

The Steklov spectrum and coarse discretizations of manifolds with boundary

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Abstract: Given $\kappa, r_0 > 0$ and $n \in \mathbb{N}$, we consider the class $\mathcal{M} = \mathcal{M}(\kappa, r_0, n)$ of compact n -dimensional Riemannian manifolds with cylindrical boundary, Ricci curvature bounded below by $-(n-1)\kappa$ and injectivity radius bounded below by r_0 away from the boundary. For a manifold $M \in \mathcal{M}$ we introduce a notion of discretization, leading to a graph with boundary which is roughly isometric to M , with constants depending only on κ, r_0, n . In this context, we prove a uniform spectral comparison inequality between the Steklov eigenvalues of a manifold $M \in \mathcal{M}$ and those of its discretization. Some applications to the construction of sequences of surfaces with boundary of fixed length and with arbitrarily large Steklov spectral gap $\sigma_2 - \sigma_1$ are given. In particular, we obtain such a sequence for surfaces with connected boundary. The applications are based on the construction of graph-like surfaces which are obtained from sequences of graphs with good expansion properties.

1. Introduction

Let M be a smooth compact Riemannian manifold of dimension $n \geq 2$, with smooth boundary $\Sigma = \partial M$. The *Steklov problem* on M is to find all $\sigma \in \mathbb{R}$ for which there exists a non-zero function u such that

$$\begin{cases} \Delta u = 0 & \text{in } M, \\ \partial_\nu u = \sigma u & \text{on } \Sigma, \end{cases}$$

where Δ is the Laplace-Beltrami operator acting on functions on M , and ∂_ν is the outward normal derivative along the boundary Σ . It is standard that the Steklov spectrum is discrete:

$$0 = \sigma_1(M) \leq \sigma_2(M) \leq \cdots \nearrow \infty.$$

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Our goal in this paper is to understand the interplay between the Steklov eigenvalues $\sigma_j(M)$ and large scale geometric features of the space M . Discretization methods which are classical in the context of the Laplace operator are developed in the context of manifolds with boundary. In order to single out large scale phenomena, we restrict our attention to a class of manifolds with bounded geometry. Throughout the paper, we therefore assume the existence of constants $\kappa \geq 0$ and $r_0 \in (0, 1)$ such that

- (H1) The boundary Σ admits a neighbourhood which is isometric to the cylinder $[0, 1] \times \Sigma$, with the boundary corresponding to $\{0\} \times \Sigma$;
- (H2) The Ricci curvature of M is bounded below by $-(n-1)\kappa$;
- (H3) The Ricci curvature of Σ is bounded below by $-(n-2)\kappa$;
- (H4) For each point $p \in M$ such that $d(p, \Sigma) > 1$, $\text{inj}_M(p) > r_0$;
- (H5) For each point $p \in \Sigma$, $\text{inj}_\Sigma(p) > r_0$.

The class of compact n -dimensional Riemannian manifolds with smooth boundary satisfying these hypotheses is denoted $\mathcal{M} = \mathcal{M}(\kappa, r_0, n)$.

Remark 1. The conditions defining the class \mathcal{M} are natural. Indeed we use methods from [3] on the boundary Σ of manifolds $M \in \mathcal{M}$ which require curvature and injectivity lower bounds. The product structure near the boundary is used to avoid situations where the Steklov eigenvalues become very small and could not be detected through a discretization. For instance, it is known that if two boundary components become close to each others, then each σ_j tends to zero. This could happen for a very short cylinder (see Lemma 6.1 of [6]) or for a surface which has a “thin passage” as described by Figure 3 of [11, Section 4]). It is also of interest to study the effect of drastic perturbations of the the Riemannian metric away from the boundary. This is the subject [7, 5]. The above conditions ensure that none of these situations will occur for manifolds in \mathcal{M} .

1.1. Discretization and spectral comparison

Our aim is to study spectral properties of manifolds in the class \mathcal{M} up to rough isometries.

Definition 2. A *rough isometry* between two metric spaces X and Y is a map $\Phi : X \rightarrow Y$ such that, there exist constants $a \geq 1, b \geq 0, \tau \geq 0$ satisfying

$$(1) \quad a^{-1}d(x_1, x_2) - b \leq d(\Phi(x_1), \Phi(x_2)) \leq a d(x_1, x_2) + b$$

for every $x_1, x_2 \in X$ and which satisfies

$$\bigcup_{x \in X} B(\Phi(x), \tau) = Y.$$

Given $\epsilon \in (0, r_0/4)$, we will define a discretization procedure

$$\mathcal{M}(\kappa, r_0, n) \longrightarrow \text{Graphs with boundary}$$

such that the ϵ -discretization Γ_M of a manifold M is roughly isometric to M with constants a, b, τ controlled in terms of the geometric constraints defining the class $\mathcal{M}(\kappa, r_0, n)$. A graph with boundary is simply a graph $\Gamma = (\mathbf{V}, \mathbf{E})$ with a distinguished set of vertices $\mathbf{B} \subset \mathbf{V}$ that is treated as a boundary. In Section 3 we will introduce a natural notion of Steklov spectrum on graphs with boundary:

$$0 = \sigma_1(\Gamma, \mathbf{B}) \leq \sigma_2(\Gamma, \mathbf{B}) \leq \dots \leq \sigma_k(\Gamma, \mathbf{B}),$$

where $k = |\mathbf{B}|$ is the number of vertices in the boundary.

Our main goal is to establish a relation between the Steklov eigenvalues of M and the Steklov eigenvalues of the discretization of M .

Theorem 3. *Given $\epsilon \in (0, r_0/4)$, there exist numbers $A, B > 0$ depending on κ, r_0, n and ϵ such that any ϵ -discretization $(\Gamma_M, \mathbf{V}_\Sigma)$ of a manifold $M \in \mathcal{M}(\kappa, r_0, n)$ satisfies*

$$(2) \quad A < \frac{\sigma_2(M)}{\sigma_2(\Gamma, \mathbf{V}_\Sigma)} < B.$$

Moreover, if $\sigma_k(M)$ is small enough, a similar result holds for for each $k \leq |\mathbf{V}_\Sigma|$: there exists a constant $C > 0$ (which depends at most on κ, ϵ, n) such that, if $\sigma_k(M) \leq C/k$, then

$$(3) \quad A < \frac{\sigma_k(M)}{\sigma_k(\Gamma, \mathbf{V}_\Sigma)} < B.$$

Without this last hypothesis, the following weaker estimate holds for each $k \leq |\mathbf{V}_\Sigma|$:

$$(4) \quad \frac{A}{k} < \frac{\sigma_k(M)}{\sigma_k(\Gamma, \mathbf{V}_\Sigma)} < B.$$

Comments and discussion Coarse discretizations have been used for a long time in spectral geometry of the Laplace operator on closed Riemannian manifolds. They were used by Buser [2] to construct compact hyperbolic surfaces with large area and uniformly positive λ_2 . They were also used by Brooks [1] to study the first non-zero eigenvalue of towers of covering. For the eigenvalues of the Laplace operator, a result similar to our Theorem 3 appeared in the work of Mantuano [16]. Our proof will be in the same spirit, but some serious technical difficulties occur because of the important role played by the boundary. As in [16], many of the tools that we use are from Chavel’s book [3], particularly from Section VI.5.

It is important not to confuse this type of discretization with those used in numerical analysis. Our goal is to discretize in a coarse sense, which is not sensitive to the local geometry. In particular, the interesting applications of our method are performed using a fixed value of the “mesh parameter” $\epsilon \in (0, r_0/4)$. It is worth observing that any compact manifold is roughly isometric to a point. The emphasis is on the control of the constants a, b, τ . In fact, if we let $\epsilon \rightarrow 0$, the control of the constants is lost: the manifold is not approximated in a better way for smaller values of ϵ . Note also that in the present context any ϵ -discretization is a finite graph.

1.2. Applications and examples

We will give three applications of our method to the construction of sequences of surfaces with large Steklov eigenvalues. Each of these are in the same spirit: a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ will describe a pattern to be used in the construction of a corresponding surface $\Omega_{\mathbf{G}}$. Roughly speaking, a finite set D_1, \dots, D_m of *fundamental pieces* is given. These are used to build a surface by associating a copy of one of the D_i ’s to each vertex, and the graph structure prescribes the pattern to follow for gluing the various fundamental pieces together. It is often natural to expect the geometric and spectral properties of the initial graph to be related to those of its induced surface. We will consider a sequence of graphs which displays some “spectral expansion” and show using our discretization results (Theorem 3 and Proposition 16) how to transplant these expansion properties to the induced surfaces. This method is classical. See [8, 9, 1]. Related methods were also introduced in [20].

Application 1. The following surprising fact follows from Theorem 1.3 of [6]: let $\Omega_l \subset \mathbb{R}^n$ be a sequence of domains with smooth boundary Σ_l , with $l \in \mathbb{N}$. If $n \geq 3$ and if the isoperimetric ratio

$$I(\Omega_l) := \frac{\text{Vol}_{n-1}(\Sigma_l)}{\text{Vol}_n(\Omega_l)^{\frac{n-1}{n}}}$$

tends to ∞ , then the normalized Steklov eigenvalues $\sigma_k(\Omega_l)\text{Vol}_{n-1}(\Sigma_l)^{1/(n-1)}$ tend to 0 as $l \nearrow \infty$. We will prove that the condition $n \geq 3$ is necessary.

Theorem 4. *There exists a sequence of planar domains $\Omega_l \subset \mathbb{R}^2$ with smooth boundary Σ_l , with $l \in \mathbb{N}$, such that*

1. *The isoperimetric ratio $I(\Omega_l) \rightarrow \infty$ as $l \rightarrow \infty$;*
2. *There exists a constant $C > 0$ (independent of l), such that for each l , $\sigma_2(\Omega_l)|\Sigma_l| \geq C$.*

Application 2. In Theorem 2 of [6], it was shown that if $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary Σ , then

$$(5) \quad \sigma_k(\Omega)\text{Vol}_{n-1}(\Sigma)^{1/(n-1)} \leq C(n)k^{2/n},$$

where $C(n)$ is a positive constant depending only on the dimension n . As the domains under consideration are Euclidean, it is a natural question to decide whether or not a similar estimate holds for more general flat Riemannian manifolds. The following Theorem shows that it is not the case.

Theorem 5. *There exists a sequence $\{\Omega_l\}_{l \in \mathbb{N}}$ of compact flat Riemannian surfaces with boundary Σ_l and a constant $C > 0$ (independent of l) such that for each $l \in \mathbb{N}$, $\text{genus}(\Omega_l) = 1 + l$, and*

$$\sigma_2(\Omega_l)L(\Sigma_l) \geq Cl.$$

Remark 6. One could use the present method to give an alternative proof of Theorem 1 from [8], or a version of Theorem 5 for surfaces of constant curvature -1 .

Application 3. In [8] two of the authors constructed surfaces modelled on regular graphs and developed an ad hoc spectral comparison inequality which allowed the construction of a sequence of surfaces Ω_l with boundary Σ_l such that

$$\lim_{l \rightarrow \infty} \sigma_2(\Omega_l)L(\Sigma_l) = +\infty.$$

This sequence also satisfies $\lim_{l \rightarrow \infty} \text{genus}(\Omega_l) = +\infty$, which is a necessary condition since it is known [15] that

$$\sigma_2(\Omega)L(\Sigma_l) \leq 8\pi(\text{genus}(\Omega) + 1).$$

In the construction proposed in [8], the number of boundary components of the surface Ω_l is also proportional to l . It is natural to ask if we can reduce

the number of boundary components. By studying this construction in the context of discretizations, we will prove that it is possible to choose each Ω_l to have exactly one boundary component.

