rotationally symmetric Riemannian manifolds satisfying hypotheses in Theorem 1.3.

The SWIF convergence has been applied to study sequences of warped product type Riemannian manifolds with non-negative scalar curvature in various interesting problems, see Lee-Sormani [LS14], LeFloch-Sormani [LS15], and Allen-Hernandez-Vazquez-Parise-Payne-Wang [AHPPW18]. Especially, LeFloch and Sormani [LS15] proved a compactness theorem for Hawking mass in the rotationally symmetric setting. They proved that for a sequence of three dimensional oriented Riemannian manifolds $M_j^3$ with boundary, with nonnegative scalar curvature and certain bounds including a bound on Hawking mass, a subsequence converges in the volume preserving SWIF distance to a limit space with nonnegative generalized scalar curvature in a generalized sense. This theorem is proved by showing $H^1_{loc}$ convergence of a subsequence of the manifolds with a well chosen gauge and showing that the $H^1_{loc}$ limit coincides with a $\mathcal{F}$ limit using Theorem 5.1. In general it is unknown whether $H^1_{loc}$ convergence implies $\mathcal{F}$ convergence, but the monotonicity of the Hawking mass allows for the implication in this setting. The limit space is a rotationally symmetric manifold with a metric tensor $g \in H^1_{loc}$ and it is possible to define generalized notions of nonnegative scalar curvature in a weak sense.

The organization of this paper is as follows. In Section 2 we derive some basic consequences from the hypotheses on volume, diameter, scalar curvature, and MinA. In Section 3 some interesting examples of SWIF convergence are given to better illustrate this notion in the context of the paper. In Section 4 we prove uniform convergence of $f_j$ to a limit function $f_\infty$ and construct the limit space $M_\infty$ (Theorem 4.1). Then we show that $M_\infty$ has Euclidean tangent cones almost everywhere (Theorem 4.6), and nonnegative generalized scalar curvature (Theorem 4.9). Here, we use the notion of distributional scalar cuvature, which is studied by LeFloch and Mardare in [LM07]. Finally, in Section 5 we prove the SWIF convergence of $M_j$ to $M_\infty$ after taking a subsequence (Theorem 5.6). The proof relies on the technique of identifying large diffeomorphic regions on $M_j$ and $M_\infty$, introduced by Lakzian and Sormani [LS13].

It remains an interesting open question to prove or disprove the prism inequality on the limit space. This question is so challenging even for smooth metric spaces that it was only recently settled by Li in [Li17].

## 2. Basic consequences of the hypotheses

In this section, we derive basic consequences from the hypotheses in Theorem 1.3. Recall that $M_j$ is diffeomorphic to $\mathbb{S}^3$ and equipped with a smooth rotationally symmetric Riemannian metric $g_j = ds^2 + f_j(s)^2 g_{\mathbb{S}^2}$, where $0 \le s \le D_j$, $f_j > 0$ on $(0, D_j)$, $f_j(0) = f_j(D_j) = 0$, $f_j'(0) = 1$, and $f_j'(D_j) = -1$, along with the bounds

$$\operatorname{Diam}(M_j) \le D, \tag{17}$$

$$\operatorname{Scalar}_j \ge 0, \tag{18}$$

and

$$\operatorname{MinA}(M_j) \ge A > 0. \tag{19}$$

### 2.1. The upper bound on diameter

**Lemma 2.1.** *$\operatorname{Diam}(M_j) = D_j \le D$.*

*Proof.* Let $d$ denote the distance function on $M_j$, and $N$, $S$ the north pole (corresponding to $s = 0$) and south pole (corresponding to $s = D_j$) respectively. First we check that $d(N, S) = D_j$. Fix $\theta \in \mathbb{S}^2$ and let $\gamma : [0, D_j] \to [0, D_j] \times \mathbb{S}^2$ be a path defined as $\gamma(t) = (t, \theta)$. Then $\gamma$ is a path connecting $N$ and $S$, and has length $D_j$. Let $\delta : [a, b] \to [0, D_j] \times \mathbb{S}^2$, $\delta(t) = (s(t), \theta(t))$ be an arbitrary path connecting $N$ and $S$, that is, $\delta(a) = N$ and $\delta(b) = S$. Let $L_j(\delta)$ be the length of $\delta$. Then we have $L_j(\delta) \geq D_j$. Indeed,

$$\begin{aligned} L_j(\delta) &= \int_a^b \sqrt{|s'(t)|^2 + f_j^2(s(t))|\theta'(t)|^2_{g_{\mathbb{S}^2}}}\, dt \\ &\geq \int_a^b |s'(t)|\ dt \\ &\geq \left|\int_a^b s'(t)\ dt\right| \\ &= |s(b) - s(a)| = D_j. \end{aligned} \tag{20}$$

Because $\delta$ is arbitrary, we obtain $d(N, S) = D_j$.

For any $p, q \in M_j$, let $d(p, N) = d_1$, $d(p, S) = d_2$, $d(q, N) = d_3$, and $d(q, S) = d_4$. Then

$$d_1 + d_2 = d_3 + d_4 = d(N, S) = D_j. \tag{21}$$

Indeed, let $p = (t_p, \theta_p) \in M_j$. By a similar argument as in (20), one can easily see that the path $\gamma_1 : [0, t_p] \to [0, D_j] \times \mathbb{S}^2$ defined as $\gamma_1(t) = (t, \theta_p)$ realizes the distance between $N$ and $p$, and so $d(N, p) = t_p$. Similarly, the path $\gamma_2 : [0, D_j - t_p] \to [0, D_j] \times \mathbb{S}^2$ defined as $\gamma_2(t) = (t_p + t, \theta_p)$ realizes the distance between $p$ and $S$, and so $d(p, S) = D_j - t_p$. Thus,

$$d_1 + d_2 = d(N, p) + d(p, S) = t_p + (D_j - t_p) = D_j. \tag{22}$$

Similarly, $d_3 + d_4 = d(N, q) + d(q, S) = D_j$.

We observe that

$$d(p, q) \leq d(p, N) + d(N, q) = d_1 + d_3, \tag{23}$$

and similarly, $d(p, q) \leq d_2 + d_4$. Since $(d_1 + d_3) + (d_2 + d_4) = (d_1 + d_2) + (d_3 + d_4) = 2D_j$, it follows that either $(d_1 + d_3)$ or $(d_2 + d_4)$ is at most $D_j$. Therefore, $d(p, q) \leq D_j$. Thus $\mathrm{Diam}(M_j) = D_j$, and so $D_j \leq D$, since $\mathrm{Diam}(M_j) \leq D$. □

### 2.2. The upper bound on volume

**Lemma 2.2.** $\mathrm{Vol}(M_j) = 4\pi \|f_j\|^2_{L^2([0,D_j])}$.

*Proof.* The volume is given by

$$\text{Vol}(M_j) = \omega_2 \int_0^{D_j} f_j^2(s)\ ds = \omega_2 \|f_j\|^2_{L^2([0,D_j])}, \tag{24}$$

where $\omega_2 = 4\pi$ is the volume of the unit sphere $\mathbb{S}^2$. □

**Lemma 2.3.** *By extending $f_j$ as 0 on $[D_j, D]$,*

$$||f_j||_{L_2([0,D])} \leq \sqrt{\frac{V}{\omega_2}}, \tag{25}$$

*and a subsequence of the $f_j$ converges to some $f \in L^2([0,D])$ weakly.*

*Proof.* By Lemma 2.2, $\text{Vol}(M_j) \leq V$ implies a uniform bound on the $L^2$ norm of $f_j$,

$$\|f_j\|_{L^2([0,D])} = \left(\int_0^D f_j^2\right)^{1/2} = \left(\int_0^{D_j} f_j^2\right)^{1/2} \leq \sqrt{\frac{V}{\omega_2}}, \tag{26}$$

and thus, a subsequence of $f_j$ converges to some $f \in L^2([0,D])$ weakly. □

### 2.3. Scalar curvature bounded below

**Lemma 2.4.** $\text{Scalar}_j = -\dfrac{4f_j''}{f_j} + \dfrac{2(1-(f_j')^2)}{f_j^2}$.

*Proof.* The scalar curvature of the metric $ds^2 + f_j(s)^2 g_{\mathbb{S}^{n-1}}$ is given by

$$\text{Scalar}_j = -2(n-1)\frac{f_j''}{f_j} + (n-1)(n-2)\frac{1-(f_j')^2}{f_j^2} \tag{27}$$

