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Monotone iterative technique for delayed evolution equation periodic problems in Banach spaces*

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Abstract: In this paper, we deal with the existence of ω-periodic mild solutions for the abstract evolution equation with delay in an ordered Banach space E

$$u'(t) + Au(t) = F(t, u(t), u(t-\tau)), \quad t \in \mathbb{R},$$

where $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$, $F : \mathbb{R} \times E \times E \to E$ is a continuous mapping which is ω-periodic in t, and $\tau \geq 0$ is a constant. Under some weaker assumptions, we construct monotone iterative method for the delayed evolution equation periodic problems, and obtain the existence of maximal and minimal periodic mild solutions. The results obtained generalize the recent conclusions on this topic. Finally, we present two applications to illustrate the feasibility of our abstract results.

Keywords: Evolution equations with delay, upper and lower solutions, existence, monotone iterative technique, positive C_0-semigroup.

1. Introduction

Let E be an ordered Banach space, whose positive cone $K = \{u \in E | \, u \geq \theta\}$ is normal with normal constant N. In this paper, we use a monotone iterative technique in the presence of the lower and upper solutions to discuss the existence of the extremal periodic mild solutions to the periodic problem of first order semilinear evolution equation with delay in ordered Banach space E

$$u'(t) + Au(t) = F(t, u(t), u(t-\tau)), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a positive strongly continuous semigroup (C_0-semigroup, in short) $T(t)(t \geq 0)$

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in E, the nonlinear function $F : \mathbb{R} \times E \times E \to E$ is a continuous mapping and for every $x, y \in E$, $F(t, x, y)$ is ω-periodic in t and τ is positive constant which denotes the time delay.

The theory of partial differential equations with delays is an important branch of differential equation theory, which has extensive physical, biological, economical, engineering background and realistic mathematical model. Hence, the theory of partial differential equations with delays has been emerging as an important area of investigation in the last few decades, and the numerous properties of their solutions have been studied, see [16, 37] and references therein.

The problem concerning periodic solutions of partial differential equations with delay is an important area of investigation since they can take into account seasonal fluctuations occurring in the phenomena appearing in the models, and have been studied by some researchers in recent years. There has been a significant development in periodic solution of evolution equation with delay in Banach spaces, we refer to the references [5, 38, 29, 30, 31, 26, 18, 20, 36, 27, 28].

In [5], Burton and Zhang obtained the existence of periodic solutions for an abstract evolution equation with infinite delay. In [38], under the assumption that the corresponding initial value problem has a priori estimate, Xiang and Ahmed showed an existence result of periodic solution to the delay evolution equations in Banach spaces. In [29, 30, 31], Liu derived periodic solutions from bounded solutions or ultimate bounded solutions for finite or infinite delay evolution equations in Banach spaces. In [26], Li discussed the periodic solutions of the evolution equation with delays and presented essential conditions on the nonlinearity to guarantee that the equation has periodic solutions. In [18], M. Kpoumiè et al. studied the existence of a periodic solution for some partial functional differential equations with infinite delay in Banach spaces. Recently, some authors also discussed the periodic solutions for some nonautonomous delay impulsive evolutionary equations (see [27, 28, 36]). They established some existence results on periodic solutions to the equations under the ultimate boundedness of the solutions of the corresponding initial value problem.

In fact, in previous works, evolution equation periodic problems with delay have been studied by many authors using different tools, such as Granas's fixed theorem, Banach contraction mapping principal, Schauder's fixed-point theorem, Horn's fixed point theorem, Sadovskii's fixed point theorem and so on. However, to the best of our knowledge, few results yet exist for the periodic problems with delay by using the method of the lower and upper solutions coupled with the monotone iterative technique.

It is well known that the monotone iterative technique of the lower and upper solutions is an effective and flexible mechanism. It yields monotone sequences of the lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Early on, Du and Lakshmikantham [13], Sun and Zhao [35] investigated the existence of extremal solutions to the initial value problem of ordinary differential equations without delay by using the method of the lower and upper solutions coupled with the monotone iterative technique. Later, Li [22] applied lower and upper solutions method to periodic solution problems for semilinear evolution equations without delay in ordered Banach spaces, and obtained the existence of maximal and minimal periodic solutions using the characteristics of positive operators semigroups and the monotone iteration scheme. For the abstract evolution equations, there are more results involving monotone iterative techniques and operator semigroups theory, we can see [6, 7, 8, 9, 10].

Recently, in [19] we dealt with the second-order delayed ordinary differential equation periodic problem in ordered Banach spaces. With the nonlinear function satisfying quasi-monotonicity, we obtained the existence of the minimal and maximal periodic solutions by monotone iterative technique of the lower and upper solutions. And in [20], we also applied operator semigroup theory and monotone iterative technique of lower and upper solutions to obtain the existence and uniqueness of periodic mild solutions of the abstract evolution equation under some quasi-monotone conditions.

Motivated by the papers mentioned above, the purpose of this paper is to construct the general principle for lower and upper solutions coupled with the monotone iterative technique for the evolution equation periodic problems with delay, and obtain the existence of maximal and minimal periodic mild solutions, which will make up the research in this area blank.

The paper is organized as follows. In Section 2, some notions, definitions, and preliminary facts are introduced, which are used through this paper. Under the different assumptions, the existence results of the extremal periodic solutions of Equation (1.1) are given in Section 3. In Section 4, we give two examples to illustrate our main results in Section 3.

2. Preliminaries

In this section, we introduce some notions, definitions, and preliminary facts which are used through this paper.

Throughout this paper, we assume that E is an ordered Banach space, whose positive cone $K = \{u \in E | u \geq \theta\}$ is normal with normal constant N.

Let $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in E. For the theory of semigroups of linear operators we refer to [33]. We only recall here some notions and properties that are essential for us. For a general C_0-semigroup $T(t)(t \geq 0)$, there exist $M \geq 1$ and $\nu \in \mathbb{R}$ such that (see [33])

$$\|T(t)\| \leq Me^{\nu t}, \quad t \geq 0. \tag{2.1}$$

Let

$$\nu_0 = \inf\{\nu \in \mathbb{R} | \text{ There exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t}, \ \forall t \geq 0\}, \tag{2.2}$$

then ν_0 is called the growth exponent of the semigroup $T(t)(t \geq 0)$. Furthermore, ν_0 can be also obtained by the following formula

$$\nu_0 = \limsup_{t \to +\infty} \frac{\ln \|T(t)\|}{t}.$$

Definition 2.1 ([4])**.** A C_0-semigroup $T(t)(t \geq 0)$ on E is said to be positive, if the order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta$, $x \in E$, and $t \geq 0$.

It is easy to see that for any $C \geq 0$, $-(A + CI)$ also generates a C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ in E. And $S(t)(t \geq 0)$ is a positive C_0-semigroup if $T(t)(t \geq 0)$ is a positive C_0-semigroup. For more details of the properties of the operator semigroups and positive C_0-semigroup, we refer to the monographs [32, 34] and the paper [21].