Theorem 7. *There exist a sequence $\{\Omega_l\}_{l \in \mathbb{N}}$ of compact surfaces with connected boundary and a constant $C > 0$ such that for each $l \in \mathbb{N}$, $\text{genus}(\Omega_l) = 1 + l$, and*

$$\sigma_2(\Omega_l)L(\partial\Omega_l) \geq Cl.$$

Remark 8. The surfaces that we construct in the above three applications are not necessarily in the class \mathcal{M} , but they are uniformly quasi-isometric to such manifolds. For instance, if a sequence of manifolds M_l is replaced by manifolds X_l which are quasi-isometric to M_l with the same constants a, b for each $l \in \mathbb{N}$, then $\sigma_2(M_l)$ tends to 0 if and only if $\sigma_2(X_l)$ does. This will play a crucial role in Section 6.

1.3. Notations

Some of the constants appearing in various results will have to be reused later on. These are numbered successively as C_1, C_2, \dots . Each C_j is used precisely once in the paper. These constants can depend on the bounds κ, r_0 and on the dimension n and parameter $\epsilon > 0$. This dependence will not be stated explicitly each time. The symbol f is used for the averaging operator on its domain. Given a function $F \in C^\infty(M)$, we write

$$\|F\|_\Sigma := \|F|_\Sigma\|_{L^2(\Sigma)}$$

We will write ∇^Σ for the gradient operator on $C^\infty(\Sigma)$ and also

$$\|\nabla^\Sigma F\|_\Sigma := \|\nabla^\Sigma(F|_\Sigma)\|_{L^2(\Sigma)}.$$

When the volume form is clear from the context, it will be omitted. Given a graph $\mathbf{\Gamma} = (\mathbf{V}, \mathbf{E})$, the set of all real valued functions on the vertices V is written $\ell^2(V)$, understood with its natural ℓ^2 inner product.

1.4. Plan of the paper

In Section 2 we introduce a coarse discretization of manifolds in the class $\mathcal{M}(\kappa, r_0, n)$ and study its basic properties. This is followed in Section 3 by the introduction of Steklov eigenvalues for graphs with boundary, and a spectral comparison for roughly isometric graphs is proved in Proposition 16. In the

next section several tools are introduced: a comparison inequality for the Dirichlet energy of a function and its restriction to the boundary (Lemma 19), a local Poincaré inequality on cylinders (Lemma 21). This is followed by the introduction of discretization of smooth functions, and the smoothing of discrete functions. In Section 5 the proof of the main comparison inequality (Theorem 3) is presented. Finally, in the last section we present the three applications to surfaces with large Steklov eigenvalue σ_2 .

2. Discretization of compact manifolds with boundary

In this section, the geometric discretization of the manifold M is introduced. Because we are considering the Steklov problem, the boundary plays a crucial role. It is therefore natural that the discretization will lead to a *graph with boundary*.

Definition 9. A graph with boundary is a pair (Γ, \mathbf{B}) where $\Gamma = (\mathbf{V}, \mathbf{E})$ is a graph and $\mathbf{B} \subset \mathbf{V}$ is a distinguished set of vertices. The path-distance on \mathbf{V} is defined as follows: given $x, y \in \mathbf{V}$, the distance $d_\Gamma(x, y)$ is the length of the shortest path between x and y , where two adjacent vertices are at distance 1. There is a natural graph structure on \mathbf{B} defined by $\mathbf{E}_\mathbf{B} \subset \mathbf{E}$, where $e \in \mathbf{E}_\mathbf{B}$ iff the edge e joins two vertices of \mathbf{B} . However, on the graph $(\mathbf{B}, \mathbf{E}_\mathbf{B})$, we consider the extrinsic distance $d_\Gamma(x, y)$.

Given $0 < \epsilon < r_0/4$, let \mathbf{V}_Σ be a maximal ϵ -separated set in Σ . This means that for $p, q \in \mathbf{V}_\Sigma$ distinct, the balls $B(p, \epsilon)$ and $B(q, \epsilon)$ are disjoint, and for each point $p \in \Sigma$ there exists $q \in \mathbf{V}_\Sigma$ such that $d(p, q) < \epsilon$.

Let \mathbf{V}'_Σ be a copy of \mathbf{V}_Σ located 4ϵ away from the boundary:

$$\mathbf{V}'_\Sigma := \{4\epsilon\} \times \mathbf{V}_\Sigma \subset M.$$

Let \mathbf{V}_I be a maximal ϵ -separated set in $M \setminus [0, 4\epsilon) \times \Sigma$ such that $\mathbf{V}'_\Sigma \subset \mathbf{V}_I$. (See Figure 1.) The set $\mathbf{V} = \mathbf{V}_\Sigma \cup \mathbf{V}_I$ is given the structure of a graph Γ by declaring

- any two $v, w \in \mathbf{V}$ adjacent whenever $d_M(v, w) < 3\epsilon$ and $v \neq w$;
- any $v \in \mathbf{V}_\Sigma$ adjacent to $v' = (4\epsilon, v) \in \mathbf{V}'_\Sigma \subset \mathbf{V}_I$.

The graph $\Gamma = (\mathbf{V}, \mathbf{E})$ together with boundary $\mathbf{B} = \mathbf{V}_\Sigma$ is called an ϵ -*discretization of M* . The path-distance on Γ is d_Γ .

Remark 10. The key features of this discretization are:

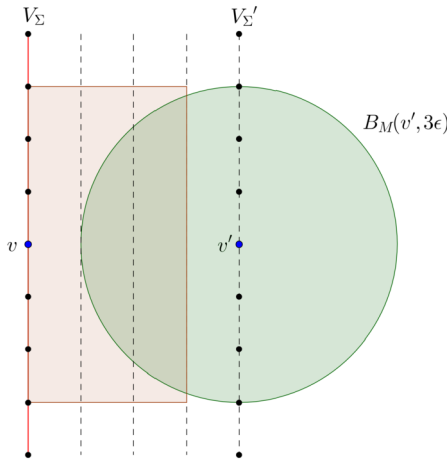


Figure 1: The discretization close to the boundary.

- The regions $C_v := [0, 3\epsilon) \times B_{\Sigma}(v, 3\epsilon)$ and $B_M(v', 3\epsilon)$ (which are shaded in Figure 1) have a large enough intersection. This will be important in the proof of Lemma 23, which controls the energy of the discretization of a function.
- Because the interior vertices are separated by a strip of width 4ϵ from the boundary, $B_M(v, 4\epsilon) \cap \Sigma = \emptyset$ for each $v \in \mathbf{V}_I$. This will allow the use of Kanai’s inequality for functions $f \in C^\infty(B_M(v, 4\epsilon))$ (See Lemma 21).
- The interaction between the boundary of the graph and its interior is simple, and it reflects the product structure of the manifold M . This will lead to the existence of a suitable partition of unity, in Section 4.4.
- The results are not sensitive to the specific definition of the discretization that we use. For instance different constants could be used, but the proofs would need to be adjusted accordingly.

Remark 11. It follows from the Bishop–Gromov theorem that the degree of each vertex $v \in \mathbf{V}$ is bounded above in terms of κ and ϵ . The total number of vertices can also be controlled. See [3, p. 147].

Lemma 12. *For any $0 < \epsilon < r_0/4$, and any ϵ -discretization $(\Gamma, \mathbf{V}_{\Sigma})$ of M , the natural inclusion $\mathbf{V} \subset M$ is a rough isometry. Indeed, the following stronger estimate holds:*

$$\frac{\epsilon}{4}d_{\Gamma}(x, y) - 10 \leq d_M(x, y) \leq 4\epsilon d_{\Gamma}(x, y).$$

Proof. Let $x, y \in V$ with $d_{\Gamma}(x, y) = k$. This means that there exists a sequence of vertices

$$x = x_0, x_1, \dots, x_k = y \in \mathbf{V}$$

with $d_{\Gamma}(x_i, x_{i+1}) = 1$ which represents a shortest path between x and y . By construction of the discretization, we have $d_M(x_i, x_{i+1}) \leq 4\epsilon$, and by compactness, there exists a path on M between x_i and x_{i+1} of length $\leq 4\epsilon$. By concatenation of these paths, we find a path joining x and y on M whose length $\leq 4\epsilon k$. This implies

$$d_M(x, y) \leq 4\epsilon k = 4\epsilon d_{\Gamma}(x, y),$$

which completes the proof of the second inequality.

To prove the left-hand-side inequality, consider $x, y \in \mathbf{V}$ with $d_M(x, y) = \alpha$. There are three cases to consider.

Case 1: $x, y \in \mathbf{V}_I$, that is x and y are not points of the boundary $\mathbf{B} = \mathbf{V}_{\Sigma}$ of the graph Γ . Because of the product structure near the boundary, there exists a geodesic parametrised by arc length

$$\gamma : [0, \alpha] \rightarrow M$$

between x and y on M whose length is α . It follows from convexity that

$$Im(\gamma) \subset M \setminus [0, 4\epsilon) \times \Sigma :$$

a geodesic entering $[0, 4\epsilon) \times \Sigma$ would be trapped.

Let $k \in \mathbb{N}$ be such that $k - 1 < \frac{\alpha}{\epsilon} \leq k$ and define $t_i = \frac{i\alpha}{k}$, $i = 0, \dots, k$. and $x_i = \gamma(t_i) \in M$. Note that $x_0, x_k \in \mathbf{V}$, but that this is in general not the case for the other points x_i . However, by construction, $\bigcup_{x \in \mathbf{V}_I} B(x, \epsilon) = M \setminus [0, 4\epsilon) \times \Sigma$ so that there exist $y_0 = x_0, y_1, \dots, y_{k-1}, y_k = y \in \mathbf{V}$ with $d_M(y_i, x_i) \leq \epsilon$ and

$$d_M(y_{i+1}, y_i) \leq d_M(y_{i+1}, x_{i+1}) + d_M(x_{i+1}, x_i) + d_M(y_i, x_i) \leq 3\epsilon$$

so that y_i and y_{i+1} are connected in Γ and $d_{\Gamma}(y_i, y_{i+1}) \leq 1$. It follows that

$$d_{\Gamma}(x, y) \leq k \leq \left(\frac{\alpha}{\epsilon} + 1\right) = \frac{1}{\epsilon} d_M(x, y) + 1.$$

Case 2: $x \in \mathbf{B} = \mathbf{V}_{\Sigma}$ and $y \notin B$: this means that there is exactly one point $x_1 \in \mathbf{V}_{\Sigma}^*$ which is connected to $x_0 = x$ and $d_{\Gamma}(x_0, x_1) = 1$. Now, x_1 and y are

as in the first step, so that we have

$$\begin{aligned} d_{\Gamma}(x, y) &\leq d_{\Gamma}(x, x_1) + d_{\Gamma}(x_1, y) \\ &\leq 1 + \frac{1}{\epsilon}d_M(x_1, y) + 1 \\ &\leq 2 + \frac{1}{\epsilon}(d_M(x_1, x) + d_M(x, y)) = 2 + \frac{1}{\epsilon}(4\epsilon + d_M(x, y)) = 6 + \frac{1}{\epsilon}d_M(x, y). \end{aligned}$$

Case 3: $x, y \in \mathbf{B} = \mathbf{V}_{\Sigma}$: we do as in the second step and get

$$d_{\Gamma}(x, y) \leq 10 + \frac{1}{\epsilon}d_M(x, y).$$

End of the proof: we have to show that $\bigcup_{x \in V} B(x, 4\epsilon) = M$. We already know that this is true for points in $M \setminus [0, 4\epsilon) \times \Sigma$ because \mathbf{V}_I is a maximal separated set in this space.