(see [LS14] or [Pet16], section 3.2.3). In our case we have $n = 3$. □

**Lemma 2.5.** $\text{Scalar}_j \geq 0$ *is equivalent to*

$$h_j'' \leq \frac{3}{4} h_j^{-1/3}, \tag{28}$$

*where $h_j(s) = f_j^{3/2}(s)$.*

*Proof.* The lemma follows from substituting $f_j = h_j^{2/3}$. Then we have

$$f_j' = (2/3)h_j^{-1/3}h_j'' \tag{29}$$

and

$$f_j'' = -(2/9)h_j^{-4/3}(h_j')^2 + (2/3)h_j^{-1/3}h_j''. \tag{30}$$

Substituting into Lemma 2.4 and simplifying gives the desired result. □

### 2.4. Lipschitz bound from nonnegative scalar curvature

**Lemma 2.6.** *If* $\mathrm{Scalar}_j \geq 0$ *then* $|f_j'| \leq 1$ *on* $(0, D_j)$.

*Proof.* Suppose to the contrary that $|f_j'(x_0)| > 1$ for some $x_0 \in (0, D_j)$. We may assume that $f_j'(x_0) > 1$. Let $x_1 = \inf\{y \in (0, x_0] \mid f_j'(x) > 1\ \forall x \in (y, x_0]\}$. Then $x_1 < x_0$ and $f_j'(x_1) = 1$ by definition of $x_1$ (if $x_1 = 0$, we understand that $f_j'(0) = 1$ from smoothness of the metric). Since $f_j'$ is $\mathcal{C}^\infty$ on $(0, D_j)$ and continuous at 0, by the mean value theorem there exists $x_2 \in (x_1, x_0)$ such that $f_j''(x_2)(x_0 - x_1) = f_j'(x_0) - f_j'(x_1) > 0$, where the last inequality follows from that $f_j'(x_0) > 1 = f_j'(x_1)$. On the other hand, $x_1 < x_2 < x_0$ implies that $f_j'(x_2) > 1$. Therefore $1 - (f_j'(x_2))^2 < 0$. By Lemma 2.4, $\mathrm{Scalar}_j \geq 0$ implies that

$$f_j''(x_2)(x_0 - x_1) \leq \frac{1 - (f_j'(x_2))^2}{2f_j(x_2)}(x_0 - x_1) < 0. \tag{31}$$

Hence we have that $0 < f_j''(x_2)(x_0 - x_1) < 0$, a contradiction. Therefore $f_j' \leq 1$ on $(0, D_j)$. A similar argument gives that $f_j' \geq -1$. □

### 2.5. The minimum area of minimal surfaces bounded below

**Lemma 2.7.** *If* $f_j'(s_0) = 0$, *then* $\{s = s_0\}$ *is a minimal surface.*

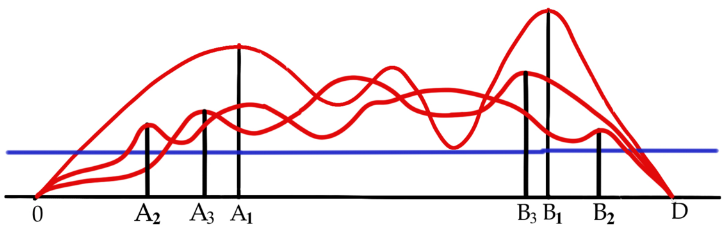
*Proof.* Define $\Sigma_s$ as the level set of the coordinate function $s$. Then for all $s \in (0, D_j)$, $\Sigma_s$ is an embedded submanifold with mean curvature

$$H_j = \frac{2f_j'(s)}{f_j(s)}. \tag{32}$$

Therefore $H_j(s) = 0$ if and only if $\Sigma_s$ is minimal. □

**Definition 2.8.** Define $0 < A_j \le B_j < D_j$ as

$$A_j = \sup\{s \mid f_j \text{ is increasing on } [0,s]\}, \tag{33}$$
$$B_j = \inf\{s \mid f_j \text{ is decreasing on } [s,D_j]\}. \tag{34}$$




**Lemma 2.9.** *$f_j'(A_j) = 0$ and $f_j'(B_j) = 0$. Moreover, $4\pi f_j^2(A_j) \ge \mathrm{MinA}(M_j)$ and $4\pi f_j^2(B_j) \ge \mathrm{MinA}(M_j)$.*

*Proof.* Suppose $f_j'(A_j) > 0$. Then $f_j'(s) > 0$ for $s \in [A_j - \varepsilon, A_j + \varepsilon]$ for some $\varepsilon > 0$. Hence $f_j$ is increasing on the interval $[0, A_j + \varepsilon]$, a contradiction. Similarly, $f_j'(A_j) < 0$ cannot hold, which proves that $f_j'(A_j) = 0$. By the same argument $f_j'(B_j) = 0$. By Lemma 2.7 there is a minimal surface at $s = A_j$ and $s = B_j$, which have areas $4\pi f_j^2(A_j)$ and $4\pi f_j^2(B_j)$ respectively, and hence the conclusion follows from the MinA bound. □

**Lemma 2.10.** *If $A_j < B_j$, then $4\pi f_j^2(s) \ge \mathrm{MinA}(M_j)$ for all $s \in [A_j, B_j]$.*

*Proof.* If $A_j < B_j$, then by continuity there exists $s_0 \in [A_j, B_j]$ such that $f_j(s) \ge f_j(s_0)$ for all $s \in [A_j, B_j]$. By Lemma 2.7 there is a minimal surface at $s = s_0$. As a result, by definition of $\mathrm{MinA}(M_j)$, we have

$$4\pi f_j^2(s) \ge 4\pi f_j^2(s_0) \ge \mathrm{MinA}(M_j) \tag{35}$$

for all $s \in [A_j, B_j]$. □

**Lemma 2.11.**

$$\sqrt{\frac{A}{4\pi}} \le D_j \le D. \tag{36}$$

*Proof.* In Lemma 2.1 we have obtained $D_j \le D$. On the other hand, from Lemma 2.6 and the definition of $A_j$, we have $f_j(A_j) \le A_j \le D_j$. Combining this with Lemma 2.9 gives $\dfrac{A}{4\pi} \le D_j^2$. □

## 3. Examples

**Example 3.1.** We will construct a family of 3-dimensional smooth closed rotationally symmetric Riemannian manifolds $M_j$, which are isometric to three-spheres with the Riemannian metrics $g_j = ds^2 + f_j^2(s)g_{\mathbb{S}^2}$, where $s \in [0, D_j]$, such that

$$\text{Scalar}_j \geq 0, \quad \text{for all} \;\; j \in \mathbb{N}, \tag{37}$$

$$\text{MinA}(M_j) \to 0, \quad \text{as} \;\; j \to \infty, \tag{38}$$

and

$$M_j \xrightarrow{\mathcal{F}} \mathbf{0}, \quad \text{as} \;\; j \to \infty, \tag{39}$$

where $\mathbf{0}$ is the zero current, since $\text{Vol}(M_j) \to 0$ as $j \to \infty$.

Let $\phi_j$ be a sequence of smooth functions defined on $[0, D]$ satisfying:

(a) $\phi_j(0) = 1$, and $\phi_j(D) = -1$;
(b) $\phi_j$ is monotone non-increasing; that is, $\phi_j' \leq 0$;
(c) $\phi_j$ is symmetric about the point $(D/2, 0)$; that is, $\phi_j(s) = -\phi_j(D - s)$ for all $s \in D$;
(d) $\lim_{j\to\infty} \int_0^{\frac{D}{2}} \phi_j(s)\, ds = 0$.

Define functions $f_j$ on $[0, D]$ as

$$f_j(s) := \int_0^s \phi_j(t)\, dt. \tag{40}$$

For example, we can set $D = 2$, and

$$\phi_j(s) = (1 - s)^{2j+1}, \quad \text{defined on} \;\; [0, 2]. \tag{41}$$

Then $f_j = \dfrac{1}{2j + 2} - \dfrac{(1 - s)^{2j+2}}{2j + 2}$ on [0, 2].

From the above properties of $\phi_j$, we have

(a′) $f_j'(0) = 1$, and $f_j'(D) = -1$;
(b′) $f_j'' = \phi_j' \leq 0$, and $|f_j'| = |\phi_j| \leq 1$;
(c′) $f_j \geq 0$ with $f_j(0) = f_j(D) = 0$ and $f_j > 0$ everywhere else;
(d′) $\max_{s\in[0,D]} \{f_j(s)\} = f_j(D/2) \to 0$ as $j \to \infty$.

By (a′) and (c′) above, $g_j = ds^2 + f_j^2(s) g_{\mathbb{S}^2}$ is a smooth Riemannian metric on $\mathbb{S}^3$.

By (b′), $g_j$ have nonnegative scalar curvatures. Indeed, (b′) implies

$$2 f_j(s) f_j''(s) \leq 0 \leq 1 - (f_j'(s))^2 \tag{42}$$

for all $s \in [0, D]$. This further implies

$$\text{Scalar}_j = -\frac{4 f_j''}{f_j} + \frac{2(1-(f_j')^2)}{f_j^2} \geq 0. \tag{43}$$

Finally, by (d′), we have

$$\text{MinA}(M_j) \leq \text{Vol}(\{D/2\} \times \mathbb{S}^2) = 4\pi f_j^2(D/2) \to 0, \tag{44}$$

and

(45)

$$\text{Vol}(M_j) = \int_0^D \int_{\mathbb{S}^2} f_j^2(s)\, d\text{vol}_{g_{\mathbb{S}^2}} = 4\pi \int_0^D f_j^2(s)\, ds \leq 4\pi f_j^2(D/2) D \to 0,$$

as $j \to \infty$.

