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Unfolding of orbifold LG B-models: a case study

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Abstract: In this note we explore the variation of Hodge structures associated to the orbifold Landau-Ginzburg B-model whose superpotential has two variables. We extend the Getzler-Gauss-Manin connection to Hochschild chains twisted by group action. As an application, we provide explicit computations for the Getzler-Gauss-Manin connection on the universal (noncommutative) unfolding of $\mathbb{Z}_2$-orbifolding of A-type singularities. The result verifies an example of deformed version of Mckay correspondence.

1. Introduction

Associated to a triple (A, W, G), where A is an associative algebra over $\mathbb{C}$ with a compatible G-action and W is a G-invariant central element of A, we consider a curved algebra $A_W[G] := A \rtimes \mathbb{C}[G]$ with W as a curvature. In this note, we investigate the deformation theory and Hodge structures for a certain type of such curved algebras.

In [11], we have shown that the compact type Hochschild cohomology $\mathrm{HH}^\bullet_c(A_W[G], A_W[G])$ is isomorphic to the G-invariant subspace $\mathrm{HH}^\bullet_c(A_W, A_W[G])^G$ as Gerstenhaber algebras. As a consequence, the deformation of $A_W[G]$ is controlled by the differential graded Lie algebra (dgLa) of the Hochschild cochains $(C^\bullet_c(A_W, A_W[G])^G, \delta_b, \{\ ,\ \})$. In this paper, we study the polynomial algebra in two variables $A = \mathbb{C}[x, y]$. This includes orbifold ADE singularities as our main interest in this paper. Obstruction theoretical computation shows that the relevant dgLa is un-obstructed, leading to a smooth formal moduli space $\mathcal{M}$ which is locally parameterized by the Hochschild cohomology $\mathrm{HH}^\bullet(A_W[G], A_W[G])$.

Our study of this moduli space $\mathcal{M}$ is motivated by Saito's work [21] on isolated singularities, which is related to so-called Landau-Ginzburg (LG) B-models in modern terminology. In [21], it was shown that the deformation space of an isolated singularity carries a version of variation of polarized Hodge structures with semi-infinite filtrations. It leads to an integrable structure on the tangent bundle of the moduli, which is nowadays called Frobenius

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manifold. This Frobenius manifold structure plays a central role in topological field theories, especially in Gromov-Witten type theories. For example, the data of Frobenius manifold on the deformation of isolated polynomial singularities is mirror to the data of counting solutions of Witten's equation on Riemann surfaces, a theory known as Fan-Jarvis-Ruan-Witten (FJRW) theory [7] for Landau-Ginzburg A-models. However, Saito's construction only involves 'un-orbifold' cases $(A, W, G = \langle 1 \rangle)$, while the full mirror symmetry between Landau-Ginzburg models asks for all orbifold groups. This requires the construction and computation of Frobenius manifold structure on the aforementioned moduli space $\mathcal{M}$.

Barannikov [1, 2] and Barannikov-Kontsevich [4] introduced the important notion of (polarized) variation of semi-infinite Hodge structures (VSHS), generalizing Saito's framework to many other geometric contexts and noncommutative world [1, 14]. Following this route, we shall consider the period cyclic homology of a deformed algebra of $A_W[G]$, with a Hodge filtration induced by the cyclic parameter u and the flat Gauss-Manin connection constructed by Getzler [10]. They give rise to a flat bundle over the moduli space $\mathcal{M}$, carrying important data of Hodge filtration. In this note, we establish a version of the Getzler-Gauss-Manin connection via operations of G-twisted cochains $C_c^{\bullet}(A, A[G])^G$ acting on the G-twisted chains $C_{\bullet}^c(A, A[G])_G$. This encodes the same information as the Getzler-Gauss-Manin connection on the deformation space of the algebra $A[G]$, but is easier to compute in practice. As an application, we perform a case study for orbifold A-type singularity $(A_{2n-1}, \mathbb{Z}_2)$. We find (see Theorem 4.1),

Theorem. *Consider an orbifold LG B-model (A, W, G) with $A = \mathbb{C}[x, y]$, W invertible and a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ acting diagonally on $\mathbb{C}^2$. The moduli space $\mathcal{M}$ of miniversal deformations of $A_W[G]$ is smooth, equipped with a variation of semi-infinite Hodge structures (VSHS) given by a flat vector bundle of period cyclic homologies. In this fashion, there is an isomorphism between the moduli spaces associated to $(\mathbb{C}[x, y], x^{2n} + y^2, \mathbb{Z}_2)$ and $(\mathbb{C}[z, w], z^n + zw^2, \langle 1 \rangle)$, which is compatible with the VSHS's on them.*

It can be seen as an example of Mckay correspondence for LG models [20], but involves the deformation data. Here, $(\mathbb{C}[x, y], x^{2n} + y^2, \mathbb{Z}_2)$ is associated to the A_{2n-1}-singularity $W = x^{2n} + y^2$ on an orbifold $X = \mathbb{C}^2/\mathbb{Z}_2$ and $(\mathbb{C}[z, w], z^n + zw^2, \langle 1 \rangle)$ is associated to the D_{n+1}-singularity $\tilde{W} = z^n + zw^2$ on $\mathbb{C}^2 \xhookrightarrow{i} Y$, where $\pi\colon Y \to X$ is the minimal resolution (so it is crepant) and $\tilde{W} = i^* \circ \pi^*(W)$.

There are three directions of generalizations of such a correspondence. One is for more general triples $(A = \mathcal{O}(X), W, G)$ as long as the crepant resolution of X/G exists and the lifting superpotential W has good Hodge theoretical properties (see [17] for a recent discussion on this model and references therein). The second is to establish the correspondence between VSHS's via crepant resolutions and related mirror symmetry. This involves a combination of LG/CY correspondence and mirror symmetry. Thirdly, there is a categorical approach to the orbifold LG models, which is called the equivariant matrix factorization (see, for example, [6, 19, 23]). It would be very interesting to compare the categorical deformation theory with our calculations. We hope to investigate these problems in future works.

2. Preliminary

In this note, $\mathbb{C}$ is taken as the base field for convenience. For a $\mathbb{Z}$ or $\mathbb{Z}_2$-graded vector space A, we denote by sA its suspension, where $(sA)_k := (A[-1])_k = A_{k-1}$. We use the Koszul sign convention and regard s as a degree 1 element. Given two graded vector spaces A and M, the spaces of Hochschild (co)chains and compact type Hochschild (co)chains are defined as

$$C^\bullet(A,M) := \prod_{p\geqslant 0} \mathrm{Hom}((sA)^{\otimes p}, sM)[1],$$
$$C^\bullet_c(A,M) := \bigoplus_{p\geqslant 0} \mathrm{Hom}((sA)^{\otimes p}, sM)[1],$$
$$C_\bullet(A,M) := \bigoplus_{p\geqslant 0} sM\otimes (sA)^{\otimes p}[1], \qquad C^c_\bullet(A,M) := \prod_{p\geqslant 0} sM\otimes (sA)^{\otimes p}[1].$$

We will write $[a_1|\cdots|a_p]$ for an element in $(sA)^{\otimes p}$, $m[a_1|\cdots|a_p]$ an element in $M\otimes(sA)^{\otimes p}$ and $\phi[a_1|\cdots|a_p]$ the value of $\phi\in C^p(A,M)$ acting on $[a_1|\cdots|a_p]$.

Remark. For each $\phi\in C^p(A,M)$, we can associate $\phi\circ s^{\otimes p}\in \mathrm{Hom}(A^{\otimes p}, M)$ as

$$\phi\circ s^{\otimes p}(a_1\otimes\cdots\otimes a_p) := (-1)^{\sum\limits_{k=1}^{p-1}(p-k)|a_k|}\phi[a_1|\cdots|a_p].$$

This fixes our sign conventions for Hochschild (co)chains.