If $x \in [0, 4\epsilon) \times \Sigma$, there is $y \in \Sigma$ or $y \in \{4\epsilon\} \times \Sigma$ with $d_M(x, y) \leq 2\epsilon$. By construction, there is a point $z \in \mathbf{V}_{\Sigma}$ in the first case, or a point $z \in \mathbf{V}'_{\Sigma}$ in the second case such that $d_M(y, z) \leq \epsilon$, and we deduce that $d_M(z, x) \leq 4\epsilon$. \square

3. Spectrum of roughly isometric graphs

Let (Γ, \mathbf{B}) be a graph with boundary. The *Dirichlet energy* of a function $f : \mathbf{V} \rightarrow \mathbb{R}$ is

$$q(f) := \sum_{v \sim w} (f(v) - f(w))^2.$$

To simplify notations, we will write $\|f\|_{\mathbf{B}}$ for $\|f|_{\mathbf{B}}\|_{\ell^2(\mathbf{B})}$. Similarly,

$$q_{\mathbf{B}}(f) := \sum_{v \sim w, v, w \in \mathbf{B}} (f(v) - f(w))^2.$$

Definition 13. For each $j = 1, \dots, |\mathbf{B}|$, the j -th *Steklov eigenvalue* of (Γ, \mathbf{B}) is defined by

$$(6) \quad \sigma_j(\Gamma, \mathbf{B}) = \min_E \max_{f \in E} \frac{q(f)}{\|f\|_{\mathbf{B}}^2}.$$

where the minimum is over all j -dimensional linear subspaces E of $\ell^2(\mathbf{V})$.

In particular, $\sigma_1(\Gamma, \mathbf{B}) = 0$ is realized by locally constant functions on \mathbf{V} .

Remark 14. Note that for $\mathbf{B} = \mathbf{V}$, $\sigma_j = \lambda_j(\mathcal{L})$ is the j -th eigenvalue of the graph Laplacian \mathcal{L} . See [16]. Steklov eigenvalues on graphs have also been studied in [18, 12].

The notion of rough isometry between metric spaces was presented in Definition 2. A specialized version will be useful.

Definition 15. A rough isometry Φ between two graphs with boundary (Γ_1, \mathbf{B}_1) and (Γ_2, \mathbf{B}_2) is a rough isometry of the underlying graphs which sends \mathbf{B}_1 to \mathbf{B}_2 . In other words, the restriction of Φ to \mathbf{B}_1 is a rough isometry $\mathbf{B}_1 \rightarrow \mathbf{B}_2$ when considering extrinsic distances on \mathbf{B}_1 and \mathbf{B}_2 .

The easiest situation in which spectral comparison occur is for rough isometries between graphs. This will be useful for applications in Section 6.

Proposition 16. *Given $a \geq 1$ and $b, \tau \geq 0$, there exist constants A, B depending only on a, b, τ and on the maximal degree of vertices, such that any two graphs with boundary (Γ_1, \mathbf{B}_1) and (Γ_2, \mathbf{B}_2) which are roughly isometric (through Φ) with constants a, b, τ satisfies*

$$A \leq \frac{\sigma_k(\Gamma_1, \mathbf{B}_1)}{\sigma_k(\Gamma_2, \mathbf{B}_2)} \leq B$$

for each $k \leq \min\{|\mathbf{B}_1|, |\mathbf{B}_2|\}$.

This proposition is in the same spirit as Theorem 2.1 of [16]. Nevertheless, the presence of a boundary brings many new difficulties. For instance, the following inequality for $f : \mathbf{V}(\Gamma_2) \rightarrow \mathbb{R}$ was proved in [3, Lemma VI.5.4]

$$\|f\|_{\mathbf{B}_2}^2 \leq \|f\|_{\Gamma_2}^2 \leq Cq(f) + C'\|\Phi^*f\|_{\Gamma_1}^2,$$

for constants C and C' which only depend on a, b, τ and the maximal degree of both graphs. Here and elsewhere in the paper, $\Phi^*f = f \circ \Phi$ is the pullback of f by Φ . It is used in [16] to obtain a lower bound on $\|\Phi^*f\|_{\Gamma_1}^2$. Because we consider Steklov eigenvalues, a lower bound on $\|\Phi^*f\|_{\mathbf{B}_1}^2$ is needed.

Lemma 17. *Given a rough isometry $\Phi : (\Gamma_1, \mathbf{B}_1) \rightarrow (\Gamma_2, \mathbf{B}_2)$ between two graphs with boundary, there exist constants C_1, C_2 which depend only on the constants a, b, τ of the quasi-isometry and on the maximal degree of the graphs, such that any function $f : \mathbf{V}(\Gamma_2) \rightarrow \mathbb{R}$ satisfies*

$$(7) \quad \|f\|_{\mathbf{B}_2}^2 \leq C_1q(f) + C_2\|\Phi^*f\|_{\mathbf{B}_1}^2.$$

Proof of Lemma 17. We have to adapt Lemma VI.5.4 of [3] to graphs with boundary. The rough isometry Φ has a rough inverse (See the introduction of [13] for a discussion.) That is a rough isometry Ψ from (Γ_2, \mathbf{B}_2) to (Γ_1, \mathbf{B}_1) , with $\Psi(\mathbf{B}_2) \subset \mathbf{B}_1$ such that any $x \in \mathbf{V}_1$ and $y \in \mathbf{V}_2$ satisfy

$$d(x, \Psi \circ \Phi(x)) \leq K; \quad d(y, \Phi \circ \Psi(y)) \leq K$$

where K is a constant depending on the constants a, b, τ of the rough isometry Φ . For $y \in \mathbf{B}_2$,

$$(8) \quad f^2(y) \leq 2(f(y) - (\Phi \circ \Psi)^* f(y))^2 + 2(\Phi \circ \Psi)^* f(y)^2.$$

In order to bound $\sum_{y \in \mathbf{B}_2} f(y)^2$, the two terms on the right-hand side of this inequality will be estimated. As $d(y, \Phi \circ \Psi(y)) \leq K$, there is a path $y_1 = y, y_2, \dots, y_n = \Phi \circ \Psi(y)$ of length at most K , and it follows from triangle and Cauchy-Schwartz Inequality that

$$((f(y) - (\Phi \circ \Psi)^* f(y))^2 \leq K^2 \sum_{i=1}^{n-1} (f(y_{i+1}) - f(y_i))^2.$$

This sum only involves points that are at a distance at most K from $y \in \mathbf{B}_2$. As the number of points in a ball of radius K is bounded above in terms of the maximal degree, there exists a constant C_1 (depending only on K and on the maximal degree) such that

$$2 \sum_{y \in \mathbf{B}_2} (f(y) - (\Phi \circ \Psi)^* f(y))^2 \leq C(K)q(f).$$

We also have to estimate $\sum_{y \in \mathbf{B}_2} ((\Phi \circ \Psi)^* f)^2(y)$. Observe that

$$\sum_{y \in \mathbf{B}_2} ((\Phi \circ \Psi)^* f)^2(y) = \sum_{y \in \mathbf{B}_2} (\Phi^* f)^2(\Psi(y)).$$

But $\Psi(y) \in \mathbf{B}_1$, which may be the image of at most a finite controlled number of vertices $y \in \mathbf{B}_2$. Therefore there exist another constant C_2 such that

$$2 \sum_{y \in \mathbf{B}_2} ((\Phi \circ \Psi)^* f)^2(y) \leq C_2 \sum_{z \in \mathbf{B}_1} (\Phi^* f)^2(z).$$

The results follows by substitution of the previous two inequalities in (8). \square

The previous Lemma will be the main tool used in the following proof.

Proof of Proposition 16. Let Φ be a rough isometry between (Γ_1, \mathbf{B}_1) and (Γ_2, \mathbf{B}_2) , with constants a, b, τ . Given a function $f : \mathbf{V}_2 \rightarrow \mathbb{R}$, define $\Phi^* f : \mathbf{V}_1 \rightarrow \mathbb{R}$ by $\Phi^* f(x) = f(\Phi(x))$. The following inequality follow from Lemma VI.5.2 of [3]:

$$(9) \quad q(\Phi^* f) \leq C_3 q(f),$$

where C_3 depend on the maximal degree and on a, b, τ .

We will treat separately the situations where $\sigma_k(\Gamma_2)$ is smaller or larger than $\frac{1}{2C_1}$.

Situation 1. Suppose that $\sigma_k(\Gamma_2) \leq \frac{1}{2C_1}$.

Let f_1, \dots, f_k be eigenfunctions corresponding to $\sigma_1(\Gamma_2, \mathbf{B}_2), \dots, \sigma_k(\Gamma_2, \mathbf{B}_2)$ respectively. The space $E_k := \text{span}(\Phi^* f_1, \dots, \Phi^* f_k)$ will be used in the variational characterization of $\sigma_k(\Gamma_1, \mathbf{B}_1)$. So, if $f \in E_k$ and $g = \Phi^* f$, we have in particular to show that the restriction g to the boundary \mathbf{B}_1 of Γ_1 is large enough. More precisely, we have by (7),

$$\begin{aligned} \|g\|_{\mathbf{B}_1}^2 &= \|\Phi^* f\|_{\mathbf{B}_1}^2 \geq C_2^{-1} \left(\|f\|_{\mathbf{B}_2}^2 - C_1 q(f) \right) \\ &\geq C_2^{-1} \|f\|_{\mathbf{B}_2}^2 (1 - C_1 \sigma_k(\Gamma_2)) \geq \frac{1}{2C_2} \|f\|_{\mathbf{B}_2}^2 > 0. \end{aligned}$$

Using (9) and the above inequality leads to $\frac{\sigma_k(\Gamma_1)}{\sigma_k(\Gamma_2)} \leq 2C_3 C_2$.

Situation 2. Suppose that $\sigma_k(\Gamma_2) \geq \frac{1}{2C_1}$.

In this case, $\sigma_k(\Gamma_1) \leq 2C_1 \sigma_k(\Gamma_1) \sigma_k(\Gamma_2)$ and

$$\frac{\sigma_k(\Gamma_1)}{\sigma_k(\Gamma_2)} \leq 2C_1 \sigma_k(\Gamma_1).$$

The conclusion now follows from the fact that $\sigma_k(\Gamma_1)$ is bounded above uniformly in terms of the maximal degree of the graph. \square

Remark 18. This proof is typical. In fact, the proof of Theorem 3 has a similar structure to that of Proposition 16. Nevertheless, the techniques are much more involved in this latter case.

4. Preliminary results

In this section some results which will be used in the proof of Theorem 3 are presented.

4.1. The Dirichlet energy on a manifold and on its boundary

On a graph with boundary (Γ, \mathbf{B}) one immediately sees that

$$q_{\mathbf{B}}(f) := \sum_{\substack{v \sim w \\ v, w \in \mathbf{B}}} (f(v) - f(w))^2 \leq q(f).$$

That is, restricting a function to the boundary reduces its energy. There is no such simple formula for compact manifolds with boundary. Nevertheless, under more restrictive hypothesis, some control can still be granted.