**Example 3.2** (Example 5.9 in [Lak16])**.** In Example 5.9 in [Lak16], Lakzian has shown that there are metrics $g_j$ on the sphere $\mathbb{S}^3$ with positive scalar curvature such that the family of rotationally symmetric Riemannian manifolds $M_j = (\mathbb{S}^3, g_j)$ has the SWIF limit round sphere $\mathbb{S}^3$, and the Gromov-Hausdorff limit $\mathbb{S}^3 \sqcup [0,1]$, the round sphere $\mathbb{S}^3$ with an interval of length 1 attached to it. Actually, these $M_j$ satisfy all hypotheses in Theorem 1.3. Lakzian has shown that $\text{Scalar}_j > 0$ and $\text{Diam}(M_j) \leq \pi + 3$. Moreover, one can easily check that $\text{MinA}(M_j) = 4\pi$. Lakzian's examples are $\mathbb{S}^3$ with a spline of finite length and arbitrary small width attached to it, and have positive scalar curvature. In Lakzian's construction, he employs some ideas related to the mass of rotationally symmetric manifolds. He first finds an admissible Hawking mass function which provides a three dimensional manifold embedded in $\mathbb{R}^4$ which is a hemisphere to which a spline of finite length and small width is attached. Then a standard hemisphere is attached along its boundary to get $M_j$. For more details about this example we refer to [Lak16].

**Example 3.3** (Example 3.12 in [AS18])**.** In Example 3.12 in [AS18], Allen and Sormani construct a sequence of warped product metrics on $\mathbb{S}^1 \times \mathbb{S}^2$ where the warping functions converge to 1 on a dense set. However, the metrics converge in Gromov-Hausdorff and SWIF sense to a metric space which is not a Riemannian manifold. In fact, no local tangent cone on this limit is isometric to the Euclidean space. It is possible to construct warped product metrics on $\mathbb{S}^3$ by cutting $\mathbb{S}^1$ to get an interval and capping off with hemispheres. Then the tangent cones in the middle region are not Euclidean. For details we refer to [AS18].

## 4. Properties of the limit space

In this section, we will define the limit space with the continuous limit metric, and show that it has Euclidean tangent cones almost everywhere. We will also show that the metric is $H^1$ and has nonnegative scalar curvature in the sense that it satisfies (28) as a distribution.

From now on, we extend the warping functions $f_j$ defined on $[0, D_j]$ to functions defined on $[0, D]$ by setting $f_j = 0$ on $[D_j, D]$. Then $f_j$ are continuous on $[0, D]$ and smooth everywhere on $(0, D)$ except at $D_j$.

Take $0 < A_j \leq B_j < D$ as in Definition 2.8. There is a subsequence such that $A_j \to A_\infty$ and $B_j \to B_\infty$ where

$$0 \leq A_\infty \leq B_\infty \leq D. \tag{46}$$

**Theorem 4.1.** *A subsequence of $f_j$ converge uniformly to a Lipschitz function $f_\infty$ on $[0, D]$, which has Lipschitz constant 1 and satisfies the following properties.*

- (i) *$f_\infty(0) = 0$ and $f_\infty$ is nondecreasing on $[0, A_\infty]$,*
- (ii) *$f_\infty \geq \sqrt{A/4\pi}$ on $[A_\infty, B_\infty]$ if $A_\infty \neq B_\infty$,*
- (iii) *$f_\infty(D) = 0$ and $f_\infty$ is nonincreasing on $[B_\infty, D]$.*

*Proof.* By Lemma 2.6 all functions $f_j$ are Lipschitz with Lipschitz constant 1 on the interval $[0, D]$. Indeed, take any $x < y \in [0, D]$. If $x, y \in [0, D_j]$, then Lemma 2.6 implies $|f_j(x) - f_j(y)| \leq |x - y|$. If $x, y \in [D_j, D]$, then $f_j(x) = f_j(y) = 0$ so $|f_j(x) - f_j(y)| \leq |x - y|$. Finally if $x \leq D_j < y \leq D$, then since $f_j(D_j) = f_j(y) = 0$, we have

$$\frac{|f_j(x) - f_j(y)|}{|x - y|} = \frac{|f(x) - f(D_j)|}{|x - y|} \leq \frac{|f(x) - f(D_j)|}{|x - D_j|} \leq 1. \tag{47}$$

By combining with Arzelà-Ascoli theorem we obtain the uniform convergence. (ii) is then immediate from Lemma 2.10. (i) and (iii) follows from the monotonicity of $f_j$ on $[0, A_j] \cup [B_j, D]$. □

**Lemma 4.2.** *Given sufficiently large $k > 0$, the set*

$$I_k = \left\{ x \,\middle|\, f_\infty(x) \geq \frac{1}{k} \right\} \tag{48}$$

*is a connected interval, $I_k = [a_k, b_k]$.*

*Proof.* $I_k$ is closed since $f_\infty$ is continuous. If $A_\infty \neq B_\infty$, by Lemma 2.10, $f_j > \sqrt{A/4\pi}$ on $[A_j, B_j]$ for all $j$. Since $A_j \to A_\infty$ and $B_j \to B_\infty$ as $j \to \infty$, if $j$ is large then $f_j > \sqrt{A/16\pi}$ on $[A_\infty, B_\infty]$. Take $k$ large enough so that

$$\frac{1}{k} \leq \min \left\{ f_\infty(A_\infty), f_\infty(B_\infty), \sqrt{\frac{A}{16\pi}} \right\}. \tag{49}$$

If $A_\infty = B_\infty$, then take $k$ large enough so that $\frac{1}{k} \leq f_\infty(A_\infty)$. Then by Theorem 4.1, $I_k$ is a connected interval containing $[A_\infty, B_\infty]$. □

Note that $f_\infty(a_k) = f_\infty(b_k) = \dfrac{1}{k}$. We set

$$a_\infty := \sup\{ s \mid f_\infty(t) = 0 \text{ on } [0, s] \} \in [0, A_\infty] \tag{50}$$

and

$$b_\infty := \inf\{ s \mid f_\infty(t) = 0 \text{ on } [s, D] \} \in [B_\infty, D]. \tag{51}$$

Then we immediately have the following lemma.

**Lemma 4.3.** *Let $a = \inf\{a_k \mid k > 0\}$ and $b = \sup\{b_k \mid k > 0\}$ so that*

$$(a, b) = \bigcup_{k>0} I_k. \tag{52}$$

*Then $(a, b) = \{x : f_\infty(x) > 0\}$ so $a = a_\infty$ and $b = b_\infty$.*

**Proposition 4.4.** $f_\infty(a) = f_\infty(b) = 0.$

*Proof.* Since $f_\infty$ is continuous,

$$f_\infty(a) = \lim_{k\to\infty} f_\infty(a_k) = \lim_{k\to\infty} \frac{1}{k} = 0. \tag{53}$$

Similarly $f_\infty(b) = 0$. □

**Definition 4.5.** The limit space

$$M_\infty = [a_\infty, b_\infty] \times \mathbb{S}^2 \tag{54}$$

is a warped product Riemannian manifold, diffeomorphic with $\mathbb{S}^3$, with the continuous metric tensor

$$g_\infty = ds^2 + f_\infty^2(s) g_{\mathbb{S}^2}. \tag{55}$$

**Theorem 4.6.** *The local tangent cones of $M_\infty$ are $\mathbb{E}^3$ almost everywhere.*

*Proof.* Since $f_\infty$ is Lipschitz, it follows that $f_\infty$ is differentiable almost everywhere on $[a_\infty, b_\infty]$; that is, the limit $a_p = \lim_{s\to s_p} \dfrac{f_\infty(s_i) - f_\infty(s_p)}{s_i - s_p}$ exists at almost every $p = (s_p, \theta_p)$. Let $l(s) = f(s_p) + a_p(s - s_p)$ be the linear function that best approximates $f(s)$ at $s = s_p$. Then the tangent cone of $M_\infty$ at $p$ is the warped product $ds^2 + l(s)^2 g_{\mathbb{S}^2}$, which is isometric to the Euclidean space. □

**Remark 4.7.** The scalar curvature control, which in turn gave Lipschitzness of $f_\infty$, is of crucial importance in this argument; compare with Example 3.3.

**Theorem 4.8.** *The sequence $h_j = f_j^{3/2}$, after possibly passing to a subsequence, converges in $H^1_{loc}$ to $h_\infty \in H^1(I)$, where $I$ is the open interval $(a_\infty, b_\infty)$. Defining $f_\infty = h_\infty^{2/3}$, a subsequence of $f_j$ also converges in $H^1_{loc}$ to $f_\infty \in H^1(I)$.*

*Proof.* First we will show that when $k$ is large enough, there is a uniform bound on the variation of $h'_j$, that is,

$$\sup_j \left\| h'_j \right\|_{BV(I_k)} < \infty. \tag{56}$$

By definition of $h_j$ we have

$$h'_j(s) = \frac{3}{2} f_j^{1/2}(s) f'_j(s), \tag{57}$$

for $s \in [0, D_j)$. By Lemma 2.6 we have $|f'_j(s)| \leq 1$ for all $j$ and for all $s \in [0, D_j]$, hence we have $0 \leq f_j(s) \leq D_j/2 \leq D/2$ for all $j$ and all $s \in [0, D_j]$. As a result, $|h'_j(s)| \leq D/2$ for all $s \in [0, D_j)$. Recall that $f_j$ has been extended as 0 on $[D_j, D]$, so $h_j = 0$ on $[D_j, D]$, and $h'_j = 0$ on $(D_j, D]$. Thus by the same argument as in proof of Theorem 4.1, $\{h_j\}$ is uniformly Lipschitz. Then by Arzelà-Ascoli we have that a subsequence of $h_j$ converges to some $h_\infty$ uniformly in $[0,D]$. The limit function $h_\infty$ is also Lipschitz, and $h_\infty = (f_\infty)^{3/2}$ (since we only need this pointwise). Since a Lipschitz function defined on an interval is actually $W^{1,\infty}$, we have $h_\infty \in W^{1,\infty}(I)$. Since for each large enough $j$, $h_j'$ is smooth on $I_k$, we have

$$\begin{aligned}\left\|h_j'\right\|_{BV(I_k)} &= \int_{I_k} \left|h_j'\right| \, ds + \int_{I_k} \left|h_j''\right| \, ds \\ &= \int_{I_k} \left|h_j'\right| \, ds + \int_{\{s\in I_k | h_j''(s)\geq 0\}} h_j'' \, ds - \int_{\{s\in I_k | h_j''(s)< 0\}} h_j'' \, ds.\end{aligned} \tag{58}$$