There are two different gradings for these (co)chains, the tensor grading and the internal grading, which are determined by the grading of A and M. We denote by $|\cdot|$ the internal grading. For example, for a homogeneous (with respect to both gradings) cochain $\phi\in C^p(A,M)$,

$$|\phi| = |\phi[a_1|\cdots|a_p]| - |a_1| - \cdots - |a_p| - p. \tag{2.1}$$

In [25], Gerstenhaber introduced the brace structure by higher operations, the braces on (compact type) Hochschild cochains. For homogeneous $\phi \in C^p(A, B)$ and $\phi_k \in C^{p_k}(B, C)$, we can define for $m = p + p_1 + \cdots + p_n - n$,

$$\begin{aligned} &\phi\{\phi_1, \cdots, \phi_n\}[a_1|\cdots|a_m] \\ &:= (-1)^{\sum_{k=1}^{n} \epsilon_{i_k}(|\phi_k|+1)} \phi[a_1|\cdots|a_{i_k}|\phi_k[a_{i_k+1}|\cdots|a_{i_k+p_k}]|\cdots|a_m], \end{aligned} \tag{2.2}$$

where

$$\epsilon_i = \sum_{j=1}^{i} |a_j| - i.$$

Notice that there is a one-shifted Lie algebraic structure on (compact type) Hochschild cochains $C^\bullet(A, A)$ (or $C_c^\bullet(A, A)$). It is defined [8] as the commutator of Gerstenhaber product (the brace operation with only one input),

$$\{\phi_1, \phi_2\} := \phi_1\{\phi_2\} - (-1)^{(|\phi_1|+1)(|\phi_2|+1)} \phi_2\{\phi_1\}. \tag{2.3}$$

In this note, we will work with 2-dimensional orbifold Landau-Ginzburg models (A_W, G). Here A_W is denoted for a curved algebra (A, W), where $A = \mathbb{C}[x, y]$ and W is an invertible polynomial. G with the identity e is a finite group acting diagonally on $\mathbb{C}^2$, which can be extended to an equivariant action on A. W is asked to be G-invariant. (See [15, 7, 11] for details.) We will regard A_W as a $\mathbb{Z}/2\mathbb{Z}$-graded A_∞-algebra concentrated in degree zero with $b_0 = -W$, $b_2[a_1|a_2] = (-1)^{|a_1|} a_1 a_2 = a_1 a_2$ and $b_i = 0, \forall i \neq 0, 2$. Similarly, the G-twisted curved algebra $A_W[G]$ is also regarded as a curved algebra on $A \otimes_{\mathbb{C}} \mathbb{C}[G]$ with $b_0 = -W\mathsf{e}$ and $b_2[a_1 g_1|a_2 g_2] = a_1{}^{g_1} a_2 g_1 g_2$ (Thus, $A_W[G] = A[G]_{W\mathsf{e}}$).

For an A_∞-algebra A, we can define boundary operators on the (compact type) Hochschild (co)chains as follows. For $a_0[a_1|\cdots|a_p] \in C_p(A, A)$ (or $C_p^c(A, A)$),

$$\begin{aligned} \partial_b(a_0[a_1|\cdots|a_p]) :=& \sum_{l=1}^{p+1} \sum_{k=p+1-l}^{p} (-1)^{\varepsilon_k(\varepsilon_p - \varepsilon_k)} b_l[a_{k+1}|\cdots|a_0|\cdots][a_{k+l-p}|\cdots|a_k] \\ &+ \sum_{l=0}^{p} \sum_{k=0}^{p-l} (-1)^{\varepsilon_k} a_0[\cdots|b_l[a_{k+1}|\cdots|a_{k+l}]|\cdots|a_p], \end{aligned} \tag{2.4}$$

where

$$\varepsilon_k := |a_0| + \cdots + |a_k| + k + 1, \tag{2.5}$$

and for $\phi \in C^p(A, A)$ (or $C^p_c(A, A)$),

$$\delta_b(\phi) := \{b, \phi\}. \tag{2.6}$$

In our cases,

$$\partial_b = \partial_{b_0} + \partial_{b_2}, \tag{2.7}$$

with

$$\begin{cases} \partial_{b_2}(a_0[a_1|\cdots|a_p]) := & a_0a_1[a_2|\cdots|a_p] + (-1)^p a_p a_0[a_1|\cdots|a_{p-1}] \\ & +\displaystyle\sum_{k=0}^{p-2}(-1)^{k+1}a_0[a_1|\cdots|a_{k+1}a_{k+2}|\cdots|a_p], \\ \partial_{b_0}(a_0[a_1|\cdots|a_p]) := & \displaystyle\sum_{k=0}^{p-1}(-1)^k a_0[a_1|\cdots|a_k|W|a_{k+1}|\cdots|a_p]. \end{cases}$$

and

$$\delta_b\phi = \delta_{b_0}\phi + \delta_{b_2}\phi := (-1)^{p-1}\phi\{W\} + \{b_2, \phi\}. \tag{2.8}$$

The (compact type) Hochschild homology and cohomology are defined as the homology and cohomology of the (compact type) chains and cochains with differentials δ_b and ∂_b respectively.

While A is augmented, we may consider the reduced Hochschild (co)chains defined on $\bar{A} = A/\mathbb{C}$ (see [16] for details).

3. Deformation theory

as we have shown in [11], we can define higher operations on the G-twisted version of (compact type) Hochschild cochains. Thus, there is a Gerstenhaber algebra structure on $\mathrm{HH}^\bullet_c(A_W, A_W[G])^G$, which is isomorphic to $\mathrm{HH}^\bullet_c(A_W[G], A_W[G])$.

Theorem 3.1. *Consider orbifold LG B-models (A, W, G) with $A = \mathbb{C}[x, y]$, W invertible and a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ (the Calabi-Yau condition) acting diagonally on $\mathbb{C}^2$. Then the shifted dgLa (differential graded Lie algebra)*

$$(C^\bullet_c(A, A[G])^G, \delta_b, \{\ ,\ \})$$

is homological abelian.

Proof. Use the cochain version of the explicit homotopy retraction constructed in appendix A,

$$\circlearrowright (C_c^\bullet(\bar{A}, Ag)^G, \delta_b) \rightleftarrows (\mathrm{Jac}(W_g)^G[-l_g], \mathsf{0}),$$

where $W_g := W|_{\mathrm{Fix}(g)}$ and

$$l_g = \begin{cases} 0, g = \mathsf{e}, \\ 2, g \neq \mathsf{e}. \end{cases}$$

By homotopy transfer theorem, we can define a shifted L_∞-structure on the later, such that there exists a quasi-isomorphism between shifted L_∞-algebras,

$$(C_c^\bullet(\bar{A}, A[G])^G, \delta_b, \{\ ,\ \}) \simeq (\mathrm{Jac}(W, G), \mathsf{0}, \ell_2 + \ell_3 + \cdots),$$

where

$$\mathrm{Jac}(W, G) := \bigoplus_{g \in G} \mathrm{Jac}(W_g)^G[-l_g].$$

Notice that the degrees of ℓ_k are all odd, while $\mathrm{Jac}(W, G)$ is concentrated in even degrees. Hence, all of those ℓ_k's are zero and $(C_c^\bullet(A, A[G])^G, \delta_b, \{\ ,\ \})$ is homological abelian. □

We can solve the Maurer-Cartan equation

$$\delta_b(\phi) + \frac{1}{2}\{\phi, \phi\} = 0, \tag{3.1}$$

by the quasi-isomorphism defined above with homotopy transfer. With respect to the tensor grading, the homotopies on Hochschild and Koszul cochains are all of degree -1, and the homotopy on polyvector fields shall not give terms of degree greater than 2. Therefore, Maurer-Cartan elements are in the form of $\phi = \phi_0 + \phi_2 \in C_c^{\mathrm{even}}(A_W, A_W[G])^G$, where ϕ_0 gives a deformation of b_0 and ϕ_2 gives a deformation of b_2. The miniversal deformation space of $A_W[G]$ is denoted by $\mathrm{Def}(A_W, G)$.