Let J denote the infinite interval $[0, +\infty)$ and $h : J \to E$, consider the initial value problem of the linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, \\ u(0) = x_0. \end{cases} \tag{2.3}$$

It is well known [33, Chapter 4, Theorem 2,9], when $x_0 \in D(A)$ and $h \in C^1(J, E)$, the initial value problem (2.3) has a unique classical solution $u \in C^1(J, E) \cap C(J, E_1)$ expressed by

$$u(t) = T(t)x_0 + \int_0^t T(t - s)h(s)ds, \tag{2.4}$$

where $E_1 = D(A)$ is Banach space with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A \cdot\|$. Generally, for $x_0 \in E$ and $h \in C(J, E)$, the function u given by (2.4) belongs to $C(J,E)$ and it is called a mild solution of the linear evolution equation (2.3).

Let $C_\omega(\mathbb{R},E)$ denote the Banach space $\{u \in (\mathbb{R},E)|\ u(t) = u(t+\omega),\ t \in \mathbb{R}\}$ endowed the maximum norm $\|u\|_C = \max_{t\in[0,\omega]} \|u(t)\|$. Evidently, $C_\omega(\mathbb{R},E)$ is an order Banach space with the partial order "$\leq$" induced by the positive cone $K_C = \{u \in C_\omega(\mathbb{R},E)|\ \ u(t) \geq \theta,\ t \in \mathbb{R}\}$ and K_C is also normal with the normal constant N. For $v,w \in C_\omega(\mathbb{R},E)$ with $v \leq w$, we use $[v,w]$ to denote the order interval $\{u|v \leq u \leq w\}$ in $C_\omega(\mathbb{R},E)$, and $[v(t),w(t)]$ to denote the order interval $\{u(t)|v(t) \leq u(t) \leq w(t), t \in \mathbb{R}\}$ in E.

Given $h \in C_\omega(\mathbb{R},E)$, for the following linear evolution equation corresponding to Eq. (1.1)

$$u'(t) + Au(t) = h(t), \quad t \in \mathbb{R}, \tag{2.5}$$

we have the following result.

Lemma 2.2 ([25])**.** *If $-A$ generates an exponentially stable positive C_0-semigroup $T(t)(t \geq 0)$ in E, that is $\nu_0 < 0$, then for $h \in C_\omega(\mathbb{R},E)$, the linear evolution equation (2.5) exists a unique ω-periodic mild solution u, which can be expressed by*

$$u(t) = (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)h(s)ds := (Ph)(t), \tag{2.6}$$

and the solution operator $P : C_\omega(\mathbb{R},E) \to C_\omega(\mathbb{R},E)$ is a positive bounded linear operator.

Proof. For any $\nu \in (0, |\nu_0|)$, there exists $M > 0$ such that

$$\|T(t)\| \leq Me^{-\nu t} \leq M, \quad t \geq 0.$$

In E, define the equivalent norm $|\cdot|$ by

$$|x| = \sup_{t\geq 0} \|e^{\nu t}T(t)x\|,$$

then $\|x\| \leq |x| \leq M\|x\|$. By $|T(t)|$ we denote the norm of $T(t)$ in $(E, |\cdot|)$, then for $t \geq 0$, it is easy to obtain that $|T(t)| < e^{-\nu t}$. Hence, $(I - T(\omega))$ has bounded inverse operator

$$(I - T(\omega))^{-1} = \sum_{n=0}^{\infty} T(n\omega),$$

and its norm satisfies

$$|(I-T(\omega))^{-1}| \leq \frac{1}{1-|T(\omega)|} \leq \frac{1}{1-e^{-\nu\omega}}. \tag{2.7}$$

Set

$$x_0 = (I-T(\omega))^{-1}\int_0^\omega T(t-s)h(s)ds := Bh, \tag{2.8}$$

then the mild solution $u(t)$ of the linear initial value problem (2.3) given by (2.4) satisfies the periodic boundary condition $u(0)=u(\omega)=x_0$. For $t\in\mathbb{R}^+$, by (2.4) and the properties of the semigroup $T(t)(t\geq 0)$, we have

$$\begin{aligned} u(t+\omega) &= T(t+\omega)u(0) + \int_0^{t+\omega} T(t+\omega-s)h(s)ds \\ &= T(t)\Big(T(\omega)u(0) + \int_0^\omega T(\omega-s)h(s)ds\Big) + \int_0^t T(t-s)h(s-\omega)ds \\ &= T(t)u(0) + \int_0^t T(t-s)h(s)ds \ = u(t). \end{aligned}$$

Therefore, the ω-periodic extension of u on $\mathbb{R}$ is a unique ω-periodic mild solution of Eq. (2.5). By (2.4) and (2.8), the ω-periodic mild solution can be expressed by

$$\begin{aligned} u(t) &= T(t)B(h) + \int_0^t T(t-s)h(s)ds \\ &= (I-T(\omega))^{-1}\int_{t-\omega}^t T(t-s)h(s)ds := (Ph)(t). \end{aligned} \tag{2.9}$$

It is easy to see that $P: C_\omega(\mathbb{R},E)\to C_\omega(\mathbb{R},E)$. Finally, by the positivity of semigroup $T(t)(t\geq 0)$, we can obtain that $(I-T(\omega))^{-1}\geq\theta$, it follows that $Ph\geq\theta$ for any $h\in C_\omega(\mathbb{R},E)$ and $h\geq\theta$. Therefore, $P: C_\omega(\mathbb{R},E)\to C_\omega(\mathbb{R},E)$ is a positive bounded linear operator. This completes the proof of Lemma 2.2. □

Next, we recall some properties of measure of noncompactness that will be used in the proof of our main results. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [3, 11, 14]. For any $B\subset C_\omega(\mathbb{R},E)$ and $t\in\mathbb{R}$, set $B(t)=\{u(t)|u\in B\}\subset E$. If B is bounded in $C_\omega(\mathbb{R},E)$, then $B(t)$ is bounded in E, and $\alpha(B(t))\leq\alpha(B)$.

The following lemmas are needed in our arguments.

Lemma 2.3 ([3, 14, 15]). *Let E be a Banach space and let $B \subset C(J,E)$ be bounded and equicontinuous, where J is a finite closed interval in $\mathbb{R}$. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)).$$

Lemma 2.4 ([17]). *Let E be a Banach space, $B = \{u_n\} \subset C(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integrable on J, and*

$$\alpha\Big(\Big\{\int_J u_n(s)ds\Big\}\Big) \le 2\int_J \alpha(B(t))dt.$$

Lemma 2.5 ([23]). *Let E be a Banach space and $D \subset E$ be bounded. Assume that $Q : E \to E$ is linear bounded operator, then*

$$\alpha(Q(D)) \le \|Q\|\alpha(D).$$

3. Main results

Now, we are in the position to state and prove our main results. We will apply monotone iterative method of the lower and upper ω-periodic solutions to obtain the existence of ω-periodic mild solution for Eq. (1.1). To this end, we define the ω-periodic lower and upper solutions of Eq. (1.1).