Note that

$$h_j'' = \frac{3}{4} f_j^{-1/2}(s)(f_j'(s))^2 + \frac{3}{2} f_j^{1/2}(s) f_j''(s). \tag{59}$$

Let $j$ be so large that

$$|h_j(s) - h_\infty(s)| < \frac{1}{3}\left(\frac{1}{k}\right)^{3/2} \tag{60}$$

for all $s \in (a_k, b_k)$. Then by definition of $I_k$, we have

$$\frac{2}{3}\left(\frac{1}{k}\right)^{3/2} \leq h_\infty(s) - \frac{1}{3}\left(\frac{1}{k}\right)^{3/2} \leq h_j(s) \tag{61}$$

for all $s \in (a_k, b_k)$. Moreover, by Lemma 2.5, we have

$$h_j''(s) \leq \frac{3}{4} h_j(s)^{-1/3} \leq \frac{3}{4}\left(\frac{2}{3}\left(\frac{1}{k}\right)^{3/2}\right)^{-1/3} \tag{62}$$

for $j$ large enough and for all $s \in I_k$. As a result, we have when $j$ is large,

$$\int_{\{s\in I_k | h_j''(s)\geq 0\}} h_j'' \, ds \leq \frac{3}{4}\left(\frac{2}{3}\left(\frac{1}{k}\right)^{3/2}\right)^{-1/3} (b_k - a_k). \tag{63}$$

Moreover, since

(64)

$$\int_{\{s\in I_k | h_j''(s)\geq 0\}} h_j'' \, ds + \int_{\{s\in I_k | h_j''(s)< 0\}} h_j'' \, ds = \int_{a_k}^{b_k} h_j''(s) \, ds = h_j'(b_k) - h_j'(a_k),$$

we have

$$
\begin{aligned}
\left\|h_j'\right\|_{BV(I_k)} &= \int_{I_k} \left|h_j'\right| \, ds + \int_{\{s\in I_k | h_j''(s)\geq 0\}} h_j'' \, ds - \int_{\{s\in I_k | h_j''(s)<0\}} h_j'' \, ds \\
&\leq \int_{I_k} \left|h_j'\right| \, ds + 2\int_{\{s\in I_k \ | \ h_j''(s)\geq 0\}} h_j'' \, ds + h_j'(a_k) - h_j'(b_k) \\
&\leq \int_{I_k} \left|h_j'\right| \, ds + \frac{3}{2}\left(\frac{2}{3}\left(\frac{1}{k}\right)^{3/2}\right)^{-1/3} (b_k - a_k) + h_j'(a_k) - h_j'(b_k) \\
&\leq \frac{D}{2}\cdot D + \frac{3}{2}\left(\frac{2}{3}\left(\frac{1}{k}\right)^{3/2}\right)^{-1/3} (b_k - a_k) + \frac{D}{2} + \frac{D}{2}
\end{aligned}
\tag{65}
$$

for all $j$ large enough.

As a result, by Theorem 5.5 in [EG15] we have that $h_j'$ converges to some $\phi \in BV(I_k)$ in $L^1(I_k)$ norm. It is easy to show that $\phi = h_\infty'$ in the weak sense by a density argument. Moreover, since $h_\infty \in W^{1,\infty}(I)$ and

$$
\sup_j \left\|h_j'\right\|_{L^\infty(I_k)} < \infty, \tag{66}
$$

we have $h_j' \to h_\infty'$ in $L^2(I_k)$ norm. Note that by the Hölder inequality,

$$
\int_{I_k} \left|h_j' - h_\infty'\right|^2 \leq \left\|h_j' - h_\infty'\right\|_{L^1(I_k)} \left\|h_j' - h_\infty'\right\|_{L^\infty(I_k)}. \tag{67}
$$

As a result $h_j \to h_\infty$ in $H^1_{loc}(I_k)$ norm.

Now we turn to the convergence of $f_j$. First note that the function $f(\xi) = \xi^{2/3}$ is $C^1$ with $f'(\xi)$ bounded when $\xi \geq \varepsilon > 0$ for some $\varepsilon \in \mathbb{R}$. By the chain rule for weak derivatives we know that the weak derivative of $f_\infty$ exists, and that

$$
f_\infty' = \frac{2}{3} h_\infty^{-1/3} h_\infty'. \tag{68}
$$

Since $h_j \to h_\infty$ uniformly on $[0, D]$, we have $h_j \geq \frac{1}{2}\left(\frac{1}{k}\right)^{3/2} > 0$ on $I_k$ for large $j$. Therefore,

$$
|f_j' - f_\infty'| \leq \frac{2}{3}\left(\frac{1}{2}\left(\frac{1}{k}\right)^{3/2}\right)^{-1/3} |h_j' - h_\infty'| \tag{69}
$$

for large $j$. Since $h_j \to h_\infty$ in $H^1_{loc}(I)$, it follows that $f_j \to f_\infty$ in $H^1_{loc}(I)$. ☐

**Theorem 4.9.** *$g_\infty$ has nonnegative generalized scalar curvature on the interior $\mathring{M}_\infty = M_\infty \setminus (\{a_\infty\} \times \mathbb{S}^2 \cup \{b_\infty\} \times \mathbb{S}^2)$ of $M_\infty$, in the sense that $f_\infty$ satisfies (28) as a distribution on $\mathring{M}_\infty$.*

*Proof.* Fix a large $k$. For $j$ large enough, and for any $u \in C_c^\infty(I_k)$ such that $u \geq 0$, by Lemma 2.4 after some calculation we have

$$\int_{I_k} (1 + (f_j')^2) u \, ds \geq 2 \int_{I_k} (f_j' f_j)' u \, ds. \tag{70}$$

After integration by part on the right hand side, we get

$$\int_{I_k} (1 + (f_j')^2) u \, ds \geq -2 \int_{I_k} (f_j' f_j) u' \, ds. \tag{71}$$

Since

$$\begin{aligned} &\left| \int_{I_k} \left( \left(f_j'\right)^2 - f_\infty'^2 \right) u \, ds \right| \\ &\leq \|u\|_{L^\infty(I_k)} \cdot \left| \left\|f_j'\right\|^2_{L^2(I_k)} - \|f_\infty'\|^2_{L^2(I_k)} \right| \\ &\leq \|u\|_{L^\infty(I_k)} \cdot \left( \left\|f_j'\right\|_{L^2(I_k)} + \|f_\infty'\|_{L^2(I_k)} \right) \cdot \left| \left\|f_j'\right\|_{L^2(I_k)} - \|f_\infty'\|_{L^2(I_k)} \right| \\ &\leq \|u\|_{L^\infty(I_k)} \cdot \left( \left\|f_j'\right\|_{L^2(I_k)} + \|f_\infty'\|_{L^2(I_k)} \right) \cdot \left\|f_j' - f_\infty'\right\|_{L^2(I_k)}, \end{aligned} \tag{72}$$

and

$$\begin{aligned} &\left| \int_{I_k} \left( f_j' f_j - f_\infty' f_\infty \right) u' \, ds \right| \\ &\leq \int_{I_k} \left| f_j' f_j - f_j' f_\infty \right| \cdot |u'| \, ds + \int_{I_k} \left| f_j' f_\infty - f_\infty' f_\infty \right| \cdot |u'| \, ds \\ &\leq \left\|f_j'\right\|_{L^2(I_k)} \cdot \|f_j - f_\infty\|_{L^2(I_k)} \cdot \|u'\|_{L^\infty(I_k)} \\ &\quad + \|f_\infty\|_{L^2(I_k)} \cdot \left\|f_j' - f_\infty'\right\|_{L^2(I_k)} \cdot \|u'\|_{L^\infty(I_k)}, \end{aligned} \tag{73}$$

by Theorem 4.8 and density, we have

$$\int_{I_k} \left(1 + (f_\infty')^2\right) u \, ds \geq -2 \int_{I_k} (f_\infty' f_\infty) \, u' \, ds. \tag{74}$$

Which means for any $u \in C_c^\infty(I)$ with $u \geq 0$, we have

$$\int_I \left(1 + (f_\infty')^2\right) u \, ds \geq 2 \int_I (f_\infty' f_\infty)' \, u \, ds, \tag{75}$$

where we think of $(f'_\infty f_\infty)'$ as a distribution on $I$. Define $\tilde{u}$ on $\mathring{M}_\infty$ by

$$\tilde{u}(s,\theta) := u(s). \tag{76}$$

Then by the previous argument we have

$$\int_{\mathring{M}_\infty} \mathrm{Scalar}_\infty\, \tilde{u}\; d\mathrm{vol}_\infty \geq 0 \tag{77}$$

in the sense of distribution. Here $\mathrm{Scalar}_\infty = \frac{-4(f'_\infty f_\infty)' + 2(1+(f'_\infty)^2)}{f_\infty^2}$ is viewed as a distribution on $\mathring{M}_\infty$, and $d\mathrm{vol}_\infty = f_\infty^2 ds d\mathrm{vol}_{\mathbb{S}^2}$ is the volume form on $\mathring{M}_\infty$.