Lemma 19. *For each Steklov eigenfunction $F \in C^\infty(M)$ corresponding to $\sigma < 1/4$, the following holds:*

$$(10) \quad \|\nabla^\Sigma F\|_\Sigma^2 \leq \frac{1}{2} \|\nabla F\|_M^2.$$

Proof. Let $(f_k) \subset L^2(\Sigma)$ be an orthonormal basis corresponding to the eigenvalues λ_k of the Laplacian on Σ . Let $F \in C^\infty(M)$ be a Steklov eigenfunction corresponding to σ . On the cylindrical neighborhood $\Sigma \times [0, 1]$ the Fourier decomposition of F is

$$F = \sum_{k=0}^{\infty} a_k(r) f_k(\theta) \quad r \in [0, 1], \quad \theta \in \Sigma.$$

As the function F is harmonic on M , and hence on the cylinder, the following holds

$$\Delta F(r, \theta) = \sum_{k=0}^{\infty} [-a_k''(r) f_k(\theta) + a_k(r) \lambda_k f_k(\theta)] = 0$$

which implies that $a_k''(r) = \lambda_k a_k(r)$ and hence

$$a_k(r) = a_k(0) \cosh(\sqrt{\lambda_k} r) + \frac{1}{\sqrt{\lambda_k}} a_k'(0) \sinh(\sqrt{\lambda_k} r).$$

Moreover, on the boundary Σ ,

$$0 = \sigma F(\theta) - \frac{\partial F}{\partial n}(\theta) = \sum_k (\sigma a_k(0) + a_k'(0)) f_k(\theta),$$

which implies $\sigma a_k(0) + a_k'(0) = 0$ whence by substitution leads to

$$a_k(r) = a_k(0) \left[\cosh(\sqrt{\lambda_k} r) - \frac{\sigma}{\sqrt{\lambda_k}} \sinh(\sqrt{\lambda_k} r) \right]$$

and

$$a_k'(r) = a_k(0) \sqrt{\lambda_k} \left[\sinh(\sqrt{\lambda_k} r) - \frac{\sigma}{\sqrt{\lambda_k}} \cosh(\sqrt{\lambda_k} r) \right]$$

The Dirichlet energy on the boundary and on the cylinder are expressed by

$$\|\nabla^\Sigma F\|_\Sigma^2 = \sum_k a_k^2(0) \lambda_k$$

and

$$(11) \quad \|\nabla F\|_{\Sigma \times (0,1)}^2 = \sum_k \int_0^1 [(a'_k)^2 + a_k^2 \lambda_k] dr.$$

Now using $x = \sqrt{\lambda_k}$ leads to

$$a_k^2(r) = a_k^2(0) \left[\cosh^2(xr) + \frac{\sigma^2}{x^2} \sinh^2(xr) - \frac{\sigma}{x} \sinh(2xr) \right]$$

and

$$(a'_k)^2(r) = a_k^2(0)x^2 \left[\sinh^2(xr) + \frac{\sigma^2}{x^2} \cosh^2(xr) - \frac{\sigma}{x} \sinh(2xr) \right]$$

Substitution in equation (11) and evaluation of the integrals give

$$\begin{aligned} \|\nabla F\|_{\Sigma \times (0,1)}^2 &= \sum_k a_k^2(0)x^2 \int_0^1 \left[\left(1 + \frac{\sigma^2}{x^2}\right) \cosh(2xr) - \frac{2\sigma}{x} \sinh(2xr) \right] dr \\ &= \sum_k a_k^2(0)x^2 \left[\left(1 + \frac{\sigma^2}{x^2}\right) \frac{\sinh(2x)}{2x} - \frac{\sigma}{x^2} (\cosh(2x) - 1) \right]. \end{aligned}$$

Moreover, for $\sigma < 1/4$, it follows from $\tanh(x) \leq x \leq x/4\sigma$ that $\cosh(x) - \frac{2\sigma}{x} \sinh(x) \geq \frac{1}{2} \cosh(x)$, whence

$$\begin{aligned} &\left(1 + \frac{\sigma^2}{x^2}\right) \frac{\sinh(2x)}{2x} - \frac{\sigma}{x^2} (\cosh(2x) - 1) \\ &= \frac{\sinh(x)}{x} \left(\cosh(x) - 2\frac{\sigma}{x} \sinh(x) + \frac{\sigma^2}{x^2} \cosh(x) \right) \\ &\geq \frac{\sinh(x)}{x} \left(\frac{1}{2} \cosh(x) + \frac{\sigma^2}{x^2} \right) \\ &= \left(\frac{\sinh(x) \cosh(x)}{2x} + \frac{\sigma^2 \sinh(x)}{x^2 x} \right) \geq \frac{1}{2}. \quad \square \end{aligned}$$

In order to use the above Lemma in estimations of higher eigenvalues, one needs to consider linear combinations of eigenfunctions.

Corollary 20. *Let $F \in C^\infty(M)$ be a linear combination of the first k Steklov eigenfunctions F_1, \dots, F_k . If $\sigma_k \leq \frac{1}{4}$ then $F = a_1 F_1 + \dots + a_k F_k$ satisfies the*

following:

$$\|\nabla^\Sigma F\|_\Sigma^2 \leq \frac{k}{8} \|\nabla F\|_M^2.$$

Proof. It follows from Green's formula that

$$\int_M \langle \nabla F_i, \nabla F_j \rangle = \int_M \Delta F_i F_j + \int_\Sigma \partial_\nu F_i F_j = \int_\Sigma \sigma_i F_i F_j, = \sigma_i \delta_{ij}.$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \|\nabla^\Sigma F\|_\Sigma^2 &= \left| \int_\Sigma \langle \nabla^\Sigma F, \nabla^\Sigma F \rangle \right| \\ &= \left| \sum_{i,j=1}^k a_i a_j \int_\Sigma \langle \nabla^\Sigma F_i, \nabla^\Sigma F_j \rangle \right| \\ &\leq \left| \sum_{i,j=1}^k a_i a_j \left(\int_\Sigma \langle \nabla^\Sigma F_i, \nabla^\Sigma F_i \rangle \right)^{1/2} \left(\int_\Sigma \langle \nabla^\Sigma F_j, \nabla^\Sigma F_j \rangle \right)^{1/2} \right| \\ &\leq \sum_{i,j=1}^k |a_i| |a_j| \|\nabla^\Sigma F_i\|_\Sigma \|\nabla^\Sigma F_j\|_\Sigma. \end{aligned}$$

It follows from Lemma 19 that

$$\|\nabla^\Sigma F\|_\Sigma^2 \leq \frac{1}{2} \sum_{i,j=1}^k |a_i| |a_j| \|\nabla F_i\|_M \|\nabla F_j\|_M = \frac{1}{2} \sum_{i,j=1}^k |a_i| |a_j| \sqrt{\sigma_i} \sqrt{\sigma_j}.$$

We also have

$$\|\nabla F\|_M^2 = \sum_{i,j=1}^k a_i a_j \langle \nabla F_i, \nabla F_j \rangle = \sum_{i=1}^k a_i^2 \sigma_i.$$

Now, in general, for $\alpha_i, \alpha_j \geq 0$, it follows from the Cauchy-Schwarz inequality that

$$\sum_{i,j=1}^k \alpha_i \alpha_j = \left(\sum_i \alpha_i \right)^2 \leq k \sum_i \alpha_i^2.$$

Setting $\alpha_i = |a_i| \sqrt{\sigma_i}$, and using $\sigma_i \leq \frac{1}{4}$ it follows that

$$\|\nabla^\Sigma F\|_\Sigma^2 \leq \frac{k}{2} \sum_{i=1}^k a_i^2 \sigma_i \leq \frac{k}{8} \sum_{i=1}^k a_i^2 \sigma_i = \frac{k}{8} \|\nabla F\|_M^2. \quad \square$$

4.2. Local Poincaré-type inequality on products

The following Lemma is similar to Lemma 8 of [14]. See also Lemma Vi.5.5 in [3, p. 177].

Lemma 21. *For each $\delta > 0$, there exists a constant $C_4 = C_4(n, \kappa, \delta)$ with the following properties: Given $p \in \Sigma$, let $C = B_\Sigma(p, \delta) \times [0, \delta]$. Then any smooth function $F \in C^\infty(\bar{C})$ satisfies*

$$\int_C |F - F_C| \leq C_4 \int_C |\nabla F|,$$

where $F_C = \mathop{\frown}\limits_C F$ is the average of F on C .

Proof of Lemma 21. Using $(x, t) \in C$ as coordinates, the integral is split

$$\begin{aligned} (12) \quad \int_C |F - F_C| &\leq \int_C \left| F(x, t) - \mathop{\frown}\limits_0^\delta F(x, s) ds \right| + \int_C \left| \mathop{\frown}\limits_0^\delta F(x, s) ds - F_C \right| \\ &= \int_{B_\Sigma(p, \delta)} \int_0^\delta \left| F(x, t) - \mathop{\frown}\limits_0^\delta F(x, s) ds \right| \\ &\quad + \int_0^\delta \int_{B_\Sigma(p, \delta)} \left| \mathop{\frown}\limits_0^\delta F(x, s) ds - F_C \right|. \end{aligned}$$

We will estimate the two terms in the right-hand side of this inequality separately. It follows from Kanai’s inequality (Lemma 8 in [14]) that

$$\int_0^\delta \left| F(x, t) - \mathop{\frown}\limits_0^\delta F(x, s) ds \right| \leq C_4 \int_0^\delta |\partial_t F|.$$

Moreover, the average of the function $G : B \rightarrow \mathbb{R}$ defined by

$$G(x) = \mathop{\frown}\limits_0^\delta F(x, s) ds$$

is F_C . Therefore, it follows (again from Kanai’s inequality) that

$$\begin{aligned} \int_{B_\Sigma(p, \delta)} \left| \mathop{\frown}\limits_0^\delta F(x, s) ds - F_C \right| &= \int_{B_\Sigma(p, \delta)} |G - F_C| \leq C_4 \int_{B_\Sigma(p, \delta)} |\nabla^\Sigma G| \\ &= C_4 \int_{B_\Sigma(p, \delta)} \left| \mathop{\frown}\limits_0^\delta \nabla^\Sigma F \right|. \end{aligned}$$

Substitution in (12) now leads to

$$\int_C |F - F_C| \leq C_4 \int_C |\partial_t F| + |\nabla^\Sigma F|. \quad \square$$

4.3. Discretization of smooth functions

Let $M \in \mathcal{M}(\kappa, r_0, n)$. Given $\epsilon \in (0, r_0/2)$ let $\mathbf{\Gamma}$ be an ϵ -discretization of M with boundary \mathbf{V}_Σ . The *discretization* $f = DF : \mathbf{V} \rightarrow \mathbb{R}$ of a smooth function $F \in C^\infty(M)$ is defined as

$$(13) \quad f(v) = \begin{cases} \mathring{f}_{B_\Sigma(v, 3\epsilon)} F & \text{if } v \in \mathbf{V}_\Sigma, \\ \mathring{f}_{B_M(v, 3\epsilon)} F & \text{if } v \in \mathbf{V}_I. \end{cases}$$

The symbol \mathring{f} is used for the averaging operator on its domain.

Remark 22. Throughout, we follow Chavel’s convention from [3] that functions on M are denoted with upper case F , while functions on (vertices of) the graph $\mathbf{\Gamma}$ are denoted with lower case f .