Definition 3.1. If a function $v_0 \in C^1_\omega(\mathbb{R},E) \cap C_\omega(\mathbb{R},E_1)$ satisfies

$$v_0'(t) + Av_0(t) \le F(t, v_0(t), v_0(t-\tau)),\ t \in \mathbb{R}, \tag{3.1}$$

we call it an ω-periodic lower solution of Eq. (1.1). If the inequality of (3.1) is inverse, we call it an ω-periodic upper solution of the Eq. (1.1).

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone K is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive compact semigroup $T(t)(t \ge 0)$, let $f : \mathbb{R} \times E \times E \to E$ be a continuous mapping which is ω-periodic in t. Assume Eq. (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R},E) \cap C_\omega(\mathbb{R},E_1)$ with $v_0 \le w_0$. If the following condition*

(H1) there exists a constant $C \ge 0$ such that for all $t \in \mathbb{R}$, $v_0(t) \le x_1 \le x_2 \le w_0(t)$, $v_0(t-\tau) \le y_1 \le y_2 \le w_0(t-\tau)$,

$$F(t, x_2, y_2) - F(t, x_1, y_1) \ge -C(x_2 - x_1)$$

holds, then the periodic problem (1.1) has minimal and maximal ω-periodic mild solution $\underline{u}, \overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Proof. Obviously, the periodic problem of evolution equation with delay (1.1) is equal to the following periodic problem

$$u'(t) + Au(t) + Cu(t) = F(t, u(t), u(t-\tau)) + Cu(t), \quad t \in \mathbb{R}, \tag{3.2}$$

where the constant C is decided by the condition (H1).

Let $C > |\nu_0|$ (otherwise replace C with $C + |\nu_0|$), then $-(A + CI)$ generates an exponentially stable, compact and positive C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ in E, whose growth exponent is $-C + \nu_0$. By Lemma 2.2, it follows that the following linear evolution equation periodic problem

$$u'(t) + Au(t) + Cu(t) = F(t, h(t), h(t-\tau)) + Ch(t), \quad t \in \mathbb{R} \tag{3.3}$$

exists unique ω-periodic mild solution

$$u(t) = (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, h(s), h(s-\tau)) + Ch(s))ds := Ph(t). \tag{3.4}$$

From Definition 3.1, it is clear that $[v_0, w_0] \subset C_\omega(\mathbb{R}, E)$ and $v_0(t) \leq w_0(t)$ for any $t \in \mathbb{R}$. Define a mapping $\mathcal{F} : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ by

$$\mathcal{F}(u)(t) = F(t, u(t), u(t-\tau)) + Cu(t), \quad u \in C_\omega(\mathbb{R}, E), t \in \mathbb{R}, \tag{3.5}$$

By the continuity of F, $\mathcal{F} : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is continuous. Define an operator $Q : [v_0, w_0] \to C_\omega(\mathbb{R}, X)$ as follows:

$$Qu = (P \circ \mathcal{F})u, \tag{3.6}$$

then we have

$$Qu(t) = (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, u(s), u(s-\tau)) + Cu(s))ds, \quad t \in \mathbb{R}, \tag{3.7}$$

and $Q : [v_0, w_0] \to C_\omega(\mathbb{R}, X)$ is continuous. Therefore, by the definition of P, we can assert $u \in [v_0, w_0]$ is the ω-periodic mild solution of Eq. (1.1) if and only if u is the fixed point of the compound operator Q.

Now, we complete the proof by four steps.

Step 1. We show that the following properties of the operator Q defined by (3.6).

(i) $v_0 \le Qv_0$ and $Qw_0 \le w_0$,
(ii) $Qu_1 \le Qu_2$ for any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$.

Since $v_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ is an ω-periodic lower solution of Eq. (1.1), thus

$$v_0'(t) + Av_0(t) + Cv_0(t) \le F(t, v_0(t), v_0(t-\tau)) + Cv_0(t), \quad t \in \mathbb{R}. \tag{3.8}$$

Set $h(t) = v_0'(t) + Av_0(t) + Cv_0(t)$, by Lemma 2.2 and the positivity of semigroup $S(t)(t \ge 0)$, one can obtain that

$$v_0(t) = Ph(t) \le P(F(t, v_0(t), v_0(t-\tau)) + Cv_0(t)) \le Qv_0(t), \ t \in \mathbb{R}, \tag{3.9}$$

which implies that $v_0 \le Qv_0$. Similarly, it can be shown that $Qw_0 \le w_0$.

For any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$ and $t \in \mathbb{R}$, we have $v_0(t) \le u_1(t) \le u_2(t) \le w_0(t)$, $v_0(t-\tau) \le u_1(t-\tau) \le u_2(t-\tau) \le w_0(t-\tau)$. By the condition (H1) and the positivity of the operator P,

$$\begin{aligned} Qu_1(t) &= P(F(t, u_1(t), u_1(t-\tau)) + C_1 u(t)) \\ &\le P(F(t, u_2(t), u_2(t-\tau)) + Cu_2(t)) = Qu_2(t), \end{aligned} \tag{3.10}$$

it follows that $Qu_1 \le Qu_2$.

Therefore, $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuous increasing operator.

Step 2. We define two sequences $\{v_i\}$ and $\{w_i\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_i = Qv_{i-1}, \quad w_i = Qw_{i-1}, \qquad\qquad i = 1, 2, \cdots . \tag{3.11}$$

Then from the monotonicity of the operator Q, it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_i \le \cdots \le w_i \le \cdots \le w_2 \le w_1 \le w_0, \tag{3.12}$$

and $\{v_i\}, \{w_i\} \subset [v_0, w_0]$ are equicontinuous in $\mathbb{R}$.

In fact, for any $u \in [v_0, w_0]$, by the periodicity of u, we consider it on $[0, \omega]$. Set $0 \le t_1 < t_2 \le \omega$, we get that

$$\begin{aligned} &Qu(t_2) - Qu(t_1) \\ &= (I - T(\omega))^{-1} \int_{t_2-\omega}^{t_2} S(t_2 - s)(F(s, u(s), u(s-\tau)) + Cu(s))ds \end{aligned}$$

$$
\begin{aligned}
&- (I-T(\omega))^{-1}\int_{t_1-\omega}^{t_1} S(t_1-s)(F(s,u(s),u(s-\tau))+Cu(s))ds\\
&= (I-T(\omega))^{-1}\int_{t_2-\omega}^{t_1} (S(t_2-s)-S(t_1-s))(F(s,u(s),u(s-\tau))+Cu(s))ds\\
&\quad - (I-T(\omega))^{-1}\int_{t_1-\omega}^{t_2-\omega} S(t_1-s)(F(s,u(s),u(s-\tau))+Cu(s))ds\\
&\quad + (I-T(\omega))^{-1}\int_{t_1}^{t_2} S(t_2-s)(F(s,u(s),u(s-\tau))+Cu(s))ds\\
&:= I_1+I_2+I_3.
\end{aligned}
$$