For a general $\tilde{v} \in C_c^\infty(\mathring{M}_\infty)$ with $\tilde{v} \geq 0$, define

$$v(s) := \int_{\mathbb{S}^2} \tilde{v}(s,\theta)\; d\theta. \tag{78}$$

Since $\mathbb{S}^2$ is compact, differentiation by $s$ commutes with integration. As a result, $v \in C_c^\infty(I)$. By the previous argument we have

$$\int_{\mathring{M}_\infty} \mathrm{Scalar}_\infty\, \tilde{v}\; d\mathrm{vol}_\infty \geq 0 \tag{79}$$

in the sense of distribution. □

**Remark 4.10.** When $\mathrm{Scalar}_j \geq 0$ is replaced by $\mathrm{Scalar}_j \geq H > 0$, as mentioned in Remark 1.4, we can still use $\mathrm{Scalar}_j \geq H > 0$ to get uniform convergence to a Lipschitz function (as in Lemma 2.6 and Theorem 4.1) and $H^1_{loc}$ convergence (as in Theorem 4.8). Then we can use a similar argument as in Theorem 4.9 to show $\mathrm{Scalar}_\infty \geq H > 0$ in the sense of distributions.

In Theorem 4.9 we have obtained nonnegativity of scalar curvature of $g_\infty$ restricted on $\mathring{M}_\infty$ in the sense of distributions. Now we consider generalized scalar curvature in the sense of small volumes at the two poles of $M_\infty$, which are $p_{a_\infty} = (a_\infty, \theta)$ and $p_{b_\infty} = (b_\infty, \theta)$. Recall that on a 3-dimensional smooth Riemannian manifold $(M, g)$ the scalar curvature at a point $p \in M$ can be expressed as

$$\mathrm{Scalar}_g(p) = 30 \cdot \lim_{r\to 0} \frac{\frac{4\pi}{3} r^3 - \mathrm{Vol}(B(p,r))}{\frac{4\pi}{3} r^5}, \tag{80}$$

where $B(p,r)$ is the ball in $M$ centered at $p$ of radius $r$. Thus we will show that $g_\infty$ has nonnegative generalized scalar curvature at points $p_{a_\infty}$ and $p_{b_\infty}$ in the sense of satisfying the following inequalities.

**Proposition 4.11.** *The limit metric $g_\infty$ satisfies*

$$\liminf_{r\to 0} \frac{\frac{4\pi}{3}r^3 - \mathrm{Vol}(B(p_{a_\infty}, r))}{\frac{4\pi}{3}r^5} \geq 0, \tag{81}$$

*and*

$$\liminf_{r\to 0} \frac{\frac{4\pi}{3}r^3 - \mathrm{Vol}(B(p_{b_\infty}, r))}{\frac{4\pi}{3}r^5} \geq 0. \tag{82}$$

*Proof.* We will prove the inequality (81). Using polar coordinates,

$$\mathrm{Vol}(B(p_{a_\infty}, r)) = \int_{a_\infty}^{r+a_\infty} f_\infty(s)^2 \mathrm{Area}(\mathbb{S}^2)\, ds = 4\pi \int_{a_\infty}^{r+a_\infty} f_\infty^2(s) ds. \tag{83}$$

Therefore, to prove (81) it suffices to show

$$\liminf_{r\to 0} \frac{\frac{4\pi}{3}r^3 - 4\pi \int_{a_\infty}^{r+a_\infty} f_\infty^2(s) ds}{\frac{4\pi}{3}r^5} \geq 0. \tag{84}$$

Note that this limit can be written as

$$\liminf_{r\to 0} \frac{3\int_{a_\infty}^{r+a_\infty} ((s-a_\infty)^2 - f_\infty^2(s)) ds}{r^5}. \tag{85}$$

We claim $0 \leq f_\infty(s) \leq s - a_\infty$ for $s \in [a_\infty, a_\infty + \epsilon]$ for small $\epsilon > 0$, from which (81) would follow. Indeed, for any $s \in [a_\infty, a_\infty + \epsilon]$ with a fixed small $\epsilon > 0$, by applying Lebesgue form of the fundamental theorem of calculus for the Lipschitz function $f_\infty$, we have

$$|f_\infty(s)| \leq \int_{a_\infty}^{s} |f'(t)| dt \leq s - a_\infty. \tag{86}$$

Here we used that $f_\infty$ is Lipschitz with Lipschitz constant 1 with $f_\infty(a_\infty) = 0$. Therefore $0 \leq f_\infty(s) \leq s - a_\infty$. The inequality (82) can be shown similarly. ☐

**Remark 4.12.** Whether Proposition 4.11 is true everywhere on $M_\infty$ is an interesting question that we have not been able to answer so far. If it is true, it gives another way of generalizing nonnegativity of scalar curvature to the possibly singular space $M_\infty$, as "small infinitesimal volumes".

## 5. Intrinsic flat convergence to the limit

In this section we will prove that there exists a subsequence of $M_j$ that converges to $M_\infty$ in the sense of the SWIF distance.

Recall that in Theorem 4.1 we obtained the uniform convergence (possibly passing to a subsequence)

$$f_j \to f_\infty \text{ on } I_k \subset (a,b) \subset [0,D]. \tag{87}$$

For each $k > 0$ we define the following sets

$$W_j = W_j^k := \{(s,\theta) \in M_j : s \in I_k,\, \theta \in \mathbb{S}^2\} \subset M_j, \tag{88}$$

$$W_\infty = W_\infty^k := \{(s,\theta) \in M_\infty : s \in I_k,\, \theta \in \mathbb{S}^2\} \subset M_\infty, \tag{89}$$

which are diffeomorphic to $W := I_k \times \mathbb{S}^2$ by diffeomorphisms

$$\psi_j : W \to W_j \quad \text{and} \quad \psi_\infty : W \to W_\infty, \tag{90}$$

with metric tensors induced from $M_j$ and $M_\infty$, respectively. We will use the uniform convergence of the metric tensors $g_j \to g_\infty$ on $W$ to prove the SWIF convergence. To do so, we will apply the following theorem of Lakzian-Sormani [LS13]:

**Theorem 5.1** (Theorem 4.6 in [LS13]). *Suppose $(M_j, g_j)$ and $(M_\infty, g_\infty)$ are oriented precompact Riemannian manifolds with diffeomorphic subregions $W_j \subset M_j$ and $W_\infty \subset M_\infty$. Identify $W_j = W_\infty = W$ by diffeomorphisms $\psi_j : W \to W_j$ and $\psi_\infty : W \to W_\infty$. Assume that on $W$ with induced metrics by $\psi_j$ and $\psi_\infty$ we have*

$$g_j \leq (1+\varepsilon)^2 g_\infty \quad \textit{and} \quad g_\infty \leq (1+\varepsilon)^2 g_j. \tag{91}$$

*Then the SWIF distance satisfies*

$$\begin{aligned} d_{\mathcal{F}}(M_j, M_\infty) \leq \left(2\bar{h} + \alpha\right) \Big[ &\mathrm{Vol}(W_j) + \mathrm{Vol}(W_\infty) + \mathrm{Area}(\partial W_j) \\ &+ \mathrm{Area}(\partial W_\infty)\Big] + \mathrm{Vol}(M_j \setminus W_j) + \mathrm{Vol}(M_\infty \setminus W_\infty), \end{aligned} \tag{92}$$

*where*

$$\bar{h} = \max\left\{h, D_0\sqrt{\varepsilon^2 + 2\varepsilon}\right\}. \tag{93}$$

*Here, $\alpha$, $h$, and $D_0$ are defined as follows:*

$$\max\{\mathrm{Diam}(M_j), \mathrm{Diam}(M_\infty)\} \leq D_0, \tag{94}$$

$$\lambda := \sup_{x,y\in W} \left| d_{M_j}(\psi_j(x), \psi_j(y)) - d_{M_\infty}(\psi_\infty(x), \psi_\infty(y)) \right|, \tag{95}$$

$$h = \sqrt{\lambda(D_0 + \lambda/4)} \leq \sqrt{2\lambda D_0}, \tag{96}$$

$$\alpha \geq \frac{\arccos(1+\varepsilon)^{-1}}{\pi} D_0. \tag{97}$$

First note that by the uniform convergence proven in Theorem 4.1 for any $k > 0$ we can take $j$ large enough to have (91).

## 5.1. Small volumes

Next we show that the volumes of $M_j \setminus W_j$ are small.