Corollary 3.2. *Under the same assumption as Theorem 3.1, a basis of* $\mathrm{HH}_c^\bullet(A_W, A_W[G])^G$ *will give a parametrization of a formal neighbourhood of the origin in* $\mathrm{Def}(A_W, G)$*.*

Notice that there is a decomposition,

$$\mathrm{HH}_c^\bullet(A_W, A_W[G])^G = \Big(\bigoplus_{g \neq \mathsf{e}} \mathrm{HH}_c^\bullet(A_W, A_W g)\Big)^G \oplus \mathrm{HH}_c^\bullet(A_W, A_W \mathsf{e})^G.$$

We call the first summand the twisted sector and the second the untwisted sector.

We can identify the formal neighbourhood of the origin in $\mathrm{Def}(A_W, G)$ with $\mathrm{Spec}(\mathbb{C}[[\boldsymbol{\tau}, \boldsymbol{s}]])$, where variables $\boldsymbol{\tau}$ parameterize deformations in the untwisted sector and $\boldsymbol{s}$ parameterize deformations in the twisted sectors. Let $\mathcal{A}(\boldsymbol{\tau}, \boldsymbol{s})$ denote the corresponding deformed curved algebra, with $b(\boldsymbol{\tau}, \boldsymbol{s})$ the deformed A_∞–products. Miniversality says that the following Kodaira-Spencer map is an isomorphism

$$\begin{aligned} \mathsf{KS}\colon \mathrm{Span}_{\mathbb{C}}\Big\{\frac{\partial}{\partial \boldsymbol{\tau}}, \frac{\partial}{\partial \boldsymbol{s}}\Big\} &\to \mathrm{HH}^\bullet_c(A_W, A_W[G])^G, \\ v \quad &\mapsto \big[v(b(\boldsymbol{\tau}, \boldsymbol{s}))|_{\boldsymbol{\tau}=\boldsymbol{s}=\mathbf{0}}\big]. \end{aligned}$$

Also notice that the first order deformation along the untwisted sector deforms the superpotential, while that along the twisted sector deforms the product of the semi-direct product polynomial ring $\mathbb{C}[x, y][G]$.

Remark. In [18], Nadaud introduced three different forms of "q-Moyal products" as q-deformations of $\mathbb{C}[x, y]$. By choosing different homotopy retractions between the Hochschild cochains and Koszul cochains, one can recover these q-Moyal products and his rigidity indicates that our deformation is in fact somewhat canonical. One should also notice that our deformation is equivalent to that constructed by Halbout, Oudom and Tang in [13]. Especially, their modified superpotential is the same as our deformed b_0 via homotopy transfer.

E. Getzler [10] introduced higher operations $\boldsymbol{b}$ and $\boldsymbol{B}$ on $C^\bullet(A, A)$ with values in $\mathrm{End}(C_\bullet(\bar{A}, A))$ for any A_∞-algebra A, as extensions of Hochschild differential ∂_b and Connes operator B. Here $\bar{A} = \mathbb{C}[x, y]/\mathbb{C}$. For homogeneous $\phi_1, \cdots, \phi_n \in C^\bullet(A, A)$ and $a_0[a_1|\cdots|a_m] \in C_m(\bar{A}, A)$, he defined for $n \geqslant 1$,

$$\begin{aligned} &\boldsymbol{b}\{\phi_1, \cdots, \phi_n\}(a_0[a_1|\cdots|a_m]) \\ &\quad := \sum_{\boldsymbol{J}\in\mathcal{J}} \sum_{l\geqslant 1} (-1)^{\eta_{\boldsymbol{J}}} b_l[a_{j_0+1}|\cdots|\phi_1[a_{j_1+1}|\cdots]| \\ &\qquad\quad \cdots|\phi_n[a_{j_n+1}|\cdots]|\cdots|a_0|\cdots][\cdots|a_{j_0}], \end{aligned} \tag{3.2}$$

where

$$\mathcal{J} = \left\{ \boldsymbol{J} = (j_0, \cdots, j_n) \;\middle|\; \begin{cases} m - (l-1) - \sum_{k=1}^{n} |\phi_k| + n \leqslant j_0 \leqslant j_1, j_n + |\phi_n| \leqslant m, \\ j_k + |\phi_k| \leqslant j_{k+1}, \forall 1 \leqslant k \leqslant n-1. \end{cases} \right\},$$

and

$$\eta_{\boldsymbol{J}} = \varepsilon_{j_0}(\varepsilon_m - 1) + \sum_{k=1}^{n}(|\phi_k| - 1)(\varepsilon_{j_k} - \varepsilon_{j_0}), \tag{3.3}$$

and

$$\begin{aligned} &\boldsymbol{B}\{\phi_1, \cdots, \phi_n\}(a_0[a_1|\cdots|a_m]) \\ &\quad := \sum_{\boldsymbol{J}\in\mathcal{J}} \sum_{l\geqslant 1} (-1)^{\eta_{\boldsymbol{J}}} 1[a_{j_0+1}|\cdots|\phi_1[a_{j_1+1}|\cdots]| \\ &\qquad \cdots|\phi_n[a_{j_n+1}|\cdots]|\cdots|a_0|\cdots|a_{j_0}]. \end{aligned} \tag{3.4}$$

For $n=0$, $\boldsymbol{b}\{\} := \partial_b$ and $\boldsymbol{B}\{\} := B$. Using $\boldsymbol{b}$ and $\boldsymbol{B}$, he defined the Getzler-Gauss-Manin system for formal deformations of an A_∞-algebra.

We will first extend his constructions to the case of G-twisted chains $C^c_\bullet(\bar{A}, A[G])_G$, which is much smaller than the space of reduced chains $C^c_\bullet(\overline{A[G]}, A[G])$. This will greatly simplify the calculation of the connection in practice.

Consider chain maps Ψ_* and Γ_* between $C^c_\bullet(A, A[G])_G$ and $C^c_\bullet(A[G], A[G])$ defined as follows. For $a_0g_0[a_1g_1|\cdots|a_pg_p] \in C_p(A[G], A[G])$ and $a_0g_0[a_1|\cdots|a_p] \in C_p(A, A[G])$,

$$\Gamma_* \circ \pi(a_0g_0[a_1|\cdots|a_p]) := \frac{1}{|G|}\sum_{g\in G} {}^g a_0 g g_0 g^{-1}[{}^g a_1 e|\cdots|{}^g a_p e], \tag{3.5}$$

$$\Psi_*(a_0g_0[a_1g_1|\cdots|a_pg_p]) := \pi\big({}^{g_1\cdots g_p}a_0 g_1\cdots g_p g_0[a_1|{}^{g_1}a_2|\cdots|{}^{g_1\cdots g_{p-1}}a_p]\big). \tag{3.6}$$

Here, $\pi\colon C_\bullet(A, A[G]) \to C_\bullet(A, A[G])_G$ is the natural projection. One can easily check that Γ_* is well-defined.

A Getzler-Gauss-Manin connection is a connection defined in terms of the mixed complex $(C_\bullet, \partial, B)$ of a deformed A_∞-algebra, for example, on

$$\big(C^c_\bullet(\overline{\mathcal{A}(\boldsymbol{\tau},\boldsymbol{s})}, \mathcal{A}(\boldsymbol{\tau},\boldsymbol{s})), \partial_{b(\boldsymbol{\tau},\boldsymbol{s})}, B\big) = \big(C^c_\bullet(\overline{A[G]}, A[G])[[\boldsymbol{\tau},\boldsymbol{s}]], \partial_{b(\boldsymbol{\tau},\boldsymbol{s})}, B\big).$$