It is clear that

$$\|Qu(t_2)-Qu(t_1)\| \le \|I_1\|+\|I_2\|+\|I_3\|. \tag{3.13}$$

Thus, we only need to check $\|I_i\|$ tend to 0 independently of $u\in[v_0,w_0]$ when $t_2-t_1\to 0, i=1,2,3$. For any $u\in[v_0,w_0]$, from the condition (H1), it follows that

$$
\begin{aligned}
F(t,v_0(t),v_0(t-\tau))+Cv_0(t) &\le F(t,u(t),u(t-\tau))+Cu(t)\\
&\le F(t,w_0(t),w_0(t-\tau))+Cw_0(t).
\end{aligned}
$$

By the normality of the cone K, there exists M_2 such that

$$\Big\|F(t,u(t),u(t-\tau))+Cu(t)\Big\| \le M_2, \quad t\in\mathbb{R}, u\in[v_0,w_0]. \tag{3.14}$$

By the compactness of $S(t)(t\ge 0)$, it follows that $S(t)$ is continuous in the uniform operator topology for $t>0$. Hence, it is easy to check $\|I_i\|$ tend to 0 independently of $u\in[v_0,w_0]$ when $t_2-t_1\to 0 (i=1,2,3)$, which means that $Q([v_0,w_0])$ is equicontinuous.

Step 3. $\{v_i(t)\}$ and $\{w_i(t)\}$ are precompact on E for any $t\in\mathbb{R}$.

Let $B_1=\{v_i\}$, $B_2=\{w_i\}$ and $B_1^0=B_1\cup\{v_0\}$, $B_2^0=B_2\cup\{w_0\}$. Obviously, $B_1(t)=(QB_1^0)(t)$ and $B_2(t)=(QB_2^0)(t)$ for $t\in\mathbb{R}$.

We define a set $(Q_\varepsilon B_1)(t)$ by

$$(Q_\varepsilon B_1^0)(t) := \{(Q_\varepsilon v_i)(t) \mid v_i\in B_1^0,\ 0<\varepsilon<\omega,\ t\in\mathbb{R}\}, \tag{3.15}$$

where

$$Q_\varepsilon v_i(t) = (I-S(\omega))^{-1}\int_{t-\omega}^{t-\varepsilon} S(t-s)\Big(F(s,v_{i-1}(t),v_{i-1}(s-\tau)+Cv_i(s)\Big)ds$$

$$= (I - S(\omega))^{-1} S(\varepsilon) \int_{t-\omega}^{t-\varepsilon} S(t-s-\varepsilon) \times \Big(F(s, v_{i-1}(t), v_{i-1}(s-\tau) + Cv_i(s)\Big) ds.$$

Then the set $(Q_\varepsilon B_1^0)(t)$ is relatively compact in E since the operator $S(\varepsilon)$ is compact in E ($S(t) = e^{-Ct}T(t)(t \geq 0)$ is compact semigroup). For any $v_i \in B_1^0$ and $t \in \mathbb{R}$, from the following inequality

$$\begin{aligned}
&\|Qv_i(t) - Q_\varepsilon v_i(t)\| \\
&\leq \Big\|(I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)\Big(F(s, v_{i-1}(t), v_{i-1}(s-\tau)) + Cv_{i-1}(s)\Big) ds \\
&\quad - (I - S(\omega))^{-1} \int_{t-\omega}^{t-\varepsilon} S(t-s)\Big(F(s, v_{i-1}(t), v_{i-1}(s-\tau)) + Cv_{i-1}(s)\Big) ds\Big\| \\
&\leq \|(I - S(\omega))^{-1}\| \int_{t-\varepsilon}^{t} \Big\|S(t-s)\Big(F(s, v_{i-1}(t), v_{i-1}(s-\tau)) + Cv_{i-1}(s)\Big)\Big\| ds \\
&\leq \|(I - S(\omega))^{-1}\| M_2 \int_{t-\varepsilon}^{t} \|S(t-s)\| ds,
\end{aligned}$$

one can obtain that the set $(QB_1^0)(t)$ is relatively compact, which implies that $\{v_i(t)\} = B_1(t) = (QB_1^0)(t)$ is relatively compact in E for $t \in \mathbb{R}$. Similarly, it can be shown that $\{w_i(t)\}$ is relatively compact in E for $t \in \mathbb{R}$.

Therefore, $\{v_i\}$ and $\{w_i\}$ are relatively compact in $C_\omega(\mathbb{R}, E)$ by the Arzela-Ascoli Theorem, so there are convergent subsequences in $\{v_i\}$ and $\{w_i\}$, respectively. Combining this with the monotonicity and the normality of the cone K_C, we can easily prove that $\{v_i\}$ and $\{w_i\}$ themselves are convergent, i.e., there are $\underline{u}, \overline{u} \in C_\omega(\mathbb{R}, E)$ such that $\lim_{i\to\infty} v_i = \underline{u}$ and $\lim_{i\to\infty} w_i = \overline{u}$.

Taking limit in (3.11), we have

$$\underline{u} = Q\underline{u}, \qquad \overline{u} = Q\overline{u}. \tag{3.16}$$

Therefore $\underline{u}, \overline{u} \in C_\omega(\mathbb{R}, X)$ are fixed points of Q, and they are the ω-periodic mild solutions of the periodic problem (1.1).

Step 4. We prove the minimal and maximal properties of $\underline{u}, \overline{u}$.

Assume that $\widetilde{u}$ is a fixed point of Q with $\widetilde{u} \in [v_0, w_0]$, then for every $t \in \mathbb{R}$, $v_0(t) \leq \widetilde{u}(t) \leq w_0(t)$,

$$v_1(t) = (Qv_0)(t) \leq (Q\widetilde{u})(t) = \widetilde{u}(t) \leq (Qw_0)(t) = w_1(t),\ t \in \mathbb{R}. \tag{3.17}$$

Similarly, $v_1(t) \leq \widetilde{u}(t) \leq w_1(t)$, $t \in \mathbb{R}$. In general

$$v_i \leq \widetilde{u} \leq w_i, \quad i = 1, 2, \cdots. \tag{3.18}$$

Taking limit in (3.18) as $i \to \infty$, we get $\underline{u} \le \widetilde{u} \le \overline{u}$. Therefore $\underline{u}, \overline{u}$ are minimal and maximal ω-periodic mild solutions of Eq. (1.1), and $\underline{u}, \overline{u}$ can be obtained by the iterative sequences defined in (3.11) starting from v_0 and w_0. This completes the proof of Theorem 3.1. $\square$

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone K is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous C_0-semigroup $T(t)(t \ge 0)$, let $F : \mathbb{R} \times E \times E \to E$ be a continuous mapping which is ω-periodic in t. Assume the periodic problem (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ with $v_0 \le w_0$. If the condition (H1) and the following condition*

(H2) There exists a constant $c \in [0, 1/4\omega C_s M_S)$ such that for all $t \in \mathbb{R}$ and monotonic sequences $\{u_n\} \subset [v_0, w_0]$,

$$\alpha(\{F(t, u_n(t), u_n(t-\tau)) + Cu_n(t)\}) \le c(\alpha(\{u_n(t)\}) + \alpha(\{u_n(t-\tau)\}))$$

hold, then the periodic problem (1.1) has minimal and maximal ω-periodic mild solution $\underline{u}, \overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, where $C_S = \|(I - S(\omega))^{-1}\|$, $M_S = \sup\{\|S(t)\| \mid t \ge 0\}$.