**Lemma 5.2.** *For each fixed large $k > 0$, if $j$ is sufficiently large then the following bounds hold.*

$$\mathrm{Vol}(M_j \setminus W_j) \leq \frac{32\pi D}{k^2}, \tag{98}$$

$$\mathrm{Vol}(M_\infty \setminus W_\infty) \leq \frac{4\pi D}{k^2}. \tag{99}$$

*Proof.* We choose and fix a large $k > 0$ so that $I_k$ defined in Lemma 4.2 is a connected interval, $I_k = [a_k, b_k]$. Then,

$$M_j \setminus W_j = \{(s,\theta) \mid s \in [0, a_k) \cup (b_k, D], \quad \theta \in \mathbb{S}^2\} \subset M_j. \tag{100}$$

Because $f_j$ converges to $f_\infty$ uniformly on $[0, D]$ (by passing to a subsequence, if necessary), there exists a large $j_0$ such that for all $j > j_0$,

$$|f_j(s) - f_\infty(s)| < \frac{1}{k}, \qquad \forall s \in [0, D]. \tag{101}$$

In particular, for any $j > j_0$ and any $s \in [0, a_k) \cup (b_k, D]$, we have

$$f_j(s) < f_\infty(s) + \frac{1}{k} < \frac{2}{k}. \tag{102}$$

Thus,

$$\begin{aligned} \mathrm{Vol}(M_j \setminus W_j) &= \int_0^{a_k} \int_{\mathbb{S}^2} f_j^2(s)\ d\mathrm{vol}_{g_{\mathbb{S}^2}} ds + \int_{b_k}^{D} \int_{\mathbb{S}^2} f_j^2(s)\ d\mathrm{vol}_{g_{S^2}} ds \\ &= 4\pi \int_0^{a_k} f_j^2(s)\ ds + 4\pi \int_{b_k}^{D} f_j^2(s)\ ds \\ &< 4\pi \frac{4}{k^2} a_k + 4\pi \frac{4}{k^2}(D - b_k) \\ &\leq \frac{32\pi D}{k^2}. \end{aligned} \tag{103}$$

This completes the proof of the first inequality. Similarly, note that

$$M_\infty \setminus W_\infty = \{(s,\theta) \mid s \in [a_\infty, a_k) \cup (b_k, b_\infty],\quad \theta \in S^2\} \subset M_\infty. \tag{104}$$

Thus, similarly we have

$$\begin{aligned} \mathrm{Vol}(M_\infty \setminus W_\infty) &= \int_{a_\infty}^{a_k} \int_{\mathbb{S}^2} f_\infty^2(s)\ d\mathrm{vol}_{g_{\mathbb{S}^2}} ds + \int_{b_k}^{b_\infty} \int_{\mathbb{S}^2} f_\infty^2(s)\ d\mathrm{vol}_{g_{S^2}} ds \\ &< 4\pi \frac{1}{k^2}(a_k - a_\infty) + 4\pi \frac{1}{k^2}(b_\infty - b_k) \\ &\leq \frac{4\pi D}{k^2}. \end{aligned} \tag{105}$$

This completes the proof of the second inequality. □

### 5.2. Uniformly bounded volumes and areas

**Lemma 5.3.** *For each fixed $k > 0$ we have uniform upper bounds on volumes of the diffeomorphic regions:*

$$\mathrm{Vol}(W_j) \leq \mathrm{Vol}(M_j) \leq 4\pi D^3, \tag{106}$$

*and*

$$\mathrm{Vol}(W_\infty) \leq 4\pi D^3. \tag{107}$$

*Proof.* The first half of the first inequality is clear, since $W_j$ is an embedded Riemannian submanifold of $M_j$. Since $f_j$ converges to $f_\infty$ on $[0, D]$ and

$$0 \leq f_j(s) \leq D, \qquad \forall s \in [0, D], \tag{108}$$

it follows that

$$0 \le f_\infty(s) \le D, \qquad \forall s \in [0, D]. \tag{109}$$

Thus,

$$\mathrm{Vol}(M_j) = \int_0^D \int_{\mathbb{S}^2} f_j^2(s)\ d\mathrm{vol}_{g_{\mathbb{S}^2}} ds \le 4\pi \int_0^D D^2\ ds = 4\pi D^3. \tag{110}$$

Similarly,

$$\mathrm{Vol}(W_\infty) = \int_{a_k}^{b_k} \int_{\mathbb{S}^2} f_\infty^2(s)\ d\mathrm{vol}_{g_{\mathbb{S}^2}} ds \le 4\pi \int_0^D D^2\ ds = 4\pi D^3. \tag{111}$$ □

**Lemma 5.4.** *For each fixed $k > 0$ we have uniform upper bounds on the areas of the boundaries,*

$$\mathrm{Area}(\partial W_j) \le 8\pi D^2, \tag{112}$$
$$\mathrm{Area}(\partial W_\infty) \le 8\pi D^2. \tag{113}$$

*Proof.* For each fixed $k > 0$,

$$\partial W_j = (\{a_k\} \times \mathbb{S}^2, f_j^2(a_k) g_{\mathbb{S}^2}) \cup (\{b_k\} \times \mathbb{S}^2, f_j^2(b_k) g_{\mathbb{S}^2}). \tag{114}$$

Thus,

$$\mathrm{Area}(\partial W_j) = 4\pi f_j^2(a_k) + 4\pi f_j^2(b_k) \le 4\pi D^2 + 4\pi D^2 = 8\pi D^2. \tag{115}$$

Moreover,

$$\partial W_\infty = (\{a_k\} \times \mathbb{S}^2, f_\infty^2(a_k) g_{\mathbb{S}^2}) \cup (\{b_k\} \times \mathbb{S}^2, f_\infty^2(b_k) g_{\mathbb{S}^2}). \tag{116}$$

Thus,

$$\mathrm{Area}(\partial W_\infty) = 4\pi f_\infty^2(a_k) + 4\pi f_\infty^2(b_k) \le 4\pi D^2 + 4\pi D^2 = 8\pi D^2. \tag{117}$$ □

### 5.3. Diameter and distance bounds

In this subsection we prove the SWIF convergence to $M_\infty$. We begin by estimating $h, \bar{h}, \lambda, h, \alpha$ appearing in Theorem 5.1. Following Theorem 5.1, define $D_0$ by

$$D_0 = D + \mathrm{Diam}(M_\infty) + 4\pi. \tag{118}$$

**Lemma 5.5.** *For each fixed large $k > 0$, there exists $j_0(k)$ such that for all $j > j_0(k)$,*

$$\lambda = \sup_{x,y\in W} \left| d_{M_j}(\psi_j(x), \psi_j(y)) - d_{M_\infty}(\psi_\infty(x), \psi_\infty(y)) \right| \le \frac{D_0}{k}. \tag{119}$$

*Thus $h$ and $\bar{h}$ can be made arbitrarily small.*

*Proof.* Because $f_j$ uniformly converges to $f_\infty$ on $[0, D]$, passing to a subsequence if necessarily, for any fixed large $k$, there exists $j_0(k)$ such that for all $j > j_0(k)$ and $s \in [0, D]$ we have

$$|f_j(s) - f_\infty(s)| < \frac{1}{k(k+1)}. \tag{120}$$

Thus, for all $s \in \overline{[0, D] \setminus I_k} = [0, a_k] \cup [b_k, D]$, and all $j > j_0(k)$,

$$f_j(s) < f_\infty(s) + \frac{1}{k(k+1)} \le \frac{1}{k} + \frac{1}{k(k+1)} < \frac{2}{k}. \tag{121}$$

Moreover, because $f_\infty(s) \ge \dfrac{1}{k}$ for all $s \in I_k$, we have that for all $j > j_0(k)$ and $s \in I_k$,

$$f_j(s) < f_\infty(s) + \frac{1}{k(k+1)} \le f_\infty(s) + f_\infty(s)\frac{1}{k+1} \le \left(1 + \frac{1}{k}\right) f_\infty(s), \tag{122}$$

on the other hand,

$$\begin{aligned} f_j(s) > f_\infty(s) - \frac{1}{k(k+1)} \ge f_\infty(s) - f_\infty(s)\frac{1}{k+1} &= \frac{k}{k+1} f_\infty(s) \\ &= \frac{1}{1+\frac{1}{k}} f_\infty(s). \end{aligned} \tag{123}$$

Thus, on $W = I_k \times \mathbb{S}^2$, for all $j > j_0(k)$, we have

$$\left(\frac{1}{1+\frac{1}{k}}\right)^2 g_\infty \le g_j \le \left(1 + \frac{1}{k}\right)^2 g_\infty. \tag{124}$$

Then for any $x, y \in W$, and any piecewise smooth path $\gamma(t)$ lying in $W$ connecting $x$ and $y$, we have

$$\begin{aligned} \left(1 - \frac{\frac{1}{k}}{1+\frac{1}{k}}\right) \sqrt{g_\infty(\gamma'(t), \gamma'(t))} &\le \sqrt{g_j(\gamma'(t), \gamma'(t))} \\ &\le \left(1 + \frac{1}{k}\right) \sqrt{g_\infty(\gamma'(t), \gamma'(t))}. \end{aligned} \tag{125}$$

Therefore

$$-\frac{1}{k}\sqrt{g_\infty(\gamma'(t),\gamma'(t))} \leq \sqrt{g_j(\gamma'(t),\gamma'(t))} - \sqrt{g_\infty(\gamma'(t),\gamma'(t))} \leq \frac{1}{k}\sqrt{g_\infty(\gamma'(t),\gamma'(t))}, \tag{126}$$

that is,

$$|\sqrt{g_j(\gamma'(t),\gamma'(t))} - \sqrt{g_\infty(\gamma'(t),\gamma'(t))}| \leq \frac{1}{k}\sqrt{g_\infty(\gamma'(t),\gamma'(t))}. \tag{127}$$

Now let $L_j(\gamma)$ and $L_\infty(\gamma)$ denote the length of the path $\gamma$ with respect to Riemannian metrics on $(M_j, g_j)$ and $(M_\infty, g_\infty)$ respectively. Then by (127), we have

$$|L_j(\gamma) - L_\infty(\gamma)| \leq \frac{1}{k}L_\infty(\gamma), \tag{128}$$

for any path $\gamma$ in $W$.