On twisted chains, we can also define such a mixed complex by defining

$$\tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})} := \Psi_* \circ \partial_{b(\boldsymbol{\tau},\boldsymbol{s})} \circ \Gamma_*, \tag{3.7}$$

$$\tilde{B} := \Psi_* \circ B \circ \Gamma_*. \tag{3.8}$$

More explicitly, if we write $b(\boldsymbol{\tau},\boldsymbol{s}) \in C^\bullet(A, A[G])$ as

$$b(\boldsymbol{\tau},\boldsymbol{s}) = \sum_{h\in G}(b_2^h h - W^h h),$$

with $b_2^h \in C^2(A,A)[\boldsymbol{s}]$ and $W^h \in A[\boldsymbol{\tau},\boldsymbol{s}]$, we can write

$$\tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})} = \tilde{\partial}_{b_2(\boldsymbol{\tau},\boldsymbol{s})} + \tilde{\partial}_{b_0(\boldsymbol{\tau},\boldsymbol{s})},$$

where

$$\begin{aligned}
&\tilde{\partial}_{b_2(\boldsymbol{\tau},\boldsymbol{s})} \circ \pi(a_0 g_0[a_1|\cdots|a_p]) \\
&= \sum_{h\in G} \pi\Big(b_2^h[a_0|{}^{g_0}a_1]hg_0[a_2|\cdots|a_p] + (-1)^p b_2^h[a_p|a_0]hg_0[a_1|\cdots|a_{p-1}] \\
&\quad + \sum_{k=0}^{p-2}(-1)^{k+1} {}^h a_0 hg_0[a_1|\cdots|b_2^h[a_{k+1}|a_{k+2}]|{}^h a_{k+3}|\cdots|{}^h a_p]\Big),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\tilde{\partial}_{b_0(\boldsymbol{\tau},\boldsymbol{s})} \circ \pi(a_0 g_0[a_1|\cdots|a_p]) \\
&= \sum_{h\in G} \pi\Big(\sum_{k=0}^{p-2}(-1)^{k} {}^h a_0 hg_0[\cdots|a_k|W^h|{}^h a_{k+1}|\cdots|{}^h a_p]\Big).
\end{aligned} \tag{3.10}$$

Similarly,

$$\tilde{B} \circ \pi(a_0 g_0[a_1|\cdots|a_p]) := \pi\Big(\sum_{k=0}^{p}(-1)^{kp} 1 g_0[a_k|\cdots|a_p|a_0|{}^{g_0}a_1|\cdots|{}^{g_0}a_{k-1}]\Big). \tag{3.11}$$

One can directly check that

$$(C^c_\bullet(\bar{A}, A[G])_G[[\boldsymbol{\tau},\boldsymbol{s}]], \tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})}, \tilde{B})$$

is a mixed complex. Furthermore, Ψ_* will induce a morphism between mixed complexes,

$$\Psi_*\colon (C^c_\bullet(\overline{A[G]}, A[G])[[\boldsymbol{\tau},\boldsymbol{s}]], \partial_{b(\boldsymbol{\tau},\boldsymbol{s})}, B) \to (C^c_\bullet(\bar{A}, A[G])_G[[\boldsymbol{\tau},\boldsymbol{s}]], \tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})}, \tilde{B}).$$

Lemma 3.3. *$\tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})}$ and $\tilde{B}$ defined above can be extended to higher operations $\tilde{\boldsymbol{b}}(\boldsymbol{\tau},\boldsymbol{s})$ and $\tilde{\boldsymbol{B}}$ on $C_c^\bullet(A, A[G])^G[[\boldsymbol{\tau},\boldsymbol{s}]]$ with values in $\mathrm{End}(C^c_\bullet(\bar{A}, A[G])_G)[[\boldsymbol{\tau}, \boldsymbol{s}]]$, such that these higher operations are also compatible with Ψ_*.*

Proof. Similar as above, for homogeneous $\phi_1, \cdots, \phi_n \in C^\bullet(A, A[G])^G$, we define

$$\tilde{\boldsymbol{b}}(\boldsymbol{\tau}, \boldsymbol{s})\{\phi_1, \cdots, \phi_n\} := \Psi_* \circ \boldsymbol{b}(\boldsymbol{\tau}, \boldsymbol{s})\{\Psi^*(\phi_1), \cdots, \Psi^*(\phi_n)\} \circ \Gamma_*, \tag{3.12}$$

$$\tilde{\boldsymbol{B}}\{\phi_1, \cdots, \phi_n\} := \Psi_* \circ \boldsymbol{B}\{\Psi^*(\phi_1), \cdots, \Psi^*(\phi_n)\} \circ \Gamma_*. \tag{3.13}$$

Here, $\boldsymbol{b}(\boldsymbol{\tau}, \boldsymbol{s})$ and $\boldsymbol{B}$ are the higher operations on $C_c^\bullet(A[G], A[G])[[\boldsymbol{\tau}, \boldsymbol{s}]]$ with values in $\mathrm{End}(C_\bullet^c(\overline{A[G]}, A[G])[[\boldsymbol{\tau}, \boldsymbol{s}]]$ extending $\partial_{b(\boldsymbol{\tau}, \boldsymbol{s})}$ and B and Ψ^* is the cochain maps we constructed in [11] such that

$$\Psi^*(\phi)[a_1 g_1 | \cdots | a_p g_p] = \phi[a_1 |^{g_1} a_2 | \cdots |^{g_1 \cdots g_{p_1}} a_p] g_1 \cdots g_p. \tag{3.14}$$

One can also directly check that

$$\begin{aligned} \tilde{\boldsymbol{b}}(\boldsymbol{\tau}, \boldsymbol{s})\{\phi_1, \cdots, \phi_n\} \circ \Psi_* =& \Psi_* \circ \boldsymbol{b}(\boldsymbol{\tau}, \boldsymbol{s})\{\Psi^*(\phi_1), \cdots, \Psi^*(\phi_n)\}, \\ \tilde{\boldsymbol{B}}\{\phi_1, \cdots, \phi_n\} \circ \Phi_* =& \Psi_* \circ \boldsymbol{B}\{\Psi^*(\phi_1), \cdots, \Psi^*(\phi_n)\}. \end{aligned} \qquad \square$$

By Getzler's computation [10], there is a connection flat up to homotopy defined as

$$\begin{aligned} \nabla \colon \mathbb{C}[[\boldsymbol{\tau}, \boldsymbol{s}]]\Big[\frac{\partial}{\partial \boldsymbol{\tau}}, \frac{\partial}{\partial \boldsymbol{s}}\Big] &\to \mathrm{End}_{\mathbb{C}}\Big(C_\bullet^c(\overline{A[G]}, A[G])[[\boldsymbol{\tau}, \boldsymbol{s}]]((u))\Big), \\ \nabla_v &:= v - \frac{1}{u}\boldsymbol{b}(\boldsymbol{\tau}, \boldsymbol{s})\{v(b(\boldsymbol{\tau}, \boldsymbol{s}))\} - \boldsymbol{B}\{v(b(\boldsymbol{\tau}, \boldsymbol{s}))\}, \end{aligned} \tag{3.15}$$

which induces a flat connection on

$$\mathrm{HP}_\bullet^c(\mathcal{A}(\boldsymbol{\tau}, \boldsymbol{s})) := \mathrm{H}_\bullet\Big(C_\bullet^c(\overline{A[G]}, A[G])[[\boldsymbol{\tau}, \boldsymbol{s}]]((u)), \partial_{b(\boldsymbol{\tau}, \boldsymbol{s})} + uB\Big).$$

∇ is flat up to homotopy on chain level, which means that ∇ is flat on homologies. Thus, there is a flat connection

$$\begin{aligned} \nabla \colon \mathbb{C}[[\boldsymbol{\tau}, \boldsymbol{s}]]\Big[\frac{\partial}{\partial \boldsymbol{\tau}}, \frac{\partial}{\partial \boldsymbol{s}}\Big] &\to \mathrm{End}_{\mathbb{C}}\Big(\mathrm{HP}_\bullet^c(\mathcal{A}(\boldsymbol{\tau}, \boldsymbol{s}))\Big), \\ v &\mapsto [\nabla_v(-)]. \end{aligned} \tag{3.16}$$

We can give a similar definition for ∇ on G-twisted chains just by replacing $\boldsymbol{b}$ and $\boldsymbol{B}$ with $\tilde{\boldsymbol{b}}$ and $\tilde{\boldsymbol{B}}$. This will also induce a flat connection on

$$\mathrm{H}_\bullet(C_\bullet^c(\bar{A}, A[G])_G[[\boldsymbol{\tau}, \boldsymbol{s}]]((u)), \tilde{\partial}_{b(\boldsymbol{\tau}, \boldsymbol{s})} + u\tilde{B}),$$

by the same reason. Lemma 3.3 and the fact that Ψ_* is a quasi-isomorphism [11] explain why these two constructions define the same flat connection on the periodic cyclic homology compatible with the Hodge filtration.