Proof. From the proof of Theorem 3.1, we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuous increasing operator and $v_0 \le Qv_0$, $Qw_0 \le w_0$. Hence, the iterative sequences v_i and w_i defined by (3.11) satisfy (3.12). By $T(t)(t \ge 0)$ is an equicontinuous C_0-semigroup, it follows that $S(t)(t \ge 0)$ is also an equicontinuous C_0-semigroup. From the proof of Theorem 3.1, we obtain that $\{v_i\}$, $\{w_i\}$ are equicontinuous in $\mathbb{R}$.

Next, we show that $\{v_i\}$, $\{w_i\}$ are convergent in $C_\omega(\mathbb{R}, X)$.

Obviously, $\{v_i\}$ is a bounded countable set. By Lemma 2.4, Lemma 2.5 and the condition (H2), one can obtain that

$$\begin{aligned}
&\alpha(\{v_i(t)\}) = \alpha(\{Qv_{i-1}(t)\}) \\
&= \alpha\Big(\Big\{(I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, v_{i-1}(s), v_{i-1}(s-\tau)) + Cv_{i-1}(s))ds\Big\}\Big) \\
&\le 2\|(I - S(\omega))^{-1}\| \cdot \int_{t-\omega}^{t} \|S(t-s)\| \\
&\quad \cdot \alpha(\{F(s, v_{i-1}(s), v_{i-1}(s-\tau)) + Cv_{i-1}(s)\})ds \\
&\le 2c\|(I - S(\omega))^{-1}\| \cdot \int_{t-\omega}^{t} \|S(t-s)\| \cdot (\alpha(\{v_{i-1}(s)\}) + \alpha(\{v_{i-1}(s-\tau)\})ds
\end{aligned}$$

$$\leq 2cC_SM_S \int_{t-\omega}^{t} \alpha(\{v_{i-1}(s)\}) + \alpha(\{v_{i-1}(s-\tau)\})ds,$$

from the periodicity of v_i and definition of measure of noncompactness, it follows that $\alpha(\{v_{i-1}(s)\}) = \alpha(\{v_{i-1}(s-\tau)\})$, thus,

$$\alpha(\{v_i(t)\}) \leq 4c\omega C_SM_S \cdot \max_{t\in[0,\omega]} \alpha(\{v_i(t)\}) \leq 4c\omega C_SM_S \cdot \alpha_C(\{v_i\}). \tag{3.19}$$

Since $\{v_i\}$ is equicontinuous, from Lemma 2.3, it follows that

$$0 \leq \alpha_C(\{v_i\}) \leq 4c\omega C_SM_S \cdot \alpha_C(\{v_i\}),$$

While $4c\omega C_SM_S < 1$, hence $\alpha_C(\{v_i\}) = 0$. Similarly, we can prove $\alpha_C(\{w_i\}) = 0$. Therefore, $\{v_i\}$, $\{w_i\}$ are relatively compact in $C_\omega(\mathbb{R}, X)$, so there are convergent subsequences in $\{v_i\}$ and $\{w_i\}$, respectively. Combining this with the monotonicity and the normality of the cone K_C, we can easily prove that $\{v_i\}$ and $\{w_i\}$ themselves are convergent, i.e., there are $\underline{u}, \overline{u} \in C_\omega(\mathbb{R}, E)$ such that $\lim_{i\to\infty} v_i = \underline{u}$ and $\lim_{i\to\infty} w_i = \overline{u}$.

Therefore, from the proof of Theorem 3.1, $\underline{u}, \overline{u}$ are minimal and maximal ω-periodic mild solutions of the periodic problem with delay (1.1) in $[v_0, w_0]$. □

In the application of partial differential equations, we often choose Banach space $L^p(\Omega)(1 \leq p < \infty)$ as working space, which is weakly sequentially complete space. Next, we discuss the existence of mild solutions for the periodic problem with delay (1.1) in weakly sequentially complete Banach space.

Theorem 3.3. *Let E be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous C_0-semigroup $T(t)(t \geq 0)$ in E, let $F : \mathbb{R} \times E \times E \to E$ be a continuous mapping which is ω-periodic in t. Assume the periodic problem (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ with $v_0 \leq w_0$. If the condition (H1) holds, then the periodic problem (3.1) has minimal and maximal ω-periodic mild solution $\underline{u}, \overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.*

Proof. From the proof of Theorem 3.1, it follows that the iterative sequences v_i and w_i defined by (3.11) satisfy (3.12). Hence, for any $t \in \mathbb{R}$, $\{v_i(t)\}$ and $\{w_i(t)\}$ are monotone and order-bounded sequences in E. Noticing that E is a weakly sequentially complete Banach space, from Theorem 2.2 in [12], one can get that $\{v_i(t)\}$ and $\{w_i(t)\}$ are precompact in E for any $t \in \mathbb{R}$. Combining this with the monotonicity (3.12), it follows that $\{v_i(t)\}$ and $\{w_i(t)\}$ are uniformly convergent in E. Denote

$$\underline{u}(t) = \lim_{n\to\infty} v_n(t), \qquad \overline{u}(t) = \lim_{n\to\infty} w_n(t), \quad t \in \mathbb{R}. \tag{3.20}$$

Obviously, $\{v_n(t)\}$, $\{w_n(t)\} \subset C_\omega(\mathbb{R}, X)$, and $v_0(t) \leq \underline{u}(t) \leq \overline{u}(t) \leq w_0(t)(t \in \mathbb{R})$. By (3.7), we have

$$\begin{aligned} v_i(t) &= Qv_{i-1}(t) \\ &= (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, v_{i-1}(s), v_{i-1}(s-\tau)) + Cv_{i-1}(s))ds, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} w_i(t) &= Qw_{i-1}(t) \\ &= (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, w_{i-1}(s), w_{i-1}(s-\tau)) + Cw_{i-1}(s))ds. \end{aligned} \tag{3.22}$$

Taking limit in (3.21) and (3.22) as $i \to \infty$, from the Lebesgue dominated convergence theorem, one can obtain

$$\underline{u}(t) = (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, \underline{u}(s), \underline{u}(s-\tau)) + C\underline{u}(s))ds, \quad t \in \mathbb{R}, \tag{3.23}$$

and

$$\overline{u}(t) = (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, \overline{u}(s), \overline{u}(s-\tau)) + C\overline{u}(s))ds, \quad t \in \mathbb{R}. \tag{3.24}$$

which implies that $\underline{u}, \overline{u} \in C_\omega(\mathbb{R}, X)$. Similar with the proof of Theorem 3.1, we know that the $\underline{u}, \overline{u}$ are minimal and maximal ω-periodic mild solutions of the periodic problem with delay (1.1) in $[v_0, w_0]$. □

Remark 1. Analytic semigroup and differentiable semigroup are continuous by operator norm for every $t > 0$ (see [33]). In the application of partial differential equations, such as parabolic equations and strongly damped wave equations, the corresponding solution semigroup is analytic semigroup. Therefore, Theorem 3.2 and Theorem 3.3 in this paper has broad applicability.