Let $\gamma_j : [0,1] \to M_j = [0, D_j] \times \mathbb{S}^2$ be the path that realizes the distance of $\psi_j(x)$ and $\psi_j(y)$ in $(M_j, g_j)$ so that $L_j(\gamma_j) = d_{M_j}(\psi_j(x), \psi_j(y))$. Let $\gamma_\infty$ be the path in $M_\infty = I_\infty \times \mathbb{S}^2$ that realizes the distance of $\psi_\infty(x)$ and $\psi_\infty(y)$ in $(M_\infty, g_\infty)$ so that $L_\infty(\gamma_\infty) = d_{M_\infty}(\psi_\infty(x), \psi_\infty(y))$.

Note $\gamma_j$ may not entirely lie in $W$ even though its endpoints are in $W$. The boundary $\partial W = \{a_k\} \times \mathbb{S}^2 \cup \{b_k\} \times \mathbb{S}^2$ of $W$ has two connected components. If $\gamma_j$ does not entirely lie in $W$, there are two possibilities: either the first and the last intersecting points of $\gamma_j$ and $\partial W$ are in the same component, or the first and the last intersecting points of $\gamma_j$ and $\partial W$ are in different components.

In the first case, without loss of generality, we may assume that the first and the last intersecting points of $\gamma_j$ and $\partial W$ are $p = \gamma_j(t_1)$ and $q = \gamma_j(t_2)$ and both in $\{a_k\} \times \mathbb{S}^2$. Here $t_1 < t_2$. Now we replace $\gamma_j|_{[t_1,t_2]}$ by the shortest geodesic $\delta_j$ connecting $p$ and $q$ in $\{a_k\} \times \mathbb{S}^2$, whose length is less than $\frac{2\pi}{k}$, since the diameter of $\{a_k\} \times \mathbb{S}^2$ is less than $\frac{2\pi}{k}$. Then we obtain a piece-wise smooth curve $\overline{\gamma}_j = \gamma_j|_{[0,t_1]} \cup \delta_j \cup \gamma_j|_{[t_2,1]}$, which is in $W$.

In the second case, without loss of generality, we say that the first intersecting point of $\gamma_j$ and $\partial W$ is $p = \gamma_j(t_1) \in \{a_k\} \times \mathbb{S}^2$ and the last intersecting point of $\gamma_j$ and $\partial W$ is $q = \gamma_j(t_4) \in \{b_k\} \times \mathbb{S}^2$. Let $p' = \gamma_j(t_2)$ be the last intersecting point of $\gamma_j$ and $\{a_k\} \times \mathbb{S}^2$, and $q' = \gamma_j(t_3)$ be the first intersecting point of $\gamma_j$ and $\{b_k\} \times \mathbb{S}^2$ after $p' = \gamma_j(t_2)$. Note $0 \leq t_1 \leq t_2 < t_3 \leq t_4 \leq 1$. Now we replace $\gamma_j|_{[t_1,t_2]}$ by the shortest geodesic $\delta_j$ connecting $p$ and $p'$ in $\{a_k\} \times \mathbb{S}^2$, whose length is less than $\frac{2\pi}{k}$. Similarly, we replace $\gamma_j|_{[t_3,t_4]}$ by the shortest geodesic $\delta_j'$ connecting $q$ and $q'$ in $\{b_k\} \times \mathbb{S}^2$, whose length is also less than $\frac{2\pi}{k}$. Then we obtain a piece-wise smooth curve $\overline{\gamma}_j = \gamma_j|_{[0,t_1]} \cup \delta_j \cup \gamma_j|_{[t_2,t_3]} \cup \delta_j' \cup \gamma_j|_{[t_4,1]}$, which is in $W$.

Similarly, if $\gamma_\infty$ is not entirely in $W$, then we do the same thing as above for $\gamma_\infty$ to obtain $\overline{\gamma}_\infty$, which is in $W$. Clearly, $\overline{\gamma}_j, \overline{\gamma}_\infty \subset W$. Moreover, by the construction of $\overline{\gamma}_j$ and $\overline{\gamma}_\infty$, we can easily obtain

$$\begin{aligned} L_j(\gamma_j) &\leq L_j(\overline{\gamma}_j), & L_\infty(\gamma_\infty) &\leq L_\infty(\overline{\gamma}_\infty), \\ L_j(\overline{\gamma}_j) - \frac{4\pi}{k} &\leq L_j(\gamma_j), & L_\infty(\overline{\gamma}_\infty) - \frac{4\pi}{k} &\leq L_\infty(\gamma_\infty). \end{aligned} \tag{129}$$

Here, in the last inequality we used the fact that

$$f_\infty(a_k) = f_\infty(b_k) = \frac{1}{k}. \tag{130}$$

If $\gamma_j$ (or $\gamma_\infty$) already entirely lies in $W$, then we keep it and simply say $\overline{\gamma}_j = \gamma_j$ (or $\overline{\gamma}_\infty = \gamma_\infty$) so the inequalities in (129) still hold.

Finally, by using inequalities in (128) and (129), we have

$$\begin{aligned} & d_{M_j}(\psi_j(x), \psi_j(y)) - d_{M_\infty}(\psi_\infty(x), \psi_\infty(y)) \\ &= L_j(\gamma_j) - L_\infty(\gamma_\infty) \\ &\leq L_j(\gamma_j) - L_\infty(\overline{\gamma}_\infty) + \frac{4\pi}{k} \\ &\leq L_j(\gamma_j) - \left(\frac{k}{k+1}\right) L_j(\overline{\gamma}_\infty) + \frac{4\pi}{k} \\ &\leq L_j(\gamma_j) - \left(\frac{k}{k+1}\right) L_j(\gamma_j) + \frac{4\pi}{k} \\ &= \frac{1}{k+1} L_j(\gamma_j) + \frac{4\pi}{k} \\ &\leq \frac{\mathrm{Diam}(M_j)}{k+1} + \frac{4\pi}{k} \\ &\leq \frac{D}{k+1} + \frac{4\pi}{k} \\ &\leq \frac{D + \mathrm{Diam}(M_\infty) + 4\pi}{k}, \end{aligned} \tag{131}$$

on the other hand,

$$\begin{aligned} & d_{M_j}(\psi_j(x),\psi_j(y)) - d_{M_\infty}(\psi_\infty(x),\psi_\infty(y)) \\ & = L_j(\gamma_j) - L_\infty(\gamma_\infty) \\ & \geq L_j(\overline{\gamma}_j) - \frac{4\pi}{k} - L_\infty(\gamma_\infty) \\ & \geq \frac{k-1}{k} L_\infty(\overline{\gamma}_j) - \frac{4\pi}{k} - L_\infty(\gamma_\infty) \\ & \geq \frac{k-1}{k} L_\infty(\gamma_\infty) - \frac{4\pi}{k} - L_\infty(\gamma_\infty) \\ & = -\frac{1}{k} L_\infty(\gamma_\infty) - \frac{4\pi}{k} \\ & \geq -\frac{\mathrm{Diam}(M_\infty)+4\pi}{k} \\ & \geq -\frac{D+\mathrm{Diam}(M_\infty)+4\pi}{k} \end{aligned} \tag{132}$$

Since $D_0 + D + \mathrm{Diam}(M_\infty) + 4\pi$, this completes the proof. □

Finally we can prove the following theorem applying these lemmas and carefully balancing the choice of $k$ and taking $j$ large enough.