4. An example: $(A_{2n-1}, \mathbb{Z}_2)$ cases

In this section, we will write down the Getzler-Gauss-Manin system on the miniversal deformation of A_{2n-1} type orbifold explicitly. Here $W = x^{2n} + y^2$ $(n \geqslant 2)$, the orbifold group is $G = \mathbb{Z}_2$, whose generator σ acts on x, y by ${}^{\sigma}x = -x, {}^{\sigma}y = -y$. The result turns out to coincide with the Gauss-Manin system on the miniversal deformation of D_{n+1} singularity. This establishes an example of crepant resolution conjecture for LG B-models over miniversal deformations. In fact, if we lift the superpotential W to the crepant resolution of $\mathbb{C}^2/\mathbb{Z}_2$, which is the total space of $\mathcal{O}_{\boldsymbol{P}^1}(-2)$, it will have an isolated singularity of D_{n+1} type on the exceptional $\boldsymbol{P}^1$.

Computation of A-type orbifolds: $W = x^{2n} + y^2, G = \mathbb{Z}_2$

Since $\mathrm{Jac}(W, G) = \mathrm{Jac}(W)^G e \oplus \mathbb{C}\sigma[-2]$, the formal neighbourhood of the origin in $\mathrm{Def}(A_W, G)$ can be parameterized as $\mathrm{Spec}(\mathbb{C}[[\tau_0, \tau_1, \cdots, \tau_{n-1}, s]])$ and the deformed curved algebra $\mathcal{A}(\boldsymbol{\tau}, s)$ is $\mathbb{C}[[\boldsymbol{\tau}, s]][x, y] \otimes \mathbb{C}[G]$ with $b(\boldsymbol{\tau}, s)$ given by

$$\begin{cases} b_l(\boldsymbol{\tau}, s) = 0, \text{ if } l \neq 0, 2, \\ b_2(\boldsymbol{\tau}, s) = b_2 + s\partial_x^\sigma \partial_y^\sigma, \\ b_0(\boldsymbol{\tau}, s) = -W_{\boldsymbol{\tau}} := -x^{2n} - y^2 - \sum_{k=0}^{n-1} \tau_k x^{2k}. \end{cases} \tag{4.1}$$

Here ∂_x^σ and ∂_y^σ are the quantum differential operators defined in [11]. As a Hochschild cochain,

$$\partial_x^\sigma \partial_y^\sigma [x^{a_1} y^{b_1} | x^{a_2} y^{b_2}] := \begin{cases} (-1)^{a_2} x^{a_1+a_2-1} y^{b_1+b_2-1}\sigma, & \text{if } a_1, b_2 \text{ odd}, \\ 0, & \text{else}. \end{cases}$$

Henceforth, we will write

$$b(\boldsymbol{\tau}, s) = b_2 + s b_\sigma + b_0(\boldsymbol{\tau}), \tag{4.2}$$

with $b_\sigma = \partial_x^\sigma \partial_y^\sigma$ and $b_0(\boldsymbol{\tau}) := b_0(\boldsymbol{\tau}, s)$. Hence, the deformed Hochschild differential is

$$\tilde{\partial}_{b(\boldsymbol{\tau},s)} = \tilde{\partial}_{b_2} + s\tilde{\partial}_{b_\sigma} + \tilde{\partial}_{b_0(\boldsymbol{\tau})}. \tag{4.3}$$

In these cases, $\mathrm{HP}^c_\bullet(\mathcal{A}(\boldsymbol{\tau}, s))$ equals to

$$(\mathrm{Jac}(W)_G[2] \oplus \mathrm{Jac}(W_\sigma))\,[[\boldsymbol{\tau}, s]]((u)) = \Big(\bigoplus_{k=0}^{n-1} \mathbb{C}[x^{2k}][2] \oplus \mathbb{C}[1\sigma]\Big)[[\boldsymbol{\tau}, s]]((u)), \tag{4.4}$$

as a $\mathbb{Z}_2$-graded $\mathbb{C}[[\boldsymbol{\tau}, s]]((u))$-module with a flat connection ∇ we have defined in the last section.

Using perturbed homotopy retraction [5] constructed by homotopy retractions defined in the appendix, the chain representations for $[x^{2k}][2] \in \mathrm{Jac}(W)_G[2]$ and $[1_\sigma] \in \mathrm{Jac}(W_\sigma)$ can be written in the following forms,

$$\begin{cases} \alpha_{2k} & = \alpha_{2k}^{(2)} + \alpha_{2k}^{(4)} + \cdots, \\ \beta & = \beta^{(0)} + \beta^{(2)} + \cdots, \end{cases}$$

where $\alpha_k^{(p)}, \beta^{(p)} \in C_p(\bar{A}, A[G])_G[\boldsymbol{\tau}, s, u]$ are defined as follows. For $\forall 0 \leqslant k \leqslant n-1$, take $\alpha_{2k}^{(2)} = x^{2k}[x|y] - x^{2k}[y|x]$ and $\beta^{(0)} = 1\sigma$; and for $l \geqslant 2$, we define

$$\alpha_{2k}^{(2l)} = -\sum_{i\geqslant 0} \big(-s(\mathsf{H}_C + \Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_\sigma}\big)^i (\mathsf{H}_C + \Phi\mathsf{H}_K\Upsilon)(\tilde{\partial}_{b_0(\boldsymbol{\tau})} + u\tilde{B})\alpha_{2k}^{(2l-2)}. \tag{4.5}$$

Similarly, $\forall l \geqslant 1$, we define

$$\beta^{(2l)} = -\sum_{i\geqslant 0} \big(-s(\mathsf{H}_C + \Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_\sigma}\big)^i (\mathsf{H}_C + \Phi\mathsf{H}_K\Upsilon)(\tilde{\partial}_{b_0(\boldsymbol{\tau})} + u\tilde{B})\beta^{(2l-2)}. \tag{4.6}$$

The lower degree terms of α_{2k} and β which will be used in further calculation are given by

$$\alpha_{2k}^{(2)} = x^{2k}[x \wedge y], \tag{4.7}$$

$$\begin{aligned} \alpha_{2k}^{(4)} = &\sum_{j=1}^{n}\sum_{i=0}^{2j-2} \tau_j\big(x^{i+2k}[x|x^{2j-i-1}|x\wedge y] + x^{i+2k}[x\wedge y|x^{2j-i-1}|x]\big) \\ &+ x^{2k}[x\wedge y|y|y] + x^{2k}[y|y|x\wedge y] + u\sum_{i=0}^{2k-2}\big(x^i[x\wedge y|x^{2k-i-1}|x] \\ &+ x^i[x|x^{2k-i-1}|x\wedge y]\end{aligned} \tag{4.8}$$

$$\beta^{(0)} = 1\sigma, \tag{4.9}$$

$$\beta^{(2)} = \sum_{j=1}^{n}\sum_{i=0}^{2j-2} \tau_j x^i \sigma[x^{2j-i-1}|x] + 1\sigma[y|y], \tag{4.10}$$

$$\begin{aligned}\beta^{(4)} = & -u\mathsf{H}_C\tilde{B}(\beta^{(2)}) \\ & + \sum_{j,j'=1}^{n}\sum_{i=0}^{2j-2}\sum_{i'=0}^{2j'-2} \tau_j\tau_{j'}x^{i+i'}\sigma[x^{2j-i-1}|x|x^{2j'-i'-1}|x] \\ & + \sum_{j=1}^{n}\sum_{i=0}^{2j-2} \tau_j\big(x^i\sigma[x^{2j-i-1}|x|y|y] - x^i\sigma[x^{2j-i-1}|y|x\wedge y] \\ & + \sum_{j=1}^{n}\sum_{i=0}^{2j-2} \tau_j\big(x^i\sigma[y|x^{2j-i-1}|x\wedge y] + x^i\sigma[y|y|x^{2j-i-1}|x]\big) \\ & + 1\sigma[y|y|y|y].\end{aligned} \tag{4.11}$$

Here, for the sake of simplicity, we denote $\tau_n = 1$ and $\cdots|x\wedge y|\cdots = \cdots|x|y|\cdots - \cdots|y|x|\cdots$.