In the above works, the key assumption (H1) (the monotone on the third variable of the nonlinear function) is employed. However, we hope that the nonlinear function is quasi-monotonicity. In this case, the results have more extensive application background.

In fact, we find that if the periodic problem (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ with $v_0 \leq w_0$ and

(H3) there is a sufficiently small constant $C_1 > 0$, such that

$$u_2(t) - u_1(t) \geq C_1(u_2(t-\tau) - u_1(t-\tau)), \quad t \in \mathbb{R},$$

for any $u_1, u_2 \in [v_0, w_0]$ with $u_2 \geq u_1$,

then the condition (H1) can be replaced by the following condition

(H4) there are nonnegative constants C_2, C_3, such that

$$F(t, x_2, y_2) - F(t, x_1, y_1) \geq -C_2(x_2 - x_1) - C_3(y_2 - y_1),$$

for all $t \in \mathbb{R}$, $x_1, x_2, y_1, y_2 \in E$ with $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, $v_0(t-\tau) \leq y_1 \leq y_2 \leq w_0(t-\tau)$.

In fact, for every $t \in \mathbb{R}$ and $u_1, u_2 \in [v_0, w_0]$ with $u_1 \leq u_2$, one can obtain that $v_0(t) \leq u_1(t) \leq u_2(t) \leq w_0(t)$, $v_0(t-\tau) \leq u_1(t-\tau) \leq u_2(t-\tau) \leq w_0(t-\tau)$. By the conditions (H3) and (H4), it follows that

$$\begin{aligned}
&F(t, u_2(t), u_2(t-\tau)) - F(t, u_1(t), u_1(t-\tau)) \\
\geq &-C_2(u_2(t) - u_1(t)) - C_3(u_2(t-\tau) - u_1(t-\tau)) \\
\geq &-C_2(u_2(t) - u_1(t)) - \frac{C_3}{C_1}(u_2(t) - u_1(t)) \\
= &-(C_2 + \frac{C_3}{C_1})(u_2(t) - u_1(t)) \\
:= &-C(u_2(t) - u_1(t)).
\end{aligned}$$

Hence, we can obtain the following results form Theorem 3.1 and Theorem 3.2, respectively.

Theorem 3.4. *Let E be an ordered Banach space, whose positive cone K is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive compact semigroup $T(t)(t \geq 0)$, let $f : \mathbb{R} \times E \times E \to E$ be a* *continuous mapping which is ω-periodic in t. Assume Eq. (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ with $v_0 \le w_0$. If the conditions (H3) and (H4) hold, then the periodic problem (1.1) has minimal and maximal ω-periodic mild solution $\underline{u}, \overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.*

Theorem 3.5. *Let E be an ordered Banach space, whose positive cone K is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous C_0-semigroup $T(t)(t \ge 0)$, let $f : \mathbb{R} \times E \times E \to E$ be a continuous mapping which is ω-periodic in t. Assume the periodic problem (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ with $v_0 \le w_0$. If the conditions (H2–H4) hold, then the periodic problem (1.1) has minimal and maximal ω-periodic mild solution $\underline{u}, \overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.*

Remark 2. Obviously, the condition (H3) is easy to satisfy, and the condition (H4) weakens the condition (H1). Hence, Theorem 3.3 and Theorem 3.4 partially improve Theorem 3.1 and Theorem 3.2.

Next, we discuss the uniqueness of the ω-periodic mild solution for the periodic problem (1.1) under $T(t)(t \ge 0)$ is an equicontinuous C_0-semigroup.

Theorem 3.6. *Let E be an ordered Banach space, whose positive cone K is normal cone with normal constant N, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous C_0-semigroup $T(t)(t \ge 0)$, let $f : \mathbb{R} \times E \times E \to E$ be a continuous mapping which is ω-periodic in t. Assume Eq. (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^1_\omega(\mathbb{R}, E) \cap C_\omega(\mathbb{R}, E_1)$ with $v_0 \le w_0$. If the conditions (H3), (H4) and*

(H5) there exist constants $L_1, L_2 > 0$, such that for every $t \in \mathbb{R}$ and $x_1, x_2, y_1, y_2 \in X$, satisfying $v_0(t-\tau) \le y_1 \le y_2 \le w_0(t-\tau)$, $v_0(t) \le x_1 \le x_2 \le w_0(t)$,

$$F(t, x_2, y_2) - F(t, x_1, y_1) \le L_1(x_2 - x_1) + L_2(y_2 - y_1),$$

(H6) $N\Big(C_2 + \frac{C_3}{C_1} + L_1 + C_1 L_2\Big) C_S M_S \omega < 1,$

hold, then the periodic problem (1.1) has a unique ω-periodic mild solution $u^* \in [v_0, w_0]$, where $C_S = \|(I - S(\omega))^{-1}\|$, $M_S = \sup\{\|S(t)\| \mid t \ge 0\}$.

Proof. From Theorem 3.1 and Theorem 3.5, one can obtain that the iterative sequences v_i and w_i defined by (3.11) satisfy (3.12). For any $t \in \mathbb{R}$, by the conditions (H3), (H5), (3.7), (3.12), it is easy to see

$$\begin{aligned}
\theta \leq w_i(t) - v_i(t) &= Qw_{i-1}(t) - Qv_{i-1}(t) \\
&= (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, w_{i-1}(s), w_{i-1}(s-\tau)) + Cw_{i-1}(s))ds \\
&\quad - (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(F(s, v_{i-1}(s), v_{i-1}(s-\tau)) + Cv_{i-1}(s))ds \\
&\leq (I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)((L_1 + C)(w_{i-1}(s) - v_{i-1}(s)) \\
&\quad + L_2(w_{i-1}(s-\tau) - v_{i-1}(s-\tau)))ds \\
&\leq (L_1 + C + L_2C_1)(I - S(\omega))^{-1} \int_{t-\omega}^{t} S(t-s)(w_{i-1}(s) - v_{i-1}(s))ds,
\end{aligned}$$

where $C = C_2 + \frac{C_3}{C_1}$. By the normality of the cone K, it follows that

$$\|w_i(t) - v_i(t)\| \leq N(L_1 + C + L_2C_1)C_S M_S \omega \|w_{i-1} - v_{i-1}\|_C, \quad t \in \mathbb{R},$$

namely

$$\|w_i - v_i\|_C \leq N(L_1 + C + L_2C_1)C_S M_S \omega \|w_{i-1} - v_{i-1}\|_C, \tag{3.25}$$

by the condition (H6), we can obtain that

$$\|w_i - v_i\|_C \leq \Big(N(L_1 + C + L_2C_1)C_S M_S \omega\Big)^i \|w_0 - v_0\|_C \to 0, \quad i \to \infty.$$