Lemma 23. *There exists a constant C_5 which only depends on κ, r_0, n with the following property. Let $F \in C^\infty(M)$ be a linear combination of the first k Steklov eigenfunctions F_1, \dots, F_k . If $\sigma_k(M) < 1/4$, then the discretization $f = DF$ satisfy*

$$q(f) \leq C_5 k \|\nabla F\|_M^2.$$

Proof. Given a boundary vertex $v \in \mathbf{V}_\Sigma$, write $v' = (4\epsilon, v) \in \mathbf{V}_I$ for the corresponding interior vertex. The energy $q(f)$ is the sum of the following three quantities

$$\begin{aligned} E_1 &= \sum_{\substack{v \sim w \\ v, w \in \mathbf{V}_I}} |f(v) - f(w)|^2, \\ E_2 &= \sum_{\substack{v \sim w \\ v, w \in \mathbf{V}_\Sigma}} |f(v) - f(w)|^2, \\ E_3 &= \sum_{v \in \mathbf{V}_\Sigma} |f(v) - f(v')|^2. \end{aligned}$$

The first term E_1 is bounded using the same argument as in [3, D:iii,p. 178]: there exists a constant A such that

$$E_1 \leq A \|\nabla F\|_M^2.$$

To bound E_2 one also uses [3, \mathcal{D} :iii,p. 178] to obtain a bound in terms of $\|\nabla^\Sigma F\|$ and then in terms of the interior Dirichlet energy using Corollary 20: there exists a constant B such that

$$E_2 \leq Bk\|\nabla F\|_M^2.$$

Let us bound E_3 . Let $\delta = 3\epsilon$. Given $v \in \mathbf{V}_\Sigma$, define $\hat{f}(v)$ to be the average of F on the cylinder $C_v := [0, \delta) \times B_\Sigma(v, \delta) \subset M$:

$$\hat{f}(v) := \int_{C_v} F.$$

It follows that

$$(f(v) - f(v'))^2 \leq 2 \left((f(v) - \hat{f}(v))^2 + (f(v') - \hat{f}(v))^2 \right).$$

The fundamental theorem of calculus leads to

$$\begin{aligned} |f(v) - \hat{f}(v)| &= \left| \int_{C_v} [F(x, 0) - F(x, t)] dA(x) dt \right| \\ &\leq \int_{C_v} \left(\int_0^\delta |\partial_r F(x, r)| dr \right) dA(x) dt \\ &= \int_{B_\Sigma(v, \delta)} \int_0^\delta |\partial_r F(x, r)| dr dA(x) = \frac{1}{|B(p, \delta)|} \int_{C_v} |\partial_r F|. \end{aligned}$$

The argument used to bound $|f(v') - \hat{f}(v)|$ is similar to that of \mathcal{D} :i, [3, p. 178]. Let $\beta = |B_M(v', 3\epsilon) \cap C_v|$ and observe that

$$\begin{aligned} |f(v') - \hat{f}(v)| &= \int_{B_M(v', 3\epsilon) \cap C_v} |f(v') - \hat{f}(v)| \\ &\leq \int_{B_\Sigma(w, 3\epsilon) \cap C_v} |F - f(v')| + |F - \hat{f}(v)| \\ &\leq \frac{1}{\beta} \left(\int_{C_v} |F - \hat{f}(v)| + \int_{B_\Sigma(w, 2\epsilon)} |F - f(w)| \right). \end{aligned}$$

These two terms are bounded using Lemma 21 and Kanai's inequality ([3, p.177]), so that

$$|f(w) - \hat{f}(v)| < \frac{1}{\beta} (C_4 + C_6) \int_C |\nabla F|.$$

The crucial point is that β is bounded below in terms of the geometry. Indeed, let $p = (\{2\epsilon\}, v)$. The ball $B_M(p, \epsilon) \subset B_M(v', 3\epsilon) \cap C_v$ and it follows from Croke's inequality [Croke1980] (See also Proposition V.2.3, [3, 136]) that

$$\beta \geq |B_M(p, \epsilon)| \geq C' \epsilon^n,$$

where C' is a constant which depends only on the dimension n . It follows that

$$E_3 \leq C \|\nabla F\|_M^2.$$

The bounds on E_1, E_2 and E_3 lead to

$$q(f) \leq (A + Bk + C) \|\nabla F\|_M^2 \leq (A + B + C)k \|\nabla F\|_M^2.$$

The proof is complete, with $C_5 = A + B + C$. □

4.4. Smoothing of discrete functions

The balls $B(v, 3\epsilon)$ for $v \in \mathbf{V}$ form an open cover of the manifold M . Indeed, it follows from the fact that \mathbf{V}_I is a maximal ϵ -separated set in $M \setminus [0, 4\epsilon) \times \Sigma$ that

$$M \setminus [0, 4\epsilon) \times \Sigma \subset \bigcup_{x \in \mathbf{V}_I} B(x, 3\epsilon).$$

Now, if $x \in [0, 4\epsilon) \times \Sigma$, there is $y \in \Sigma$ or $y \in \{4\epsilon\} \times \Sigma$ with $d(x, y) \leq 2\epsilon$. But, by construction, there is a point $z \in \mathbf{V}_\Sigma$ in the first case, or a point $z \in \mathbf{V}'_\Sigma$ in the second case such that $d(y, z) \leq \epsilon$, and we deduce that $d(z, x) \leq 3\epsilon$.

The existence of a partition of unity with controlled energy is a standard tool in geometric analysis. Nevertheless, we could not locate a construction completely adapted to our present context. For the sake of completeness, we therefore proved the following result.

Lemma 24. *There exists a smooth partition of unity $\{\phi_v\}_{v \in \mathbf{V}} \subset C^\infty(M)$ subordinate to the cover $B(v, 4\epsilon)$ which satisfy the pointwise bound*

$$|\nabla \phi_v| \leq A/\epsilon$$

where $A = A(\kappa_M)$ is a constant which depends only on the lower bounds on Ricci curvature.

Proof. We will construct a partition of unity $\{\phi_v\}$ subordinated to the open cover $\{B(v, 4\epsilon)\}_{v \in \mathbf{V}}$. Choose $4\epsilon < r_0 < \text{inj}(M)$. Note that with this choice, the covering is uniformly locally finite. Indeed, it follows from the theorem

of Bishop-Gromov that a point is contained in a finite number of balls of the covering, and because $\epsilon < 1$, this number is bounded above by a constant $A_1(\kappa)$ depending only on the lower bound κ of the Ricci curvature.

Let $\chi : \mathbb{R}_+ \rightarrow [0, 1]$ be a smooth function with $\chi(t) = 1$ if $t \leq \frac{9}{10}$ and $\chi(t) = 0$ if $t \geq 1$. The family of functions $\{\psi_v\}_{v \in \mathbf{V}}$ is defined by

$$\psi_v(x) = \chi\left(\frac{3}{10\epsilon}d(x, v)\right).$$

Note that ψ_v is of class C^∞ : for $x = v$, d is not smooth, but ψ_v is constant, and around the cut locus and after, where again d may not be smooth, ψ_v is constant, equal to 0. The function ψ_v is equal to 1 on $B(v, 3\epsilon)$ and to 0 outside of the ball $B(v, \frac{10\epsilon}{3})$.

As partition of unity, we choose $\{\phi_v\}_{v \in \mathbf{V}}$, with

$$\phi_v(x) = \frac{\psi_v(x)}{\sum_{w \in \mathbf{V}} \psi_w(x)}.$$

Control of the derivative of ϕ_v . We have

$$\nabla \phi_v(x) = \frac{\nabla \psi_v(x) \sum_{w \in \mathbf{V}} \psi_w(x) - \psi_v(x) \sum_{w \in \mathbf{V}} \nabla \psi_w(x)}{(\sum_{w \in \mathbf{V}} \psi_w(x))^2}$$

with

$$|\nabla \psi_v(x)| = (\max |\chi'|) \frac{3}{10\epsilon} |\nabla d| \leq \frac{3A_2}{10\epsilon}$$

where $A_2 = \max |\chi'|$. As $\sum_{w \in \mathbf{V}} \psi_w(x)^2 \geq 1$, it follows that

$$\begin{aligned} |\nabla \phi_v(x)| &\leq |\nabla \psi_v(x)| \sum_{w \in \mathbf{V}} \psi_w(x) + \psi_v(x) \sum_{w \in \mathbf{V}} |\nabla \psi_w(x)| \\ &\leq \frac{6A_1(\kappa)A_2}{10\epsilon}. \end{aligned} \quad \square$$

The *smoothing* $F = S^M f \in C^\infty(M)$ of a function $f : \mathbf{V} \rightarrow \mathbb{R}$ is defined by

$$F = \sum_{v \in \mathbf{V}} f(v)\phi_v.$$

The following two Lemmas are very simple, but fundamental since they will allow the use of some arguments from [3].

Lemma 25. *The functions $\phi_v|_\Sigma$ with $v \in \mathbf{V}_\Sigma$ form a partition of unity on the boundary Σ . This allows the definition of the smoothing operator*

$$S^\Sigma : \ell^2(\mathbf{V}_\Sigma) \rightarrow C^\infty(\Sigma),$$

which commutes with restriction:

$$(14) \quad S^\Sigma \left(f \Big|_{\mathbf{V}_\Sigma} \right) = (Sf) \Big|_\Sigma, \quad \forall f \in \ell^2(\mathbf{V}).$$

Proof. This follows directly from the fact that a function ϕ_v for which $v \in \mathbf{V}_I$ is supported away from Σ . □

Lemma 26. *Let f be a function on \mathbf{V} . If $v \in \mathbf{V}_\Sigma$, we have*

$$(\mathcal{D}Sf)(v) = (\mathcal{D}^\Sigma \mathcal{S}^\Sigma f)(v)$$

where \mathcal{D}^Σ and \mathcal{S}^Σ denote the discretization and the smoothing on Σ with the induced Riemannian metric.

If F is a differentiable function on M and $x \in \Sigma$, we have

$$(\mathcal{S}DF)(x) = (\mathcal{S}^\Sigma \mathcal{D}^\Sigma F)(x)$$

Proof. The proof is a direct consequence of the definition of the discretization we associate to M and of the way to discretize and to smooth.

The restriction to Σ of the smoothing of f takes only account of the values of f at point of \mathbf{V}_Σ and the restriction of the partition of unity we have defined is a partition of unity on Σ . So, for $x \in \Sigma$, we have $(Sf)(x) = (\mathcal{S}^\Sigma f)(x)$.

By definition, the same is true for the discretization because in order to discretize a function F at points of \mathbf{V}_Σ , we take the mean of F restricted to balls of Σ . So, for $v \in \mathbf{V}_\Sigma$, we have $(\mathcal{D}F)(v) = (\mathcal{D}^\Sigma F)(v)$.

From these two facts, we deduce immediately the lemma. □

Lemma 27. *There exist constants C_7, C_8, C_9 depending only on κ, r_0, n such that any $f \in \ell^2(\mathbf{V})$ satisfies*

$$\begin{aligned} \|Sf\|_M^2 &\leq C_7 \|f\|_{\mathbf{V}}^2, \\ \|\nabla Sf\|_M^2 &\leq C_8 q(f), \\ \|Sf\|_\Sigma^2 &\leq C_9 \|f\|_{\mathbf{V}_\Sigma}^2. \end{aligned}$$

Proof. The proof of the two first inequalities follows exactly the same arguments as that of paragraph ($\mathcal{S} : ii$) and ($\mathcal{S} : iii$) in [3, section VI.5.2].