Computation of D-type: $W = z^n + zw^2, G = \{1\}$

For G trivial, D. Shklyarov had shown in [22] that the Gauss-Manin system via similar non-commutative methods is equivalent to that given by Saito's singularity theory [21]. Denote by $B_{\hat{W}} = \mathbb{C}[z,w]_{z^n+zw^2}$ a curved polynomial algebra with a curvature $\hat{W} = z^n + zw^2$ and consider its deformation as

$$\mathcal{B}(\boldsymbol{\tau}, s) := B_{\hat{W}(\boldsymbol{\tau},s)}, \text{ with } \hat{W}(\boldsymbol{\tau}, s) := z^n + \sum_{j=0}^{n-1}\tau_j z^j + zw^2 - sw.$$

$\mathrm{HP}^c_\bullet(\mathcal{B}(\boldsymbol{\tau}, s))$ equals to $\Big(\bigoplus_{k=0}^{n-1}\mathbb{C}[z^k][2]\oplus\mathbb{C}[w][2]\Big)[[\boldsymbol{\tau}, s]]((u))$ while regarded as a $\mathbb{Z}_2$-graded $\mathbb{C}[[\boldsymbol{\tau}, s]]((u))$-module with a flat connection ∇. Similar as above, we can also find chain representations for $[z^k]$ and $[w]$ in the following forms,

$$\begin{cases}\hat{\alpha}_{2k} & = \hat{\alpha}_{2k}^{(2)} + \hat{\alpha}_{2k}^{(4)} + \cdots, \\ \hat{\beta} & = \hat{\beta}^{(2)} + \hat{\beta}^{(4)} + \cdots,\end{cases}$$

where $\hat{\alpha}_k^{(p)}, \hat{\beta}^{(p)} \in C_p(B,B)[\boldsymbol{\tau}, s, u]$ satisfies that

$$\hat{\alpha}_{2k}^{(2)} = z^k[z|w] - z^k[w|z], \forall 0 \leqslant k \leqslant n, \tag{4.12}$$

$$\hat{\beta}^{(2)} = w[z|w] - w[w|z]. \tag{4.13}$$

Consider the bundle map Λ on $\mathrm{Spec}(\mathbb{C}[[\boldsymbol{\tau}, s]])$, which maps sections of $\mathrm{HP}^c_\bullet(\mathcal{A}(\boldsymbol{\tau}, s))$ to sections of $\mathrm{HP}^c_\bullet(\mathcal{B}(\boldsymbol{\tau}, s))$ induced by $\alpha_{2k} \mapsto \hat{\alpha}_{2k}$ and $\beta \mapsto \hat{\beta}$.

Theorem 4.1. *Viewed as bundles over* $\mathrm{Spec}(\mathbb{C}[[\boldsymbol{\tau}, s]])$, *the periodic cyclic homology* $\mathrm{HP}^c_\bullet(\mathcal{A}(\boldsymbol{\tau}, s))$ *of the deformed curved algebra* $\mathcal{A}(\boldsymbol{\tau}, s)$ *associated to the orbifold LG B-model* $(\mathbb{C}[x, y], x^{2n}+y^2, \mathbb{Z}_2)$ *and that of the deformed curved algebra* $\mathcal{B}(\boldsymbol{\tau}, s)$ *associated to the LG B-model* $(\mathbb{C}[z, w], z^n+zw^2)$ *are isomorphic via the bundle map* Λ *defined above. Furthermore, this isomorphism is compatible with the Getzler-Gauss-Manin connections on both bundles. To be explicit, for any* $v \in \mathrm{Der}_{\mathbb{C}}\mathbb{C}[[\boldsymbol{\tau}, s]]$ *and any section* θ *of the bundle* $\mathrm{HP}^c_\bullet(\mathcal{A}(\boldsymbol{\tau}, s))$, *we have*

$$\Lambda(\nabla_v(\theta)) = \nabla_v(\Lambda(\theta)). \tag{4.14}$$

Proof. We will prove this by direct calculation using tools coming from homotopy perturbations. Notice that we only need to show (4.14) for $v = \frac{\partial}{\partial \tau_j}$ or $\frac{\partial}{\partial s}$ and $\theta = [\alpha_{2k}], 0 \leqslant k \leqslant n-1$ or $[\beta]$.

Case 1: $v = \frac{\partial}{\partial s}$ and $\theta = [1\sigma]$. We have

$$[\nabla_{\frac{\partial}{\partial s}}(\hat{\beta})] = -\frac{1}{u}\big(-\sum_{j=1}^{n} j\tau_j[\hat{\alpha}_{2j-2}]\big).$$

By (4.9), (4.10) and (4.11), we have

$$\begin{aligned}\nabla_{\frac{\partial}{\partial s}}(\beta) = &-\frac{1}{u}\Big(\sum_{j=1}^{n}\tau_j\sum_{i=0}^{2j-2}(-1)^{i+1}(x^i[x^{2j-i-1}|y] - x^i[y|x^{2j-i-1}])\Big)\\ &+ (\text{order } \geqslant 4 \text{ terms}),\end{aligned}$$

While acted on the above by the perturbed projection,

$$\begin{aligned}\sum_{l\geqslant 0} p\Pi\Upsilon\Big(&-(s\tilde{\partial}_{b_\sigma} + \tilde{\partial}_{b_0(\boldsymbol{\tau},s)-b_0} + u\tilde{B})\sum_{i\geqslant 0}\big(-(\mathsf{H}_C + \Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_0}\big)^i\\ &(\mathsf{H}_C + \Phi\mathsf{H}_K\Upsilon + \Phi\Theta\mathsf{H}_\Omega\Pi\Upsilon)\Big)^l,\end{aligned}$$

the section represented by the above is given by

$$-\frac{1}{u}\sum_{j=1}^{n}\tau_j\Big(\sum_{i=0}^{2j-2}(-1)^{i+1}(2j-i-1)\Big)[\alpha_{2j-2}] = -\frac{1}{u}\big(-\sum_{j=1}^{n} j\tau_j[\alpha_{2j-2}]\big).$$

Hence

$$\Lambda(\nabla_{\frac{\partial}{\partial s}}([\beta])) = \nabla_{\frac{\partial}{\partial s}}(\Lambda([\beta])). \tag{4.15}$$

Case 2: $v = \frac{\partial}{\partial s}$ and $\theta = [\alpha_{2k}]$ for $0 \leqslant k \leqslant n-1$.

$$\nabla_{\frac{\partial}{\partial s}}([\hat{\alpha}_{2k}]) = \begin{cases} -\dfrac{1}{u}[\hat{\beta}], & k = 0, \\ -\dfrac{1}{u}(\dfrac{s}{2}[\hat{\alpha}_{2k-2}]), & 0 < k \leqslant n-1. \end{cases}$$

And similarly,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}}(\alpha_{2k}) = &-\frac{1}{u}(x^{2k}\sigma + x^{2k}\sigma[y|y] + \sum_{j=1}^{n}\sum_{i=0}^{2j-2}\tau_j(-1)^i x^{2k+i}\sigma[x|x^{2j-i-1}] \\ &+ u\sum_{i=0}^{2k-2}(-1)^i x^i\sigma[x|x^{2k-i-1}]) + (\text{order } \geqslant 4 \text{ terms}), \end{aligned} \tag{4.16}$$

so after acted by the perturbed projection

$$\nabla_{\frac{\partial}{\partial s}}([\alpha_{2k}]) = \begin{cases} -\dfrac{1}{u}[\beta], & k = 0, \\ -\dfrac{1}{u}(\dfrac{s}{2}[\alpha_{2k-2}]), & 0 < k \leqslant n-1. \end{cases}$$