Thus, there is a unique ω-periodic mild solution $u^* \in C_\omega(\mathbb{R}, X)$, such that $\lim\limits_{i\to\infty} w_i = \lim\limits_{i\to\infty} v_i = u^*$. Hence, taking limit in (3.11) as $i \to \infty$, we get $u^* = Qu^*$, which implies that u^* is unique ω-periodic mild solution $u^* \in [v_0, w_0]$ of the periodic problem (1.1). □

4. Application

In this section, we present two examples, which do not aim at generality but indicate how our abstract results can be applied to concrete problems.

Example 4.1. Periodic solutions of delay parabolic equations in $\mathbb{R}^n (n \geq 1)$.

Let $\overline{\Omega} \in \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Let

$$A(x,D)u = -\sum_{i,j=1}^{N} a_{ij}(x)D_iD_ju + \sum_{j=1}^{N} a_j(x)D_ju + a_0(x)u, \tag{4.1}$$

be a uniformly elliptic differential operator in $\overline{\Omega}$, whose coefficients $a_{ij}(x)$, $a_j(x)$ $(i,j = 1,\cdots,n)$ and $a_0(x)$ are Höder-continuous on $\overline{\Omega}$, and $a_0(x) \geq 0$. We let $B = B(x,D)$ be a boundary operator on $\partial\Omega$ of the form:

$$Bu := b_0(x)u + \delta\frac{\partial u}{\partial \beta}, \tag{4.2}$$

where either $\delta = 0$ and $b_0(x) \equiv 1$ (Dirichlet boundary operator), or $\delta = 1$ and $b_0(x) \geq 0$ (regular oblique derivative boundary operator; at this point, we further assume that $a_0(x) \not\equiv 0$ or $b_0(x) \not\equiv 0$), β is an outward pointing, nowhere tangent vector field on $\partial\Omega$. Let λ_1 be the first eigenvalue of elliptic operator $A(x,D)$ under the boundary condition $Bu = 0$. It is well known ([1, Theorem 1.16]), that $\lambda_1 > 0$.

Under the above assumptions, we discuss the existence and uniqueness of ω-periodic solutions of the semilinear parabolic equation boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t),u(x,t-\tau)), \; x \in \Omega, \; t \in \mathbb{R}, \\ Bu = 0, \quad x \in \partial\Omega, \end{cases} \tag{4.3}$$

where $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a local Hölder-continuous function which is ω-periodic in t, $\tau > 0$ denotes the time delay.

Theorem 4.1. *Let $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a local Hölder-continuous function which is ω-periodic in t. If the following conditions*

(H7) $f(x,t,0,0) \geq 0$ for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, and there is a function $0 \leq w = w(x,t) \in C^{2,1}(\Omega \times \mathbb{R})$ which is ω-periodic in t, such that

$$\begin{cases} \frac{\partial}{\partial t}w(x,t) + A(x,D)w(x,t) \geq f(x,t,w(x,t),w(x,t-\tau)), (x,t) \in \Omega \times \mathbb{R}, \\ Bw = 0, \quad x \in \partial\Omega, \end{cases}$$

(H8) there exists a constant $c > 0$, such that for any $x \in \Omega$, $t \in \mathbb{R}$ and $0 \leq y_1 \leq y_2 \leq w(x,t)$, $0 \leq z_1 \leq z_2 \leq w(x,t-\tau)$,

$$f(x,t,y_2,z_2) - f(x,t,y_1,z_1) \geq -C(y_2 - y_1),$$

hold, then the semilinear delayed parabolic equation boundary value problem (4.3) has minimal and maximal ω*-periodic solution* $\underline{u}, \overline{u} \in C^{2,1}(\overline{\Omega}\times\mathbb{R})$ *between* 0 *and* w*, which can be obtained by monotone iterative sequences starting from* 0 *and* w*.*

Proof. Let $E = L^p(\Omega)(p > 1)$, $K = \{u \in E|\ u(x) \geq 0\ a.e.\ x \in \Omega\}$, then E is an ordered Banach space, whose positive cone K is a normal regeneration cone. Define an operator $A : D(A) \subset E \to E$ by:

$$D(A) = \{u \in W^{2,p}(\Omega)|\ B(x, D)u = 0,\ x \in \partial\Omega\}, \quad Au = A(x, D)u. \tag{4.4}$$

If $a_0(x) \geq 0$, then $-A$ generates an exponentially stable analytic semigroup $T_p(t)(t \geq 0)$ in E (see [2]). By the maximum principle of elliptic operators, we know that $(\lambda I + A)$ has a positive bounded inverse operator $(\lambda I + A)^{-1}$ for $\lambda > 0$, hence $T_p(t)(t \geq 0)$ is a positive semigroup (see [21]). From the operator $A(x, D)$ has compact resolvent in $L^p(\Omega)$, we obtain $T_p(t)(t \geq 0)$ is also a compact semigroup (see [33]).

Denote $u(t) = u(\cdot, t)$, and $F(t, u(t), u(t-\tau)) = f(\cdot, t, u(\cdot, t), u(\cdot, t-\tau))$, then parabolic boundary value problem (4.3) can be reformulated as the abstract evolution (1.1) in E. By the condition (H7), it follows that $v_0 \equiv 0$ and $w_0 = w(x, t)$ are time ω-periodic lower solution and time ω-periodic upper solution of the problem (4.3), and $v_0 \leq w_0$. By the condition (H8), it follows that the condition (H1) holds. Hence, form Theorem 3.1, one can see the delayed parabolic boundary value problem (4.3) has minimal and maximal ω-periodic mild solution $\underline{u}, \overline{u}$, which can be obtained by monotone iterative sequences starting from 0 and w, respectively.