For the last inequality, we use the observation of Lemma 25: using again paragraph ($\mathcal{S} : ii$) and ($\mathcal{S} : iii$) in [3, section VI.5.2], we get that

$$\|S^\Sigma(f|_{\mathbf{V}_\Sigma})\|_\Sigma \leq C_{10}\|f\|_{\mathbf{V}_\Sigma}.$$

But we have

$$S^\Sigma(f|_{\mathbf{V}_\Sigma}) = (Sf)|_\Sigma$$

and

$$\|fs\|_{\mathbf{V}_\Sigma} = \|f|_{\mathbf{V}_\Sigma}\|_{\mathbf{V}}. \quad \square$$

5. Proof of the main comparison inequality

The proof of Theorem 3 is broken down between Proposition 28 and Proposition 30.

Proposition 28. *There exists a constant C_{11} depending only on κ, r_0, n such that*

$$\sigma_k(\mathbf{\Gamma}, \mathbf{V}_\Sigma) \leq C_{11}k\sigma_k(M).$$

Let F_1, \dots, F_k be Steklov eigenfunctions on M . The space

$$E_k := \text{span}(DF_1, \dots, DF_k)$$

will be used in the variational characterization of $\sigma_k(\mathbf{\Gamma}, \mathbf{V}_\Sigma)$. One first needs to ensure that this space is k -dimensional. This is not true in general, but will hold for low energy functions thanks to the following quantitative injectivity property.

Lemma 29. *There is a constant $0 < C_{12} < 1/4$ with the usual dependance such that $\sigma_k(M)k < C_{12}$ implies*

$$(15) \quad \|DF\|_{\mathbf{V}_\Sigma} \geq \frac{C_7^{-1}}{2}\|F\|_\Sigma.$$

Proof of Lemma 29. Observe that Lemma 27 and the triangle inequality implies

$$(16) \quad \|DF\|_{\mathbf{V}_\Sigma} \geq C_7^{-1}\|SDF\|_\Sigma \geq C_7^{-1}(\|F\|_\Sigma - \|F - SDF\|_\Sigma).$$

The argument in [3, p. 183] shows that

$$\|F - SDF\|_{\Sigma} \leq C_{13}\|\nabla F\|_{\Sigma},$$

for some constant C_{13} . Together with inequality (16) and Corollary 20 this leads to

$$\begin{aligned} (17) \quad & \|DF\|_{\mathbf{V}_{\Sigma}} \geq C_7^{-1}(\|F\|_{\Sigma} - C_{13}\|\nabla^{\Sigma}F\|_{\Sigma}) \\ (18) \quad & = C_7^{-1}\|F\|_{\Sigma}\|(1 - \sqrt{k}C_{13}\frac{\|\nabla^{\Sigma}F\|_M}{\|F\|_{\Sigma}})\| \geq \|F\|_{\Sigma}C_7^{-1}\|(1 - C_{13}\sqrt{k}\sigma_k)\|. \end{aligned}$$

One can therefore take $C_{12} = \min((2C_{13})^{-2}, 1/4)$. □

Everything is now in place for the proof of Proposition 28

Proof of Proposition 28. In the situation that $\sigma_k(M)k < C_{12}$, it follows from Lemma 29 that the space E_k is k -dimensional and because $C_9 < 1/4$, Lemma 23 leads to

$$\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}) \leq \frac{q(DF)}{\|DF\|_{\Sigma}^2} \leq 4C_5C_7^2 \frac{\|\nabla F\|_M^2}{\|F\|_{\Sigma}^2} \leq 4C_5C_7^2\sigma_k(M).$$

On the other hand, if $\sigma_k(M)k \geq C_{12}$, then one has

$$\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}) \leq kC_{12}^{-1}\sigma_k(M)\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}).$$

Now, let ν be the maximal degree of a vertex $v \in \mathbf{V}_{\Sigma}$. Then for $k = |\mathbf{V}_{\Sigma}|$, one has $\sigma_k(\mathbf{\Gamma}) \leq \nu$, so that

$$\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}) \leq C_{12}^{-1}k\nu\sigma_k(M).$$

It follows that

$$\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}) \leq (4C_5C_7^2 + C_{12}^{-1}k\nu)\sigma_k(M) \leq (4C_5C_7^2 + C_{12}^{-1}\nu)k\sigma_k(M)$$

and one concludes by setting $C_{11} = (4C_5C_7^2 + C_{12}^{-1}\nu)$. □

We are now ready to move on to the second comparison inequality.

Proposition 30. *There exists a constant B such that the following holds for each $k \leq |\mathbf{V}_{\Sigma}|$:*

$$\sigma_k(M) \leq B\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}).$$

Let f_1, \dots, f_k be Steklov eigenfunctions on the graph $(\Gamma, \mathbf{V}_\Sigma)$. The space

$$E_k := \text{span}(Sf_1, \dots, Sf_k)$$

will be used in the variational characterization of $\sigma_k(M)$. As in the smooth case, we show that this space is k -dimensional.

Lemma 31. *There exists constants $C_{14}, C_{15} > 0$ such that $\sigma_k(\Gamma, \mathbf{V}_\Sigma) < C_{14}$ implies*

$$\|Sf\|_\Sigma^2 \geq C_{15} \|f\|_{\mathbf{V}_\Sigma}^2 \quad \forall f \in E_k$$

Proof of Lemma 31. Let $f = a_1 f_1 + \dots + a_k f_k$ and $F = Sf = a_1 Sf_1 + \dots + a_k Sf_k \in E_k$.

We denote by F^Σ and f^Σ the restriction of F to Σ and of f to \mathbf{V}_Σ , respectively. Because we are working in the closed manifold Σ , we can use point (8) from the paper [16], which states in our situation that

$$\|\mathcal{D}^\Sigma F^\Sigma\|_{\mathbf{V}_\Sigma} \leq C \|F^\Sigma\|_\Sigma,$$

for some constant C depending only on κ, n, r_0 . It follows that

$$\|F^\Sigma\|_\Sigma \geq \frac{1}{C} \|\mathcal{D}^\Sigma F^\Sigma\|_{\mathbf{V}_\Sigma} \geq \frac{1}{C} (\|f^\Sigma\|_{\mathbf{V}_\Sigma} - \|f - D^\Sigma S^\Sigma f^\Sigma\|_{\mathbf{V}_\Sigma}).$$

Working on Σ and using point (10) of [16]:

$$\|f - D^\Sigma S^\Sigma f^\Sigma\|_{\mathbf{V}_\Sigma} \leq C' q^\Sigma(f) \leq C' q(f),$$

for another constant C' . Here $q^\Sigma(f)$ is the energy of f restricted to the boundary $\mathbf{V}_\Sigma \subset V$. Because $f \in \text{Span}(f_1, \dots, f_k)$, we have

$$\frac{q(f)}{\|f^\Sigma\|_{\mathbf{V}_\Sigma}^2} \leq \sigma_k(\Gamma, \mathbf{V}_\Sigma).$$

Setting $C_{14} = \frac{1}{2C'}$, observe that the inequality $\sigma_k(\Gamma, \mathbf{V}_\Sigma) \leq C_{14}$ implies $\|F\|_\Sigma^2 \geq \frac{1}{2C'} \|f\|_{\mathbf{V}_\Sigma}^2$. The proof is complete, with $C_{15} = \frac{1}{2C}$. \square

We are now ready to finish the proof of the comparison inequality.

Proof of Proposition 30. In the situation that $\sigma_k(\Gamma, \mathbf{V}_\Gamma) < C_{14}$, it follows from Lemma 31 that the space $E_k = \text{span}(Sf_1, \dots, Sf_k)$ is k -dimensional. In combination with Lemma 27, this leads for each $F \in E_k$ to

$$(19) \quad \sigma_k(M) \leq \frac{\|\nabla F\|_M^2}{\|F\|_M^2} \leq C_8 C_{15}^{-1} \frac{q(f)}{\|f\|_{\mathbf{V}_\Sigma}} \leq C_8 C_{15}^{-1} \sigma_k(\Gamma, \mathbf{V}_\Sigma).$$

On the other hand, if $\sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\mathbf{\Gamma}}) \geq C_{14}$, then one has

$$\sigma_k(M) \leq C_{14}^{-1} \sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}) \sigma_k(M).$$

The proof is completed by giving a rough upper bound on $\sigma_k(M)$. Because $k \leq |\mathbf{V}_{\Sigma}|$ and $\mathbf{V}_{\Sigma} \subset \Sigma$ is ϵ -separated, there exists k disjoint balls $B_1, \dots, B_k \subset \Sigma$ of radius ϵ . Let f_j be a first eigenfunction corresponding to the first Dirichlet eigenvalue $\lambda_1(B_j)$. Each function f_j is extended to a function $\phi_j : B_j \times [0, 1] \subset M \rightarrow \mathbb{R}$ defined on this set by

$$\phi_j(x, t) = f_j(x)(1 - t),$$

and then extended by 0 elsewhere in M . It follows that

$$\frac{\|\nabla \phi_j\|_M^2}{\|f_j\|_{\Sigma}^2} \leq \frac{\|\nabla^{\Sigma} f_j\|_{\Sigma}^2 + \|f_j\|_{\Sigma}^2}{\|f_j\|_{\Sigma}^2} = \lambda_1(B_j) + 1.$$

Now, Cheng's theorem [4] states that $\lambda_1(B_j) \leq C$, for some constant $C = C(n, \kappa, \epsilon)$. As the functions ϕ_j are compactly supported in the disjoint domains $B_j \times [0, 1]$, the min-max characterization of σ_k implies that $\sigma_k(M) \leq C + 1$. Hence

$$\sigma_k(M) \leq (C + 1) C_{14}^{-1} \sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}).$$

Together with Inequality (19), this implies that

$$\sigma_k(M) \leq B \sigma_k(\mathbf{\Gamma}, \mathbf{V}_{\Sigma}),$$

for $B = \max\{(C + 1)C_{14}^{-1}, C_8 C_{15}^{-1}\}$. □

6. Applications

In this section, we present the three applications which were mentioned in the introduction. They are similar to each others. The general strategy is as follows:

- Construct a sequence of graphs $\{\mathbf{G}_l\}_{l \in \mathbb{N}}$ with some desired spectral property;
- Construct a sequence of surfaces $\{\Omega_l\}_{l \in \mathbb{N}}$ associated to \mathbf{G}_l ;
- Obtain an ϵ -discretization $(\mathbf{\Gamma}_l, \mathbf{B}_l)$ of Ω_l ;
- Prove that $\mathbf{\Gamma}_l$ and \mathbf{G}_l are roughly isometric;
- Conclude using Theorem 3 and Proposition 16.

6.1. Spectral stability under quasi-isometries

In order to compare the Steklov spectrum of our manifolds with the discrete Steklov spectrum of a discretization, it was necessary to suppose that a neighborhood of the boundary Σ is isometric to the product $\Sigma \times [0, 1[$. This is a strong hypothesis, which however can be relaxed to having a quasi-isometry with the product $\Sigma \times [0, 1[$, with uniform control on constants.