Hence,

$$\Lambda(\nabla_{\frac{\partial}{\partial s}}([\alpha_{2k}])) = \nabla_{\frac{\partial}{\partial s}}(\Lambda([\alpha_{2k}])). \tag{4.17}$$

Case 3: $v = \frac{\partial}{\partial \tau_k}$ for $0 \leqslant k \leqslant n-1$ and $\theta = [\beta]$. By

$$[\nabla_{\frac{\partial}{\partial s}}, \nabla_{\frac{\partial}{\partial \tau_k}}] = 0,$$

and Case 2, we have obviously

$$\Lambda(\nabla_{\frac{\partial}{\partial \tau_k}}([\beta])) = \nabla_{\frac{\partial}{\partial \tau_k}}(\Lambda([\beta])). \tag{4.18}$$

Case 4: $v = \frac{\partial}{\partial \tau_l}$ and $\theta = [\alpha_{2k}]$ for $0 \leqslant k, l \leqslant n-1$. $\nabla_{\frac{\partial}{\partial \tau_l}}(\Lambda([\alpha_{2k}]))$ is given by the homology class of

$$-\frac{1}{u}\big(-z^{k+l}[z\wedge w]\big) + (\text{order } \geqslant 4 \text{ terms}),$$

and $\nabla_{\frac{\partial}{\partial \tau_l}}([x^{2k}])$ is given by the homology class of

$$-\frac{1}{u}\big(-x^{2k+2l}[x\wedge y]\big)+(\text{order }\geqslant 4\text{ terms}),$$

so we can show the statement (4.14) by induction on $k+l$. In cases $k+l\leqslant n-1$, (4.14) is obvious and we can assume that (4.14) holds for $k+l\leqslant m-1$ with $2n-2\geqslant m\geqslant n$. Then for $k+l=m$, $\nabla_{\frac{\partial}{\partial \tau_l}}(\Lambda([\alpha_{2k}]))$ equals to the homology class of

$$-\frac{1}{u}\Big(\sum_{j=1}^{n-1}\frac{j}{n}\tau_j\big(z^{m-n+j}[z\wedge w]\big)+\frac{1}{2n}s\big(z^{m-n}w[z\wedge w]\big)$$
$$+u\frac{2m-2n+1}{2n}\big(z^{m-n}[z\wedge w]\big)\Big).$$

And similarly, $\nabla_{\frac{\partial}{\partial \tau_l}}([\alpha_{2k}])$ equals to the homology class of

$$-\frac{1}{u}\Big(\sum_{j=1}^{n-1}\frac{j}{n}\tau_j\big(x^{2m-2n+2j}[x\wedge y]\big)+\frac{1}{2n}sx^{2m-2n}\sigma$$
$$+u\frac{2m-2n+1}{2n}\big(x^{2m-2n}[x\wedge y]\big)\Big),$$

By the same calculation in the above cases and our assumption, (4.14) holds in these cases. $\square$

Appendix A. Constructions of the homotopies

Since we are working on the G-twisted chains, a direct calculation on the 'G-twisted version' of periodic homology of those deformed algebras is needed. This can be done by constructing an explicit special homotopy retraction.

Firstly, consider the Koszul chains,

$$K_\bullet(A,A[G]):=\bigoplus_{p\geqslant 0}A[G]\otimes\mathbb{C}[e_1,e_2],$$

with G-action given by

$$g.(ahe_{i_1}\cdots e_{i_p})={}^g a ghg^{-1}{}^g e_{i_1}\cdots{}^g e_{i_p}. \tag{A.1}$$

Here e_i are the odd parameters with respect to x_i. On Koszul chains, we can define a differential call a Koszul differential as

$$\partial_K(age_{\boldsymbol{I}}) := \sum_{k=1}^{p}(-1)^{k-1}({}^g x_{i_k} - x_{i_k})age_{\boldsymbol{I}\backslash\{i_k\}}, \tag{A.2}$$

where $\boldsymbol{I} = \{i_1 < \cdots < i_p\} \subseteq \{1,2\}$. In [24], Shepler and Witherspoon introduced two chain maps Φ and Υ and in [11], we construct a homotopy H_C such that Φ, Υ are both G-equivariant and we have a special homotopy retraction

$$\mathsf{H}_C \circlearrowright (C_\bullet(\bar{A}, A[G])_G, \tilde{\partial}_{b_2}) \underset{\Phi}{\overset{\Upsilon}{\rightleftarrows}} (K_\bullet(A, A[G])_G, \partial_K).$$

Equivalently, $(\Phi, \Upsilon, \mathsf{H}_C)$ satisfies that

$$\begin{cases} \Upsilon \circ \Phi = \mathsf{id}, \mathsf{id} - \Phi \circ \Upsilon = [\tilde{\partial}_{b_2}, \mathsf{H}_C]; \\ \mathsf{H}_C \circ \mathsf{H}_C = 0, \mathsf{H}_C \circ \Phi = 0, \Upsilon \circ \mathsf{H}_C = 0. \end{cases}$$

Secondly, we can also construct a special homotopy retraction

$$\mathsf{H}_K \circlearrowright (K_\bullet(A, A[G])_G, \partial_K) \underset{\Theta}{\overset{\Pi}{\rightleftarrows}} \Big(\bigoplus_{g\in G} \Omega^\bullet(\mathrm{Fix}(g))_G, 0\Big).$$

Here, chain maps Π and Θ between $\big(K_\bullet(A, A[G]), \partial_K\big)$ and $\big(\bigoplus_{g\in G} \Omega^\bullet(\mathrm{Fix}(g)), 0\big)$ are defined as

$$\Pi(x_1^{\gamma_1}x_2^{\gamma_2}ge_{\boldsymbol{I}}) := \begin{cases} (x_1^{\gamma_1}x_2^{\gamma_2})\,|_{\mathrm{Fix}(g)}\mathsf{d}x_{\boldsymbol{I}}, & \text{if } \boldsymbol{I}_g \cap \boldsymbol{I} = \emptyset, \\ 0, & \text{else}, \end{cases}$$

where $\boldsymbol{I}_g = \{i = 1,2 \mid \lambda_i \neq 1\}$, and for a differential form $x_1^{\gamma_1}x_2^{\gamma_2}\mathsf{d}x_{\boldsymbol{I}} \in \Omega^\bullet(\mathrm{Fix}(g))$,

$$\Theta(x_1^{\gamma_1}x_2^{\gamma_2}\mathsf{d}x_{\boldsymbol{I}}) := x_1^{\gamma_1}x_2^{\gamma_2}ge_{\boldsymbol{I}}.$$

We can define the weight of a Koszul chain as

$$\mathsf{wt}(x_1^{\gamma_1}x_2^{\gamma_2}ge_{\boldsymbol{I}}) = \sum_{k,\lambda_k\neq 1}\gamma_k + \sum_{i\in\boldsymbol{I},\lambda_i\neq 1} 1. \tag{A.3}$$

Then for a chain κ with $\mathsf{wt}(\kappa) \neq 0$, we can define the homotopy H_K as

$$\text{(A.4)} \qquad \mathsf{H}_K \colon \kappa = x_1^{\gamma_1} x_2^{\gamma_2} g e_{\boldsymbol{I}} \mapsto \sum_{i \in \boldsymbol{I}, \lambda_i \neq 1} \frac{1}{\mathsf{wt}(\kappa)} \frac{1}{\lambda_i - 1} \frac{\partial}{\partial x_i} (x_1^{\gamma_1} x_2^{\gamma_2})\, g e_i \wedge e_{\boldsymbol{I}},$$

where ${}^g x_i = \lambda_i x_i$. If $\mathsf{wt}(\kappa) = 0$, we will ask $\mathsf{H}_K(\kappa) = 0$. Notice that we also have Π, Θ and H_K are all G-equivariant and they give a special homotopy retraction. The former is obvious and the later is by