By the analyticity of the semigroup $T_p(t)(t \geq 0)$ and the regularization method used in [2], we can see that $\underline{u}, \overline{u} \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$ are time ω-periodic solutions of the problem(4.3). This completes the proof of the theorem. □

Furthermore, if the following condition

(H9) there is a constant $C_1 > 0$*, such that*

$$u_2(x, t) - u_1(x, t) \geq C_1(u_2(x, t-\tau) - u_1(x, t-\tau)),$$

for $(x, t) \in \overline{\Omega} \times \mathbb{R}, u_1, u_2 \in [0, w(x, t)], u_2 \geq u_1,$ then the condition (H8) can be replaced by

(H10) there exist nonnegative constants C_2, C_3 *such that*

$$f(x, t, y_2, z_2) - f(x, t, y_1, z_1) \geq -C_2(y_2 - y_1) - C_3(z_2 - z_1).$$

for $(x, t) \in \overline{\Omega} \times \mathbb{R}$ *and* $x_1, x_2, y_1, y_2 \in X$ *with* $0 \leq x_1 \leq x_2 \leq w_0(x, t)$*,* $0 \leq y_1 \leq y_2 \leq w_0(x, t-\tau)$*.*

Thus, according to Theorem 3.4, we have the following result

Theorem 4.2. *Let $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a local Hölder-continuous function which is ω-periodic in t. If the conditions (H7),(H9) and (H10) hold, then the semilinear delayed parabolic equation boundary value problem (4.3) has minimal and maximal ω-periodic solution $\underline{u}, \overline{u} \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$ between 0 and w, which can be obtained by monotone iterative sequences starting from 0 and w.*

Example 4.2. Doubly periodic problems of first order partial differential equation with delay.

Let $f : \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, which is 2π-periodic in t and x. We are concerned with the existence of solutions for the semilinear first order partial differential equation with delay in $\mathbb{R}^2$:

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}u(x,t) = f(x,t,u(x,t),u(x,t-\tau)), \quad (x,t) \in \mathbb{R}^2, \tag{4.5}$$

with doubly periodic boundary conditions

$$u(x+2\pi,t) = u(x,t+2\pi) = u(x,t), \quad (x,t) \in \mathbb{R}^2, \tag{4.6}$$

where $\tau > 0$ denotes the time delay.

Theorem 4.3. *Let $f(x,t,u,v) \in C^1(\mathbb{R}^4)$, and f is 2π-periodic in t and x. Assume $f(x,t,0,0) \geq 0$, there is a function $w(x,t) \in C^1(\mathbb{R}^2)$ and w is 2π-periodic in t and x satisfying $w(x,t) \geq 0$, such that*

$$\frac{\partial}{\partial t}w(x,t) + \frac{\partial}{\partial x}w(x,t) \geq f(x,t,w(x,t),w(x,t-\tau)), \qquad (x,t) \in \mathbb{R}^2.$$

If the following conditions

(H11) for any $x,t \in \mathbb{R}$ and $u_1,u_2 \in C(\mathbb{R}^2)$, $0 \leq u_1(x,t) \leq u_2(x,t) \leq w(x,t)$,

$$\begin{aligned} f(x,t,u_2(x,t),u_2(x,t-\tau)) - f(x,t,u_1(x,t),u_1(x,t-\tau)) \\ \geq -(u_2(x,t) - u_1(x,t)), \end{aligned}$$

(H12) there is a constant $c \in [0, \frac{e^{2\pi}-1}{8\pi e^{2\pi}})$, such that for any $x,t \in \mathbb{R}$ and monotone sequence $\{u_n(x,t)\} \in [0,w(x,t)]$,

$$\begin{aligned} \alpha(\{f(x,t,u_n(x,t),u_n(x,t-\tau)) + u_n(x,t)\}) \\ \geq c(\alpha(\{u_n(x,t)\}) + \alpha(\{u_n(x,t-\tau)\})), \end{aligned}$$

hold, then the doubly periodic problems of first order partial differential equation (4.5)–(4.6) has minimal and maximal classical solutions $\underline{u}, \overline{u} \in C^1(\mathbb{R}^2)$ between 0 and w.

Proof. Let $C_{2\pi}(\mathbb{R})$ denote the Banach space $\{u \in C(\mathbb{R}) | u(x+2\pi) = u(x), x \in \mathbb{R}\}$ endowed the maximum norm $\|u\|_C = \max_{x\in[0,2\pi]} \|u(x)\|$. Denote $E = C_{2\pi}(\mathbb{R})$, let

$$D(A) = C^1_{2\pi}(\mathbb{R}), \quad A = \frac{\partial u}{\partial x}. \tag{4.7}$$

From [24, Lemma 2.1], if $\lambda \neq 0$, we know that $(\lambda I + A)$ has a bounded inverse operator $(\lambda I + A)^{-1}$ in E and

$$(\lambda I + A)^{-1} h(x) = \int_{x-2\pi}^{x} r(s-y)h(y)dy, \quad h \in E, \tag{4.8}$$

where

$$r(x) = \frac{e^{-\lambda x}}{1 - e^{-2\pi\lambda}}, \qquad x \in [0, 2\pi].$$

By (4.8), it follows that $(\lambda I + A)^{-1}$ is positive operator for $\lambda > 0$, and its norm $\|(\lambda I + A)^{-1}\| \leq \frac{1}{\lambda}$. Form Hille-Yosida Theorem and exponential formula of semigroup (see [33]), we can obtain that $-A$ generates a contractive and positive C_0-semigroup $T(t)(t \geq 0)$, whose growth exponent $\nu_0 \leq 0$. Thus, $-(A+I)$ generates a contractive and positive C_0-semigroup $S(t) = e^{-t}T(t)(t \geq 0)$ in E, and the growth exponent $\nu_1 = -1 + \nu_0 \leq -1$, which implies that $S(t)(t \geq 0)$ is an exponentially stable, positive C_0-semigroup and $\|S(2\pi)\| \leq e^{-2\pi}$, $\|(I - S(2\pi))^{-1}\| \leq \frac{e^{2\pi}}{e^{2\pi}-1}$.

Set $u(t)(x) = u(x,t), u(t-\tau)(x) = u(x, t-\tau)$, and

$$F(t, u(t), u(t-\tau))(x) = f(t, u(x,t), u(x, t-\tau)), \tag{4.9}$$

then the doubly periodic problems (4.5)–(4.6) can be reformulated as following

$$u'(t) + Au(t) = F(t, u(t), u(t-\tau)), \quad t \in \mathbb{R}, \tag{4.10}$$

where $F : \mathbb{R} \times E \times E \to E$ is C_1-mapping which is 2π-periodic in t.

It is easy to see that $v_0(t) \equiv 0$ and $w_0(\cdot, t) = w(x,t)$ are 2π-periodic lower solution and 2π-periodic upper solution of Eq. (4.10). From the condition (H11), (H12) and Theorem 3.2, one can obtain that Eq. (4.10) has minimal and maximal 2π-periodic mild solution $\underline{u}, \overline{u}$ between 0 and w_0, which can be obtained by monotone iterative sequences starting from 0 and w_0. Since F is a C^1-mapping, from regularity of solutions for the semilinear evolution equations (see [33]), we know that

$$\underline{u}, \overline{u} \in C^1_{2\pi}(\mathbb{R}, X) \cap C_{2\pi}(\mathbb{R}, D(A)), \tag{4.11}$$

namely $\underline{u}, \overline{u}$ are minimal and maximal 2π-periodic classical solutions, respectively. Therefore, by the definition of A, it follows that $\underline{u}, \overline{u}$ are 2π-doubly periodic classical solutions of the doubly periodic problems (4.5)–(4.6). □

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