**Theorem 5.6.** *Under the assumptions in Theorem 1.3, there exists a subsequence of $\{M_j\}$ that converges to $M_\infty$ in SWIF sense.*

*Proof.* By Lemma 5.3 and Lemma 5.4, for all $j$,

$$\mathrm{Vol}(W_j) \leq 4\pi D^3, \qquad \mathrm{Area}(\partial W_j) \leq 8\pi D^2, \tag{133}$$

and

$$\mathrm{Vol}(W_\infty) \leq 4\pi D^3, \qquad \mathrm{Area}(\partial W_\infty) \leq 8\pi D^2. \tag{134}$$

Set

$$D_0 = D + \mathrm{Diam}(M_\infty) + 4\pi. \tag{135}$$

Choose a sufficiently large integer $k$ such that $\arccos\left(\frac{1}{1+\frac{1}{k}}\right) < \frac{2}{\sqrt{k}}$, which implies $\arccos\left(\frac{1}{1+\frac{1}{k+i}}\right) < \frac{2}{\sqrt{k+i}}$, for all positive integers $i$. Now take

$$\alpha = \frac{2D_0}{\pi\sqrt{k}}. \tag{136}$$

By Lemmas 5.2 and 5.5, there exists a subsequence $\{M_1^{(1)}, M_2^{(1)}, M_3^{(1)}, \cdots\}$ of $\{M_j\}$ such that

$$\mathrm{Vol}(M_j^{(1)} \setminus (W_j^{(1)})^k) \leq \frac{32\pi D}{k^2}, \qquad \mathrm{Vol}(M_\infty \setminus W_\infty^k) \leq \frac{4\pi D}{k^2}, \tag{137}$$

and

$$\lambda \leq \frac{D_0}{k}. \tag{138}$$

Then

$$h \leq \sqrt{2\lambda D_0} \leq \frac{2D_0}{\sqrt{k}}, \tag{139}$$

and

$$\bar{h} = \max\left\{h, D_0\sqrt{\frac{1}{k^2}+\frac{2}{k}}\right\} \leq \frac{2D_0}{\sqrt{k}}. \tag{140}$$

By Theorem 5.1,

$$\begin{aligned} d_{\mathcal{F}}(M_j^{(1)}, M_\infty) &\leq \left(2\bar{h}+\alpha\right)\Big[\mathrm{Vol}((W_j^{(1)})^k) + \mathrm{Vol}(W_\infty^k) + \mathrm{Area}(\partial(W_j^{(1)})^k) \\ &\quad + \mathrm{Area}(\partial W_\infty^k)\Big] + \mathrm{Vol}(M_j^{(1)} \setminus (W_j^{(1)})^k) + \mathrm{Vol}(M_\infty \setminus W_\infty^k) \\ &\leq \left(\frac{4D_0}{\sqrt{k}} + \frac{2D_0}{\pi\sqrt{k}}\right)(8\pi D^3 + 16\pi D^2) + \frac{36\pi D}{k^2} \\ &\leq \left(\frac{1}{\sqrt{k}}\right)\left[\left(4D_0 + \frac{2D_0}{\pi}\right)(8\pi D^3 + 16\pi D^2) + 36\pi D\right], \end{aligned} \tag{141}$$

for all $j$.

Similarly, we can take a subsequence

$$\{M_1^{(2)}, M_2^{(2)}, M_3^{(2)}, \cdots\} \subset \{M_1^{(1)}, M_2^{(1)}, M_3^{(1)}, \cdots\} \tag{142}$$

such that

$$d_{\mathcal{F}}(M_j^{(2)}, M_\infty) \le \left(\frac{1}{\sqrt{k+1}}\right)\left[\left(4D_0 + \frac{2D_0}{\pi}\right)(8\pi D^3 + 16\pi D^2) + 36\pi D\right], \tag{143}$$

for all $j$.

Repeating this process, we have for each positive integer $i$, there are subsequence $\{M_1^{(i+1)}, M_2^{(i+1)}, M_3^{(i+1)}, \cdots\} \subset \{M_1^{(i)}, M_2^{(i)}, M_3^{(i)}, \cdots\}$ such that

$$d_{\mathcal{F}}(M_j^{(i+1)}, M_\infty) \le \left(\frac{1}{\sqrt{k+i}}\right)\left[\left(4D_0 + \frac{2D_0}{\pi}\right)(8\pi D^3 + 16\pi D^2) + 36\pi D\right], \tag{144}$$

for all $j$.

Finally, we take the subsequence $\{M_1^{(1)}, M_2^{(2)}, M_3^{(3)}, \cdots\}$. For any $\varepsilon > 0$, there exists a positive integer $i_0$ such that for all $i > i_0$,

$$\left(\frac{1}{\sqrt{k+i}}\right)\left[\left(4D_0 + \frac{2D_0}{\pi}\right)(8\pi D^3 + 16\pi D^2) + 36\pi D\right] < \varepsilon. \tag{145}$$

Thus, $d_{\mathcal{F}}(M_i^{(i)}, M_\infty) < \varepsilon$ for all $i > i_0$. This completes the proof. □

## Acknowledgements

This project began at the Summer School for Geometric Analysis as a part of the Thematic Program on Geometric Analysis in July 2017 at the Fields Institute for Research in Mathematical Sciences. The authors would like to thank Spyros Alexakis, Walter Craig, Robert Haslhofer, Spiro Karigiannis, and McKenzie Wang for organizing the program. The authors also thank the Fields Institute for providing an excellent research environment. The authors would like to express their gratitude to Christina Sormani for giving an inspiring course on the subject in the summer school, grouping them together for the project, and her constant support and guidance through numerous helpful discussions. The authors are grateful to Brian Allen, Christian Ketterer, Chen-Yun Lin, and Raquel Perales for serving as teaching assistants during Sormani's course. The authors would like to thank Hanci Chi for many useful discussions. The authors participated in workshops to work on this project, which were funded by Sormani's NSF grant DMS-1309360 and DMS-1612049. The authors would like to thank referees for carefully reading the manuscript and providing many helpful suggestions.

## References


[AHPPW18] Allen, B., Hernandez-Vazquez, L., Parise, D., Payne, A, Wang, S.: *Warped tori with almost non-negative scalar curvature*, Geom. Dedicata (2018). https://doi.org/10.1007/s10711-018-0365-y. MR3956190

[AK00] Ambrosio, L., Kirchheim, B.: *Currents in metric spaces*, Acta Math., **185**(1) (2000), 1–80. MR1794185

[AS18] Allen, B., Sormani, C.: *Contrasting various notions of convergence in geometric analysis*, Preprint. arXiv:1803.06582.

[Bam16] Bamler, R.: *A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature*, Mathematical Research Letters **23**(2) (2016), 325–337. MR3512888

[BDS17] Basilio, J., Dodziuk, J., Sormani, C.: *Sewing Riemannian Manifolds with Positive Scalar Curvature*, J. Geom. Anal. **28**(4) (2018), 3553–3602. https://doi.org/10.1007/s12220-017-9969-y. MR3881982

[EG15] Evans, L., Gariepy, R.: *Measure Theory and Fine Properties of Functions*, Revised edition, Textbooks in Mathematics, CRC Press, 2015. MR3409135

[FF60] Federer, H., Fleming, W.H.: *Normal and integral currents*, Ann. Math. **72**(2) (1960), 458–520. MR0123260

[Gro14a] Gromov, M.: *Plateau-Stein manifolds*, Cent. Eur. J. Math. **12**(7) (2014), 923–951. MR3188456

[Gro14b] Gromov, M.: *Dirac and Plateau billiards in domains with corners*, Cent. Eur. J. Math., **12**(8) (2014), 1109–1156. MR3201312

[GS18] Gromov, M., Sormani, C.: *Emerging Topics: Scalar Curvature and Convergence*, Institute for Advanced Study Emerging Topics Report (2018).

[Lak16] Lakzian, S.: *On diameter controls and smooth convergence away from singularities*, Differential Geom. Appl., **47** (2016), 99–129. MR3504922

[Li17] Li, C.: *A polyhedron comparison theorem for 3-manifolds with positive scalar curvature*, Preprint. arXiv:1710.08067.

[LM07] LeFloch, P. G., Mardare, C.: *Definition and stability of Lorentizian manifold with distributional curvature*, Preprint. arXiv:0712.0122. MR2374400

[LS13] Lakzian, S., Sormani, C.: *Smooth convergence away from singular sets*, Comm. Anal. Geom., **21**(1) (2013), 39–104. MR3046939

[LS14] Lee, D. A., Sormani, C.: *Stability of the positive mass theorem for rotationally symmetric Riemannian manifolds*, J. reine angew. Math. (Crelle's Journal), **686** (2014), 187–220. MR3176604

[LS15] LeFloch, P. G., Sormani, C.: *The nonlinear stability of rotationally symmetric spaces with low regularity*, J. Funct. Anal., **268**(7) (2015), 2005–2065. MR3315585

[Pet16] Petersen, P.: *Riemannian geometry*, Third edition, Graduate Texts in Mathematics, **171**, Springer, 2016. MR3469435

[Sor17] Sormani, C.: *Scalar curvature and intrinsic flat convergence*, Measure Theory in Non-Smooth spaces, De Gruyter Press, edited by Nicola Gigli. (2017) 288–338. MR3701743

[SW10] Sormani, C., Wenger, S.: *Weak convergence and cancellation*, Calc. Var. Partial Differential Equations, **38**(1–2) (2010), 183–206. MR2610529

[SW11] Sormani, C., Wenger, S.: *The intrinsic flat distance between Riemannian manifolds and other integral current spaces*, J. Differential Geom., **87**(1) (2011), 117–199. MR2786592

[Wen11] Wenger, S.: *Compactness for manifolds and integral currents with bounded diameter and volume*, Calc. Var. Partial Differential Equations **40**(3-4) (2011), 423–448. MR2764913

Jiewon Park
182 Memorial Drive
Cambridge
Massachusetts 02139
United States
E-mail: jiewon@mit.edu

Wenchuan Tian
619 Red Cedar Road, C212 Wells Hall
East Lansing
Michigan 48824
United States
E-mail: tian.wenchuan@gmail.com

Changliang Wang
Max-Planck-Institut für Mathematik
Vivatsgasse 7
Bonn 53111
Germany
E-mail: cwangmath@outlook.com