(1) $\Pi \circ \Theta = \mathsf{id}$,
(2) $\mathsf{id} - \Theta \circ \Pi = [\mathsf{H}_K, \partial_K]$, because for $\kappa = age_{\boldsymbol{I}}$ with $\mathrm{wt}(\kappa) \neq 0$, we have

$$\begin{aligned} [\mathsf{H}_K, \partial_K](\kappa) = & \sum_{i \in \boldsymbol{I} \bigcap \boldsymbol{I}_g} \frac{1}{\mathsf{wt}(\kappa)} \frac{\partial}{\partial x_i} (x_i a)\, g e_{\boldsymbol{I}} + \sum_{i \in \boldsymbol{I}_g \setminus \boldsymbol{I}} \frac{1}{\mathsf{wt}(\kappa)} x_i \frac{\partial}{\partial x_i} (a)\, g e_{\boldsymbol{I}} \\ = & \kappa. \end{aligned}$$

and for κ with $\mathsf{wt}(\kappa) = 0 \iff \kappa = \Theta \circ \Pi(\kappa)$, we have $[\mathsf{H}_K, \partial_K](\kappa) = 0$.
(3) $\mathsf{H}_K \circ \mathsf{H}_K = 0$, $\mathsf{H}_K \circ \Theta = 0$ and $\Pi \circ \mathsf{H}_K = 0$ for obvious reasons.

As a direct corollary, we have the following homotopy retraction,

$$\mathsf{H}_C + \Phi \circ \mathsf{H}_K \circ \Upsilon \circlearrowright (C_\bullet(\bar{A}, A[G])_G, \tilde{\partial}_{b_2}) \underset{\Phi \circ \Theta}{\overset{\Pi \circ \Upsilon}{\rightleftarrows}} \Big(\bigoplus_{g \in G} \Omega^\bullet(\mathrm{Fix}(g))_G, 0 \Big).$$

By taking the compact type, we can regard $\tilde{\partial}_{b_0}$ as a small perturbation of $\tilde{\partial}_{b_2}$ to get a perturbed special homotopy retraction. (See for example [5].)

Lemma A.1. *The induced differential on* $\Omega^\bullet(\mathrm{Fix}(g))_G$ *by the perturbation with respect to* $\tilde{\partial}_{b_0}$ *is given by* $\mathsf{d}W_g \wedge$.

Proof. By definition, the induced differential is given by

$$\Pi\Upsilon \circ \tilde{\partial}_{b_0} \circ \Phi\Theta + \sum_{i \geqslant 1} (-1)^i \Pi\Upsilon \circ \tilde{\partial}_{b_0} \big((\mathsf{H}_C + \Phi \mathsf{H}_K \Upsilon) \tilde{\partial}_{b_0} \big)^i \circ \Phi\Theta.$$

Notice that

$$\begin{aligned} & \Pi\Upsilon \circ \tilde{\partial}_{b_0} \circ \Phi\Theta|_{\Omega^\bullet(\mathrm{Fix}(g))_G} = \mathsf{d}W_g \wedge, \text{ and} \\ \text{(A.5)} \qquad & \Pi\Upsilon \circ \tilde{\partial}_{b_0} = (\mathsf{d}W_g \wedge) \circ \Pi\Upsilon \text{ on } \Omega^\bullet(\mathrm{Fix}(g))_G, \end{aligned}$$

so the higher order terms in the differential on the g-sector $\Omega^\bullet(\mathrm{Fix}(g))_G$ can be written as

$$\begin{aligned}
&\sum_{i\geqslant 1}(-1)^i \Pi\Upsilon \circ \tilde{\partial}_{b_0}\big((\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_0}\big)^i \circ \Phi\Theta \\
=&\sum_{i\geqslant 1}(-1)^i (\mathsf{d}W_g\wedge)\circ \Pi\Upsilon\big((\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_0}\big)^i \circ \Phi\Theta \\
=&0.
\end{aligned}$$

The last equality is because $\Upsilon\mathsf{H}_c=0$ and $\Pi\Upsilon\Phi\mathsf{H}_K=\Pi\mathsf{H}_K=0$. ☐

In summary, we have the following homotopy retraction,

$$\circlearrowleft\ (C^c_\bullet(\bar{A},A[G])_G,\tilde{\partial}_b) \rightleftarrows \Big(\bigoplus_{g\in G}\mathrm{Jac}(W_g)_G[\overline{2-|\boldsymbol{I}_g|}],0\Big). \tag{A.6}$$

Here the homotopy is given by

$$\begin{aligned}
&\sum_{n\geqslant 0}\big(-(\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_0}\big)^n(\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon) \\
&+\sum_{m\geqslant 0}\big(-(\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_0}\big)^m\Phi\Theta\mathsf{H}_\Omega\Pi\Upsilon \\
=&\sum_{n\geqslant 0}\big(-(\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon)\tilde{\partial}_{b_0}\big)^n(\mathsf{H}_C+\Phi\mathsf{H}_K\Upsilon+\Phi\Theta\mathsf{H}_\Omega\Pi\Upsilon).
\end{aligned}\tag{A.7}$$

where H_Ω is some appropriate homotopy on $\bigoplus_{g\in G}\Omega^\bullet(\mathrm{Fix}(g))_G$, such that

$$\mathsf{H}_\Omega\circlearrowleft\ \Big(\bigoplus_{g\in G}\Omega^\bullet(\mathrm{Fix}(g))_G,\bigoplus_{g\in G}\mathsf{d}W_g\Big) \underset{i}{\overset{p}{\rightleftarrows}} \Big(\bigoplus_{g\in G}\mathrm{Jac}(W_g)_G[\overline{2-|\boldsymbol{I}_g|}],0\Big),$$

gives a special homotopy retraction. We can further require that $\mathsf{H}_\Omega|_{\Omega^\bullet(\mathrm{Fix}(g))_G}=0$ for $g\neq e$ in 2-dimensional Calabi-Yau cases. Then (A.6) gives a special homotopy retraction.

Remark. There are many choices for (i,p,H_Ω) and none is canonical. For example, for $W=x^n+y^m$ with G trivial on $\mathbb{C}^2$, we can select such an H_Ω as

$$\mathsf{H}_\Omega(x^ay^b\mathsf{d}x\wedge\mathsf{d}y):=\begin{cases}\dfrac{1}{n}x^{a-n+1}y^b\mathsf{d}y, & a\geqslant n-1,\\ -\dfrac{1}{m}x^ay^{b-m+1}\mathsf{d}x, & a<n-1, b\geqslant m-1,\\ 0, & \text{else},\end{cases}\tag{A.8}$$

$$\mathsf{H}_\Omega(x^a y^b \mathsf{d}x) := \begin{cases} \frac{1}{n} x^{a-n+1} y^b, & a \geqslant n-1, \\ 0, & a < n-1, \end{cases} \tag{A.9}$$

$$\mathsf{H}_\Omega(x^a y^b \mathsf{d}y) := 0, \tag{A.10}$$

with the easiest inclusion i and projection p.

Finally, we can regard the deformed differential $\tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})} + u\tilde{B}$ as a perturbation of $\tilde{\partial}_b$. Then if G satisfies the Calabi-Yau condition that $G \subset SL(2,\mathbb{C})$, we have a special homotopy retraction,

$$\begin{aligned} &\circlearrowright (C^c_\bullet(\bar{A}, A[G])_G[[\boldsymbol{\tau},\boldsymbol{s}]]((u)), \tilde{\partial}_{b(\boldsymbol{\tau},\boldsymbol{s})} + u\tilde{B}) \\ &\rightleftarrows \Big(\bigoplus_{g\in G} \mathrm{Jac}(W_g)_G[\overline{2-|\boldsymbol{I}_g|}][[\boldsymbol{\tau},\boldsymbol{s}]]((u)), 0\Big). \end{aligned} \tag{A.11}$$

The induced differential on the right complex is zero because it is concentrated in even degrees.

Remark. We only construct the homotopies in two dimensional cases. However, they all can be generalized in any dimensions.

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