Proposition 32. *Let M^n be a compact manifold with smooth boundary Σ and let g_1, g_2 be two Riemannian metrics on M . Suppose the existence of a constant $A \geq 1$ such that for each $x \in M$ and $0 \neq v \in T_x M$ we have*

$$\frac{1}{A} \leq \frac{g_1(x)(v, v)}{g_2(x)(v, v)} \leq A.$$

Then the Steklov spectrum with respect to g_1 and g_2 satisfies the

$$\frac{1}{A^{2n+1}} \leq \frac{\sigma_k(M, g_1)}{\sigma_k(M, g_2)} \leq A^{2n+1},$$

and for the normalized Steklov spectrum, we have

$$\frac{1}{A^{2n+2}} \leq \frac{\tilde{\sigma}_k(M, g_1)}{\tilde{\sigma}_k(M, g_2)} \leq A^{2n+2}.$$

Proof. This follows directly from the variational characterization of the Steklov eigenvalues σ_k . The earliest paper where this principle was used extensively is that of Dodziuk [10], where the spectrum of the Laplace-Beltrami operator acting on forms is studied. □

6.2. Planar domains with large eigenvalues

The proof of Theorem 4 is based on gluing copies of three different planar building blocks.

Definition 33. A *planar fundamental piece* is a domain $D \subset \mathbb{R}^2$ bounded by smooth successive arcs $\gamma_1, \Gamma_1, \dots, \gamma_{n-1}, \Gamma_{n-1}, \gamma_n = \gamma_1$ meeting orthogonally (see Figure 2) such that the arc Γ_i is a straight segment of length 1 and in a neighbourhood of each Γ_i , D is isometric to $\Gamma_i \times [0, 1]$ (a square of side 1).

Let us denote by $N(\delta)$ the δ -neighbourhood of $\gamma := \cup_i \gamma_i$, that is

$$N(\delta) = \{x \in D : d(x, \gamma) \leq \delta\}.$$

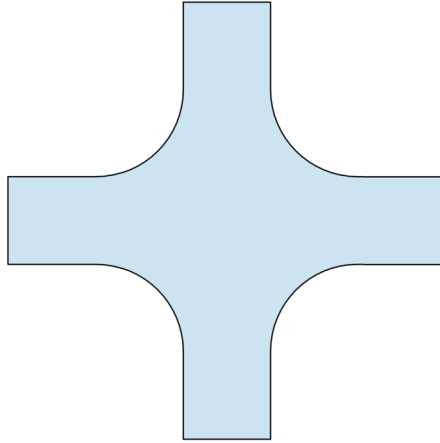


Figure 2: A flat cross domain.

Lemma 34. *Let $D \subset \mathbb{R}^2$ be a flat fundamental piece. Let g_0 be the Euclidean metric. There exist numbers $K, \delta > 0$ and a Riemannian metric g on D such that*

- *($N(\delta), g$) is isometric to $\gamma \times [0, 1]$;*
- *The Riemannian metrics g_0 and g are quasi-isometric with constant K ;*
- *The Riemannian metrics g and g_0 are homothetic on $D \setminus N(3\delta)$ and on the square ends of D .*

Proof. Let s be the arclength parameter along γ . Using the distance t to γ as a second parameter leads to the Fermi parallel coordinates, which are defined in a neighborhood $\mathcal{O} \subset D$ of γ . In this coordinate system, the Euclidean metric is expressed by

$$g_0(s, t) = \phi(s, t)ds^2 + dt^2,$$

where the smooth function ϕ satisfy $\phi(s, 0) = 1$ (Gauss Lemma). Let $\delta > 0$ be such that the restriction of ϕ to $N(3\delta)$ is smaller than 2. On this neighborhood the Euclidean metric g_0 is quasi-isometric with ratio 2 to the product metric g' , which in Fermi coordinates is expressed by $g'(s, t) = ds^2 + dt^2$. Let $\chi : [0, 3\delta] \rightarrow [0, 1]$ be a smooth increasing function taking the value 0 on $[0, \delta]$ and the value 1 on $[2\delta, 3\delta]$. Using the Fermi coordinates again, define a new metric on $N(3\delta)$ by

$$g_\delta(s, t) = \chi(t)g_0(s, t) + (1 - \chi(t))g'(s, t).$$

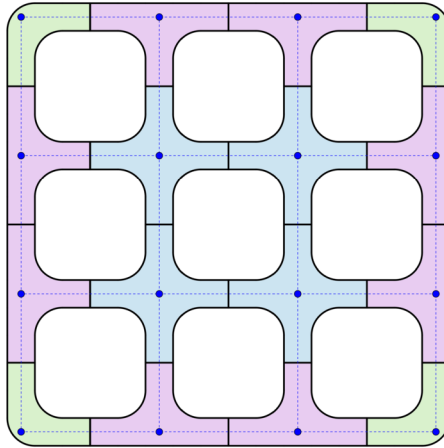


Figure 3: A puzzle domain with three fundamental pieces.

On $N(\delta)$ this metric coincide with the product metric g' , while on $N(3\delta) \setminus N(2\delta)$ it coincides with the euclidean metric g_0 . It can therefore be extended to a metric (still denoted g_δ) which is defined on the full domain D .

Note that on the square ends of the fundamental piece D , one has $g_\delta = g' = g_0$. In order to obtain a cylindrical boundary of length one, define the metric

$$g = \frac{1}{\delta^2} g_\delta.$$

This metric satisfy all the required condition. Indeed it is 2 quasi-isometric to $\frac{1}{\delta^2} g_0$ by construction, and equal to $\frac{1}{\delta^2} g_0$ on $D \setminus N(3\delta)$ and on the square ends of D . \square

Let D_1, \dots, D_m be planar fundamental pieces. Let $\Omega \subset \mathbb{R}^2$ be a planar domain of the form

$$\Omega = \text{interior} \overline{\dot{\bigcup}_{i=1}^m \Omega_i}$$

where each Ω_i is one of the fundamental piece, with the external boundary γ of each piece included in the boundary $\partial\Omega$. We call such a domain a *puzzle domain* (See Figure 3).

Corollary 35. *Let D_1, \dots, D_m be planar fundamental pieces. There exist constant $K, \kappa, r_0 > 0$ such that any puzzle domain Ω based on D_1, \dots, D_m is K -quasi-isometric to a Riemannian surface (Ω', g) in the class $\mathcal{M}(\kappa, r_0, 2)$.*

Proof. This follows from Lemma 34 and the fact that a finite number of fundamental pieces are used. \square

Remark 36. One way to think of Ω_l intuitively is that it is a “thickening” of the graph \mathbf{G}_l (perceived as a subset of the usual lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$), that is by considering a tubular neighbourhood of the corresponding lattice graph. Nevertheless, it is easier to analyse the situation precisely when we build Ω_l by gluing fundamental pieces.

We are now ready to proceed with the proof of our first application.

Proof of Theorem 4. For each $l \in \mathbb{N}$, consider the graph $\mathbf{G}_l = (\mathbf{V}_l, \mathbf{E}_l)$ defined through the usual lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ as follows:

$$\mathbf{V}_l = \{(a, b) \in \mathbb{Z}^2 \subset \mathbb{R}^2 : 0 \leq a, b \leq l\}$$

and declare the vertices (a_1, b_1) and (a_2, b_2) to be adjacent if they are related in the lattice \mathbb{Z}^2 . In order to obtain a graph with boundary, a distinguished subset of vertices has to be chosen. For the present application, chose $\mathbf{B}_l = \mathbf{V}_l$. In this very special situation where the boundary coincide with the full graph, the spectrum of the discrete Steklov problem on coincide with the spectrum of the combinatorial Laplacian of the graph (See Definition 13 and Remark 14).

Vertices of \mathbf{G}_l are either of degree 2,3 or 4. To each of these degree, their correspond one planar fundamental piece: D_2, D_3 and D_4 , as illustrated in Figure 3. For each $l \in \mathbb{N}$, these are used to construct a puzzle domain $\Omega_l \subset \mathbb{R}^2$ as follows:

- To each vertex $v \in \mathbf{V}_l$, a congruent copy Ω_v of the fundamental piece $D_{\deg v}$ is attached;
- If $v, w \in \mathbf{V}$ are adjacent in the graph $\mathbf{G}_l = (\mathbf{V}_l, \mathbf{E}_l)$ then Ω_v and Ω_w are placed so as to touch along a common straight Γ -segment of their boundary.

It follows from Corollary 35 that there exists a constant $K > 0$ independent of $l \in \mathbb{N}$ such that each domain $\Omega_l \subset \mathbb{R}^2$ is quasi-isometric (in the Riemannian sense) with constant K to a Riemannian surface (M_l, g_l) in the class $\mathcal{M}(\kappa, r_0, 2)$ for some constants $\kappa, r_0 > 0$ which are also independent of l . Moreover, it is clear from the construction that (M_l, g) is roughly isometric to the graph \mathbf{G}_l , with uniform constants a, b, τ .

Let $\epsilon \in (0, r_0/4)$ and consider an ϵ -discretization $(\Gamma(M_l), \mathbf{V}_\Sigma(M_l))$ of the surface M_l . This graph is roughly-isometric to M_l by Corollary 12.

We are now in position to conclude. It follows from Proposition 16 that there exists a constant $A_1 \geq 1$ not depending on l such that

$$\frac{1}{A_1} \leq \frac{\sigma_2(\mathbf{G}_l)}{\sigma_2(\Gamma_l)} \leq A_1.$$

From our main comparison result (Theorem 3) we get a constant $A_2 \geq 1$ not depending on l such that

$$\frac{1}{A_2} \leq \frac{\sigma_2(\Omega_l)}{\sigma_2(\Gamma_l)} \leq A_2.$$

As for the graph \mathbf{G}_l we have $\mathbf{B}_l = \mathbf{V}_l$, we have $\sigma_2(\mathbf{G}_l) = \lambda_2(\mathbf{G}_l)$. It is well known that the first non-zero eigenvalue of the the discrete Laplacian on this so called lattice graph is $8 \sin^2(\pi/l)$ (see [17]). In particular, this implies the existence of a constant A_3 such that

$$\sigma_2(\mathbf{G}_l) = \lambda_2(\mathbf{G}_l) \geq \frac{A_3}{l^2}$$

The conclusion is that there exists a constant C depending on A, A_1, A_2, A_3 but not depending on l such that

$$\sigma_2(\Omega_l) \geq \frac{C}{l^2}.$$

Now it is obvious by construction that there exists $B_1, B_2 > 0$ with

$$B_1 l^2 \leq \text{Vol}(\Omega_l) \leq B_2 l^2, \quad B_1 l^2 \leq L(\Sigma_l) \leq B_2 l^2.$$

This implies first that

$$\sigma_2(\Omega_l)L(\Sigma_l) \geq B_1 C.$$

and that

$$I(\Omega_l) = \frac{L(\Sigma_l)}{\text{Vol}(\Omega_l)^{1/2}} \geq \frac{B_1}{\sqrt{B_2}} l.$$

So that $I(\Omega_l) \rightarrow \infty$ as $l \rightarrow \infty$ and $\sigma_2(\Omega_l)L(\Sigma_l) \geq B_1 C$. □

6.3. Flat surfaces

The proof of the second application is similar.

Proof of Theorem 5. It follows from the classical probabilistic method that there exist a an expanding family $\{\mathbf{G}_l\}_{l \in \mathbb{N}}$ of 4-regular graphs such that the number of vertices $|\mathbf{V}(\mathbf{G}_l)| = l$ (See [19]). In particular, $\lim_{l \rightarrow \infty} |\mathbf{V}(\mathbf{G}_l)| = +\infty$ and $\lambda_1(\mathbf{G}_l)$ is uniformly bounded below by a positive constant c . In order to obtain a surface Ω_l from the graph \mathbf{G}_l , a “flat cross domain” is used (see Figure 2). This domain is a fundamental planar piece in the sense of

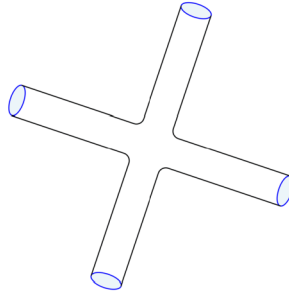


Figure 4: The fundamental piece M_0 for $k = 4$.

Definition 33. The construction of Ω_l is similar to that used for the previous application, but it is simpler since only one building block is required: a flat cross Ω_v is associated to each vertex $v \in \mathbf{V}(\mathbf{G}_l)$, and the graph structure is used to glue them together. It follows exactly as in Corollary 35 that the surface Ω_l is K -quasi-isometric to a surface M_l in some class $\mathcal{M}(\kappa, r_0, 2)$ for some constants K, κ, r_0 which are independent of $l \in \mathbb{N}$.

Given $\epsilon \in (0, r_0/4)$, let $(\mathbf{\Gamma}_{M_l}, \mathbf{V}_{\Sigma_l})$ be an ϵ -discretization of M_l . Lemma 12 says that M_l is roughly-isometric to $(\mathbf{\Gamma}_{M_l}, \mathbf{V}_{\Sigma_l})$ for some constants a, b, τ independent of $l \in \mathbb{N}$.

The end of the proof is exactly as that of the previous Theorem 4. \square

6.4. Surfaces with large eigenvalues and connected boundary

The basic idea behind the proof of Theorem 7 is similar to the two previous examples and to the construction of [8], but some care has to be taken to guarantee the presence of only one boundary component. Intuitively, we proceed as follows: consider a family \mathbf{G}_l of expander graphs of degree 4 and genus $N + 1$ as in the previous paragraph, and use it to construct a sequence S_l of closed surfaces, using exactly one building block of the form given in Figure 4. Consider a *maximal tree* on the graph $\mathbf{G}_l = (\mathbf{V}_l, \mathbf{E}_l)$. One would like to embed this tree in the surface S_l and remove a neighbourhood from O_l the closed surface S_l . Because this neighbourhood is a topological ball in S_l , the surface $\Omega_l := S_l \setminus O_l$ has exactly one boundary component Σ_l , which is spread out in the surface: it visits each and every fundamental piece that was used to construct S_l , and its length is proportional to l . Nevertheless, it is not easy to embed the graph \mathbf{G}_l in the closed surface S_l while controlling a quasi-isometry class for O_l . To get around this difficulty, four different fundamental pieces will be used (See Figure 5). These fundamental pieces already carry

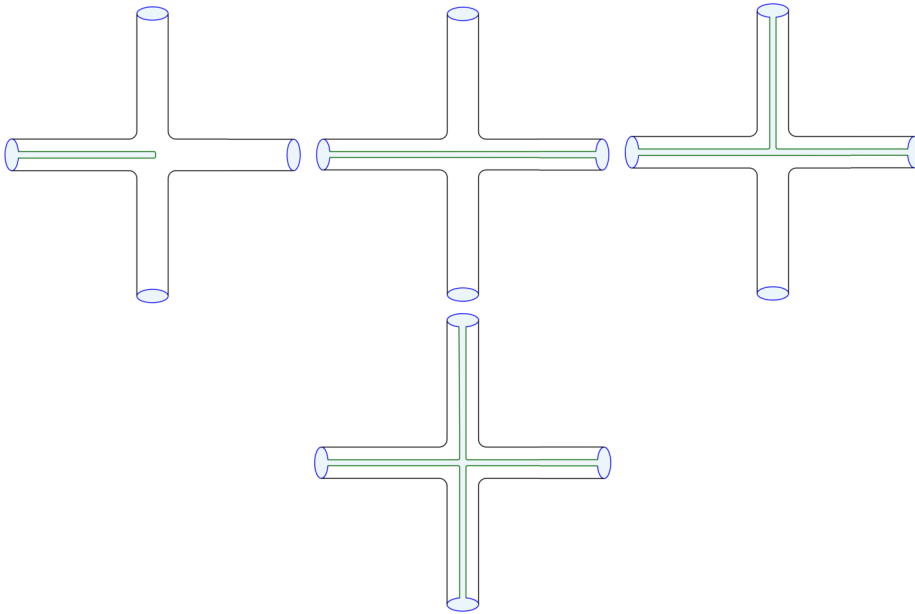


Figure 5: Carved fundamental pieces.

part of the boundary of Ω_l , which is built using the maximal tree and the five pieces which correspond to the degree of each vertex in the tree. What we have gained through this construction is that each fundamental piece has a fixed geometry, and this implies the existence of a quasi-isometry from a neighbourhood of the boundary Σ_l to $\Sigma_l \times [0, 1)$ with constant not depending on N . The rest of the proof is essentially the same as those of the two previous examples.

Proof of Theorem 7. Let M_0 be the smooth surface of genus 0 with 4 boundary components illustrated in Figure 4. Each boundary component B_1, \dots, B_4 has a neighbourhood which is isometric to the cylinder $[0, 1] \times S^1$, with boundary corresponding to $\{0\} \times S^1$. Moreover, the surface M_0 is symmetric with respect to rotations of 90 degrees. From M_0 we build 4 new surfaces by carving out smooth curves as illustrated in figure 5. These surfaces are such that they match together seamlessly, so that they can be used like the basic building blocks of a puzzle. Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a finite connected regular graph of degree 4. Let \mathbf{T} be a maximal tree in \mathbf{G} . To each vertex $v \in \mathbf{V}$, a copy M_v of one of the four building blocks is associated, according to the degree of v in the maximal tree T . These blocks are then glued together to obtain a

surface $\Omega_{\mathbf{G},\mathbf{T}}$, with one boundary component $\Sigma_{\mathbf{G},\mathbf{T}}$. It is clear from this construction that each $\Omega_{\mathbf{G},\mathbf{T}}$ is K -quasi-isometric to a surface $M_{\mathbf{G},\mathbf{T}}$ in the class $\mathcal{M}(\kappa, r_0, 2)$ for some constants K, κ, r_0 which are independent of the graph \mathbf{G} and maximal tree \mathbf{T} . See Lemma 34 for details in the planar case.

Exactly as in the previous two applications, it follows from Proposition 12 that any ϵ -discretization $(\Gamma_{M_{\mathbf{G}}}, \mathbf{V}_{\Sigma_{\mathbf{G},\mathbf{T}}})$ of $M_{\mathbf{G},\mathbf{T}}$ is roughly isometric to $M_{\mathbf{G},\mathbf{T}}$, with constants a, b, τ independent of the graph \mathbf{G} and of the maximal tree \mathbf{T} . Moreover, the discretization is roughly isometric to the original graph (\mathbf{G}, \mathbf{T}) .

The rest of the proof is also exactly as that of the previous Theorem 5, using an expander sequence of graphs \mathbf{G}_l and applying the spectral comparison Proposition 16 and Theorem 3. \square

Remark 37. In all construction presented above, a choice is involved when we glue various building blocks together: we did not specify which boundary component of one fundamental piece is glued to the other. In the first construction (planar domains), there was no ambiguity, since we are using congruent copies of the fundamental pieces in the plane. For the last two applications, the choice involved does not affect the end result, despite the surface not being uniquely defined by the procedure. One way to resolve this non-uniqueness is to label the edges emanating from each vertex and label the gluing boundaries. See [8] for more details.

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References

- [1] R. BROOKS. The first eigenvalue in a tower of coverings. *Bull. Amer. Math. Soc. (N.S.)*, **13**(2):137–140, 1985. [MR0799796](#)
- [2] P. BUSER. On the bipartition of graphs. *Discrete Appl. Math.*, **9**(1):105–109, 1984. [MR0754431](#)
- [3] I. CHAVEL. *Isoperimetric inequalities*, volume 145 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. Differential geometric and analytic perspectives. [MR1849187](#)

- [4] S. Y. CHENG. Eigenvalue comparison theorems and its geometric applications. *Math. Z.*, **143**(3):289–297, 1975. [MR0378001](#)
- [5] D. CIANCI and A. GIROUARD. Large spectral gaps for Steklov eigenvalues under volume constraints and under localized conformal deformations. *Ann. Global Anal. Geom.*, **54**(4):529–539, 2018.. [MR3878841](#)
- [6] B. COLBOIS, A. EL SOUFI, and A. GIROUARD. Isoperimetric control of the Steklov spectrum. *J. Funct. Anal.*, **261**(5):1384–1399, 2011. [MR2807105](#)
- [7] B. COLBOIS, A. EL SOUFI, and A. GIROUARD. Compact manifolds with fixed boundary and large Steklov eigenvalues. *Proc. Amer. Math. Soc.*, **147**(9):3813–3827, 2019. [MR3993774](#)
- [8] B. COLBOIS and A. GIROUARD. The spectral gap of graphs and Steklov eigenvalues on surfaces. *Electron. Res. Announc. Math. Sci.*, **21**:19–27, 2014. [MR3181630](#)
- [9] B. COLBOIS and A.-M. MATEI. On the optimality of J. Cheeger and P. Buser inequalities. *Differential Geom. Appl.*, **19**(3):281–293, 2003. [MR2013096](#)
- [10] J. DODZIUK. Eigenvalues of the Laplacian on forms. *Proc. Amer. Math. Soc.*, **85**(3):437–443, 1982. [MR0656119](#)
- [11] A. GIROUARD and I. POLTEROVICH. Spectral geometry of the Steklov problem. *J. Spectr. Theory*, **7**(2):321–359, 2017. [MR3662010](#)
- [12] B. HUA, Y. HUANG, and Z. WANG. First eigenvalue estimates of Dirichlet-to-Neumann operators on graphs. *Calc. Var. Partial Differential Equations*, **56**(6):Art. 178, 21, 2017. [MR3722072](#)
- [13] M. KANAI. Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds. *J. Math. Soc. Japan*, **37**(3):391–413, 1985. [MR0792983](#)
- [14] M. KANAI. Rough isometries and the parabolicity of Riemannian manifolds. *J. Math. Soc. Japan*, **38**(2):227–238, 1986. [MR0833199](#)
- [15] G. KOKAREV. Variational aspects of Laplace eigenvalues on Riemannian surfaces. *Adv. Math.*, **258**:191–239, 2014. [MR3190427](#)
- [16] T. MANTUANO. Discretization of compact Riemannian manifolds applied to the spectrum of Laplacian. *Ann. Global Anal. Geom.*, **27**(1):33–46, 2005. [MR2130531](#)

- [17] B. MOHAR. The Laplacian spectrum of graphs. In *Graph theory, combinatorics, and applications. Vol. 2* (Kalamazoo, MI, 1988), Wiley-Intersci. Publ., pages 871–898. Wiley, New York, 1991. [MR1170831](#)
- [18] H. PERRIN. Lower bounds for the first eigenvalue of the Steklov problem on graphs. *Calc. Var. Partial Differential Equations*, **58**(2):Art. 67, 12, 2019. [MR3925558](#)
- [19] M. S. PINSKER. On the complexity of a concentrator. In: *7th International Teletraffic Conference*, pages 318/1–318/4, 1973.
- [20] O. POST. *Spectral analysis on graph-like spaces*, volume 2039 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012. [MR2934267](#)

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